# Torsion theories and Auslander-Reiten sequences

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*The Road goes ever on and on*
*Down from the door where it began.*
*Now far ahead the Road has gone,*
*And I must follow, if I can,*
*Pursuing it with eager feet,*
*Until it joins some larger way*
*Where many paths and errands meet.*
*And whither then? I cannot say.*

*J. R. R. Tolkien*

*The Lord of the Rings*

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Abstract

Chapter 0 gives a gentle background to the thesis. It begins with some general notions and concepts from homological algebra. For example, not only are the notions of *universal property* and of *duality* central to the flavour of the subject, they are also suggestive in understanding mathematics at another depth. In category theory, *objects* and *morphisms* are the two main elements in a category, and notions such as *kernels* and *cokernels* are defined in terms of objects together with morphisms. In accordance with it, the morphisms are given a very subtle significance within a category. The chapter then introduces the notion of a *triangulated category*, where due to the lack of uniqueness of certain morphisms described in the axioms, is allowed to be far from an *abelian category*. A few examples of triangulated categories are given, the *homotopy category*, the *derived category* and certain *stable categories*. The chapter ends with a little description of an *Auslander-Reiten quiver* defined on a *Krull-Schmidt category*, as well as the notions of *Serre functor* and of *Auslander-Reiten triangles* in subcategories. The introduction chapter selects lemmas and theorems not only to be referenced in later chapters, but also those which can induce good intuition on the reader, for example, in their capacity of being analogues to each other, in the interplay between them and in their different suggestiveness in approximating or generalizing concepts in different ways and directions.

Chapter 1 studies *torsion pairs* in abelian categories and *torsion theories* with *torsion theory triangles* in triangulated categories. It then gives a necessary and sufficient condition for the existence of certain *adjoint functors* in triangulated categories. Intuitively, they are all different expressions of subcategories approximating their ambient categories. The chapter goes on to introduce two special cases of torsion theories, namely *t-structures* and *split torsion theories*, and finishes with a characterization of a split torsion theory and a classification of split torsion theories in a chosen derived category.

There is a very close and subtle relationship between the existence of torsion theory triangles and the existence of Auslander-Reiten triangles. Chapter 2 studies the existence of *Auslander-Reiten sequences* in subcategories of $\mathrm{mod}(\Lambda)$, where $\Lambda$ is a finite-dimensional $k$-algebra over the field $k$, based on the theory of the existence of Auslander-Reiten triangles in subcategories developed by Jørgensen. The existence theorems strengthen the results by Auslander and Smalø and by Kleiner.

Chapter 3 sees that quotients of certain triangulated categories are triangulated and are in addition derived categories, appealing to a theorem which is a slight variation of the results by Rickard and Keller. In this chapter, the Auslander-Reiten triangles play a predominant role in reflecting the tri-

angulation structure of a triangulated category, and the Auslander-Reiten triangles can be read off from the Auslander-Reiten quiver.

The *cluster category $\mathcal{D}$ of Dynkin type* $A_\infty$ was introduced by Jørgensen. One of its several definitions, which is completely analogous to the definition of the cluster category of type $A_n$, motivates us to say that $\mathcal{D}$ is a cluster category of type $A_\infty$. In the result by Holm and Jørgensen, the cluster tilting subcategories of $\mathcal{D}$ were shown to be in bijection with certain maximal sets of non-crossing arcs connecting non-neighbouring integers. Chapter 4 generalizes the result by giving a bijection between torsion theories in $\mathcal{D}$ and certain configurations of arcs connecting non-neighbouring integers. Finally, a few examples, characterizing all t-structures and co-t-structures in $\mathcal{D}$, are given.

# Contents

# Chapter 0

# Introduction

The chapter serves as a prelude to the thesis. The definitions, lemmas and theorems are not introduced once and for all, but they will be reincarnated again and again in the subsequent chapters, so that each time they appear with a unique signification and acquire a life of their own, depending on the context where they manifest themselves, thus altogether a unique phenomenon each time. The totality of the thesis can only be attained when all the chapters are put together like an orchestra, serving a greater whole. Let us now begin.

## 0.1 Homological algebra

In this section, the reader is suggested to refer to [18] for the background.

**Definition 0.1.1.** A *category* $\mathfrak{C}$ consists of

(i) a class of *objects* $X, Y, Z, \ldots$,

(ii) for any pair of objects $X$, $Y$ in $\mathfrak{C}$, a set $\mathrm{Hom}_{\mathfrak{C}}(X, Y)$, or simply $(X, Y)$, when the underlying category is understood, of *morphisms* from $X$ to $Y$,

(iii) for objects $X$, $Y$, $Z$ in $\mathfrak{C}$, a *composition relation* $(X, Y) \times (Y, Z) \to (X, Z)$.

The morphism $f$ from $X$ to $Y$ is written $f : X \to Y$. The set $(X, Y) \times (Y, Z)$ consists of pairs $(f, g)$ where $f : X \to Y$, $g : Y \to Z$ and the composition of $f$ and $g$ is written $gf$. In addition, $\mathfrak{C}$ is to satisfy the following axioms.

(i) the sets $(X_1, Y_1)$ and $(X_2, Y_2)$ are disjoint unless $X_1 = X_2$ and $Y_1 = Y_2$,

(ii) given the morphisms $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$, the relation $h(gf) = (hg)f$,

(iii) for each object $X$ there is an *identity morphism* $1_X : X \to X$, sometimes written $\mathrm{id}_X$, such that for any $f : X \to Y$ and $g : W \to X$, we have $f1_X = f$ and $1_X g = g$. When the object $X$ is understood, the identity morphism is simply written 1 or id.

Given a category $\mathfrak{C}$, let $\mathfrak{C}^{op}$ be the *opposite category*. The objects of $\mathfrak{C}^{op}$ and $\mathfrak{C}$ are the same with $\mathrm{Hom}_{\mathfrak{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathfrak{C}}(Y, X)$. The composition relation in $\mathfrak{C}^{op}$ follows naturally from $\mathfrak{C}$, i.e. given morphisms $f^{op}$ in $\mathrm{Hom}_{\mathfrak{C}^{op}}(Z, Y)$, $g^{op}$ in $\mathrm{Hom}_{\mathfrak{C}^{op}}(Y, X)$, with the corresponding morphisms $f$ in $\mathrm{Hom}_{\mathfrak{C}}(Y, Z)$, $g$ in $\mathrm{Hom}_{\mathfrak{C}}(X, Y)$, the composition $g^{op} f^{op}$ in $\mathrm{Hom}_{\mathfrak{C}^{op}}(Z, X)$ takes the value $fg$ in $\mathrm{Hom}_{\mathfrak{C}}(X, Z)$.

A morphism $f : X \to Y$ in $\mathfrak{C}$ is an *isomorphism* if and only if there is a morphism $g : Y \to X$ in $\mathfrak{C}$ such that $gf = 1_X$ and $fg = 1_Y$, and we write $g = f^{-1}$ and $X \cong Y$. A morphism $f : X \to X$ in $\mathfrak{C}$ is an *endomorphism* and an isomorphism $f : X \to X$ in $\mathfrak{C}$ is an *automorphism*. A *zero object* 0 in $\mathfrak{C}$ is one such that, for any object $X$ in $\mathfrak{C}$, the sets $(X, 0)$ and $(0, X)$ both consist of precisely one element. Any zero objects in $\mathfrak{C}$, if they exist, are isomorphic.

Any class of morphisms $\mathfrak{M}$ together with a composition relation is sufficient to define a category $\mathfrak{C}$. Then a subclass of morphisms $\mathfrak{M}_0$ of $\mathfrak{M}$ is to define a *subcategory* $\mathfrak{C}_0$ of $\mathfrak{C}$ if for every $f, g$ in $\mathfrak{M}_0$, if $gf$ is defined in $\mathfrak{M}$, then $gf$ is in $\mathfrak{M}_0$, and given any identity morphism $e$ in $\mathfrak{M}$ and any morphism $f$ in $\mathfrak{M}_0$, and if either $ef$ or $fe$ is defined in $\mathfrak{M}$, then $e$ is in $\mathfrak{M}_0$. In particular, a *full subcategory* $\mathfrak{C}_0$ of a category $\mathfrak{C}$ satisfies $\mathrm{Hom}_{\mathfrak{C}_0}(X, Y) = \mathrm{Hom}_{\mathfrak{C}}(X, Y)$ for any objects $X, Y$ of $\mathfrak{C}_0$. Therefore to give a full subcategory $\mathfrak{C}_0$ of $\mathfrak{C}$, it suffices to specify its objects. This notion illustrates the importance of morphisms in a category. For example, if a morphism is an isomorphism in $\mathfrak{C}$, then it is also an isomorphism in the full subcategory $\mathfrak{C}_0$ of $\mathfrak{C}$.

A *covariant (resp. contravariant) functor* $F : \mathfrak{C} \to \mathfrak{D}$ assigns an object $FX$ in $\mathfrak{D}$ for each object $X$ in $\mathfrak{C}$, and a morphism $Ff$ in $\mathrm{Hom}_{\mathfrak{D}}(FX, FY)$ (resp. $\mathrm{Hom}_{\mathfrak{D}}(FY, FX)$) for each morphism $f$ in $\mathrm{Hom}_{\mathfrak{C}}(X, Y)$, such that $F(fg) = (Ff)(Fg)$ (resp. $F(fg) = (Fg)(Ff)$) and $F(1_A) = 1_{FA}$ for each object $A$ in $\mathfrak{C}$. A (covariant) functor $F : \mathfrak{C} \to \mathfrak{D}$ is *full* if given objects $X, Y$ in $\mathfrak{C}$, the map $F_{XY} : \mathrm{Hom}_{\mathfrak{C}}(X, Y) \to \mathrm{Hom}_{\mathfrak{D}}(FX, FY)$ is onto, and is *faithful* if given objects $X, Y$ in $\mathfrak{C}$, the map $F_{XY} : \mathrm{Hom}_{\mathfrak{C}}(X, Y) \to \mathrm{Hom}_{\mathfrak{D}}(FX, FY)$ is injective. Finally, $F$ is a *full embedding* if $F$ is full, faithful and injective on objects.

Let $\Lambda$ be a finite-dimensional $k$-algebra over the field $k$. A *left module over* $\Lambda$, or a $\Lambda$-*left-module*, is an abelian group $A$ with a *scalar multiplication* $\cdot$

such that given $\lambda$ in $\Lambda$ and $a$ in $A$, the element $\lambda \cdot a$ is in $A$, and given $a, a_1, a_2$ in $A$, $\lambda, \lambda_1, \lambda_2$ in $\Lambda$, the following axioms are satisfied.

(i) $(\lambda_1 + \lambda_2) \cdot a = \lambda_1 \cdot a + \lambda_2 \cdot a$,

(ii) $(\lambda_1 \lambda_2) \cdot a = \lambda_1 \cdot (\lambda_2 \cdot a)$,

(iii) $1_\Lambda \cdot a = a$, where $1_\Lambda \neq 0$ is the unity element of $\Lambda$,

(iv) $\lambda \cdot (a_1 + a_2) = \lambda \cdot a_1 + \lambda \cdot a_2$.

A $\Lambda$-*left-submodule* is a subgroup $A'$ of $A$ with $\lambda \cdot a'$ in $A'$ for all $\lambda$ in $\Lambda$ and $a'$ in $A'$. A $\Lambda$-*right-(sub)module* is defined similarly. A $\Lambda$-*bi-module* is a module which is simultaneously a $\Lambda$-left-module and a $\Lambda$-right-module.

Let $A$ and $B$ be $\Lambda$-left-modules. A map $f : A \to B$ is a *homomorphism* if given $u, v$ in $A$ and $\lambda$ in $\Lambda$, we have $f(u+v) = f(u)+f(v)$ and $f(\lambda u) = \lambda f(u)$. Let $S$ be a subset of $A$, and let $A_0$ be the set of all elements $a$ in $A$ of the form $a = \Sigma \lambda_s s$ where $\lambda_s$ is in $\Lambda$ and $\lambda_s \neq 0$ for only a finite number of elements $s$ in $S$. It is trivial that $A_0$ is a submodule of $A$. If the submodule $A_0$ is equal to $A$, then $S$ is a set of *generators* of $A$. If $A$ admits a finite set of generators, it is said to be *finitely generated.* A set $S$ of generators of $A$ is said to be a *basis* of $A$ if every element $a$ of $A$ can be expressed uniquely in the form $a = \Sigma \lambda_s s$ where $\lambda_s$ is in $\Lambda$ and $\lambda_s \neq 0$ for only a finite number of elements $s$ in $S$. If $S$ is a basis of $A$, then $A$ is said to be *free on the set* $S$. If $A$ is free on some subset, then $A$ is said to be *free.*

*Example* 0.1.2. The category $\mathrm{Mod}(\Lambda)$ is a category with $\Lambda$-left-modules as objects and homomorphisms as morphisms. The category $\mathrm{mod}(\Lambda)$ is a category with finitely generated $\Lambda$-left-modules as objects and homomorphisms as morphisms.

Below are a few definitions in an *additive category.* They are deeply involved in an *abelian category.* Both notions will be introduced in a little while. A *kernel* of a morphism $\varphi : A \to B$, denoted by $\ker\varphi$, is a morphism $\mu : K \to A$ such that $\mu\varphi = 0$ and if $\varphi\psi = 0$ for $\psi : X \to A$, then there is a unique $\psi' : X \to K$ such that $\psi = \mu\psi'$. This is suggestive of the following dual notion. A *cokernel* of a morphism $\varphi : A \to B$, denoted by $\mathrm{coker}\varphi$, is a morphism $\mu : B \to C$ such that $\mu\varphi = 0$ and if $\psi\varphi = 0$ for $\psi : B \to X$, then there is a unique $\psi' : C \to X$ such that $\psi = \psi'\mu$. The *image* of a morphism $\varphi : A \to B$, denoted by $\mathrm{im}\varphi$, is the kernel of the cokernel of $\varphi$. For example, consider $\Lambda$-left-modules $A$ and $B$ and the homomorphism $\varphi : A \to B$. Then simply $\ker\varphi = \{a \in A \mid \varphi a = 0\}$, $\mathrm{im}\varphi = \varphi A = \{b \in B \mid b = \varphi a$ for some $a \in A\}$ and $\mathrm{coker}\varphi = B/\mathrm{im}\varphi$.

A morphism $\mu : A \rightarrowtail B$ is a *monomorphism* if $\mu\alpha = \mu\beta$ implies $\alpha = \beta$ for all morphisms $\alpha$, $\beta$. This lends to the dual notion. A morphism $\varepsilon : A \twoheadrightarrow B$ is an *epimorphism* if $\alpha\varepsilon = \beta\varepsilon$ implies $\alpha = \beta$ for all morphisms $\alpha$, $\beta$.

An object $P$ is *projective* if given any epimorphism $B \overset{\varepsilon}{\twoheadrightarrow} C$, the map $(P, \varepsilon) : (P, B) \to (P, C)$ is surjective.

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] \\ B \arrow[r, two heads, "\varepsilon"] & C \end{tikzcd}$$

An *injective* object is defined dually, i.e. an object $I$ is *injective* if given any monomorphism $A \overset{\iota}{\hookrightarrow} B$, the map $(\iota, I) : (B, I) \to (A, I)$ is surjective.

$$\begin{tikzcd} A \arrow[r, hook, "\iota"] \arrow[d] & B \arrow[dl, dashed] \\ I & \end{tikzcd}$$

Given a family $\{X_i\}$ of objects of a category $\mathfrak{C}$, the *product* of the $X_i$ is $(X, p_i)$, where $X$ is an object and the morphisms $p_i : X \to X_i$ are the *projections* with the *universal property*, this is to say, given any object $Y$ and morphisms $f_i : Y \to X_i$, there is a unique morphism $f = \{f_i\} : Y \to X$ such that $p_i f = f_i$. The notion of a *coproduct* is defined dually, i.e. given a family $\{X_i\}$ of objects of a category $\mathfrak{C}$, the *coproduct* of the $X_i$ is $(X, q_i)$, written $X = \coprod X_i$, where $X$ is an object and the morphisms $q_i : X_i \to X$ are the *injections* with the *universal property*, this is to say, given any object $Y$ and morphisms $f_i : X_i \to Y$, there is a unique morphism $f = \langle f_i \rangle : X \to Y$ such that $f q_i = f_i$.

In the category of $\Lambda$-left-modules, the product and the coproduct are simply the *direct product* and the *direct sum*, written $\prod X_i$ and $\bigoplus X_i$ respectively.

This reveals the notion of *duality* in category theory, see [18, II.3.] for an account on it. For example, a morphism $\varphi$ is a monomorphism in a category $\mathfrak{C}$ if and only if $\varphi$ is an epimorphism in $\mathfrak{C}^{op}$, and the notions are said to be dual to each other. Similarly, the diagram illustrating the notion of a projective object, by reversing the arrows and changing *epimorphism* to *monomorphism*, is precisely the diagram illustrating the notion of an injective object, and vice versa, thus the notions are dual to each other, and so are the notions of a product and a coproduct. Since $(\mathfrak{C}^{op})^{op} = \mathfrak{C}$, any statement involving certain notions which is true in a category $\mathfrak{C}$ remains true if they are replaced by their respective dual notions, with the necessary adjustments entailed by the changes. Though the duality pair, when it occurs, is written in its entirety more often in the introduction chapter, the duals of the statements and proofs are usually left to the reader.

**Definition 0.1.3.** The finite-dimensional $k$-algebra $\Lambda$ over the field $k$ is said to be *right hereditary* if any right ideal of $\Lambda$ is projective as a $\Lambda$-module. A *left hereditary* algebra is defined dually, i.e. the algebra $\Lambda$ is said to be left hereditary if any left ideal of $\Lambda$ is projective as a $\Lambda$-module. Since the algebra $\Lambda$ is right hereditary if and only if it is left hereditary, we simply say a hereditary algebra, see the remark after [1, Theorem VII.1.4].

Let $M$ be a $\Lambda$-right-module, $N$ be a $\Lambda$-left-module and consider the set $B = \{m \otimes n \mid m \in M, n \in N\}$, and let $V$ be a $k$-vector space with $B$ as the underlying set. Then the *tensor product* of $M$ and $N$, written $M \otimes_\Lambda N$, is defined to be the $k$-vector space $V$ modulo the following relations.

(i) $m_1 \otimes n + m_2 \otimes n = (m_1 + m_2) \otimes n$,

(ii) $m \otimes n_1 + m \otimes n_2 = m \otimes (n_1 + n_2)$,

(iii) $m\lambda \otimes n = m \otimes \lambda n$, $\lambda \in \Lambda$.

The tensor product $M \otimes_\Lambda N$ does not necessarily have a module structure.

**Lemma 0.1.4.** *Let $M$ be a $\Lambda$-right-module, $N$ be a $\Lambda$-left-module and $J$ be a $\Lambda$-bi-module. Then*

*(i) $M \otimes_\Lambda \Lambda \cong M$ and $\Lambda \otimes_\Lambda N \cong N$,*

*(ii) $J \otimes_\Lambda N$ is injective if $J$ is injective as a $\Lambda$-left-module and $N$ is projective.*

A category $\mathfrak{C}$ is *additive* if $\mathfrak{C}$ has a zero object, any two objects in $\mathfrak{C}$ have a product in $\mathfrak{C}$ and given objects $A, B, C$ in $\mathfrak{C}$, the morphism sets $(A, B)$ are abelian groups such that the composition $(A, B) \times (B, C) \to (A, C)$ is bilinear. An additive category $\mathfrak{C}$ is *abelian* if every morphism has a kernel and a cokernel, every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel and finally, every morphism can be expressed as the composite of an epimorphism and a monomorphism. The categories $\mathrm{Mod}(\Lambda)$ and $\mathrm{mod}(\Lambda)$ are examples of additive and abelian categories, see [18, II.9.]. Finally, let $F : \mathfrak{C} \to \mathfrak{D}$ be a functor between the additive categories. Then $F$ is an *additive functor* if given $X, Y$ in $\mathfrak{C}$, the map $F_{XY} : \mathrm{Hom}_{\mathfrak{C}}(X, Y) \to \mathrm{Hom}_{\mathfrak{D}}(FX, FY)$ is a homomorphism.

A *sequence* $A_0 \xrightarrow{j_0} A_1 \to \ldots \to A_n \xrightarrow{j_n} A_{n+1}$ in a category $\mathfrak{C}$ satisfies $j_{i+1}j_i = 0$ for $0 \le i \le n-1$. A sequence $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ in $\mathrm{Mod}(\Lambda)$ is *exact* at $B$ if $\ker\psi = \mathrm{im}\varphi$. A sequence $A_0 \to A_1 \to \ldots \to A_n \to A_{n+1}$ in $\mathrm{Mod}(\Lambda)$ is exact if it is exact at all the $A_i, 1 \le i \le n$. The exact sequence $0 \to A \rightarrowtail B \twoheadrightarrow C \to 0$ in $\mathrm{Mod}(\Lambda)$ is a *short exact sequence.* The notion of exactness can readily be generalized in any abelian categories, with some delicate considerations.

A covariant functor $F$ is *left exact* if given the exact sequence $0 \to A \to B \to C$, the sequence $0 \to FA \to FB \to FC$ is exact. A contravariant functor $F$ is *left exact* if given the exact sequence $A \to B \to C \to 0$, the sequence $0 \to FC \to FB \to FA$ is exact. The *right exactness* of a functor $F$, covariant or contravariant, is defined similarly.

The notions of a monomorphism and an epimorphism are generalizations of the following.

**Definition 0.1.5.** Let $\alpha : A \to B$ be a morphism in a category $\mathfrak{C}$. Then $\alpha$ is a *split monomorphism* if there is a morphism $\beta : B \to A$ such that $\beta\alpha = 1_A$. A *split epimorphism* is defined dually, i.e. $\alpha$ is a split epimorphism if there is a morphism $\beta : B \to A$ such that $\alpha\beta = 1_B$.

In a different manner, the following generalizes the notions of a monomorphism and an epimorphism.

**Definition 0.1.6.** Let $A$ and $B$ be in a category $\mathfrak{C}$. A morphism $\alpha : A \to B$ is *right minimal* if $\alpha f = \alpha$ for a morphism $f : A \to A$ implies that $f$ is an automorphism. Dually, a morphism $\alpha : A \to B$ is *left minimal* if $f\alpha = \alpha$ for a morphism $f : B \to B$ implies that $f$ is an automorphism.

Let $\mathfrak{A}$ be an abelian category in this chapter. The following lemma is standard.

**Lemma 0.1.7.** *(Five Lemma) In the following commutative diagram, considered in $\mathfrak{A}$,*

$$\begin{tikzcd}
{} \arrow[r] & A_1 \arrow[r] \arrow[d, "\varphi_1"] & A_2 \arrow[r] \arrow[d, "\varphi_2"] & A_3 \arrow[r] \arrow[d, "\varphi_3"] & A_4 \arrow[r] \arrow[d, "\varphi_4"] & A_5 \arrow[r] \arrow[d, "\varphi_5"] & {} \\
{} \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \arrow[r] & {}
\end{tikzcd},$$

*assume that the rows are exact. If $\varphi_1$, $\varphi_2$, $\varphi_4$ and $\varphi_5$ are isomorphisms, then $\varphi_3$ is also an isomorphism.*

Given a category $\mathfrak{C}$, any functor $F : \mathfrak{C} \to \mathfrak{C}$ is an *endofunctor*. If there is a functor $G : \mathfrak{C} \to \mathfrak{C}$ such that $FG = GF = \mathrm{id}$, then $F$ is an *automorphism* and written $G = F^{-1}$. Given functors $F, G : \mathfrak{C} \to \mathfrak{D}$ and objects $X$, $Y$ in $\mathfrak{C}$, a *natural transformation* $t$ from $F$ to $G$ is a collection of morphisms $t_X : FX \to GX$ in $\mathfrak{D}$ such that for any morphism $f : X \to Y$ in $\mathfrak{C}$, there is the relation $G(f)t_X = t_Y F(f)$, as shown in the following commutative diagram.

$$\begin{tikzcd}
FX \arrow[r, "t_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\
FY \arrow[r, "t_Y"] & GY
\end{tikzcd}$$

If $t_X$ is an isomorphism for each $X$, then $t$ is a *natural equivalence* and we write $F \simeq G$. Let $F : \mathfrak{C} \to \mathfrak{D}$, $G : \mathfrak{D} \to \mathfrak{C}$ be functors such that $GF \simeq \mathrm{id} : \mathfrak{C} \to \mathfrak{C}$ and $FG \simeq \mathrm{id} : \mathfrak{D} \to \mathfrak{D}$. Then $F$ (and also $G$) is an *equivalence* and they are *quasi-inverses* of each other, and the categories $\mathfrak{C}$ and $\mathfrak{D}$ are *equivalent categories.* If $\mathfrak{C} = \mathfrak{D}$, then $F$ (and also $G$) is an *autoequivalence.* For example, an automorphism is an antoequivalence.

*Example* 0.1.8. (i) ([18, Exercise II.4.1.]) Let $\mathfrak{C}_0$ be a full subcategory of $\mathfrak{C}$ such that given any object $A$ in $\mathfrak{C}$, there is precisely one object $A_0$ in $\mathfrak{C}_0$ with $A_0 \cong A$. Then $\mathfrak{C}_0$ is equivalent to $\mathfrak{C}$, and the subcategory $\mathfrak{C}_0$ is said to be a *skeleton* of $\mathfrak{C}$. Along with this, any two skeletons of $\mathfrak{C}$ are isomorphic.

(ii) ([31, Theorem IV.4.1]) Consider the functor $F : \mathfrak{C} \to \mathfrak{D}$. Then $F$ is a natural equivalence of categories if and only if $F$ is full, faithful and essentially surjective, this is to say, for all $Y$ in $\mathfrak{D}$, there is some $X$ in $\mathfrak{C}$ such that $FX \cong Y$.

The object $X$ in an additive category $\mathfrak{C}$ is *indecomposable* if $X$ is non-zero and if $X \cong X_1 \oplus X_2$, then either $X_1 \cong 0$ or $X_2 \cong 0$. If $\mathfrak{C}$ is the module category $\mathrm{Mod}(\Lambda)$, then $\mathrm{End}(X)$ is also a $k$-algebra, where $\mathrm{End}(X)$ is the endomorphism ring $\mathrm{Hom}(X, X)$. A $k$-algebra $A$ is a *local algebra* if $A$ has a unique maximal right ideal. A characterization of a finite-dimensional $k$-algebra $A$ as a local algebra is given in [1, Lemma I.4.6].

**Lemma 0.1.9.** *([1, Corollary I.4.8]) Let $M$ be a $\Lambda$-left-module. Then $M$ is indecomposable if and only if the algebra End$(M)$ is local.*

Let $k$ be a field and let $X, Y, Z$ be objects in a category $\mathfrak{C}$. Then $\mathfrak{C}$ is said to be *$k$-linear* if it is additive, each Hom set $(X, Y)$ is a $k$-vector space and each composition map $(X, Y) \times (Y, Z) \to (X, Z)$ is bilinear. It is *Hom finite* if given any $X, Y$ in $\mathfrak{C}$, the morphism space $(X, Y)$ is a finite-dimensional $k$-vector space.

A category $\mathfrak{C}$ is *finite-dimensional $k$-additive* if $\mathfrak{C}$ is $k$-linear and all Hom sets $(X, Y)$ are finite-dimensional $k$-vector spaces. If all idempotents split, that is, given $e = e^2$ in $(X, X)$ for an object $X$ in $\mathfrak{C}$, there are maps $\mu : Y \to X$ and $\rho : X \to Y$ with $\rho\mu = 1_Y$ and $\mu\rho = e$, then the endomorphism ring $\mathrm{End}(X)$ of any indecomposable object $X$ of $\mathfrak{C}$ is a local ring, so that the category $\mathfrak{C}$ is to be a *Krull-Schmidt* category, see the following definition in [41, 2.2].

**Definition 0.1.10.** *(c.f. Example 0.1.8)* Let $\mathfrak{C}$ be a finite-dimensional $k$-linear category. Then $\mathfrak{C}$ is *Krull-Schmidt* if given indecomposable objects $x_i, x_j$, $1 \le i \le s$, $1 \le j \le t$, in $\mathfrak{C}$ such that $\bigoplus_{i=1}^{s} x_i \cong \bigoplus_{j=1}^{t} y_j$, then $s = t$ and there is a permutation $\pi$ of $\{1, \dots, s\}$ such that $x_i \cong y_{\pi(i)}$ for all $i$.

For example, the category mod($\Lambda$) is Krull-Schmidt, where we remind the reader that $\Lambda$ is a finite-dimensional $k$-algebra over the field $k$.

*Notation* 0.1.11. Let $\mathfrak{C}$ be a Krull-Schmidt category, and let $\{X_i\}$ be a complete set of representatives from the isomorphism classes of indecomposable objects in $\mathfrak{C}$. The full subcategory of $\mathfrak{C}$, with objects the $X_i$, is denoted by $\operatorname{ind}\mathfrak{C}$. Let $\mathfrak{C}$ be any additive category and $\mathfrak{C}_0$ a subcategory of $\mathfrak{C}$. Then denote by $\operatorname{add}\mathfrak{C}_0$ the smallest subcategory of $\mathfrak{C}$ containing all the objects in $\mathfrak{C}_0$, which is closed under direct sums and direct summands. The category $\operatorname{add}\mathfrak{C}_0$ is said to be the *additive closure* of $\mathfrak{C}_0$, or informally, the *add* of objects in $\mathfrak{C}_0$.

**Definition 0.1.12.** Let $\mathfrak{C}$ be an additive category. A *chain complex* $C = \{C_n, \partial_n\}$ over $\mathfrak{C}$ is

$$C = \cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \longrightarrow \cdots,$$

where the $C_i$ are in $\mathfrak{C}$ and $\partial_i \partial_{i+1} = 0$.

The following is the dual notion.

A *cochain complex* $C = \{C^n, \partial^n\}$ over $\mathfrak{C}$ is

$$C = \cdots \longrightarrow C^{n-2} \xrightarrow{\partial^{n-2}} C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \longrightarrow \cdots,$$

where the $C^i$ are in $\mathfrak{C}$ and $\partial^{i+1}\partial^i = 0$.

*Remark* 0.1.13. Contexts written in terms of chain complexes can be translated suitably in the context of cochain complexes, where the prefix "co-" is to be added when it makes sense, and vice versa. This is to be seen readily as subscripts are used for chain complexes and superscripts for cochain complexes.

A chain complex $C = \{C_n, \partial_n\}$ is said to be *bounded below* if $C_n = 0$ for $n \ll 0$. Similarly, there is the notion of a chain complex which is *bounded above* or *bounded on both sides.* An object $M$ in $\mathfrak{C}$ is *identified* with the chain complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

with $M$ in degree 0 when there is no confusion.

**Definition 0.1.14.** Let $\mathfrak{C}$ be an additive category and let $C = \{C_n, \partial_n\}$ and $D = \{D_n, \tilde{\partial}_n\}$ be chain complexes over $\mathfrak{C}$. Then a chain map $\varphi : C \to D$ is a family $\{\varphi_n : C_n \to D_n\}$ of morphisms such that the following diagram,

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "\varphi_n"'] & C_{n-1} \arrow[d, "\varphi_{n-1}"] \\ D_n \arrow[r, "\tilde{\partial}_n"] & D_{n-1}, \end{tikzcd}$$

is commutative for each value of $n$.

Given an additive category $\mathfrak{C}$, the category with (co)chain complexes over $\mathfrak{C}$ as objects and (co)chain maps as morphisms is denoted by $C(\mathfrak{C})$. The category of (co)chain complexes over $\mathrm{Mod}(\Lambda)$ is denoted by $C(\Lambda)$.

**Definition 0.1.15.** Let $\mathfrak{C}$ be an additive category and let $C = \{C_n, \partial_n\}$ and $D = \{D_n, \tilde{\partial}_n\}$ be chain complexes over $\mathfrak{C}$. Given a chain map $\varphi : C \to D$, let the *mapping cone* of $\varphi$ be the chain complex $E = E(\varphi) = \{E_n, \tilde{\tilde{\partial}}_n\}$, where $E_n = C_{n-1} \oplus D_n$ and $\tilde{\tilde{\partial}}_n(a, b) = (-\partial_{n-1}a, \varphi_{n-1}a + \tilde{\partial}_n b)$ for $a \in C_{n-1}$ and $b \in D_n$.

**Definition 0.1.16.** Let $C = \{C_n, \partial_n\}$ be a chain complex in $C(\Lambda)$. Let $\mathrm{H}_n(C) = \ker\partial_n / \mathrm{im}\partial_{n+1}$ be the *$n$-th homology module* of $C$ and write $\mathrm{H}(C) = \{H_n(C)\}$. Also $Z_n = Z_n(C) = \ker\partial_n$.

Let $C = \{C_n, \partial_n\}$ and $D = \{D_n, \tilde{\partial}_n\}$ be chain complexes in $C(\Lambda)$ and consider the chain map $\varphi : C \to D$. Then it is natural to define $\mathrm{H}_n(\varphi) : \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ to be $\mathrm{H}_n(\varphi)(x_n + \mathrm{im}\partial_{n+1}) = \varphi_n(x_n) + \mathrm{im}\tilde{\partial}_{n+1}$ for $x_n$ in $\ker\partial_n$. Since $x_n$ is in $\ker\partial_n$ and $\tilde{\partial}_n\varphi_n(x_n) = \varphi_{n-1}\partial_n(x_n) = 0$, therefore $\varphi_n(x_n)$ is indeed in $\ker\tilde{\partial}_n$.

The map $\mathrm{H}_n(\varphi)$ is well-defined. Consider the following diagram,

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\varphi_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "\varphi_n"'] & C_{n-1} \arrow[d, "\varphi_{n-1}"] \\
D_{n+1} \arrow[r, "\tilde{\partial}_{n+1}"] & D_n \arrow[r, "\tilde{\partial}_n"] & D_{n-1}.
\end{tikzcd}$$

Let $x_n$ and $x'_n$ be in $\ker\partial_n$ such that $x_n + \mathrm{im}\partial_{n+1} = x'_n + \mathrm{im}\partial_{n+1}$, which is to say $x_n - x'_n$ is in $\mathrm{im}\partial_{n+1}$. Therefore $x_n - x'_n = \partial_{n+1}(u)$ for some $u$ in $C_{n+1}$. Hence $\varphi_n x_n - \varphi_n x'_n = \varphi_n(x_n - x'_n) = \varphi_n\partial_{n+1}(u) = \tilde{\partial}_{n+1}\varphi_{n+1}(u)$, which gives $\varphi_n x_n - \varphi_n x'_n$ in $\mathrm{im}\tilde{\partial}_{n+1}$ indeed.

*Example* 0.1.17. Functors can be defined on different levels. Here are a few examples.

(i) A group can be viewed as a category with a single object. The morphisms of the category are the elements of the group, which are all invertible. A functor between two groups is then a homomorphism between them.

(ii) Consider the chain complex $X = \{X_n, \partial_n\}$. The *suspension functor* $\Sigma$ is defined by shifting the complex $X$ one place to the left, and changing the sign of the differential $\partial$, i.e. for the chain complex $\Sigma(X) = \{\Sigma(X)_n, \tilde{\partial}_n\}$, let $\Sigma(X)_n = X_{n-1}$ and $\tilde{\partial}_n = -\partial_{n-1}$.

(iii) Let $C$ and $D$ be two chain complexes in $C(\Lambda)$ with a chain map $\varphi : C \to D$. This induces a well-defined morphism $\mathrm{H}(\varphi) : \mathrm{H}(C) \to \mathrm{H}(D)$ and the homology functor $\mathrm{H}(-)$ correspondingly, by the little description after Definition 0.1.16. The chain map $\varphi$ is a *quasi-isomorphism* if the induced homomorphisms $\mathrm{H}_n(\varphi)$ are isomorphisms.

(iv) Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary categories. A pair of functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ are *adjoints* if there is an isomorphism $\tau : (LA, B) \overset{\cong}{\to} (A, RB)$ for all $A$ in $\mathcal{A}$ and $B$ in $\mathcal{B}$, such that $\tau$ is *natural* in $A$ and $B$. For the isomorphism $\tau$ to be natural in $A$ is to say that given a morphism $f : A_1 \to A_2$ in $\mathcal{A}$, the following diagram is commutative.

$$\begin{tikzcd} (LA_2, B) \arrow[r, "\tau"] \arrow[d] & (A_2, RB) \arrow[d] \\ (LA_1, B) \arrow[r, "\tau"] & (A_1, RB) \end{tikzcd}$$

Similarly, for the isomorphism $\tau$ to be natural in $B$ is to say that given a morphism $g : B_1 \to B_2$ in $\mathcal{B}$, the following diagram is commutative.

$$\begin{tikzcd} (LA, B_1) \arrow[r, "\tau"] \arrow[d] & (A, RB_1) \arrow[d] \\ (LA, B_2) \arrow[r, "\tau"] & (A, RB_2) \end{tikzcd}$$

$L$ is the *left adjoint* and $R$ is the *right adjoint* of the pair of functors.

In Example 0.1.17(iv), given $A$ in $\mathcal{A}$ and $B$ in $\mathcal{B}$, the map $\eta_A = \tau(\mathrm{id}_{LA}) : A \to RLA$ is the *unit* of the adjunction, and the map $\varepsilon_B = \tau^{-1}(\mathrm{id}_{RB}) : LRB \to B$ is the *counit* of the adjunction. If $\mathcal{A}$ and $\mathcal{B}$ are abelian, then $L$ is right exact and $R$ is left exact, see [42, Theorem 2.14]. In this way, adjoints approximate exactness as well.

Intuitively, adjoints are approximations of inverse functors. Indeed suppose the functors $L$ and $R$ are inverse functors. Then their actions can be retrieved through $\tau$. For example, let $A_1$ and $A_2$ be in $\mathcal{A}$ and consider the morphism $f : A_1 \to A_2$. Suppose $A_2 = RB_2$ for some $B_2$ in $\mathcal{B}$. Then the morphism $Lf : LA_1 \to LA_2$, where $LA_2 = LRB_2 = B_2$, takes the value $\tau^{-1}f$.

Any two left adjoints of a functor $R$ are naturally equivalent, dually, any two right adjoints of a functor $L$ are naturally equivalent, see [11, A5.2.1]. A functor with a left adjoint respects products, and a functor with a right adjoint respects coproducts. Any invertible functor has both a right and a left adjoint, see [35, Proposition 1.1.6].

The following lemma follows from the definition of a quasi-isomorphism.

**Lemma 0.1.18.** *Let $X$ be a chain complex in $C(\Lambda)$ and consider the morphism $f : 0 \to X$ in $C(\Lambda)$. Then $f$ is a quasi-isomorphism if and only if $X$ is exact.*

**Lemma 0.1.19.** *([18, Theorem IV.2.1]) Let $\mathfrak{C}$ be an abelian category and let $A \rightarrowtail B \twoheadrightarrow C$ be a short exact sequence of chain complexes over $\mathfrak{C}$, which is equivalent to saying $A_n \rightarrowtail B_n \twoheadrightarrow C_n$ is a short exact sequence in $\mathfrak{C}$ for each value of $n$. Then there is a long exact sequence*

$$\cdots \longrightarrow \mathrm{H}_n(A) \longrightarrow \mathrm{H}_n(B) \longrightarrow \mathrm{H}_n(C) \longrightarrow \mathrm{H}_{n-1}(A) \longrightarrow \cdots$$

*in $\mathfrak{C}$.*

**Lemma 0.1.20.** *Consider a chain map $\varphi : C \to D$ in $C(\Lambda)$ and the mapping cone $E = E(\varphi)$ in Definition 0.1.15. Let $\Sigma$ be the suspension functor. Then there is a short exact sequence $D \overset{\imath}{\rightarrowtail} E(\varphi) \overset{\rho}{\twoheadrightarrow} \Sigma C$ in $C(\Lambda)$, where $\imath$ is the canonical inclusion and $\rho$ is the canonical surjection. In addition,*

*(i) By Lemma 0.1.19, there is a long exact sequence*

$$\cdots \longrightarrow \mathrm{H}_n(C) \longrightarrow \mathrm{H}_n(D) \longrightarrow \mathrm{H}_n(E(\varphi)) \longrightarrow \mathrm{H}_{n-1}(C) \longrightarrow \cdots$$

*in $\mathrm{Mod}(\Lambda)$. Therefore $\mathrm{H}(E(\varphi)) = 0$ if and only if $\mathrm{H}(\varphi) : \mathrm{H}(C) \overset{\cong}{\to} \mathrm{H}(D)$.*

*(ii) Let $X$ be a chain complex in $C(\Lambda)$ and consider the chain map $\mu : X \to E(\varphi)$ such that $\rho\mu = 0$. Then there is a unique chain map $\mu' : X \to D$ such that $\imath\mu' = \mu$.*

$$\begin{tikzcd}
 & & X \arrow[dl, dashed, "\mu'"'] \arrow[d, "\mu"] \arrow[dr, "0"] & & \\
0 \arrow[r] & D \arrow[r, "\imath"] & E(\varphi) \arrow[r, "\rho"] & \Sigma C \arrow[r] & 0
\end{tikzcd}$$

In Lemma 0.1.20(i), the map $\varphi$ is a quasi-isomorphism if and only if the mapping cone $E = E(\varphi)$ is exact.

The construction of a projective resolution is a subtle way of encoding a module $A$ in $\mathrm{Mod}(\Lambda)$ into a complex in $C(\Lambda)$.

**Definition 0.1.21.** Let $A$ be in $\mathrm{Mod}(\Lambda)$. A *projective resolution* $P(A)$ of $A$ is a chain complex

$$P(A) = \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \longrightarrow \cdots$$

in $C(\Lambda)$ where $P_i$ is projective for all $i \geq 0$, $\mathrm{H}_i(P) = 0$ for all $i \geq 1$ and $\mathrm{H}_0(P) \cong A$.

**Definition 0.1.22.** Let $A$ be in $\mathrm{Mod}(\Lambda)$. An *injective resolution* $I(A)$ of $A$ is a cochain complex

$$I(A) = \cdots \longrightarrow 0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

in $C(\Lambda)$ where $I^i$ is injective for all $i \geq 0$, $\mathrm{H}^i(I) = 0$ for all $i \geq 1$ and $\mathrm{H}^0(I) \cong A$.

*Example* 0.1.23. Below are some examples of projective resolutions.

(i) Let $k$ be a field. Consider the ring $R = k[X]/(X^2)$, and turn $k$ into an $R$-module by letting $(k_0 + k_1X)m = k_0m$ for $m$ in $k$. Then the projective resolution $P(M)$ of $k$ is

$$P(M) = \cdots \longrightarrow k[X]/(X^2) \xrightarrow{\cdot X} k[X]/(X^2) \longrightarrow 0 \longrightarrow \cdots \qquad .$$

(ii) Let $k$ be a field. Consider the ring $R = k[X]$, and turn $k$ into an $R$-module by letting $(k_0 + k_1X + \ldots + k_nX^n)m = k_0m$ for $m$ in $k$. Then the projective resolution $P(M)$ of $k$ is

$$P(M) = \cdots \longrightarrow 0 \longrightarrow k[X] \xrightarrow{\cdot X} k[X] \longrightarrow 0 \longrightarrow \cdots \qquad .$$

**Definition 0.1.24.** ([18, VII.7.]) The *global dimension* of $\Lambda$, denoted by *gl. dim.* $\Lambda$, is less than or equal to $m$ if for all $\Lambda$-modules $A$ and all projective resolutions $P(A)$ of $A$, $\ker(P_{m-1} \to P_{m-2})$ is projective. Then gl. dim. $\Lambda$ $= m$ if gl. dim. $\Lambda \leq m$ but gl. dim. $\Lambda \nleq m-1$.

*Example* 0.1.25. ([1, Theorem VII.1.4]) The algebra $\Lambda$ is hereditary if and only if the global dimension of $\Lambda$ is at most one.

**Definition 0.1.26.** Let $\mathfrak{C}$ be an additive category and let $C = \{C_n, \partial_n\}$ and $D = \{D_n, \tilde{\partial}_n\}$ be chain complexes over $\mathfrak{C}$. A *homotopy* $h : \varphi \to \psi$ between chain maps $\varphi, \psi : C \to D$ is a collection of morphisms $\{h_n : C_n \to D_{n+1}\}$ such that $\varphi_n - \psi_n = \tilde{\partial}_{n+1}h_n + h_{n-1}\partial_n$.

$$\begin{tikzcd}
C = \cdots \arrow[r] & C_{n+1} \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[dl, "h_n"] & C_{n-1} \arrow[r] \arrow[dl, "h_{n-1}"] & \cdots \\
D = \cdots \arrow[r] & D_{n+1} \arrow[r, "\tilde{\partial}_{n+1}"] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

If there is a homotopy $h : \varphi \to \psi$, then $\varphi, \psi$ are *homotopic* and we write $\varphi \simeq \psi$. The homotopy relation is an equivalence relation, so that given a chain map $\varphi$, let $\overline{\varphi}$ be the equivalence class of $\varphi$. The chain complexes $C$ and $D$ are *homotopy equivalent* if there are chain maps $\varphi : C \to D$ and $\psi : D \to C$ such that $\psi\varphi \simeq \mathrm{id}$ and $\varphi\psi \simeq \mathrm{id}$.

A chain complex $C$ in $C(\Lambda)$ is *projective* if $C_n$ is projective for all $n \geq 0$ and $C_n = 0$ for all $n \leq -1$. A cochain complex $D$ in $C(\Lambda)$ is *injective* if $D^n$ is injective for all $n \geq 0$ and $D^n = 0$ for all $n \leq -1$. A (co)chain complex $C$ in $C(\Lambda)$ is *acyclic* if ($\mathrm{H}^n(C) = 0$) $\mathrm{H}_n(C) = 0$ for $n \geq 1$.

**Lemma 0.1.27.** *(i) ([18, Theorem IV.4.1]) Let $C$ be a projective chain complex and $D$ be an acyclic chain complex in $C(\Lambda)$. Then for every homomorphism $\varphi : \mathrm{H}_0(C) \to \mathrm{H}_0(D)$, there is a chain map $\tilde{\varphi}$ inducing $\varphi$ and any two such chain maps are homotopic.*

*(ii) ([18, Theorem IV.4.4]) Let $C$ be an acyclic cochain complex and $D$ be an injective cochain complex in $C(\Lambda)$. Then for every homomorphism $\varphi : \mathrm{H}^0(C) \to \mathrm{H}^0(D)$, there is a chain map $\tilde{\varphi}$ inducing $\varphi$ and any two such chain maps are homotopic.*

*Example* 0.1.28. For any $M$ in $\mathrm{Mod}(\Lambda)$, the projective resolution $P(M)$ always exists, see [18, Lemma IV.4.2]. Consider the isomorphism $f_0$ : $\mathrm{H}^0 P(M) \overset{\cong}{\to} M$. By Lemma 0.1.27(i), the morphism $f_0$ can be extended to a chain map $f : P(M) \to M$. The chain map $f$ is a quasi-isomorphism. It follows from Lemma 0.1.27(i) that any two projective resolutions of $M$ are homotopy equivalent.

**Lemma 0.1.29.** *Let $P$ be a chain complex in $C(\Lambda)$. If $P$ is projective, then* $\mathrm{H}(P) = 0$ *if and only if* $1 \simeq 0 : P \to P$.

*Proof. (if)* Since $1 \simeq 0 : P \to P$, there are morphisms $h_n : P_n \to P_{n+1}$ such that $1 = \partial_{n+1} h_n + h_{n-1}\partial_n$.

$$\begin{tikzcd}
P = \cdots \arrow[r] & P_{n+1} \arrow[r] & P_n \arrow[r, "\partial_n"] \arrow[dl, "h_n"] & P_{n-1} \arrow[r] \arrow[dl, "h_{n-1}"] & \cdots \arrow[r] & P_0 \\
P = \cdots \arrow[r] & P_{n+1} \arrow[r, "\partial_{n+1}"] & P_n \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_0
\end{tikzcd}$$

Let $y \in \ker\partial_n$. Therefore $y = (\partial_{n+1}h_n + h_{n-1}\partial_n)y = (\partial_{n+1}h_n)y$ and so $y \in \mathrm{im}\partial_{n+1}$. *(only if)* Consider the morphism $\pi : \mathrm{H}_0(P) \to \mathrm{H}_0(P)$. Since $\mathrm{H}_0(P) = 0$, therefore $\pi = 1 = 0$. Lemma 0.1.27(i) then gives $1 \simeq 0 : P \to P$. □

**Lemma 0.1.30.** *([18, Exercise IV.3.1.]) Let $C$ and $D$ be chain complexes in $C(\Lambda)$ and let $\varphi$ and $\psi$ be homotopic chain maps : $C \longrightarrow D$. Then the mapping cones $E(\varphi)$ and $E(\psi)$ are isomorphic.*

The *homotopy category* $K(\Lambda)$ is a category whose objects are the chain complexes in $C(\Lambda)$, and whose morphisms are the homotopy equivalence classes of chain maps between chain complexes, see [17, Chapter I. §2.]. This is possible because the null homotopic morphisms form an ideal in $C(\Lambda)$. Let $K_+(\Lambda)$, $K_-(\Lambda)$ and $K_b(\Lambda)$ be the full subcategories of $K(\Lambda)$ consisting of the complexes bounded below, bounded above and bounded on both sides, respectively.

**Lemma 0.1.31.** *Let $X$ and $Y$ be chain complexes in $K(\Lambda)$ and let $\varphi$ and $\psi$ be homotopic chain maps : $X \to Y$. Let $F : K(\Lambda) \to K(\Lambda)$ be an additive functor and consider the homology functor* $\mathrm{H}(-)$. *Then*

*(i)* $\mathrm{H}(\varphi) = \mathrm{H}(\psi)$,

*(ii)* $F\varphi \simeq F\psi$,

*(iii) The morphism $f : 0 \to X$ in $K(\Lambda)$ is an isomorphism if and only if $X$ is null homotopic, i.e. $X \simeq 0$.*

*Proof.* This is trivial. □

By virtue of Lemma 0.1.31, the homology functor $\mathrm{H}(-)$ and subsequently quasi-isomorphisms are well-defined on $K(\Lambda)$.

**Lemma 0.1.32.** *Let $P$ be a projective module in* $\mathrm{Mod}(\Lambda)$ *and let $X$ be in $C(\Lambda)$. Then* $\mathrm{Hom}_\Lambda(P, \mathrm{H}_0X) \cong \mathrm{H}_0(\mathrm{Hom}_\Lambda(P, X))$.

*Proof.* Let the chain complex $X = \{X_n, \partial_n\}$ be

$$X = \cdots \longrightarrow X_2 \longrightarrow X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\partial_0} X_{-1} \longrightarrow X_{-2} \longrightarrow \cdots \quad ,$$

and the chain complex $(P, X)$ be

$$(P, X) = \cdots \longrightarrow (P, X_1) \xrightarrow{d_1} (P, X_0) \xrightarrow{d_0} (P, X_{-1}) \longrightarrow \cdots \quad ,$$

where $d_i = (P, \partial_i)$.

Let $\theta : \mathrm{Hom}_\Lambda(P, \mathrm{H}_0X) \to \mathrm{H}_0(\mathrm{Hom}_\Lambda(P, X))$ and $\theta' : \mathrm{H}_0(\mathrm{Hom}_\Lambda(P, X)) \to \mathrm{Hom}_\Lambda(P, \mathrm{H}_0X)$ be maps between the two Hom sets, which are going to be defined. Let $p$ be in $P$. Given a morphism $f : P \to \mathrm{H}_0X$ where $f(p) = x_p + \mathrm{im}\partial_1$ for some $x_p$ in $\ker\partial_0$, consider the canonical morphism $\pi_X : \ker\partial_0 \twoheadrightarrow \mathrm{H}_0(X)$. Since $P$ is projective, there is a morphism $g : P \to \ker\partial_0$ such that $\pi_X g = f$.

$$\begin{tikzcd} \ker\partial_0 \arrow[r, two heads, "\pi_X"] & \mathrm{H}_0(X) \\ P \arrow[u, dashed, "g"] \arrow[ur, "f"'] & \end{tikzcd}$$

This gives a morphism $g : P \to X_0$. Since $\partial_0 g = 0$, therefore $g$ is in $\ker d_0$ and define $\theta(f) = g + \operatorname{im} d_1$ in $\mathrm{H}_0(\mathrm{Hom}_\Lambda(P, X))$. Conversely, consider the morphism $g + \operatorname{im} d_1$ in $\mathrm{H}_0(\mathrm{Hom}_\Lambda(P, X))$ where $g : P \to X_0$ and $\partial_0 g = 0$. Given $p$ in $P$, define $\theta'(g + \operatorname{im} d_1) = f$ where $f(p) = g(p) + \operatorname{im}\partial_1$. Finally, one can verify that $\theta$ and $\theta'$ are well-defined and that $\theta\theta' = \theta'\theta = 1$.

□

## 0.2 Triangulated categories

This section is in reminiscence of [17] and [44], though the reader can also seek guidance from [24], [34], [35] and [25].

Unlike an abelian category, whose properties are inherent in the category, a triangulated category is a category with extra structure layered on it. Let us begin with the definition.

**Definition 0.2.1.** ([17, Chapter I. §1.]) A *triangulated category* is an additive category $\mathfrak{T}$, together with

(a) an automorphism $\Sigma : \mathfrak{T} \to \mathfrak{T}$ known as the *translation functor*, and

(b) a collection of sextuples $(x, y, z, u, v, w)$ known as the *distinguished triangles* of $\mathfrak{T}$, written $x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} \Sigma x$, or simply $x \xrightarrow{u} y \xrightarrow{v} z \to$. Here $x, y, z$ are objects and $u, v, w$ are morphisms of $\mathfrak{T}$.

A morphism from the distinguished triangle $(x, y, z, u, v, w)$ to the distinguished triangle $(x', y', z', u', v', w')$ is a commutative diagram,

$$\begin{tikzcd}
x \arrow[r, "u"] \arrow[d, "f"] & y \arrow[r, "v"] \arrow[d, "g"] & z \arrow[r, "w"] \arrow[d, "h"] & \Sigma x \arrow[d, "\Sigma f"] \\
x' \arrow[r, "u'"] & y' \arrow[r, "v'"] & z' \arrow[r, "w'"] & \Sigma x'.
\end{tikzcd}$$

If the morphisms $f$, $g$ and $h$ are isomorphisms, the distinguished triangles $(x, y, z, u, v, w)$ and $(x', y', z', u', v', w')$ are then said to be *isomorphic*.

The above is required to satisfy

(i) (TR1) Every sextuple $(x, y, z, u, v, w)$ isomorphic to a distinguished triangle is a distinguished triangle. Every morphism $u : x \to y$ can be imbedded in a distinguished triangle $(x, y, z, u, v, w)$. For each $x$, the sextuple $(x, x, 0, \mathrm{id}_x, 0, 0)$ is a distinguished triangle.

(ii) (TR2) $(x, y, z, u, v, w)$ is a distinguished triangle if and only if $(y, z, \Sigma(x), v, w, -\Sigma(u))$ is a distinguished triangle.

(iii) (TR3) Given distinguished triangles $(x, y, z, u, v, w)$ and $(x', y', z', u', v', w')$, and morphisms $f$ and $g$ such that $gu = u'f$, there is a morphism $h$ such that the following diagram is commutative.

$$\begin{tikzcd} x \arrow[r, "u"] \arrow[d, "f"] & y \arrow[r, "v"] \arrow[d, "g"] & z \arrow[r, "w"] \arrow[d, dashed, "h"] & \Sigma x \arrow[d, "\Sigma f"] \\ x' \arrow[r, "u'"] & y' \arrow[r, "v'"] & z' \arrow[r, "w'"] & \Sigma x' \end{tikzcd}$$

(iv) (TR4) *The octahedral axiom.* Given two composable morphisms $f : x \to y$ and $g : y \to u$, as in the following commutative diagram,

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "gf"'] & y \arrow[d, "g"] \\ u \arrow[r, equal] & u, \end{tikzcd}$$

we can extend it to the following commutative diagram,

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "gf"'] & y \arrow[r] \arrow[d, "g"] & z \arrow[r] \arrow[d] & \Sigma x \arrow[d] \\ u \arrow[r, equal] \arrow[d] & u \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \Sigma u \arrow[d] \\ v \arrow[r] \arrow[d] & w \arrow[r] \arrow[d] & \Sigma z \arrow[r] \arrow[d, equal] & \Sigma v \arrow[d] \\ \Sigma x \arrow[r] & \Sigma y \arrow[r] & \Sigma z \arrow[r] & \Sigma^2 x, \end{tikzcd}$$

where all the rows and columns are distinguished triangles.

The automorphism $\Sigma : \mathfrak{T} \to \mathfrak{T}$ is an additive functor. By abuse of notation, given a distinguished triangle $x \overset{u}{\to} y \overset{v}{\to} z \overset{w}{\to} \Sigma x$, the object $z$ is the *mapping cone* of the morphism $u : x \to y$, see Definition 0.1.15. By (TR1), any object $x$ in a triangulated category $\mathfrak{T}$ yields a distinguished triangle $(x, x, 0, \mathrm{id}_x, 0, 0)$. This is similar to the way any object in a category yields a chain complex, see Definition 0.1.12.

The following is a slight variation of (TR3).

(TR3') Given distinguished triangles $(x, y, z, u, v, w)$ and $(x', y', z', u', v', w')$, and morphisms $f$ and $h$ such that $\Sigma f \circ w = w'h$, there is a morphism $g$ such that the following diagram is commutative.

$$\begin{tikzcd}
x \arrow[r, "u"] \arrow[d, "f"] & y \arrow[r, "v"] \arrow[d, dashed, "g"] & z \arrow[r, "w"] \arrow[d, "h"] & \Sigma x \arrow[d, "\Sigma f"] \\
x' \arrow[r, "u'"] & y' \arrow[r, "v'"] & z' \arrow[r, "w'"] & \Sigma x'
\end{tikzcd}$$

This follows from (TR2) and (TR3).

A full additive subcategory $\mathfrak{S}$ in $\mathfrak{T}$ is a *triangulated subcategory* if it is closed under isomorphisms and the translation functor $\Sigma$, and for any distinguished triangle $x \to y \to z \to \Sigma x$, if $x$ and $y$ are in $\mathfrak{S}$, then $z$ is in $\mathfrak{S}$. Let $\mathfrak{T}$ and $\mathfrak{T}'$ be triangulated categories with translation functors $\Sigma$ and $\Sigma'$ respectively. A *triangulated functor* from $\mathfrak{T}$ to $\mathfrak{T}'$ is a pair $(F, \sigma)$ where $F : \mathfrak{T} \to \mathfrak{T}'$ is an additive functor and $\sigma : F\Sigma \overset{\cong}{\to} \Sigma' F$, such that given a distinguished triangle $x \overset{\alpha}{\to} y \overset{\beta}{\to} z \overset{\gamma}{\to} \Sigma x$ in $\mathfrak{T}$, the image $Fx \overset{F\alpha}{\to} Fy \overset{F\beta}{\to} Fz \overset{\sigma_x F\gamma}{\to} \Sigma' Fx$ is a distinguished triangle in $\mathfrak{T}'$. If a triangulated functor $F : \mathfrak{T} \to \mathfrak{T}'$ is an equivalence, then $F$ is an *triangle equivalence* and $\mathfrak{T}$ and $\mathfrak{T}'$ are *triangle equivalent.*

**Lemma 0.2.2.** *(i) Let $x \overset{\alpha}{\to} y \overset{\beta}{\to} z \to \Sigma x$ be a distinguished triangle. Then the composition $\beta\alpha$ is zero.*

*(ii) In (TR3), if the morphisms $f$ and $g$ are isomorphisms, then the morphism $h$ is an isomorphism.*

*(iii) Let $x \overset{\alpha}{\to} y \to z \to \Sigma x$ be a distinguished triangle. Then $\alpha$ is an isomorphism if and only if $z \cong 0$.*

*(iv) If $x_1 \to y_1 \to z_1 \to \Sigma x_1$ and $x_2 \to y_2 \to z_2 \to \Sigma x_2$ are distinguished triangles, then $x_1 \oplus x_2 \to y_1 \oplus y_2 \to z_1 \oplus z_2 \to \Sigma x_1 \oplus \Sigma x_2$ is a distinguished triangle. In particular, $x \to x \oplus z \to z \overset{0}{\to} \Sigma x$ is a distinguished triangle.*

*(v) The distinguished triangles $x \to y \to z \overset{\gamma}{\to} \Sigma x$ and $x \to x \oplus z \to z \overset{0}{\to} \Sigma x$ are isomorphic if and only if $\gamma = 0$.*

*(vi) Let $x \overset{\alpha}{\to} y \overset{\beta}{\to} z \to \Sigma x$ be a distinguished triangle. If there is a morphism $f : m \to y$ such that $\beta f = 0$, then there is a morphism $f' : m \to x$ such that $\alpha f' = f$.*

$$\begin{tikzcd}
& m \arrow[dl, dashed, "f'"'] \arrow[d, "f"] \arrow[dr, "0"] & & \\
x \arrow[r, "\alpha"] & y \arrow[r, "\beta"] & z \arrow[r] & \Sigma x
\end{tikzcd}$$

*Proof.* (i) and (ii): This is [17, Proposition I.1.1.]. (iii) This is [35, Corollary 1.2.6]. (iv) This is [35, Proposition 1.2.3]. (v) This is [35, Corollary 1.2.7]. (vi) By (TR1) and (TR2), there is a distinguished triangle $0 \to m \xrightarrow{\mathrm{id}} m \to 0$, and by (TR2), there is a distinguished triangle $\Sigma^{-1}z \to x \xrightarrow{\alpha} y \xrightarrow{\beta} z$. By (TR3'), there is a morphism $f' : m \to x$ such that the following diagram is commutative.

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & m \arrow[r, "\mathrm{id}"] \arrow[d, dashed, "f'"'] & m \arrow[r] \arrow[d, "f"'] & 0 \arrow[d, "0"] \\
\Sigma^{-1}z \arrow[r] & x \arrow[r, "\alpha"] & y \arrow[r, "\beta"] & z
\end{tikzcd}
$$

This gives $\alpha f' = f$. □

Lemma 0.2.2(vi) suggests that a triangulated category might not permit kernels, since the morphism $f'$ is not unique. A distinguished triangle of the form $x \to x \oplus z \to z \xrightarrow{0} \Sigma x$ is said to be a *split distinguished triangle.*

**Lemma 0.2.3.** *([14, Remark 3.2]) Let $x \xrightarrow{\alpha} y \xrightarrow{\beta} z \xrightarrow{\gamma} \Sigma x$ be a distinguished triangle. Then the following are equivalent.*

*(i)* $\gamma \neq 0$,

*(ii) $\alpha$ is not a split monomorphism,*

*(iii) $\beta$ is not a split epimorphism.*

**Lemma 0.2.4.** *([30, Lemma 2.5]) Let $x \xrightarrow{\alpha} y \xrightarrow{\beta} z \xrightarrow{\gamma} \Sigma x$ be a distinguished triangle. Then $\beta$ is right minimal if and only if $\alpha$ is left minimal.*

In each of the next three little sections a different flavour of triangulated categories is introduced. They show how triangulation is distilled and shaped in different categories.

### 0.2.1 The homotopy category

The homotopy category $K(\Lambda)$ is triangulated, see [17, Chapter I. §2.]. This is described as follows. Let the translation functor $\Sigma$ take the value of the suspension functor. Given a morphism $\overline{\varphi} : X \to Y$ in $K(\Lambda)$, complete it to a diagram $X \xrightarrow{\overline{\varphi}} Y \to E(\varphi) \to \Sigma X$, where $E(\varphi)$ is the mapping cone and the morphisms are viewed in $K(\Lambda)$. By Lemma 0.1.30, the mapping cone $E(\varphi)$ is independent of the choice of $\varphi$ up to isomorphism. Let the distinguished triangles in $K(\Lambda)$ be either diagrams of such a form or diagrams isomorphic to them.

The following lemma is essentially a verification of (TR2) in the category $K(\Lambda)$.

**Lemma 0.2.5.** *Consider the identity chain map $1_X : X \to X$ in $C(\Lambda)$. Then the mapping cone $E(1_X)$ is isomorphic to $0$ in $K(\Lambda)$.*

*Proof.* For $(x_{n-1}, x_n)$ in $X_{n-1} \oplus X_n$, define $h_n : X_{n-1} \oplus X_n \to X_n \oplus X_{n+1}$ to be $h_n(x_{n-1}, x_n) = (x_n, 0)$. Consider the following diagram,

$$\begin{tikzcd}
E(1_X) = \cdots \arrow[r] & X_{n+1} \oplus X_{n+2} \arrow[r, "\partial_{n+2}"] \arrow[d, equal] & X_n \oplus X_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, equal] \arrow[dl, "h_{n+1}"'] & X_{n-1} \oplus X_n \arrow[r, "\partial_n"] \arrow[d, equal] \arrow[dl, "h_n"'] & \cdots \\
E(1_X) = \cdots \arrow[r] & X_{n+1} \oplus X_{n+2} \arrow[r, "\partial_{n+2}"] & X_n \oplus X_{n+1} \arrow[r, "\partial_{n+1}"] & X_{n-1} \oplus X_n \arrow[r, "\partial_n"] & \cdots
\end{tikzcd}.$$

Since $(\partial_{n+2}h_{n+1}+h_n\partial_{n+1})(x_n, x_{n+1}) = (\partial_{n+2}h_{n+1})(x_n, x_{n+1})+(h_n\partial_{n+1})(x_n, x_{n+1}) = \partial_{n+2}(x_{n+1}, 0) + h_n(-\partial_n x_n, x_n + \partial_{n+1}x_{n+1}) = (-\partial_{n+1}x_{n+1}, x_{n+1}) + (x_n + \partial_{n+1}x_{n+1}, 0) = (x_n, x_{n+1})$, therefore $\partial_{n+2}h_{n+1} + h_n\partial_{n+1} = 1$. Consequently, the identity chain map on $E(1_X)$ is homotopic to 0, and so $\mathrm{Hom}_{K(\Lambda)}(E(1_X), E(1_X)) = 0$ whence $E(1_X) \cong 0$ in $K(\Lambda)$. $\square$

**Lemma 0.2.6.** *Let $P$ be a projective module in* $\mathrm{Mod}(\Lambda)$ *and let $X$ be in $K(\Lambda)$. Then* $\mathrm{Hom}_{K(\Lambda)}(P, X) \cong \mathrm{Hom}_\Lambda(P, \mathrm{H}_0(X))$.

*Proof.* Let the chain complex $X = \{X_n, \partial_n\}$ be

$$X = \cdots \longrightarrow X_2 \longrightarrow X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\partial_0} X_{-1} \longrightarrow X_{-2} \longrightarrow \cdots \qquad .$$

Let $\theta : \mathrm{Hom}_{K(\Lambda)}(P, X) \to \mathrm{Hom}_\Lambda(P, \mathrm{H}_0(X))$ and $\theta' : \mathrm{Hom}_\Lambda(P, \mathrm{H}_0(X)) \to \mathrm{Hom}_{K(\Lambda)}(P, X)$ be maps between the two Hom sets, which are going to be defined. Consider a morphism $f$ in $\mathrm{Hom}_\Lambda(P, \mathrm{H}_0(X))$ and the canonical morphism $\pi_X : \ker\partial_0 \twoheadrightarrow \mathrm{H}_0(X)$. Since $P$ is projective, there is a morphism $f_0'' : P \to \ker\partial_0$ such that $\pi_X f_0'' = f$.

$$\begin{tikzcd}
\ker\partial_0 \arrow[r, two heads, "\pi_X"] & \mathrm{H}_0(X) \\
P \arrow[u, dashed, "f_0''"] \arrow[ur, "f"']
\end{tikzcd}$$

Extend $f_0''$ to a morphism $f_0' : P \to X_0$, which can readily be extended to a chain map $f'$ in $\mathrm{Hom}_{C(\Lambda)}(P, X)$ and subsequently to a morphism $\overline{f'}$ in $\mathrm{Hom}_{K(\Lambda)}(P, X)$ so that we define $\theta'(f) = \overline{f'}$.

$$\begin{tikzcd}
P = \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "f'_0"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
X = \cdots \arrow[r] & X_2 \arrow[r, "\partial_2"] & X_1 \arrow[r, "\partial_1"] & X_0 \arrow[r, "\partial_0"] & X_{-1} \arrow[r, "\partial_{-1}"] & X_{-2} \arrow[r] & \cdots
\end{tikzcd}$$

Conversely, suppose a morphism $\overline{g}$ in $\mathrm{Hom}_{K(\Lambda)}(P, X)$ is given. In particular, consider the morphism $g_0 : P \to X_0$ obtained from the chain map $g$ in $\mathrm{Hom}_{C(\Lambda)}(P, X)$.

$$\begin{tikzcd}
P = \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "g_0"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
X = \cdots \arrow[r] & X_2 \arrow[r, "\partial_2"] & X_1 \arrow[r, "\partial_1"] & X_0 \arrow[r, "\partial_0"] & X_{-1} \arrow[r, "\partial_{-1}"] & X_{-2} \arrow[r] & \cdots
\end{tikzcd}$$

Let us define $\theta(\overline{g}) = h$ where $h(p) = g_0(p) + \mathrm{im}\partial_1$ for $p$ in $P$. Finally, one can verify that $\theta$ and $\theta'$ are well-defined and that $\theta\theta' = \theta'\theta = 1$. $\square$

*Example* 0.2.7. Below is a special case in Lemma 0.2.6. Let $P$ be the projective module $\Lambda$ and consider the following diagram.

$$\begin{tikzcd}
P = \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] \arrow[dl, dashed] & \Lambda \arrow[r] \arrow[d] \arrow[dl, dashed] & 0 \arrow[r] \arrow[d] \arrow[dl, dashed] & 0 \arrow[r] \arrow[d] \arrow[dl, dashed] & \cdots \\
X = \cdots \arrow[r] & X_2 \arrow[r, "\partial_2"] & X_1 \arrow[r, "\partial_1"] & X_0 \arrow[r, "\partial_0"] & X_{-1} \arrow[r, "\partial_{-1}"] & X_{-2} \arrow[r] & \cdots
\end{tikzcd}$$

Then $\mathrm{Hom}_{K(\Lambda)}(P, X) \cong \ker\partial_0/\mathrm{im}\partial_1 = \mathrm{H}_0(X) \cong \mathrm{Hom}_\Lambda(P, \mathrm{H}_0(X))$.

The following lemma is a little consequence of Lemma 0.2.6.

**Lemma 0.2.8.** *Let $P$ be a projective module in* $\mathrm{Mod}(\Lambda)$ *and let $\overline{\alpha} : A \to B$ be a quasi-isomorphism in $K(\Lambda)$. Then* $\mathrm{Hom}_{K(\Lambda)}(P, \overline{\alpha}) : \mathrm{Hom}_{K(\Lambda)}(P, A) \xrightarrow{\cong} \mathrm{Hom}_{K(\Lambda)}(P, B)$.

*Proof.* Consider the chain complexes $A = \{A_n, \partial_n^A\}$, $B = \{B_n, \partial_n^B\}$ and the quasi-isomorphism $\alpha : A \to B$ in the following diagram.

$$\begin{tikzcd}
A = \cdots \arrow[r] & A_2 \arrow[r, "\partial_2^A"] \arrow[dd, "\alpha_2"] & A_1 \arrow[r, "\partial_1^A"] \arrow[dd, "\alpha_1"] & A_0 \arrow[r, "\partial_0^A"] \arrow[dd, "\alpha_0"] & A_{-1} \arrow[r, "\partial_{-1}^A"] \arrow[dd, "\alpha_{-1}"] & A_{-2} \arrow[r] \arrow[dd, "\alpha_{-2}"] & \cdots \\
 & & P \arrow[ur, "g_0"] \arrow[dr, "h_0"'] & & & & \\
B = \cdots \arrow[r] & B_2 \arrow[r, "\partial_2^B"] & B_1 \arrow[r, "\partial_1^B"] & B_0 \arrow[r, "\partial_0^B"] & B_{-1} \arrow[r, "\partial_{-1}^B"] & B_{-2} \arrow[r] & \cdots
\end{tikzcd}$$

The lemma is to say that given a morphism $\overline{h} : P \to B$ in $\mathrm{Hom}_{K(\Lambda)}(P, B)$, there is a unique morphism $\overline{g} : P \to A$ such that $\overline{\alpha}\,\overline{g} = \overline{h}$. One can see the existence of $\overline{g}$, and here we shall only show its uniqueness.

Suppose there are morphisms $\overline{g}$ and $\overline{g'}$ in $\mathrm{Hom}_{K(\Lambda)}(P, A)$ such that $\overline{\alpha}\,\overline{g} = \overline{\alpha}\,\overline{g'} = \overline{h}$ in $\mathrm{Hom}_{K(\Lambda)}(P, B)$. By Lemma 0.2.6, there is an isomorphism $\theta : \mathrm{Hom}_{K(\Lambda)}(P, B) \overset{\cong}{\to} \mathrm{Hom}_\Lambda(P, \mathrm{H}_0(B))$. Hence $\theta(\,\overline{\alpha}\,\overline{g}) = \theta(\,\overline{\alpha}\,\overline{g'})$, where $\theta(\,\overline{\alpha}\,\overline{g}) = f$ with $f(p) = \alpha_0 g_0(p) + \mathrm{im}\partial_1^B$ and where $\theta(\,\overline{\alpha}\,\overline{g'}) = f'$ with $f'(p) = \alpha_0 g_0'(p) + \mathrm{im}\partial_1^B$ for $p$ in $P$. Hence $\alpha_0 g_0(p) + \mathrm{im}\partial_1^B = \alpha_0 g_0'(p) + \mathrm{im}\partial_1^B$, which gives $g_0(p) + \mathrm{im}\partial_1^A = g_0'(p) + \mathrm{im}\partial_1^A$, so that $\overline{g} = \overline{g'}$ again by Lemma 0.2.6. $\square$

The following lemma is a triangulated version of Lemma 0.1.19.

**Lemma 0.2.9.** *([17, Proposition I.1.1.]) Given a distinguished triangle $(X, Y, Z, u, v, w)$ and an object $M$ in $K(\Lambda)$, there is a long exact sequence*

$$\cdots \longrightarrow (M, \Sigma^{-1}Z) \to (M, X) \longrightarrow (M, Y) \longrightarrow (M, Z) \longrightarrow (M, \Sigma X) \longrightarrow \cdots$$

*in* $\mathrm{Mod}(\mathbb{Z})$*, where all the Hom groups are considered in $K(\Lambda)$. In particular, if*

$$M = \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \Lambda \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \qquad ,$$

*then there is the long exact sequence*

$$\cdots \longrightarrow H_1(Z) \longrightarrow H_0(X) \longrightarrow H_0(Y) \longrightarrow H_0(Z) \longrightarrow H_{-1}(X) \longrightarrow \cdots$$

*by Lemma 0.2.6.*

In Lemma 0.2.9, the functor $(M, -)$ is said to be a *homological* functor. The functor $(-, M)$ is also a homological functor.

**Corollary 0.2.10.** *Let $X \overset{f}{\to} Y \to Z \to$ be a distinguished triangle in $K(\Lambda)$. Then by Lemma 0.2.9, $f$ is a quasi-isomorphism if and only if $Z$ is exact.*

**Lemma 0.2.11.** *([29, Proposition 2.5.3]) Let $X \overset{f}{\to} Y \to Z \to$ be a distinguished triangle in $K(\Lambda)$. Then $Z$ is quasi-isomorphic to the mapping cone $E(f)$.*

### 0.2.2 The derived category

This section introduces one example of a *quotient category*. A collection $S$ of morphisms in a category $\mathfrak{C}$ is a *multiplicative system* if it satisfies the following axioms, see [17, Chapter I. §3.].

(i) (FR1) If $f, g$ are in $S$, then $fg$ is in $S$. For any object $X$ in $\mathfrak{C}$, $\mathrm{id}_X$ is in $S$.

(ii) (FR2) Any diagram
$$\begin{tikzcd} & Z \arrow[d, "s"] \\ X \arrow[r, "u"] & Y \end{tikzcd}$$
with $s$ in $S$ can be completed to a commutative diagram,
$$\begin{tikzcd} W \arrow[r, "v"] \arrow[d, "t"'] & Z \arrow[d, "s"] \\ X \arrow[r, "u"] & Y, \end{tikzcd}$$
with $t$ in $S$. Similarly for the opposite statement.

(iii) (FR3) If $f, g : X \longrightarrow Y$ are morphisms in $\mathfrak{C}$, then the following conditions are equivalent.

  (i) There is an $s : Y \longrightarrow Y'$ in $S$ such that $sf = sg$,
  
  (ii) There is a $t : X \longrightarrow X'$ in $S$ such that $ft = gt$.

The *category of fractions* $S^{-1}\mathfrak{C}$ is the *localization* of $\mathfrak{C}$ with respect to $S$. Namely, $S^{-1}\mathfrak{C}$ has the same objects as $\mathfrak{C}$, and a morphism from $X$ to $Y$ in $S^{-1}\mathfrak{C}$ is represented by the following diagram,
$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y, \end{tikzcd}$$
where $s : Z \to X$ is a morphism in $S$ and $f : Z \to Y$ is a morphism in $\mathfrak{C}$. The equivalence relation on and composition of morphisms in $S^{-1}\mathfrak{C}$ are given in [17, Chapter I. §3.], and are reproduced here.

(i) (*Equivalence relation on morphisms*) Given morphisms from $X$ to $Y$ in $S^{-1}\mathfrak{C}$ represented by the following diagrams,
$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "a"] & \\ X & & Y \end{tikzcd}$$

and

$$\begin{tikzcd}
& X'' \arrow[dl, "t"'] \arrow[dr, "b"] & \\
X & & Y,
\end{tikzcd}$$

the diagrams represent the same morphism if and only if there is a morphism $u : X''' \to X$ in $S$ and morphisms $f : X''' \to X'$, $g : X''' \to X''$ in $\mathfrak{C}$ such that $sf = u = tg$ and $af = bg$, as shown in the following diagram.

$$\begin{tikzcd}
& & X''' \arrow[dl, "f"'] \arrow[dr, "g"] & & \\
& X' \arrow[dl, "s"'] \arrow[drrr, "a" near start] & & X'' \arrow[dlll, "t"' near start] \arrow[dr, "b"] & \\
X & & & & Y
\end{tikzcd}$$

(ii) (*Composition of morphisms*) To compose morphisms

$$\begin{tikzcd}
& X' \arrow[dl, "s"'] \arrow[dr, "a"] & \\
X & & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
& Y' \arrow[dl, "t"'] \arrow[dr, "b"] & \\
Y & & Z,
\end{tikzcd}$$

consider the following commutative diagram given by (FR2),

$$\begin{tikzcd}
& & X'' \arrow[dl, "t'"'] \arrow[dr, "c"] & & \\
& X' \arrow[dl, "s"'] \arrow[dr, "a"] & & Y' \arrow[dl, "t"'] \arrow[dr, "b"] & \\
X & & Y & & Z,
\end{tikzcd}$$

and then take

$$\begin{tikzcd}
& X'' \arrow[dl, "st'"'] \arrow[dr, "bc"] & \\
X & & Z
\end{tikzcd}$$

to be the composition.

The canonical functor $\pi : \mathfrak{C} \to S^{-1}\mathfrak{C}$ maps a morphism $f : X \to Y$ in $\mathfrak{C}$ to the morphism

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

in $S^{-1}\mathfrak{C}$.

**Lemma 0.2.12.** *Let $\mathfrak{C}$ be a category and $S$ be a multiplicative system in $\mathfrak{C}$. Consider the canonical functor $\pi : \mathfrak{C} \to S^{-1}\mathfrak{C}$. Then*

*(i) Given a morphism $s : X \to Y$ in $S$, the morphism $\pi(s)$ is invertible.*

*(ii) If $S$ consists of isomorphisms, then $\pi$ is an equivalence of categories.*

*Proof.*

(i) By definition, $\pi(s) =$

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "s"] & \\ X & & Y, \end{tikzcd}$$

and then naturally $\pi(s)^{-1} =$

$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "\mathrm{id}"] & \\ Y & & X. \end{tikzcd}$$

(ii) Consider the canonical functor $\pi : \mathfrak{C} \to S^{-1}\mathfrak{C}$ and a morphism $f : X \to Y$ in $\mathfrak{C}$. Then $\pi(f) =$

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] & \\ X & & Y. \end{tikzcd}$$

Also define the functor $G : S^{-1}\mathfrak{C} \to \mathfrak{C}$ by

$$G\left(\begin{tikzcd} & Z \arrow[dl, "t"'] \arrow[dr, "d"] & \\ X & & Y \end{tikzcd}\right) = dt^{-1} : X \to Y.$$

Accordingly, $G$ is well-defined and is a quasi-inverse to $F = \pi$. $\square$

Now let $\mathfrak{D}$ be a category and consider a functor $F : \mathfrak{C} \to \mathfrak{D}$. Suppose $F(s)$ is an isomorphism for any $s$ in $S$. Then there is a unique functor $\tilde{F} : S^{-1}\mathfrak{C} \to \mathfrak{D}$ such that the following diagram commutes, i.e. $\tilde{F}\pi = F$.

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "\pi"] \arrow[d, "F"'] & S^{-1}\mathfrak{C} \arrow[dl, dashed, "\tilde{F}"] \\ \mathfrak{D} & \end{tikzcd}$$

This is the *universal property* of $S^{-1}\mathfrak{C}$, see [17, Chapter I. §3.].

Let $\mathfrak{C}$ be a triangulated category with translation functor $\Sigma$, and let $S$ be a multiplicative system in $\mathfrak{C}$ such that, other than (FR1) - (FR3), the following two axioms are also satisfied.

(iv) (FR4) $s$ is in $S$ if and only if $\Sigma(s)$ is in $S$,

(v) (FR5) The same as (TR3), but where $f, g$ are assumed to be in $S$, and $h$ is required to be in $S$.

The multiplicative system $S$ is then said to be *compatible* with the triangulation, see [17, Chapter I. §3.]. Then by [17, Proposition I.3.2], the category $S^{-1}\mathfrak{C}$ has a *unique* triangulated structure such that the canonical functor $\pi : \mathfrak{C} \to S^{-1}\mathfrak{C}$ is *triangulated*. Given a morphism from $X$ to $Y$ in $S^{-1}\mathfrak{C}$,

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y, \end{tikzcd}$$

complete the morphism $f : Z \to Y$ in $\mathfrak{C}$ to a distinguished triangle $Z \xrightarrow{f} Y \to C \to \Sigma Z$ in $\mathfrak{C}$. Then take $\pi X \xrightarrow{\theta} \pi Y \to \pi C \to \Sigma\pi X$, where $\theta = (\pi f)(\pi s)^{-1}$, to be a distinguished triangle in $S^{-1}\mathfrak{C}$.

**Definition 0.2.13.** Let $\mathfrak{C}$ be a category and $S$ be a multiplicative system in $\mathfrak{C}$. An object $P$ in $\mathfrak{C}$ is *$K$-projective* if given a morphism $s : X \to Y$ in $S$, the map $(P, s) : (P, X) \to (P, Y)$ is bijective.

**Lemma 0.2.14.** *Let $\mathfrak{C}$ be a category and $S$ be a multiplicative system in $\mathfrak{C}$. Let $P$ in $\mathfrak{C}$ be $K$-projective. Then $\mathrm{Hom}_{S^{-1}\mathfrak{C}}(P, Y) \cong \mathrm{Hom}_{\mathfrak{C}}(P, Y)$ for any $Y$ in $\mathfrak{C}$.*

*Proof.* Let $\theta : \mathrm{Hom}_{S^{-1}\mathfrak{C}}(P, Y) \to \mathrm{Hom}_{\mathfrak{C}}(P, Y)$ and $\theta' : \mathrm{Hom}_{\mathfrak{C}}(P, Y) \to \mathrm{Hom}_{S^{-1}\mathfrak{C}}(P, Y)$ be maps between the two Hom sets, which are going to be defined. Given a morphism $g : P \to Y$ in $\mathfrak{C}$, define $\theta'(g)$ to be the following morphism

$$\begin{tikzcd} & P \arrow[dl, "\mathrm{id}"'] \arrow[dr, "g"] & \\ P & & Y \end{tikzcd}$$

in $S^{-1}\mathfrak{C}$. Conversely, given the following morphism $\mu =$

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ P & & Y \end{tikzcd}$$

in $\mathrm{Hom}_{S^{-1}\mathfrak{C}}(P, Y)$, where $f : Z \to Y$ is any morphism in $\mathfrak{C}$, there is the bijective map $(P, s) : (P, Z) \overset{\cong}{\to} (P, P)$, since $s : Z \to P$ is a morphism in $S$ and $P$ is $K$-projective. In particular, consider the identity morphism $\mathrm{id} : P \to P$ in $(P, P)$. Then there is a unique $p$ in $(P, Z)$ such that $sp = id$, and $\theta(\mu)$ is defined to be $fp$ in $\mathrm{Hom}_{\mathfrak{C}}(P, Y)$.

Finally, one can verify that $\theta$ and $\theta'$ are well-defined and that $\theta\theta' = 1$. On the other hand, since the diagrams

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ P & & Y \end{tikzcd}$$

and

$$\begin{tikzcd} & P \arrow[dl, "sp=\mathrm{id}"'] \arrow[dr, "fp"] & \\ P & & Y \end{tikzcd}$$

represent the same morphism in $\mathrm{Hom}_{S^{-1}\mathfrak{C}}(P, Y)$, therefore $\theta'\theta$ does indeed equal 1. $\square$

Consider the category $K(\Lambda)$. By [17, Proposition I.4.1], the class of quasi-isomorphisms forms a multiplicative system $S$ in $K(\Lambda)$ compatible with the triangulation. The category of fractions $S^{-1}K(\Lambda)$ is defined to be the *derived category* $D(\Lambda)$ of $\Lambda$. Similarly, let $S_+$ be the class of quasi-isomorphisms inside $K_+(\Lambda)$. Then $S_+$ forms a multiplicative system in $K_+(\Lambda)$, and we define $D_+(\Lambda) = (S_+)^{-1}K_+(\Lambda)$, $D_-(\Lambda)$ and $D_b(\Lambda)$. They are full subcategories of $D(\Lambda)$, and $D_+(\Lambda) \cap D_-(\Lambda) = D_b(\Lambda)$, see [17, Chapter I. §4.].

Since the homology functor sends quasi-isomorphisms to isomorphisms, it induces a well-defined functor on $D(\Lambda)$ by

$$\mathrm{H}_i\left(\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}\right) = \mathrm{H}_i(f)\mathrm{H}_i(s)^{-1}.$$

**Lemma 0.2.15.** *Let $P \to Q \to R \to$ be a distinguished triangle in $K(\Lambda)$. If $P$ and $Q$ are $K$-projective, then $R$ is $K$-projective.*

*Proof.* Let $s : A \to B$ be a quasi-isomorphism in $K(\Lambda)$. Since $P$ and $Q$ are $K$-projective, $\Sigma P$ and $\Sigma Q$ are $K$-projective. This gives the following commutative diagram,

$$\begin{tikzcd} (\Sigma Q, A) \arrow[r] \arrow[d, "\cong"] & (\Sigma P, A) \arrow[r] \arrow[d, "\cong"] & (R, A) \arrow[r] \arrow[d] & (Q, A) \arrow[r] \arrow[d, "\cong"] & (P, A) \arrow[d, "\cong"] \\ (\Sigma Q, B) \arrow[r] & (\Sigma P, B) \arrow[r] & (R, B) \arrow[r] & (Q, B) \arrow[r] & (P, B). \end{tikzcd}$$

Therefore the map $(R, A) \to (R, B)$ is an isomorphism by Lemma 0.1.7 and $R$ is $K$-projective. $\square$

**Lemma 0.2.16.** *Let $P$ and $Q$ be $K$-projective in $K(\Lambda)$. If $p : P \to Q$ is a quasi-isomorphism, then $p$ is an isomorphism.*

*Proof.* The proof given here helps understanding of some of the lemmas mentioned. Since $K(\Lambda)$ is triangulated, complete $p : P \to Q$ to a distinguished triangle $P \xrightarrow{p} Q \to R \to$. Then $R$ is $K$-projective by Lemma 0.2.15. The triangle induces a long exact sequence

$$\cdots \longrightarrow (R, P) \xrightarrow{\cong} (R, Q) \longrightarrow (R, R) \longrightarrow (R, \Sigma P) \xrightarrow{\cong} (R, \Sigma Q) \longrightarrow \cdots$$

by Lemma 0.2.9. Therefore $(R, R) = 0$ and $R \cong 0$, and $p$ is an isomorphism by Lemma 0.2.2(iii). $\square$

*Remark* 0.2.17. Alternatively, by Lemma 0.2.16, the distinguished triangle $P \xrightarrow{p} Q \to R \to$ induces the long exact sequence $\mathrm{H}_i(P) \xrightarrow{\cong} \mathrm{H}_i(Q) \to \mathrm{H}_i(R) \to \mathrm{H}_{i-1}(P) \xrightarrow{\cong} \mathrm{H}_{i-1}(Q)$. Therefore $\mathrm{H}_i(R) = 0$ and $R$ is exact, and the morphism $f : 0 \to R$ is a quasi-isomorphism by Lemma 0.1.18. Since $R$ is also $K$-projective, $(R, 0) \cong (R, R)$. Then there is a unique morphism $g : R \to 0$ such that $fg = 1_R$ where $1_R : R \to R$ is the identity morphism, as shown in the following diagram.

$$\begin{tikzcd} & 0 \arrow[d, "f"] \\ R \arrow[ur, dashed, "g"] \arrow[r, "1_R"'] & R \end{tikzcd}$$

Therefore $1_R = fg = 0$ as it factors through the zero object, and this gives $R \cong 0$.

**Definition 0.2.18.** Let $\mathfrak{C}$ be a category and $S$ be a multiplicative system in $\mathfrak{C}$. Let $X$ be in $\mathfrak{C}$. A *$K$-projective resolution* of $X$ is a $K$-projective $Q$ in $\mathfrak{C}$, together with a morphism $s : Q \to X$ in $S$.

For example, given $X$ in $K(\Lambda)$, a $K$-projective resolution of $X$ is a $K$-projective $Q$ in $K(\Lambda)$, together with a quasi-isomorphism $q : Q \to X$.

*Example* 0.2.19. Let $M$ be in $\mathrm{Mod}(\Lambda)$. Then the projective resolution $P(M)$ is also a $K$-projective resolution. For all $X$ in $K(\Lambda)$, there is a $K$-projective resolution of $X$, see the (unnumbered) remark after [8, Proposition 2.12] and [43, Corollary 3.5].

**Lemma 0.2.20.** *Let $M$ and $N$ be in $\mathrm{Mod}(\Lambda)$, and let $p : P \to M$ be a $K$-projective resolution of $M$. Then $\mathrm{Hom}_{D(\Lambda)}(M, \Sigma^i N) \cong \mathrm{Hom}_{K(\Lambda)}(P, \Sigma^i N)$.*

*Proof.* Let $\theta : \mathrm{Hom}_{D(\Lambda)}(M, \Sigma^i N) \to \mathrm{Hom}_{K(\Lambda)}(P, \Sigma^i N)$ and $\theta' : \mathrm{Hom}_{K(\Lambda)}(P, \Sigma^i N) \to \mathrm{Hom}_{D(\Lambda)}(M, \Sigma^i N)$ be maps between the two Hom sets.

Given the following morphism $\mu =$

$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & \Sigma^i N \end{tikzcd}$$

in $\mathrm{Hom}_{D(\Lambda)}(M, \Sigma^i N)$, where $f : X \to \Sigma^i N$ is any morphism in $\mathrm{Hom}_{K(\Lambda)}(X, \Sigma^i N)$, there is the bijective map $(P, s) : (P, X) \xrightarrow{\cong} (P, M)$, since $s : X \to M$ is a quasi-isomorphism and $P$ is $K$-projective. In particular, consider the morphism $p : P \to M$ in $(P, M)$. Then there is a unique $p'$ in $(P, X)$ such that $sp' = p$. Let $\theta(\mu)$ be $fp'$ in $\mathrm{Hom}_{K(\Lambda)}(P, \Sigma^i N)$.

Conversely, given a morphism $g : P \to \Sigma^i N$ in $K(\Lambda)$, let $\theta'(g)$ be the following morphism

$$\begin{tikzcd} & P \arrow[dl, "p"'] \arrow[dr, "g"] & \\ M & & \Sigma^i N \end{tikzcd}$$

in $\mathrm{Hom}_{D(\Lambda)}(M, \Sigma^i N)$.

Finally, one can verify that $\theta$ and $\theta'$ are well-defined and that $\theta\theta' = \theta'\theta = 1$. $\square$

*Remark* 0.2.21. Alternatively, Lemma 0.2.20 can also been seen directly from Lemma 0.2.12(i) and Lemma 0.2.14.

Let $P_\Lambda$ be the full subcategory of $K(\Lambda)$ consisting of $K$-projectives, and let $S$ be the class of quasi-isomorphisms in $K(\Lambda)$. Then $S \cap P_\Lambda$ forms a multiplicative system in $P_\Lambda$. Consider the following diagram,

$$\begin{tikzcd} P_\Lambda \arrow[r, "\pi_P"] \arrow[d, hook, "i"'] & (S \cap P_\Lambda)^{-1}P_\Lambda \arrow[d, dashed, "\tilde{i}"] \\ K(\Lambda) \arrow[r, "\pi_C"] & S^{-1}K(\Lambda), \end{tikzcd}$$

where $\pi_P, \pi_C$ are the canonical functors. By Lemma 0.2.12(i), quasi-isomorphisms in $K(\Lambda)$ become isomorphisms in $S^{-1}K(\Lambda) = D(\Lambda)$, hence there is a unique functor $\tilde{i} : (S \cap P_\Lambda)^{-1}P_\Lambda \to S^{-1}K(\Lambda)$ making the diagram commutative by the universal property of $(S \cap P_\Lambda)^{-1}P_\Lambda$. By Example 0.2.19, [17, Proposition I.3.3] and Example 0.1.8(ii), the functor $\tilde{i}$ is an equivalence. By Lemma 0.2.12(ii) and Lemma 0.2.16, the functor $\pi_P : P_\Lambda \to (S \cap P_\Lambda)^{-1}P_\Lambda$ is also an equivalence. Hence the functor $\tilde{i}\pi_P : P_\Lambda \to S^{-1}K(\Lambda)$ is an equivalence. Therefore $D(\Lambda)$ can be viewed either as the category of fractions $S^{-1}K(\Lambda)$, or as $P_\Lambda$, the full subcategory inside $K(\Lambda)$.

By virtue of Lemma 0.1.27(i), define the functor $P(-) : \mathrm{Mod}(\Lambda) \to K(\Lambda)$ which sends $M$ in $\mathrm{Mod}(\Lambda)$ to the projective resolution $P(M)$ of $M$ in $K(\Lambda)$. Similarly, define the functor $I(-) : \mathrm{Mod}(\Lambda) \to K(\Lambda)$ which sends $M$ in $\mathrm{Mod}(\Lambda)$ to the injective resolution $I(M)$ of $M$ in $K(\Lambda)$.

The notations $P(-)$ and $I(-)$ will be used for the rest of this section.

**Lemma 0.2.22.** *Given a short exact sequence* $0 \to M' \xrightarrow{\mu'} M \xrightarrow{\mu''} M'' \to 0$ *in* $\mathrm{Mod}(\Lambda)$, *the diagram* $P(M') \xrightarrow{P(\mu')} P(M) \xrightarrow{P(\mu'')} P(M'') \xrightarrow{\partial} \Sigma P(M')$ *is a distinguished triangle in* $K(\Lambda)$, *where the connecting morphism* $\partial : P(M'') \to \Sigma P(M')$ *can be described explicitly.*

*Proof.* Consider the morphism $\mu' : M' \to M$ in $\mathrm{Mod}(\Lambda)$ and then the corresponding morphism $P(\mu') : P(M') \to P(M)$ in $K(\Lambda)$. Complete the morphism $P(\mu')$ to a distinguished triangle $P(M') \xrightarrow{P(\mu')} P(M) \to C \to \Sigma P(M')$ in $K(\Lambda)$, where $C$ is the mapping cone $E(P(\mu'))$ of $P(\mu')$, since $K(\Lambda)$ is triangulated.

This is portrayed in the following diagram, where the morphism $\imath$ is the canonical inclusion and the morphism $\rho$ is the canonical surjection.

$$\begin{tikzcd}
P(M) \arrow[d, "\imath"'] & = \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
C \arrow[d, "\rho"'] & = \arrow[r] & P_2 \oplus P_1' \arrow[r] \arrow[d] & P_1 \oplus P_0' \arrow[r] \arrow[d] & P_0 \oplus 0 \arrow[r] \arrow[d] & 0 \oplus 0 \arrow[r] \arrow[d] & {} \\
\Sigma P(M') & = \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & {}
\end{tikzcd}$$

It can be readily seen that $C$ is a projective resolution of $M''$. Consider the following diagram in $K(\Lambda)$,

$$\begin{tikzcd}
P(M') \arrow[r, "P(\mu')"] \arrow[d, equal] & P(M) \arrow[r, "\imath"] \arrow[d, equal] & C \arrow[r, "\rho"] \arrow[d, "\alpha"] & \Sigma P(M') \arrow[d, equal] \\
P(M') \arrow[r, "P(\mu')"] & P(M) \arrow[r, "P(\mu)"] & P(M'') \arrow[r, "\partial"] & \Sigma P(M'),
\end{tikzcd}$$

where the first row is the distinguished triangle constructed and the second row is the diagram described in the lemma. Since both $C$ and $P(M'')$ are projective resolutions of $M''$, by Lemma 0.1.27(i), there is a chain map $\alpha : C \to P(M'')$ which is an isomorphism by Lemma 0.2.16. Let the connecting morphism $\partial : P(M'') \to \Sigma P(M')$ be $\rho\alpha^{-1}$, and since $\alpha\imath = P(\mu)$ by Lemma 0.1.27(i), the diagram on the second row is then a distinguished triangle by (TR1). $\square$

The description of the left derived functors and right derived functors is an abridgment of [18, IV.5., 6., 7., 8.]. Let $\Lambda_1$ and $\Lambda_2$ be finite-dimensional $k$-algebras over the field $k$.

Consider an additive covariant functor $F : \mathrm{Mod}(\Lambda_1) \to \mathrm{Mod}(\Lambda_2)$, and let $F_C : C(\Lambda_1) \to C(\Lambda_2)$ be the lift of $F$ from the module category to the category of chain complexes, i.e. given a chain complex $M = \cdots \to M_1 \overset{\partial_1}{\to} M_0 \overset{\partial_0}{\to} M_{-1} \to \cdots$ in $C(\Lambda_1)$, define the chain complex $F_C(M) = \cdots \to FM_1 \overset{F\partial_1}{\to} FM_0 \overset{F\partial_0}{\to} FM_{-1} \to \cdots$ in $C(\Lambda_2)$. Let $f, g : M \to N$ be chain maps in $C(\Lambda_1)$ such that $f \simeq g$. Then it is immediate that $F_C f \simeq F_C g$ in $C(\Lambda_2)$. Therefore the functor $F_C$ is well-defined on the homotopy category $K(\Lambda_1)$ as well. Let $F_K : K(\Lambda_1) \to K(\Lambda_2)$ be the lift of $F_C$ from the category of chain complexes to the homotopy category. It is also a triangulated functor.

**Definition 0.2.23.** In the above description, let the *$i$-th left derived functor* of $F$ be $L_i F = \mathrm{H}_i \circ F_K \circ P$.

$$\begin{tikzcd} K(\Lambda_1) \arrow[r, "F_K"] & K(\Lambda_2) \arrow[d, "H_i"] \\ \mathrm{Mod}(\Lambda_1) \arrow[u, "P"] \arrow[r, dashed, "L_iF"] & \mathrm{Mod}(\Lambda_2) \end{tikzcd}$$

The following lemma views the left derived functor $L_iF$ as generalizing the functor $F$.

**Lemma 0.2.24.** *Let us assume that $F$ is right exact, i.e. given a short exact sequence $0 \to M' \overset{\mu}{\to} M \to M'' \to 0$ in $\mathrm{Mod}(\Lambda_1)$, the sequence $FM' \overset{F\mu}{\to} FM \to FM'' \to 0$ is right exact in $\mathrm{Mod}(\Lambda_2)$. Then*

*(i)* $L_iF = 0$ *for* $i < 0$,

*(ii)* $L_0F \simeq F$,

*(iii) There is a long exact sequence*

$$\cdots \longrightarrow L_2F(M'') \longrightarrow L_1F(M') \longrightarrow L_1F(M) \longrightarrow L_1F(M'') \longrightarrow$$
$$L_0F(M') \longrightarrow L_0F(M) \longrightarrow L_0F(M'') \longrightarrow 0$$

*in* $\mathrm{Mod}(\Lambda_2)$.

In Lemma 0.2.24, the functor $F$ fails to be an exact functor, and (iii) expresses $\ker F\mu$ in terms of the long exact sequence. Consider the functor $i : P_{\Lambda_1} \hookrightarrow K(\Lambda_1)$ and the equivalence $\tilde{i}\pi_P : P_{\Lambda_1} \to S^{-1}K(\Lambda_1)$ described above by taking $\Lambda = \Lambda_1$, the functors $F$ and $F_K$ in Definition 0.2.23 and the canonical functors $\pi_i : K(\Lambda_i) \to D(\Lambda_i)$. Let $LF$ be the functor $\pi_2 F_K i(\tilde{i}\pi_P)^{-1} : D(\Lambda_1) \to D(\Lambda_2)$.

$$\begin{tikzcd} K(\Lambda_1) \arrow[r, shift left, "\pi_1"] \arrow[d, "F_K"'] & D(\Lambda_1) \arrow[l, shift left, "i(\tilde{i}\pi_P)^{-1}"] \arrow[d, dashed, "LF"] \\ K(\Lambda_2) \arrow[r, "\pi_2"] & D(\Lambda_2) \end{tikzcd}$$

For example, given $M$ in $\mathrm{Mod}(\Lambda)$, we have

$$\begin{aligned} \mathrm{H}_i LF(M) &= \mathrm{H}_i \pi F_K(P(M)) \\ &= \mathrm{H}_i F_K(P(M)) \\ &= L_iF(M). \end{aligned}$$

In turn, the description of the *right derived functors* of some special functors, namely, the functors $\mathrm{Hom}_\Lambda(-,N)$ and $\mathrm{Hom}_\Lambda(N,-)$, where $N$ is in $\mathrm{Mod}(\Lambda)$, is given.

Both the contravariant functor $\mathrm{Hom}_\Lambda(-,N)$ and the covariant functor $\mathrm{Hom}_\Lambda(N,-)$ are left exact. This allows us to give the following definition.

**Definition 0.2.25.** Let the *i-th right derived functor* $R^i\mathrm{Hom}_\Lambda(-,N)$ of $\mathrm{Hom}_\Lambda(-,N)$, written $\mathrm{Ext}^i_\Lambda(-,N)$, be $\mathrm{Ext}^i_\Lambda(-,N) = \mathrm{H}^i\mathrm{Hom}_\Lambda(P(-),N) : \mathrm{Mod}(\Lambda) \to \mathrm{Mod}(\Lambda)$. Similarly, let the *i-th right derived functor* $R^i\mathrm{Hom}_\Lambda(N,-)$ of $\mathrm{Hom}_\Lambda(N,-)$, written $\overline{\mathrm{Ext}}^i_\Lambda(N,-)$, be $\overline{\mathrm{Ext}}^i_\Lambda(N,-) = \mathrm{H}^i\mathrm{Hom}_\Lambda(N,I(-)) : \mathrm{Mod}(\Lambda) \to \mathrm{Mod}(\Lambda)$.

**Lemma 0.2.26.** *Let $A$ and $B$ be in $\mathrm{Mod}(\Lambda)$. Then*

*(i) ([18, Proposition IV.7.2]) Given a projective module $P$ and an injective module $I$ in $\mathrm{Mod}(\Lambda)$, we have $Ext^i_\Lambda(P,B) = 0 = Ext^i_\Lambda(A,I)$, where $i \geq 1$,*

*(ii) ([18, Proposition IV.8.1]) The functors $Ext^i_\Lambda(-,-)$ and $\overline{Ext}^i_\Lambda(-,-)$, where $i \geq 0$, are naturally equivalent.*

**Lemma 0.2.27.** *(c.f. Lemma 0.2.24) Consider a right exact sequence $M' \to M \to M'' \to 0$ in $\mathrm{Mod}(\Lambda)$. Then*

*(i) $Ext^i_\Lambda(-,N) = 0$ for $i < 0$,*

*(ii) $Ext^0_\Lambda(-,N) \simeq (-,N)$,*

*(iii) There is a long exact sequence*

$$\begin{array}{c} 0 \longrightarrow (M'',N) \longrightarrow (M,N) \longrightarrow (M',N) \longrightarrow \\ Ext^1_\Lambda(M'',N) \longrightarrow Ext^1_\Lambda(M,N) \longrightarrow Ext^1_\Lambda(M',N) \longrightarrow Ext^2_\Lambda(M'',N) \longrightarrow \cdots \end{array}$$

*in $\mathrm{Mod}(\Lambda)$.*

The following is consistent with Definition 0.1.24.

**Definition 0.2.28.** ([18, Exercise IV.8.8.]) The *global dimension* of $\Lambda$, denoted by *gl. dim.* $\Lambda$, is less than or equal to $m$ if for all $\Lambda$-modules $A, B$, $\mathrm{Ext}^q_\Lambda(A,B) = 0$ for all $q > m$. The smallest $m$ with gl. dim. $\Lambda \leq m$ is the global dimension of $\Lambda$.

The following realizes the value of $\mathrm{Ext}^i_\Lambda(-,N)$ as the Hom space in the derived category.

Let $M$ and $N$ be in $\mathrm{Mod}(\Lambda)$.

By Lemma 0.2.6 and Lemma 0.2.20, we have

$$\begin{aligned}
\mathrm{Hom}_{D(\Lambda)}(M, \Sigma^i N) &\cong \mathrm{Hom}_{K(\Lambda)}(P_M, \Sigma^i N), \\
&\quad\ \text{where } P_M \text{ is a } K\text{-projective resolution of } M, \\
&\cong \mathrm{H}_0\mathrm{Hom}_\Lambda(P_M, \Sigma^i N) \\
&\cong \mathrm{H}_0\Sigma^i\mathrm{Hom}_\Lambda(P_M, N) \\
&\cong \mathrm{H}^i\mathrm{Hom}_\Lambda(P_M, N) \\
&\equiv \mathrm{Ext}^i_\Lambda(M, N).
\end{aligned}$$

**Lemma 0.2.29.** *([17, Chapter I. §6.]) Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of chain complexes in $C(\Lambda)$. Then there is a morphism $\zeta : Z \to \Sigma X$ in $D(\Lambda)$ such that $X \to Y \to Z \xrightarrow{\zeta} \Sigma X$ is a distinguished triangle in $D(\Lambda)$.*

### 0.2.3 The stable category

The stable category described in this section is a different flavour of quotient category. For example, the left or right triangulated structure of stable categories of an artin algebra is studied in [6]. In the interim, the following naïve introduction serves as a minute interlude.

The stable categories described below are attainable because given $A_1$ and $A_2$ in $\mathrm{mod}(\Lambda)$, the direct sum $A_1 \oplus A_2$ is injective (resp. projective) if and only if $A_1$ and $A_2$ are injective (resp. projective).

**Definition 0.2.30.** Given $A$ and $B$ in $\mathrm{mod}(\Lambda)$, let $\mathcal{I}(A, B)$ be the set of homomorphisms from $A$ to $B$ which factor through an injective module.

**Definition 0.2.31.** Given $A$ and $B$ in $\mathrm{mod}(\Lambda)$, let $\mathcal{P}(A, B)$ be the set of homomorphisms from $A$ to $B$ which factor through a projective module.

**Definition 0.2.32.** The *(injective) stable category* $\overline{\mathrm{mod}}(\Lambda)$ of $\mathrm{mod}(\Lambda)$ has the same objects as $\mathrm{mod}(\Lambda)$, while the morphism set $\overline{\mathrm{Hom}}(A, B)$ in $\overline{\mathrm{mod}}(\Lambda)$ is defined to be $\mathrm{Hom}(A, B)/\mathcal{I}(A, B)$ for all $A, B$ in $\overline{\mathrm{mod}}(\Lambda)$.

**Definition 0.2.33.** The *(projective) stable category* $\underline{\mathrm{mod}}(\Lambda)$ of $\mathrm{mod}(\Lambda)$ has the same objects as $\mathrm{mod}(\Lambda)$, while the morphism set $\underline{\mathrm{Hom}}(A, B)$ in $\underline{\mathrm{mod}}(\Lambda)$ is defined to be $\mathrm{Hom}(A, B)/\mathcal{P}(A, B)$ for all $A, B$ in $\underline{\mathrm{mod}}(\Lambda)$.

The following lemma shows that objects which are not isomorphic in $\mathrm{mod}(\Lambda)$ can be isomorphic in $\underline{\mathrm{mod}}(\Lambda)$. For example, the ring $\Lambda$ is isomorphic to 0 in $\underline{\mathrm{mod}}(\Lambda)$.

**Lemma 0.2.34.** *(c.f. Lemma 0.1.29) Let* $0$ *be the zero object and let* $X$ *be an object in* $\mathrm{mod}(\Lambda)$*. Then* $\underline{f} : 0 \to X$ *is an isomorphism in* $\underline{\mathrm{mod}}(\Lambda)$ *if and only if* $X$ *is projective.*

*Proof.* (*if*) This is immediate. (*only if*) Let $\underline{g} : X \to 0$ be a morphism in $\underline{\mathrm{mod}}(\Lambda)$ such that $\underline{f}\ \underline{g} = \underline{1_X}$. This means $fg - 1_X$ is in $\mathcal{P}(X,X)$. Since $fg = 0$, therefore $1_X$ is in $\mathcal{P}(X,X)$, i.e. there is a projective module $P$ such that $1_X$ factorizes as $1_X = \beta\alpha$ as follows.

$$\begin{tikzcd} X \arrow[rr, "1_X"] \arrow[dr, "\alpha"'] & & X \\ & P \arrow[ur, "\beta"'] & \end{tikzcd}$$

To see that $X$ is projective, consider an epimorphism $\varepsilon : A \twoheadrightarrow B$ and a morphism $\theta : X \to B$ in $\mathrm{mod}(\Lambda)$. We need to find a morphism $\pi' : X \to A$ such that $\varepsilon\pi' = \theta$.

$$\begin{tikzcd} A \arrow[rr, two heads, "\varepsilon"] & & B \\ & X \arrow[ul, dashed, "\pi'"'] \arrow[ur, "\theta"'] \arrow[d, shift right, "\alpha"'] & \\ & P \arrow[uul, dashed, "\pi"] \arrow[u, shift right, "\beta"'] & \end{tikzcd}$$

Now consider the morphism $\theta\beta : P \to B$. Since $P$ is projective, there is a morphism $\pi : P \to A$ such that $\varepsilon\pi = \theta\beta$. Therefore $\varepsilon\pi\alpha = \theta\beta\alpha = \theta 1_X = \theta$. Finally, $\pi' = \pi\alpha$ is the required morphism. $\square$

## 0.3 Auslander-Reiten quiver

For this section, the reader is suggested to read [14] and [15] for a recapitulation. Let $\mathfrak{C}$ be any category, $\mathfrak{A}$ an abelian category and $\mathfrak{T}$ a triangulated category.

Let $\mathfrak{C}'$ be a full subcategory of $\mathfrak{C}$.

**Definition 0.3.1.** A morphism $g : B \to C$ in the subcategory $\mathfrak{C}'$ is said to be a *right almost split* morphism in $\mathfrak{C}'$ if

(i) $g$ is not a split epimorphism,

(ii) if $h : C' \to C$ in $\mathfrak{C}'$ is not a split epimorphism, then there is an $h' : C' \to B$ such that $gh' = h$.

The notion of a left almost split morphism in the subcategory $\mathfrak{C}'$ is defined dually, i.e. a morphism $f : A \to B$ in the subcategory $\mathfrak{C}'$ is said to be a *left almost split* morphism in $\mathfrak{C}'$ if

(i) $f$ is not a split monomorphism,

(ii) if $h : A \to A'$ in $\mathfrak{C}'$ is not a split monomorphism, then there is an $h' : B \to A'$ such that $h'f = h$.

**Lemma 0.3.2.** *([30, Lemma 2.3]) Let the subcategory $\mathfrak{C}'$ be additive. Then given a left almost split morphism $g : B \to C$ in the subcategory $\mathfrak{C}'$, the endomorphism ring End$(B)$ is a local ring.*

**Definition 0.3.3.** A morphism $h : X \to Y$ in $\mathfrak{A}$ is *irreducible* if $h$ is neither a split monomorphism nor a split epimorphism, and if $h = h_2h_1$ then either $h_1$ is a split monomorphism or $h_2$ is a split epimorphism.

Given objects $X$ and $Y$ in a Krull-Schmidt category $\mathfrak{C}$, the subspace $\text{rad}(X,Y)$ of $(X,Y)$ is defined to be $\text{rad}(X,Y) = \{h \in (X,Y) \mid 1_X - gh \text{ is invertible for any } g \in (Y,X)\}$. Then for $m \geq 1$, the subspace $\text{rad}^m(X,Y) \subseteq \text{rad}(X,Y)$ of $\text{rad}(X,Y)$ consists of all finite sums of morphisms of the form $X = X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} X_2 \to \cdots \to X_{m-1} \xrightarrow{h_m} X_m = Y$, where $h_i$ is in $\text{rad}(X_{i-1}, X_i)$. Let $d_{XY}$ be the dimension $\dim_k \text{Irr}(X,Y)$, where $\text{Irr}(X,Y) = \text{rad}(X,Y)/\text{rad}^2(X,Y)$.

If $X$ and $Y$ are indecomposable, then $f : X \to Y$ is irreducible if and only if $f$ is in $\text{rad}(X,Y)$ but not in $\text{rad}^2(X,Y)$. Therefore there is an irreducible morphism from $X$ to $Y$ if and only if $\text{Irr}(X,Y) \neq 0$.

**Definition 0.3.4.** Let $\mathfrak{C}$ be a Krull-Schmidt category. Then the *Auslander-Reiten quiver* of $\mathfrak{C}$ is a quiver where the vertices are the isomorphism classes $[X]$ of the indecomposable objects $X$ of $\mathfrak{C}$, and the quiver has $d_{XY}$ arrows from $[X]$ to $[Y]$.

The following definition can be found in [14, 3.1] and in [22, Definition 1.3].

**Definition 0.3.5.** *(c.f. Definition 2.2.2)* A distinguished triangle $x \xrightarrow{\alpha} y \xrightarrow{\beta} z \to$, with $x$, $y$ and $z$ in the subcategory $\mathfrak{T}'$ of $\mathfrak{T}$, is an *Auslander-Reiten triangle* in $\mathfrak{T}'$ if

(i) the triangle is not split,

(ii) if $x'$ is in $\mathfrak{T}'$, then each morphism $x \to x'$ which is not a split monomorphism factors through $\alpha$,

(iii) if $z'$ is in $\mathfrak{T}'$, then each morphism $z' \to z$ which is not a split epimorphism factors through $\beta$.

Consider the Auslander-Reiten triangle $x \xrightarrow{\alpha} y \xrightarrow{\beta} z \to$ in Definition 0.3.5. By Lemma 0.2.3, the morphism $\alpha$ is left almost split and the morphism $\beta$ is right almost split.

Let us recollect some standard results with regard to Auslander-Reiten triangles.

**Lemma 0.3.6.** *([30, Lemma 2.6]) Let $x \xrightarrow{\alpha} y \xrightarrow{\beta} z \to$ be a distinguished triangle in $\mathfrak{T}$. Suppose $\beta$ is right almost split. Then the following are equivalent.*

*(i) End$(x)$ is local,*

*(ii) $\beta$ is right minimal,*

*(iii) $\alpha$ is left almost split,*

*(iv) The distinguished triangle is an Auslander-Reiten triangle.*

The following lemma shows how the Auslander-Reiten triangles can be inherent in the Auslander-Reiten quiver.

**Lemma 0.3.7.** *Let $\mathfrak{T}$ be Krull-Schmidt. If $x \to y \to z \to$ is an Auslander-Reiten triangle in $\mathfrak{T}$, then given an indecomposable object $y_i$ in $\mathfrak{T}$, the following are equivalent.*

*(i) There is an irreducible morphism $x \to y_i$,*

*(ii) There is an irreducible morphism $y_i \to z$,*

*(iii) $y_i$ is an indecomposable direct summand of $y$.*

*Proof.* This is described in [15, 4.8]. $\square$

**Definition 0.3.8.** [39, I.1.] Let $\mathfrak{A}$ be $k$-linear and Hom finite. A *right Serre functor* is an additive functor $S : \mathfrak{A} \to \mathfrak{A}$, together with isomorphisms

$$\varphi_{A,B} : (A, B) \xrightarrow{\cong} (B, SA)^*$$

for any $A, B \in \mathfrak{A}$, which are natural in $A$ and in $B$, and where $(-)^* = \mathrm{Hom}_k(-, k)$. If $S$ is an autoequivalence, then it is a *Serre functor*.

Let $\varphi_{A,A}(\mathrm{id}_A)$ be denoted by $\varphi_A$. Any two right Serre functors are isomorphic, and if $\epsilon$ is an autoequivalence of $\mathfrak{A}$, then $\epsilon S \cong S\epsilon$, see [32, Section 3].

**Definition 0.3.9.** The triangulated category $\mathfrak{T}$ is said to *have right (resp. left) Auslander-Reiten triangles* if given any indecomposable $z$ (resp. $x$) in $\mathfrak{T}$, there is an Auslander-Reiten triangle $x \to y \to z \to \Sigma x$. It is then said to *have Auslander-Reiten triangles* if it has both right and left Auslander-Reiten triangles.

**Lemma 0.3.10.** *The following are equivalent.*

*(i)* $\mathfrak{T}$ *has a Serre functor,*

*(ii)* $\mathfrak{T}$ *has Auslander-Reiten triangles.*

*Proof.* This is [39, Theorem I.2.4]. In addition, the Auslander-Reiten triangle is of the form $S\Sigma^{-1}z \to y \to z \to Sz$, where $S$ is the Serre functor and $\Sigma$ is the translation functor of $\mathfrak{T}$. □

This leads us to the following definition.

**Definition 0.3.11.** The functor $\tau \cong S\Sigma^{-1}$ is the *Auslander-Reiten translation.*

# Chapter 1

# The approximation properties of subcategories

## 1.1 Introduction

The idea of approximation is very frequent in mathematics indeed. Several notions in Chapter 0 are inspired in such a way. For example, given $A$ in $\mathrm{Mod}(\Lambda)$, a projective resolution $P(A)$ of $A$ intuitively approximates how projective $A$ is. The right derived functor $\mathrm{Ext}^i_\Lambda(-, N)$ of $\mathrm{Hom}_\Lambda(-, N)$ in Section 0.2.2 can also be understood to approximate the notion of a cokernel. The projective stable category of a given category in Section 0.2.3 approximates the original category by identifying the projective objects with the zero objects.

Paradoxical as it seems, the greater the degree of accuracy to be attained, the greater the strength and the more varieties of approximation needed. In this chapter, torsion pairs in abelian categories and torsion theories in triangulated categories are introduced. The existence of certain adjoint functors in triangulated categories is also studied. Intuitively, they are all different expressions of subcategories approximating their ambient categories. The chapter goes on to introduce some examples of torsion theories, namely t-structures and split torsion theories, and finishes with a characterization of a split torsion theory and a classification of split torsion theories in a chosen derived category.

## 1.2 Existence of certain adjoint functors

Let $\Lambda$ be a finite-dimensional $k$-algebra over the field $k$. As usual, let $\mathrm{Mod}(\Lambda)$ be the category of $\Lambda$-left-modules. In this section, let $\mathcal{T}$ be a triangulated category with translation functor $\Sigma$, and $\mathcal{X}$ and $\mathcal{Y}$ be full additive subcategories of $\mathcal{T}$. The subcategories $\mathcal{X}$ and $\mathcal{Y}$ are assumed to be closed under isomorphisms.

Let us begin with torsion pairs in abelian categories.

**Definition 1.2.1.** [1, Definition VI.1.1] A pair $(\mathcal{M}, \mathcal{N})$ of full subcategories of $\mathrm{Mod}(\Lambda)$ is a *torsion pair* if

(i) $\mathrm{Hom}(M, N) = 0$ for all $M \in \mathcal{M}$, $N \in \mathcal{N}$,

(ii) $\mathrm{Hom}(M, -)|_{\mathcal{N}} = 0$ implies $M \in \mathcal{M}$,

(iii) $\mathrm{Hom}(-, N)|_{\mathcal{M}} = 0$ implies $N \in \mathcal{N}$.

Here, $\mathcal{M}$ is the *torsion class* and $\mathcal{N}$ is the *torsion-free class*. By virtue of (ii) and (iii), $\mathcal{M}$ and $\mathcal{N}$ uniquely determine each other.

**Definition 1.2.2.** [1, Proposition VI.1.4] Let $U$ be in $\mathrm{Mod}(\Lambda)$ and $tU$ be the *trace* of $\mathcal{M}$ in $U$, that is, the sum of images of all homomorphisms from modules in $\mathcal{M}$ to $U$. Since $\mathcal{M}$ is closed under images and direct sums, $tU$ is the largest submodule of $U$ that lies in $\mathcal{M}$.

*Remark* 1.2.3. ([1, Propositions VI.1.4, 1.5]) Intuitively, the functor $t$ in Definition 1.2.2 is an indicator which measures how far $U$ is from being in $\mathcal{M}$ or in $\mathcal{N}$. It has the following significance.

(i) *(approximation)* It approximates $U$ to give $tU$ in $\mathcal{M}$,

(ii) *(characterization)* It characterizes the torsion and torsion-free classes, since $\mathcal{M} = \{M \in \mathrm{Mod}(\Lambda) \mid tM = M\}$ and $\mathcal{N} = \{N \in \mathrm{Mod}(\Lambda) \mid tN = 0\}$,

(iii) *(existence and uniqueness)* It gives existence and uniqueness of short exact sequences of the form $0 \to tU \to U \to \frac{U}{tU} \to 0$.

Instead of the decomposition of an object into direct summands, the following description of torsion theories in triangulated categories intuitively decomposes a category into a pair of subcategories. The definition given below is the one in [21, Definition 2.2], except that we do not assume $\mathcal{X}$ and $\mathcal{Y}$ to be closed under direct sums and direct summands.

**Definition 1.2.4.** Let $\mathcal{X}^{\perp} = \{t \in \mathcal{T} | (x,t) = 0 \text{ for all } x \text{ in } \mathcal{X}\}$ and ${}^{\perp}\mathcal{X} = \{t \in \mathcal{T} | (t,x) = 0 \text{ for all } x \text{ in } \mathcal{X}\}$.

**Definition 1.2.5.** $(\mathcal{X}, \mathcal{Y})$ forms a *torsion theory* if

(i) $(\mathcal{X}, \mathcal{Y}) = 0$,

(ii) $\mathcal{T} = \mathcal{X} * \mathcal{Y}$, i.e. given $t$ in $\mathcal{T}$, there is a *torsion theory triangle* $x \to t \to y \to \Sigma x$ with $x$ in $\mathcal{X}$ and $y$ in $\mathcal{Y}$.

*Remark* 1.2.6. (i) In Definition 1.2.5, (i) gives $\mathcal{X} \cap \mathcal{Y} = 0$. Otherwise a non-zero $U$ in $\mathcal{X} \cap \mathcal{Y}$ would give $\mathrm{id}_U$ in $(\mathcal{X}, \mathcal{Y})$, which is not possible. Accordingly, there are no non-zero $U$ in $\mathcal{X}$ and $V$ in $\mathcal{Y}$ such that $U \cong V$.

(ii) Let $x \xrightarrow{\mu} t \xrightarrow{\nu} y \to \Sigma x$ be the torsion theory triangle described in (ii). Then by Lemma 0.2.2(i), the composition $\nu\mu$ is zero, which accords with (i). By Lemma 0.2.2(iv), given $t \cong x \oplus y$ with $t, x, y$ in $\mathcal{T}$, $\mathcal{X}$, $\mathcal{Y}$ respectively, there is always a distinguished triangle $x \to t \to y \to \Sigma x$, thus satisfying (ii) for the chosen $t$.

The following lemma gives the situations where $\mathcal{X}$ and $\mathcal{Y}$ uniquely determine each other.

**Lemma 1.2.7.** *Let $(\mathcal{X}, \mathcal{Y})$ be a torsion theory.*

*(i) If $\Sigma\mathcal{X} \subseteq \mathcal{X}$, then $\mathcal{X}^{\perp} = \mathcal{Y}$ and ${}^{\perp}\mathcal{Y} = \mathcal{X}$.*

*(ii) If $\mathcal{X}$ and $\mathcal{Y}$ are closed under direct summands, then $\mathcal{X}^{\perp} = \mathcal{Y}$ and ${}^{\perp}\mathcal{Y} = \mathcal{X}$.*

*Proof.* (i) This is [7, Remark I.2.2].

(ii) It is immediate that $\mathcal{Y} \subseteq \mathcal{X}^{\perp}$. To see that $\mathcal{X}^{\perp} \subseteq \mathcal{Y}$, consider $t$ in $\mathcal{X}^{\perp}$. Since $(\mathcal{X}, \mathcal{Y})$ is a torsion theory, there is a distinguished triangle $x \to t \to y \to \Sigma x$ with $x$ in $\mathcal{X}$ and $y$ in $\mathcal{Y}$, where the morphism $x \to t$ is zero. Then by Lemma 0.2.2(v), the distinguished triangle $t \to y \to \Sigma x \to \Sigma t$ splits, i.e. $y \cong t \oplus \Sigma x$. Since $\mathcal{Y}$ is closed under direct summands and isomorphisms, $t$ is in $\mathcal{Y}$. Hence $\mathcal{X}^{\perp} \subseteq \mathcal{Y}$ and so $\mathcal{X}^{\perp} = \mathcal{Y}$. Similarly, ${}^{\perp}\mathcal{Y} = \mathcal{X}$.

□

Torsion theories in triangulated categories are analogues of torsion pairs in abelian categories. Given a torsion pair $(\mathcal{M}, \mathcal{N})$, the functor $t$ described in Definition 1.2.2 characterizes the subcategories $\mathcal{M}$ and $\mathcal{N}$, see Remark 1.2.3.

The functor $t$ involves the notion of images, which does not necessarily make sense in triangulated categories. Following on from this, an analogue of the functor $t$ in torsion theories is not immediate, and this leads to the study of certain adjoint functors in triangulated categories.

**Definition 1.2.8.** An $\mathcal{X}$*-precover* for an object $t$ in $\mathcal{T}$ is a morphism $\alpha : x \to t$ for some $x$ in $\mathcal{X}$, such that for all $x'$ in $\mathcal{X}$, each morphism $x' \to t$ factorizes through $\alpha$. An $\mathcal{X}$*-cover* is an $\mathcal{X}$-precover which is right minimal. The notion of an $\mathcal{X}$*-(pre)envelope* is defined dually.

**Definition 1.2.9.** $\mathcal{X}$ is said to be a *(pre)covering* for $\mathcal{T}$ if every object in $\mathcal{T}$ has an $\mathcal{X}$-(pre)cover. The notion of a *(pre)enveloping* for $\mathcal{T}$ is defined dually.

*Example* 1.2.10. (i) In Remark 1.2.3(i), $tU \to U$ is an $\mathcal{M}$-cover.

(ii) Let $(\mathcal{X}, \mathcal{Y})$ be a torsion theory. Then $\mathcal{X}$ is a precovering and $\mathcal{Y}$ is a preenveloping for $\mathcal{T}$. This is because given $t$ in $\mathcal{T}$, there is a distinguished triangle $x \to t \to y \to \Sigma x$ with $x$ in $\mathcal{X}$, $y$ in $\mathcal{Y}$, where $x \to t$ is an $\mathcal{X}$-precover and $t \to y$ is a $\mathcal{Y}$-preenvelope (Lemma 0.2.2(vi)).

The notion of adjoints is given in Example 0.1.17(iv).

**Lemma 1.2.11.** *Let $L$ be the embedding functor $\imath : \mathcal{X} \hookrightarrow \mathcal{T}$. Assume the right adjoint $R : \mathcal{T} \to \mathcal{X}$ exists. Let $x$ be in $\mathcal{X}$ and $t$ be in $\mathcal{T}$. Consider the isomorphism $\tau : (x, t) \overset{\cong}{\to} (x, Rt)$ and the morphism $\varepsilon_t = \tau^{-1}(\mathrm{id}_{Rt})$. Then given $f : x \to t$, there is a unique morphism $f' : x \to Rt$ such that the following diagram commutes.*

$$\begin{tikzcd} & x \arrow[dl, dashed, "f'"'] \arrow[d, "f"] \\ Rt \arrow[r, "\varepsilon_t"] & t \end{tikzcd}$$

*In addition to this, the morphism $\varepsilon_t = \tau^{-1}(\mathrm{id}_{Rt})$ is right minimal.*

*Proof.* By the naturality of $\tau$, the morphism $f' = \tau(f)$ is the required unique morphism, and this is also conceived of as the *universal property* of the counit $\varepsilon_t : Rt \to t$. The morphism $\varepsilon_t$ is also right minimal. Indeed suppose $\varepsilon_t g = \varepsilon_t$ for any given $g : Rt \to Rt$ and consider the following commutative diagram obtained from the naturality of $\tau$,

$$\begin{tikzcd} (Rt, Rt) \arrow[r, "g^*"] \arrow[d, "\tau^{-1}"'] & (Rt, Rt) \arrow[d, "\tau^{-1}"] \\ (Rt, t) \arrow[r, "g^*"] & (Rt, t). \end{tikzcd}$$

Therefore $(\tau^{-1}(\mathrm{id}_{Rt}))g = \tau^{-1}(\mathrm{id}_{Rt}g)$, which gives $\varepsilon_t g = \tau^{-1}(g)$ and then $\tau^{-1}(g) = \tau^{-1}(\mathrm{id}_{Rt})$. Since $\tau^{-1}$ is an isomorphism, $g = id_{Rt}$ which is an automorphism. Hence $\varepsilon_t$ is right minimal. $\square$

*Remark* 1.2.12. In Lemma 1.2.11, $\mathcal{X}$ is covering in $\mathcal{T}$. Intuitively, $Rt$ in $\mathcal{X}$ is the approximation of $t$ in $\mathcal{X}$, i.e. the right adjoint $R$ can be understood as measuring how close $\mathcal{X}$ is in approximating $\mathcal{T}$.

**Definition 1.2.13.** ([5]) Let $(\mathcal{X}, \mathcal{Y})$ be a torsion theory. If $\Sigma\mathcal{X} \subseteq \mathcal{X}$, then $(\mathcal{X}, \mathcal{Y})$ is a *t-structure.*

*Remark* 1.2.14. Let $(\mathcal{X}, \mathcal{Y})$ be a t-structure. Then by Lemma 1.2.7(i), $\mathcal{X}$ and $\mathcal{Y}$ uniquely determine each other.

**Lemma 1.2.15.** *Let $(\mathcal{X}, \mathcal{Y})$ be a t-structure. Then $\Sigma^{-1}\mathcal{Y} \subseteq \mathcal{Y}$.*

*Proof.* This is [7, Remark I.2.2]. $\square$

**Lemma 1.2.16.** *Let $(\mathcal{X}, \mathcal{Y})$ be a t-structure. Then for each $t$ in $\mathcal{T}$, there is a distinguished triangle $x \to t \to y \to \Sigma x$, with $x$ in $\mathcal{X}$ and $y$ in $\mathcal{Y}$, and the third morphism $y \to \Sigma x$ is uniquely determined.*

*Proof.* Suppose there are distinguished triangles $x \xrightarrow{\alpha} t \xrightarrow{\beta} y \xrightarrow{\gamma_1} \Sigma x$ and $x \xrightarrow{\alpha} t \xrightarrow{\beta} y \xrightarrow{\gamma_2} \Sigma x$. Then consider the following diagram,

$$\begin{tikzcd}
x \arrow[r, "\alpha"] \arrow[d, equal] & t \arrow[r, "\beta"] \arrow[d, equal] & y \arrow[r, "\gamma_1"] \arrow[d, dashed, "g"'] & \Sigma x \arrow[d, equal] \\
x \arrow[r, "\alpha"] & t \arrow[r, "\beta"] & y \arrow[r, "\gamma_2"] & \Sigma x.
\end{tikzcd}$$

By (TR3), there is a morphism $g : y \to y$ such that $\gamma_1 = \gamma_2 g$ and $g\beta = \beta$. Let $f = \mathrm{id}_y - g : y \to y$. Then $f\beta = (\mathrm{id}_y - g)\beta = \beta - g\beta = 0$.

Now consider the following diagram,

$$\begin{tikzcd}
t \arrow[r, "\beta"] \arrow[d] & y \arrow[r, "\gamma_1"] \arrow[d, "f"] & \Sigma x \arrow[r, "-\Sigma\alpha"] \arrow[d, dashed, "h"'] & \Sigma t \arrow[d] \\
0 \arrow[r] & y \arrow[r, "\mathrm{id}"] & y \arrow[r] & 0.
\end{tikzcd}$$

Since $f\beta = 0$, by (TR3) there is $h : \Sigma x \to y$ such that $h\gamma_1 = \mathrm{id}_y f = f$. Since $(\Sigma x, y) = 0$, $h = 0$, and so $f = \mathrm{id}_y - g = 0$, $g = \mathrm{id}_y$ and $\gamma_1 = \gamma_2$. $\square$

**Lemma 1.2.17.** *Let $\mathcal{U} = \mathcal{X} * \mathcal{Y}$. If $\Sigma\mathcal{X} \subseteq \mathcal{X}$ and $\Sigma\mathcal{Y} \subseteq \mathcal{Y}$, then $\Sigma\mathcal{U} \subseteq \mathcal{U}$.*

*Proof.* This is immediate. □

**Definition 1.2.18.** A subcategory $\mathcal{X}$ is *closed under extensions* if given $t$ in $\mathcal{T}$ and a distinguished triangle $x_1 \to t \to x_2 \to \Sigma x_1$ with $x_1$ and $x_2$ in $\mathcal{X}$, then $t$ is in $\mathcal{X}$.

**Lemma 1.2.19.** *The subcategories $\mathcal{X}^\perp$ and ${}^\perp\mathcal{Y}$ are closed under extensions.*

*Proof.* Let $u$ be in $\mathcal{Y}$. Given $t$ in $\mathcal{T}$, consider a distinguished triangle $x_1 \to t \to x_2 \to \Sigma x_1$ with $x_1$ and $x_2$ in ${}^\perp\mathcal{Y}$. Then by Lemma 0.2.9, the sequence $(x_2, u) \to (t, u) \to (x_1, u)$ is exact. Since $(x_2, u) = (x_1, u) = 0$, it follows that $(t, u) = 0$ and so $t$ is in ${}^\perp\mathcal{Y}$ and ${}^\perp\mathcal{Y}$ is closed under extensions. Similarly, $\mathcal{X}^\perp$ is closed under extensions. □

*Example* 1.2.20. Let ($\mathcal{X}$, $\mathcal{Y}$) be a t-structure. By Remark 1.2.14, $\mathcal{X} = {}^\perp\mathcal{Y}$ which, by Lemma 1.2.19, is closed under extensions. For example, given $t$ in $\mathcal{T}$ and a distinguished triangle $x_1 \to x_2 \to t \to \Sigma x_1$ with $x_1$ and $x_2$ in $\mathcal{X}$, then $t$ is in $\mathcal{X}$.

*Remark* 1.2.21.

(i) Let $\mathcal{X}$ be closed under extensions. Given $x$ in $\mathcal{X}$ and the isomorphism $x' \cong x$, embed the isomorphism into a distinguished triangle $x \xrightarrow{\cong} x' \to z \to \Sigma x$ in $\mathcal{T}$. Then by Lemma 0.2.2(iii), $z \cong 0$. Since $\mathcal{X}$ is closed under isomorphisms, $z$ is in $\mathcal{X}$. Since both $x$ and $z$ are in $\mathcal{X}$ and $\mathcal{X}$ is closed under extensions, $x'$ is in $\mathcal{X}$. The two conditions, that $\mathcal{X}$ is closed under extensions, and that $\mathcal{X}$ is closed under isomorphisms, are compatible.

(ii) If $\mathcal{X}$ is closed under extensions, then $\mathcal{X}$ is closed under direct sums. Let $x_1$ and $x_2$ be in $\mathcal{X}$. Then by Lemma 0.2.2(iv), there is a distinguished triangle $x_1 \to x_1 \oplus x_2 \to x_2 \to \Sigma x_1$. Since $\mathcal{X}$ is closed under extensions, $x_1 \oplus x_2$ is in $\mathcal{X}$.

**Definition 1.2.22.** Let $u$ and $v$ be in $\mathcal{T}$. A morphism $f : u \to v$ is an *$\mathcal{X}$-left phantom* if given any $x$ in $\mathcal{X}$ and any morphism $g : x \to u$, the composition $fg$ is zero. Dually, a morphism $f : u \to v$ is a *$\mathcal{Y}$-right phantom* if given any $y$ in $\mathcal{Y}$ and any morphism $g : v \to y$, the composition $gf$ is zero.

For the rest of the chapter, given a torsion theory ($\mathcal{X}$, $\mathcal{Y}$), assume $\mathcal{X}$ and $\mathcal{Y}$ to determine each other uniquely. By Lemma 1.2.19, the subcategories $\mathcal{X}$ and $\mathcal{Y}$ are both closed under extensions and direct sums (Remark 1.2.21(ii)). The following propositions give necessary and sufficient conditions for the existence of certain adjoint functors in triangulated categories.

**Proposition 1.2.23.** *(c.f. [27, 1.1], [7, Proposition I.2.3]) Let ($\mathcal{X}$, $\mathcal{Y}$) be a torsion theory. Then the inclusion $\imath : \mathcal{X} \hookrightarrow \mathcal{T}$ admits a right adjoint $R : \mathcal{T} \to \mathcal{X}$ if and only if for each $t$ in $\mathcal{T}$, there is a distinguished triangle*

*$x \to t \to y \to \Sigma x$, where $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, and the morphism $h$: $\Sigma^{-1}y \to x$ is an $\mathcal{X}$-left phantom.*

$$\begin{tikzcd} x' \arrow[d] \arrow[dr, dashed, "0"] & \\ \Sigma^{-1}y \arrow[r] & x \end{tikzcd}$$

*Proof.* (*if*) Let $t$ be in $\mathcal{T}$. Then there is a distinguished triangle $x \to t \to y \to \Sigma x$, with $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, and the morphism $h : \Sigma^{-1}y \to x$ is an $\mathcal{X}$-left phantom. Let $x'$ be in $\mathcal{X}$. Applying the homological functor $(x', -)$ to the distinguished triangle, there is the long exact sequence $(x', \Sigma^{-1}y) \to (x', x) \to (x', t) \to (x', y)$ with $(x', y) = 0$ (Lemma 0.2.9). The first map is zero since the morphism $h$: $\Sigma^{-1}y \to x$ is an $\mathcal{X}$-left phantom. Hence $(x', x) \cong (x', t)$ and $R$ exists by defining $R(t) = x$. (*only if*) Since the right adjoint $R$ exists, there is the isomorphism $\tau^{-1} : (x, Rt) \xrightarrow{\cong} (x, t)$ for all $x$ in $\mathcal{X}$, $t$ in $\mathcal{T}$. Let $\alpha = \tau^{-1}(\mathrm{id}_{Rt}) : Rt \to t$. Then by Lemma 1.2.11, $\alpha$ is right minimal. Embed $\alpha$ into a distinguished triangle $Rt \xrightarrow{\alpha} t \to y \to \Sigma Rt$. By [22, Lemma 2.1], $y$ is in $\mathcal{Y}$. Let $x'$ be in $\mathcal{X}$. Applying the homological functor $(x', -)$ to the distinguished triangle, there is the exact sequence $(x', \Sigma^{-1}y) \to (x', Rt) \to (x', t)$ (Lemma 0.2.9). By Lemma 1.2.11, the second map is an isomorphism, and so the first map is zero. Hence the morphism $h$: $\Sigma^{-1}y \to Rt$ is an $\mathcal{X}$-left phantom. $\square$

The following is the dual.

**Proposition 1.2.24.** *Let ($\mathcal{X}$, $\mathcal{Y}$) be a torsion theory. Then the inclusion $\imath : \mathcal{Y} \hookrightarrow \mathcal{T}$ admits a left adjoint $L : \mathcal{T} \to \mathcal{Y}$ if and only if for each $t$ in $\mathcal{T}$, there is a distinguished triangle $x \to t \to y \to \Sigma x$, where $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, and the morphism $h$: $y \to \Sigma x$ is a $\mathcal{Y}$-right phantom.*

$$\begin{tikzcd} y \arrow[r] \arrow[dr, dashed, "0"'] & \Sigma x \arrow[d] \\ & y' \end{tikzcd}$$

*Proof.* This is the dual of Proposition 1.2.23. $\square$

*Example* 1.2.25. Let ($\mathcal{X}$, $\mathcal{Y}$) be a t-structure. Then the inclusion $\imath : \mathcal{X} \hookrightarrow \mathcal{T}$ admits a right adjoint $R : \mathcal{T} \to \mathcal{X}$. Similarly, the inclusion $\imath : \mathcal{Y} \hookrightarrow \mathcal{T}$ admits a left adjoint $L : \mathcal{T} \to \mathcal{Y}$.

*Proof.* Given $t$ in $\mathcal{T}$, there is a distinguished triangle $x \to t \to y \to \Sigma x$, where $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. By Lemma 1.2.15, $\Sigma^{-1}y$ is in $\mathcal{Y}$ and so $(x', \Sigma^{-1}y) = 0$ for any $x'$ in $\mathcal{X}$ and $h : \Sigma^{-1}y \to x$ is trivially an $\mathcal{X}$-left phantom. Therefore by Proposition 1.2.23, the inclusion $\imath : \mathcal{X} \hookrightarrow \mathcal{T}$ admits a right adjoint $R : \mathcal{T} \to \mathcal{X}$. Similarly, by Proposition 1.2.24, the inclusion $\imath : \mathcal{Y} \hookrightarrow \mathcal{T}$ admits a left adjoint $L : \mathcal{T} \to \mathcal{Y}$. □

*Remark* 1.2.26. In Example 1.2.25, given $t$ in $\mathcal{T}$, there is a unique torsion theory triangle of the form $Rt \to t \to Lt \to \Sigma Rt$. This makes sense, since any two right (resp. left) adjoints of the same functor are naturally equivalent.

**Corollary 1.2.27.** *Let $(\mathcal{X}, \mathcal{Y})$ be a torsion theory. Suppose the right adjoint $R$ in Proposition 1.2.23 and the left adjoint $L$ in Proposition 1.2.24 exist. Then $\mathcal{X} = \{t \in \mathcal{T} \mid Lt = 0\}$ and $\mathcal{Y} = \{t \in \mathcal{T} \mid Rt = 0\}$.*

*Proof.* Let $x$ be in $\mathcal{X}$ and $t$ be in $\mathcal{T}$. If $Rt = 0$, then $t \in \mathcal{X}^{\perp} = \mathcal{Y}$, since $(\imath x, t) \cong (x, Rt) = (x, 0) = 0$. On the other hand, given $t \in \mathcal{Y} = \mathcal{X}^{\perp}$, since $0 = (\imath x, t) \cong (x, Rt)$ for each $x$ in $\mathcal{X}$, therefore $Rt = 0$. Hence $\mathcal{Y} = \{t \in \mathcal{T} \mid Rt = 0\}$. Similarly, $\mathcal{X} = \{t \in \mathcal{T} \mid Lt = 0\}$. □

By virtue of Example 1.2.25, Remark 1.2.26 and Corollary 1.2.27 are summarized in the following.

**Property 1.2.28.** *(c.f. Remark 1.2.3)* Let $(\mathcal{X}, \mathcal{Y})$ be a t-structure and consider the right adjoint $R$ and the left adjoint $L$ in Example 1.2.25. Then

(i) *(existence and uniqueness)* For each $t$ in $\mathcal{T}$, there is a torsion theory triangle $Rt \to t \to Lt \to \Sigma Rt$, and it is unique up to a unique isomorphism,

(ii) *(characterization)* $\mathcal{X} = \{t \in \mathcal{T} \mid Lt = 0\}$ and $\mathcal{Y} = \{t \in \mathcal{T} \mid Rt = 0\}$.

*Example* 1.2.29. Let $(\mathcal{X}, \mathcal{Y})$ be a t-structure with $\mathcal{X}$, $\mathcal{Y}$ closed under direct summands. Let $t$ be in $\mathcal{X}^{\perp}$. Then since $(x, 0) \cong (x, t) = 0$, therefore $Rt = 0$. This gives the distinguished triangle $0 \to t \to Lt \to 0$. By Lemma 0.2.2(iii), $t \cong Lt$, and the above distinguished triangle is isomorphic to $t \xrightarrow{id} t \to 0 \to \Sigma t$. Alternatively, let $x \to t \to y \to \Sigma x$ be the torsion theory triangle, where $x$ is in $\mathcal{X}$ and $y$ is in $\mathcal{Y}$. Since $t$ is in $\mathcal{X}^{\perp}$, therefore $y \cong t \oplus \Sigma x$ by Lemma 0.2.2(v). It follows that $\Sigma x$ is in $\mathcal{Y}$. On the other hand, since $(\mathcal{X}, \mathcal{Y})$ is a t-structure, $\Sigma x$ is in $\mathcal{X}$ as well, therefore $\Sigma x$ is in $\mathcal{X} \cap \mathcal{Y} = 0$. This gives $\Sigma x \cong 0$ and $t \cong y$. Then the torsion theory triangle is isomorphic to the distinguished triangle $0 \to t \xrightarrow{\alpha} t \to 0$ where $\alpha$ is an isomorphism.

*Remark* 1.2.30. *(c.f. Remark 1.2.3, Remark 1.2.12)* Intuitively, a torsion theory $(\mathcal{X}, \mathcal{Y})$ can be understood as measuring how far each $t$ in $\mathcal{T}$ is from being in $\mathcal{X}$ or in $\mathcal{Y}$. Since $(\mathcal{X}, \mathcal{Y})$ is a torsion theory, $\mathcal{X} \cap \mathcal{Y} = 0$ and so there is no contradiction (Remark 1.2.6(i)).

## 1.3 Examples of torsion theories

In this section, let $k$ be a field and $\mathcal{T}$ be a $k$-linear Hom finite and Krull-Schmidt triangulated category with translation functor $\Sigma$ and Serre functor $S$, see Definition 0.3.8. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be full additive subcategories of $\mathcal{T}$. They are assumed to be closed under isomorphisms.

Let us begin with a few lemmas.

**Lemma 1.3.1.** *(i) If $S^{-1}\Sigma\mathcal{X} \subseteq \mathcal{X}$, $\mathcal{X} \subseteq S^{-1}\mathcal{X}$, then $\Sigma\mathcal{X} \subseteq \mathcal{X}$.*

*(ii) If $S^{-1}\Sigma\mathcal{X} \subseteq \mathcal{X}$, $\mathcal{X} \subseteq \Sigma\mathcal{X}$, then $S^{-1}\mathcal{X} \subseteq \mathcal{X}$.*

*(iii) If $\Sigma\mathcal{X} \subseteq \mathcal{X}$, $S^{-1}\mathcal{X} \subseteq \mathcal{X}$, then $S^{-1}\Sigma\mathcal{X} \subseteq \mathcal{X}$.*

*(iv) If $\mathcal{Y} \subseteq S^{-1}\Sigma\mathcal{Y}$, $\Sigma\mathcal{X} \subseteq \mathcal{Y}$, then $S\mathcal{X} \subseteq \mathcal{Y}$.*

*Proof.* (i) This is because $\Sigma\mathcal{X} \subseteq \Sigma S^{-1}\mathcal{X} = S^{-1}\Sigma\mathcal{X} \subseteq \mathcal{X}$.

(ii) This is because $S^{-1}\mathcal{X} \subseteq \Sigma^{-1}\mathcal{X} \subseteq \mathcal{X}$.

(iii) This is because $S^{-1}\Sigma\mathcal{X} \subseteq S^{-1}\mathcal{X} \subseteq \mathcal{X}$.

(iv) Since $\Sigma\mathcal{X} \subseteq \mathcal{Y} \subseteq S^{-1}\Sigma\mathcal{Y}$, therefore $\mathcal{X} \subseteq S^{-1}\mathcal{Y}$, $S\mathcal{X} \subseteq \mathcal{Y}$.

□

**Lemma 1.3.2.** *Suppose $\mathcal{X}^{\perp} = \mathcal{Y}$, $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $(\mathcal{X}, \mathcal{Y}) = 0$. Then $S\mathcal{Y} \subseteq \mathcal{Y}$ if and only if $S^{-1}\mathcal{X} \subseteq \mathcal{X}$.*

*Proof. (only if)* Let $X$ be in $\mathcal{X}$ and $Y$ be in $\mathcal{Y}$. Since $S\mathcal{Y} \subseteq \mathcal{Y}$, therefore $SY = Y'$ for some $Y'$ in $\mathcal{Y}$. Hence

$$\begin{aligned} (S^{-1}X, Y) &\cong (X, SY) \\ &= (X, Y') \\ &= 0. \end{aligned}$$

Therefore $S^{-1}X \in {}^{\perp}\mathcal{Y} = \mathcal{X}$. *(if)* Similar. □

**Lemma 1.3.3.** *Suppose $\mathcal{X}^{\perp} = \mathcal{Y}$, $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $(\mathcal{X}, \mathcal{Y}) = 0$. Then $S\mathcal{X} \subseteq \mathcal{X}$ if and only if $S^{-1}\mathcal{Y} \subseteq \mathcal{Y}$.*

*Proof.* Similar to Lemma 1.3.2. □

Consider $t_1$ and $t_2$ in $\mathcal{T}$ with $t_1 \cong x_1 \oplus y_1$ and $t_2 \cong x_2 \oplus y_2 \oplus z$, where $x_1$ and $x_2$ are in $\mathcal{X}$, $y_1$ and $y_2$ are in $\mathcal{Y}$ and $z$ is in $\mathcal{Z}$. In both cases objects from both $\mathcal{X}$ and $\mathcal{Y}$ are present. However, there might or might not be the two subcategories $\mathcal{U}_1$ and $\mathcal{U}_2$ of $\mathcal{T}$ such that each $t$ in $\mathcal{T}$ can be written $t \cong u_1 \oplus u_2$ with $u_1$ in $\mathcal{U}_1$ and $u_2$ in $\mathcal{U}_2$. A split torsion theory is an expression for this.

**Definition 1.3.4.** A *split torsion theory* is a torsion theory where all the torsion theory triangles split.

**Theorem 1.3.5.** *(Characterizations of split torsion theories) Let $(\mathcal{X}, \mathcal{Y})$ be a torsion theory. Then the following statements are equivalent.*

*(i)* $(\mathcal{X}, \mathcal{Y})$ *is a split torsion theory,*

*(ii)* $(Y, \Sigma X) = 0$ *for all* $X \in \mathcal{X}$, $Y \in \mathcal{Y}$,

*(iii)* $S^{-1}\Sigma\mathcal{X} \subseteq \mathcal{X}$,

*(iv)* $S\Sigma^{-1}\mathcal{Y} \subseteq \mathcal{Y}$.

*Proof.* Let $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

(i) $\Rightarrow$ (ii): Consider $f : y \to \Sigma x$. Complete it to a distinguished triangle $x \to t \to y \xrightarrow{f} \Sigma x$ in $\mathcal{T}$. Then $f = 0$ by definition of a split torsion theory (Lemma 0.2.2(v)).

(ii) $\Rightarrow$ (i): This is true by definition (Lemma 0.2.2(v)).

(ii) $\Rightarrow$ (iii): Since $(S^{-1}\Sigma x, y) \cong (y, \Sigma x)^* = 0$, therefore $S^{-1}\Sigma x \in {}^{\perp}\mathcal{Y} = \mathcal{X}$.

(iii) $\Rightarrow$ (ii): For $x' = S^{-1}\Sigma x \in \mathcal{X}$, we have $(y, \Sigma x) \cong (\Sigma x, Sy)^* \cong (S^{-1}\Sigma x, y)^* = (x', y)^* = 0$.

(ii) $\Rightarrow$ (iv): Since $(x, S\Sigma^{-1}y) \cong (\Sigma^{-1}y, x)^* \cong (y, \Sigma x)^* = 0$, therefore $S\Sigma^{-1}y \in \mathcal{X}^{\perp} = \mathcal{Y}$.

(iv) $\Rightarrow$ (ii): For $y' = S\Sigma^{-1}y \in \mathcal{Y}$, we have $(y, \Sigma x) \cong (\Sigma^{-1}y, x) \cong (x, S\Sigma^{-1}y)^* = (x, y')^* = 0$. □

*Remark* 1.3.6. The composition $S\Sigma^{-1}$ is the *Auslander-Reiten translation* given in Definition 0.3.11.

*Example* 1.3.7. *(c.f. Example 1.2.25)* Let $(\mathcal{X}, \mathcal{Y})$ be a split torsion theory. Then the inclusion $\imath : \mathcal{X} \hookrightarrow \mathcal{T}$ admits a right adjoint $R : \mathcal{T} \to \mathcal{X}$. Similarly, the inclusion $\imath : \mathcal{Y} \hookrightarrow \mathcal{T}$ admits a left adjoint $L : \mathcal{T} \to \mathcal{Y}$.

*Proof.* Consider the morphism $h : \Sigma^{-1}y \to x$ in Proposition 1.2.23. Since $(\Sigma^{-1}y, x) = (y, \Sigma x) = 0$ by Theorem 1.3.5, $h$ is trivially an $\mathcal{X}$-left phantom. Therefore by Proposition 1.2.23, the inclusion $\imath : \mathcal{X} \hookrightarrow \mathcal{T}$ admits a right adjoint $R : \mathcal{T} \to \mathcal{X}$. Similarly, by Proposition 1.2.24, the inclusion $\imath : \mathcal{Y} \hookrightarrow \mathcal{T}$ admits a left adjoint $L : \mathcal{T} \to \mathcal{Y}$. □

*Remark* 1.3.8. By virtue of Example 1.3.7, Property 1.2.28 is also true for split torsion theories.

*Remark* 1.3.9. Consider the diagram in Proposition 1.2.23. If the torsion theory $(\mathcal{X}, \mathcal{Y})$ is a t-structure, then the morphism $x' \to \Sigma^{-1}y$ is zero and the right adjoint exists (Example 1.2.25). On the other hand, Theorem 1.3.5 describes the behaviour of the morphism $\Sigma^{-1}y \to x$, and characterizes another example of a torsion theory, the split torsion theory, where the right adjoint exists as well (Example 1.3.7).

**Lemma 1.3.10.** *Let $(\mathcal{X}, \mathcal{Y})$ be a split torsion theory. Then for each indecomposable $t$ in $\mathcal{T}$, either $t$ is in $\mathcal{X}$ or $t$ is in $\mathcal{Y}$.*

*Proof.* Since $(\mathcal{X}, \mathcal{Y})$ is a torsion theory, $\mathcal{X} \cap \mathcal{Y} = 0$ and so $t$ cannot be in both $\mathcal{X}$ and $\mathcal{Y}$ (Remark 1.2.6(i)). Given an indecomposable $t$ in $\mathcal{T}$, there is a distinguished triangle $x \to t \to y \to \Sigma x$ for some $x$ in $\mathcal{X}$ and $y$ in $\mathcal{Y}$. Since the torsion theory is split, $t \cong x \oplus y$. Since $t$ is indecomposable, either $x \cong 0$ or $y \cong 0$, which gives either $t \cong y$ or $t \cong x$. Therefore either $t$ is in $\mathcal{X}$ or $t$ is in $\mathcal{Y}$, since $\mathcal{X}$ and $\mathcal{Y}$ are closed under isomorphisms. □

*Remark* 1.3.11. The converse of Lemma 1.3.10 is not true. Suppose $(\mathcal{X}, \mathcal{Y})$ is a torsion theory, and that for each indecomposable $t$ in $\mathcal{T}$, either $t$ is in $\mathcal{X}$ or $t$ is in $\mathcal{Y}$. Then $(\mathcal{X}, \mathcal{Y})$ does not need to be a split torsion theory. This is because the existence of a torsion theory triangle is not unique.

**Lemma 1.3.12.** *Let $(\mathcal{X}, \mathcal{Y})$ be a split torsion theory. Then $(\mathcal{X}, \mathcal{Y})$ is a t-structure.*

*Proof.* Consider an indecomposable $x$ in $\mathcal{X}$. By Lemma 1.3.10, either $\Sigma x$ is in $\mathcal{X}$ or $\Sigma x$ is in $\mathcal{Y}$. Suppose $\Sigma x$ is in $\mathcal{Y}$. By Theorem 1.3.5, $\mathcal{Y} \subseteq S^{-1}\Sigma\mathcal{Y}$, which gives $Sx \in \mathcal{Y}$ (Lemma 1.3.1(iv)). By Lemma 0.3.10, there is the Auslander-Reiten triangle $u \to v \to x \to Sx$ in $\mathcal{T}$. However, the triangle would split, since $x$ is in $\mathcal{X}$ and $Sx$ is in $\mathcal{Y}$, which is not possible. Therefore $\Sigma x$ is in $\mathcal{X}$, i.e. $\Sigma\mathcal{X} \subseteq \mathcal{X}$. □

## 1.4 The finite derived category of Dynkin type

Let $\triangle$ be a quiver (a directed graph). Given an arrow $\alpha$, the initial point (resp. end point) of $\alpha$ is denoted by $s(\alpha)$ (resp. $e(\alpha)$). A *path* of length

$l \geq 1$ from a vertex $x$ to a vertex $y$ is of the form $(x \mid \alpha_1, \ldots, \alpha_l \mid y)$ with arrows $\alpha_i$ satisfying $e(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i < l$, $s(\alpha_1) = x$ and $e(\alpha_l) = y$. For any vertex $x$ in $\triangle$, a path of length 0 is denoted by $(x \mid x)$. Let the *path algebra* $k\triangle$ be the $k$-vector space with basis the set of all paths of length $l \geq 0$ in $\triangle$. The product of two paths $p_1 = (x_1 \mid \alpha_1, \ldots, \alpha_m \mid y_1)$ and $p_2 = (x_2 \mid \beta_1, \ldots, \beta_n \mid y_2)$ is $p_1 p_2 = (x_1 \mid \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \mid y_2)$ if $y_1 = x_2$. Otherwise it is zero. The global dimension of a path algebra $k\triangle$ is either 0 or 1 (Example 0.1.25).

Below are the cases where $\triangle$ is a *Dynkin graph*.

The Dynkin graph of type $A_n$:

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } \cdots \text{ --- } n-1 \text{ --- } n$$

The Dynkin graph of type $D_n$, $n \geq 4$:

$$\begin{tikzcd}[arrows=dash]
 & & & & & n-1 \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n-2 \arrow[ur] \arrow[dr] & \\
 & & & & & n
\end{tikzcd}$$

The Dynkin graph of type $E_6$:

$$\begin{tikzcd}[arrows=dash]
 & & 6 & & \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[u] \arrow[r] & 4 \arrow[r] & 5
\end{tikzcd}$$

The Dynkin graph of type $E_7$:

$$\begin{tikzcd}[arrows=dash]
 & & 7 & & & \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[u] \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6
\end{tikzcd}$$

The Dynkin graph of type $E_8$:

$$\begin{tikzcd}[arrows=dash]
 & & 8 & & & & \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[u] \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7
\end{tikzcd}$$

The finite derived category $D^b(\operatorname{mod} k\triangle)$ is a $k$-linear Hom finite triangulated category. It is Krull-Schmidt as well. By [14, Theorem 3.6], the finite derived category $D^b(\operatorname{mod} k\triangle)$ has Auslander-Reiten triangles, and by Lemma 0.3.10 also a Serre functor.

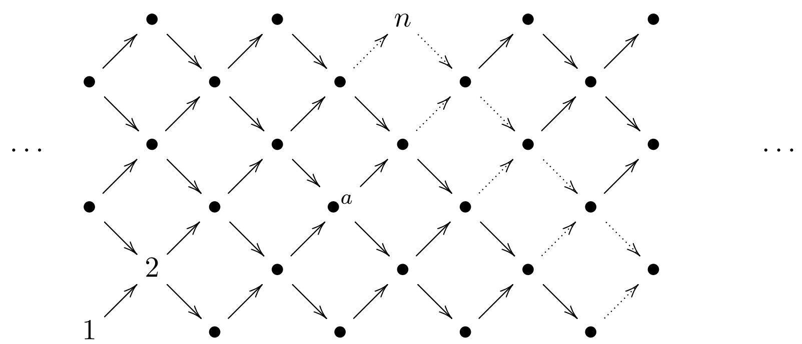
Given a quiver $\triangle$, there is the *translation quiver* $\mathbb{Z}\triangle$ naturally induced by $\triangle$. The vertices of $\mathbb{Z}\triangle$ is the set $\mathbb{Z} \times \triangle_0$, where $\triangle_0$ is the set of vertices of $\triangle$. The number of arrows from $(n, x)$ to $(m, y)$ in $\mathbb{Z} \times \triangle_0$ is the number of arrows from $x$ to $y$ (resp. from $y$ to $x$) in $\triangle_0$ if $n = m$ (resp. $m = n + 1$), and there are no arrows otherwise. If $\triangle$ is a Dynkin graph, then $\mathbb{Z}\triangle$ does not depend on the orientation of $\triangle$, see [15, I.5.6], and the Auslander-Reiten quiver of the finite derived category $D^b(\operatorname{mod} k\triangle)$ is $\mathbb{Z}\triangle$. This is given in [14, 4.5].

**Lemma 1.4.1.** *Let $x$ and $y$ be indecomposable objects of the finite derived category $D^b(\operatorname{mod} k\triangle)$, where $\triangle$ is a Dynkin graph. Then by [14, 4.6], any non-zero morphism $f$ from $x$ to $y$ is a linear combination of morphisms, written $f = \Sigma\alpha_i f_i$, where the $\alpha_i$ are scalars and the $f_i : x \to y$ are compositions of irreducible morphisms.*

Let $\triangle$ be a Dynkin graph. Then the category $\operatorname{ind} D^b(\operatorname{mod} k\triangle)$ is equivalent to the *mesh category* of $\mathbb{Z}\triangle$. This means that given an Auslander-Reiten triangle $a \stackrel{\alpha}{\to} b \stackrel{\beta}{\to} c \to$ in the finite derived category $D^b(\operatorname{mod} k\triangle)$, which can be read off from the Auslander-Reiten quiver, there is the relation $\beta\alpha = 0$ (Lemma 0.2.2(i)), and only relations of this kind are present. The above is only a gentle and informal initiation, and the reader will be able to seek guidance from [14, 4.6].

Together with Lemma 1.4.1, the finite derived category $D^b(\operatorname{mod} k\triangle)$, where $\triangle$ is a Dynkin graph, is then said to be *standard*.

For example, the Auslander-Reiten quiver of $D^b(\operatorname{mod} kA_n)$ is




The following coordinate system on the quiver is employed. Suppose the indecomposable object $a$ has coordinates $(i,j)$. The coordinates of some of its surrounding objects are given in the following diagram.

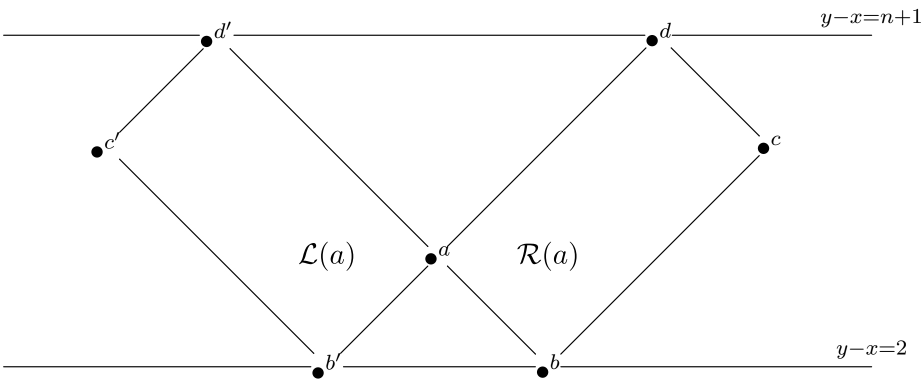
$$\begin{tikzcd}[column sep=tiny]
 & (i-2,j-1) \arrow[dr] & & (i-1,j) \arrow[dr] & & (i,j+1) \arrow[dr] & \\
(i-2,j-2) \arrow[ur] \arrow[dr] & & (i-1,j-1) \arrow[ur] \arrow[dr] & & (i,j) \arrow[ur] \arrow[dr] & & (i+1,j+1) \\
 & (i-1,j-2) \arrow[ur] & & (i,j-1) \arrow[ur] & & (i+1,j) \arrow[ur] &
\end{tikzcd}$$

The coordinates on the bottom line of the quiver satisfy the equation $y-x = 2$, and those on the top line satisfy $y - x = n + 1$.

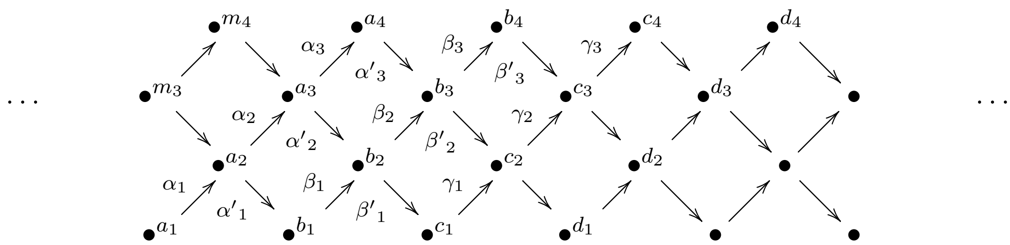
Let $\Sigma$ be the translation functor of $D^b(\operatorname{mod} kA_n)$. Then the action of $\Sigma$ is given by $\Sigma(i,j) = (j-1, i+n+2)$ ([25, Example 2.8]), and the action of the Auslander-Reiten translation $\tau$ by $\tau(i,j) = (i-1, j-1)$.

**Definition 1.4.2.** Let $a$ be an indecomposable object of $D^b(\operatorname{mod} kA_n)$, and let $\mathcal{L}(a)$ be the set of indecomposable objects with non-zero morphisms to $a$. Dually, let $\mathcal{R}(a)$ be the set of indecomposable objects to which there are non-zero morphisms from $a$.

**Sketch 1.4.3.** The two regions $\mathcal{L}(a)$ and $\mathcal{R}(a)$ are depicted below. The coordinates of the corners of $\mathcal{R}(a)$ are given by $a = (i,j)$, $b = (j-2, j)$, $c = (j-2, i+n+1)$ and $d = (i, i+n+1)$.




*Example* 1.4.4. Consider the Auslander-Reiten quiver of $D^b(\operatorname{mod} kA_4)$ and the indecomposable object $c_2$.

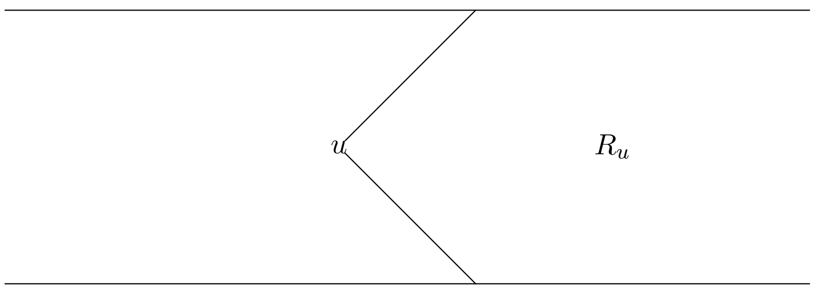



Since $b_2 \to b_3 \oplus c_1 \to c_2 \to$ is an Auslander-Reiten triangle, $\beta_2'\beta_2 + \gamma_1\beta_1' = 0$, and so by Lemma 1.4.1, $(b_2, c_2)$ is one-dimensional. In general, given indecomposable objects $x$ and $y$ of $D^b(\operatorname{mod} kA_4)$, if $(x, y)$ is non-zero, then $(x, y)$ is one-dimensional. Since $b_1 \to b_2 \to c_1 \to$ is an Auslander-Reiten triangle, $\beta_1'\beta_1 = 0$. Therefore by Lemma 1.4.1, $(b_1, c_1) = 0$ and so $(b_1, c_2) = 0$, and then $(a_2, c_2) = (m_3, c_2) = 0$. Similarly, since $m_4 \to a_3 \to a_4 \to$ is an Auslander-Reiten triangle, by Lemma 1.4.1, $(m_4, a_4) = 0$ and so $(m_4, c_2) = 0$. Accordingly, $\mathcal{L}(c_2) = \{a_3, a_4, b_2, b_3, c_1, c_2\}$. Dually, $\mathcal{R}(c_2) = \{c_2, c_3, c_4, d_1, d_2, d_3\}$.

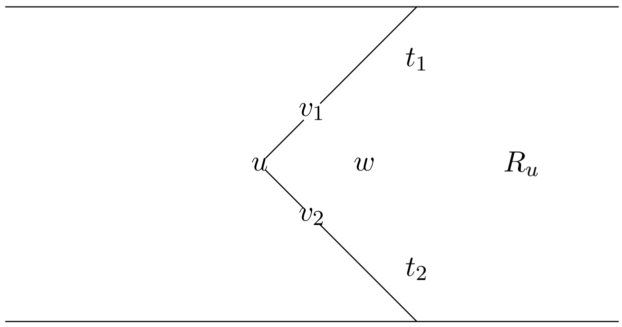
For the rest of this section, let $\mathcal{X}, \mathcal{Y}$ be full additive subcategories of the finite derived category $D^b(\operatorname{mod} kA_n)$, denoted by $\mathcal{T}$, with translation functor $\Sigma$, Serre functor $S$ and Auslander-Reiten translation $\tau \cong S\Sigma^{-1}$. The subcategories are assumed to be closed under isomorphisms and direct summands.

**Definition 1.4.5.** A *zig zag* $Z$ in the Auslander-Reiten quiver of the finite derived category $D^b(\operatorname{mod} kA_n)$ is a set of coordinates $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ where $y_1 - x_1 = 2$, $y_n - x_n = n+1$ and $(x_{i+1} - x_i, y_{i+1} - y_i) \in \{(-1, 0), (0, 1)\}$ for $1 \leq i \leq n-1$.

**Proposition 1.4.6.** *(c.f. Lemma 1.3.10) Let $(\mathcal{X}, \mathcal{Y})$ be a split torsion theory. If the indecomposable $u = (g, h)$ is in $\mathcal{X}$, then the region $R_u = \{(x, y) | g \leq x, h \leq y$ and $2 \leq y - x \leq n+1\}$ is in $\mathcal{X}$.*

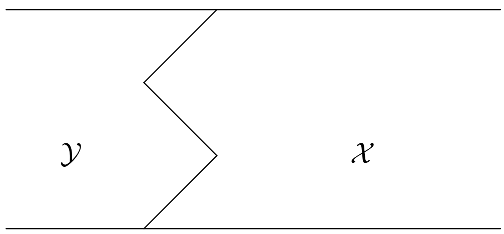


*Proof.* Since $(\mathcal{X}, \mathcal{Y})$ is a torsion theory, $\mathcal{X}$ is closed under extensions. In the following diagram, let $v_1 = (g, h+1), v_2 = (g+1, h)$ and $w = (g+1, h+1)$. Then there is the Auslander-Reiten triangle $u \to v_1 \oplus v_2 \to w \to$. Since $\tau^{-1}\mathcal{X} \subseteq \mathcal{X}$ by Theorem 1.3.5, $w = \tau^{-1}(u)$ is in $\mathcal{X}$. Since $\mathcal{X}$ is closed under extensions and direct summands, $v_1$ and $v_2$ are in $\mathcal{X}$. By repeating a (similar) argument on $v_1$ and $v_2$ and so on, the two line segments $t_1$ and $t_2$ are in $\mathcal{X}$. Eventually, the region $R_u$ is in $\mathcal{X}$, since $\tau^{-1}\mathcal{X} \subseteq \mathcal{X}$.




□

**Corollary 1.4.7.** *Let $(\mathcal{X}, \mathcal{Y})$ be a split torsion theory. If $\mathcal{X}$ is neither zero nor all of $\mathcal{T}$, then there is a zig zag $Z$ such that ind$\mathcal{X} = \tau^{-i}Z$ for $i \geq 0$ and ind$\mathcal{Y} = \tau^{j}Z$ for $j > 0$. One example is as follows.*




*Proof.* Consider a horizontal line $y - x = k$ with $3 \leq k \leq n$ in the Auslander-Reiten quiver of $\mathcal{T}$. If there are objects from $\mathcal{X}$ on this line, then there is a leftmost such object. Otherwise there are objects of $\mathcal{X}$ arbitrarily far to the left on $y - x = k$, so all objects on $y - x = k$ are in $\mathcal{X}$ because $\tau^{-1}\mathcal{X} \subseteq \mathcal{X}$ by Theorem 1.3.5. In the following diagram, let $d_1 = (u_1, u_2)$ and $d_2 = (u_1+1, u_2-1)$ be objects on the lines $y-x = k+1$ and $y-x = k-1$ respectively. Then there is the Auslander-Reiten triangle $d_0 \to d_1 \oplus d_2 \to d_0' \to$, where $d_0 = (u_1, u_2 - 1)$ and $d_0' = (u_1 + 1, u_2)$. Since $d_0$ and $d_0'$ both lie on the line $y - x = k$ which is in $\mathcal{X}$, it follows that $d_1$ and $d_2$ are in $\mathcal{X}$, since $\mathcal{X}$ is closed under extensions and direct summands. Therefore the two neighbouring lines $y - x = k + 1$ and $y - x = k - 1$ are in $\mathcal{X}$.

Repeating the argument for other (horizontal) lines, $\mathcal{X}$ has to contain all the indecomposable objects of $\mathcal{T}$, i.e. $\mathcal{X}$ has to be all of $\mathcal{T}$. The cases where $k = 2$ and $k = n + 1$ are similar.

$$\begin{tikzcd}
 & d_1 \arrow[dr] & \\
d_0 \arrow[ur] \arrow[dr] & & d_0' \\
 & d_2 \arrow[ur] &
\end{tikzcd}$$

Now on the line $y - x = k$, $2 \leq k \leq n+1$, let $a_k$ be the leftmost object which is in $\mathcal{X}$, as shown in the following diagram. Suppose that for some $k$ the object $a_{k+1}$ was neither $a_k^+$ nor $a_k^-$. If $a_{k+1}$ were left of $a_k^-$, then its region $R_{a_{k+1}} \subseteq \mathcal{X}$ (Proposition 1.4.6) would contain an object to the left of $a_k$ on $y - x = k$, which is a contradiction. Also $a_{k+1}$ could not have been to the right of $a_k^+$ since $a_k^+$ is in the region $R_{a_k} \subseteq \mathcal{X}$ (Proposition 1.4.6). Therefore $a_{k+1}$ can only be $a_k^+$ or $a_k^-$.

$$\begin{tikzcd}
a_k^- \arrow[dr] & & a_k^+ \\
 & a_k \arrow[ur] &
\end{tikzcd}$$

Finally, the result follows by applying Proposition 1.4.6 on all such leftmost objects. $\square$

**Theorem 1.4.8.** *(classification of split torsion theories) There is a bijection between the set of zig zag Z's and the set of split torsion theories $(\mathcal{X}, \mathcal{Y})$ in the finite derived category $D^b(\operatorname{mod} kA_n)$, where $\mathcal{X}$ and $\mathcal{Y}$ are separated by $Z$ such that ind$\mathcal{X} = \tau^{-i}Z$ for $i \geq 0$ and ind$\mathcal{Y} = \tau^{j}Z$ for $j > 0$.*

*Proof.* If $(\mathcal{X}, \mathcal{Y})$ is a split torsion theory, then by Corollary 1.4.7, there is a zig zag $Z$ which separates $\mathcal{X}$ and $\mathcal{Y}$ on the Auslander-Reiten quiver. Conversely, suppose there is a zig zag $Z$ which separates $\mathcal{X}$ and $\mathcal{Y}$ on the

Auslander-Reiten quiver. Then by inspection, $\mathcal{Y}$ (resp. $\mathcal{X}$) is precisely $\mathcal{X}^{\perp}$ (resp. ${}^{\perp}\mathcal{Y}$) (Sketch 1.4.3). Therefore by Lemma 1.2.19, $\mathcal{X}$ is closed under extensions. The subcategory $\mathcal{X}$ is trivially precovering since $(\mathcal{X},\ \mathcal{Y}) = 0$. Therefore $(\mathcal{X}, \mathcal{Y})$ is a torsion theory by [21, Proposition 2.3]. Since it can be readily seen by considering the Auslander-Reiten quiver that $\tau^{-1}\mathcal{X} \subseteq \mathcal{X}$, by Theorem 1.3.5, $(\mathcal{X}, \mathcal{Y})$ is a split torsion theory. □

The following lemma is a special case of Lemma 1.3.12. This is because the finite derived category $D^b(\operatorname{mod} kA_n)$ has a Serre functor.

**Lemma 1.4.9.** *Let $(\mathcal{X}, \mathcal{Y})$ be a split torsion theory. Then $(\mathcal{X}, \mathcal{Y})$ is a t-structure.*

*Proof.* Let $u = (g, h)$ be in $\mathcal{X}$. Then by Proposition 1.4.6, the region $R_u$ is in $\mathcal{X}$. Thus $\Sigma u = \Sigma(g, h) = (h-1, g+n+2)$ is in $R_u = \{(x, y) | g \leq x, h \leq y$ and $2 \leq y - x \leq n+1\} \subseteq \mathcal{X}$. □

An alternative proof is possible by inspecting the action of the Serre functor $S$ and by appealing to Lemma 1.3.1(i). First we need a little lemma.

**Lemma 1.4.10.** *Let $(\mathcal{X}, \mathcal{Y})$ be a split torsion theory. Then $S\mathcal{X} \subseteq \mathcal{X}$.*

*Proof.* Remember $S \cong \Sigma\tau \cong \tau\Sigma$ (Definition 0.3.11). Let $u = (g, h) \in \mathcal{X}$ and consider the region $R_u$ in Proposition 1.4.6. Thus $Sx = S(g, h) = \tau\Sigma(g, h) = \tau(h-1, g+n+2) = (h-2, g+n+1)$ is in $R_u = \{(x, y) | g \leq x, h \leq y$ and $2 \leq y - x \leq n+1\} \subseteq \mathcal{X}$. □

An alternative proof of Lemma 1.4.9 is then given.

*Proof.* Since $(\mathcal{X}, \mathcal{Y})$ is a split torsion theory, $S^{-1}\Sigma\mathcal{X} \subseteq \mathcal{X}$ by Theorem 1.3.5 and $S\mathcal{X} \subseteq \mathcal{X}$ by Lemma 1.4.10. Finally, by Lemma 1.3.1(i), $S^{-1}\Sigma\mathcal{X} \subseteq \mathcal{X}$ and $S\mathcal{X} \subseteq \mathcal{X}$ give $\Sigma\mathcal{X} \subseteq \mathcal{X}$. □

The following lemma is true not only for the subcategory $\mathcal{X}$ of the finite derived category $D^b(\operatorname{mod} kA_n)$, but simply of any $k$-linear Hom finite and Krull-Schmidt triangulated category. It is suggestive of the situation when the notions of split torsion theories and t-structures coincide.

**Lemma 1.4.11.** *Suppose $S\mathcal{X} \cong \mathcal{X}$. Then the following are equivalent.*

*(i)* $S^{-1}\Sigma\mathcal{X} \subseteq \mathcal{X}$,

*(ii)* $\Sigma\mathcal{X} \subseteq \mathcal{X}$.

*Proof.* This follows by Lemma 1.3.1(i) and Lemma 1.3.1(iii). □

*Remark* 1.4.12. Suppose a torsion theory satisfies the condition in Lemma 1.4.11. Then the one characterized by (i) corresponds to a split torsion theory by Theorem 1.3.5, and the one characterized by (ii) corresponds to a t-structure by definition.

The following is an illustration of Lemma 1.4.11.

*Example* 1.4.13. Consider the finite derived category $D^b(k) = D^b(\operatorname{mod} kA_n)$ where $n = 1$. By [15, Section 5.2], an indecomposable $X$ in $D^b(k)$ is of the form

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow k \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots,$$

where $k$ is at any $i$-th position.

By Lemma 0.3.6 and Lemma 0.1.9, the end terms of an Auslander-Reiten triangle are indecomposable. Therefore an Auslander-Reiten triangle in $D^b(k)$ is of the form

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma \neq 0} \Sigma X,$$

where $X$ and $Z$ are indecomposable, and $\Sigma^{-1}Z \cong X$ since $(Z, \Sigma X)$ is one-dimensional and the morphism $\gamma : Z \to \Sigma X$ is non-zero. Therefore $Y \cong 0$ by Lemma 0.2.2(iii) and an Auslander-Reiten triangle in $D^b(k)$ is of the form

$$X \longrightarrow 0 \longrightarrow Z \xrightarrow{\gamma \neq 0} \Sigma X.$$

Together with $\tau Z \cong S\Sigma^{-1}Z \cong X$ this gives $SX \cong X$.

The Auslander-Reiten quiver of $D^b(k)$ is as follows.

$$\cdots \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots$$

*Remark* 1.4.14. Let $(\mathcal{X}_1, \mathcal{Y}_1)$ be a torsion theory in $D^b(k)$ (Example 1.4.13). Then by Lemma 1.4.11, the torsion theory is a split torsion theory if and only if it is a t-structure (c.f. Remark 1.4.12).

## 1.5 Anecdote

The chapter concludes with the following informal recapitulation. As usual, let $k$ be a field. Let $\mathcal{T}$ be a $k$-linear Hom finite and Krull-Schmidt triangulated category with translation functor $\Sigma$, Serre functor $S$ and Auslander-Reiten translation $\tau \cong S\Sigma^{-1}$. Let $\mathcal{X}$, $\mathcal{Y}$ be full additive subcategories of $\mathcal{T}$. They do not need to be triangulated, but are assumed to be closed under isomorphisms.

Three different types of distinguished triangles have been hitherto described.

(i) split distinguished triangles,

(ii) torsion theory triangles,

(iii) Auslander-Reiten triangles.

Torsion theory triangles can be split. On the other hand, Auslander-Reiten triangles are not split.

**Lemma 1.5.1.** *Let $(\mathcal{X}, \mathcal{Y})$ be a torsion theory. If $\tau^{-1}\mathcal{X} \subseteq \mathcal{X}$, then there are no distinguished triangles (other than the one with all terms zero) $\epsilon$ where $\epsilon$ is both a torsion theory triangle and an Auslander-Reiten triangle.*

*Proof.* Suppose $\epsilon : x \to t \to y \to \Sigma x$ is such a triangle. Then by Theorem 1.3.5, $\epsilon$ is split, but by definition Auslander-Reiten triangles are not split. Alternatively, $\tau^{-1}\mathcal{X} \subseteq \mathcal{X}$ implies that $y$ is in $\mathcal{X}$, but then $y$ would have to be in $\mathcal{X} \cap \mathcal{Y} = 0$ (Remark 1.2.6(i)). Similarly, $x$ would also have to be 0 since $\tau\mathcal{Y} \subseteq \mathcal{Y}$ by Theorem 1.3.5. □

Below are a few comparisons between torsion theory triangles and Auslander-Reiten triangles.

*Comparison* 1.5.2. *(nature of restrictions)* Let $(\mathcal{X}, \mathcal{Y})$ be a torsion theory, and $\epsilon : u \xrightarrow{\mu} v \xrightarrow{\nu} w \xrightarrow{\xi} \Sigma u$ be a distinguished triangle in $\mathcal{T}$. If $\epsilon$ is a torsion theory triangle $\epsilon_1$, then it is defined in terms of the membership of $u$ and $w$ in the given subcategories $\mathcal{X}$ and $\mathcal{Y}$ respectively, and the morphisms $\mu$ and $\nu$ can be intuitively perceived as approximations of the canonical inclusion and the canonical projection respectively (Remark 1.2.30). On the other hand, if $\epsilon$ is an Auslander-Reiten triangle $\epsilon_2$, then it is defined in terms of the morphisms $\mu$, $\nu$ and $\xi$, and in the example given in Proposition 1.4.6, the (first three) objects $u$, $v_1 \oplus v_2$ and $w$ in the Auslander-Reiten triangle $u \to v_1 \oplus v_2 \to w \to$ are all in $\mathcal{X}$.

*Comparison* 1.5.3. *(construction)* A torsion theory triangle is of the form $Rt \to t \to Lt \to \Sigma Rt$ when the right adjoint $R$ and the left adjoint $L$ exist (Property 1.2.28). An Auslander-Reiten triangle is of the form $S\Sigma^{-1}z \to y \to z \to Sz$ when the Serre functor $S$ exists (Lemma 0.3.10).

*Comparison* 1.5.4. *(right minimal morphisms)* Let $(\mathcal{X}, \mathcal{Y})$ be a torsion theory, and $\epsilon : u \xrightarrow{\mu} v \xrightarrow{\nu} w \xrightarrow{\xi} \Sigma u$ be a distinguished triangle in $\mathcal{T}$. If $\epsilon$ is an Auslander-Reiten triangle, then $\nu$ is right minimal by Lemma 0.3.6. On the other hand, any given right minimal morphism $\mu : u \to v$ in $\mathcal{T}$, with $\mu$ an $\mathcal{X}$-precover, can be extended to a torsion theory triangle $u \xrightarrow{\mu} v \to w \to \Sigma u$ with $w$ in $\mathcal{Y}$ by Proposition 1.2.23. Accordingly, the occurrence of a right minimal morphism in a distinguished triangle affects whether the distinguished triangle is an Auslander-Reiten triangle or a torsion theory triangle.

*Remark* 1.5.5.

(i) In Comparison 1.5.2, the example from Proposition 1.4.6 is given, where the (first three) objects $u$, $v_1 \oplus v_2$ and $w$ in the (Auslander-Reiten) triangle $u \xrightarrow{\alpha} v_1 \oplus v_2 \to w \to$ are all in $\mathcal{X}$. This can also be true in some distinguished triangles other than Auslander-Reiten triangles. For example, consider the morphism $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ in the above Auslander-Reiten triangle. The objects $u$ and $v_1$ are in $\mathcal{X}$, and in Section 3.2.2, the mapping cone $y$ of the morphism $\alpha_1 : u \to v_1$ is shown to be in $\mathcal{X}$ as well. The distinguished triangle $u \to v_1 \to y \to$ is not however an Auslander-Reiten triangle.

(ii) Let $(\mathcal{X}, \mathcal{Y})$ be a torsion theory. Suppose for each $t$ in $\mathcal{T}$, there is a torsion theory triangle $x \to t \to y \to \Sigma x$, where $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and the morphism $h$: $\Sigma^{-1} y \to x$ is an $\mathcal{X}$-left phantom. Then by Lemma 1.2.23, the right adjoint $R : \mathcal{T} \to \mathcal{X}$ of the inclusion $\imath : \mathcal{X} \hookrightarrow \mathcal{T}$ is $R(t) = x$, i.e. $x = \Sigma^{-1} u$ where $u$ is the mapping cone of the $\mathcal{Y}$-preenvelope $t \to y$. The action of the right adjoint $R$ can be seen on the Auslander-Reiten quiver, if it is possible to recognize the $\mathcal{Y}$-preenvelope, the mapping cone construction as well as the action of $\Sigma$ on the quiver. There is more description about mapping cone constructions in some selected situations in Section 3.2.2, Section 3.3.2 and Section 3.4.3, and an illustration is given in Example 4.5.6.

# Chapter 2

# Existence of Auslander-Reiten sequences in subcategories

This chapter is also written in the form of a paper which is accepted for publication in the Journal of Pure and Applied Algebra ([36]).

## 2.1 Introduction

The reader is reminded of the notion of Auslander-Reiten triangles in subcategories in Definition 0.3.5. The octahedral axiom of a triangulated category described in Section 0.2 is not necessarily true when it is confined to Auslander-Reiten triangles only. There will be more examples of applications of the octahedral axiom in Section 3.4.3, where not all the distinguished triangles induced by the octahedral axiom are Auslander-Reiten triangles. This leads us to other conditions of the existence of Auslander-Reiten triangles, which is very much related to the existence of Auslander-Reiten sequences described in this chapter.

Let $\mathcal{X}, \mathcal{Y}$ be full subcategories of a triangulated category $\mathcal{T}$. The notions of a torsion theory and torsion theory triangles are given in Definition 1.2.5. Also the notions of an $\mathcal{X}$-(pre)cover and of $\mathcal{X}$ as (pre)covering for $\mathcal{T}$ in Definition 1.2.8 and in Definition 1.2.9. Suppose $(\mathcal{X}, \mathcal{Y})$ is a torsion theory. The connection between torsion theory triangles and Auslander-Reiten triangles is very subtle, and it has been described in Comparison 1.5.4.

The following offers another facet of the interrelationship. Since $(\mathcal{X}, \mathcal{Y})$ is a torsion theory, $\mathcal{X}$ is a precovering for $\mathcal{T}$. If $\mathcal{X}$ is also a covering for $\mathcal{T}$, then given $t$ in $\mathcal{T}$, there is an $\mathcal{X}$-cover of the form $x \xrightarrow{\alpha} t$ which can be extended to a torsion theory triangle $x \xrightarrow{\alpha} t \to y \to$ with $y$ in $\mathcal{Y}$ by Proposition 1.2.23.

This is very similar to the existence of Auslander-Reiten triangles in subcategories described in [22, Theorem 3.1]. Suppose $\mathcal{T}$ is a skeletally small $k$-linear Hom finite triangulated category with split idempotents, and $\mathcal{X}$ is a full subcategory of $\mathcal{T}$ closed under extensions and direct summands. The theorem is restated as follows.

**Theorem.** *([22, Theorem 3.1]) Let $x$ be in $\mathcal{X}$ and let $u \to v \to x \to$ be an Auslander-Reiten triangle in $\mathcal{T}$. Then the following are equivalent.*

*(i) $u$ has an $\mathcal{X}$-cover of the form $a \to u$,*

*(ii) There is an Auslander-Reiten triangle $a \to b \to x \to$ in $\mathcal{X}$.*

Therefore there is a very close and subtle relationship between the existence of torsion theory triangles and the existence of Auslander-Reiten triangles. In this chapter, we study the existence of Auslander-Reiten sequences in subcategories of $\mathrm{mod}(\Lambda)$, where $\Lambda$ is a finite-dimensional $k$-algebra over the field $k$, based on the theory of the existence of Auslander-Reiten triangles in subcategories developed in [22].

In the above theorem, when $u$ is in $\mathcal{X}$, (i) and (ii) are true trivially and independently. Therefore intuitively, (i) and (ii) are two equivalent forms in disguise for measuring how far for each $u \notin \mathcal{X}$ it is from $\mathcal{X}$. Accordingly, $\mathcal{X}$-covers and Auslander-Reiten triangles in $\mathcal{X}$ are different expressions of the subcategory $\mathcal{X}$ approximating $\mathcal{T}$.

## 2.2 Notations

Let $\Lambda$ be a finite-dimensional $k$-algebra over the field $k$, and $\Lambda^{op}$ be the *opposite algebra*. The elements of $\Lambda$ and $\Lambda^{op}$ are the same, and given multiplication $\cdot$ and elements $\lambda_1$, $\lambda_2$ in $\Lambda^{op}$, let $\lambda_1 \cdot \lambda_2 = \lambda_2\lambda_1$, where $\lambda_2\lambda_1$ is considered in $\Lambda$. As usual, let $\mathrm{mod}(\Lambda)$ (resp. $\mathrm{mod}(\Lambda^{op})$) be the category of finitely generated $\Lambda$-left-modules (resp. $\Lambda^{op}$-left-modules or $\Lambda$-right-modules).

A finite-dimensional $k$-algebra $\Lambda$ is said to be *representation-finite* if the number of isomorphism classes of indecomposable finite-dimensional $\Lambda$-left-modules is finite. Otherwise, it is said to be *representation-infinite*.

Let D be the usual duality functor $\mathrm{D}(-) = \mathrm{Hom}_k(-, k) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Lambda^{op})$. Given a $\Lambda$-left-module $M$ in $\mathrm{mod}(\Lambda)$, the set $\mathrm{Hom}_k(M, k)$ is a $k$-vector space, and is also a $\Lambda$-right-module. Indeed given $\mu$ in $\mathrm{Hom}_k(M, k)$

and $\lambda$ in $\Lambda$, define $(\mu \cdot \lambda)(m) = \mu(\lambda \cdot m)$ for $m$ in $M$. In particular, $\mathrm{D}\Lambda = \mathrm{Hom}_k(\Lambda, k)$ is the $k$-linear dual of $\Lambda$ which is a $\Lambda$-bi-module.

The duality functor $\mathrm{D}(-)$ is an exact, contravariant functor and it induces a contravariant equivalence between the categories $\mathrm{mod}(\Lambda)$ and $\mathrm{mod}(\Lambda^{op})$, so that the category $\mathrm{mod}(\Lambda^{op})$ is equivalent to the opposite category of $\mathrm{mod}(\Lambda)$.

**Lemma 2.2.1.** *Let $M$ be in* $\mathrm{mod}(\Lambda)$. *Then*

*(i) $M$ is indecomposable if and only if* $\mathrm{D}M$ *is indecomposable,*

*(ii) If $M$ is projective (resp. injective), then* $\mathrm{D}M$ *is injective (resp. projective).*

Subsequently, the duality $\mathrm{D}(-) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Lambda^{op})$ induces a duality $\mathrm{D}(-) : \underline{\mathrm{mod}}(\Lambda) \to \overline{\mathrm{mod}}(\Lambda^{op})$.

Let $\mathcal{M}$ be a full subcategory of $\mathrm{mod}(\Lambda)$, and remember the notions of a right almost split morphism and of a left almost split morphism in Definition 0.3.1.

**Definition 2.2.2.** *(c.f. Definition 0.3.5)* An exact sequence $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ with $A, B, C$ in the subcategory $\mathcal{M}$ is an *Auslander-Reiten sequence* in $\mathcal{M}$ if $g$ is left almost split and $f$ is right almost split in $\mathcal{M}$.

Following the notations of [22], let $\mathrm{K}(\mathrm{Inj}\Lambda)$ be the homotopy category of complexes of injective $\Lambda$-left-modules. Let $\mathcal{T}$ be the full subcategory of $\mathrm{K}(\mathrm{Inj}\Lambda)$ consisting of complexes $X$ for which each $X^i$ is finitely generated, and where $\mathrm{H}^i(X) = 0$ for $i \gg 0$ and $\mathrm{H}^i\mathrm{Hom}_\Lambda(\mathrm{D}\Lambda, X) = 0$ for $i \ll 0$.

The category $\mathcal{T}$ is a skeletally small $k$-linear Hom finite triangulated category with split idempotents. It is Krull-Schmidt as well. Let the suspension functor $\Sigma$ be the translation functor of the triangulated category $\mathcal{T}$. Let $\mathcal{C}$ be the full subcategory of $\mathcal{T}$ which consists of injective resolutions of modules in $\mathrm{mod}(\Lambda)$. The category $\mathcal{C}$ is closed under extensions and direct summands. The reader can refer to [22, Remark 1.2] and [22, Lemma 4.3] for more details.

Let $\mathcal{M}$ be a full subcategory of $\mathrm{mod}(\Lambda)$ closed under extensions and direct summands, and let $\mathcal{C}'$ consist of the injective resolutions of the $M$ in $\mathcal{M}$. The subcategories $\mathcal{C}'$ and $\mathcal{M}$ need not be abelian.

*Remark* 2.2.3. (i) $\mathcal{C}$ is equivalent to $\mathrm{mod}(\Lambda)$, by the functor $F : \mathrm{mod}(\Lambda) \to \mathcal{C}$ which sends $X$ in $\mathrm{mod}(\Lambda)$ to its injective resolution $C$ in $\mathcal{C}$. Similarly, $\mathcal{C}'$ and $\mathcal{M}$ are equivalent.

(ii) By [22, Lemma 4.5], $\mathcal{C}$ is covering in $\mathcal{T}$. If $\mathcal{C}'$ is precovering in $\mathcal{C}$, then $\mathcal{C}'$ is precovering in $\mathcal{T}$.

By Remark 2.2.3(i), properties in the category $\mathrm{mod}(\Lambda)$ can be passed on to the category $\mathcal{C}$, and vice versa. For example, $\mathrm{mod}(\Lambda)$ is closed under extensions and direct summands if and only if $\mathcal{C}$ is closed under extensions and direct summands. Also $\mathcal{C}'$ is (pre)covering in $\mathcal{C}$ if and only if $\mathcal{M}$ is (pre)covering in $\mathrm{mod}(\Lambda)$.

The setup described above could be summarized in the following diagram,

$$\begin{tikzcd}
\mathcal{M} \arrow[r, hook] \arrow[d, dash, "\simeq"] & \mathrm{mod}(\Lambda) \arrow[d, dash, "\simeq"] & & \\
\mathcal{C}' \arrow[r, hook] & \mathcal{C} \arrow[r, hook] & \mathcal{T} \arrow[r, hook] & K(\mathrm{Inj}\Lambda).
\end{tikzcd}$$

*Remark* 2.2.4. Intuitively, $\mathcal{C}$ stays away from $\mathcal{T}$ far enough to be abelian ($\mathrm{mod}(\Lambda)$ is abelian). On the other hand, it needs to stay close enough so that there are $\mathcal{C}$-covers of objects in $\mathcal{T}$.

Finally, the bridge between Auslander-Reiten sequences and Auslander-Reiten triangles in subcategories is shown in the following lemma.

**Lemma 2.2.5.** $0 \to X \to Y \to Z \to 0$ *is an Auslander-Reiten sequence in* $\mathcal{M}$ *if and only if* $A \to B \to C \to$ *is an Auslander-Reiten triangle in* $\mathcal{C}'$*, where* $A, B, C$ *are injective resolutions of* $X, Y, Z$ *respectively.*

*Proof.* (*if*) This is Lemma 0.2.9. (*only if*) This is the dual of Lemma 0.2.22. Essentially, $\mathcal{M}$ and $\mathcal{C}'$ are equivalent so that short exact sequences in $\mathcal{M}$ correspond to distinguished triangles in $\mathcal{C}'$. □

## 2.3 Weakened notions of precovers (preenvelopes)

Let us begin this section with some notations and definitions.

Let $(-)^*$ be the functor $\mathrm{Hom}_\Lambda(-, \Lambda) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Lambda^{op})$. Given a $\Lambda$-left-module $M$ in $\mathrm{mod}(\Lambda)$, the set $\mathrm{Hom}_\Lambda(M, \Lambda)$ is a $k$-vector space, and is also a $\Lambda$-right-module. Indeed given $\mu$ in $\mathrm{Hom}_\Lambda(M, \Lambda)$ and $\lambda$ in $\Lambda$, define $(\mu \cdot \lambda)(m) = \mu(m) \cdot \lambda$ for $m$ in $M$. In particular, $\mathrm{Hom}_\Lambda(\Lambda, \Lambda) = \Lambda$.

The functor $(-)^*$ is a left exact contravariant functor. However, unlike the duality functor $\mathrm{D}(-)$, the functor $(-)^*$ does not induce an equivalence between the categories $\mathrm{mod}(\Lambda)$ and $\mathrm{mod}(\Lambda^{op})$. Therefore, the functor $(-)^*$ is not a duality and given $N$ in $\mathrm{mod}(\Lambda^{op})$, there is not necessarily an $M$ in $\mathrm{mod}(\Lambda)$ such that $M^* = N$.

**Lemma 2.3.1.** *(c.f. Lemma 2.2.1) Let* $M$ *be in* $\mathrm{mod}(\Lambda)$.

*(i) If $M$ is free, then* $\mathrm{Hom}_\Lambda(M,\Lambda)$ *is also free,*

*(ii) If $M$ is projective, then* $\mathrm{Hom}_\Lambda(M,\Lambda)$ *is also projective,*

*(iii) If $M$ is projective, then $M^{**} \cong M$.*

Let $M$ be in $\mathcal{M}$. Let

$$P = \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \longrightarrow \cdots$$

be a projective resolution of $M$. The $P_i$ do not need to be in $\mathcal{M}$.

The sequence

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is right exact, and the functor $(-)^*$ induces the following left exact sequence,

$$0 \longrightarrow M^* \longrightarrow P_0^* \longrightarrow P_1^*.$$

Since the functor $(-)^*$ is not an exact functor, this leads us to the following definition.

**Definition 2.3.2.** The *transpose* $\mathrm{Tr}M$ of $M$ is defined to be the cokernel of the map $P_0^* \to P_1^*$.

The transpose $\mathrm{Tr}M$ is a $k$-vector space and is also a $\Lambda$-right-module in $\mathrm{mod}(\Lambda^{op})$. The transpose does not induce a duality $\mathrm{Tr}(-) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Lambda^{op})$, and in general not even a functor from $\mathrm{mod}(\Lambda)$ to $\mathrm{mod}(\Lambda^{op})$ since $\mathrm{Tr}(-) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Lambda^{op})$ is well-defined on objects but in general not well-defined on morphisms.

**Lemma 2.3.3.** *([2, Proposition IV.1.7]) Let $M$ be in* $\mathrm{mod}(\Lambda)$*. Then*

*(i)* $\mathrm{Tr}M = 0$ *if and only if $M$ is projective,*

*(ii) If $M$ is not projective, then $M$ is indecomposable if and only if* $\mathrm{Tr}M$ *is indecomposable.*

Subsequently, the transpose $\mathrm{Tr}(-)$ induces a well-defined contravariant functor and along with it a duality $\mathrm{Tr}(-) : \underline{\mathrm{mod}}(\Lambda) \to \underline{\mathrm{mod}}(\Lambda^{op})$.

Now let $P_i$ be a projective module in $\mathrm{mod}(\Lambda)$. The tensor product $\mathrm{D}\Lambda \otimes_\Lambda P_i$ is naturally equipped with the structure of a $\Lambda$-left-module. Given $d$ in $\mathrm{D}\Lambda$, $p$ in $P_i$ and $\lambda$ in $\Lambda$, define $\lambda \cdot (d \otimes p) = (\lambda \cdot d) \otimes p$. By Lemma 0.1.4(ii), $\mathrm{D}\Lambda \otimes_\Lambda P_i$ is injective.

Since $\mathrm{D}\Lambda \otimes_\Lambda P_i \cong \mathrm{Hom}_k(\mathrm{Hom}_\Lambda(P_i,\Lambda),k) = \mathrm{D}(P_i^*)$, we write

$$\mathrm{D}\Lambda \otimes_\Lambda P = \mathrm{D}(P^*) = \cdots \longrightarrow \mathrm{D}(P_2^*) \longrightarrow \mathrm{D}(P_1^*) \xrightarrow{d_1} \mathrm{D}(P_0^*) \longrightarrow 0 \longrightarrow \cdots \quad .$$

*Remark* 2.3.4. Even though the $\mathrm{D}\Lambda \otimes_\Lambda P_i$ are injective, the complex $\mathrm{D}\Lambda \otimes_\Lambda P$ is far from being injective, let alone an injective resolution.

From the exact sequence

$$0 \longrightarrow M^* \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \mathrm{Tr}M \longrightarrow 0,$$

and the consideration that the functor $\mathrm{D}(-)$ preserves exactness, there is the following exact sequence,

$$0 \longrightarrow \mathrm{DTr}M \longrightarrow \mathrm{D}(P_1^*) \longrightarrow \mathrm{D}(P_0^*) \longrightarrow \mathrm{D}(M^*) \longrightarrow 0.$$

Accordingly, $\mathrm{DTr}M$ and $\mathrm{D}(M^*)$ are the kernel and cokernel of the map $d_1$.

*Remark* 2.3.5.
(i) Even though $M$ is in $\mathcal{M}$, the module $\mathrm{DTr}M$ might not be in $\mathcal{M}$,

(ii) The composition $\mathrm{DTr}(-) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Lambda)$ induces an equivalence $\mathrm{DTr}(-) : \underline{\mathrm{mod}}(\Lambda) \to \overline{\mathrm{mod}}(\Lambda)$ of categories with inverse equivalence $\mathrm{TrD}(-) : \overline{\mathrm{mod}}(\Lambda) \to \underline{\mathrm{mod}}(\Lambda)$,

(iii) The right exact sequence $P_0^* \to P_1^* \to \mathrm{Tr}M \to 0$ induces a left exact sequence $0 \to (\mathrm{Tr}M)^* \to P_1^{**} \to P_0^{**}$. Since $P_i^{**} \cong P_i$ for $i = 0, 1$ by Lemma 2.3.1(iii), there is the exact sequence $0 \to (\mathrm{Tr}M)^* \to P_1 \to P_0 \to M \to 0$. The module $(\mathrm{Tr}M)^*$, which is the kernel of the morphism $P_1 \to P_0$, is known as the *second syzygy of* $M$, denoted by $\Omega_2 M$.

Now we are ready to introduce a weakened notion of an $\mathcal{M}$-precover, i.e. an $\mathcal{M}$-precover with error term, and discover its relationship with a $\mathcal{C}'$-precover.

**Definition 2.3.6.** Let $M$ and $N$ be in $\mathcal{M}$. Then $\nu : N \to DTrM$ is said to be an *$\mathcal{M}$-precover with error term* if for all $L$ in $\mathcal{M}$, each morphism $s' : L \to DTrM$ factors through $\nu$ up to an error term, i.e. there is a morphism $\nu' : L \to N$ such that $\nu\nu' - s'$ factors through $f_2$ in the following way: $L \to D(P_2^*) \xrightarrow{f_2} DTrM$, as indicated in the following diagram.

$$\begin{tikzcd}
 & & L \arrow[d, dashed, "\nu'"] \arrow[dl, dotted, bend right] \arrow[ddd, bend right, "s'"'] & & & & \\
 & & N \arrow[dd, "\nu"] & & & & \\
\Sigma^{-1}D(P^*) = {} \arrow[r] & D(P_2^*) \arrow[rr, "d_2"] \arrow[dr, "f_2"'] & & D(P_1^*) \arrow[r, "d_1"] & D(P_0^*) \arrow[r] & 0 \arrow[r] & {} \\
 & & DTrM \arrow[ur, hook, "j_1"'] & & & &
\end{tikzcd}$$

*Remark* 2.3.7. In Definition 2.3.6, there is a unique morphism $f_2 : D(P_2^*) \to$ DTr$M$ since DTr$M$ is the kernel of the map $d_1$.

Similarly, consider the subcategory $D\mathcal{M}$ of mod$(\Lambda^{op})$. The subcategory $D\mathcal{M}$ is equivalent to the opposite category of $\mathcal{M}$. Let $M$ be in $\mathcal{M}$. Let

$$Q = \cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow 0 \longrightarrow \cdots$$

be a projective resolution of $DM$.

**Definition 2.3.8.** Let $M$ and $N$ be in $\mathcal{M}$. Then $Df : DN \to DTr(DM)$ is said to be a *$D\mathcal{M}$-precover with error term* if for all $L$ in $\mathcal{M}$, each morphism $Ds' : DL \to DTr(DM)$ factors through $Df$ up to an error term, i.e. there is a morphism $Df' : DL \to DN$ such that $DfDf' - Ds'$ factors through $D(Q_2^*)$ in the following way: $DL \to D(Q_2^*) \overset{D(g_2)}{\to} DTr(DM)$, as indicated in the following diagram.

$$\begin{tikzcd}
 & & DL \arrow[d, dashed, "Df'"] \arrow[ddl, dotted, bend right] \arrow[dd, bend right, "Ds'"'] & & & & \\
 & & DN \arrow[dd, "Df"] & & & & \\
\Sigma^{-1}D(Q^*) = \arrow[r] & D(Q_2^*) \arrow[rr] \arrow[dr, "D(g_2)"'] & & D(Q_1^*) \arrow[r] & D(Q_0^*) \arrow[r] & 0 \arrow[r] & {} \\
 & & DTr(DM) \arrow[ur, hook] & & & &
\end{tikzcd}$$

There is also the dual version.

**Definition 2.3.9.** Let $M$ and $N$ be in $\mathcal{M}$. Then $f : Tr(DM) \to N$ is said to be an *$\mathcal{M}$-preenvelope with error term* if for all $L$ in $\mathcal{M}$, each morphism $s' : Tr(DM) \to L$ factors through $f$ up to an error term, i.e. there is a morphism $f' : N \to L$ such that $f'f - s'$ factors through $Q_2^*$ in the following way: $Tr(DM) \to Q_2^* \to L$.

**Lemma 2.3.10.** *Let $M$, $N$ be in $\mathcal{M}$ and $\nu : N \to DTrM$ be given. Let $J$ in $\mathcal{C}'$ be the injective resolution of $N$. Then there is a chain map $\lambda : J \to \Sigma^{-1}D(P^*)$ induced by $\nu$ as indicated in the following diagram.*

$$\begin{tikzcd}
J = \arrow[r] \arrow[dd, dashed, "\lambda"'] & 0 \arrow[rr] \arrow[dd, dashed] & & J^0 \arrow[r, "a^0"] \arrow[dd, dashed, "\lambda_0"'] & J^1 \arrow[r] \arrow[dd, dashed, "\lambda_1"'] & J^2 \arrow[r] \arrow[dd, dashed] & {} \\
 & & N \arrow[ur, hook, "i"] \arrow[dd, "\nu"] & & & & \\
\Sigma^{-1}D(P^*) = \arrow[r] & D(P_2^*) \arrow[rr, "d_2"] \arrow[dr, "f_2"'] & & D(P_1^*) \arrow[r, "d_1"'] & D(P_0^*) \arrow[r] & 0 \arrow[r] & {} \\
 & & DTrM \arrow[ur, hook, "j_1"'] & & & &
\end{tikzcd}$$

*Proof.* This is Lemma 0.1.27(ii), but we shall show it explicitly here. The $J^i$ here do not need to be in $\mathcal{M}$.

$$\begin{tikzcd}
J = \arrow[r] \arrow[dd, dashed, "\lambda"'] & 0 \arrow[rr, "a^{-1}"] \arrow[dd, dashed] & & J^0 \arrow[rr, "a^0"] \arrow[dr, "h_1"] \arrow[dd, dashed] & & J^1 \arrow[r] \arrow[dd, dashed] & J^2 \arrow[r] \arrow[dd, dashed] & {} \\
& & N \arrow[ur, hook, "i"'] \arrow[dd, "\nu"] & & \sigma N \arrow[ur, hook, "i_1"'] \arrow[dd] & & & \\
\Sigma^{-1}D(P^*) = \arrow[r] & D(P_2^*) \arrow[rr, "d_2"] \arrow[dr, "f_2"'] & & D(P_1^*) \arrow[rr, "d_1"] \arrow[dr, "f_1"'] & & D(P_0^*) \arrow[r] & 0 \arrow[r] & {} \\
& & DTrM \arrow[ur, hook, "j_1"'] & & C \arrow[ur, "j_2"'] & & &
\end{tikzcd}$$

In the above diagram, let $\sigma N$ and $C$ be the cokernels of $i$ and of $j_1$ respectively. Since $a^0a^{-1} = 0$ and $\sigma N$ is the cokernel, there is a unique $i_1 : \sigma N \to J^1$ such that $a^0 = i_1h_1$. Similarly, there is a unique $j_2 : C \to \mathrm{D}(P_0^*)$ such that $d_1 = j_2f_1$.

Since $\mathrm{Z}^0(J) = N$ and $\mathrm{D}(P_1^*)$ is injective, there is a morphism $\lambda_0 : J^0 \to \mathrm{D}(P_1^*)$ such that $\lambda_0 i = j_1\nu$. Since $f_1\lambda_0 i = f_1j_1\nu = 0$, there is a unique morphism $\mu : \sigma N \to C$ such that $\mu h_1 = f_1\lambda_0$, since $C$ is the cokernel. Finally, there is a morphism $\lambda_1 : J^1 \to \mathrm{D}(P_0^*)$ such that $\lambda_1 i_1 = j_2\mu$ since $\mathrm{D}(P_0^*)$ is injective. Hence $\lambda_1 a^0 = d_1\lambda_0$ and there is a chain map $\lambda : J \to \Sigma^{-1}D(P^*)$, which is only dependent on the morphism $\nu : N \to \mathrm{DTr}M$. $\square$

*Remark* 2.3.11. Conversely, given a chain map $\lambda : J \to \Sigma^{-1}D(P^*)$, since $\mathrm{Z}^0(J) = N$, therefore $d_1\lambda_0 i = \lambda_1 a^0 i = 0$, and so there is a unique morphism $\nu : N \to \mathrm{DTr}M$ such that $j_1\nu = \lambda_0 i$, since $\mathrm{DTr}M$ is the kernel of the map $d_1$.

**Lemma 2.3.12.** *Let $M$, $N$ be in $\mathcal{M}$ and $\nu : N \to DTrM$ be given. Let $J$ in $\mathcal{C}'$ be the injective resolution of $N$ and let $\lambda : J \to \Sigma^{-1}D(P^*)$ be a chain map induced by $\nu$ as indicated in the following diagram.*

$$\begin{tikzcd}
J = \arrow[r] \arrow[dd, "\lambda"'] & 0 \arrow[rr] \arrow[dd] & & J^0 \arrow[rr, "a^0"] \arrow[dd, "\lambda_0"'] \arrow[ddll, dotted, bend right, "\varphi^0"'] & & J^1 \arrow[r] \arrow[dd, "\lambda_1"'] \arrow[ddll, dotted, bend right, "\varphi^1"'] & J^2 \arrow[r] \arrow[dd] & {} \\
& & N \arrow[ur, hook, "i"'] \arrow[dd, "\nu"] & & & & & \\
\Sigma^{-1}D(P^*) = \arrow[r] & D(P_2^*) \arrow[rr, "d_2"] \arrow[dr, "f_2"'] & & D(P_1^*) \arrow[rr, "d_1"'] & & D(P_0^*) \arrow[r] & 0 \arrow[r] & {} \\
& & DTrM \arrow[ur, hook, "j_1"'] & & & & &
\end{tikzcd}$$

*If $\lambda$ is null homotopic, then $\nu$ factorizes as $f_2\varphi^0 i$ for some $\varphi^0$.*

*Proof.* Since $\lambda$ is null homotopic, $\lambda_0 = d_2\varphi^0 + \varphi^1a^0 = j_1f_2\varphi^0 + \varphi^1a^0$ for some $\varphi^0, \varphi^1$. Hence $\lambda_0 i = j_1f_2\varphi^0 i + \varphi^1a^0 i = j_1f_2\varphi^0 i$ since $a^0i = 0$. Since

$\lambda_0 i = j_1\nu$ (by construction), we have $j_1\nu = j_1 f_2\varphi^0 i$. Since $j_1$ is injective, $\nu = f_2\varphi^0 i$. □

**Lemma 2.3.13.** *Let $M$, $N$ be in $\mathcal{M}$ and $\nu : N \to DTrM$ be given. Let $J$ in $\mathcal{C}'$ be the injective resolution of $N$ and let $\lambda : J \to \Sigma^{-1}D(P^*)$ be a chain map induced by $\nu$ as indicated in the following diagram.*

$$\begin{tikzcd}
J = \arrow[r] \arrow[dd, "\lambda"'] & 0 \arrow[rr] \arrow[dd] & & J^0 \arrow[rr, "a^0"] \arrow[dr, "h_1"] \arrow[dd, "\lambda_0"'] \arrow[ddll, bend right, "\varphi^0"'] & & J^1 \arrow[r, "a^1"] \arrow[dd, "\lambda_1"] \arrow[ddll, dotted, "\varphi^1"'] & J^2 \arrow[r] \arrow[dd] \arrow[ddl, dotted, "\varphi^2"] & {} \\
& & N \arrow[ur, hook, "i"'] \arrow[dd, "\nu"] & & \sigma N \arrow[ur, hook, "i_1"'] \arrow[dl, dotted, "g"] & & & \\
\Sigma^{-1}D(P^*) = \arrow[r] & D(P_2^*) \arrow[rr, "d_2"] \arrow[dr, "f_2"'] & & D(P_1^*) \arrow[rr, "d_1"'] & & D(P_0^*) \arrow[r] & 0 \arrow[r] & {} \\
& & DTrM \arrow[ur, hook, "j_1"'] & & & & &
\end{tikzcd}$$

*If $\nu$ factorizes as $f_2\varphi^0 i$ for some $\varphi^0$, then the chain map $\lambda$ is null homotopic.*

*Proof.* Since $\lambda_0 i = j_1\nu$, we have $\lambda_0 i = j_1 f_2\varphi^0 i = d_2\varphi^0 i$. Hence $(\lambda_0 - d_2\varphi^0)i = 0$. Let $\sigma N$ be the cokernel of $i$, then there is a unique $g$: $\sigma N \to \mathrm{D}(P_1^*)$ such that $\lambda_0 - d_2\varphi^0 = gh_1$. Since $\mathrm{D}(P_1^*)$ is injective, there is $\varphi^1 : J^1 \to \mathrm{D}(P_1^*)$ such that $\varphi^1 i_1 = g$. Hence $\lambda_0 - d_2\varphi^0 = \varphi^1 i_1 h_1$ so $\lambda_0 = d_2\varphi^0 + \varphi^1 i_1 h_1 = d_2\varphi^0 + \varphi^1 a^0$. Similarly, we obtain a map $\varphi^2 : J^2 \to \mathrm{D}(P_0^*)$ such that $\lambda_1 = d_1\varphi^1 + \varphi^2 a^1$. □

The following proposition is a slight variation of Lemma 0.1.27(ii).

**Proposition 2.3.14.** *Let $M$ be in $\mathcal{M}$. Then $DTrM$ has an $\mathcal{M}$-precover with error term if and only if $\Sigma^{-1}D(P^*)$ has a $\mathcal{C}'$-precover in $\mathcal{T}$.*

*Proof.* Since $\Sigma^{-1}\mathrm{D}(P^*)$ need not be in $\mathcal{C}$, we cannot appeal to Remark 2.2.3. We begin by giving a useful diagram.

$$\begin{tikzcd}
K = \arrow[r] \arrow[dd, dashed, "r"] \arrow[dddd, bend right, "s"'] & 0 \arrow[rr] \arrow[dd] & & K^0 \arrow[r] \arrow[dd, dashed, "r_0"'] & K^1 \arrow[r] \arrow[dd, dashed, "r_1"'] & K^2 \arrow[r] \arrow[dd] & {} \\
& & L \arrow[ur, hook, "i_l"'] \arrow[dd, dashed, "\nu'"] \arrow[dddd, bend right, "s'"'] & & & & \\
J = \arrow[r] \arrow[dd, "\lambda"] & 0 \arrow[rr] \arrow[dd] & & J^0 \arrow[r, "a^0"] \arrow[dd, "\lambda_0"'] & J^1 \arrow[r, "a^1"] \arrow[dd, "\lambda_1"'] & J^2 \arrow[r] \arrow[dd] & {} \\
& & N \arrow[ur, hook, "i"'] \arrow[dd, "\nu"] & & & & \\
\Sigma^{-1}D(P^*) = \arrow[r] & D(P_2^*) \arrow[rr] \arrow[dr, "f_2"'] \arrow[uuuurr, dotted, bend left, "\varphi"] & & D(P_1^*) \arrow[r, "d_1"] & D(P_0^*) \arrow[r] & 0 \arrow[r] & {} \\
& & DTrM \arrow[ur, hook, "j_1"'] & & & &
\end{tikzcd}$$

The following discussion is with reference to the diagram.

(*only if*) Let $\nu : N \to \mathrm{DTr}M$ be an $\mathcal{M}$-precover with error term. Let $J$ be an injective resolution of $N$ and extend $\nu$ to a chain map $\lambda : J \to \Sigma^{-1}\mathrm{D}(P^*)$ by Lemma 2.3.10. Then $\lambda$ is a $\mathcal{C}'$-precover. First of all, $J$ is in $\mathcal{C}'$ since $N$ is in $\mathcal{M}$. Suppose $K$ is in $\mathcal{C}'$ with a chain map $s : K \to \Sigma^{-1}\mathrm{D}(P^*)$. Then there is the induced map $s' : \mathrm{Z}^0(K) = L \to \mathrm{DTr}M$. Since $L$ is in $\mathcal{M}$ and $\nu$ is an $\mathcal{M}$-precover with error term, there is a morphism $\nu' : L \to N$ such that $\nu\nu' - s' = f_2\varphi i_l$ for some $\varphi : K^0 \to \mathrm{D}(P_2^*)$. By Lemma 2.3.10, extend $\nu'$ to a chain map $r : K \to J$. By Lemma 2.3.13, $\lambda r - s$ is null homotopic, that is, $\lambda r = s$ in $\mathcal{T}$ and $\lambda$ is a $\mathcal{C}'$-precover. (*if*) Suppose $\lambda : J \to \Sigma^{-1}\mathrm{D}(P^*)$ is a $\mathcal{C}'$-precover. Then there is a morphism $\nu : \mathrm{Z}^0(J) = N \to \mathrm{DTr}M$, and $\nu$ is an $\mathcal{M}$-precover with error term. Suppose we are given $s' : L \to \mathrm{DTr}M$ where $L$ is in $\mathcal{M}$. By Lemma 2.3.10, extend $s'$ to a chain map $s : K \to \Sigma^{-1}\mathrm{D}(P^*)$ where $K$ is an injective resolution of $L$. Since $\lambda$ is a $\mathcal{C}'$-precover, there is $r : K \to J$ such that $\lambda r = s$. This $r$ induces a homomorphism $\nu' : L \to N$. By Lemma 2.3.12, $\nu\nu' - s'$ factorizes as $f_2\varphi^0 i$ for some $\varphi^0$ and $\nu$ is therefore an $\mathcal{M}$-precover with error term. □

Here we turn to study precovers in the stable category. First we need a little lemma.

**Lemma 2.3.15.** *Let $U$ in $\mathrm{mod}(\Lambda)$ be a finitely generated injective $\Lambda$-module. Consider the complex $D(P^*)$ from Section 2.3. Then $(U, D(P_2^*)) \to (U, D(P_1^*)) \to (U, D(P_0^*))$ is exact.*

*Proof.* First consider the case when $U = \mathrm{D}\Lambda$. Then the sequence becomes $(\mathrm{D}\Lambda, \mathrm{D}(P_2^*)) \to (\mathrm{D}\Lambda, \mathrm{D}(P_1^*)) \to (\mathrm{D}\Lambda, \mathrm{D}(P_0^*))$, which is isomorphic to the sequence $(P_2^*, \Lambda) \to (P_1^*, \Lambda) \to (P_0^*, \Lambda)$ since the duality functor $\mathrm{D}(-)$ is a contravariant equivalence, which is the same as the sequence $P_2^{**} \to P_1^{**} \to P_0^{**}$, which by Lemma 2.3.1(iii) is isomorphic to the exact sequence $P_2 \to P_1 \to P_0$. Finally, any finitely generated injective is a direct summand in a sum of copies of $\mathrm{D}\Lambda$. □

The following proposition shows that an $\mathcal{M}$-precover with error term is equivalent to an $\overline{\mathcal{M}}$-precover in the stable category.

**Proposition 2.3.16.** *Let $N$ be in $\mathcal{M}$. Then $\nu : N \to DTrM$ is an $\mathcal{M}$-precover with error term in $\mathrm{mod}(\Lambda)$ if and only if its class $\overline{\nu} = \nu + \mathcal{I}(N, DTrM)$ is an $\overline{\mathcal{M}}$-precover in the stable category $\overline{\mathrm{mod}}(\Lambda)$.*

*Proof.* (*only if*) Suppose $\nu : N \to \mathrm{DTr}M$ is an $\mathcal{M}$-precover with error term. Then $\overline{\nu} = \nu + \mathcal{I}(N, \mathrm{DTr}M)$ is an $\overline{\mathcal{M}}$-precover in the stable category. Suppose we are given $\overline{s'} : L \to \mathrm{DTr}M$ in the stable category with $L$ in $\overline{\mathcal{M}}$, i.e. $s' : L \to \mathrm{DTr}M$ in $\mathrm{mod}(\Lambda)$. Since $\nu$ is an $\mathcal{M}$-precover with error term, there is $\nu' : L \to N$ such that $\nu\nu' - s' = f_2\psi$ for some $\psi : L \to \mathrm{D}(P_2^*)$. Hence $s' = \nu\nu' - f_2\psi$ so $\overline{s'} = \overline{\nu}\ \overline{\nu'}$ - $\overline{f_2}\ \overline{\psi} = \overline{\nu}\ \overline{\nu'}$, since $\overline{f_2} = \overline{0}$.

(*if*) Suppose $\overline{\nu}$ is an $\overline{\mathcal{M}}$-precover in the stable category. Then $\nu$ is an $\mathcal{M}$-precover with error term. Suppose we are given $s' : L \to \mathrm{DTr}M$ with $L$ in $\mathcal{M}$. Consider its class $\overline{s'}$ in the stable category. Since $\overline{\nu}$ is an $\overline{\mathcal{M}}$-precover in the stable category, we have $\overline{\nu'} : L \to N$ such that $\overline{s'} = \overline{\nu}\ \overline{\nu'}$ for some $\overline{\nu'}$, i.e. $\nu\nu' - s'$ factors through an injective $U$, say $\nu\nu' - s' = u_2u_1$. We would like $\nu\nu' - s'$, however, to factor through $\mathrm{D}(P_2^*) \overset{f_2}{\to} \mathrm{DTr}M$ instead.

$$\begin{tikzcd}
L \arrow[rr, "\nu\nu' - s'"] \arrow[dr, "u_1"] \arrow[ddr, dashed] & & DTrM \\
& U \arrow[ur, "u_2"'] \arrow[d, dashed] & \\
& D(P_2^*) \arrow[uur, "f_2"'] &
\end{tikzcd}$$

Consider the morphism $j_1u_2 : U \to \mathrm{D}(P_1^*)$, where $j_1$ is as in Definition 2.3.6.

$$\begin{tikzcd}
& U \arrow[dl, dashed, "g"'] \arrow[d, "j_1u_2"] \arrow[dr, "0"] & \\
D(P_2^*) \arrow[r, "d_2"'] & D(P_1^*) \arrow[r, "d_1"'] & D(P_0^*)
\end{tikzcd}$$

Since $d_1j_1u_2 = 0$, by Lemma 2.3.15, there is $g$ such that $d_2g = j_1u_2$, which gives $j_1f_2g = j_1u_2$. Since $j_1$ is injective, $f_2g = u_2$ and $\nu\nu' - s = u_2u_1 = f_2gu_1$. $\square$

**Proposition 2.3.17.** *Let $N$ be in $\mathcal{M}$. Then $Df : DN \to DTr(DM)$ is a $D\mathcal{M}$-precover with error term in* $\mathrm{mod}(\Lambda^{op})$ *if and only if the class $\underline{f} = f + \mathcal{P}(Tr(DM), N)$ is an $\underline{\mathcal{M}}$-preenvelope in the stable category* $\underline{\mathrm{mod}}(\Lambda)$.

*Proof.* Similar. $\square$

**Proposition 2.3.18.** *Let $N$ be in $\mathcal{M}$. Then $f : Tr(DM) \to N$ is an $\mathcal{M}$-preenvelope with error term in* $\mathrm{mod}(\Lambda)$ *if and only if the class $\overline{Df} = Df + \mathcal{I}(DN, DTr(DM))$ is a $\overline{D\mathcal{M}}$-precover in the stable category* $\overline{\mathrm{mod}}(\Lambda^{op})$.

*Proof.* Similar. $\square$

## 2.4 Existence of Auslander-Reiten sequences in subcategories

Let $N$ be in $\mathrm{mod}(\Lambda)$. The functors $\mathrm{Ext}^i_\Lambda(-,N)$ and $\overline{\mathrm{Ext}}^i_\Lambda(N,-)$, the right derived functors of $\mathrm{Hom}_\Lambda(-,N)$ and $\mathrm{Hom}_\Lambda(N,-)$ respectively, are described in Section 0.2.2. By Lemma 0.2.26(ii), the functors $\mathrm{Ext}^i_\Lambda(-,-)$ and $\overline{\mathrm{Ext}}^i_\Lambda(-,-)$, where $i \geq 0$, are naturally equivalent.

**Definition 2.4.1.** ([18, III.1.], [45, 3.4]) Let $A$, $B$, $X$ and $X'$ be in $\mathrm{mod}(\Lambda)$. An *extension* of $A$ by $B$ in $\mathrm{mod}(\Lambda)$ is an exact sequence $0 \to B \to X \to A \to 0$ in $\mathrm{mod}(\Lambda)$. Two extensions $\varepsilon$ and $\varepsilon'$ are *equivalent* if there is the following commutative diagram in $\mathrm{mod}(\Lambda)$.

$$\begin{tikzcd}
\varepsilon : 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\cong"] & A \arrow[r] \arrow[d, equal] & 0 \\
\varepsilon' : 0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

An extension is *split* if it is equivalent to $0 \to B \to A \oplus B \to A \to 0$.

Let $\mathrm{E}(A,B)$ be the set of equivalence classes of extensions of $A$ by $B$.

The following lemmas are standard.

**Lemma 2.4.2.** *([45, Lemma 3.4.1]) Let $A$ and $B$ be in* $\mathrm{mod}(\Lambda)$. *If* $\mathrm{E}(A,B) = 0$*, then every extension of $A$ by $B$ is split.*

**Lemma 2.4.3.** *([18, Theorem III.2.4], [18, Proposition IV.7.1], [45, Theorem 3.4.3]) Let $A$ and $B$ be in* $\mathrm{mod}(\Lambda)$*, and consider the extension $\varepsilon : 0 \to B \to X \to A \to 0$ in* $\mathrm{mod}(\Lambda)$. *The functors $Ext^i_\Lambda(A,-)$ induce the exact sequence $(A,X) \to (A,A) \xrightarrow{\partial} Ext^1(A,B)$ in* $\mathrm{mod}(\Lambda)$. *Then $\theta : \mathrm{E}(A,B) \to Ext^1(A,B)$, given by $\theta(\varepsilon) = \partial(\mathrm{id}_A)$, is a one-one correspondence, in which the split extension corresponds to the element $0$ in $Ext^1(A,B)$.*

**Lemma 2.4.4.** *Let $\varepsilon : 0 \to B \to X \to A \to 0$ be an extension in* $\mathrm{mod}(\Lambda)$. *If either $A$ is projective or $B$ is injective, then the extension $\varepsilon$ is split.*

*Proof.* This is true by Lemma 0.2.26(i), Lemma 2.4.2 and Lemma 2.4.3. □

Let us reminisce the following with regard to Auslander-Reiten sequences in $\mathrm{mod}(\Lambda)$.

**Lemma 2.4.5.** *([2, Theorem V.1.16]) Let $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ and $0 \to A' \xrightarrow{g'} B' \xrightarrow{f'} C' \to 0$ be two Auslander-Reiten sequences in* $\mathrm{mod}(\Lambda)$. *Then the following are equivalent.*

*(i)* $A \cong A'$,

*(ii)* $C \cong C'$,

*(iii)* *The sequences are isomorphic, which is to say there is a commutative diagram,*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "\cong"] & B \arrow[r, "f"] \arrow[d, "\cong"] & C \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & A' \arrow[r, "g'"] & B' \arrow[r, "f'"] & C' \arrow[r] & 0,
\end{tikzcd}$$

*where all the vertical morphisms are isomorphisms.*

**Lemma 2.4.6.** *([2, Proposition V.1.14]) Let* $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ *be an exact sequence in* $\mathrm{mod}(\Lambda)$*. Then the following are equivalent.*

*(i)* *The sequence is an Auslander-Reiten sequence,*

*(ii)* *The module $A$ is indecomposable and the morphism $f$ is right almost split,*

*(iii)* *The module $C$ is indecomposable and the morphism $g$ is left almost split,*

*(iv)* $A \cong \mathrm{DTr}C$ *and the morphism $f$ is right almost split,*

*(v)* $C \cong \mathrm{TrD}A$ *and the morphism $g$ is left almost split.*

By Lemma 2.4.5 and Lemma 2.4.6, the existence of Auslander-Reiten sequences in $\mathrm{mod}(\Lambda)$ implies that given an indecomposable $A$ in $\mathrm{mod}(\Lambda)$, there is up to isomorphism a unique indecomposable $C$ in $\mathrm{mod}(\Lambda)$ such that $\mathrm{DTr}C \cong A$. Similarly, given an indecomposable $C$ in $\mathrm{mod}(\Lambda)$, there is up to isomorphism a unique indecomposable $A$ in $\mathrm{mod}(\Lambda)$ such that $\mathrm{TrD}A \cong C$.

Originally in [3, Theorem 2.4], Auslander and Smalø developed a theory for the existence of Auslander-Reiten sequences in subcategories of $\mathrm{mod}(\Lambda)$. Then in [28, Corollary 2.8], Kleiner gave a new proof of their existence theorems without the use of the theory of dualizing $R$-varieties. These are rephrased below.

**Theorem.** *Let $\mathcal{C}$ be a precovering of* $\mathrm{mod}(\Lambda)$ *closed under extensions and direct summands, and let $C$ be an indecomposable module in $\mathcal{C}$ such that $Ext^1(C, \tilde{C}) \neq 0$ for some $\tilde{C}$ in $\mathcal{C}$. Then there is an Auslander-Reiten sequence* $0 \to A \to B \to C \to 0$ *in $\mathcal{C}$.*

**Theorem.** *Let $\mathcal{A}$ be a preenveloping of* $\operatorname{mod}(\Lambda)$ *closed under extensions and direct summands, and let $A$ be an indecomposable module in $\mathcal{A}$ such that $Ext^1(\tilde{A}, A) \neq 0$ for some $\tilde{A}$ in $\mathcal{A}$. Then there is an Auslander-Reiten sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$.*

In this section, we strengthen Auslander-Smalø's and Kleiner's results by providing necessary and sufficient conditions, based on the theory of Auslander-Reiten triangles developed in [22]. The proof here is more subtle than the proof of [22, Theorem 3.1], since we cannot replace $\mathcal{C}$ by $\mathcal{C}'$ and $\mathcal{T}$ by $\mathcal{C}$, because $\mathcal{T}$ is triangulated and $\mathcal{C}$ is not triangulated.

It is conceivable that the necessary condition of the existence of Auslander-Reiten sequences in subcategories provided here is weaker, because in [22, Theorem 3.1], for the subcategory $\mathcal{C} \subseteq \mathcal{T}$ to have Auslander-Reiten triangles, only certain objects $X$ in $\mathcal{T}$ are required to have $\mathcal{C}$-covers. This naturally leads us to give a characterization of the existence of Auslander-Reiten sequences in subcategories not in terms of a global condition but in terms of a condition which describes locally at the level of objects.

Now we state the existence theorem for (right) Auslander-Reiten sequences in subcategories.

**Theorem 2.4.7.** *Let $\mathcal{M}$ be a subcategory of* $\operatorname{mod}(\Lambda)$ *closed under extensions and direct summands, and let $M$ be an indecomposable module in $\mathcal{M}$ such that $Ext^1(M, \tilde{M}) \neq 0$ for some $\tilde{M}$ in $\mathcal{M}$. Then the following are equivalent.*

*(i) $DTrM$ has an $\overline{\mathcal{M}}$-precover in the injective stable category $\overline{\operatorname{mod}}(\Lambda)$,*

*(ii) There is an Auslander-Reiten sequence $0 \to X \to Y \to M \to 0$ in $\mathcal{M}$.*

*Proof.* (i) $\Rightarrow$ (ii): Let $P$ and $C$ be a projective and an injective resolution of $M$. By Proposition 2.3.16, $\mathrm{DTr}M$ has an $\mathcal{M}$-precover with error term, and then by Proposition 2.3.14, $\Sigma^{-1}\mathrm{D}(P^*)$ has a $\mathcal{C}'$-precover. By [22, Theorem 4.6], there is an Auslander-Reiten triangle $A \to B \to C \to$ in $\mathcal{C}'$, and finally an Auslander-Reiten sequence $0 \to \mathrm{H}^0(A) \to \mathrm{H}^0(B) \to \mathrm{H}^0(C) \to 0$ in $\mathcal{M}$, where $M$ is retrieved through the isomorphism $\mathrm{H}^0(C) \cong M$ (Lemma 2.2.5).

(ii) $\Rightarrow$ (i): Let $P$ and $C$ be a projective and an injective resolution of $M$. Following the argument in Theorem 4.6 in [22], there is an Auslander-Reiten triangle $\Sigma^{-1}\mathrm{D}(P^*) \to Y' \to C \to$ in $\mathcal{T}$. Since there is an Auslander-Reiten sequence $0 \to X \to Y \to M \to 0$ in $\mathcal{M}$, by Remark 2.2.5, there is an Auslander-Reiten triangle $A \to B \to C \to$ in $\mathcal{C}'$. By Theorem 3.1 in [22], $\Sigma^{-1}\mathrm{D}(P^*)$ has a $\mathcal{C}'$-precover. By Proposition 2.3.14, $\mathrm{DTr}M$ has an $\mathcal{M}$-precover with error term. Finally by Proposition 2.3.16, $\mathrm{DTr}M$ has an $\overline{\mathcal{M}}$-precover in the stable category $\overline{\operatorname{mod}}(\Lambda)$. □

*Remark* 2.4.8. (i) In the proof of Theorem 2.4.7, (i) $\Rightarrow$ (ii), the Auslander-Reiten sequence not only exists, but also takes the form $0 \to \mathrm{H}^0(A) \to \mathrm{H}^0(B) \to \mathrm{H}^0(C) \to 0$ in $\mathcal{M}$.

(ii) In the proof of Theorem 2.4.7, (ii) $\Rightarrow$ (i), the Auslander-Reiten triangle $\Sigma^{-1}\mathrm{D}(P^*) \to Y' \to C \to$ in $\mathcal{T}$ induces the exact sequence $\cdots \to 0 \to \mathrm{H}^0(\Sigma^{-1}\mathrm{D}(P^*)) \to \mathrm{H}^0(Y') \to M \to \mathrm{H}^1(\Sigma^{-1}\mathrm{D}(P^*)) \to \cdots$ in $\mathrm{mod}(\Lambda)$ by Lemma 0.2.9, but does not render any Auslander-Reiten sequences in $\mathrm{mod}(\Lambda)$.

*Remark* 2.4.9. (i) The necessary condition (i) in Theorem 2.4.7 is weaker than that of Auslander-Smalø and Kleiner in two senses. Firstly, only precovers in the stable category are considered, and secondly, only precovers of the $\mathrm{DTr}M$ in $\mathrm{mod}(\Lambda)$ with $M$ in $\mathcal{M}$ are considered. As $\mathcal{M}$ is precovering in $\mathrm{mod}(\Lambda)$ if and only if $\mathcal{C}'$ is precovering in $\mathcal{C}$, correspondingly we do not require as much as $\mathcal{C}'$ to be precovering in $\mathcal{C}$ either.

(ii) In the proof of Theorem 2.4.7, (ii) $\Rightarrow$ (i), even though $\mathrm{D}(P^*) \to \Sigma Y' \to \Sigma C \to$ is also an Auslander-Reiten triangle in $\mathcal{T}$, the object $\Sigma C$ is not necessarily in $\mathcal{C}$ since $\mathcal{C}$ is not triangulated.

Before we give the existence theorem for (left) Auslander-Reiten sequences in subcategories, we need the following.

**Lemma 2.4.10.** *Let $\mathcal{L}$ be a subcategory of* $\mathrm{mod}(\Lambda)$. *Then the following are equivalent.*

*(i)* $0 \to X \to Y \to Z \to 0$ *is an Auslander-Reiten sequence in $\mathcal{L}$,*

*(ii)* $0 \to DZ \to DY \to DX \to 0$ *is an Auslander-Reiten sequence in $D\mathcal{L}$.*

*Proof.* Please refer to the remark after [2, Proposition V.1.13]. □

**Proposition 2.4.11.** *Let $\mathcal{L}$ be a subcategory of* $\mathrm{mod}(\Lambda)$. *Denote $D\mathcal{L}$ by $\mathcal{M}$. Let $L$ be in $\mathcal{L}$. Then the following are equivalent.*

*(i) $DL$ has an $\overline{\mathcal{M}}$-precover in the injective stable category* $\overline{\mathrm{mod}}(\Lambda^{op})$,

*(ii) $L$ has an $\underline{\mathcal{L}}$-preenvelope in the projective stable category* $\underline{\mathrm{mod}}(\Lambda)$.

*Proof.* This can be shown by standard arguments. However, we shall only consider the special case $L = \mathrm{TrD}L'$ for some $L'$ in $\mathcal{L}$, since this is what is needed in Theorem 2.4.13. Let $N$ be in $\mathcal{L}$. Then by Proposition 2.3.17, $\underline{f} = f + \mathcal{P}(\mathrm{TrD}L', N)$ is an $\underline{\mathcal{L}}$-preenvelope in the projective stable category $\underline{\mathrm{mod}}(\Lambda)$ if and only if $Df : DN \to \mathrm{DTrD}L'$ is a $D\mathcal{M}$-precover with error term in $\mathrm{mod}(\Lambda^{op})$. This is equivalent to saying $f : \mathrm{TrD}L' \to N$ is an $\mathcal{L}$-preenvelope with error term in $\mathrm{mod}(\Lambda)$, which by Proposition 2.3.18 is equivalent to saying $\overline{Df} = Df + \mathcal{I}(DN, \mathrm{DTrD}L')$ is an $\overline{\mathcal{M}}$-precover in the injective stable category $\overline{\mathrm{mod}}(\Lambda^{op})$. $\square$

**Proposition 2.4.12.** *Let $\mathcal{L}$ be a subcategory of* $\mathrm{mod}(\Lambda)$ *closed under extensions and direct summands. Denote $D\mathcal{L}$ by $\mathcal{M}$. Let $L$ be an indecomposable module in $\mathcal{L}$ such that $Ext^1(DL, D\tilde{L}) \neq 0$ for some $\tilde{L}$ in $\mathcal{L}$. Then the following are equivalent.*

*(i) $DTrDL$ has an $\overline{\mathcal{M}}$-precover in the injective stable category* $\overline{\mathrm{mod}}(\Lambda^{op})$*,*

*(ii) There is an Auslander-Reiten sequence* $0 \to DA \to DB \to DL \to 0$ *in $\mathcal{M}$, where $A$ and $B$ are in $\mathcal{L}$.*

*Proof.* Since $\mathcal{M} = \mathrm{D}\mathcal{L}$, and $\mathcal{L}$ is closed under extensions and direct summands, $\mathcal{M}$ is a subcategory of $\mathrm{mod}(\Lambda^{op})$ closed under extensions and direct summands. Since $L$ is an indecomposable module in $\mathcal{L}$, $\mathrm{D}L$ is an indecomposable module in $\mathcal{M}$. The rest follows from the right module version of Theorem 2.4.7. $\square$

Finally, we give the dual of Theorem 2.4.7.

**Theorem 2.4.13.** *Let $\mathcal{L}$ be a subcategory of* $\mathrm{mod}(\Lambda)$ *closed under extensions and direct summands, and let $L$ be an indecomposable module in $\mathcal{L}$ such that $Ext^1(\tilde{L}, L) \neq 0$ for some $\tilde{L}$ in $\mathcal{L}$. Then the following are equivalent.*

*(i) $TrDL$ has an $\underline{\mathcal{L}}$-preenvelope in the projective stable category* $\underline{\mathrm{mod}}(\Lambda)$*,*

*(ii) There is an Auslander-Reiten sequence* $0 \to L \to B \to A \to 0$ *in $\mathcal{L}$.*

*Proof.* Let $\mathcal{M} = \mathrm{D}\mathcal{L}$. By Proposition 2.4.11, $\mathrm{TrD}L$ has an $\underline{\mathcal{L}}$-preenvelope in $\underline{\mathrm{mod}}(\Lambda)$ if and only if $\mathrm{DTrD}L$ has an $\overline{\mathcal{M}}$-precover in $\overline{\mathrm{mod}}(\Lambda^{op})$. Also $\mathrm{Ext}^1(\tilde{L}, L) \neq 0$ if and only if $\mathrm{Ext}^1(\mathrm{D}L, \mathrm{D}\tilde{L}) \neq 0$. Hence the result follows from Proposition 2.4.12, with the help of Lemma 2.4.10. $\square$

*Remark* 2.4.14. Theorem 2.4.13 can also be shown directly from [22, Theorem 3.2].

*Remark* 2.4.15. Theorem 2.4.7 and Theorem 2.4.13 together give back the results by Auslander-Smalø and by Kleiner. For example, suppose $\mathcal{M}$ is precovering in $\mathrm{mod}(\Lambda)$. Then given $M$ in $\mathcal{M}$, the module $\mathrm{DTr}M$ has an $\mathcal{M}$-precover in $\mathrm{mod}(\Lambda)$ and then an $\overline{\mathcal{M}}$-precover in the injective stable category $\overline{\mathrm{mod}}(\Lambda)$. Finally, by Theorem 2.4.7, there is an Auslander-Reiten sequence $0 \to X \to Y \to M \to 0$ in $\mathcal{M}$.

## 2.5 An example

In this section, examples of Theorem 2.4.7 and Theorem 2.4.13 are given. First, we need a little lemma.

**Lemma 2.5.1.** *Let $\mathcal{M}$ be a full subcategory of* $\mathrm{mod}(\Lambda)$ *closed under direct sums and direct summands. Let $M$ be in* $\mathrm{mod}(\Lambda)$*. Suppose there are only finitely many indecomposable modules in $\mathcal{M}$ with non-zero maps to $M$. Then $M$ has an $\mathcal{M}$-precover in* $\mathrm{mod}(\Lambda)$.

*Proof.* Since there are only finitely many indecomposable modules in $\mathcal{M}$ with non-zero maps to $M$, let them be denoted by $K_1, K_2, \ldots, K_n$.

(i) Suppose $n = 1$.

If $(K_1, M)$ is one-dimensional with basis $\{\kappa\}$, then $K_1 \xrightarrow{\kappa} M$ is an $\mathcal{M}$-precover. Let a morphism $m : K' \to M$, with $K'$ an indecomposable in $\mathcal{M}$, be given.

$$\begin{tikzcd} & K' \arrow[d, "m"] \\ K_1 \arrow[r, "\kappa"] & M \end{tikzcd}$$

If $K' \neq K_1$, then $m$ is zero. If $K' = K_1$, then $m$ factors through $\kappa$ by a scalar multiple of $\kappa$. On the other hand, if $(K_1, M)$ is $q$-dimensional with basis $\{\kappa_1, \kappa_2, \ldots, \kappa_q\}$, then $\underbrace{K_1 \oplus K_1 \oplus \ldots \oplus K_1}_{q} \xrightarrow{\kappa} M$ is an $\mathcal{M}$-precover, where $\kappa = \begin{pmatrix} \kappa_1 & \kappa_2 & \ldots & \kappa_q \end{pmatrix}$.

(ii) Suppose $n = 2$.

If $(K_1, M)$ has basis $\{\kappa_1^1, \kappa_2^1, \ldots, \kappa_s^1\}$ and $(K_2, M)$ has basis $\{\kappa_1^2, \kappa_2^2, \ldots, \kappa_t^2\}$, then $\underbrace{K_1 \oplus K_1 \oplus \ldots \oplus K_1}_{s} \oplus \underbrace{K_2 \oplus K_2 \oplus \ldots \oplus K_2}_{t} \xrightarrow{\kappa} M$ is an $\mathcal{M}$-precover, where $\kappa = \begin{pmatrix} \kappa_1^1 & \kappa_2^1 & \ldots & \kappa_s^1 & \kappa_1^2 & \kappa_2^2 & \ldots & \kappa_t^2 \end{pmatrix}$. Let a morphism $m : K' \to M$, with $K'$ an indecomposable in $\mathcal{M}$, be given. Without loss of generality, assume $K' = K_1$.

$$\begin{tikzcd} & K_1 \arrow[dl, dashed, "g"'] \arrow[d, "m"] \\ K_1 \oplus K_1 \oplus \ldots \oplus K_1 \oplus K_2 \oplus K_2 \oplus \ldots \oplus K_2 \arrow[r, "\kappa"] & M \end{tikzcd}$$

If $m = \alpha_1\kappa_1^1 + \alpha_2\kappa_2^1 + \ldots + \alpha_s\kappa_s^1$, then $m$ can be factorized as $m = \kappa g$, where $g = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_s \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Thus $M$ has an $\mathcal{M}$-precover in $\mathrm{mod}(\Lambda)$.

The case of $n > 2$ is similar. $\square$

In Lemma 2.5.1, an $\mathcal{M}$-precover is given by construction. For example, in case (ii), $K_1 \oplus K_1 \oplus \ldots \oplus K_1 \oplus K_2 \oplus K_2 \oplus \ldots \oplus K_2 \xrightarrow{\kappa} M$ is an $\mathcal{M}$-precover. This is not the minimal construction when, for example, each $\kappa_i^1$ factors through $\kappa_j^2$, some $j$, in which case $K_2 \oplus K_2 \oplus \ldots \oplus K_2 \to M$ will be sufficient to be an $\mathcal{M}$-precover.

*Example* 2.5.2. Let $\Lambda$ be a representation-infinite hereditary algebra. Let $\mathcal{M}$ be any full subcategory of $\mathrm{mod}(\Lambda)$ which consists of postprojective modules, which is closed under extensions and direct summands, see [1, Definition VIII.2.2]. Let $M$ be an indecomposable module in $\mathcal{M}$ such that $\mathrm{Ext}^1(M, \tilde{M}) \neq 0$ for some $\tilde{M}$ in $\mathcal{M}$. Then there is an Auslander-Reiten sequence $0 \to X \to Y \to M \to 0$ in $\mathcal{M}$.

*Proof.* By [1, Lemma VIII.2.5], there are only finitely many indecomposable modules in $\mathcal{M}$ which have non-zero maps to $\mathrm{DTr}M$. Therefore by Lemma 2.5.1, $\mathrm{DTr}M$ has an $\mathcal{M}$-precover in $\mathrm{mod}(\Lambda)$ and then an $\overline{\mathcal{M}}$-precover in the stable category $\overline{\mathrm{mod}}(\Lambda)$. The existence of the Auslander-Reiten sequence $0 \to X \to Y \to M \to 0$ in $\mathcal{M}$ follows from Theorem 2.4.7. $\square$

*Remark* 2.5.3. Dually, let $\mathcal{L}$ be the full subcategory of $\mathrm{mod}(\Lambda)$ consisting of preinjective modules over $\Lambda$ and is closed under extensions and direct summands. Let $L$ be an indecomposable module in $\mathcal{L}$ such that $\mathrm{Ext}^1(\tilde{L}, L) \neq 0$ for some $\tilde{L}$ in $\mathcal{L}$. Then the existence of the Auslander-Reiten sequence $0 \to L \to Y \to X \to 0$ in $\mathcal{L}$ follows from Theorem 2.4.13.

# Chapter 3

# Quotients of certain triangulated categories as derived categories

## 3.1 Introduction

Let us begin with the notion of a *pretriangulated* category. This is an additive category with an endofunctor $\sigma$ and a class of diagrams of the form $x \to y \to z \to \sigma x$ known as *distinguished right triangles*, and another endofunctor $\omega$ and a class of diagrams of the form $\omega z \to x \to y \to z$ known as *distinguished left triangles*. The distinguished right triangles have to satisfy the axioms of a triangulated category, except that $\sigma$ does not need to be an equivalence. Similarly for the distinguished left triangles. It is worth acknowledging that $(\sigma, \omega)$ is an adjoint pair of functors, which is reminiscent of the intuition that adjoints are approximations of inverse functors.

Let $k$ be a field and $\mathcal{T}$ be a $k$-linear triangulated category with translation functor $\Sigma$ and Serre functor $S$, which is Hom finite and Krull-Schmidt. Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$ which is both precovering and preenveloping, and let $\mathcal{T}_{\mathcal{X}}$ be the quotient category defined in [23, Introduction]. By [23, Theorem 1.2], the quotient category $\mathcal{T}_{\mathcal{X}}$ is pretriangulated. The quotient category $\mathcal{T}_{\mathcal{X}}$ is triangulated if and only if $\tau\mathcal{X} = \mathcal{X}$, where $\tau$ is the Auslander-Reiten translation, see [23, Theorem 2.3]. This means the subcategory $\mathcal{X}$ is a union of orbits of $\tau$.

The definition of a *Frobenius category* is given in [14]. A subcategory of a triangulated category is *thick* if it is triangulated and closed under direct summands. A thick subcategory generated by $X$, where $X$ is an object of $\mathcal{T}$, is denoted by *thick*$(X)$. A triangulated category $\mathcal{T}$ is *of algebraic origin* if it is equivalent, as a triangulated category, to the quotient category $\mathcal{E}_{\mathcal{P}}$ of a $k$-linear Frobenius category $\mathcal{E}$, where $\mathcal{P}$ is the class of projective objects of $\mathcal{E}$, see [23, Section 3]. By [23, Theorem 3.2], if $\mathcal{T}$ is of algebraic origin, then $\mathcal{T}_{\mathcal{X}}$ is also of algebraic origin.

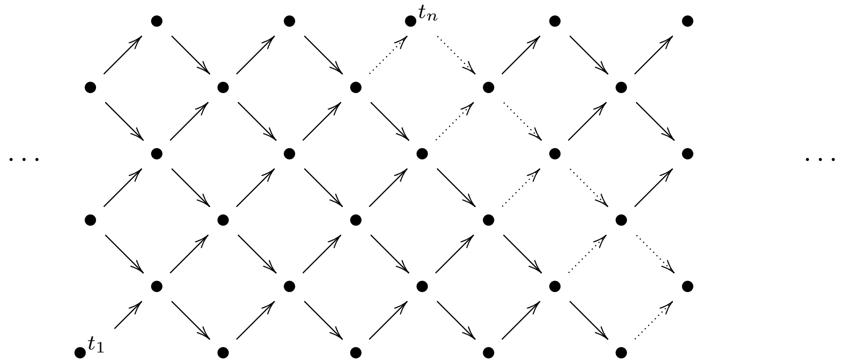
Let $\Lambda$ be a finite-dimensional $k$-algebra over the field $k$. A *compact derived category* $D^c(\Lambda)$ is the full subcategory of the derived category $D(\Lambda)$ consisting of *compact* objects, i.e. those finitely built from the algebra $\Lambda$ using suspensions, distinguished triangles, direct sums and direct summands. If $\Lambda$ is Noetherian and of finite global dimension, then the compact derived category $D^c(\Lambda)$ is equivalent to the finite derived category $D^b(\Lambda)$. Let $D^c(A)$ be the compact derived category generated by $A$.

In this chapter, quotients of certain triangulated categories are triangulated and are in addition derived categories, appealing to the following theorem which is a slight variation of [40, Theorem 6.4] and of [26, 3.3].

**Theorem 3.1.1.** *Let $k$ be a field and $\mathcal{U}$ be a $k$-linear triangulated category of algebraic origin such that $\mathcal{U} = thick(X)$ for some $X$ in $\mathcal{U}$. Let $\Sigma$ be the translation functor of the triangulated category $\mathcal{U}$ and let $A = End_{\mathcal{U}}(X)$. Suppose* $\mathrm{Hom}_{\mathcal{U}}(X, \Sigma^i X) = 0$ *for $i \neq 0$. Then there is an equivalence of triangulated categories $f : \mathcal{U} \xrightarrow{\simeq} D^c(A)$ where $f(X) = A$.*

## 3.2 The finite derived category $D^b(\mathrm{mod}\, kA_n)$

In this section, let $\mathcal{T}$ be the finite derived category $D^b(\mathrm{mod}\, kA_n)$ described in Section 1.4. The category $\mathcal{T}$ is a $k$-linear Hom finite triangulated category. It is Krull-Schmidt as well. The Auslander-Reiten quiver of $D^b(\mathrm{mod}\, kA_n)$ is reproduced here, with the coordinate system described in Section 1.4.




Given an indecomposable object $(i, j)$ of $D^b(\mathrm{mod}\, kA_n)$, the action of $\Sigma$ is given by $\Sigma(i,j) = (j-1, i+n+2)$, and the action of $\tau$ by $\tau(i,j) = (i-1, j-1)$ (Section 1.4).

The following is an example of Lemma 1.4.1 and the narration after it, and is rewritten below to remind the reader.

**Lemma 3.2.1.** *Let $x$ and $y$ be indecomposable objects of $D^b(\operatorname{mod} kA_n)$. Then by [14, 4.6], any non-zero morphism $f$ from $x$ to $y$ is a linear combination of morphisms, written $f = \Sigma\alpha_i f_i$, where the $\alpha_i$ are scalars and the $f_i : x \to y$ are compositions of irreducible morphisms. The finite derived category $D^b(\operatorname{mod} kA_n)$ is also standard.*

The Auslander-Reiten quiver of $D^b(\operatorname{mod} kA_n)$ contains the following local configuration throughout, which gives rise to Auslander-Reiten triangles and commutativity relations.

**Configuration 3.2.2.**

$$\begin{tikzcd} & \circ^{b} \arrow[dr, "t"] & \\ \circ^{a} \arrow[ur, "s"] \arrow[dr, "q"'] & & \circ^{d} \\ & \circ^{c} \arrow[ur, "r"'] & \end{tikzcd}$$

In Configuration 3.2.2, since $a \to b \oplus c \to d \to$ is an Auslander-Reiten triangle, $ts + rq = 0$. This gives $ts = -rq$, which means the different morphisms from $a$ to $d$ are equal up to signs by Lemma 3.2.1.

Given an indecomposable object $a$ of $D^b(\operatorname{mod} kA_n)$, the notions of $\mathcal{L}(a)$ and of $\mathcal{R}(a)$, and the sketches of the two respective regions, are described in Section 1.4. Given indecomposable objects $x$ and $y$ of $D^b(\operatorname{mod} kA_n)$, if $(x,y)$ is non-zero, then by Lemma 3.2.1 and by virtue of the little remark after Configuration 3.2.2, $(x,y)$ is one-dimensional.

With the following lemma we end the section.

**Lemma 3.2.3.** *Let $a$, $b$ and $c$ be indecomposable objects of $D^b(\operatorname{mod} kA_n)$. Let $c$ be in $\mathcal{R}(a)$, and let $f : a \to b$ and $g : b \to c$ be non-zero morphisms. Then the composition $gf : a \to c$ is non-zero.*

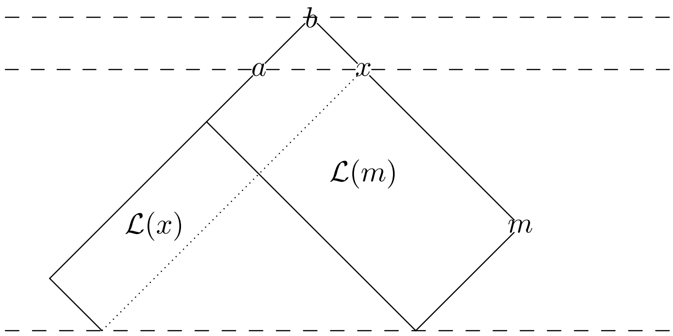
*Proof.* Assume $gf$ is zero. Let $h$ be a non-zero morphism in $(a,c)$. Since $(a,c)$ is non-zero, it is one-dimensional. Thus the morphism $h$ is a composition of irreducible morphisms. Furthermore, the morphism $h$ can be written $h = \alpha\tilde{g}f$ for some scalar $\alpha$ and some non-zero $\tilde{g} : b \to c$ by the commutativity relation described above. Since $(b,c)$ is one-dimensional, $\tilde{g} = \beta g$ for some scalar $\beta$. However, this would give $h = \alpha\beta gf = 0$, which is a contradiction. $\square$

### 3.2.1 Lemmas

Consider the indecomposable objects in a band of width $n'$ vertices along the top of the Auslander-Reiten quiver of $\mathcal{T}$. Henceforth, let $\mathcal{X}$ be add of them.

**Lemma 3.2.4.** *In the following sketch, the indecomposable objects of $\mathcal{X}$ lie in the region bounded by and including the two upper lines. Let $m$ be an indecomposable object in $\mathcal{T}$. Then $m$ has an $\mathcal{X}$-cover $g : x \to m$. Similarly, $m$ has an $\mathcal{X}$-envelope.*

*Proof.* If $m$ is in $\mathcal{X}$, then the lemma is trivial. The following sketch shows the case where $m$ is not in $\mathcal{X}$.




Let $x$ be an object in $\mathcal{X}$ which lies on the bottom line of the upper band, and is on the upper right hand boundary of the region $\mathcal{L}(m)$. Then the non-zero morphism $g : x \to m$ is an $\mathcal{X}$-precover of $m$. Indeed consider an indecomposable $y$ in $\mathcal{X}$ with a non-zero morphism $h : y \to m$. Then $y$ is in the region $abx$ which is the intersection of $\mathcal{X}$ and $\mathcal{L}(m)$. Since the region $abx$ is inside $\mathcal{L}(x)$, there is a non-zero morphism $f : y \to x$. By Lemma 3.2.3, the composition $gf$ is non-zero. Since $h$ and $gf$ are both in $(y, m)$ which is one-dimensional, $h = k(gf)$, where $k$ is a scalar, and so $h$ factors through $g$.

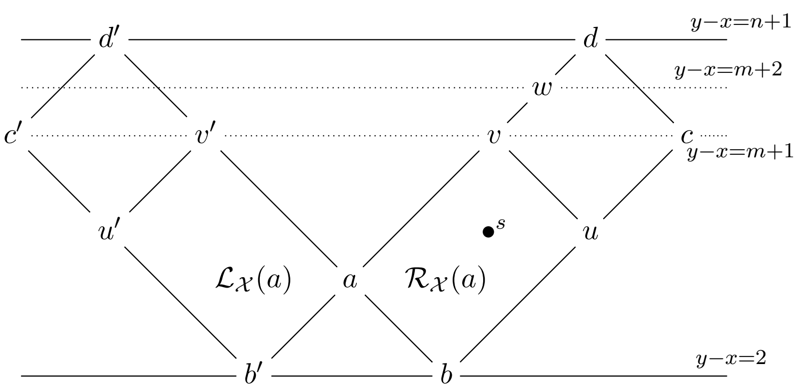
Furthermore, $g$ is an $\mathcal{X}$-cover. Suppose $\varphi : x \to x$ is a morphism such that $g\varphi = g$. Then again since $(x, x)$ is one-dimensional, $\varphi = \alpha \cdot \mathrm{id}_x$ where $\alpha$ is a scalar, so $g\varphi = g$ gives $\alpha \cdot g = g$ but $g \neq 0$ so $\alpha = 1$. Therefore $\varphi$ is invertible. The dual argument gives an $\mathcal{X}$-envelope. $\square$

*Remark* 3.2.5. In Lemma 3.2.4, the region $abx$, which is the intersection of $\mathcal{X}$ and $\mathcal{L}(m)$, has only finitely many objects. Therefore an $\mathcal{X}$-precover of $m$ can be obtained in the same fashion as in Lemma 2.5.1. The above shows how an $\mathcal{X}$-precover can be strengthened into an $\mathcal{X}$-cover.

By Lemma 3.2.4 and [23, Theorem 1.2], the quotient category $\mathcal{T}_{\mathcal{X}}$ is pretriangulated. Since evidently $\tau\mathcal{X} = \mathcal{X}$, therefore $\mathcal{T}_{\mathcal{X}}$ is in addition triangulated by [23, Theorem 2.3]. By [23, Theorem 3.2], the Auslander-Reiten quiver of $\mathcal{T}_{\mathcal{X}}$ is obtained by deleting the upper $n'$ lines of vertices and the incident arrows from the Auslander-Reiten quiver of $\mathcal{T}$. Suppose the Auslander-Reiten quiver of $\mathcal{T}_{\mathcal{X}}$ is of width $m$ vertices.

**Definition 3.2.6.** Let $a$ be an indecomposable object in $\mathcal{T}_{\mathcal{X}}$. Similarly to Definition 1.4.2, let $\mathcal{L}_{\mathcal{X}}(a)$ be the set of indecomposable objects with non-zero morphisms to $a$ in $\mathcal{T}_{\mathcal{X}}$. Dually, let $\mathcal{R}_{\mathcal{X}}(a)$ be the set of indecomposable objects to which there are non-zero morphisms from $a$ in $\mathcal{T}_{\mathcal{X}}$.

**Lemma 3.2.7.** *Suppose the Auslander-Reiten quiver of $\mathcal{T}$ lies in the following sketch in the region bounded by $y - x = 2$ and $y - x = n + 1$, and the indecomposable objects of $\mathcal{X}$ form the upper band, which is the region bounded by $y - x = m + 2$ and $y - x = n + 1$. The Auslander-Reiten quiver of $\mathcal{T}_{\mathcal{X}}$ lies in the lower band, which is the region bounded by $y - x = 2$ and $y - x = m + 1$. Here the lower band is of width $m$ vertices.*

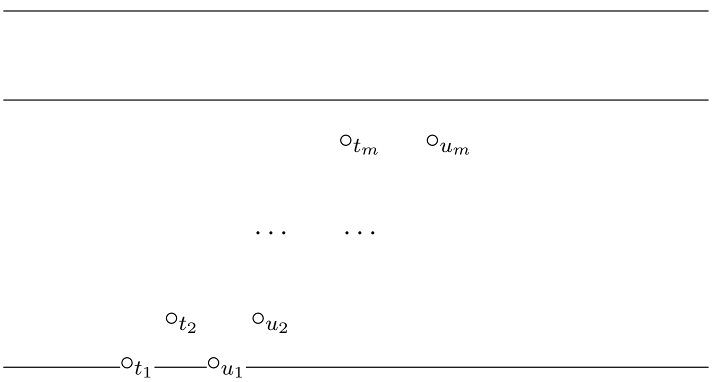



*Then $\mathcal{L}_{\mathcal{X}}(a)$ is the region $ab'u'v'$ and $\mathcal{R}_{\mathcal{X}}(a)$ is the region $abuv$.*

*Proof.* Since $\mathcal{R}(a)$ is the region $abcd$ (Sketch 1.4.3), $\mathcal{R}_{\mathcal{X}}(a)$ is at most the region $abuv$, since any path from $a$ to an object in $\mathcal{R}(a)$ outside $abuv$ can be written as a path which factors through the object $w$ in $\mathcal{X}$, by virtue of the little remark after Configuration 3.2.2. Now let $s$ be an object within the region $abuv$. Since $s$ is in $\mathcal{R}(a)$, there is a non-zero morphism $f : a \to s$ in $\mathcal{T}$, and the corresponding morphism $\overline{f} : a \to s$ in $\mathcal{T}_{\mathcal{X}}$ is non-zero in $\mathcal{T}_{\mathcal{X}}$. Suppose it is not. Then $f$ is a morphism in $\mathcal{T}$ which factors through a (not necessarily indecomposable) object $t$ in $\mathcal{X}$, i.e. $f$ factorizes as $f : a \to t \to s$ in $\mathcal{T}$. By Lemma 3.2.4, $a$ has the $\mathcal{X}$-preenvelope $a \to w$, therefore $f$ further factorizes as $f : a \to w \to t \to s$ in $\mathcal{T}$. However, $s$ is not in $\mathcal{R}(w)$, which contradicts $f$ non-zero. This gives $\overline{f}$ non-zero in $\mathcal{T}_{\mathcal{X}}$. Also Lemma 3.2.3 is implicitly used throughout. The situation for $\mathcal{L}_{\mathcal{X}}(a)$ is similar. $\square$

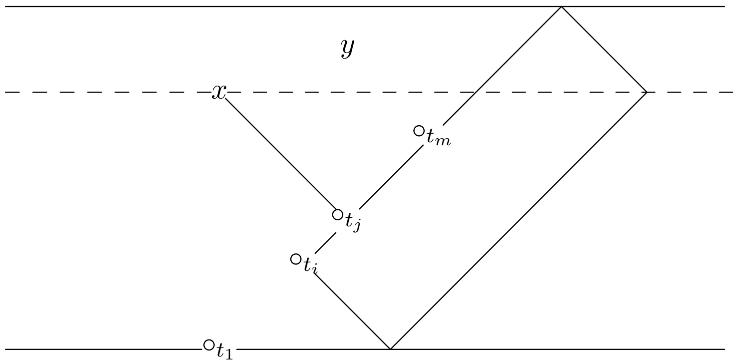
The following lemma shows how a triangulated subcategory can be recovered from a finite set of indecomposable objects by virtue of the Auslander-Reiten triangles alone.

**Lemma 3.2.8.** *The following sketch shows some indecomposable objects in $\mathcal{T}_{\mathcal{X}}$. Consider $t = \bigoplus_{i=1}^{m} t_i$ in $\mathcal{T}_{\mathcal{X}}$. Then the thick subcategory thick$(t)$ of $\mathcal{T}_{\mathcal{X}}$, generated by $t$, is in fact equal to $\mathcal{T}_{\mathcal{X}}$.*




*Proof.* This makes sense since thick$(t)$ is triangulated by definition and $\mathcal{T}_{\mathcal{X}}$ is triangulated by [23, Theorem 2.3]. For $1 \leq i \leq m$, $t_i$ is in thick$(t)$, since thick$(t)$ is closed under direct summands. By [23, Theorem 3.2], $\mathcal{T}_{\mathcal{X}}$ has Auslander-Reiten triangles, and they can be read off from the Auslander-Reiten quiver. Since $t_1 \to t_2 \to u_1 \to$ is an Auslander-Reiten triangle, $u_1$ is in thick$(t)$. Similarly, for $2 \leq i \leq m-1$, $t_i \to t_{i+1} \oplus u_{i-1} \to u_i \to$ is an Auslander-Reiten triangle, and so $u_i$ is in thick$(t)$. Finally, since $t_m \to u_{m-1} \to u_m \to$ is an Auslander-Reiten triangle, $u_m$ is in thick$(t)$. Subsequently, all the $u_i$ immediately next to the $t_i$ on the right hand side are in thick$(t)$. By induction, all the indecomposable objects on the right hand side of $t_i$ are in thick$(t)$. By a mirror argument, all the indecomposable objects on the left hand side of $t_i$ are in thick$(t)$. It follows that all the indecomposable objects of $\mathcal{T}_{\mathcal{X}}$ are in thick$(t)$, and thick$(t)$ is equal to $\mathcal{T}_{\mathcal{X}}$. $\square$

**Lemma 3.2.9.** *Consider $t = \bigoplus_{i=1}^{m} t_i$ in $\mathcal{T}_{\mathcal{X}}$ in the following sketch. Then $End_{\mathcal{T}_{\mathcal{X}}}(t) \cong End_{\mathcal{T}}(t)$.*




*Proof.* The idea is that for $1 \le i, j \le m$, there are no non-zero morphisms $h : t_i \to t_j$ in $\mathcal{T}$ which factor through an object in $\mathcal{X}$. Assume $i \le j$ since it is apparent that $t_j$ is not in $\mathcal{R}(t_i)$ for $j < i$. Suppose there is a non-zero $h : t_i \to t_j$ such that $h$ factors through an object $y$ in $\mathcal{X}$ as $h : t_i \to y \to t_j$. In the above sketch, $y$ is assumed to be indecomposable, though this assumption is not necessary. By Lemma 3.2.4, $t_j$ has an $\mathcal{X}$-cover $x \to t_j$, hence the morphism $h$ further factorizes as $h : t_i \to y \to x \to t_j$. But $x$ is above and behind $t_j$ in the quiver, so that $x$ is not in $\mathcal{R}(t_i)$. Therefore the only morphism from $t_i$ to $t_j$ factoring through an object in $\mathcal{X}$ is the zero morphism, and the isomorphism exists. $\square$

*Remark* 3.2.10. The path algebra $kA_n$ is isomorphic to its opposite algebra $(kA_n)^\circ$. Suppose the quiver $A_n$ is $x_1 \xrightarrow{f_1} x_2 \to \cdots \to x_{n-1} \xrightarrow{f_{n-1}} x_n$, then an isomorphism $\varphi$ could be obtained by $\varphi(x_1) = x_n$, $\varphi(x_2) = x_{n-1}$, $\varphi(f_1) = f_{n-1}$, and so on. The distinguishment between $kA_n$ and $(kA_n)^\circ$ is nevertheless made explicitly, since this is not true for an arbitrary algebra.

**Lemma 3.2.11.** *Let $k$ be a field. Suppose that in a $k$-linear category $\mathcal{S}$, objects $p_1, \ldots, p_m$ satisfy*

$$(p_i, p_j) \cong \begin{cases} k & \textit{for } i \le j, \\ 0 & \textit{otherwise.} \end{cases} \tag{A}$$

*Suppose the non-zero morphisms $p_i \to p_j$ and $p_j \to p_l$ compose to a non-zero morphism $p_i \to p_l$. Then $End_{\mathcal{S}}(S) \cong (kA_m)^\circ$ where $S = \bigoplus_{i=1}^{m} p_i$.*

*Proof.* Let $h$ be an endomorphism in $\mathrm{End}_{\mathcal{S}}(S)$,

$$h : p_1 \oplus p_2 \oplus \cdots \oplus p_m \longrightarrow p_1 \oplus p_2 \oplus \cdots \oplus p_m.$$

Then $h$ can be viewed as a matrix,

$$\begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1m} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2m} \\ & & & \\ & \ddots & & \\ & & & \\ \rho_{m1} & \rho_{m2} & \cdots & \rho_{mm} \end{pmatrix},$$

where the $\rho_{ij}$ are morphisms $\rho_{ij} : p_j \to p_i$ for $1 \leq i, j \leq m$.

By the given condition (A) on the $(p_i, p_j)$, the above matrix has the form of a lower triangular matrix,

$$\begin{pmatrix} k_{11} & 0 & 0 & \cdots & 0 \\ k_{21} & k_{22} & 0 & \cdots & 0 \\ k_{31} & k_{32} & k_{33} & \cdots & 0 \\ & & & & \\ & & \ddots & & \\ & & & & \\ k_{m1} & k_{m2} & k_{m3} & \cdots & k_{mm} \end{pmatrix},$$

where each $k_{ji}$ for $1 \leq i \leq j \leq m$ takes a value in the field $k$. Therefore the endomorphism algebra $\mathrm{End}_{\mathcal{S}}(S)$ is isomorphic to the following lower triangular $m \times m$ matrix algebra

$$M = \begin{pmatrix} k & 0 & \cdots & 0 \\ k & k & \cdots & 0 \\ & & & \\ & \ddots & & \\ & & & \\ k & k & \cdots & k \end{pmatrix}.$$

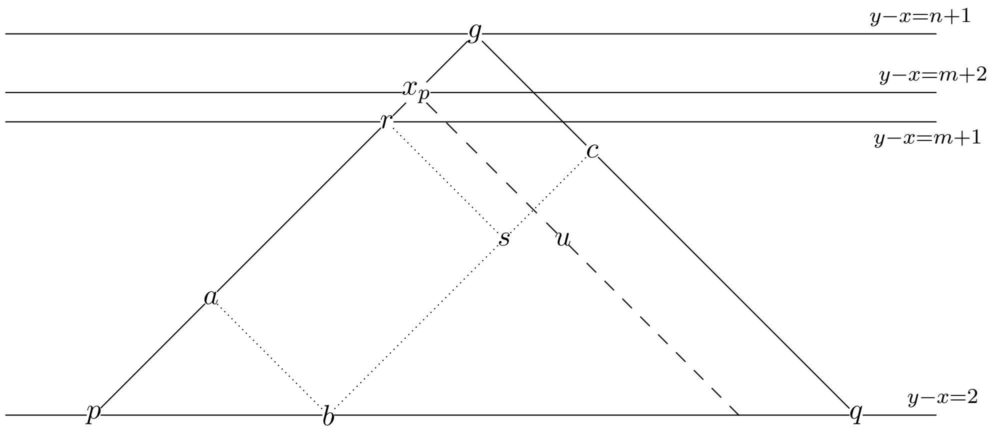
On the other hand, by [1, Lemma II.1.12], the path algebra $kA_m$ is also isomorphic to the lower triangular $m \times m$ matrix algebra $M$. This gives the isomorphisms $\mathrm{End}_{\mathcal{S}}(S) \cong kA_m \cong (kA_m)^{\circ}$ (Remark 3.2.10). □

**Corollary 3.2.12.** *Consider the categories $\mathcal{T}$, $\mathcal{X}$ and $\mathcal{T}_{\mathcal{X}}$ as usual, and the object $t = \bigoplus_{i=1}^{m} t_i$ as in Lemma 3.2.8. Then $End_{\mathcal{T}_{\mathcal{X}}}(t) \cong End_{\mathcal{T}}(t) \cong (kA_m)^{\circ}$.*

*Proof.* The first isomorphism exists by Lemma 3.2.9. The existence of the second isomorphism follows by identifying the $t_i$ with the $p_i$ in Lemma 3.2.11, with the help of Lemma 3.2.3. □

### 3.2.2 Mapping cone construction

In the following sketch, the Auslander-Reiten quiver of $\mathcal{T}$ lies in the region bounded by $y - x = 2$ and $y - x = n + 1$, and the indecomposable objects of $\mathcal{X}$ form the upper band, the region bounded by $y - x = m + 2$ and $y - x = n + 1$. The Auslander-Reiten quiver of $\mathcal{T}_{\mathcal{X}}$ lies in the lower band of width $m$ vertices, the region bounded by $y - x = 2$ and $y - x = m + 1$.




Let $i$ and $j$ be two fixed integers. Suppose the object $a$ has coordinates $(i, j)$. The coordinates of some other objects are listed: $p = (i, i + 2)$, $x_p = (i, i{+}m{+}2)$, $g = (i, i{+}n{+}1)$, $q = (i{+}n{-}1, i{+}n{+}1)$, $s = (j{-}2, i{+}m{+}1)$. Modules are identified with their coordinates. $\mathcal{R}(a)$ is the region $agcb$, and $\mathcal{R}_{\mathcal{X}}(a)$ is the region $arsb$ (Lemma 3.2.7).

By Lemma 3.2.4, $(i, j)$ has the $\mathcal{X}$-envelope $(i, j) \to (i, i + m + 2)$. A copy of the module category $\operatorname{mod}(kA_n)$ can be placed inside $D^b(\operatorname{mod} kA_n)$ in such a way that its Auslander-Reiten quiver corresponds to the region $pgq$. Then the modules on the line $x = i$ may be perceived as projective modules in $\operatorname{mod}(kA_n)$. Perceiving the projective modules as representations of the quiver $A_n$, the projective module $(i, j)$ becomes

$$0 \to 0 \to \ldots \to 0 \to k \to k \to \ldots \to k,$$

where there are $j - i - 1$ copies of $k$ at the end, and the projective module $(i, i + m + 2)$ becomes

$$0 \to 0 \to \ldots \to 0 \to k \to k \to k \to \ldots \to k,$$

where there are $m + 1$ copies of $k$ at the end.
The cokernel of the embedding $(i, j) \hookrightarrow (i, i + m + 2)$ is $(i, i + m + 2)/(i, j)$, which, when perceived as a representation, becomes

$$0 \to \ldots \to 0 \to k \to \ldots \to k \to 0 \to \ldots \to 0,$$

where $j-i-1$ copies of $k$ are removed from the end. Correspondingly, in the diagram, the cokernel lies in the same descending line as $(i, i+m+2)$, but it is $j-i-1$ steps down the line. Therefore the cokernel has coordinates $(i+(j-i-1), i+m+2) = (j-1, i+m+2)$, which is denoted by $u$ in the diagram.

There is an alternative way of realizing the mapping cone (c.f. Section 3.3.2), and the reader will be reminded again in Remark 3.3.21. An example is described in the Appendix (Lemma A.3.1 and Lemma A.3.2).

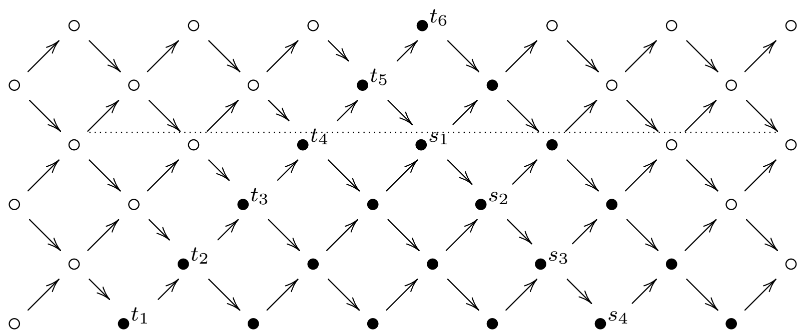
Using the same notation as in [23, Theorem 2.3], let $\sigma$ be the translation functor of the triangulated category $\mathcal{T}_{\mathcal{X}}$. The construction of the functor $\sigma$ is described in [23, Setup 1.1].

**Lemma 3.2.13.** *For $i+2 \leq j' \leq i+m+1$, we have $\sigma(i, j') = (j'-1, i+m+2)$.*

*Proof.* By the above description, given the $\mathcal{X}$-envelope $(i, j') \to (i, i+m+2)$ of $(i, j')$, extend it to the short exact sequence $0 \to (i, j') \to (i, i+m+2) \to (j'-1, i+m+2) \to 0$ in the module category $\mathrm{mod}(kA_n)$. By Lemma 0.2.29, this induces a distinguished triangle in the derived category $\mathcal{T}$, which is in the form of the distinguished triangle (2) in [23, Setup 1.1] when $M$ is replaced by $(i, j')$. Hence $\sigma(i, j') = (j'-1, i+m+2)$. $\square$

The special case when $n = 6$ and $m = 4$ is given in the following diagram. The action of $\sigma$ is $\sigma(t_i) = s_i$ for $1 \leq i \leq 4$.

**Diagram 3.2.14.**




**Corollary 3.2.15.** *For $i+2 \leq j' \leq i+m+1$ and $N \geq 0$ an integer,*

$$\sigma^N(i, j') = \begin{cases} (i + \frac{N}{2}(m+1), j' + \frac{N}{2}(m+1)) & \text{for } N \text{ even,} \\ (j' + \frac{N-1}{2}(m+1) - 1, i + \frac{N+1}{2}(m+1) + 1) & \text{for } N \text{ odd.} \end{cases}$$

*Proof.* Immediate from Lemma 3.2.13 by induction. $\square$

For example, for $i+2 \leq j' \leq i+m+1$, we have $\sigma^2(i,j') = (i+m+1, j'+m+1)$.

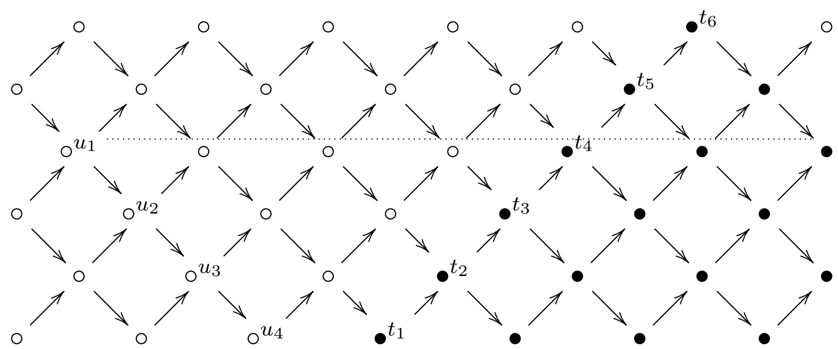
Since the translation functor $\sigma$ is an autoequivalence, there is also an explicit description of $\sigma^{-1}$.

**Lemma 3.2.16.** *For $j-m-1 \leq i' \leq j-2$, we have $\sigma^{-1}(i',j) = (j-m-2, i'+1)$.*

*Proof.* Immediate from Lemma 3.2.13. $\square$

The special case when $n = 6$ and $m = 4$ is given in the following diagram. The action of $\sigma^{-1}$ is $\sigma^{-1}(t_i) = u_i$ for $1 \leq i \leq 4$.

**Diagram 3.2.17.**




**Corollary 3.2.18.** *For $j-m-1 \leq i' \leq j-2$ and $N \geq 0$ an integer,*

$$\sigma^{-N}(i',j) = \begin{cases} (i' - \frac{N}{2}(m+1), j - \frac{N}{2}(m+1)) & \text{for } N \text{ even,} \\ (j - \frac{N+1}{2}(m+1) - 1, i' - \frac{N-1}{2}(m+1) + 1) & \text{for } N \text{ odd.} \end{cases}$$

*Proof.* Immediate from Lemma 3.2.16 by induction. $\square$

**Lemma 3.2.19.** *Let $N \geq 1$ be an integer. Consider $t = \bigoplus_{j'=i+2}^{i+m+1} (i,j')$ in $\mathcal{T}_{\mathcal{X}}$. Then $\mathrm{Hom}_{\mathcal{T}_{\mathcal{X}}}(t, \sigma^N t) = 0$.*

*Proof.* Suppose $N = 1$. The action of $\sigma$ is as given in Lemma 3.2.13. For $i+2 \leq j' \leq i+m+1$, the set $\mathcal{R}_{\mathcal{X}}(i,j')$ is the region bounded by $(i,j'), (i, i+m+1), (j'-2, i+m+1)$ and $(j'-2, j')$ (Lemma 3.2.7). The idea is that for $i+2 \leq j' \leq i+m+1$, there are no non-zero morphisms from $(i,j')$ to $(l, i+m+2)$, where $i+1 \leq l \leq i+m$. This is apparent because for $i+1 \leq l \leq i+m$, the object $(l, i+m+2)$ lies outside $\mathcal{R}_{\mathcal{X}}(i,j')$, which is bounded by $y = i+m+1$. Suppose $N > 1$. Then by similar considerations, the result is immediate from Corollary 3.2.15. $\square$

**Lemma 3.2.20.** *Let $N \geq 1$ be an integer. Consider $t = \bigoplus_{j'=i+2}^{i+m+1} (i, j')$ in $\mathcal{T}_{\mathcal{X}}$. Then $\mathrm{Hom}_{\mathcal{T}_{\mathcal{X}}}(t, \sigma^{-N} t) = 0$.*

*Proof.* Similar. $\square$

### 3.2.3 Theorem

The following is the main theorem of the section.

**Theorem 3.2.21.** *Consider the categories $\mathcal{T}$, $\mathcal{X}$ and $\mathcal{T}_{\mathcal{X}}$ as usual, and the object $t = \bigoplus_{i=1}^{m} t_i$ as in Lemma 3.2.8. Then there is an equivalence of triangulated categories $f : \mathcal{T}_{\mathcal{X}} \xrightarrow{\simeq} D^b(\mathrm{mod}\,(kA_m)^\circ)$ where $f(t) = (kA_m)^\circ$.*

*Proof.* Let $\mathcal{U}$ and $X$ in Theorem 3.1.1 be $\mathcal{T}_{\mathcal{X}}$ and $t$ respectively. By Corollary 3.2.12, $A \cong (kA_m)^\circ$. Let us take $\Sigma$ in Theorem 3.1.1 to be the translation functor $\sigma$ of the triangulated category $\mathcal{T}_{\mathcal{X}}$. Then the result follows from Lemma 3.2.8, Lemma 3.2.19 and Lemma 3.2.20. $\square$

## 3.3 The cluster category of Dynkin type $A_\infty$

In this section, let $\mathcal{T}$ be the cluster category $\mathcal{D}$ of Dynkin type $A_\infty$ described in [19, Section 1]. The category $\mathcal{D}$ is a $k$-linear Hom finite triangulated category. It is Krull-Schmidt as well.

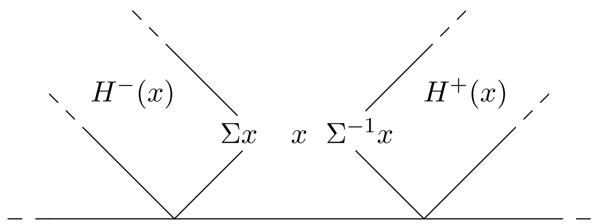
The Auslander-Reiten (AR) quiver of $\mathcal{D}$ is $\mathbb{Z}A_\infty$, endowed with the same coordinate system described in [19, Remark 1.4].

$$
\begin{tikzcd}[column sep=tiny]
& \vdots \arrow[dr] & & \vdots \arrow[dr] & & \vdots \arrow[dr] & & \vdots \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & (-4,1) \arrow[ur] \arrow[dr] & & (-3,2) \arrow[ur] \arrow[dr] & & (-2,3) \arrow[ur] \arrow[dr] & & \cdots \\
& (-4,0) \arrow[ur] \arrow[dr] & & (-3,1) \arrow[ur] \arrow[dr] & & (-2,2) \arrow[ur] \arrow[dr] & & (-1,3) \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & (-3,0) \arrow[ur] \arrow[dr] & & (-2,1) \arrow[ur] \arrow[dr] & & (-1,2) \arrow[ur] \arrow[dr] & & \cdots \\
& (-3,-1) \arrow[ur] & & (-2,0) \arrow[ur] & & (-1,1) \arrow[ur] & & (0,2) \arrow[ur] &
\end{tikzcd}
$$

Since $\mathcal{D}$ is 2-Calabi-Yau, its Serre functor is $S = \Sigma^2$ and the Auslander-Reiten translation $\tau$ is $\tau = S\Sigma^{-1} = \Sigma$, where $\Sigma$ is the translation functor of

$\mathcal{D}$. In terms of coordinates, the action of $\Sigma = \tau$ is $\Sigma(m,n) = (m-1, n-1)$, see [19, Remark 1.4].

The following lemma describes the morphism spaces between indecomposable objects of $\mathcal{D}$. Its significance is in giving the picture of the category $\mathcal{D}$ without a priori knowledge of how it originated and was derived. Let $x$ be an indecomposable object of $\mathcal{D}$. The regions $\mathrm{H}^-(x)$ and $\mathrm{H}^+(x)$ in the Auslander-Reiten quiver of $\mathcal{D}$, described in [19, Definition 2.1], are sketched as follows.




We write $\mathrm{H}(x) = \mathrm{H}^-(x) \cup \mathrm{H}^+(x)$.

**Lemma 3.3.1.** *([19, Corollary 2.3]) Let $x$ and $y$ be indecomposable objects of $\mathcal{D}$. Then the following are equivalent.*

*(i)* $(x,y) \neq 0$,

*(ii)* $(x,y) = k$,

*(iii)* $y \in \mathrm{H}(\Sigma x)$,

*(iv)* $x \in \mathrm{H}(\Sigma^{-1} y)$.

In Lemma 3.3.1, there are two different types of non-zero morphisms going from $x$ to $y$. Those morphisms with $y$ in $\mathrm{H}^+(x)$ are said to be *forward morphisms* and those morphisms with $y$ in $\mathrm{H}^-(x)$ are said to be *backward morphisms.*

*Remark* 3.3.2. (i) If $f$ is a backward morphism, then it can always be written $f = gf'$ where $f'$ is a backward morphism and $g$ is a forward morphism (Lemma 3.3.4). Therefore backward morphisms are not irreducible (can be decomposed indefinitely), and so they do not arise in the Auslander-Reiten quiver.

(ii) The analogue of Lemma 3.2.1 for $\mathcal{D}$ is false in view of the occurrence of the backward morphisms.

With the following lemmas we end the section.

**Lemma 3.3.3.** *Let $a$, $b$ and $c$ be indecomposable objects of $\mathcal{D}$. Then*

*(i) $b$ is in $\mathrm{H}^+(\Sigma a)$ if and only if $Sa$ is in $\mathrm{H}^-(\Sigma b)$,*

*(ii) $c$ is in $\mathrm{H}^-(\Sigma a)$ if and only if $Sa$ is in $\mathrm{H}^+(\Sigma c)$.*

*Proof.* This is [19, Lemma 2.6]. □

**Lemma 3.3.4.** *Let $a$, $b$ and $c$ be indecomposable objects of $\mathcal{D}$. Suppose $b$ and $c$ are in $\mathrm{H}^-(\Sigma a)$ and $c$ is in $\mathrm{H}^+(\Sigma b)$. Consider the non-zero morphism $f : b \to c$. Then each morphism $a \to c$ factors as $a \to b \xrightarrow{f} c$.*

*Proof.* This is [19, Lemma 2.7]. □

**Lemma 3.3.5.** *(c.f. Lemma 3.2.3)*

*(i) Let $a$, $b$ and $c$ be indecomposable objects of $\mathcal{D}$. Suppose $b$ and $c$ are in $\mathrm{H}^+(\Sigma a)$ and $c$ is in $\mathrm{H}^+(\Sigma b)$. Consider the non-zero morphisms $f : a \to b$ and $g : b \to c$. Then the composition $gf : a \to c$ is non-zero.*

*(ii) Let $a$, $b$ and $c$ be indecomposable objects of $\mathcal{D}$. Suppose $b$ and $c$ are in $\mathrm{H}^-(\Sigma a)$ and $c$ is in $\mathrm{H}^+(\Sigma b)$. Consider the non-zero morphisms $f : a \to b$ and $g : b \to c$. Then the composition $gf : a \to c$ is non-zero.*

*Proof.* (i) This is [19, Lemma 2.5(i)].

(ii) Assume $gf$ is zero. Let $h$ be any morphism in $(a, c)$. By Lemma 3.3.4, $h = g\tilde{f}$ for some morphism $\tilde{f} : a \to b$. Since $(a, b)$ is one-dimensional, $\tilde{f} = \alpha f$ for some scalar $\alpha$. Then $h = g(\alpha f) = \alpha(gf)$ would have to be zero. Since $h$ is arbitrary, $(a, c)$ would have to be zero as well, which is a contradiction since there is $c$ in $\mathrm{H}^-(\Sigma a)$.

□

**Lemma 3.3.6.** *Let $a$, $b$ and $c$ be indecomposable objects of $\mathcal{D}$. Suppose $b$ and $c$ are in $\mathrm{H}^+(\Sigma a)$ and $c$ is in $\mathrm{H}^+(\Sigma b)$. Consider the non-zero morphism $f : b \to c$. Then each morphism $a \to c$ factors as $a \to b \xrightarrow{f} c$.*

*Proof.* This is [19, Lemma 2.5(ii)]. □

**Lemma 3.3.7.** *Let $a$, $b$ and $c$ be indecomposable objects of $\mathcal{D}$. Suppose $b$ is in $\mathrm{H}^+(\Sigma a)$ and $c$ is in both $\mathrm{H}^-(\Sigma a)$ and $\mathrm{H}^-(\Sigma b)$. Consider the non-zero morphism $f : b \to c$. Then each morphism $a \to c$ factors as $a \to b \xrightarrow{f} c$.*

*Proof.* This is to show that $(a, f) : (a, b) \to (a, c)$ is surjective. By Serre duality, this is equivalent to $(f, Sa) : (c, Sa) \to (b, Sa)$ injective. This map sends $\xi : c \to Sa$ to the composition $b \xrightarrow{f} c \xrightarrow{\xi} Sa$. Since $c$ is in $\mathrm{H}^-(\Sigma a)$, by Lemma 3.3.3(ii), $Sa$ is in $\mathrm{H}^+(\Sigma c)$. If $\xi$ is non-zero, then it is a forward morphism. Since $b$ is in $\mathrm{H}^+(\Sigma a)$, by Lemma 3.3.3(i), $Sa$ is in $\mathrm{H}^-(\Sigma b)$. Therefore by Lemma 3.3.5(ii), the composition $\xi f$ is non-zero. □

**Lemma 3.3.8.** *(c.f. Lemma 3.2.3) Let $a$, $b$ and $c$ be indecomposable objects of $\mathcal{D}$. Suppose $b$ is in $\mathrm{H}^+(\Sigma a)$ and $c$ is in both $\mathrm{H}^-(\Sigma a)$ and $\mathrm{H}^-(\Sigma b)$. Consider the non-zero morphisms $f : a \to b$ and $g : b \to c$. Then the composition $gf : a \to c$ is non-zero.*

*Proof.* Assume $gf$ is zero. Let $h$ be any morphism in $(a, c)$. By Lemma 3.3.7, $h = g\tilde{f}$ for some morphism $\tilde{f} : a \to b$. Since $(a, b)$ is one-dimensional, $\tilde{f} = \alpha f$ for some scalar $\alpha$. Then $h = g(\alpha f) = \alpha(gf)$ would have to be zero. Since $h$ is arbitrary, $(a, c)$ would have to be zero as well, which is a contradiction since there is $c$ in $\mathrm{H}^-(\Sigma a)$. □

However, the composition of non-zero backward morphisms is always zero, and this is stated below. The proof is left to the reader.

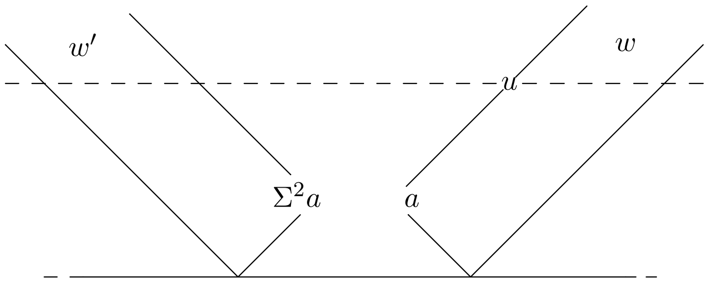
**Lemma 3.3.9.** *Let $a$, $b$ and $c$ be indecomposable objects of $\mathcal{D}$. Suppose $b$ is in $\mathrm{H}^-(\Sigma a)$ and $c$ is in both $\mathrm{H}^-(\Sigma a)$ and $\mathrm{H}^-(\Sigma b)$. Consider the non-zero morphisms $f : a \to b$ and $g : b \to c$. Then the composition $gf : a \to c$ is zero.*

### 3.3.1 Lemmas

Consider an infinite band of indecomposable objects, given by $\{(x, y) \mid y - x \geq m + 2\}$, along the top of the Auslander-Reiten quiver of $\mathcal{D}$. Henceforth, let $\mathcal{X}$ be add of them.

**Lemma 3.3.10.** *(c.f. Lemma 3.2.4) In the following sketch, the indecomposable objects of $\mathcal{X}$ lie in the region bounded below by and including the dotted boundary line $y - x = m + 2$. Let $a$ be an indecomposable object in $\mathcal{D}$. Then $a$ has an $\mathcal{X}$-preenvelope $a \to u$.*

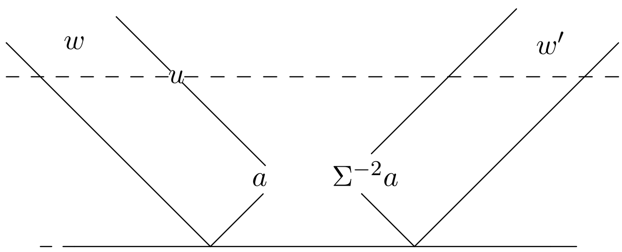
*Proof.* If $a = (i, j)$ is in $\mathcal{X}$, then the lemma is trivial. The following sketch illustrates the situation when $a$ is not in $\mathcal{X}$.




Consider $u = (i, i+m+2)$ on the dotted line $y - x = m+2$, which is the leftmost object on the line in the region $\mathrm{H}^+(\Sigma a)$. Then the non-zero morphism $g : a \to u$ is an $\mathcal{X}$-preenvelope of $a$. Indeed suppose $w$ is in $\mathrm{H}^+(\Sigma a) \cap \mathcal{X}$ which is inside $\mathrm{H}^+(\Sigma u)$. Then each morphism $a \to w$ factors as $a \xrightarrow{g} u \to w$ (c.f. Lemma 3.3.6). Now suppose $w'$ is in $\mathrm{H}^-(\Sigma a) \cap \mathcal{X}$ which is inside $\mathrm{H}^-(\Sigma u)$. To show that each morphism $a \to w'$ factors as $a \xrightarrow{g} u \to w'$ is to show that $(g, w') : (u, w') \to (a, w')$ is surjective. By Serre duality, this is equivalent to $(w', Sg) : (w', Sa) \to (w', Su)$ injective. This map sends $\xi : w' \to Sa$ to the composition $w' \xrightarrow{\xi} Sa \xrightarrow{Sg} Su$. Since $w'$ is in $\mathrm{H}^-(\Sigma a)$, by Lemma 3.3.3(ii), $Sa$ is in $\mathrm{H}^+(\Sigma w')$. If $\xi$ is non-zero, then it is a forward morphism. Since $g$ is non-zero, $Sg$ is non-zero. Since $u$ is in $\mathrm{H}^+(\Sigma a)$, $Su$ is in $\mathrm{H}^+(\Sigma Sa)$ and thus $Sg$ is also a forward morphism. Finally, $w' \xrightarrow{\xi} Sa \xrightarrow{Sg} Su$ is non-zero by Lemma 3.3.5(i) since $Su$ is in $\mathrm{H}^+(\Sigma w')$, and this holds by Lemma 3.3.3(ii) since $w'$ is in $\mathrm{H}^-(\Sigma u)$. $\square$

**Lemma 3.3.11.** *(c.f. Lemma 3.2.4) In the following sketch, the indecomposable objects of $\mathcal{X}$ lie in the region bounded below by and including the dotted boundary line $y - x = m+2$. Let $a$ be an indecomposable object in $\mathcal{D}$. Then $a$ has an $\mathcal{X}$-precover $u \to a$.*

*Proof.* If $a = (i, j)$ is in $\mathcal{X}$, then the lemma is trivial. The following sketch illustrates the situation when $a$ is not in $\mathcal{X}$.

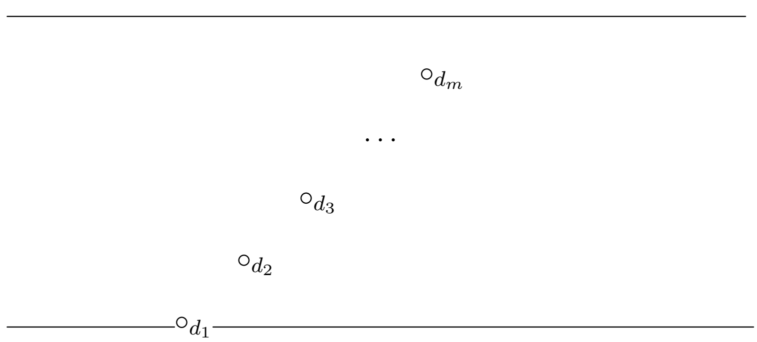


Consider $u = (j-m-2, j)$ on the dotted line $y - x = m + 2$, which is the rightmost object on the line in the region $\mathrm{H}^-(\Sigma^{-1}a)$. Then the non-zero morphism $g : u \to a$ is an $\mathcal{X}$-precover of $a$. Indeed suppose $w$ is in $\mathrm{H}^-(\Sigma^{-1}a) \cap \mathcal{X}$ which is inside $\mathrm{H}^-(\Sigma^{-1}u)$. Then by Lemma 3.3.6 each morphism $w \to a$ factors as $w \to u \xrightarrow{g} a$. This is because $a$ is in both $\mathrm{H}^+(\Sigma u)$ and $\mathrm{H}^+(\Sigma w)$, and $u$ is in $\mathrm{H}^+(\Sigma w)$ (Lemma 3.3.3(ii)). Now suppose $w'$ is in $\mathrm{H}^+(\Sigma^{-1}a) \cap \mathcal{X}$ which is inside $\mathrm{H}^+(\Sigma^{-1}u)$. To show that each morphism $w' \to a$ factors as $w \to u \xrightarrow{g} a$ is to show that $(w', g) : (w', u) \to (w', a)$ is surjective. By Serre duality, this is equivalent to $(g, Sw') : (a, Sw') \to (u, Sw')$ injective. This map sends $\xi : a \to Sw'$ to the composition $u \xrightarrow{g} a \xrightarrow{\xi} Sw'$. Since $w'$ is in $\mathrm{H}^+(\Sigma^{-1}a)$, $Sw'$ is in $\mathrm{H}^+(\Sigma a)$ by Lemma 3.3.3(i), (ii). If $\xi$ is non-zero, then it is a forward morphism. Similarly, $g$ is non-zero, and since $u$ is in $\mathrm{H}^-(\Sigma^{-1}a)$, $a$ is in $\mathrm{H}^+(\Sigma u)$ by Lemma 3.3.3(ii) and thus $g$ is also a forward morphism. Finally, $u \xrightarrow{g} a \xrightarrow{\xi} Sw'$ is non-zero by Lemma 3.3.5(i) since $Sw'$ is in $\mathrm{H}^+(\Sigma u)$, and this holds by Lemma 3.3.3(i), (ii) since $w'$ is in $\mathrm{H}^+(\Sigma^{-1}u)$. $\square$

*Remark* 3.3.12. Alternatively, in Lemma 3.3.11, each morphism $w' \to a$ factors as $w \to u \xrightarrow{g} a$ by Lemma 3.3.4, since $w'$ in $\mathrm{H}^+(\Sigma^{-1}a)$ and $\mathrm{H}^+(\Sigma^{-1}u)$ implies $a$, $u$ in $\mathrm{H}^-(\Sigma w')$ (Lemma 3.3.3(i)).

By Lemma 3.3.10, Lemma 3.3.11 and [23, Theorem 1.2], the quotient category $\mathcal{D}_{\mathcal{X}}$ is pretriangulated. Since $\tau = \Sigma$ and evidently $\tau\mathcal{X} = \mathcal{X}$, therefore $\mathcal{D}_{\mathcal{X}}$ is in addition triangulated by [23, Theorem 2.3]. By [23, Theorem 3.2], the Auslander-Reiten quiver of $\mathcal{D}_{\mathcal{X}}$ is obtained by deleting the vertices of $\mathcal{X}$ and the incident arrows from the Auslander-Reiten quiver of $\mathcal{D}$. Suppose the Auslander-Reiten quiver of $\mathcal{D}_{\mathcal{X}}$ is of width $m$ vertices. This means $\mathcal{D}_{\mathcal{X}}$ has Auslander-Reiten quiver $\mathbb{Z}A_m$.

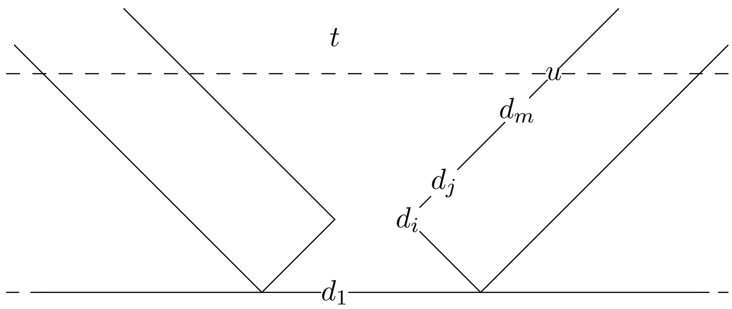
**Lemma 3.3.13.** *(c.f. Lemma 3.2.8) The following sketch shows some indecomposables in $\mathcal{D}_{\mathcal{X}}$.*




*Consider $d = \bigoplus_{i=1}^{m} d_i$ in $\mathcal{D}_{\mathcal{X}}$. Then the thick subcategory thick$(d)$ of $\mathcal{D}_{\mathcal{X}}$,* *generated by d, is in fact equal to $\mathcal{D}_{\mathcal{X}}$.*

*Proof.* Similar to Lemma 3.2.8, since $\mathcal{D}_{\mathcal{X}}$ has Auslander-Reiten triangles by [23, Theorem 3.2]. $\square$

**Lemma 3.3.14.** *(c.f. Lemma 3.2.9) Consider $d = \bigoplus_{i=1}^{m} d_i$ in $\mathcal{D}_{\mathcal{X}}$ in the following sketch.*




*Then $End_{\mathcal{D}_{\mathcal{X}}}(d) \cong End_{\mathcal{D}}(d)$.*

*Proof.* The idea is that for $1 \leq i, j \leq m$, there are no non-zero morphisms $h : d_i \to d_j$ in $\mathcal{D}$ which factor through an object in $\mathcal{X}$. Assume $i \leq j$ since it is apparent that $d_j$ is not in $\mathrm{H}(\Sigma d_i)$ for $j < i$. Suppose there is a non-zero $h : d_i \to d_j$ such that $h$ factors through a (not necessarily indecomposable) object $t$ in $\mathcal{X}$ as $h : d_i \to t \to d_j$. By Lemma 3.3.10, $d_i$ has the $\mathcal{X}$-preenvelope $d_i \to u$, thus the morphism $h$ further factorizes as $h : d_i \to u \to t \to d_j$. Since it is again apparent that $d_j$ is not in $\mathrm{H}(\Sigma u)$, the only morphism from $d_i$ to $d_j$ factoring through an object in $\mathcal{X}$ is the zero morphism, and the isomorphism exists. $\square$

**Corollary 3.3.15.** *(c.f. Corollary 3.2.12) Consider the categories $\mathcal{D}$, $\mathcal{X}$ and $\mathcal{D}_{\mathcal{X}}$ as usual, and the object $d = \bigoplus_{i=1}^{m} d_i$ as in Lemma 3.3.13. Then $End_{\mathcal{D}_{\mathcal{X}}}(d) \cong End_{\mathcal{D}}(d) \cong (kA_m)^{\circ}$.*

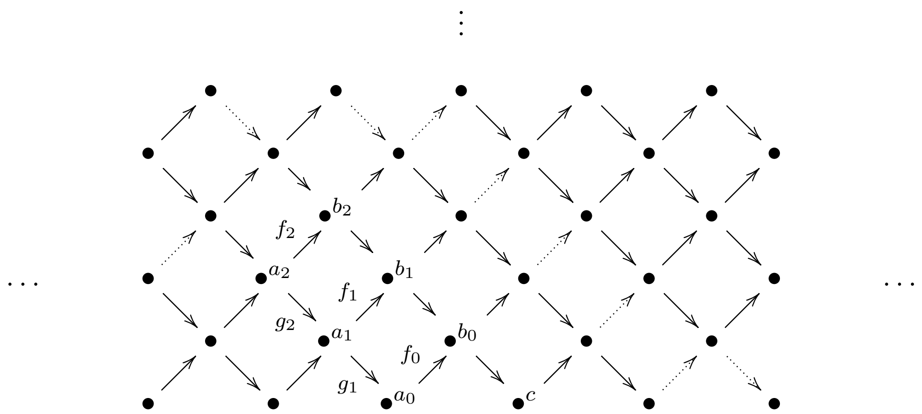
*Proof.* The first isomorphism exists by Lemma 3.3.14. The existence of the second isomorphism follows by identifying the $d_i$ with the $p_i$ in Lemma 3.2.11, with the help of Lemma 3.3.1 and Lemma 3.3.5(i). $\square$

### 3.3.2 Mapping cone construction

This section displays the significance of Auslander-Reiten triangles in determining the mapping cones of certain morphisms in the category $\mathcal{D}$, and the way Auslander-Reiten triangles alone reflect the (unique) triangulation structure of the triangulated category $\mathcal{D}$.

The mapping cone constructions realized in this section might not remain the same in other triangulated categories with Auslander-Reiten quivers which contain the same (local) configurations. This is because properties of a category, for example, the actions of the translation functor and the Auslander-Reiten translation, are inherent attributes of the quiver.

Consider the Auslander-Reiten quiver of $\mathcal{D}$ in the following diagram.




The coordinates of some of the objects are as follows: $a_0 = (i, j), b_0 = (i, j + 1), a_1 = (i - 1, j), b_1 = (i - 1, j + 1), a_2 = (i - 2, j), b_2 = (i - 2, j + 1)$ and $c = (i + 1, j + 1)$. As usual, the coordinates of objects on the bottom line satisfy the equation $y - x = 2$.

**Lemma 3.3.16.** *The mapping cones of the maps $f_n : (i-n, j) \to (i-n, j+1)$ are all isomorphic to $c = (i + 1, j + 1)$.*

*Proof.* Since $a_0 \to b_0 \to c \to$ is an Auslander-Reiten triangle, the statement is true for $n = 0$. Suppose the statement is true for $n = p$, $p \geq 0$, i.e. the mapping cone of the map $f_p : a_p \to b_p$ is $c = (i + 1, j + 1)$. To see that the statement is true for $n = p+1$, consider the following commutative diagram,

$$\begin{tikzcd}
a_{p+1} \arrow[r, "\mu_{p+1}"] \arrow[d, equal] & a_p \oplus b_{p+1} \arrow[d, "\jmath"] \\
a_{p+1} \arrow[r, "f_{p+1}"'] & b_{p+1},
\end{tikzcd}$$

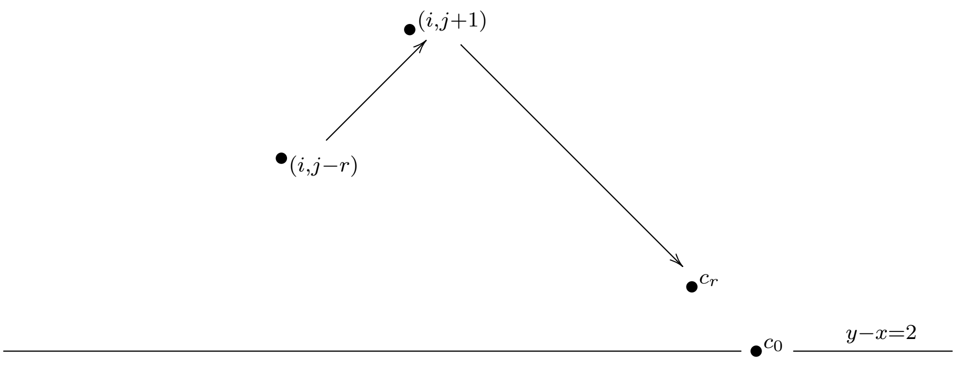
where $\mu_{p+1} = \begin{pmatrix} g_{p+1} \\ f_{p+1} \end{pmatrix}$ and the map $\jmath$ is the canonical surjection. By the octahedral axiom, it may be extended to the following commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & a_p \arrow[r, equal] \arrow[d, "\imath"] & a_p \arrow[r] \arrow[d, "f_p"] & 0 \arrow[d] \\
a_{p+1} \arrow[r, "\mu_{p+1}"] \arrow[d, equal] & a_p \oplus b_{p+1} \arrow[r] \arrow[d, "\jmath"] & b_p \arrow[r] \arrow[d] & \Sigma a_{p+1} \arrow[d, equal] \\
a_{p+1} \arrow[r, "f_{p+1}"'] \arrow[d] & b_{p+1} \arrow[r] \arrow[d] & c \arrow[r] \arrow[d] & \Sigma a_{p+1} \arrow[d] \\
0 \arrow[r] & \Sigma a_p \arrow[r, equal] & \Sigma a_p \arrow[r] & 0,
\end{tikzcd}$$

where the map $\imath$ is the canonical injection. The distinguished triangle on the second row is an Auslander-Reiten triangle. Hence the mapping cone of the map $f_{p+1} : a_{p+1} \to b_{p+1}$ is the same as the mapping cone of the map $f_p : a_p \to b_p$, which is $c = (i+1, j+1)$ by the induction hypothesis. $\square$

*Remark* 3.3.17. The connecting morphism $\partial$ in the Auslander-Reiten triangle $a_{p+1} \overset{\mu_{p+1}}{\to} a_p \oplus b_{p+1} \to b_p \overset{\partial}{\to} \Sigma a_{p+1}$ on the second row is a backward morphism.

**Lemma 3.3.18.** *Consider the following sketch.*

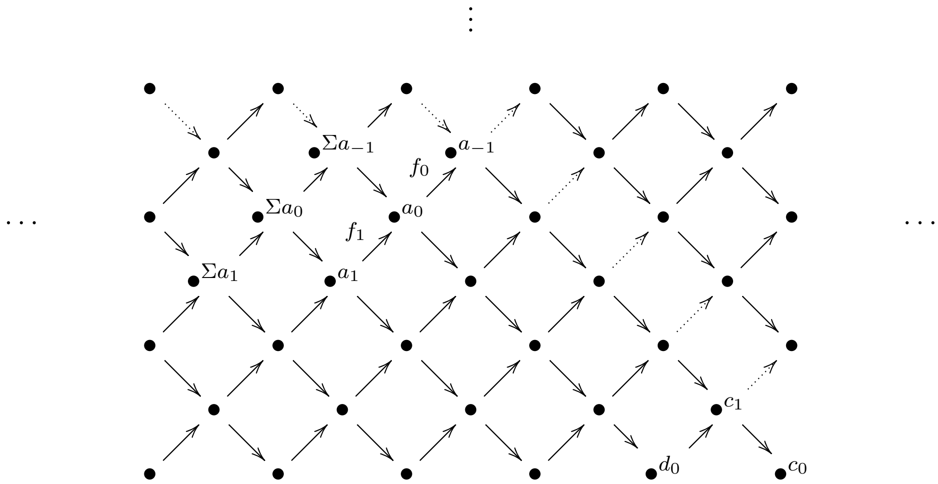



*For $-1 \le n \le j-i-2$, let $a_n = (i, j-n)$. Let $f_n$ be the morphism $f_n : (i, j-n) \to (i, j-n+1)$. Then for $0 \le r \le j-i-2$, the composition $f_0 \dots f_{r-1} f_r : (i, j-r) \to (i, j+1)$ has mapping cone $c_r = (j-1-r, j+1)$.*

*Proof.* Consider the object $c_0 = (j-1, j+1)$ on the bottom line $y - x = 2$. It lies in the same descending line as $(i, j+1)$, where the composition

$f_0 \dots f_{r-1} f_r$ maps to. The mapping cone $c_r = (j-1-r, j+1)$ lies in the same descending line as $c_0$, but it is $r$ steps up the line. This is how we understand the location of the mapping cone $c_r$.

Let us consider again the Auslander-Reiten quiver of $\mathcal{D}$.




The statement is true for $r = 0$ by Lemma 3.3.16. Suppose the statement is true for $r = p$, $p \geq 0$, and then the statement is also required to be true for $r = p + 1$.

Consider the following commutative diagram,

$$\begin{tikzcd}
a_{p+1} \arrow[r, "g"] \arrow[d, equal] & a_0 \arrow[d, "f_0"] \\
a_{p+1} \arrow[r, "f_0 g"'] & a_{-1},
\end{tikzcd}$$

where $g$ is the morphism $f_1 \dots f_{p+1} : (i, j-p-1) \to (i, j)$.

By the octahedral axiom, it may be extended to the following commutative diagram,

$$\begin{tikzcd}
a_{p+1} \arrow[r, "g"] \arrow[d, equal] & a_0 \arrow[r, "\rho"] \arrow[d, "f_0"] & d \arrow[r] \arrow[d] & \Sigma a_{p+1} \arrow[d, equal] \\
a_{p+1} \arrow[r, "f_0 g"] \arrow[d] & a_{-1} \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & \Sigma a_{p+1} \arrow[d] \\
0 \arrow[r] \arrow[d] & c_0 \arrow[r, equal] \arrow[d, "\varrho"] & c_0 \arrow[r] \arrow[d, "(\Sigma\rho)\varrho"] & 0 \arrow[d] \\
\Sigma a_{p+1} \arrow[r] & \Sigma a_0 \arrow[r, "\Sigma\rho"] & \Sigma d \arrow[r] & \Sigma^2 a_{p+1}.
\end{tikzcd}$$

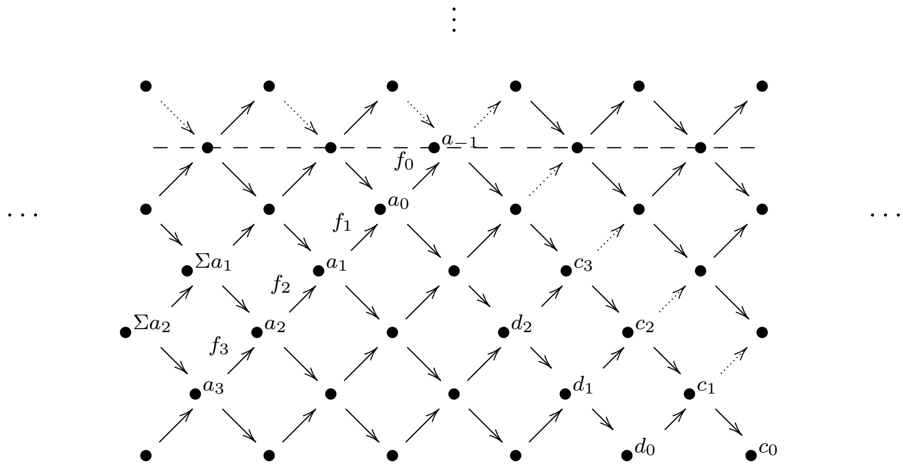
Consider the object $d_0 = (j-2, j)$ on the bottom line $y - x = 2$. It lies in the same descending line as $(i, j)$, where the morphism $g$ maps to. By the induction hypothesis, the mapping cone of the morphism $g$ lies in the same descending line as $d_0$, but it is $p$ steps up the line, i.e. it is the object $d = (j-2-p, j)$. This gives the distinguished triangle on the first row. The mapping cone of the morphism $f_0$ is $c_0 = (j-1, j+1)$ by Lemma 3.3.16, which gives the distinguished triangle in the second column.

The map $\rho : a_0 \to d$ is non-zero, as otherwise $a_{p+1} \cong \Sigma^{-1}d \oplus a_0$ by Lemma 0.2.2(v), which is not possible since $a_{p+1}$ is indecomposable. Therefore the map $\Sigma\rho : \Sigma a_0 \to \Sigma d$ is non-zero as well. Similarly, the map $\varrho : c_0 \to \Sigma a_0$ is non-zero. Therefore by Lemma 3.3.5(ii) the composition $(\Sigma\rho)\varrho : c_0 \to \Sigma d$ is non-zero, and the distinguished triangle $d \to * \to c_0 \to \Sigma d$ is non-split.

Let an object $e$ have coordinates $(j-2-p, j+1)$. By Lemma 3.3.16, the mapping cone of $d \to e$ is $c_0$. Since $(c_0, \Sigma d)$ is one-dimensional, the object $*$ is indeed equal to $e$. Therefore the mapping cone of the morphism $f_0 g$ is $e = (j-2-p, j+1) = (j-1-(p+1), j+1) = c_{p+1}$ as desired. $\square$

Using the same notation as in [23, Theorem 2.3], let $\sigma$ be the translation functor of the triangulated category $\mathcal{D}_{\mathcal{X}}$. The construction of the functor $\sigma$ is described in [23, Setup 1.1]. The following example helps understand it.

*Example* 3.3.19. Consider again the Auslander-Reiten quiver in Lemma 3.3.18 with the same coordinate system.




Consider an infinite band of indecomposable objects along the top of the Auslander-Reiten quiver bounded below by and including the dotted boundary line, i.e. the region given by $\{(x,y) \mid y-x \geq j+1-i\}$, and let $\mathcal{X}$ be add of them.

For example, the morphism $f_1 : a_1 \to a_0$ is an *$\mathcal{X}$-monomorphism* in $\mathcal{D}$, i.e. a morphism such that each morphism $a_1 \to x$ with $x$ in $\mathcal{X}$ factors through $f_1$ (c.f. Lemma 3.3.10). Extend $f_1 : a_1 \to a_0$ to the distinguished triangle $a_1 \xrightarrow{f_1} a_0 \to d_0 \xrightarrow{\partial_1} \Sigma a_1$ in $\mathcal{D}$ by Lemma 3.3.18. On the other hand, the mapping cone of the morphism $f_0 f_1$ is $c_1$ by Lemma 3.3.18, where $f_0 f_1 : a_1 \to a_{-1}$ is an $\mathcal{X}$-preenvelope of $a_1$ by Lemma 3.3.10. Then the diagram $a_1 \xrightarrow{\overline{f_1}} a_0 \to d_0 \xrightarrow{\overline{\partial_2}} c_1$, considered in $\mathcal{D}_{\mathcal{X}}$, is defined to be a distinguished triangle in $\mathcal{D}_{\mathcal{X}}$, so that $\sigma(a_1) = c_1$. The reader can refer to [23, Setup 1.1] for more details.

The following deserves attention.

(i) The connecting morphism $\partial_1$ in the distinguished triangle $a_1 \xrightarrow{f_1} a_0 \to d_0 \xrightarrow{\partial_1} \Sigma a_1$ in $\mathcal{D}$ is a backward morphism, while the connecting morphism $\partial_2$ in the distinguished triangle $a_1 \xrightarrow{\overline{f_1}} a_0 \to d_0 \xrightarrow{\overline{\partial_2}} c_1$ in $\mathcal{D}_{\mathcal{X}}$ is a forward morphism.

(ii) The morphism $f_0 f_1 : a_1 \to a_{-1}$ is an $\mathcal{X}$-preenvelope of $a_1$, hence an $\mathcal{X}$-monomorphism as well. Therefore $a_1 \xrightarrow{\overline{f_0 f_1}} a_{-1} \to c_1 \xrightarrow{\overline{\partial}} c_1$ is a distinguished triangle in $\mathcal{D}_{\mathcal{X}}$ by construction. However, $a_{-1}$ in $\mathcal{X}$ is isomorphic to zero in $\mathcal{D}_{\mathcal{X}}$, so that the connecting morphism $\overline{\partial}$ has to be an isomorphism in $\mathcal{D}_{\mathcal{X}}$.

(iii) Similarly, to suggest but a few, there is the distinguished triangle $a_2 \xrightarrow{\overline{f_1 f_2}} a_0 \to d_1 \to c_2$ in $\mathcal{D}_{\mathcal{X}}$, and also $a_3 \xrightarrow{\overline{f_1 f_2 f_3}} a_0 \to d_2 \to c_3$ in $\mathcal{D}_{\mathcal{X}}$. The distinguished triangles in $\mathcal{D}_{\mathcal{X}}$ seem to be quite symmetrical.

**Lemma 3.3.20.** *(c.f. Lemma 3.2.13) For $i+2 \leq j' \leq i+m+1$, we have $\sigma(i,j') = (j'-1, i+m+2)$.*

*Proof.* By Lemma 3.3.10, $(i,j') \to (i,i+m+2)$ is an $\mathcal{X}$-preenvelope. Extend the map $(i,j') \to (i,i+m+2)$ to the distinguished triangle $(i,j') \to (i,i+m+2) \to (j'-1,i+m+2) \to$ given in Lemma 3.3.18, which is in the form of the distinguished triangle (2) in [23, Setup 1.1] when $M$ is replaced by $(i,j')$. Hence $\sigma(i,j') = (j'-1, i+m+2)$. □

*Remark* 3.3.21. The mapping cone construction described in this section can be imitated to accommodate the previous situation in Section 3.2.2. An example in the finite derived category $D^b(\operatorname{mod} kA_7)$ is given in the Appendix (Lemma A.3.1 and Lemma A.3.2). Since there is a high resemblance between the respective locations of the $\mathcal{X}$-preenvelopes (Lemma 3.2.4 and Lemma 3.3.10), there is no surprise that Lemma 3.2.13 and Lemma 3.3.20 yield the same consequence accordingly.

**Corollary 3.3.22.** *(c.f. Corollary 3.2.15) For $i+2 \leq j' \leq i+m+1$ and $N \geq 0$ an integer,*

$$\sigma^N(i,j') = \begin{cases} (i+\frac{N}{2}(m+1), j'+\frac{N}{2}(m+1)) & \text{for } N \text{ even,} \\ (j'+\frac{N-1}{2}(m+1)-1, i+\frac{N+1}{2}(m+1)+1) & \text{for } N \text{ odd.} \end{cases}$$

*Proof.* Immediate from Lemma 3.3.20 by induction. □

For example, for $i+2 \leq j' \leq i+m+1$, we have $\sigma^2(i,j') = (i+m+1, j'+m+1)$.

Since the translation functor $\sigma$ is an autoequivalence, there is also an explicit description of $\sigma^{-1}$.

**Lemma 3.3.23.** *(c.f. Lemma 3.2.16) For $j-m-1 \leq i' \leq j-2$, we have $\sigma^{-1}(i',j) = (j-m-2, i'+1)$.*

*Proof.* Immediate from Lemma 3.3.20. □

Lemma 3.3.23 is deduced from Lemma 3.3.20, where $\mathcal{X}$ is perceived as preenveloping. Alternatively, one can obtain the value of $\sigma^{-1}$ directly if $\mathcal{X}$ is perceived as precovering. An example is given in the Appendix (Example A.3.8).

**Corollary 3.3.24.** *(c.f. Corollary 3.2.18) For $j-m-1 \leq i' \leq j-2$ and $N \geq 0$ an integer,*

$$\sigma^{-N}(i',j) = \begin{cases} (i'-\frac{N}{2}(m+1), j-\frac{N}{2}(m+1)) & \text{for } N \text{ even,} \\ (j-\frac{N+1}{2}(m+1)-1, i'-\frac{N-1}{2}(m+1)+1) & \text{for } N \text{ odd.} \end{cases}$$

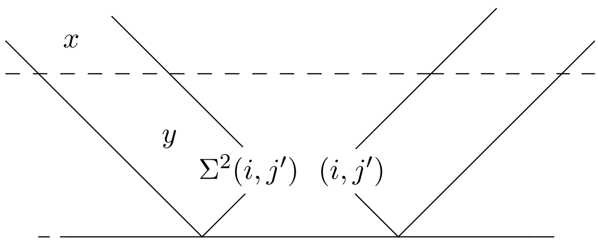
*Proof.* Immediate from Lemma 3.3.23 by induction. □

**Lemma 3.3.25.** *(c.f. Lemma 3.2.19) Let $N \geq 1$ be an integer. Consider $d = \bigoplus_{j'=i+2}^{i+m+1} (i,j')$ in $\mathcal{D}_{\mathcal{X}}$. Then $\mathrm{Hom}_{\mathcal{D}_{\mathcal{X}}}(d, \sigma^N d) = 0$.*

*Proof.* This is similar to Lemma 3.2.19. □

**Lemma 3.3.26.** *(c.f. Lemma 3.2.20) Let $N \geq 1$ be an integer. Consider $d = \bigoplus_{j'=i+2}^{i+m+1} (i,j')$ in $\mathcal{D}_{\mathcal{X}}$. Then $\mathrm{Hom}_{\mathcal{D}_{\mathcal{X}}}(d, \sigma^{-N} d) = 0$.*

*Proof.* This is similar to Lemma 3.2.20. However, for $i+2 \leq j' \leq i+m+1$, any non-zero morphism in $\mathcal{D}$ from $(i,j')$ to an object $y$ in $\mathrm{H}^-(\Sigma(i,j'))$ has to factor through an object in $\mathcal{X}$. This is true by Lemma 3.3.4, since there is always a non-zero forward morphism $u : x \to y$ where $x$ is an indecomposable object in $\mathcal{X}$ and in $\mathrm{H}^-(\Sigma(i,j'))$. An example is given in the following diagram, where the indecomposable objects of $\mathcal{X}$ lie in the region bounded below by and including the dotted boundary line.




By Lemma 3.3.1, the non-zero morphism space from $(i,j')$ to $y$ is one-dimensional. Therefore there are no non-zero morphisms in $\mathcal{D}_{\mathcal{X}}$ from $(i,j')$ to an object $y$ in $\mathrm{H}^-(\Sigma(i,j'))$. $\square$

### 3.3.3 Theorem

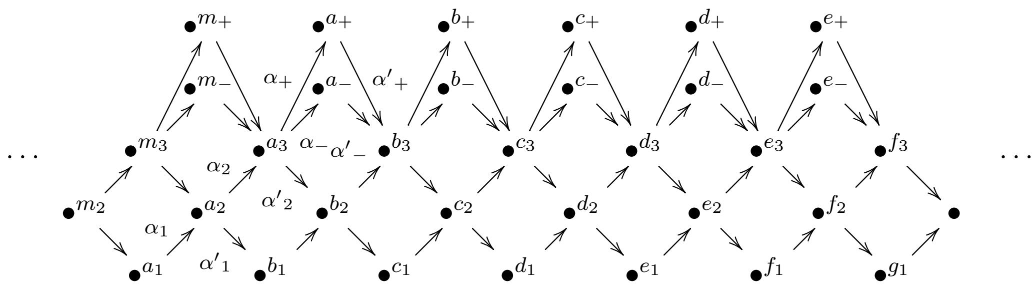
The following is the main theorem of the section.

**Theorem 3.3.27.** *Consider the categories $\mathcal{D}$, $\mathcal{X}$ and $\mathcal{D}_{\mathcal{X}}$ as usual, and the object $d = \bigoplus_{i=1}^{m} d_i$ as in Lemma 3.3.13. Then there is an equivalence of triangulated categories $f : \mathcal{D}_{\mathcal{X}} \xrightarrow{\simeq} D^b(\operatorname{mod}(kA_m)^\circ)$ where $f(d) = (kA_m)^\circ$.*

*Proof.* Let $\mathcal{U}$ and $X$ in Theorem 3.1.1 be $\mathcal{D}_{\mathcal{X}}$ and $d$ respectively. By Corollary 3.3.15, $A \cong (kA_m)^\circ$. Let us take $\Sigma$ in Theorem 3.1.1 to be the translation functor $\sigma$ of the triangulated category $\mathcal{D}_{\mathcal{X}}$. Then the result follows from Lemma 3.3.13, Lemma 3.3.25 and Lemma 3.3.26. $\square$

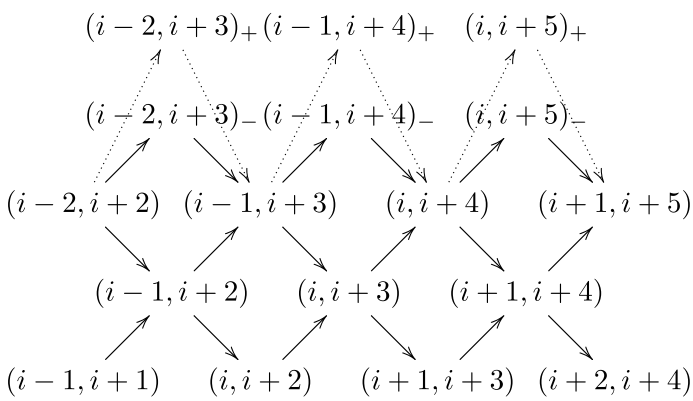
## 3.4 The finite derived category $D^b(\operatorname{mod} kD_5)$

In this section, let $\mathcal{T}$ be the finite derived category $D^b(\operatorname{mod} kD_5)$ described in Section 1.4. The category $\mathcal{T}$ is a $k$-linear Hom finite triangulated category. It is Krull-Schmidt as well. The Auslander-Reiten quiver of $D^b(\operatorname{mod} kD_5)$ is given below (Section 1.4).




Let $i \in \{1,2,3\}$ and $j \in \{2,3,+,-\}$. If there is an arrow $b_i \to b_j$, then let the arrow be $\beta_i : b_i \to b_j$ if $j \in \{2,3\}$ and let the arrow be $\beta_j : b_i \to b_j$ if $j \in \{+,-\}$. Now let $i \in \{2,3,+,-\}$ and $j \in \{1,2,3\}$. If there is an arrow $b_i \to c_j$, then let the arrow be $\beta'_{i-1} : b_i \to c_j$ if $i \in \{2,3\}$ and let the arrow be $\beta'_i : b_i \to c_j$ if $i \in \{+,-\}$. The rest of the arrows are named similarly in terms of $\gamma$'s, $\delta$'s, etc. The vertices $m_+, m_-, a_+, a_-, b_+, b_-, \ldots$ are the *exceptional vertices*. The vertices $a_3, b_3, c_3, \ldots$ are the *bridge vertices*. The exceptional vertices together with the bridge vertices are said to lie in the *exceptional part* of the Auslander-Reiten quiver. The vertices other than the exceptional vertices are the *type $A$ vertices*, and are said to lie in the *type $A$ part* of the Auslander-Reiten quiver. The exceptional and type $A$ parts of the Auslander-Reiten quiver are not disjoint. The vertices $a_3$ and $b_3$ are assigned to be one horizontal unit apart, and so are $a_+$ and $b_+$ etc.

The following coordinate system is employed. Suppose the indecomposable object $a_3$ has coordinates $(i-1, i+3)$. The coordinates of some of its surrounding objects are given in the following diagram.




The coordinates of objects of the bottom line satisfy the equation $y - x = 2$, and the coordinates of the exceptional vertices satisfy the equation $y - x = 5$.

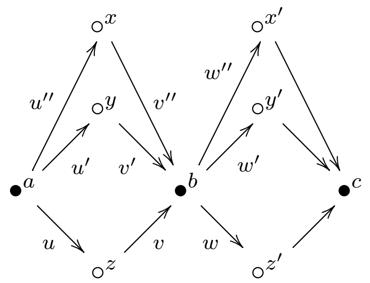
The action of the translation functor $\Sigma$ on the Auslander-Reiten quiver is given in [33, Table 4.I.]. It acts by shifting 4 units to the right and switching each pair of exceptional vertices. For example, $\Sigma(b_2) = f_2$ and $\Sigma(a_+) = e_-$.

The following is an example of Lemma 1.4.1 and the narration after it, and is rewritten below to remind the reader.

**Lemma 3.4.1.** *(c.f. Lemma 3.2.1) Let $x$ and $y$ be indecomposable objects of $D^b(\operatorname{mod} kD_5)$. Then by [14, 4.6], any non-zero morphism $f$ from $x$ to $y$ is a linear combination of morphisms, written $f = \Sigma\alpha_i f_i$, where the $\alpha_i$ are scalars and the $f_i : x \to y$ are compositions of irreducible morphisms. The finite derived category $D^b(\operatorname{mod} kD_5)$ is also standard.*

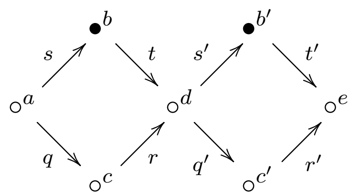
**Definition 3.4.2.** Let $a$ be an indecomposable object of $D^b(\operatorname{mod} kD_5)$, and let $\mathcal{L}(a)$ be the set of indecomposable objects with non-zero morphisms to $a$. Dually, let $\mathcal{R}(a)$ be the set of indecomposable objects to which there are non-zero morphisms from $a$.

The Auslander-Reiten quiver contains different types of local configurations which give rise to different Auslander-Reiten triangles and commutativity relations. In the configurations below, the black dots indicate the bridge vertices.




(i)

In Configuration (i), since $a \to x \oplus y \oplus z \to b \to$ is an Auslander-Reiten triangle, $v''u'' + v'u' + vu = 0$. There are also the Auslander-Reiten triangles $x \to b \to x' \to$ and $y \to b \to y' \to$, and so $w''v'' = 0$ and $w'v' = 0$.




(ii)

In Configuration (ii), since $a \to b \oplus c \to d \to$ is an Auslander-Reiten triangle, $ts + rq = 0$. There is also the Auslander-Reiten triangle $c \to d \to c' \to$, and so $q'r = 0$.

The above description permits us to give examples of the region $\mathcal{R}(x)$ for some indecomposable objects $x$ in $D^b(\operatorname{mod} kD_5)$. The way they are derived is very similar to the situation given in Example 1.4.4.

**Diagram 3.4.3.** The region $\mathcal{R}(a_3)$ is as shown by the black dots in the following diagram.

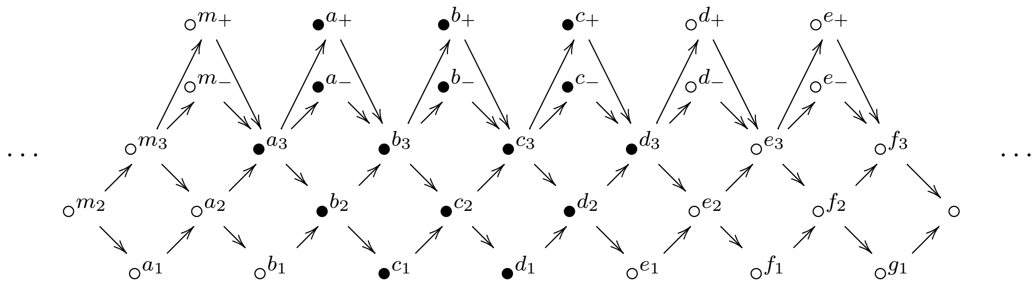



**Diagram 3.4.4.** The region $\mathcal{R}(a_-)$ is as shown by the black dots in the following diagram.

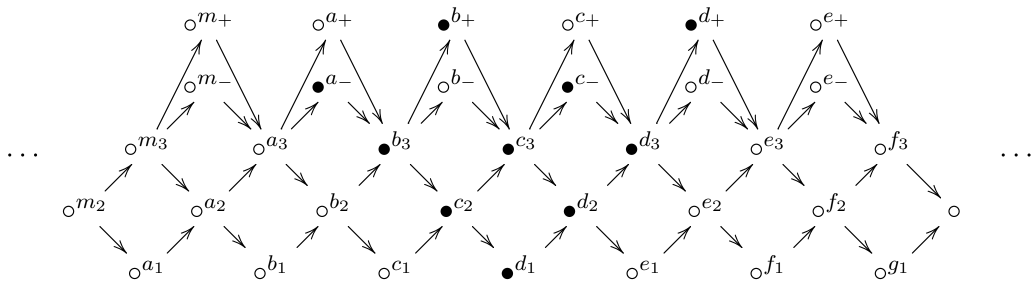
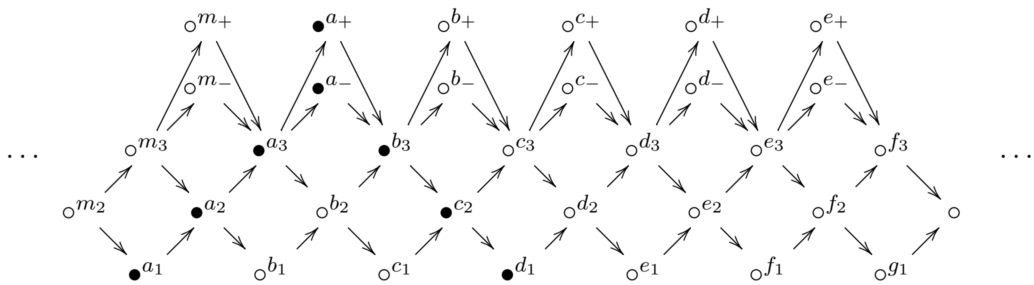



**Diagram 3.4.5.** The region $\mathcal{R}(a_1)$ is as shown by the black dots in the following diagram.




With the following lemma we end this section.

**Lemma 3.4.6.** *(c.f. Lemma 3.2.3) Let $a$, $b$ and $c$ be indecomposable objects of $D^b(\operatorname{mod} kD_5)$. Assume $(a, b)$, $(b, c)$ and $(a, c)$ are all one-dimensional. Let $f : a \to b$ and $g : b \to c$ be some non-zero morphisms. Then the composition $gf : a \to c$ is non-zero.*

*Proof.* Similar to Lemma 3.2.3. □

### 3.4.1 Hom spaces

In this section, the non-zero morphisms between certain indecomposable objects are described. This will lead to the calculations of the dimensions of some of the Hom spaces.

**Lemma 3.4.7.** *Suppose $a$ is a bridge vertex and $b$ is an exceptional vertex where $b$ is in $\mathcal{R}(a)$. Then any non-zero morphism $f : a \to b$ can be written as a linear combination of paths lying in the exceptional part of the quiver.*

*Proof.* Let $y$ be the bridge vertex with an arrow to $b$. Any non-zero morphism $f : a \to b$ can be factored as $a \xrightarrow{g} y \to b$ by Lemma 3.4.1, since $y$ is the only vertex with an arrow to $b$. Consider $g : a \to y$. Write $g = \Sigma \alpha_i p_i$, where $\alpha_i$ is a scalar and $p_i$ is a path from $a$ to $y$. It is enough to show that the path $p_i$ is a linear combination of paths lying in the exceptional part of the quiver.

If in the path $p_i$ there are consecutive arrows $q$ and $r$ placed as follows, where the black dot is a bridge vertex,

$$\begin{tikzcd} & \bullet \arrow[dr, "t"] & \\ \circ \arrow[ur, "s"] \arrow[dr, "q"'] & & \circ \\ & \circ \arrow[ur, "r"'] & \end{tikzcd} \quad ,$$

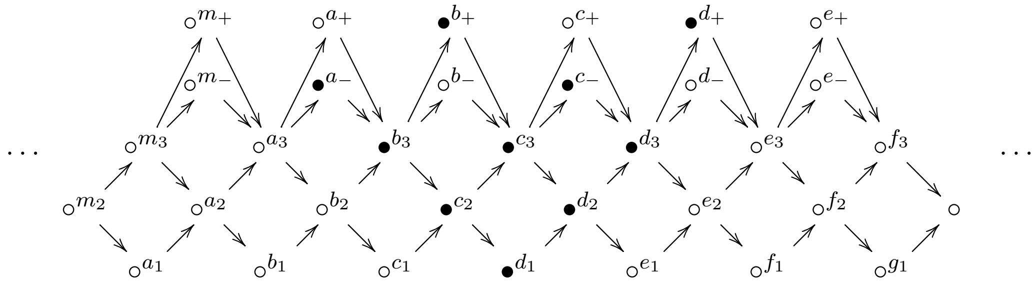
then $rq$ can be replaced by $-ts$, by virtue of the little remark after Configuration (ii). And if in the path $p_i$ there are consecutive arrows $u$ and $v$ placed as follows, where the black dots are the bridge vertices,

$$\begin{tikzcd} & \circ \arrow[ddr, "v''"] & \\ & \circ \arrow[dr, "v'"] & \\ \bullet \arrow[uur, "u''"] \arrow[ur, "u'"'] \arrow[dr, "u"'] & & \bullet \\ & \circ \arrow[ur, "v"'] & \end{tikzcd} \quad ,$$

then $vu$ can be replaced by $-(v'u' + v''u'')$, by virtue of the little remark after Configuration (i). In this way, the path $p_i$ is transformed into a linear combination of paths lying in the exceptional part of the quiver. □

**Lemma 3.4.8.** *Suppose $a$ is an exceptional vertex and $b$ is a type $A$ vertex where $b$ is in $\mathcal{R}(a)$. Then any non-zero morphism $f : a \to b$ can be written $f = \Sigma\alpha_i f_i g$, where $\alpha_i$ is a scalar, $g$ is an irreducible morphism from the vertex $a$ to a bridge vertex $y$ and $f_i : y \to b$ is a path lying in the type $A$ part of the quiver.*

*Proof.* It is sufficient to consider the example $a = a_-$. The region $\mathcal{R}(a_-)$ is given in Diagram 3.4.4, which is as shown by the black dots in the following diagram.




The bridge vertex $y$ is $b_3$ and $g = \alpha'_-$ here. The lemma is immediate when $b = b_3, c_2$ and $d_1$. Suppose $b = c_3$, then write $f = \Sigma\alpha_i p_i \alpha'_-$, where $p_i$ is a path from $b_3$ to $c_3$. Then it remains to show that $p_i\alpha'_-$ can be written $p_i\alpha'_- = f_i\alpha'_-$, where $f_i$ is a path lying in the type $A$ part of the quiver. However, $p_i$ can only be

(i) $p_i = \beta'_+\beta_+$,

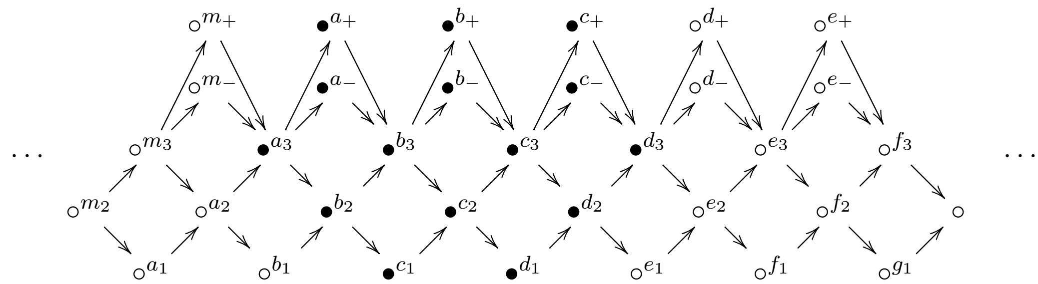
(ii) $p_i = \beta'_-\beta_-$,

(iii) $p_i = \gamma_2\beta'_2$.

For (iii), the path $p_i$ already lies in the type $A$ part of the quiver. For (ii), we have however $p_i\alpha'_- = \beta'_-\beta_-\alpha'_- = 0$ by virtue of the little remark after Configuration (i). Finally, for (i), we have $p_i\alpha'_- = \beta'_+\beta_+\alpha'_- = -(\beta'_-\beta_- + \gamma_2\beta'_2)\alpha'_- = -\gamma_2\beta'_2\alpha'_-$ by virtue of the little remark after Configuration (i), and $\gamma_2\beta'_2$ lies in the type $A$ part of the quiver. The case where $b = d_2$ or $b = d_3$ is similar. $\square$

**Corollary 3.4.9.** *Suppose $a$ is a bridge vertex and $b$ is an exceptional vertex where $b$ is in $\mathcal{R}(a)$. Then $(a, b)$ is one-dimensional.*

*Proof.* It is sufficient to consider the example $a = a_3$. The region $\mathcal{R}(a_3)$ is given in Diagram 3.4.3, which is as shown by the black dots in the following diagram.




If $b = a_+$ or $b = a_-$, then it is true by Lemma 3.4.1. Consider a non-zero morphism $f : a \to b$. Then by Lemma 3.4.7, assume $f$ to be a linear combination of paths lying in the exceptional part of the quiver. If $b = b_+$, then there is only one non-zero path $\beta_+\alpha'_-\alpha_- : a \to b$ lying in the exceptional part of the quiver. This is because the other path $\beta_+\alpha'_+\alpha_+ : a \to b$ is a zero path by virtue of the little remark after Configuration (i). The case where $b = b_-$, $b = c_+$ or $b = c_-$ is similar. $\square$

**Corollary 3.4.10.** *Suppose $a$ is an exceptional vertex and $b$ is a type $A$ vertex where $b$ is in $\mathcal{R}(a)$. Then $(a, b)$ is one-dimensional.*

*Proof.* Consider a non-zero morphism $f : a \to b$. Then by Lemma 3.4.8, $f$ can be written $f = \Sigma\alpha_i f_i g$, where $\alpha_i$ is a scalar, $g$ is an irreducible morphism from $a$ to a bridge vertex $y$ and $f_i : y \to b$ is a path lying in the type $A$ part of the quiver. And the result is immediate from the little remark after Configuration (ii). $\square$

**Corollary 3.4.11.** *Let $a$ and $b$ be two exceptional vertices where $b$ is in $\mathcal{R}(a)$. Then $(a, b)$ is one-dimensional.*

*Proof.* Consider a non-zero morphism $f : a \to b$. Then $f$ can be written $f = \Sigma\alpha_i f_i f'$, where $\alpha_i$ is a scalar, $f' : a \to y$ is a morphism from the vertex $a$ to a bridge vertex $y$, because there is only one arrow going from $a$. Since $f_i$ is then a morphism from the bridge vertex $y$ to the exceptional vertex $b$, the rest is very similar to Corollary 3.4.9. $\square$

**Corollary 3.4.12.** *The Hom space $(a_1, d_1)$ is one-dimensional.*

*Proof.* The region $\mathcal{R}(a_1)$ is given in Diagram 3.4.5. Consider a non-zero morphism $f : a_1 \to d_1$. Then write $f = \Sigma\alpha_i p_i$ as a linear combination of paths with $p_i : a_1 \to d_1$. The path $p_i$ cannot contain the vertices $b_1$, $b_2$ nor $c_1$, since they are not in $\mathcal{R}(a_1)$, therefore it has to either go through the vertex $a_+$ or the vertex $a_-$. However, the two choices are equal up to signs.

For example, consider a path $p_i$ which goes through the vertex $a_+$. Since the vertex $b_2$ is not in $\mathcal{R}(a_1)$, it can be written as a path which goes through the vertex $a_-$ by virtue of the little remark after Configuration (i). $\square$

### 3.4.2 Lemmas

Consider the indecomposable objects $a_+, b_+, c_+, \dots$ along the top line of the Auslander-Reiten quiver of $\mathcal{T}$. Henceforth, let $\mathcal{X}$ be add of them.

**Lemma 3.4.13.** *The indecomposable object $a_3$ has an $\mathcal{X}$-preenvelope $a_3 \to a_+ \oplus b_+$.*

*Proof.* The objects in $\mathcal{X}$ to which there is a non-zero morphism from $a_3$ are $a_+$, $b_+$ and $c_+$. The cases for $a_+$ and $b_+$ are trivial. Suppose a non-zero morphism $h : a_3 \to c_+$ is given. It is enough to assume that $h$ is a composition of irreducible morphisms and to show that $h$ factors through $a_3 \to a_+$. This is true by Lemma 3.4.7 and Corollary 3.4.9. $\square$

**Lemma 3.4.14.** *The indecomposable object $a_2$ has an $\mathcal{X}$-preenvelope $a_2 \to a_+ \oplus b_+$.*

*Proof.* This is because the intersection of $\mathcal{R}(a_2)$ and $\mathcal{X}$ only consists of $a_+$ and $b_+$, and that $(a_+, b_+)$ is zero by virtue of the little remark after Configuration (i). $\square$

**Lemma 3.4.15.** *The indecomposable object $a_-$ has an $\mathcal{X}$-preenvelope $a_- \to b_+$.*

*Proof.* This is because the intersection of $\mathcal{R}(a_-)$ and $\mathcal{X}$ is the same as the intersection of $\mathcal{R}(b_+)$ and $\mathcal{X}$ (the intersection consists of $b_+$ and $d_+$), and since $(a_-, d_+)$ is one-dimensional by Corollary 3.4.11, any non-zero morphism $f : a_- \to d_+$ has to factor through $b_+$. This is in the proof of Corollary 3.4.11 as well. $\square$

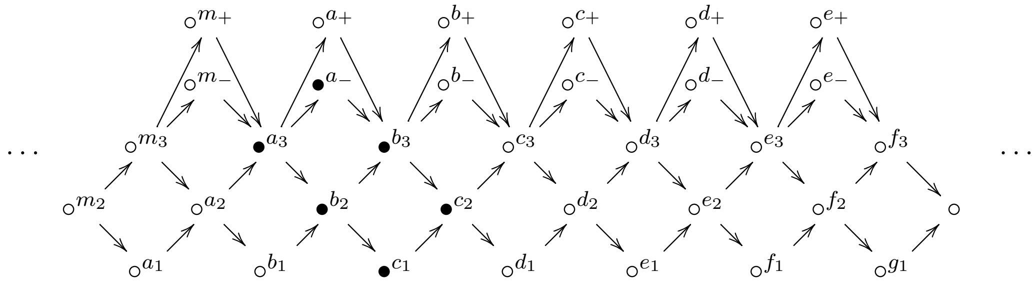
**Lemma 3.4.16.** *The indecomposable object $a_1$ has an $\mathcal{X}$-preenvelope $a_1 \to a_+$.*

*Proof.* Similar to Lemma 3.4.15. $\square$

By Lemma 3.4.13, Lemma 3.4.14, Lemma 3.4.15, Lemma 3.4.16 and [23, Theorem 1.2], the quotient category $\mathcal{T}_{\mathcal{X}}$ is pretriangulated. Since evidently $\tau\mathcal{X} = \mathcal{X}$, therefore $\mathcal{T}_{\mathcal{X}}$ is in addition triangulated by [23, Theorem 2.3]. By [23, Theorem 3.2], the Auslander-Reiten quiver of $\mathcal{T}_{\mathcal{X}}$ is obtained by deleting vertices on the top line and the incident arrows from the Auslander-Reiten quiver of $\mathcal{T}$. Therefore the two categories $\mathcal{T}_{\mathcal{X}}$ and $D^b(\operatorname{mod} kA_4)$ have the same Auslander-Reiten quiver $\mathbb{Z}A_4$.

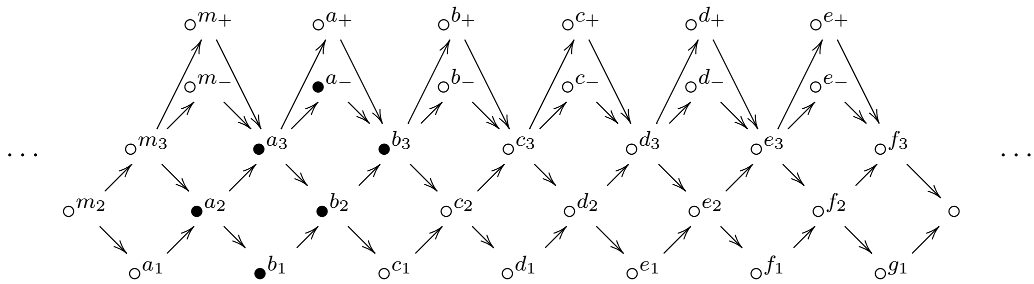
**Definition 3.4.17.** Let $a$ be an indecomposable object in $\mathcal{T}_{\mathcal{X}}$. Similarly to Definition 3.4.2, let $\mathcal{L}_{\mathcal{X}}(a)$ be the set of indecomposable objects with non-zero morphisms to $a$ in $\mathcal{T}_{\mathcal{X}}$. Dually, let $\mathcal{R}_{\mathcal{X}}(a)$ be the set of indecomposable objects to which there are non-zero morphisms from $a$ in $\mathcal{T}_{\mathcal{X}}$.

**Lemma 3.4.18.** *(c.f. Lemma 3.2.7) The region $\mathcal{R}_{\mathcal{X}}(a_3)$ is as shown by the black dots in the following diagram.*



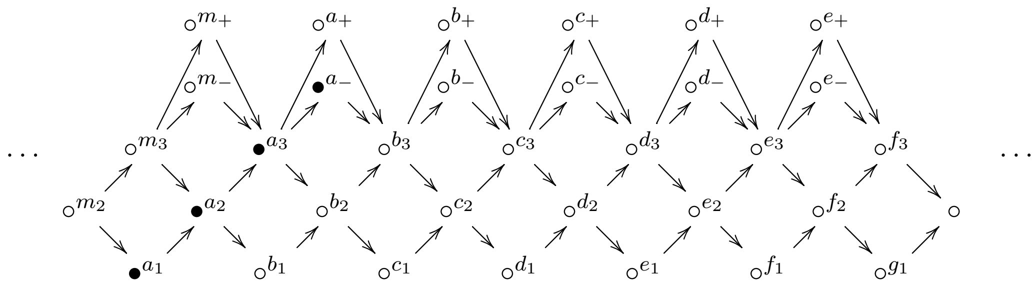

*Proof.* The philosophy of Section 3.4.1 allows us to see that the region $\mathcal{R}_{\mathcal{X}}(a_3)$ is at most the region as indicated. Now let $s$ be an object in it. Since $s$ is in $\mathcal{R}(a_3)$, there is a non-zero morphism $f : a_3 \to s$ in $\mathcal{T}$ so that the corresponding morphism $\overline{f} : a_3 \to s$ is non-zero in $\mathcal{T}_{\mathcal{X}}$. Suppose otherwise that $\overline{f}$ is zero. Then $f$ is a morphism in $\mathcal{T}$ which factors through a (not necessarily indecomposable) object $t$ in $\mathcal{X}$, i.e. $f$ factorizes as $f : a_3 \to t \to s$ in $\mathcal{T}$. By Lemma 3.4.13, $a_3$ has the $\mathcal{X}$-preenvelope $a_3 \to a_+ \oplus b_+$, therefore $f$ further factorizes as $f : a_3 \to a_+ \oplus b_+ \to t \to s$ in $\mathcal{T}$. The morphism $a_+ \oplus b_+ \to t$ is non-zero, therefore either $a_+ \to t$ is non-zero, or $b_+ \to t$ is non-zero. In the first case, $t$ is to be (finite sums of) $c_+$, but $s$ is not in $\mathcal{R}(c_+)$. In the second case, $t$ is to be (finite sums of) $d_+$, but $s$ is not in $\mathcal{R}(d_+)$ either. This gives $\overline{f}$ non-zero in $\mathcal{T}_{\mathcal{X}}$. Also Lemma 3.4.6 is implicitly used throughout. $\square$

**Lemma 3.4.19.** *(c.f. Lemma 3.2.7) The region $\mathcal{R}_{\mathcal{X}}(a_2)$ is as shown by the black dots in the following diagram.*



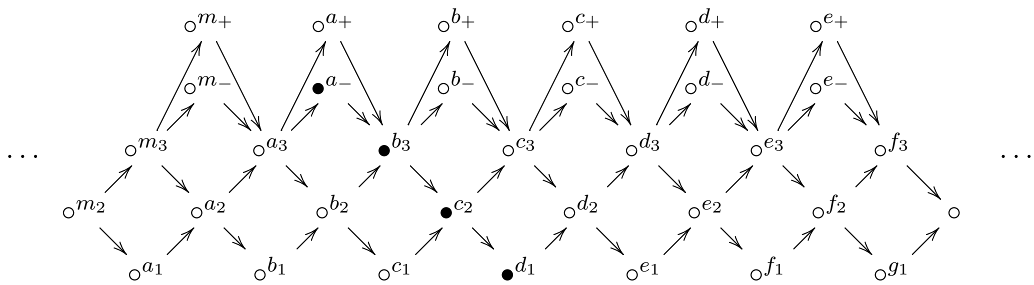
*Proof.* Similar. ☐

**Lemma 3.4.20.** *(c.f. Lemma 3.2.7) The region $\mathcal{R}_{\mathcal{X}}(a_1)$ is as shown by the black dots in the following diagram.*



*Proof.* Similar. ☐

**Lemma 3.4.21.** *(c.f. Lemma 3.2.7) The region $\mathcal{R}_{\mathcal{X}}(a_-)$ is as shown by the black dots in the following diagram.*



*Proof.* Similar. ☐

**Lemma 3.4.22.** *(c.f. Lemma 3.2.8) Consider $a = a_1 \oplus a_2 \oplus a_3 \oplus a_-$ in $\mathcal{T}_{\mathcal{X}}$. Then the thick subcategory thick$(a)$ of $\mathcal{T}_{\mathcal{X}}$, generated by $a$, is in fact equal to $\mathcal{T}_{\mathcal{X}}$.*

*Proof.* Similar to Lemma 3.2.8, since $\mathcal{T}_{\mathcal{X}}$ has Auslander-Reiten triangles by [23, Theorem 3.2]. ☐

**Lemma 3.4.23.** *(c.f. Lemma 3.2.9) Consider $a = a_1 \oplus a_2 \oplus a_3 \oplus a_-$ in $\mathcal{T}_{\mathcal{X}}$. Then $End_{\mathcal{T}_{\mathcal{X}}}(a) \cong End_{\mathcal{T}}(a)$.*

*Proof.* This is similar to Lemma 3.2.9. But surely, we shall do it again. The idea is that there are no non-zero morphisms $h : a_i \to a_j$ in $\mathcal{T}$, where $i, j \in \{1, 2, 3, -\}$, which factor through an object in $\mathcal{X}$. Only the cases where $i, j \in \{1, 2, 3\}$ or $j = -$ need to be considered, since there are no non-zero morphisms from $a_i$ to $a_j$ otherwise. Assume $i \leq j$ if $i, j \in \{1, 2, 3\}$. Suppose there is a non-zero $h : a_i \to a_j$ such that $h$ factors through an object $y$ in $\mathcal{X}$ as $h : a_i \to y \to a_j$. By Lemma 3.4.13, Lemma 3.4.14, Lemma 3.4.15 and Lemma 3.4.16, $a_i$ has an $\mathcal{X}$-preenvelope $a_i \to x$, thus the morphism $h$ further factorizes as $h : a_i \to x \to y \to a_j$. Since $x$ can only be $a_+, b_+$ or $a_+ \oplus b_+$, there are apparently no non-zero morphisms from $x$ to $a_j$. Therefore the only morphism from $a_i$ to $a_j$ factoring through an object in $\mathcal{X}$ is the zero morphism, and the isomorphism exists. □

**Corollary 3.4.24.** *(c.f. Corollary 3.2.12) Consider the categories $\mathcal{T}$, $\mathcal{X}$ and $\mathcal{T}_{\mathcal{X}}$ as usual, and the object $a = a_1 \oplus a_2 \oplus a_3 \oplus a_-$ in $\mathcal{T}_{\mathcal{X}}$ as in Lemma 3.4.22. Then $End_{\mathcal{T}_{\mathcal{X}}}(a) \cong End_{\mathcal{T}}(a) \cong (kA_4)^\circ$.*

*Proof.* The first isomorphism exists by Lemma 3.4.23. The existence of the second isomorphism follows by identifying the $a_i$ with the $p_i$ for $1 \leq i \leq 3$ and by identifying $a_-$ with $p_4$ in Lemma 3.2.11, with the help of Lemma 3.4.6. □

### 3.4.3 Mapping cone construction

**Lemma 3.4.25.** *The mapping cone of the morphism $\beta_1' : b_2 \to c_1$ is $\Sigma b_1$.*

*Proof.* This is because $b_1 \to b_2 \to c_1 \to$ is an Auslander-Reiten triangle. □

**Lemma 3.4.26.** *The mapping cone of the morphism $\beta_2' : b_3 \to c_2$ is $\Sigma b_1$.*

*Proof.* By the octahedral axiom, extend to the following commutative diagram,

$$\begin{tikzcd}
b_1 \arrow[r] \arrow[d] & b_3 \arrow[r, "\beta_2'"] \arrow[d] & c_2 \arrow[r] \arrow[d, equal] & * \\
b_2 \arrow[r] \arrow[d] & b_3 \oplus c_1 \arrow[r] \arrow[d] & c_2 \arrow[r] \arrow[d] & {} \\
c_1 \arrow[r, equal] \arrow[d] & c_1 \arrow[r] & 0 & \\
\Sigma b_1 & & &
\end{tikzcd}.$$

The distinguished triangle on the second row is an Auslander-Reiten triangle, and the distinguished triangle on the first column is given by Lemma 3.4.25. □

**Lemma 3.4.27.** *The mapping cone of the morphism $\gamma_1' \beta_2' : b_3 \to d_1$ is $\Sigma b_2$.*

*Proof.* By the octahedral axiom, extend to the following commutative diagram,

$$\begin{tikzcd}
b_1 \arrow[d] & & & \\
* \arrow[r] \arrow[d] & b_3 \arrow[r, "\gamma_1'\beta_2'"] \arrow[d] & d_1 \arrow[r] \arrow[d, equal] & \Sigma * \\
c_1 \arrow[r] \arrow[d] & c_2 \arrow[r] \arrow[d] & d_1 \arrow[r] \arrow[d] & \Sigma c_1 \\
\Sigma b_1 \arrow[r, equal] & \Sigma b_1 \arrow[r] & 0 & .
\end{tikzcd}$$

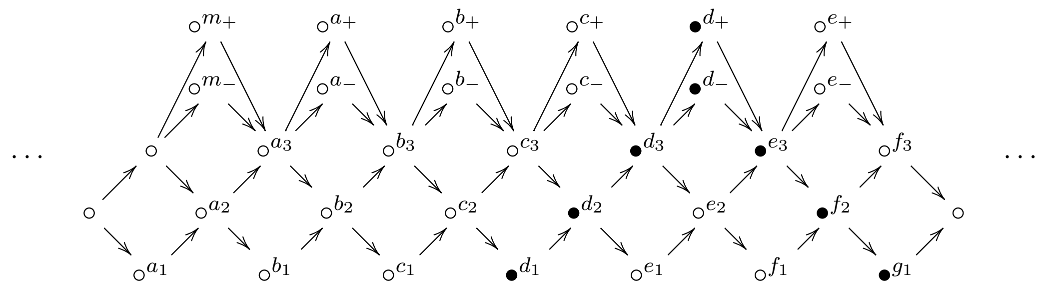
The distinguished triangle on the third row is given by Lemma 3.4.25 and the distinguished triangle in the second column is given by Lemma 3.4.26.

By Corollary 3.4.12, $(c_1, \Sigma b_1) = (c_1, f_1)$ is one-dimensional (replacing $a_1$ by $c_1$ and $d_1$ by $f_1$). Since $b_3$ is indecomposable, the distinguished triangle $b_1 \to b_3 \to c_2 \to \Sigma b_1$ is non-split, and the morphism $c_2 \to \Sigma b_1$ is non-zero. Similarly, since $d_1$ is indecomposable, the morphism $c_1 \to c_2$ is non-zero, hence the morphism $c_1 \to \Sigma b_1$ is non-zero by Lemma 3.4.6. Therefore the distinguished triangle in the first column has to be the Auslander-Reiten triangle $b_1 \to b_2 \to c_1 \to$. □

**Lemma 3.4.28.** *The mapping cone of the morphism $\mu : a_+ \to d_1$ is $d_+$.*

*Proof.* The Hom space $(a_+, d_1)$ is one-dimensional by Corollary 3.4.10, therefore the mapping cones of all non-zero morphisms $\mu : a_+ \to d_1$ are isomorphic. Such considerations permeate this section, and they will not be repeated each time, since most Hom spaces are one-dimensional.

In the following diagram, the region $\mathcal{R}(d_1)$ is as shown by the black dots.




Extend a non-zero morphism $\mu$ to the distinguished triangle $a_+ \xrightarrow{\mu} d_1 \xrightarrow{\nu} * \to$. By [16, Lemma 6.5], the mapping cone $*$ is indecomposable. The morphism $\nu$ is non-zero, otherwise the distinguished triangle $\Sigma^{-1}* \to a_+ \xrightarrow{\mu} d_1 \xrightarrow{\nu} *$ splits, but this is impossible since $a_+$ is indecomposable. Therefore the object $*$ has to be in $\mathcal{R}(d_1)$.

The object $*$ cannot be $d_1, d_2, d_3$ or $d_-$. Assume it is. Since $(a_+, d_1)$ is one-dimensional by Corollary 3.4.10, $(d_1, *)$ is one-dimensional by Lemma 3.4.1 and $(a_+, *)$ is one-dimensional by Corollary 3.4.10 and Corollary 3.4.11, the composition $\nu\mu$ is non-zero by Lemma 3.4.6. However, this is a contradiction given that the composition of two consecutive morphisms of a distinguished triangle is zero by Lemma 0.2.2(i).

Appealing to the octahedral axiom, extend to the following commutative diagram,

$$\begin{tikzcd}
\Sigma^{-1}* \arrow[r] \arrow[d] & a_+ \arrow[r, "\mu"] \arrow[d] & d_1 \arrow[r, "\nu"] \arrow[d, equal] & * \\
b_2 \arrow[r] \arrow[d] & b_3 \arrow[r] \arrow[d] & d_1 \arrow[r] \arrow[d] & \Sigma b_2 \\
b_+ \arrow[r, equal] \arrow[d] & b_+ \arrow[r] \arrow[d] & 0 & \\
* & {} & & .
\end{tikzcd}$$

The distinguished triangle on the second row is given in Lemma 3.4.27. The distinguished triangle in the second column is an Auslander-Reiten triangle. Suppose the object $*$ is $e_3$. Then the distinguished triangle on the first row is $a_3 \to a_+ \xrightarrow{\mu} d_1 \xrightarrow{\nu} e_3$. But again this is not possible for the same reason.

Consider the distinguished triangle in the first column. Since $b_2$ is indecomposable, the distinguished triangle is non-split. This means the morphism from $b_+$ to $*$ is non-zero. The object $*$ cannot be $f_2$ or $g_1$, since neither of them is in $\mathcal{R}(b_+)$. Hence the mapping cone has to be the object $d_+$. $\square$

**Corollary 3.4.29.** *The mapping cone of the morphism $\mu : b_2 \to b_+$ is $d_+$.*

*Proof.* Consider the distinguished triangle $\Sigma^{-1}* \to b_2 \to b_+ \to *$ in the first column of the commutative diagram in Lemma 3.4.28, and the result is evident. Also the Hom space $(b_2, b_+)$ is one-dimensional. $\square$

**Lemma 3.4.30.** *The mapping cone of the morphism $\mu : m_+ \to b_2$ is $b_-$.*

*Proof.* The Hom space $(m_+, b_2)$ is one-dimensional by Corollary 3.4.10. By the octahedral axiom, extend to the following commutative diagram,

$$\begin{tikzcd}
& m_+ \arrow[r, "\mu"] \arrow[d] & b_2 \arrow[r] \arrow[d] & * \arrow[d] \\
0 \arrow[r] & c_1 \arrow[r, equal] \arrow[d] & c_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
* \arrow[r] & c_+ \arrow[r] \arrow[d] & \Sigma b_1 \arrow[r] \arrow[d] & \Sigma * \\
& {} & {} & .
\end{tikzcd}$$

The distinguished triangle $m_+ \to c_1 \to c_+ \to$ is given by Lemma 3.4.28 and the distinguished triangle $b_2 \to c_1 \to \Sigma b_1 \to$ is given by Lemma 3.4.25. Finally, the mapping cone of the morphism $c_+ \to \Sigma b_1$ is $\Sigma b_-$ by Lemma 3.4.28. Also the Hom space $(c_+, \Sigma b_1)$ is one-dimensional. □

**Lemma 3.4.31.** *The mapping cone of the morphism $\alpha_+ : a_3 \to a_+$ is $d_-$.*

*Proof.* By the octahedral axiom, extend to the following commutative diagrams,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & a_- \oplus b_2 \arrow[r, equal] \arrow[d] & a_- \oplus b_2 \arrow[r] \arrow[d] & {} \\
a_3 \arrow[r] \arrow[d, equal] & a_+ \oplus a_- \oplus b_2 \arrow[r] \arrow[d] & b_3 \arrow[r] \arrow[d] & {} \\
a_3 \arrow[r, "\alpha_+"] & a_+ \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & {} \\
& {} & {} & ,
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & b_2 \arrow[r, equal] \arrow[d] & b_2 \arrow[r] \arrow[d] & 0 \arrow[d] \\
{} \arrow[r] & a_- \oplus b_2 \arrow[r] \arrow[d] & b_3 \arrow[r] \arrow[d] & * \arrow[d, equal] \\
{} \arrow[r] & a_- \arrow[r] \arrow[d] & d_1 \arrow[r] \arrow[d] & * \arrow[d] \\
& {} & {} & .
\end{tikzcd}$$

In the first diagram, the distinguished triangle on the second row is an Auslander-Reiten triangle. In the second diagram, the distinguished triangle in the middle column is given by Lemma 3.4.27. Finally, the mapping cone of the morphism $a_- \to d_1$ is $d_-$, by a mirror version of Lemma 3.4.28. $\square$

**Corollary 3.4.32.** *The mapping cone of the morphism $a_2 \to a_+$ is $c_+$.*

*Proof.* It is the same as Corollary 3.4.29 by a suitable translation of coordinates. $\square$

**Corollary 3.4.33.** *The mapping cone of the morphism $a_1 \to a_+$ is $b_-$.*

*Proof.* The Hom space $(a_1, a_+)$ is one-dimensional by Lemma 3.4.1. By the octahedral axiom, extend to the following commutative diagram,

$$\begin{tikzcd}
 & a_1 \arrow[r] \arrow[d] & a_+ \arrow[r] \arrow[d, equal] & * \arrow[r] & {} \\
\Sigma^{-1}c_+ \arrow[r] \arrow[d] & a_2 \arrow[r] \arrow[d] & a_+ \arrow[r] \arrow[d] & c_+ & \\
b_1 \arrow[r, equal] \arrow[d] & b_1 \arrow[r] & 0 & & \\
* & & & &
\end{tikzcd}.$$

The distinguished triangle $a_1 \to a_2 \to b_1 \to$ is an Auslander-Reiten triangle. The distinguished triangle on the second row is given by Corollary 3.4.32.

Since $a_+$ is indecomposable, the morphism $\Sigma^{-1}c_+ \to a_2$ is non-zero. Similarly, since $a_1$ is indecomposable, the morphism $a_2 \to b_1$ is non-zero, hence the morphism $\Sigma^{-1}c_+ \to b_1$ is non-zero by Lemma 3.4.6.

Finally, the mapping cone of the morphism $\Sigma^{-1}c_+ \to b_1$ is $b_-$, by a mirror version of Lemma 3.4.28. $\square$

**Lemma 3.4.34.** *The mapping cone of the morphism $m_+ \to b_+$ is $d_2$.*

*Proof.* The Hom space $(m_+, b_+)$ is one-dimensional by Corollary 3.4.11. By Corollary 3.4.32, the mapping cone of $m_2 \to m_+$ is $b_+$. Hence the mapping cone of $m_+ \to b_+$ is $\Sigma m_2 = d_2$. $\square$

**Lemma 3.4.35.** *The mapping cone of the morphism $a_3 \to a_+ \oplus b_+$ is $d_2$.*

*Proof.* By the octahedral axiom, extend to the following commutative diagram,

$$\begin{tikzcd}
 & & m_+ \arrow[d] & \\
0 \arrow[r] \arrow[d] & b_+ \arrow[r, equal] \arrow[d] & b_+ \arrow[r] \arrow[d, "\theta_1\ (i)"] & 0 \arrow[d] \\
a_3 \arrow[r] \arrow[d, equal] & a_+ \oplus b_+ \arrow[r] \arrow[d] & * \arrow[r, "\gamma_1"] \arrow[d, "\theta_2\ (ii)"] & e_3 \arrow[d, equal] \\
a_3 \arrow[r] & a_+ \arrow[r] & d_- \arrow[r, "\gamma_2"] & e_3.
\end{tikzcd}$$

The distinguished triangle on the last row is given by Lemma 3.4.31 and the distinguished triangle in the third column is given by Lemma 3.4.34.

Finally, rest reassured that the morphism $m_+ \to b_+$ considered is non-zero. Assume it is. Then the object $*$ would have to be isomorphic to $b_+ \oplus d_-$, and $\theta_1$ would have to be the non-zero canonical inclusion and $\theta_2$ would have to be the non-zero canonical surjection. Since square (i) is commutative, $\gamma_1$ would have to be zero although $e_3$ is in $\mathcal{R}(b_+)$. Similarly, since square (ii) is commutative, $\gamma_2$ would have to be zero although $e_3$ is in $\mathcal{R}(d_-)$. However, this is a contradiction since $a_+$ is indecomposable. Therefore the morphism $m_+ \to b_+$ considered is non-zero indeed. $\square$

*Remark* 3.4.36. In Lemma 3.4.35, it is not possible to use [16, Lemma 6.5] to show that $*$ is indecomposable, as in Lemma 3.4.28.

**Corollary 3.4.37.** *The mapping cone of the morphism $a_2 \to a_+ \oplus b_+$ is $c_3$.*

*Proof.* By the octahedral axiom, extend to the following commutative diagram,

$$\begin{tikzcd}
 & & \Sigma^{-1}c_+ \arrow[d] & \\
0 \arrow[r] \arrow[d] & b_+ \arrow[r, equal] \arrow[d] & b_+ \arrow[d] & \\
a_2 \arrow[r] \arrow[d, equal] & a_+ \oplus b_+ \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & {} \\
a_2 \arrow[r] & a_+ \arrow[r] & c_+ \arrow[r] & {}.
\end{tikzcd}$$

The distinguished triangle on the last row is given by Lemma 3.4.32. There is an Auslander-Reiten triangle $b_+ \to c_3 \to c_+ \to$, and the result is immediate since $(a_2, a_+ \oplus b_+)$ is one-dimensional.

Finally, the morphism $\Sigma^{-1}c_+ \to b_+$ considered is non-zero indeed. This is very similar to Lemma 3.4.35 where the morphism $m_+ \to b_+$ is non-zero. $\square$

**Lemma 3.4.38.** *The mapping cone of the morphism $a_- \to b_+$ is $e_1$.*

*Proof.* By a mirror version of Corollary 3.4.33, the mapping cone of the morphism $a_1 \to a_-$ is $b_+$. Hence the mapping cone of the morphism $a_- \to b_+$ is $\Sigma a_1 = e_1$. $\square$

Using the same notation as in [23, Theorem 2.3], let $\sigma$ be the translation functor of the triangulated category $\mathcal{T}_{\mathcal{X}}$. The construction of the functor $\sigma$ is described in [23, Setup 1.1].

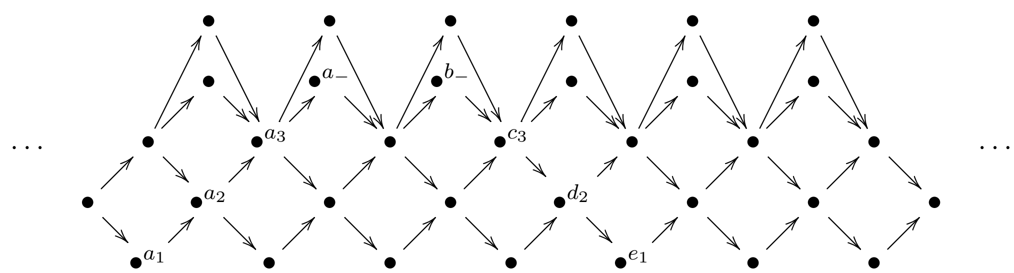
**Lemma 3.4.39.** *(c.f. Lemma 3.2.13) Let $a = (u, v)$ be an indecomposable object of $\mathcal{T}_{\mathcal{X}}$. Let $p$ and $q$ be integers and let $\lambda = v - u$. Then*

$$\sigma(u,v) = \begin{cases} (p+4, q+1) & \textit{if } (u,v) = (p,q)_-, \\ (u+\lambda, v+(5-\lambda)) = (v, u+5) & \textit{if } 1 < \lambda < 4, \\ (u+1, v+4)_- & \textit{if } \lambda = 1. \end{cases}$$

*Proof.* The first line is true by Lemma 3.4.15 and Lemma 3.4.38. The second line is true by Lemma 3.4.13, Lemma 3.4.14, Lemma 3.4.35 and Corollary 3.4.37. The last line is true by Lemma 3.4.16 and Corollary 3.4.33. $\square$

The action of $\sigma$ is shown in the following diagram.

**Diagram 3.4.40.**




The action of $\sigma$ is given by $\sigma(a_1) = b_-, \sigma(a_2) = c_3, \sigma(a_3) = d_2$ and $\sigma(a_-) = e_1$.

**Corollary 3.4.41.** *(c.f. Corollary 3.2.15) Let $a = (p, q)_-$ be an indecomposable object of $\mathcal{T}_{\mathcal{X}}$, $N \geq 0$ an integer. Then*

$$\sigma^N(a) = \begin{cases} (p + \frac{5N}{2}, q + \frac{5N}{2})_- & \textit{for } N \textit{ even}, \\ (p + \frac{5(N+1)}{2} - 1, q + \frac{5(N-1)}{2} + 1) & \textit{for } N \textit{ odd}. \end{cases}$$

*Proof.* Immediate from Lemma 3.4.39 by induction. □

**Corollary 3.4.42.** *(c.f. Corollary 3.2.15) Let $a = (u, v)$ be an indecomposable object of $\mathcal{T}_{\mathcal{X}}$. Let $\lambda = v - u$, $N \geq 0$ an integer. If $1 < \lambda < 4$, then*

$$\sigma^N(a) = \begin{cases} (u + \frac{5N}{2}, v + \frac{5N}{2}) & \textit{for } N \textit{ even,} \\ (v + \frac{5(N-1)}{2}, u + \frac{5(N+1)}{2}) & \textit{for } N \textit{ odd.} \end{cases}$$

*Proof.* Immediate from Lemma 3.4.39 by induction. □

**Corollary 3.4.43.** *(c.f. Corollary 3.2.15) Let $a = (u, v)$ be an indecomposable object of $\mathcal{T}_{\mathcal{X}}$. Let $N \geq 0$ be an integer. If $v - u = 1$, then*

$$\sigma^N(a) = \begin{cases} (u + \frac{5N}{2}, v + \frac{5N}{2}) & \textit{for } N \textit{ even,} \\ (u + \frac{5(N-1)}{2} + 1, v + \frac{5(N+1)}{2} - 1)_- & \textit{for } N \textit{ odd.} \end{cases}$$

*Proof.* Immediate from Lemma 3.4.39 by induction. □

For example, let $a = (u, v)$ be an indecomposable object. Then $\sigma^2(u, v) = (u + 5, v + 5)$.

Since the translation functor $\sigma$ is an autoequivalence, there is also an explicit description of $\sigma^{-1}$.

**Lemma 3.4.44.** *(c.f. Lemma 3.2.16) Let $a = (u, v)$ be an indecomposable object of $\mathcal{T}_{\mathcal{X}}$. Let $p$ and $q$ be integers and let $\lambda = v - u$. Then*

$$\sigma^{-1}(u, v) = \begin{cases} (p - 1, q - 4) & \textit{if } (u, v) = (p, q)_-, \\ (v - 5, u) & \textit{if } 1 < \lambda < 4, \\ (u - 4, v - 1)_- & \textit{if } \lambda = 1. \end{cases}$$

*Proof.* Immediate from Lemma 3.4.39. □

**Corollary 3.4.45.** *(c.f. Corollary 3.2.18) Let $a = (p, q)_-$ be an indecomposable object of $\mathcal{T}_{\mathcal{X}}$, $N \geq 0$ an integer. Then*

$$\sigma^{-N}(a) = \begin{cases} (p - \frac{5N}{2}, q - \frac{5N}{2})_- & \textit{for } N \textit{ even,} \\ (p - \frac{5(N-1)}{2} - 1, q - \frac{5(N+1)}{2} + 1) & \textit{for } N \textit{ odd.} \end{cases}$$

*Proof.* Immediate from Lemma 3.4.44 by induction. □

**Corollary 3.4.46.** *(c.f. Corollary 3.2.18) Let $a = (u, v)$ be an indecomposable object of $\mathcal{T}_{\mathcal{X}}$. Let $\lambda = v - u$, $N \geq 0$ an integer. If $1 < \lambda < 4$, then*

$$\sigma^{-N}(a) = \begin{cases} (u - \frac{5N}{2}, v - \frac{5N}{2}) & \text{for } N \text{ even,} \\ (v - \frac{5(N+1)}{2}, u - \frac{5(N-1)}{2}) & \text{for } N \text{ odd.} \end{cases}$$

*Proof.* Immediate from Lemma 3.4.44 by induction. □

**Corollary 3.4.47.** *(c.f. Corollary 3.2.18) Let $a = (u, v)$ be an indecomposable object of $\mathcal{T}_{\mathcal{X}}$. Let $N \geq 0$ be an integer. If $v - u = 1$, then*

$$\sigma^{-N}(a) = \begin{cases} (u - \frac{5N}{2}, v - \frac{5N}{2}) & \text{for } N \text{ even,} \\ (u - \frac{5(N+1)}{2} + 1, v - \frac{5(N-1)}{2} - 1)_- & \text{for } N \text{ odd.} \end{cases}$$

*Proof.* Immediate from Lemma 3.4.44 by induction. □

**Lemma 3.4.48.** *(c.f. Lemma 3.2.19) Let $N \geq 1$ be an integer. Consider $a = a_1 \oplus a_2 \oplus a_3 \oplus a_-$ in $\mathcal{T}_{\mathcal{X}}$. Then $\mathrm{Hom}_{\mathcal{T}_{\mathcal{X}}}(a, \sigma^N a) = 0$.*

*Proof.* Similar to Lemma 3.2.19, with the help of Lemma 3.4.18, Lemma 3.4.19, Lemma 3.4.20, Lemma 3.4.21, Corollary 3.4.41, Corollary 3.4.42 and Corollary 3.4.43. □

**Lemma 3.4.49.** *(c.f. Lemma 3.2.20) Let $N \geq 1$ be an integer. Consider $a = a_1 \oplus a_2 \oplus a_3 \oplus a_-$ in $\mathcal{T}_{\mathcal{X}}$. Then $\mathrm{Hom}_{\mathcal{T}_{\mathcal{X}}}(a, \sigma^{-N} a) = 0$.*

*Proof.* Similar. □

### 3.4.4 Theorem

The following is the main theorem of the section.

**Theorem 3.4.50.** *Consider the categories $\mathcal{T}$, $\mathcal{X}$ and $\mathcal{T}_{\mathcal{X}}$ as usual, and the object $a = a_1 \oplus a_2 \oplus a_3 \oplus a_-$ in $\mathcal{T}_{\mathcal{X}}$ as in Lemma 3.4.22. Then there is an equivalence of triangulated categories $f : \mathcal{T}_{\mathcal{X}} \xrightarrow{\simeq} D^b(\mathrm{mod}\,(kA_4)^\circ)$ where $f(a) = (kA_4)^\circ$.*

*Proof.* Let $\mathcal{U}$ and $X$ in Theorem 3.1.1 be $\mathcal{T}_{\mathcal{X}}$ and $a$ respectively. By Corollary 3.4.24, $A \cong (kA_4)^\circ$. Let us take $\Sigma$ in Theorem 3.1.1 to be the translation functor $\sigma$ of the triangulated category $\mathcal{T}_{\mathcal{X}}$. Then the result follows from Lemma 3.4.22, Lemma 3.4.48 and Lemma 3.4.49. □

# Chapter 4

# A characterization of torsion theories in the cluster category of Dynkin type $A_\infty$

This chapter is also written in the form of a paper which is submitted for publication in Homology, Homotopy and Applications ([37]).

## 4.1 Introduction

The cluster category $\mathcal{D}$ of Dynkin type $A_\infty$ was introduced in [19]. One of its several definitions, which is completely analogous to the definition of the cluster category of type $A_n$, motivates us to say that $\mathcal{D}$ is a cluster category of type $A_\infty$. Namely, it is the orbit category $D^f(\operatorname{mod}\Gamma)/S\Sigma^{-2}$. Here $\Gamma$ is a quiver of type $A_\infty$ with zigzag orientation and $S$ and $\Sigma$ are the Serre and translation functors of the finite derived category $D^f(\operatorname{mod}\Gamma)$.

There are also several other ways to realize the category $\mathcal{D}$. In brief, it is the algebraic triangulated category generated by a 2-spherical object. It is also the compact derived category $D^c(A)$ of the differential graded cochain algebra $A = C^*(S^2; k)$ where $S^2$ is the 2-sphere and $k$ is a field. Finally, $\mathcal{D}$ is the finite derived category $D^f(k[T])$ where $k[T]$ is viewed as a DG algebra with $T$ placed in homological degree 1 and zero differential. It is ubiquitous and the reader can refer to [19, Section 0] for more details.

In [19], the cluster tilting subcategories of $\mathcal{D}$ were shown to be in bijection with certain maximal sets of non-crossing arcs connecting non-neighbouring integers. One can think of these maximal sets as "triangulations of the $\infty$-gon".

A torsion theory in $\mathcal{D}$ is a pair $(\mathcal{X}, \mathcal{Y})$ of subcategories such that there are no non-zero morphisms from any object in $\mathcal{X}$ to any object in $\mathcal{Y}$, and that for each $d$ in $\mathcal{D}$, there is a distinguished triangle $x \to d \to y \to$ in $\mathcal{D}$, with $x$ in $\mathcal{X}$ and $y$ in $\mathcal{Y}$ (Definition 1.2.5). If $\mathcal{T}$ is a cluster tilting subcategory, then $(\mathcal{T},\ \Sigma\mathcal{T})$ is a torsion theory, but t-structures and co-t-structures are other examples of torsion theories.

In this chapter, the results of [19] are generalized by giving a bijection between torsion theories in $\mathcal{D}$ and certain configurations of arcs connecting non-neighbouring integers. A few examples, characterizing all t-structures and co-t-structures in $\mathcal{D}$, are given.

## 4.2 Coordinate system

Let us recollect some material from [19]. The category $\mathcal{D}$ has finite-dimensional Hom spaces over a field $k$ and split idempotents, so it is Krull-Schmidt ([19, Remark 1.2]). In this chapter, a subcategory of $\mathcal{D}$ is assumed to be a full subcategory closed under direct sums and direct summands. For subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{D}$, the set of morphisms from any $x$ in $\mathcal{X}$ to any $y$ in $\mathcal{Y}$ is denoted by $(\mathcal{X}, \mathcal{Y})$. In particular, for objects $x$ and $y$ of $\mathcal{D}$, the set of morphisms from $x$ to $y$ is denoted by $(x, y)$.

By [19, Remark 1.4], the Auslander-Reiten quiver of $\mathcal{D}$ is $\mathbb{Z}A_\infty$, and the following standard coordinate system is laid down on the Auslander-Reiten quiver.

$$\begin{tikzcd}[column sep=tiny, row sep=small]
 & \vdots \arrow[dr] & & \vdots \arrow[dr] & & \vdots \arrow[dr] & & \vdots \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & (-4,1) \arrow[ur] \arrow[dr] & & (-3,2) \arrow[ur] \arrow[dr] & & (-2,3) \arrow[ur] \arrow[dr] & & \cdots \\
 & (-4,0) \arrow[ur] \arrow[dr] & & (-3,1) \arrow[ur] \arrow[dr] & & (-2,2) \arrow[ur] \arrow[dr] & & (-1,3) \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & (-3,0) \arrow[ur] \arrow[dr] & & (-2,1) \arrow[ur] \arrow[dr] & & (-1,2) \arrow[ur] \arrow[dr] & & \cdots \\
 & (-3,-1) \arrow[ur] & & (-2,0) \arrow[ur] & & (-1,1) \arrow[ur] & & (0,2) \arrow[ur] &
\end{tikzcd}$$

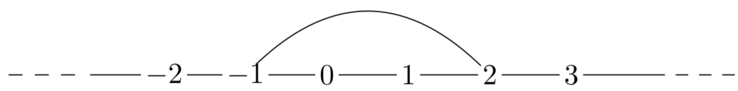
Let $\Sigma$ be the translation functor of $\mathcal{D}$. Since $\mathcal{D}$ is 2-Calabi-Yau, its Serre functor is $S = \Sigma^2$ and the Auslander-Reiten translation is $\tau = S\Sigma^{-1} = \Sigma$. In terms of coordinates, the action of $\Sigma = \tau$ is given by $\Sigma(m, n) = (m-1, n-1)$, see [19, Remark 1.4].

The definitions to follow interpret coordinate pairs in an alternative way, as arcs connecting non-neighbouring integers.

**Definition 4.2.1.** ([19, Definition 3.1]) An *arc* is a pair $(m, n)$ of integers with $n-m \geq 2$. The arc $(m, n)$ is said to *end* in each of the integers $m$ and $n$.

Two arcs $(m_1, n_1)$ and $(m_2, n_2)$ are said to *cross* if either $m_1 < m_2 < n_1 < n_2$ or $m_2 < m_1 < n_2 < n_1$. The action of $\Sigma$ makes sense on arcs as well.

In the diagrams to follow arcs are drawn on number lines which are numbered thus.




For example, in the above diagram, the arc $(-1, 2)$ is drawn as a curve between the integers $-1$ and 2. Crossing of arcs has been defined to match this geometrical picture in the natural way.

This, together with the following definition, relies heavily on the coordinate system.

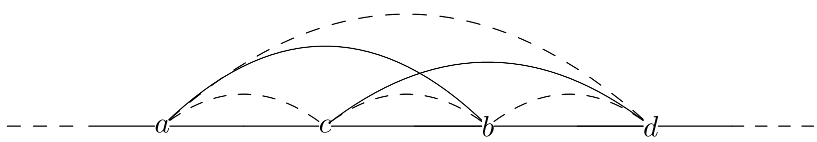
**Definition 4.2.2.** ([19, Definition 3.2]) Let $\mathfrak{A}$ be a set of arcs. If, for each integer $n$, there are only finitely many arcs in $\mathfrak{A}$ which end in $n$, then $\mathfrak{A}$ is said to be *locally finite*. A *left (resp. right) fountain* of $\mathfrak{A}$ is an integer $n$ for which there are infinitely many arcs of the form $(m, n)$ (resp. $(n, m)$) in $\mathfrak{A}$. A *fountain* of $\mathfrak{A}$ is an integer $n$ which is both a left and a right fountain of $\mathfrak{A}$. A left (resp. right) fountain $n$ of $\mathfrak{A}$ is said to be *full* if for each $m$ the arc of the form $(m, n)$ (resp. $(n, m)$) is in $\mathfrak{A}$. Finally, $\mathfrak{A}$ is said to be *non-crossing* if $\mathfrak{A}$ does not contain any pairs of crossing arcs.

Coordinate pairs $(m, n)$, $n - m \geq 2$, are identified with either indecomposable objects of $\mathcal{D}$ or with arcs. Thereupon a set $\mathfrak{A}$ of arcs induces a collection of indecomposable objects of $\mathcal{D}$, and add of them gives a subcategory $\mathcal{A}$ of $\mathcal{D}$, thus there is a bijection between sets of arcs and subcategories of $\mathcal{D}$.

It is our intention to display (in diagrams) at times both versions of interpretations, coordinate pairs as indecomposable objects of $\mathcal{D}$ on the quiver and as arcs on number lines. The reader should be able to feel so comfortable with the simultaneous visualizations in the mind's eye, so much so that any one interpretation is at once able to find an echo and a corroboration in the other.

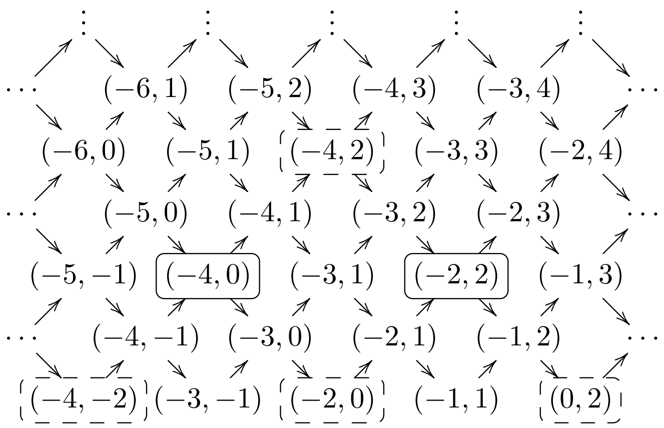
The following pairs of definitions describe some constraints on a set of arcs $\mathfrak{A}$.

**Definition 4.2.3.** A set of arcs $\mathfrak{A}$ is said to satisfy *condition* $(\star)$ if, for each pair of crossing arcs $(a, b)$ and $(c, d)$ in $\mathfrak{A}$, those of the pairs $(a, c)$, $(c, b)$, $(b, d)$ and $(a, d)$ which are arcs belong to $\mathfrak{A}$ (for instance, $(a, c)$ is only an arc if $c - a \geq 2$).

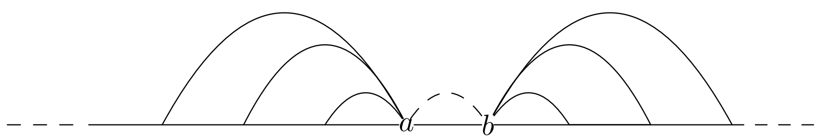



The following example visualizes condition ($\star$) on the quiver.

*Example* 4.2.4. Suppose the set of arcs $\mathfrak{A}$ satisfies condition ($\star$). Then given the arcs $(-4,0)$ and $(-2,2)$ in $\mathfrak{A}$, the arcs $(-4,-2)$, $(-2,0)$, $(0,2)$ and $(-4,2)$ are in $\mathfrak{A}$.

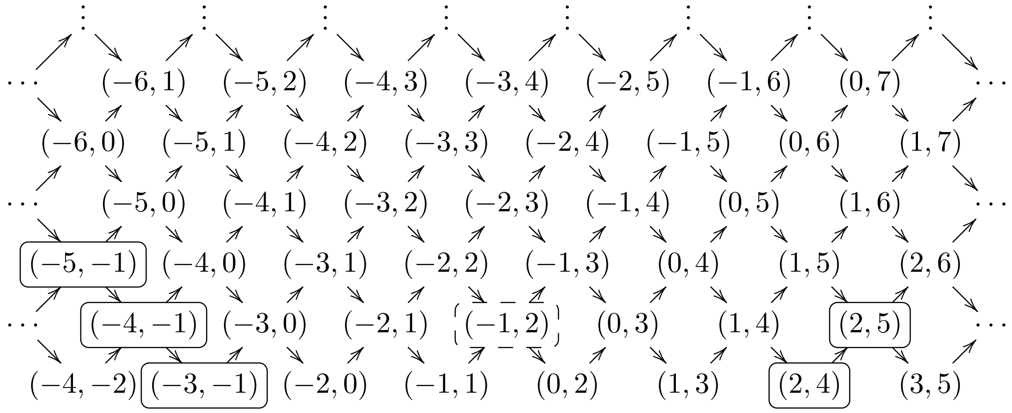



**Definition 4.2.5.** A set of arcs $\mathfrak{A}$ is said to satisfy *condition ($\star\star$)* if it has the following property: if $a$ is a left fountain but not a right fountain of $\mathfrak{A}$, $b$ is a right fountain but not a left fountain of $\mathfrak{A}$ and $b-a \geq 2$, then the arc $(a,b)$ is in $\mathfrak{A}$.




The following example visualizes condition ($\star\star$) on the quiver.

*Example* 4.2.6. Suppose the set of arcs $\mathfrak{A}$ satisfies condition ($\star\star$). Then given the left fountain (but not right fountain) $-1$ and the right fountain (but not left fountain) $2$ of $\mathfrak{A}$, the arc $(-1,2)$ is in $\mathfrak{A}$. The diagram below shows only some of the arcs from the fountains.




**Definition 4.2.7.** Given a subcategory $\mathcal{U}$ of $\mathcal{D}$, let $\mathcal{U}^{\perp} = \{d \in \mathcal{D} | (u, d) = 0$ for all $u$ in $\mathcal{U}\}$ and ${}^{\perp}\mathcal{U} = \{d \in \mathcal{D} | (d, u) = 0$ for all $u$ in $\mathcal{U}\}$.

**Definition 4.2.8.** Given a set of arcs $\mathfrak{U}$, let $\operatorname{ort}(\mathfrak{U})$ be the set of arcs $\operatorname{ort}(\mathfrak{U}) = \{\mathfrak{d} \mid \mathfrak{d}$ does not cross any arcs in $\mathfrak{U}\}$.

**Lemma 4.2.9.** *Let* $\mathfrak{U}$ *be a set of arcs. Then* $\operatorname{ort}\operatorname{ort}\operatorname{ort}(\mathfrak{U}) = \operatorname{ort}(\mathfrak{U})$.

*Proof.* That $\operatorname{ort}\operatorname{ort}(\mathfrak{U}) \supseteq \mathfrak{U}$ is immediate, so that $\operatorname{ort}\operatorname{ort}\operatorname{ort}(\mathfrak{U}) \subseteq \operatorname{ort}(\mathfrak{U})$, since ort reverses inclusions. Similarly, $\operatorname{ort}\operatorname{ort}\operatorname{ort}(\mathfrak{U}) \supseteq \operatorname{ort}(\mathfrak{U})$. Therefore $\operatorname{ort}\operatorname{ort}\operatorname{ort}(\mathfrak{U}) = \operatorname{ort}(\mathfrak{U})$. □

Lemma 4.2.9 gives the minimal level when the operation ort becomes periodic. Later on, Lemma 4.4.20 gives the equivalent condition for the equality $\operatorname{ort}\operatorname{ort}\mathfrak{A} = \mathfrak{A}$.

*Remark* 4.2.10. In addition, the operation $\operatorname{ort}^2$ satisfies the following.

(i) $\operatorname{ort}\operatorname{ort}(\mathfrak{U}) \supseteq \mathfrak{U}$,

(ii) $\operatorname{ort}^4(\mathfrak{U}) = \operatorname{ort}^2(\mathfrak{U})$, which is immediate from Lemma 4.2.9.

**Lemma 4.2.11.** *Let* $\mathfrak{U}$ *be a set of arcs. Then* $\mathfrak{U} = \operatorname{ort}(\mathfrak{C})$ *for some set of arcs* $\mathfrak{C}$ *if and only if* $\mathfrak{U} = \operatorname{ort}\operatorname{ort}(\mathfrak{U})$.

*Proof.* (*if*) This is immediate. (*only if*) Since $\mathfrak{U} = \operatorname{ort}(\mathfrak{C})$, therefore $\operatorname{ort}\operatorname{ort}(\mathfrak{U}) = \operatorname{ort}\operatorname{ort}\operatorname{ort}(\mathfrak{C}) = \operatorname{ort}(\mathfrak{C}) = \mathfrak{U}$ by Lemma 4.2.9. □

**Lemma 4.2.12.** *Let* $\mathfrak{U}$, $\mathfrak{U}'$ *be sets of arcs. Suppose* $\operatorname{ort}\operatorname{ort}(\mathfrak{U}) = \mathfrak{U}$, $\operatorname{ort}\operatorname{ort}(\mathfrak{U}') = \mathfrak{U}'$. *Then*

*(i) If* $\operatorname{ort}(\mathfrak{U}) \supseteq \operatorname{ort}(\mathfrak{U}')$, *then* $\mathfrak{U} \subseteq \mathfrak{U}'$.

*(ii) If* $\mathrm{ort}(\mathfrak{U}) = \mathrm{ort}(\mathfrak{U}')$*, then* $\mathfrak{U} = \mathfrak{U}'$*.*

*Proof.* (i) If $\mathrm{ort}(\mathfrak{U}) \supseteq \mathrm{ort}(\mathfrak{U}')$, then $\mathrm{ort}\,\mathrm{ort}(\mathfrak{U}) \subseteq \mathrm{ort}\,\mathrm{ort}(\mathfrak{U}')$ which gives $\mathfrak{U} \subseteq \mathfrak{U}'$.

(ii) This is immediate by (i).

□

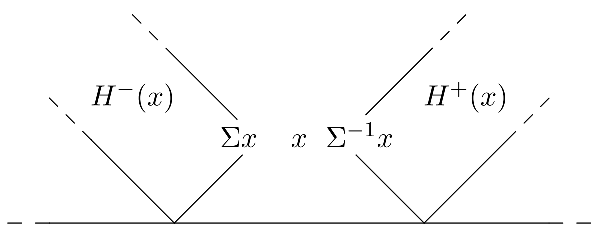
**Lemma 4.2.13.** *Let* $\mathfrak{U}$ *be a set of arcs. Then*

*(i)* $\mathfrak{U}$ *is non-crossing if and only if* $\mathfrak{U} \subseteq \mathrm{ort}(\mathfrak{U})$*.*

*(ii)* $\mathfrak{U}$ *is maximal non-crossing if and only if* $\mathfrak{U} = \mathrm{ort}(\mathfrak{U})$*.*

*Proof.* This is immediate. □

Let $x = (i, j)$ be an indecomposable object of $\mathcal{D}$, accompanied by the regions $\mathrm{H}^-(x) = \{(m, n) \mid m \leq i-1, i+1 \leq n \leq j-1\}$ and $\mathrm{H}^+(x) = \{(m, n) \mid i+1 \leq m \leq j-1, j+1 \leq n\}$ in the Auslander-Reiten quiver of $\mathcal{D}$ ([19, Definition 2.1]). They are sketched as follows.




We write $\mathrm{H}(x) = \mathrm{H}^-(x) \cup \mathrm{H}^+(x)$.

*Remark* 4.2.14. Let $(a, b)$ and $(c, d)$ be arcs. Then the two arcs cross if and only if $(c, d)$ is in $\mathrm{H}(a, b)$.

The following lemma describes morphisms in the category $\mathcal{D}$.

**Lemma 4.2.15.** *([19, Corollary 2.3]) Let $x$ and $y$ be indecomposable objects of $\mathcal{D}$. Then the following are equivalent.*

*(i)* $(x, y) \neq 0$*,*

*(ii)* $(x, y) = k$*,*

*(iii)* $y \in \mathrm{H}(\Sigma x)$*,*

*(iv)* $x \in \mathrm{H}(\Sigma^{-1} y)$*.*

*Remark* 4.2.16. By virtue of the coordinate system, the way the regions $\mathrm{H}^-(x)$ and $\mathrm{H}^+(x)$ are defined, and the above lemma, crossing of arcs is endowed with a meaningful interpretation. This is to say, given indecomposable objects $x$ and $y$ of $\mathcal{D}$, then $(x, y) \neq 0$ if and only if the arcs corresponding to $x$ and $\Sigma^{-1}y$ cross, see [19, Lemma 3.6].

Now let $\mathcal{X}$ and $\mathcal{Y}$ be two subcategories of $\mathcal{D}$ such that $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{\perp}$, accompanied by the sets of arcs $\mathfrak{X}$ and $\mathfrak{Y}$ respectively. By Remark 4.2.16, $\mathfrak{X} = \{\mathfrak{d} \mid \Sigma\mathfrak{d}$ does not cross any arcs in $\mathfrak{Y}\}$ and $\mathfrak{Y} = \{\mathfrak{d} \mid \Sigma^{-1}\mathfrak{d}$ does not cross any arcs in $\mathfrak{X}\}$. Together they determine each other.

Let $\Sigma^{-1}\mathfrak{Y} = \mathfrak{W}$. The above can be rewritten as $\mathfrak{X} = \{\mathfrak{d} \mid \mathfrak{d}$ does not cross any arcs in $\mathfrak{W}\} = \mathrm{ort}(\mathfrak{W})$ and $\mathfrak{W} = \{\mathfrak{d} \mid \mathfrak{d}$ does not cross any arcs in $\mathfrak{X}\} = \mathrm{ort}(\mathfrak{X})$.

The following lemma bridges indecomposable objects of $\mathcal{D}$ on the quiver and arcs on number lines.

**Lemma 4.2.17.** *(double orthogonal property) Let $\mathcal{X}$ and $\mathfrak{X}$ be defined as above. Then $\mathcal{X} = {}^{\perp}(\mathcal{X}^{\perp})$ is equivalent to $\mathfrak{X} = \mathrm{ort}\,\mathrm{ort}(\mathfrak{X})$.*

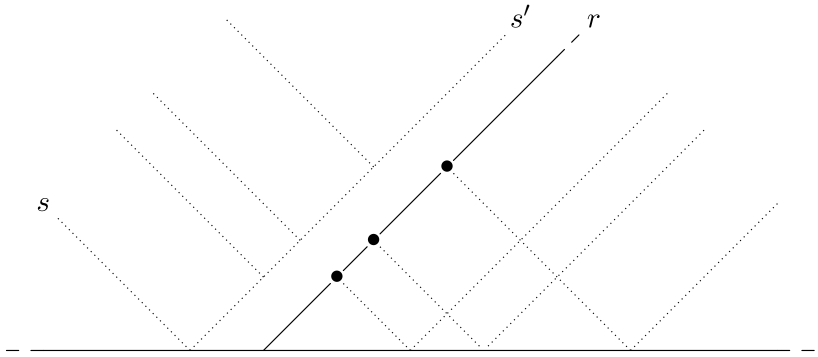
*Proof.* This is immediate by the above description. □

## 4.3 Precovering (preenveloping) subcategories

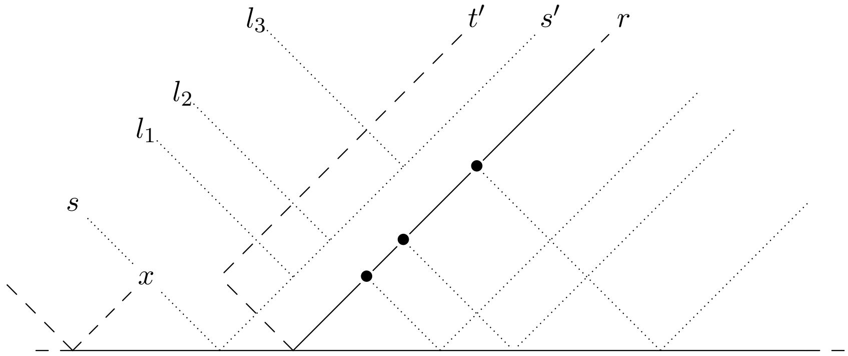
In this section, precovering and preenveloping subcategories are characterized in terms of their corresponding sets of arcs.

**Theorem 4.3.1.** *Let $\mathcal{A}$ be a subcategory of $\mathcal{D}$ and let $\mathfrak{A}$ be the corresponding set of arcs. Then $\mathcal{A}$ is precovering if and only if each right fountain of $\mathfrak{A}$ is in fact a fountain.*

*Proof.* Suppose $\mathcal{A}$ is precovering. A right fountain of $\mathfrak{A}$ is an integer $n$ for which there are infinitely many arcs of the form $(n, p)$ in $\mathfrak{A}$. The corresponding collection $P$ of indecomposable objects of $\mathcal{A}$ lies on a diagonal half line $r$ in the Auslander-Reiten quiver of $\mathcal{D}$. The following sketch shows $r$ along with some of the indecomposable objects $a_i$ in $P$ (indicated by the black dots) and, in dotted lines, their respective regions $\mathrm{H}(\Sigma a_i)$.




To show that $n$ is also a left fountain, that is, there are infinitely many arcs in $\mathfrak{A}$ of the form $(m,n)$, is the same as showing that there are infinitely many indecomposable objects of $\mathcal{A}$ which are on the half line $s$. Consider an object $x$ on the half line $s$ and its region $\mathrm{H}(\Sigma^{-1}x)$ indicated by dashed lines in the following diagram.




Let $\beta : b \to x$ be an $\mathcal{A}$-precover, where $b$ and $\beta$ are written $b_1 \oplus \cdots \oplus b_q$ and $\big(\beta_1, \ldots, \beta_q\big)$ respectively. The morphism $\beta : b \to x$ can be assumed to be non-zero on each direct summand $b_j$ of $b$, so that all the $b_j$ belong to $\mathrm{H}(\Sigma^{-1}x)$.

It is apparent that $x$ is in $\mathrm{H}^{-}(\Sigma a_i)$ for all the $a_i$ in $P$. Pick an indecomposable $a$ in $P$ and let $\alpha : a \to x$ be a non-zero morphism. Then $\alpha$ factors through $\beta$ as shown in the following diagram, where the morphism $\gamma$ is written $\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_q \end{pmatrix}$.

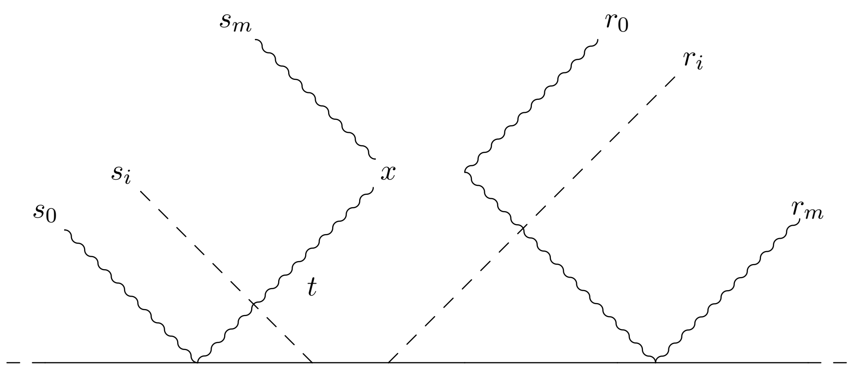
$$\begin{tikzcd}
a \arrow[d, "\gamma"'] \arrow[dr, "\alpha"] & \\
b_1 \oplus \cdots \oplus b_q \arrow[r, "\beta"'] & x
\end{tikzcd}$$

Since $\alpha$ is non-zero and $\alpha = \beta_1\gamma_1 + \cdots + \beta_q\gamma_q$, there is a term $\beta_k\gamma_k$ which is non-zero. Hence $\gamma_k : a \to b_k$ is non-zero so $b_k$ is in $\mathrm{H}(\Sigma a)$.

Therefore $b_k$ can only be on the half line $s$ above $x$, or in the infinite region inside $\mathrm{H}^+(\Sigma^{-1}x)$ bounded by $t'$, $s'$ and $l_i$ for some $i$. But it cannot be the latter because then $\beta_k$ and $\gamma_k$ would both be backward morphisms whence $\beta_k\gamma_k$ would be zero (Lemma 3.3.9). Therefore $b_k$ can only be on the half line $s$ above $x$. Repeating the argument, consider an object $x_1$ on the half line $s$ above $b_k$. Then another object $c$ in $\mathcal{A}$ on the half line $s$ above $x_1$ is revealed. Since it can be continued in this way indefinitely, there are infinitely many indecomposable objects of $\mathcal{A}$ which are on the half line $s$.

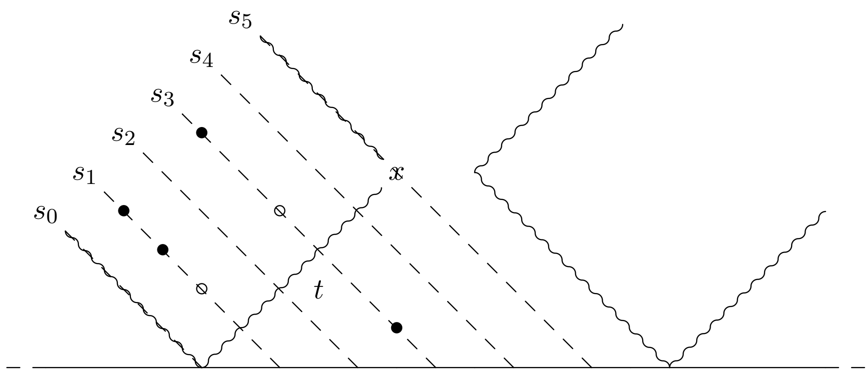
Now suppose each right fountain of $\mathfrak{A}$ is in fact a fountain.

In the following diagram, consider an object $x$ and its region $\mathrm{H}(\Sigma^{-1}x)$ in the Auslander-Reiten quiver of $\mathcal{D}$, indicated by wavy lines. As indicated, half lines (which start from the bottom line) of the form $r_i$ or $s_i$, where $i \in \mathbb{N}$ is a variable, are also introduced. Suppose the region $\mathrm{H}^-(\Sigma^{-1}x)$ (resp. $\mathrm{H}^+(\Sigma^{-1}x)$) has boundary half lines $s_0$ and $s_m$ (resp. $r_0$ and $r_m$). Then each line $s_i$ (resp. $r_i$), $0 \le i \le m$, has to pass through the region $\mathrm{H}^-(\Sigma^{-1}x)$ (resp. $\mathrm{H}^+(\Sigma^{-1}x)$) and is parallel to the boundary lines of the region $\mathrm{H}^-(\Sigma^{-1}x)$ (resp. $\mathrm{H}^+(\Sigma^{-1}x)$).




Let $S$ be the intersection of $\mathrm{H}^-(\Sigma^{-1}x)$ and the objects of $\mathcal{A}$. On each line $s_i$, $0 \le i \le m$, consider the first object $a_i$ in $\mathcal{A}$ which lies above the line segment $t$. Denote by $a_s$ the direct sum of all the $a_i$ and consider the canonical morphism $a_s \to x$. By Lemma 3.3.6, each morphism $a \to x$ with $a$ in $S$ factors as $a \to a_s \to x$.

For example, in the following diagram, let $m = 5$ and the circles and bullets indicate the first few objects of $\mathcal{A}$ on each line $s_i$. The object $a_s$ described above is the direct sum of the objects indicated by circles $\circ$.

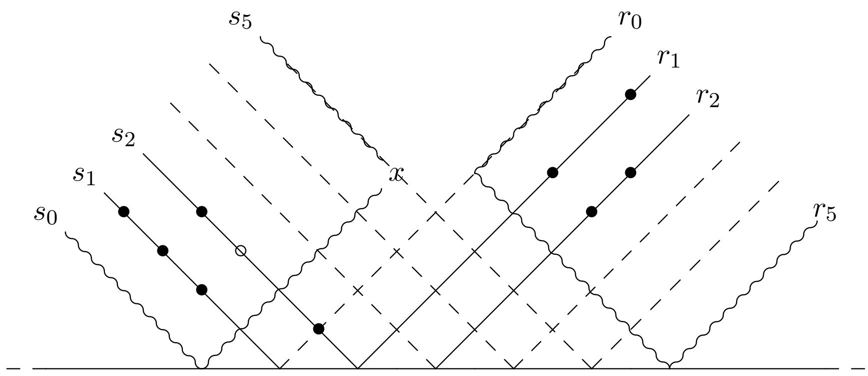



Let $R$ be the intersection of $\mathrm{H}^+(\Sigma^{-1}x)$ and the objects of $\mathcal{A}$. There is the desire of an $a_r$ in $\mathcal{A}$ with a morphism $a_r \to x$ such that each morphism $a \to x$ with $a$ in $R$ factors as $a \to a_r \to x$.

Suppose $R$ is finite. Then let $a_r$ be the direct sum of the objects in $R$ (Lemma 2.5.1). Otherwise $R$ is infinite. Since there are only finitely many $r_i$, there is a line $r_j$ which contains infinitely many objects in $\mathcal{A}$ with a non-zero morphism to $x$. Let $J \subseteq \{0, \ldots, m\}$ consist of the $j$ such that the line $r_j$ contains infinitely many objects in $\mathcal{A}$ with a non-zero morphism to $x$.

Now for each $j$ in $J$, the line $r_j$ corresponds to a right fountain of $\mathfrak{A}$, and by assumption, each right fountain is also a fountain. Hence the corresponding line $s_j$ contains infinitely many objects in $\mathcal{A}$ with a non-zero morphism to $x$. Among the $s_j$ with $j$ in $J$, take the line $s_q$ which is closest to the boundary line $s_m$, and then consider any object which is in both $\mathcal{A}$ and $\mathrm{H}^-(\Sigma^{-1}x)$ on that line. By Lemma 3.3.4, this object plays the role of $a_r$.

For example, in the following diagram, $m = 5$ and $J = \{1, 2\}$. The line $s_q$ described above is the line $s_2$ here. The bullets on the $r_i$ indicate the first few objects of $\mathcal{A}$ in $\mathrm{H}^+(\Sigma^{-1}x)$. The bullets and the circle on the $s_i$ indicate the first few objects of $\mathcal{A}$ on those half lines. The object $a_r$ described above is indicated by the circle $\circ$ (one of the infinitely many choices).




Finally, an $\mathcal{A}$-precover of $x$ can be obtained as $a_r \oplus a_s \to x$. $\square$

*Remark* 4.3.2. (i) In the *only if* part of the proof of Theorem 4.3.1, the way the position of the direct summand $b_k$ is calibrated is very similar to the descriptions found in two previous situations, by simply looking at which morphisms are zero and which are not. The first situation is in Lemma 3.2.7, where the regions $\mathcal{L}_{\mathcal{X}}(a)$ and $\mathcal{R}_{\mathcal{X}}(a)$ are determined. The other situation is in Lemma 3.4.28, where the mapping cone of the morphism $\mu : a_+ \to d_1$ is determined.

(ii) In the *if* part of the proof of Theorem 4.3.1, the occurrence of the backward morphisms begets the simultaneous occurrence of a left fountain (where it is a right fountain already) in pursuance of symmetry. Even though the introduction of the left fountain induces more objects in $\mathcal{A}$, it also provides another distribution of objects in $\mathcal{A}$ with different suggestiveness, thus permitting $\mathcal{A}$ to be precovering.

The following is the dual of Theorem 4.3.1.

**Theorem 4.3.3.** *Let $\mathcal{A}$ be a subcategory of $\mathcal{D}$ and let $\mathfrak{A}$ be the corresponding set of arcs. Then $\mathcal{A}$ is preenveloping if and only if each left fountain of $\mathfrak{A}$ is in fact a fountain.*

*Proof.* Similar. $\square$

## 4.4 Torsion theories

In this section, let $\mathfrak{A}$ be a set of arcs. A sequence of results regarding $\mathfrak{A}$ is given, and then also a checkable condition equivalent to $\mathfrak{X} = \operatorname{ort}\operatorname{ort}(\mathfrak{X})$, see Lemma 4.4.20. Finally, the main theorem of this chapter is given in Theorem 4.4.22.

**Definition 4.4.1.** Let $\mathcal{U}$ be a subcategory of $\mathcal{D}$. Then $\mathcal{U}$ is said to be *weak cluster tilting* if it satisfies $\mathcal{U} = (\Sigma^{-1}\mathcal{U})^{\perp}$ and $\mathcal{U} = {}^{\perp}(\Sigma\mathcal{U})$, and is said to be *cluster tilting* if it is weak cluster tilting, precovering and preenveloping.

In [19], the cluster tilting subcategories of $\mathcal{D}$ were shown to be in bijection with certain maximal sets of non-crossing arcs connecting non-neighbouring integers. This is rephrased as follows.

**Theorem 4.4.2.** *([19, Theorem 4.4]) Let $\mathcal{U}$ be a weak cluster tilting subcategory of $\mathcal{D}$ with $\mathfrak{U}$ as the corresponding maximal set of non-crossing arcs. Then $\mathcal{U}$ is precovering and preenveloping (that is, $\mathcal{U}$ is a cluster tilting subcategory of $\mathcal{D}$) if and only if $\mathfrak{U}$ is (i) locally finite, or (ii) has a fountain.*

In this section, the above theorem is generalized in terms of a bijection between torsion theories in $\mathcal{D}$ and certain configurations of arcs connecting non-neighbouring integers.

The following little lemma is a simple play of concepts.

**Lemma 4.4.3.** *Let $(a, b)$ be an arc (not necessarily in $\mathfrak{A}$). Then the following are equivalent.*

*(i)* $\operatorname{ort}\operatorname{ort}\mathfrak{A} = \mathfrak{A}$,

*(ii) If $(a, b)$ is an arc such that each arc crossing $(a, b)$ also crosses an arc in $\mathfrak{A}$, then $(a, b)$ is in $\mathfrak{A}$.*

*Proof.* (i) $\Rightarrow$ (ii): Suppose $\operatorname{ort}\operatorname{ort}\mathfrak{A} = \mathfrak{A}$ and let the arc $(a, b)$ be as described. Then the arc $(a, b)$ is in $\operatorname{ort}\operatorname{ort}\mathfrak{A}$, otherwise there would have to be an arc $(c, d)$ in $\operatorname{ort}\mathfrak{A}$ which crossed $(a, b)$. Subsequently, $(c, d)$ would be an arc which crossed $(a, b)$ but would not cross any arcs in $\mathfrak{A}$, which is a contradiction. Therefore the arc $(a, b)$ is in $\operatorname{ort}\operatorname{ort}\mathfrak{A} = \mathfrak{A}$.

(ii) $\Rightarrow$ (i): Suppose otherwise that $\operatorname{ort}\operatorname{ort}\mathfrak{A} \supset \mathfrak{A}$. Let $(a, b)$ be an arc in $\operatorname{ort}\operatorname{ort}\mathfrak{A}$ but not in $\mathfrak{A}$. Then there has to be an arc $(c, d)$ which crosses $(a, b)$ but does not cross any arcs in $\mathfrak{A}$. Therefore $(c, d)$ is in $\operatorname{ort}\mathfrak{A}$. However, $(c, d)$ is not in $\operatorname{ort}\operatorname{ort}\operatorname{ort}(\mathfrak{U})$, which is a contradiction since $\operatorname{ort}\operatorname{ort}\operatorname{ort}(\mathfrak{U})$ $= \operatorname{ort}(\mathfrak{U})$ by Lemma 4.2.9. Therefore $\operatorname{ort}\operatorname{ort}\mathfrak{A} \subseteq \mathfrak{A}$ which, together with $\operatorname{ort}\operatorname{ort}(\mathfrak{U}) \supseteq \mathfrak{U}$, gives $\operatorname{ort}\operatorname{ort}\mathfrak{A} = \mathfrak{A}$. $\square$

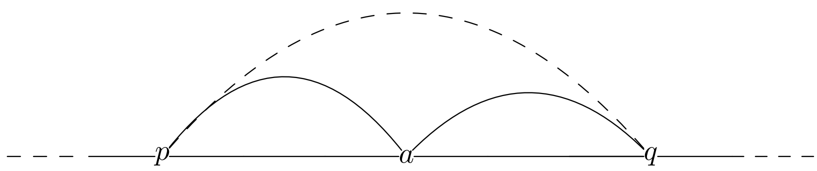
Now let us enter the spirit of this section.

**Lemma 4.4.4.** *Suppose* $\operatorname{ort}\operatorname{ort}\mathfrak{A} = \mathfrak{A}$. *Then $\mathfrak{A}$ satisfies conditions $(\star)$ and $(\star\star)$ of Definitions 4.2.3 and 4.2.5.*

*Proof.* To see that $\mathfrak{A}$ satisfies condition $(\star)$, consider the diagram in Definition 4.2.3. The arc $(a,c)$ is in ort ort $\mathfrak{A}$, otherwise there would have to be an arc $(m,n)$ in ort $\mathfrak{A}$ which crossed $(a,c)$, but the arc $(m,n)$ cannot be in ort $\mathfrak{A}$ since it crosses either the arc $(a,b)$ or the arc $(c,d)$ in $\mathfrak{A}$. Therefore the arc $(a,c)$ is in ort ort $\mathfrak{A} = \mathfrak{A}$. The rest is similar.

To see that $\mathfrak{A}$ satisfies condition $(\star\star)$, consider the diagram in Definition 4.2.5. Suppose the arc $(a,b)$ is not in ort ort $\mathfrak{A}$. Then there is an arc $(m,n)$ in ort $\mathfrak{A}$ which crosses $(a,b)$, that is, $m < a < n < b$ or $a < m < b < n$. For the first case the arc $(m,n)$ cannot however be in ort $\mathfrak{A}$, since there is always an arc $(q,a)$ with $q < m$ in $\mathfrak{A}$ which crosses $(m,n)$, and similarly for the second case. Therefore the arc $(a,b)$ is in ort ort $\mathfrak{A} = \mathfrak{A}$. $\square$

**Lemma 4.4.5.** *Suppose $\mathfrak{A}$ satisfies condition $(\star)$ and suppose there are only finitely many (but not zero) arcs in $\mathfrak{A}$ that end in $a$. Suppose there are both arcs going to the left and arcs going to the right from $a$. If $(p,a)$ is the longest arc in $\mathfrak{A}$ going to the left from $a$ and $(a,q)$ is the longest arc in $\mathfrak{A}$ going to the right from $a$, then $(p,q)$ is an arc in* ort $\mathfrak{A}$.

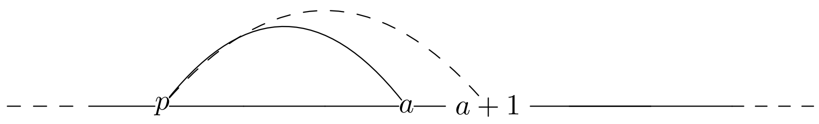



*Proof.* There are no arcs $(m,n)$ in $\mathfrak{A}$ with $m < p$, $p < n < a$, otherwise $(m,a)$ would be in $\mathfrak{A}$ by condition $(\star)$, contradicting that $(p,a)$ is the longest arc in $\mathfrak{A}$ going to the left from $a$. There are also no arcs $(m,n)$ in $\mathfrak{A}$ with $p < m < a$, $q < n$, otherwise $(a,n)$ would be in $\mathfrak{A}$ by condition $(\star)$, contradicting that $(a,q)$ is the longest arc in $\mathfrak{A}$ going to the right from $a$.

Similarly, there are no arcs $(m,n)$ in $\mathfrak{A}$ with $a < m < q$, $q < n$, and there are also no arcs $(m,n)$ in $\mathfrak{A}$ with $m < p$, $a < n < q$. By construction there are no arcs $(m,a)$ in $\mathfrak{A}$ with $m < p$, and there are also no arcs $(a,n)$ in $\mathfrak{A}$ with $q < n$.

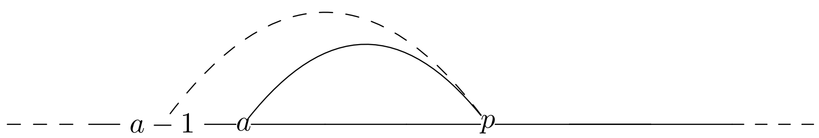
Combining all these shows that there are no arcs in $\mathfrak{A}$ crossing $(p,q)$ so $(p,q)$ has to be in ort $\mathfrak{A}$. $\square$

**Lemma 4.4.6.** *Suppose $\mathfrak{A}$ satisfies condition $(\star)$ and suppose there are only finitely many (but not zero) arcs in $\mathfrak{A}$ that end in $a$. Suppose there are only arcs going to the left from $a$. If $(p,a)$ is the longest arc in $\mathfrak{A}$ going to the left from $a$, then $(p,a+1)$ is in* ort $\mathfrak{A}$.




*Proof.* There are no arcs $(m, n)$ in $\mathfrak{A}$ with $m < p$, $p < n < a$, otherwise $(m, a)$ would be in $\mathfrak{A}$ by condition $(\star)$, contradicting that $(p, a)$ is the longest arc in $\mathfrak{A}$ going to the left from $a$. There are also no arcs $(m, n)$ in $\mathfrak{A}$ with $p < m < a$, $n > a + 1$, otherwise $(a, n)$ would be in $\mathfrak{A}$ by condition $(\star)$, contradicting that there are no arcs in $\mathfrak{A}$ going to the right from $a$. By construction there are no arcs $(m, a)$ in $\mathfrak{A}$ with $m < p$ and it is a condition that there are no arcs $(a, n)$ in $\mathfrak{A}$. Therefore $(p, a+1)$ has to be in $\operatorname{ort} \mathfrak{A}$. $\square$

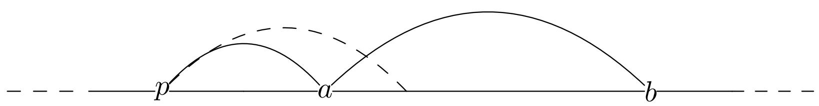
**Lemma 4.4.7.** *Suppose $\mathfrak{A}$ satisfies condition $(\star)$ and suppose there are only finitely many (but not zero) arcs in $\mathfrak{A}$ that end in $a$. Suppose there are only arcs going to the right from $a$. If $(a, p)$ is the longest arc in $\mathfrak{A}$ going to the right from $a$, then $(a - 1, p)$ is in* $\operatorname{ort} \mathfrak{A}$.




*Proof.* Similar to Lemma 4.4.6. $\square$

*Remark* 4.4.8. Suppose $(a, b)$ is an arc in $\operatorname{ort}^2(\mathfrak{A})$. Then there has to be some arc in $\mathfrak{A}$ which ends in $a$ (resp. $b$). Otherwise the arc $(a - 1, a + 1)$ (resp. $(b-1, b+1)$) is in $\operatorname{ort} \mathfrak{A}$, which is a contradiction since it crosses $(a, b)$ in $\operatorname{ort}^2(\mathfrak{A})$.

**Corollary 4.4.9.** *Suppose $\mathfrak{A}$ satisfies condition $(\star)$. Suppose that $(a, b)$ is an arc in* $\operatorname{ort}^2(\mathfrak{A})$ *and that there are only finitely many arcs in $\mathfrak{A}$ that end in $a$. Then it is not possible that all arcs in $\mathfrak{A}$ that end in $a$ are of the form $(m, a)$.*

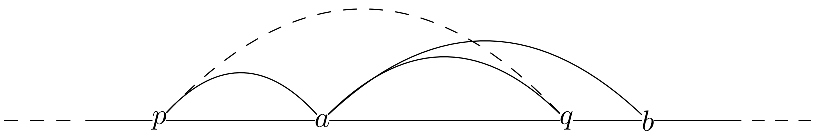



*Proof.* Suppose all arcs in $\mathfrak{A}$ that end in $a$ were of the form $(m, a)$ and let $(p, a)$ be the longest one. Then by Lemma 4.4.6 the arc $(p, a + 1)$ is in ort $\mathfrak{A}$, but this is a contradiction since it crosses $(a, b)$ which is in $\operatorname{ort}^2(\mathfrak{A})$. $\square$

**Corollary 4.4.10.** *Suppose $\mathfrak{A}$ satisfies condition ($\star$). Let $(a, b)$ be an arc in $\operatorname{ort}^2(\mathfrak{A})$ and suppose there are only finitely many arcs in $\mathfrak{A}$ that end in $a$. Then there is an arc $(a, m)$ in $\mathfrak{A}$ with $b \leq m$.*

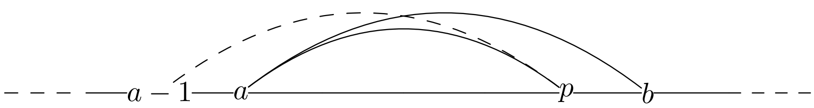
*Proof.* By Corollary 4.4.9, it is not possible that all arcs in $\mathfrak{A}$ that end in $a$ are of the form $(m, a)$. Therefore only the following two cases remain.

(i) Suppose there are arcs in $\mathfrak{A}$ going to the left and going to the right from $a$. Let $(a, q)$ be the longest arc in $\mathfrak{A}$ going to the right from $a$ and $(p, a)$ be the longest arc in $\mathfrak{A}$ going to the left from $a$. By Lemma 4.4.5, $(p, q)$ is in $\operatorname{ort}\mathfrak{A}$, so $q < b$ is not possible since then $(p, q)$ would cross $(a, b)$ in $\operatorname{ort}^2(\mathfrak{A})$.




Therefore $b \leq q$.

(ii) Suppose there are only arcs in $\mathfrak{A}$ going to the right from $a$. Let $(a, p)$ be the longest arc in $\mathfrak{A}$ going to the right from $a$. By Lemma 4.4.7, $(a-1, p)$ is in $\operatorname{ort}\mathfrak{A}$, so $p < b$ is not possible since then $(a-1, p)$ would cross $(a, b)$ in $\operatorname{ort}^2(\mathfrak{A})$.




Therefore $b \leq p$.

□

**Corollary 4.4.11.** *Suppose $\mathfrak{A}$ satisfies condition ($\star$). Suppose that $(a, b)$ is an arc in $\operatorname{ort}^2(\mathfrak{A})$ and that there are only finitely many arcs in $\mathfrak{A}$ that end in $b$. Then it is not possible that all arcs in $\mathfrak{A}$ that end in $b$ are of the form $(b, m)$.*

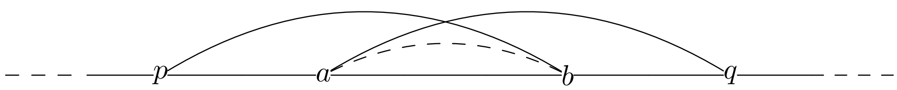
*Proof.* Similar to Corollary 4.4.9. □

**Corollary 4.4.12.** *Suppose $\mathfrak{A}$ satisfies condition ($\star$). Let $(a, b)$ be an arc in $\operatorname{ort}^2(\mathfrak{A})$ and suppose there are only finitely many arcs in $\mathfrak{A}$ that end in $b$. Then there is an arc $(m, b)$ in $\mathfrak{A}$ with $m \leq a$.*

*Proof.* Similar to Corollary 4.4.10. □

**Lemma 4.4.13.** *Suppose $\mathfrak{A}$ satisfies condition $(\star)$. Let $(a, b)$ be an arc in $\mathrm{ort}^2(\mathfrak{A})$ and suppose that each of $a$ and $b$ is only an end point of finitely many arcs in $\mathfrak{A}$. Then $(a, b)$ is in $\mathfrak{A}$.*

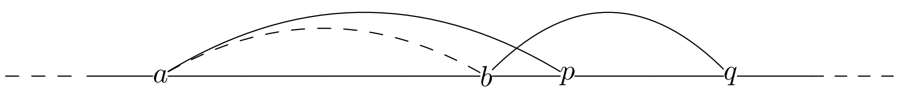
*Proof.* By Corollary 4.4.10, there is an arc $(a, q)$ in $\mathfrak{A}$ with $b \leq q$. On the other hand by Corollary 4.4.12, there is an arc $(p, b)$ in $\mathfrak{A}$ with $p \leq a$. If $q = b$ or $p = a$ already, then this is what is to be shown. Assume otherwise that $q > b$ and $p < a$. But then $(a, b)$ is in $\mathfrak{A}$ by condition $(\star)$.




□

**Lemma 4.4.14.** *Suppose $\mathfrak{A}$ satisfies condition $(\star)$. Let both $a$ and $b$ be right (resp. left) fountains of $\mathfrak{A}$ with $b - a \geq 2$. Then $(a, b)$ is in $\mathfrak{A}$.*

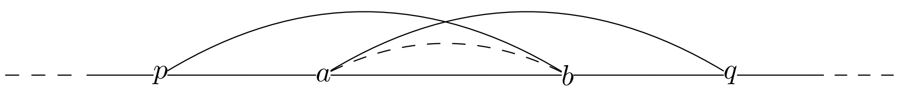
*Proof.* Suppose both $a$ and $b$ are right fountains of $\mathfrak{A}$. Choose an arc $(a, p)$ in $\mathfrak{A}$ with $b < p$ and then choose an arc $(b, q)$ in $\mathfrak{A}$ with $p < q$. Then $(a, b)$ is in $\mathfrak{A}$ by condition $(\star)$.




The other case is similar. □

**Lemma 4.4.15.** *Suppose $\mathfrak{A}$ satisfies condition $(\star)$. Let $a$ be a right fountain of $\mathfrak{A}$, $b$ be a left fountain of $\mathfrak{A}$ with $b - a \geq 2$. Then $(a, b)$ is in $\mathfrak{A}$.*

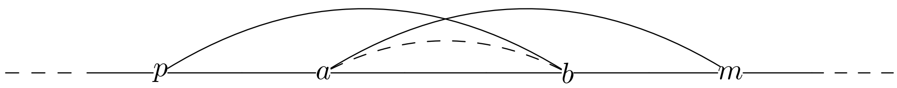
*Proof.* Choose an arc $(a, q)$ in $\mathfrak{A}$ with $b < q$ and then choose an arc $(p, b)$ in $\mathfrak{A}$ with $p < a$. Then $(a, b)$ is in $\mathfrak{A}$ by condition $(\star)$.




□

**Lemma 4.4.16.** *Suppose $\mathfrak{A}$ satisfies condition $(\star)$. Let $(a, b)$ be an arc in $\mathrm{ort}^2(\mathfrak{A})$. Suppose $b$ is a left fountain of $\mathfrak{A}$ and suppose there are only finitely many arcs in $\mathfrak{A}$ that end in $a$. Then $(a, b)$ is in $\mathfrak{A}$.*

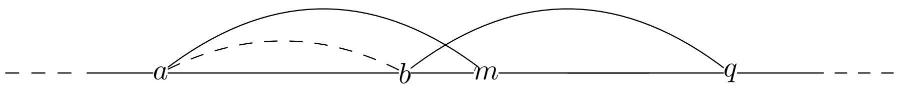
*Proof.* By Corollary 4.4.10, there is an arc $(a, m)$ in $\mathfrak{A}$ with $b \leq m$. If $m = b$ already, then this is what is to be shown. Otherwise choose an arc $(p, b)$ in $\mathfrak{A}$ with $p < a$. Then $(a, b)$ is in $\mathfrak{A}$ by condition $(\star)$.




□

**Lemma 4.4.17.** *Suppose $\mathfrak{A}$ satisfies condition $(\star)$. Let $(a, b)$ be an arc in* $\mathrm{ort}^2(\mathfrak{A})$. *Suppose $b$ is a right fountain of $\mathfrak{A}$ and suppose there are only finitely many arcs in $\mathfrak{A}$ that end in $a$. Then $(a, b)$ is in $\mathfrak{A}$.*

*Proof.* By Corollary 4.4.10, there is an arc $(a, m)$ in $\mathfrak{A}$ with $b \leq m$. If $m = b$ already, then this is what is to be shown. Otherwise choose an arc $(b, q)$ in $\mathfrak{A}$ with $m < q$. Then $(a, b)$ is in $\mathfrak{A}$ by condition $(\star)$.




□

**Lemma 4.4.18.** *Suppose $\mathfrak{A}$ satisfies condition $(\star)$. Let $(a, b)$ be an arc in* $\mathrm{ort}^2(\mathfrak{A})$. *Suppose $a$ is a right fountain of $\mathfrak{A}$ and suppose there are only finitely many arcs in $\mathfrak{A}$ that end in $b$. Then $(a, b)$ is in $\mathfrak{A}$.*

*Proof.* Similar to Lemma 4.4.16. □

**Lemma 4.4.19.** *Suppose $\mathfrak{A}$ satisfies condition $(\star)$. Let $(a, b)$ be an arc in* $\mathrm{ort}^2(\mathfrak{A})$. *Suppose $a$ is a left fountain of $\mathfrak{A}$ and suppose there are only finitely many arcs in $\mathfrak{A}$ that end in $b$. Then $(a, b)$ is in $\mathfrak{A}$.*

*Proof.* Similar to Lemma 4.4.17. □

Finally, we deliver the following lemma which is a recollection of the above lemmas.

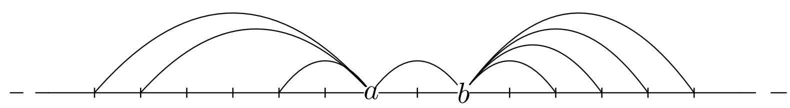
**Lemma 4.4.20.** *(c.f. Lemma 4.2.17)* $\mathrm{ort}\,\mathrm{ort}\,\mathfrak{A} = \mathfrak{A}$ *if and only if* $\mathfrak{A}$ *satisfies conditions $(\star)$ and $(\star\star)$.*

*Proof.* (*only if*) This is Lemma 4.4.4. (*if*) It is immediate that $\mathfrak{A} \subseteq \operatorname{ort}\operatorname{ort}\mathfrak{A}$. Let $(a,b)$ be an arc in $\operatorname{ort}^2(\mathfrak{A})$. Suppose that each of $a$ and $b$ is only an end point of finitely many arcs in $\mathfrak{A}$. Then $(a,b)$ is in $\mathfrak{A}$ by Lemma 4.4.13. Otherwise suppose that both $a$ and $b$ are end points of infinitely many arcs in $\mathfrak{A}$. Then $(a,b)$ is in $\mathfrak{A}$ by Lemma 4.4.14, Lemma 4.4.15 and condition $(\star\star)$. Finally, suppose that precisely one of $a$ and $b$ is an end point of finitely many arcs in $\mathfrak{A}$. Then $(a,b)$ is in $\mathfrak{A}$ by Lemma 4.4.16, Lemma 4.4.17, Lemma 4.4.18 and Lemma 4.4.19. $\square$

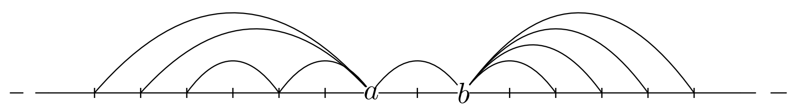
Unlike Lemma 4.4.3, Lemma 4.4.20 expresses explicitly $\operatorname{ort}\operatorname{ort}\mathfrak{A} = \mathfrak{A}$ in terms of the configuration of $\mathfrak{A}$. It can also be seen directly that the condition in Lemma 4.4.3 does imply conditions $(\star)$ and $(\star\star)$.

*Example* 4.4.21. (i) Let $\mathfrak{U}$ be a set of arcs. By Lemma 4.2.9, $\operatorname{ort}\operatorname{ort}\operatorname{ort}(\mathfrak{U}) = \operatorname{ort}(\mathfrak{U})$. Therefore $\operatorname{ort}(\mathfrak{U})$ satisfies conditions $(\star)$ and $(\star\star)$ by Lemma 4.4.20. The same is true for $\operatorname{ort}^{\mathrm{n}}(\mathfrak{U})$, $n \geq 1$.

(ii) Suppose $\mathfrak{A}$ is a set of non-crossing arcs where $\operatorname{ort}\operatorname{ort}\mathfrak{A} = \mathfrak{A}$. Then it is not necessary that $\mathfrak{A}$ is a maximal set of non-crossing arcs. For example, let $\mathfrak{A}$ consist only of a left fountain $a$, a right fountain $b$ and the arc $(a,b)$. The following diagram shows only some of the arcs from the configuration described.




By Lemma 4.4.20, $\operatorname{ort}\operatorname{ort}\mathfrak{A} = \mathfrak{A}$. However, it is certainly not a maximal set of non-crossing arcs. For example, it is possible to add an arc as in the following diagram.




Now we are ready to deliver the main theorem of this chapter.

**Theorem 4.4.22.** *Let $\mathcal{X}$ be a subcategory of $\mathcal{D}$ and let $\mathfrak{X}$ be the corresponding set of arcs. Then the following conditions are equivalent.*

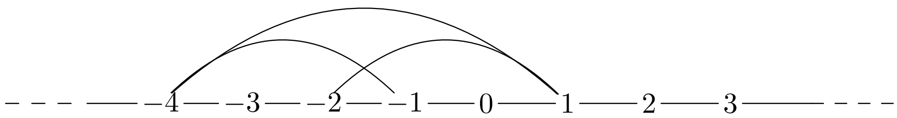
*(i) $\mathfrak{X}$ satisfies conditions $(\star)$ and $(\star\star)$, and each right fountain of $\mathfrak{X}$ is in fact a fountain,*

*(ii) The subcategory $\mathcal{X}$ is precovering and is closed under extensions,*

*(iii)* $(\mathcal{X}, \mathcal{Y})$ *is a torsion theory for some subcategory* $\mathcal{Y}$ *of* $\mathcal{D}$*. In particular,* $\mathcal{Y} = \mathcal{X}^{\perp}$.

*Proof.* (i) $\Rightarrow$ (ii): $\mathfrak{X}$ satisfying conditions $(\star)$ and $(\star\star)$ implies $\mathcal{X}$ being closed under extensions by Lemma 4.2.17 and Lemma 4.4.20 (as well as Lemma 1.2.19). Finally, $\mathcal{X}$ is precovering if and only if each right fountain of $\mathfrak{X}$ is in fact a fountain by Theorem 4.3.1. (ii) $\Leftrightarrow$ (iii): This is true by [21, Proposition 2.3]. (iii) $\Rightarrow$ (i): Since $(\mathcal{X}, \mathcal{Y})$ is a torsion theory, $\mathcal{X} = {}^{\perp}(\mathcal{X}^{\perp})$ (Lemma 1.2.7) and $\mathcal{X}$ is precovering. Therefore $\mathfrak{X}$ satisfies conditions $(\star)$ and $(\star\star)$ by Lemma 4.2.17 and Lemma 4.4.20 and each right fountain of $\mathfrak{X}$ is in fact a fountain by Theorem 4.3.1. $\square$

*Example* 4.4.23. *(c.f. Example 4.4.21(ii))* In Theorem 4.4.22, if $\mathcal{X}$ is only closed under extensions, then $\mathfrak{X}$ does not necessarily satisfy condition $(\star)$. For example, by Lemma 3.3.18, $(-4,-1) \to (-4,1) \to (-2,1) \to$ is a distinguished triangle. Suppose the arcs $(-4,-1)$, $(-4,1)$ and $(-2,1)$ are all in $\mathfrak{X}$. This however does not imply the arcs $(-4,-2)$ and $(-1,1)$ in $\mathfrak{X}$, which is required by condition $(\star)$.




*Example* 4.4.24. Let $\mathcal{U}$ be a cluster tilting subcategory with the corresponding set of arcs $\mathfrak{U}$. Then by definition $(\mathcal{U}, \Sigma\mathcal{U})$ is a torsion theory. Therefore by Theorem 4.4.22, $\mathfrak{U}$ satisfies conditions $(\star)$ and $(\star\star)$, and each right fountain of $\mathfrak{U}$ is in fact a fountain. This is compatible with the description given in Theorem 4.4.2, and is altogether a generalization of it.

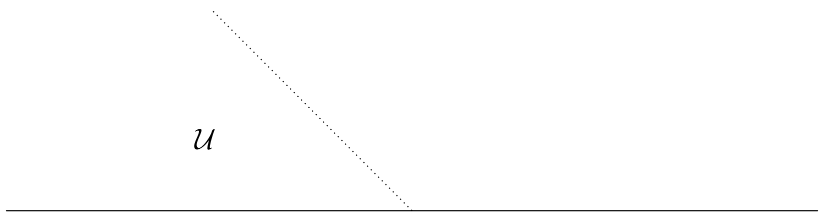
*Remark* 4.4.25. This is a little digression. The way torsion theories generalize cluster tilting subcategories reminds us of the way distinguished triangles generalize Auslander-Reiten triangles. It is good mathematics to consider (different) special cases of the given notions, and then unveil the special relationships hidden not because they are particularly enigmatic but that coincidences are simply nature's camouflage.

## 4.5 Examples

In this section, two special types of torsion theories in $\mathcal{D}$ are described, those of t-structures and co-t-structures. Given a subcategory $\mathcal{U}$ of $\mathcal{D}$, examples where the pair $(\mathcal{U}, \mathcal{U}^{\perp})$ might or might not be a torsion theory are given. In the context of a t-structure, an example is given to illustrate the torsion theory triangles explicitly on the Auslander-Reiten quiver.

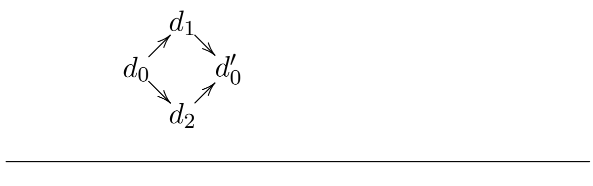
Let us first describe t-structures.

**Theorem 4.5.1.** *Let $(\mathcal{U}, \mathcal{V})$ be a t-structure in $\mathcal{D}$, that is, $(\mathcal{U}, \mathcal{V})$ is a torsion theory with $\Sigma\mathcal{U} \subseteq \mathcal{U}$. Suppose $\mathcal{U}$ is neither zero nor all of $\mathcal{D}$, then there is a half line such that the indecomposable objects of $\mathcal{U}$ are precisely the objects on the half line and to the left of it, as shown in the following diagram.*



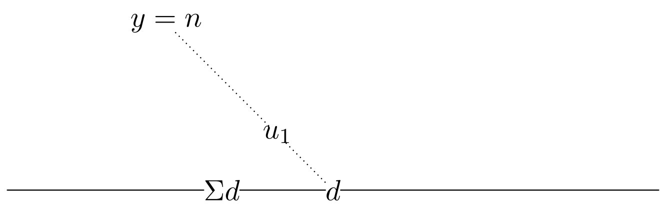

*Proof.* Let $\mathfrak{U}$ be the corresponding set of arcs for the subcategory $\mathcal{U}$.

(*Step 1*) Consider a horizontal line $y - x = k$ with $k \geq 3$ in the Auslander-Reiten quiver of $\mathcal{D}$. If there are objects from $\mathcal{U}$ on this line, then there is a rightmost such object. Namely, suppose not. Then there are objects of $\mathcal{U}$ arbitrarily far to the right on $y - x = k$, so all objects on $y - x = k$ are in $\mathcal{U}$ because $\Sigma\mathcal{U} \subseteq \mathcal{U}$. In the following diagram, let $d_1 = (u_1, u_2)$ and $d_2 = (u_1 + 1, u_2 - 1)$ be objects on the lines $y - x = k + 1$ and $y - x = k - 1$ respectively. Then there is the Auslander-Reiten triangle $d_0 \to d_1 \oplus d_2 \to d_0' \to$, where $d_0 = (u_1, u_2 - 1)$ and $d_0' = (u_1 + 1, u_2)$. Since $d_0$ and $d_0'$ both lie on the line $y - x = k$ which is in $\mathcal{U}$, it follows that $d_1$ and $d_2$ are in $\mathcal{U}$, since $\mathcal{U}$ is closed under extensions and direct summands. Therefore the two neighbouring lines $y - x = k + 1$ and $y - x = k - 1$ are in $\mathcal{U}$. Repeating the argument for other (horizontal) lines, $\mathcal{U}$ has to contain all the indecomposable objects of $\mathcal{D}$, i.e. $\mathcal{U}$ has to be all of $\mathcal{D}$. The case where $k = 2$ is similar.

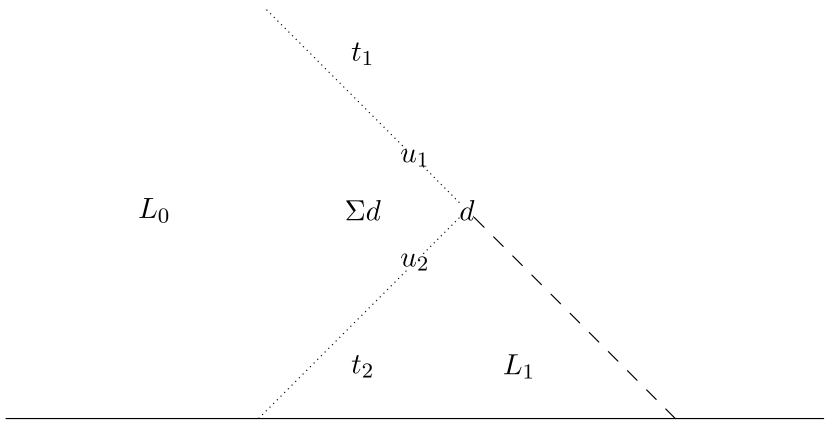



(*Step 2*) Pick an object $d = (m, n)$ in $\mathcal{U}$ and assume it is the rightmost object of $\mathcal{U}$ on the horizontal line $y - x = n - m$. Here it will be shown that the region $L = \{(x, y) | y \leq n, y - x \geq 2\}$ is in $\mathcal{U}$.

(i) Suppose $n - m = 2$. In the following diagram, let $u_1 = (m - 1, n)$. Then there is the Auslander-Reiten triangle $\Sigma d \to u_1 \to d \to$. Since $\Sigma\mathcal{U} \subseteq \mathcal{U}$, therefore $\Sigma d$ is in $\mathcal{U}$. Hence $u_1$ is in $\mathcal{U}$, since $\mathcal{U}$ is closed under extensions. By applying a (similar) argument on $u_1$ and so on, the half line $y = n$ is in $\mathcal{U}$, and so are all the half lines $y = n'$ with $n' \leq n$, since $\Sigma\mathcal{U} \subseteq \mathcal{U}$. Therefore the region $L$ is in $\mathcal{U}$.

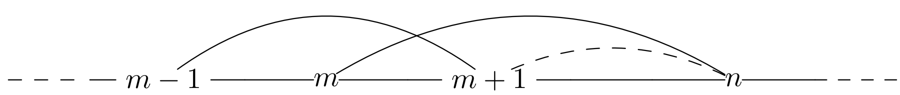



(ii) Suppose $n - m > 2$. Since $\Sigma\mathcal{U} \subseteq \mathcal{U}$, therefore $\Sigma d = (m-1, n-1)$ is in $\mathcal{U}$. In the following diagram, let $u_1 = (m-1, n)$ and $u_2 = (m, n-1)$. Then there is the Auslander-Reiten triangle $\Sigma d \to u_1 \oplus u_2 \to d \to$. Therefore $u_1$ and $u_2$ are both in $\mathcal{U}$, since $\mathcal{U}$ is closed under extensions. By applying a (similar) argument on $u_1$ and $u_2$ and so on (if possible), the two half lines $t_1$ and $t_2$ are in $\mathcal{U}$. Eventually, the region labelled $L_0$ (including the two half lines $t_1$ and $t_2$) is in $\mathcal{U}$, since $\Sigma\mathcal{U} \subseteq \mathcal{U}$.

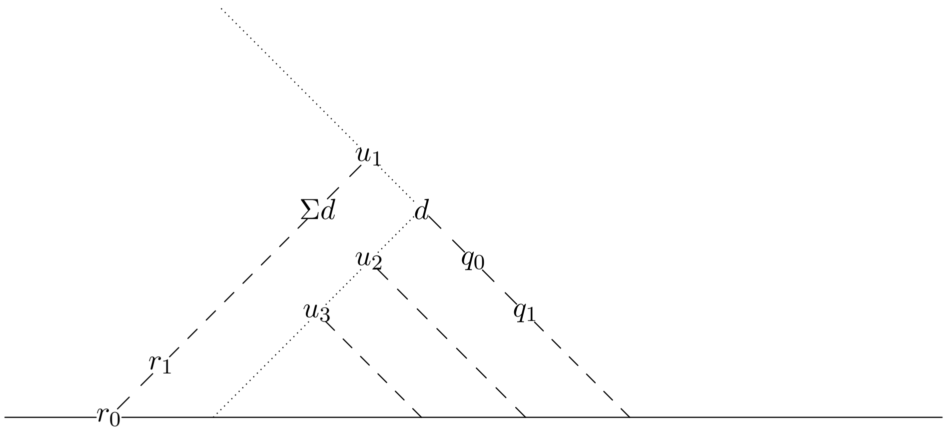



Now it remains to show that the little triangular region, $L_1 = \{(x, y) \mid m+1 \leq x \leq n-2, m+3 \leq y \leq n \text{ and } y - x \geq 2\}$, is also in $\mathcal{U}$.

Let $r_0 = (m-1, m+1)$. Since the arcs $r_0$ and $d = (m, n)$ cross, it follows that $q_0 = (m+1, n)$ is in $\mathcal{U}$ since $\mathfrak{U}$ satisfies condition $(\star)$ by Theorem 4.4.22.




Similarly, with the help of $r_1 = (m-1, m+2)$, it follows that $q_1 = (m+2, n)$ is in $\mathcal{U}$, and so on (if possible), until all the objects $(m', n)$, with $m+1 \le m' \le n-2$, are in $\mathcal{U}$. This argument is to be repeated, starting with $u_2 = (m, n-1)$, $u_3 = (m, n-2)$ and so on instead (if possible), until the region $L_1$ is in $\mathcal{U}$. Therefore $L = L_0 \cup L_1$ is in $\mathcal{U}$, and this is what is needed to be shown.

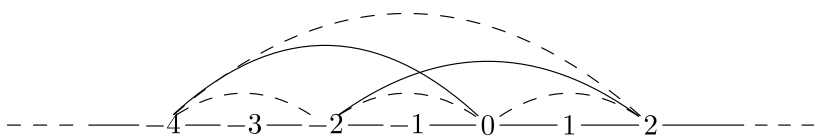



(*Step 3*) Suppose there is an indecomposable object $d' = (u, v)$ of $\mathcal{U}$ which lies outside the region $L$, i.e. $n < v$. Similar to Step 2, all the half lines $y = v'$ with $v' \le v$ will be in $\mathcal{U}$. Therefore the indecomposable object $\Sigma^{-1}d = (m+1, n+1)$ will also be in $\mathcal{U}$, contradicting that $d$ is the indecomposable object of $\mathcal{U}$ which is rightmost on the line $y - x = n - m$. Therefore the indecomposable objects of $\mathcal{U}$ are precisely the objects on the half line $y = n$ and to the left of it, i.e. the region $L$. $\square$

*Remark* 4.5.2. Consider $\mathcal{U}$ and $\mathfrak{U}$ in Theorem 4.5.1. Other possible choices of crossing of arcs within $\mathfrak{U}$ would not give us any more new arcs not contained in $\mathfrak{U}$ already (Example 4.2.4 and Remark 4.2.14).

Alternatively, one can conceive the set of arcs $\mathfrak{U}$ as a sequence of left full fountains going to the left. This means there is an integer $p$ such that $q$ is a left full fountain of $\mathfrak{U}$ for all $q \le p$ and there are no left (full) fountains $q > p$ of $\mathfrak{U}$ ($p$ is the rightmost left full fountain of $\mathfrak{U}$). Any arcs $(a, b)$ induced by crossing of arcs within $\mathfrak{U}$ (those arcs that are to be included under condition $(\star)$) are in $\mathfrak{U}$ already (i.e. $b$ has to be a left full fountain of $\mathfrak{U}$ already). Therefore the set of arcs $\mathfrak{U}$ in Theorem 4.5.1 does satisfy condition $(\star)$.

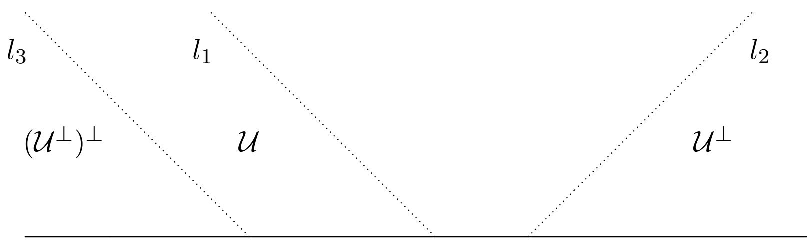
For example, let $p = 2$. The following diagram shows, amongst other arcs of $\mathfrak{U}$, the arcs $(-4, 0)$ and $(-2, 2)$ and they cross.



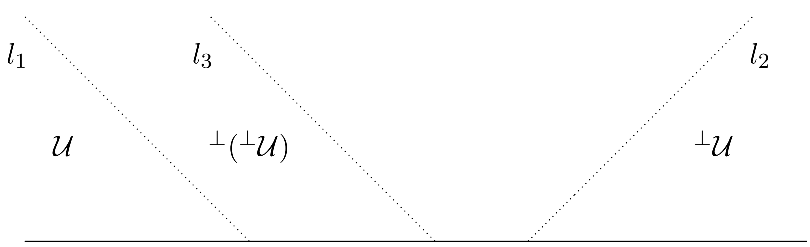
The arc $(0,2)$ has to be in $\mathfrak{U}$ since 2 is a left full fountain, and similarly the arcs $(-2,0)$, $(-4,-2)$ and $(-4,2)$ have to be in $\mathfrak{U}$, since 0, $-2$ and 2 are all left full fountains.

*Example* 4.5.3. Let $\mathcal{U}$ be a subcategory of $\mathcal{D}$. Lemma 4.2.15 is needed for the following.

(i) In the following diagram, suppose the indecomposable objects of $\mathcal{U}$ are precisely the objects on the dotted half line $l_1 : y = m$ (some $m$) and to the left of it. Then the indecomposable objects of $\mathcal{U}^\perp$ are precisely the objects on the dotted half line $l_2 : x = m-1$ and to the right of it, and the indecomposable objects of $(\mathcal{U}^\perp)^\perp$ are precisely the objects on the dotted half line $l_3 : y = m-2$ and to the left of it.




(ii) In the following diagram, suppose the indecomposable objects of $\mathcal{U}$ are precisely the objects on the dotted half line $l_1 : y = m$ (some $m$) and to the left of it. Then the indecomposable objects of ${}^\perp\mathcal{U}$ are precisely the objects on the dotted half line $l_2 : x = m+1$ and to the right of it, and the indecomposable objects of ${}^\perp({}^\perp\mathcal{U})$ are precisely the objects on the dotted half line $l_3 : y = m+2$ and to the left of it.

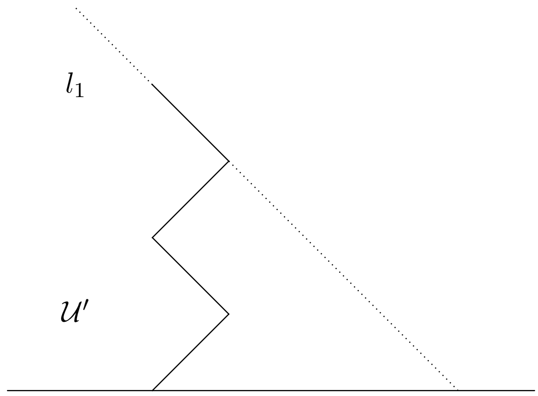



Here it is true that $\mathcal{U} = {}^\perp(\mathcal{U}^\perp)$ and that $\mathcal{U} = ({}^\perp\mathcal{U})^\perp$.

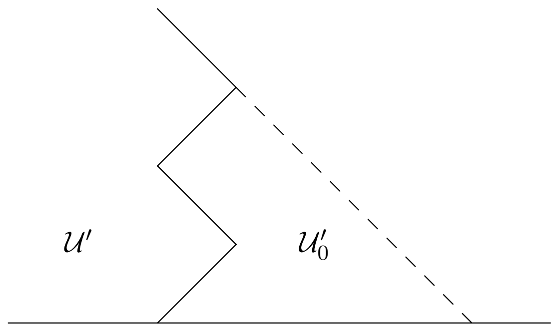
Let $\mathcal{U}$ be as described in Theorem 4.5.1. Then all the rightmost objects of $\mathcal{U}$ on their respective horizontal lines form a straight half line. This is profoundly different from Corollary 1.4.7, where given a split torsion theory

$(\mathcal{X}, \mathcal{Y})$ in the finite derived category $D^b(\operatorname{mod} kA_n)$, the leftmost objects $a_k$ (on their respective lines $y - x = k$) which are in $\mathcal{X}$ form a zig zag $Z$. It would not be true if the straight half line in Theorem 4.5.1 were replaced by a half zig zag line. One example is as follows.

*Example* 4.5.4. Let $\mathcal{U}'$ be a subcategory of $\mathcal{D}$. Suppose there is a half zig zag line such that the indecomposable objects of $\mathcal{U}'$ are precisely the objects on the half zig zag line and to the left of it, as shown in the following diagram. Here the solid line segment on the top keeps on going to the left. The dotted half line is $l_1 : y = m$ (some $m$).




In the following diagram, let $\mathcal{U}'_0$ be the region outside $\mathcal{U}'$ bounded by and including the dotted line.



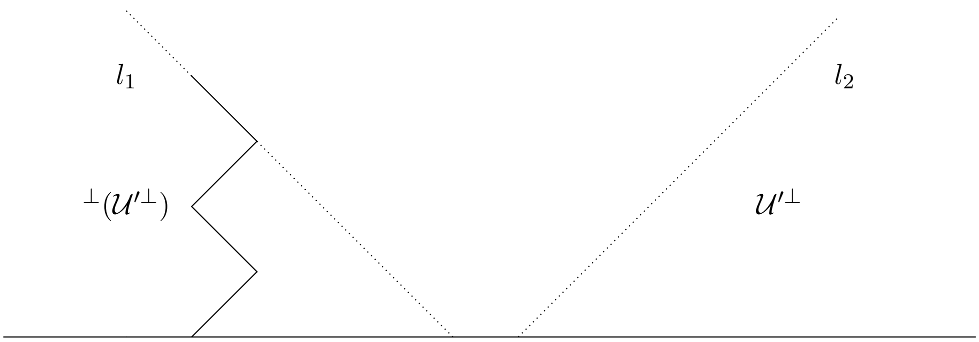

Let $\mathfrak{U}'$ be the corresponding set of arcs for the subcategory $\mathcal{U}'$.

Either interpretation, as arcs or as indecomposable objects of $\mathcal{D}$ on the quiver, gives that $\mathcal{U}'$ is not equal to ${}^{\perp}(\mathcal{U}'^{\perp})$. This is described as follows.

In terms of arcs, follow (with variations) Step 2 of the proof of Theorem 4.5.1 (since $\mathcal{U}' = {}^{\perp}(\mathcal{U}'^{\perp})$ is equivalent to $\mathfrak{U}'$ satisfying conditions $(\star)$ and $(\star\star)$ by Lemma 4.2.17 and Lemma 4.4.20, to have $\mathcal{U}' = {}^{\perp}(\mathcal{U}'^{\perp})$ is to have the region

$\mathcal{U}'_0$ in $\mathcal{U}'$ as well, similar to the way the little triangular region $L_1$ is to be in $\mathcal{U}$ in Theorem 4.5.1).

In terms of indecomposable objects of $\mathcal{D}$ on the quiver, this is simply by inspection. The regions $\mathcal{U}'^{\perp}$ and ${}^{\perp}(\mathcal{U}'^{\perp})$ are as follows, i.e. the indecomposable objects of $\mathcal{U}'^{\perp}$ are precisely the objects on the dotted half line $l_2 : x = m-1$ and to the right of it, and the indecomposable objects of ${}^{\perp}(\mathcal{U}'^{\perp})$ are precisely the objects on the dotted half line $l_1 : y = m$ and to the left of it (c.f. Example 4.5.3).

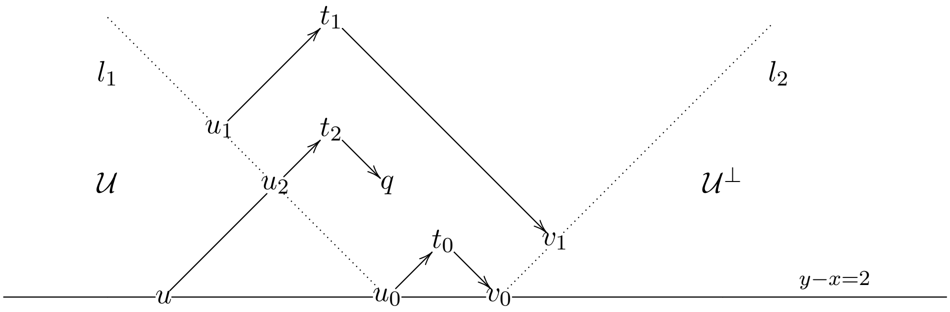



Both interpretations give $\mathcal{U}' \subset {}^{\perp}(\mathcal{U}'^{\perp})$.

*Remark* 4.5.5. In Example 4.5.4, the set of arcs $\mathfrak{U}'$ does not satisfy condition ($\star$). Therefore by Theorem 4.4.22, ($\mathcal{U}'$,$\mathcal{U}'^{\perp}$) is not a torsion theory, even though $\mathfrak{U}'$ contains no right fountains at all. This is no surprise, as $\mathcal{U}' \subset {}^{\perp}(\mathcal{U}'^{\perp})$ contradicts Lemma 1.2.7.

Given an (indecomposable) object $t$ of $\mathcal{D}$ which is neither in $\mathcal{U}'$ nor in $\mathcal{U}'^{\perp}$, there might not even be a torsion theory triangle $u' \to t \to u'' \to$, where $u'$ is in $\mathcal{U}'$ and $u''$ is in $\mathcal{U}'^{\perp}$. Nevertheless, the following example illustrates the existence of (some) torsion theory triangles in a t-structure explicitly.

*Example* 4.5.6. Consider again the t-structure $(\mathcal{U}, \mathcal{V})$ in Theorem 4.5.1, and let $\mathfrak{U}$ and $\mathfrak{V}$ be the corresponding sets of arcs for the subcategories $\mathcal{U}$ and $\mathcal{V}$ respectively. The lines $l_1 : y = m$ and $l_2 : x = m-1$, and the regions $\mathcal{U}$ and $\mathcal{U}^{\perp}$ are shown in the following diagram (Example 4.5.3(i)).




As seen in Lemma 1.2.7, $\mathcal{U}^\perp = \mathcal{V}$ and $\mathcal{U} = {}^\perp(\mathcal{U}^\perp)$. This is again no surprise (Example 4.5.3).

(i) Let $t_0 = (m-2, m+1)$. Then there is the Auslander-Reiten triangle $u_0 \to t_0 \to v_0 \to$, which is also a torsion theory triangle, where $u_0 = (m-2, m)$ is in $\mathcal{U}$ and $v_0 = (m-1, m+1)$ is in $\mathcal{U}^\perp = \mathcal{V}$.

(ii) Let $t_1 = (a_1, b_1)$ be an indecomposable object of $\mathcal{D}$ which is neither in $\mathcal{U}$ nor in $\mathcal{U}^\perp$. Then by Lemma 3.3.18, there is the distinguished triangle $u_1 \to t_1 \to v_1 \to$, which is a torsion theory triangle, where $u_1 = (a_1, m)$ is in $\mathcal{U}$ and $v_1 = (m-1, b_1)$ is in $\mathcal{U}^\perp = \mathcal{V}$.

(iii) Let $u = (m-4, m-2)$ be in $\mathcal{U}$ and let $t_2 = (m-4, m+1)$ be an indecomposable object of $\mathcal{D}$ which is neither in $\mathcal{U}$ nor in $\mathcal{U}^\perp$. Then by Lemma 3.3.18, there is the distinguished triangle $u \to t_2 \to q \to$, where $q = (m-3, m+1)$. This is however not a torsion theory triangle, since $q$ is not in $\mathcal{U}^\perp = \mathcal{V}$. But similar to (ii) there is a torsion theory triangle $u_2 \to t_2 \to v_2 \to$, where $u_2 = (m-4, m)$ is in $\mathcal{U}$ and $v_2 = v_0 = (m-1, m+1)$ is in $\mathcal{U}^\perp = \mathcal{V}$.

(iv) It can be seen directly from the diagram that $\mathcal{U}^\perp = \mathcal{V}$ is not closed under the action of $\Sigma$ (it is closed under the action of $\Sigma^{-1}$, and it is of no surprise, see Lemma 1.2.15), and it is not true that each right fountain of $\mathfrak{V}$ is in fact a fountain (it is however true that each left fountain of $\mathfrak{V}$ is in fact a fountain, though $\mathfrak{V}$ does not have any left fountains).

(v) The morphisms $u_0 \to t_0$ in (i), $u_1 \to t_1$ in (ii) and $u_2 \to t_2$ in (iii) are all $\mathcal{U}$-precovers, and the morphisms $t_0 \to v_0$ in (i), $t_1 \to v_1$ in (ii) and $t_2 \to v_2$ in (iii) are all $\mathcal{V}$-preenvelopes (Example 1.2.10(ii)). Compare Lemma 3.3.10 and Lemma 3.3.11.

(vi) This example echoes Remark 1.5.5 in Section 1.5. Since $(\mathcal{U}, \mathcal{V})$ is a t-structure, the inclusion $\imath : \mathcal{U} \hookrightarrow \mathcal{D}$ admits a right adjoint $R$ :

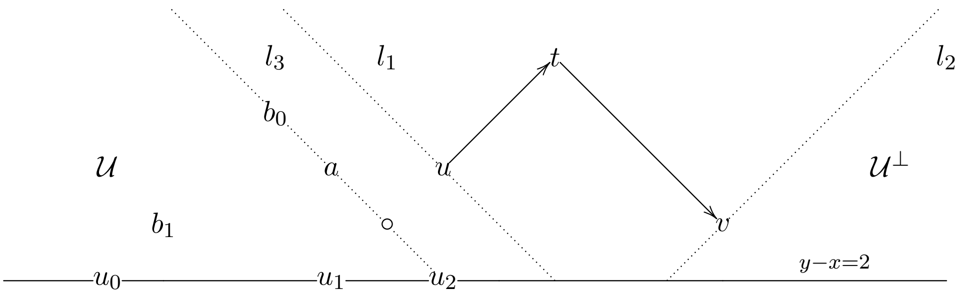
$\mathcal{D} \to \mathcal{U}$. Similarly, the inclusion $\imath : \mathcal{V} \hookrightarrow \mathcal{D}$ admits a left adjoint $L : \mathcal{D} \to \mathcal{V}$ (Example 1.2.25). Given $t$ in $\mathcal{D}$, there is a (unique) torsion theory triangle $Rt \to t \to Lt \to \Sigma Rt$ with $Rt$ in $\mathcal{U}$ and $Lt$ in $\mathcal{V}$. Accordingly, the torsion theory triangles $u_i \to t_i \to v_i \to$ in (i), (ii) and (iii) serve as examples of realizing the actions of the left and right adjoints on the Auslander-Reiten quiver. For example, in (ii), $\Sigma u_1 = \Sigma(a_1, m) = (a_1 - 1, m - 1)$ is indeed the mapping cone $c_1$ of the $\mathcal{V}$-preenvelope $t_1 \to v_1$ (Lemma A.3.7), thus the value of $Rt_1$ can be retrieved (i.e. $Rt_1 = \Sigma^{-1} c_1 = \Sigma^{-1}\Sigma u_1 = u_1$).

(vii) Simultaneously, $Rt \to t$ in (vi) is right minimal (Lemma 1.2.11), thus in (i), (ii) and (iii), $u_i \to t_i$ is a $\mathcal{U}$-cover.

The following example is a slight modification of Example 4.5.6.

*Example* 4.5.7. Let $\mathcal{U}$ be a subcategory of $\mathcal{D}$, and $\mathfrak{U}$ be the corresponding set of arcs for the subcategory $\mathcal{U}$.

In the following diagram, suppose the indecomposable objects of $\mathcal{U}$ are precisely the objects on the dotted half line $l_1 : y = m$ (some $m$) and to the left of it, except the object $\circ$. Then the indecomposable objects of $\mathcal{U}^\perp$ are precisely the objects on the dotted half line $l_2 : x = m - 1$ and to the right of it, and the indecomposable objects of ${}^\perp(\mathcal{U}^\perp)$ are precisely the objects on the dotted half line $l_1$ and to the left of it, thus including the object $\circ$ (c.f. Example 4.5.3).




Let $\circ = (m-4, m-1)$, $a = (m-5, m-1)$ be objects on the line $l_3 : y = m-1$, and $u_0 = (m-6, m-4)$, $u_1 = (m-4, m-2)$ and $u_2 = (m-3, m-1)$ be objects on the line $y - x = 2$. Even though $(\mathcal{U}, \mathcal{U}^\perp) = 0$, and as seen in Example 4.5.6(ii), that given an indecomposable object $t = (a, b)$ of $\mathcal{D}$ which is neither in $\mathcal{U}$ nor in $\mathcal{U}^\perp$ nor the object $\circ$, there is the distinguished triangle $u \to t \to v \to$ with $u = (a, m)$ in $\mathcal{U}$ and $v = (m - 1, b)$ in $\mathcal{U}^\perp$, the pair $(\mathcal{U}, \mathcal{U}^\perp)$ is not a torsion theory.

This is because there are no torsion theory triangles $\circ_u \to \circ \to \circ_v \to$ with $\circ_u$ in $\mathcal{U}$ and $\circ_v$ in $\mathcal{U}^\perp$. Assume there is. Then since the object $\circ$ is in ${}^\perp(\mathcal{U}^\perp)$, the morphism $\circ \to \circ_v$ is zero, and thus the distinguished triangle $\Sigma^{-1}\circ_v \to \circ_u \to \circ \to \circ_v$ splits, i.e. $\circ_u \cong \Sigma^{-1} \circ_v \oplus \circ$. Since $\mathcal{U}$ is closed under isomorphisms and direct summands, the object $\circ$ will have to be in $\mathcal{U}$, which is a contradiction. Therefore $(\mathcal{U},\mathcal{U}^\perp)$ is not a torsion theory.

This is again no surprise by Theorem 4.4.22, since

(i) $\mathcal{U}$ is not closed under extensions. This is because $u_1 \to \circ \to u_2 \to$ is an Auslander-Reiten triangle with $u_1$, $u_2$ in $\mathcal{U}$ but not $\circ$. Alternatively, let $b_1 = (m-6, m-3)$ and $b_0 = (m-6, m-1)$. Then by Lemma 3.3.18, $b_1 \to b_0 \to \circ \to \Sigma b_1$ is a distinguished triangle. Since both $b_0$ and $\Sigma b_1$ are in $\mathcal{U}$ but not $\circ$, this is another example where $\mathcal{U}$ fails to be closed under extensions.

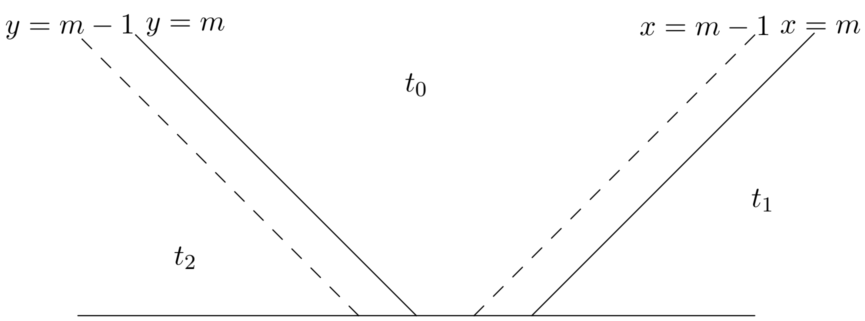
(ii) $\mathfrak{U}$ does not satisfy condition $(\star)$. Again consider the pair of crossing arcs $u_0 = (m-6, m-4)$ and $a = (m-5, m-1)$, but the arc $\circ = (m-4, m-1)$ is not in $\mathfrak{U}$.

(iii) Simply, $\mathcal{U} \subset {}^\perp(\mathcal{U}^\perp)$ (Lemma 4.2.17 and Lemma 4.4.20).

The author would like to thank Rafael Bocklandt and Vanessa Miemietz for suggesting the following example.

Let $\mathcal{U}$ be some subcategory of $\mathcal{D}$. The following gives the example where the pair $(\mathcal{U},\mathcal{U}^\perp)$ is a torsion theory but not a t-structure.

*Example* 4.5.8. Let the indecomposable objects of $\mathcal{U}$ consist precisely of the objects on the half lines $y = m$ and $x = m$. Then the indecomposable objects of $\mathcal{U}^\perp$ consist precisely of the objects on the dotted half lines $x = m-1$ and $y = m-1$ (Example 4.5.3(i)).




Then by Theorem 4.4.22, the pair $(\mathcal{U},\mathcal{U}^\perp)$ is a torsion theory, and it is readily seen from Theorem 4.5.1 that the pair is not a t-structure. The reader can also readily establish the torsion theory triangle $u_0 \to t_0 \to v_0 \to$ where $u_0$ is in $\mathcal{U}$ and $v_0$ is in $\mathcal{U}^\perp$ (c.f. Example 4.5.6). Similarly, by virtue of Lemma A.3.7, there is a torsion theory triangle $u_1 \to t_1 \to v_1 \to$ where $u_1$ is in $\mathcal{U}$ and $v_1$ is in $\mathcal{U}^\perp$, and then finally by virtue of Lemma 3.3.18, there is a torsion theory triangle $u_2 \to t_2 \to v_2 \to$ where $u_2$ is in $\mathcal{U}$ and $v_2$ is in $\mathcal{U}^\perp$.

*Remark* 4.5.9. By virtue of Remark 4.5.2, the subcategory $\mathcal{U}$ of the given form in Theorem 4.5.1 does indeed satisfy the conditions in Theorem 4.4.22 because the corresponding set of arcs $\mathfrak{U}$ contains no right fountains at all. Since such a subcategory $\mathcal{U}$ is also closed under the action of $\Sigma$, the converse of Theorem 4.5.1 has to be true.

*Remark* 4.5.10. In Example 4.5.6, let $\mathfrak{U}$ and $\mathfrak{V}$ be the corresponding sets of arcs for the subcategories $\mathcal{U}$ and $\mathcal{U}^\perp = \mathcal{V}$ respectively. Then $\mathfrak{V} = \Sigma \operatorname{ort} \mathfrak{U}$. This gives $\Sigma \operatorname{ort} \mathfrak{U} = \{\mathfrak{d} \mid \Sigma^{-1}\mathfrak{d}$ does not cross any arcs in $\mathfrak{U}\} = \{\mathfrak{d} \mid \mathfrak{d}$ does not cross any arcs in $\Sigma\mathfrak{U}\} = \operatorname{ort}(\Sigma\mathfrak{U})$. Therefore $\operatorname{ort}\operatorname{ort}(\Sigma \operatorname{ort} \mathfrak{U}) = \operatorname{ort}\operatorname{ort}\operatorname{ort}(\Sigma\mathfrak{U}) = \operatorname{ort}(\Sigma\mathfrak{U}) = \Sigma \operatorname{ort} \mathfrak{U}$ (Lemma 4.2.9). Alternatively, $\operatorname{ort}\operatorname{ort} \mathfrak{V} = \mathfrak{V}$ (Lemma 4.2.17) since $\mathcal{U}^\perp = {}^\perp((\mathcal{U}^\perp)^\perp)$ (Example 4.5.3).

On the other hand, $\mathfrak{V}$ satisfies conditions $(\star)$ and $(\star\star)$, by a mirror argument of the situation in $\mathfrak{U}$ (Remark 4.5.2 and Remark 4.5.9). This is no surprise (Lemma 4.4.20).

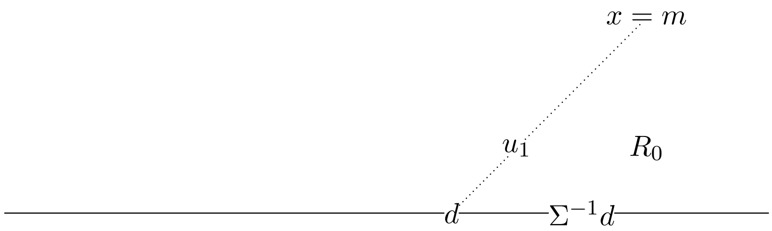
By virtue of Remark 4.5.9, there are by contrast no non-trivial co-t-structures ([10], [38]).

**Theorem 4.5.11.** *Let $(\mathcal{U}, \mathcal{V})$ be a co-t-structure in $\mathcal{D}$, that is, $(\mathcal{U}, \mathcal{V})$ is a torsion theory with $\Sigma^{-1}\mathcal{U} \subseteq \mathcal{U}$. If $\mathcal{U}$ is non-zero, then $\mathcal{U}$ has to be all of $\mathcal{D}$.*

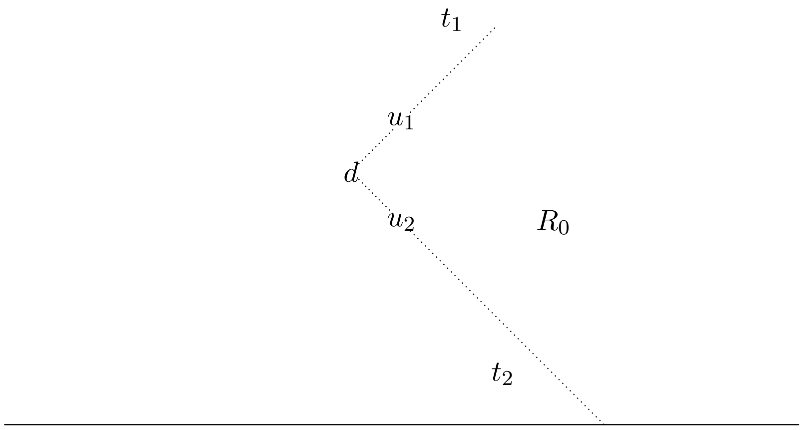
*Proof.* Let $\mathfrak{U}$ be the corresponding set of arcs for the subcategory $\mathcal{U}$. If $\mathcal{U}$ is non-zero, let $d = (m, n)$ be an indecomposable object of $\mathcal{U}$.

(*Step 1*) Here it will be shown that the region $R_0 = \{(x, y) | m \le x, n \le y$ and $y - x \ge 2\}$ is in $\mathcal{U}$.

(i) Suppose $n - m = 2$. In the following diagram, let $u_1 = (m, n+1)$. Then there is the Auslander-Reiten triangle $d \to u_1 \to \Sigma^{-1}d \to$ . Since $\Sigma^{-1}\mathcal{U} \subseteq \mathcal{U}$, therefore $\Sigma^{-1}d$ is in $\mathcal{U}$. Hence $u_1$ is in $\mathcal{U}$, since $\mathcal{U}$ is closed under extensions. By applying a similar argument on $u_1$ and so on, the half line $x = m$ is in $\mathcal{U}$, and so are all the half lines $x = m'$ with $m \le m'$, since $\Sigma^{-1}\mathcal{U} \subseteq \mathcal{U}$. Therefore the region $R_0$ is in $\mathcal{U}$.

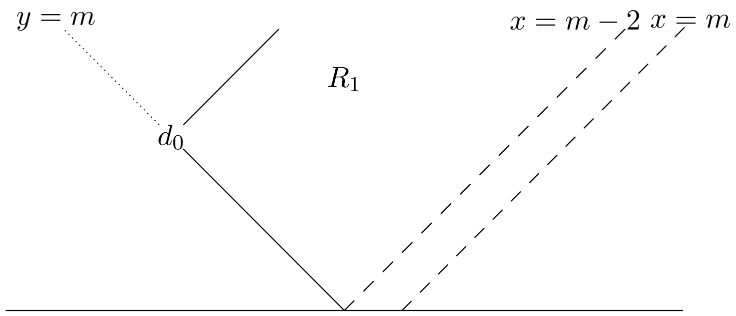



(ii) Suppose $n-m > 2$. Since $\Sigma^{-1}\mathcal{U} \subseteq \mathcal{U}$, therefore $\Sigma^{-1}d = (m+1, n+1)$ is in $\mathcal{U}$. In the following diagram, let $u_1 = (m, n+1)$ and $u_2 = (m+1, n)$. Then there is the Auslander-Reiten triangle $d \to u_1 \oplus u_2 \to \Sigma^{-1}d \to$. Therefore both $u_1$ and $u_2$ are in $\mathcal{U}$, since $\mathcal{U}$ is closed under extensions. By applying a (similar) argument on $u_1$ and $u_2$ and so on (if possible), the two half lines $t_1$ and $t_2$ are in $\mathcal{U}$. Eventually, the region $R_0$ is in $\mathcal{U}$, since $\Sigma^{-1}\mathcal{U} \subseteq \mathcal{U}$.




(*Step 2*) It follows from Step 1 that $m$ is a right fountain of $\mathfrak{U}$. By Theorem 4.4.22, it is a fountain.

In particular, choose an indecomposable object $d_0 = (p, m)$ of $\mathcal{U}$ with $m-p > 2$, as shown in the following diagram. Similar to (ii) in Step 1, the region $R_1 = \{(x, y) | p \leq x, m \leq y \text{ and } y - x \geq 2\} \supset R_0$ is in $\mathcal{U}$.




(*Step 3*) Similarly, $p$ is a right fountain of $\mathfrak{U}$, and so repeating the argument of Step 2, there is a region $R_2 \supset R_1$ in $\mathcal{U}$. Since the argument can be continued in this way indefinitely, $\mathcal{U}$ has to contain all the indecomposable objects of $\mathcal{D}$, i.e. $\mathcal{U}$ has to be all of $\mathcal{D}$.

□

At last we have arrived at the finale of the thesis. The following is an echo of Section 1.3 and Section 1.4 in Chapter 1.

*Remark* 4.5.12. Let $(\mathcal{U}, \mathcal{V})$ be a t-structure in $\mathcal{D}$.

(i) Lemma 1.3.12 and Theorem 1.4.8 together classifies t-structures in the finite derived category $D^b(\operatorname{mod} kA_n)$. Compare Theorem 4.5.1.

(ii) Remark 1.4.12 describes the condition where the notions of split torsion theories and t-structures coincide ($S\mathcal{U} \cong \mathcal{U}$). This is very far from the situation in $\mathcal{D}$ ($S = \Sigma^2$).

(iii) The distinguished triangles given in Example 4.5.6(i), (ii) and (iii) are indubitably non-split, and there are no doubt indecomposable objects $t$ in $\mathcal{D}$ which are neither in $\mathcal{U}$ nor in $\mathcal{V}$. Again $(\mathcal{U}, \mathcal{V})$ is not a split torsion theory (Lemma 1.3.10).

*Remark* 4.5.13. Let $(\mathcal{U}, \mathcal{V})$ be a co-t-structure in $\mathcal{D}$.

(i) By Theorem 1.3.5, a torsion theory in $\mathcal{D}$ is a co-t-structure if and only if it is a split torsion theory, since $\tau = S\Sigma^{-1} = \Sigma$. However, Theorem 4.5.11 differs very much from Theorem 1.4.8, where the split torsion theories in the finite derived category $D^b(\operatorname{mod} kA_n)$ are classified.

(ii) By Lemma 1.3.10 and (i), each indecomposable $d$ in $\mathcal{D}$ is either in $\mathcal{U}$ or in $\mathcal{V}$. This is vacuously true, since if $\mathcal{U}$ is non-zero, then $\mathcal{U}$ has to be all of $\mathcal{D}$ by Theorem 4.5.11.

(iii) By Lemma 1.3.12, every split torsion theory is a t-structure. Together with (i), this means $(\mathcal{U}, \mathcal{V})$ is a t-structure. However, this is again vacuously true, since there are no non-trivial co-t-structures by Theorem 4.5.11.

*"It is a land of peace," whispered Frodo, though he knew not from whence his words came. "A land of quiet, and of healing, and of bright white light. There is joy there, and there is rest."*

*"There is quiet and rest in the Shire," Merry pointed out, "or at least, there is for me."*

*"Yes," said Frodo, laughing suddenly, "and I do not want for quiet and rest while you are with me. You are the light in my darkness, the gentle strength that has carried me through the darkest of times. You are my family, and no more could I wish for until my work here is done."*

*The white gem that hung upon Frodo's neck flashed with a sudden light, and all four hobbits turned to see the sun rising above the mountains of Mordor, chasing away the lingering shadows with her healing rays of gold. The hobbits stood transfigured, silver cloaks gently rippling in the soft breeze, faces shining with light, and in each eye a vision of white shores and rolling hills of soft green grass glimmering in the golden light of day.*

*J. R. R. Tolkien*

*The Lord of the Rings*

# Appendix A

In Section A.1 and Section A.2, coherence problems in the situations of adjoint functors and Serre functors are studied. In category theory, a multitude of notions are portrayed in terms of commutative diagrams. As suggested by the word coherence, certain diagrams are considered and shown to be commutative. Problems of this nature are largely contributed by the myriad paths connecting different Hom sets, and it is not always trivial that the different paths are the same.

In Section A.3.1, some mapping cone constructions in the finite derived category $D^b(\operatorname{mod} kA_7)$ are illustrated. It is an attempt to provide an alternative to the mapping cone construction in Section 3.2.2, in the manner of Section 3.3.2.

In Section A.3.1 and Section A.3.2, mapping cones of (compositions of) downward morphisms in $D^b(\operatorname{mod} kA_7)$ and the cluster category of Dynkin type $A_\infty$ are also portrayed. They serve to enrich our répertoire of mapping cone constructions, albeit the different suggestiveness bequeathed upon them.

## A.1 Coherence problems for adjoint functors

In this section, let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary categories, and let the pair of functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ be adjoints, i.e. there is an isomorphism $\tau : (LA, B) \xrightarrow{\cong} (A, RB)$ for all $A$ in $\mathcal{A}$ and $B$ in $\mathcal{B}$, such that $\tau$ is natural in $A$ and $B$. This is given in Example 0.1.17(iv).

Let $A$ be in $\mathcal{A}$ and $B$ be in $\mathcal{B}$. Then the map $\eta_A = \tau(\mathrm{id}_{LA}) : A \to RLA$ is the unit of the adjunction, and the map $\varepsilon_B = \tau^{-1}(\mathrm{id}_{RB}) : LRB \to B$ is the counit of the adjunction.

*Example* A.1.1. Let $A$ be in $\mathcal{A}$ and $B$ be in $\mathcal{B}$. Suppose $\eta_A$ is invertible, and consider $\eta_{RB} = \tau(\mathrm{id}_{LRB}) : RB \to RLRB$. Then $\eta_{RB}^{-1} = R(\varepsilon_B) : RLRB \to$

$RB$.

*Proof.* Since $\tau$ is natural in $B$, there is the following commutative diagram.

$$\begin{tikzcd}
(LRB, LRB) \arrow[r, "\tau_{RB,LRB}"] \arrow[d, "(LRB,\varepsilon_B)"'] & (RB, RLRB) \arrow[d, "(RB,R\varepsilon_B)"] \\
(LRB, B) \arrow[r, "\tau_{RB,B}"] & (RB, RB)
\end{tikzcd}$$

$$\begin{aligned}
\text{Therefore} \quad & (RB, R\varepsilon_B)\tau_{RB,LRB}(\mathrm{id}_{LRB}) = \tau_{RB,B}(LRB, \varepsilon_B)(\mathrm{id}_{LRB}). \\
\text{Thus} \quad & (RB, R\varepsilon_B)\eta_{RB} = \tau_{RB,B}\varepsilon_B, \\
\text{and finally,} \quad & R\varepsilon_B\eta_{RB} = \mathrm{id}_{RB}.
\end{aligned}$$

Since $\eta_{RB}$ is invertible, $\eta_{RB}^{-1} = R(\varepsilon_B)$. $\square$

*Example* A.1.2. Let $A$ be in $\mathcal{A}$ and $B$ be in $\mathcal{B}$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are triangulated with translation functors $\Sigma_{\mathcal{A}}$ and $\Sigma_{\mathcal{B}}$ respectively. If $(L, \sigma^L) : \mathcal{A} \to \mathcal{B}$ is a triangulated functor where $\sigma^L : L\Sigma_{\mathcal{A}} \overset{\cong}{\Rightarrow} \Sigma_{\mathcal{B}}L$, then by [35, Lemma 5.3.6], $(R, \sigma^R)$ is also a triangulated functor where $\sigma^R : R\Sigma_{\mathcal{B}} \overset{\cong}{\Rightarrow} \Sigma_{\mathcal{A}}R$.

Consider $\varepsilon_{\Sigma_{\mathcal{B}}B} : LR\Sigma_{\mathcal{B}}B \to \Sigma_{\mathcal{B}}B$ and $\Sigma_{\mathcal{B}}(\varepsilon_B) : \Sigma_{\mathcal{B}}LRB \to \Sigma_{\mathcal{B}}B$. Since $\sigma_B^R : R\Sigma_{\mathcal{B}}B \overset{\cong}{\to} \Sigma_{\mathcal{A}}RB$, therefore $L\sigma_B^R : LR\Sigma_{\mathcal{B}}B \overset{\cong}{\to} L\Sigma_{\mathcal{A}}RB$. Also $\sigma_{RB}^L : L\Sigma_{\mathcal{A}}RB \overset{\cong}{\to} \Sigma_{\mathcal{B}}LRB$. Is the following diagram commutative? The author at the time of writing is not able to give an answer.

$$\begin{tikzcd}
LR\Sigma_{\mathcal{B}}B \arrow[r, "\varepsilon_{\Sigma_{\mathcal{B}}B}"] \arrow[d, "L(\sigma_B^R)"] & \Sigma_{\mathcal{B}}B \arrow[ddd, equal] \\
L(\Sigma_{\mathcal{A}}R)B \arrow[d, equal] & \\
(L\Sigma_{\mathcal{A}})RB \arrow[d, "\sigma_{RB}^L"] & \\
\Sigma_{\mathcal{B}}LRB \arrow[r, "\Sigma_{\mathcal{B}}(\varepsilon_B)"'] & \Sigma_{\mathcal{B}}B
\end{tikzcd}$$

## A.2 Coherence problems for Serre functors

In this section, let $\mathfrak{A}$ be a $k$-linear and Hom finite additive category with a Serre functor $S$. This is given in Definition 0.3.8. Nevertheless, it is stated as follows.

*Notation* A.2.1. A *right Serre functor* is an additive functor $S : \mathfrak{A} \to \mathfrak{A}$, together with isomorphisms

$$\varphi_{A,B} : (A, B) \xrightarrow{\cong} (B, SA)^*$$

for any $A, B \in \mathfrak{A}$, which are natural in $A$ and in $B$, and where $(-)^* = \mathrm{Hom}_k(-, k)$. If $S$ is an autoequivalence, then it is a *Serre functor*.

Let $\varphi_{A,A}(\mathrm{id}_A)$ be denoted by $\varphi_A$. The notations $\varphi_A$ and $\varphi_{A,B}$, where $A, B \in \mathfrak{A}$, will be used for the rest of the section.

Let $V$ and $W$ be $k$-vector spaces and consider $f : V \to W$. Then for $\theta$ in $W^*$ and $v$ in $V$, there is a natural way to define $f^* : W^* \to V^*$ described as follows.

$$f^*(\theta)(v) = \theta(f(v)) \tag{A.2.1}$$

Given the $k$-vector spaces $U$, $V$ and $W$, consider $f : U \to V$ and $g : V \to W$. Then by equation (A.2.1), we have

$$(gf)^* = f^* g^*. \tag{A.2.2}$$

*Notation* A.2.2. Let $U$ be a finite-dimensional $k$-vector space. Then there is a natural way to define the *double dual isomorphism* $\eta_U : U \xrightarrow{\cong} U^{**}$, i.e. for $u$ in $U$, $f$ in $U^*$,

$$\eta_U(u)(f) = f(u). \tag{A.2.3}$$

The notation $\eta_U$, simply written $\eta$, where $U$ is a finite-dimensional $k$-vector space, will be used for the rest of the section.

The isomorphism $\eta$ is *natural* in the following sense. Suppose $U$ and $V$ are finite-dimensional $k$-vector spaces. Then given $f : U \to V$, there is the following commutative diagram.

$$\begin{tikzcd} U \arrow[r, "\eta"] \arrow[d, "f"'] & U^{**} \arrow[d, "f^{**}"] \\ V \arrow[r, "\eta"] & V^{**} \end{tikzcd}$$

This is because given $\rho$ in $V^*$, $u$ in $U$, we have

$$\begin{aligned} f^{**}(\eta(u))(\rho) &= \eta(u)(f^*(\rho)) && \text{(by equation (A.2.1))} \\ &= f^*(\rho)(u) && \text{(by equation (A.2.3))} \\ &= \rho(f(u)). && \text{(by equation (A.2.1))} \end{aligned}$$

On the other hand, since

$$\eta(f(u))(\rho) = \rho(f(u)), \qquad \text{(by equation (A.2.3))}$$

therefore

$$f^{**}(\eta(u)) = \eta(f(u)). \tag{A.2.4}$$

This means the diagram is commutative.

*Example* A.2.3. Let $A$, $B$ be in $\mathfrak{A}$. Consider $\varphi_{B,SA} : (B, SA) \overset{\cong}{\to} (SA, SB)^*$, $\eta : (SA, SB) \overset{\cong}{\to} (SA, SB)^{**}$ and $f : A \to B$, $g : B \to SA$ in $\mathfrak{A}$. Then we have

$$\begin{aligned} (\varphi^*_{B,SA}\eta(Sf))(g) &= (\eta(Sf))(\varphi_{B,SA}(g)) && \text{(by equation (A.2.1))} \\ &= (\varphi_{B,SA}(g))(Sf). && \text{(by equation (A.2.3))} \end{aligned}$$

Therefore

$$(\varphi^*_{B,SA}\eta(Sf))(g) = (\varphi_{B,SA}(g))(Sf). \tag{A.2.5}$$

*Example* A.2.4. Let $A_1$, $A_2$, $B$ be in $\mathfrak{A}$. Consider $\varphi_{A_i,B} : (A_i, B) \overset{\cong}{\to} (B, SA_i)^*$, $i = 1, 2$. Consider $g : A_1 \to A_2$, $\theta : A_2 \to B$ and $f : B \to SA_1$ in $\mathfrak{A}$. Since $\varphi_{A_i,B}$ is natural in the first variable, there is the following commutative diagram.

$$\begin{tikzcd} (A_2, B) \arrow[r, "\varphi_{A_2,B}"] \arrow[d, "(g,B)"'] & (B, SA_2)^* \arrow[d, "(B,Sg)^*"] \\ (A_1, B) \arrow[r, "\varphi_{A_1,B}"'] & (B, SA_1)^* \end{tikzcd}$$

This is to say

$$\varphi_{A_1,B} \circ (g, B) = (B, Sg)^* \circ \varphi_{A_2,B}, \tag{A.2.6}$$

which gives

$$(\varphi_{A_1,B}(\theta g))(f) = \varphi_{A_2,B}(\theta)((Sg)f).$$

Suppose $A_2 = B$ and $\theta = \mathrm{id}_B$, then

$$(\varphi_{A_1,B}(g))(f) = \varphi_{B,B}(\mathrm{id}_B)((Sg)f) = \varphi_B((Sg)f). \tag{A.2.7}$$

Finally, A.2.2 and A.2.6 together give

$$(g, B)^* \circ \varphi^*_{A_1,B} = \varphi^*_{A_2,B} \circ (B, Sg)^{**}. \tag{A.2.6*}$$

*Example* A.2.5. Let $A$, $B_1$, $B_2$ be in $\mathfrak{A}$. Consider $\varphi_{A,B_i} : (A, B_i) \overset{\cong}{\to} (B_i, SA)^*$, $i = 1, 2$. Consider $g : B_1 \to B_2$, $\theta : A \to B_1$ and $f : B_2 \to SA$ in $\mathfrak{A}$. Since $\varphi_{A,B_i}$ is natural in the second variable, there is the following commutative diagram.

$$\begin{tikzcd}
(A, B_1) \arrow[r, "\varphi_{A,B_1}"] \arrow[d, "{(A,g)}"'] & (B_1, SA)^* \arrow[d, "{(g,SA)^*}"] \\
(A, B_2) \arrow[r, "\varphi_{A,B_2}"'] & (B_2, SA)^*
\end{tikzcd}$$

This is to say

$$\varphi_{A,B_2} \circ (A, g) = (g, SA)^* \circ \varphi_{A,B_1}, \tag{A.2.8}$$

which gives

$$(\varphi_{A,B_2}(g\theta))(f) = \varphi_{A,B_1}(\theta)(fg).$$

Suppose $A = B_1$ and $\theta = \mathrm{id}_{B_1}$, then

$$\varphi_{B_1,B_2}(g)(f) = \varphi_{B_1,B_1}(\mathrm{id}_{B_1})(fg) = \varphi_{B_1}(fg). \tag{A.2.9}$$

Finally, A.2.2 and A.2.8 together give

$$(A, g)^* \circ \varphi^*_{A,B_2} = \varphi^*_{A,B_1} \circ (g, SA)^{**}. \tag{A.2.8*}$$

**Lemma A.2.6.** *Let $A$, $B$ be in $\mathfrak{A}$, and consider $f : A \to B$ and $g : B \to SA$ in $\mathfrak{A}$. Then $\varphi_{B,SA}(g)(Sf) = \varphi_{A,B}(f)(g)$.*

*Proof.*

$$\begin{aligned}
\varphi_{B,SA}(g)(Sf) &= \varphi_B((Sf)g) && \text{(by equation (A.2.9))}\\
&= \varphi_{A,B}(f)(g). && \text{(by equation (A.2.7))}
\end{aligned}$$

□

The first equality corresponds to the lower commutative square, while the second equality corresponds to the upper commutative square in the following diagram.

$$\begin{tikzcd}
(A,B) \arrow[r, "\varphi_{A,B}"] & (B,SA)^* \\
(B,B) \arrow[u, "{(f,B)}"] \arrow[r, "\varphi_{B,B}"] \arrow[d, "{(B,g)}"'] & (B,SB)^* \arrow[u, "{(B,Sf)^*}"'] \arrow[d, "{(g,SB)^*}"] \\
(B,SA) \arrow[r, "\varphi_{B,SA}"'] & (SA,SB)^*
\end{tikzcd}$$

**Lemma A.2.7.** *Let $A$, $B$ be in $\mathfrak{A}$, and consider $g : A \to B$ and $\theta : B \to SA$ in $\mathfrak{A}$. Then $(\varphi_{SA,SB}(Sg))(S\theta) = \varphi_{A,B}(g)(\theta)$.*

*Proof.*

$$\begin{aligned}
(\varphi_{SA,SB}(Sg))(S\theta) &= \varphi_{SA}(S\theta Sg) && \text{(by equation (A.2.9))}\\
&= \varphi_{B,SA}(\theta)(Sg) && \text{(by equation (A.2.7))}\\
&= \varphi_B((Sg)\theta) && \text{(by equation (A.2.9))}\\
&= \varphi_{A,B}(g)(\theta). && \text{(by equation (A.2.7))}
\end{aligned}$$

$$\begin{tikzcd}
(A,B) \arrow[r, "\varphi_{A,B}"] \arrow[d, "{S(-,-)}"'] & (B,SA)^* \\
(SA,SB) \arrow[r, "\varphi_{SA,SB}"] & (SB,S^2A)^* \arrow[u, dashed, "{S(-,-)^*}"']
\end{tikzcd}$$

□

**Lemma A.2.8.** *Let $A$, $B$ be in $\mathfrak{A}$. Consider $\eta : (SA, SB) \overset{\cong}{\to} (SA, SB)^{**}$ and then $f : A \to B$ and $g : B \to SA$ in $\mathfrak{A}$. Then $(\varphi^*_{B,SA}\eta(Sf))(g) = \varphi_{A,B}(f)(g)$.*

*Proof.*

$$\begin{aligned}(\varphi^*_{B,SA}\eta(Sf))(g) &= \varphi_{B,SA}(g)(Sf) && \text{(by equation (A.2.5))}\\ &= \varphi_{A,B}(f)(g). && \text{(by Lemma A.2.6)}\end{aligned}$$

$$\begin{tikzcd}
(A, B) \arrow[r, "\varphi_{A,B}"] \arrow[d, "S(-,-)"'] & (B, SA)^* \arrow[d, "(\varphi^*_{B,SA})^{-1}"] \\
(SA, SB) \arrow[r, "\eta"] & (SA, SB)^{**}
\end{tikzcd}$$

Therefore $\varphi^*_{B,SA}\eta(Sf) = \varphi_{A,B}(f)$. □

**Lemma A.2.9.** *Let $A$, $B$ be in $\mathfrak{A}$. Consider $\eta : (SA, SB) \overset{\cong}{\to} (SA, SB)^{**}$ and then $g : A \to B$ and $f : B \to SA$ in $\mathfrak{A}$. Then $(\varphi^*_{B,SA}\eta(Sg))(f) = \varphi_{SA,SB}(Sg)(Sf)$.*

*Proof.*

$$\begin{aligned}(\varphi^*_{B,SA}\eta(Sg))(f) &= \varphi_{A,B}(g)(f) && \text{(by Lemma A.2.8)}\\ &= \varphi_{SA,SB}(Sg)(Sf). && \text{(by Lemma A.2.7)}\end{aligned}$$

$$\begin{tikzcd}
(SB, S^2A)^* \arrow[r, "(\varphi_{SA,SB})^{-1}"] \arrow[d, dashed, "S(-,-)^*"'] & (SA, SB) \arrow[d, "\eta"] \\
(B, SA)^* & (SA, SB)^{**} \arrow[l, "\varphi^*_{B,SA}"]
\end{tikzcd}$$

□

*Example* A.2.10. Consider again the diagram in Lemma A.2.7.

$$\begin{tikzcd}
(A, B) \arrow[r, "\varphi_{A,B}"] \arrow[d, "S(-,-)"'] & (B, SA)^* \\
(SA, SB) \arrow[r, "\varphi_{SA,SB}"'] & (SB, S^2A)^* \arrow[u, dashed, "S(-,-)^*"']
\end{tikzcd}$$

Consider $\eta : (SA, SB) \overset{\cong}{\to} (SA, SB)^{**}$. The diagram can be expressed as follows.

$$\begin{tikzcd}
(A,B) \arrow[r, "\varphi_{A,B}"] \arrow[d, "S(-,-)"'] & (B,SA)^* \\
(SA,SB) \arrow[r, "\eta"] \arrow[d, "\varphi_{SA,SB}"'] & (SA,SB)^{**} \arrow[u, "\varphi^*_{B,SA}"'] \arrow[d, "\varphi^*_{B,SA}"] \\
(SB,S^2A)^* \arrow[r, dashed, "S(-,-)^*"'] & (B,SA)^*
\end{tikzcd}$$

The upper square corresponds to the diagram in Lemma A.2.8, while the lower square corresponds to the diagram in Lemma A.2.9.

**Lemma A.2.11.** *Let $A$, $B$, $Y$ be in $\mathfrak{A}$, and consider $y : SY \to B$ in $\mathfrak{A}$. Then the following diagram is commutative.*

$$\begin{tikzcd}
(B,SA) \arrow[r, "\eta"] \arrow[d, "(y,SA)"'] & (B,SA)^{**} \arrow[r, "\varphi^*_{A,B}"] & (A,B)^* \arrow[dd, "(A,y)^*"] \\
(SY,SA) \arrow[d, "(S(-,-))^{-1}"'] & & \\
(Y,A) \arrow[rr, "\varphi_{Y,A}"'] & & (A,SY)^*
\end{tikzcd}$$

*Proof.* The diagram can be decomposed as follows.

$$\begin{tikzcd}
(B,SA) \arrow[r, "\eta"] \arrow[d, "(y,SA)"'] & (B,SA)^{**} \arrow[r, "\varphi^*_{A,B}"] \arrow[d, "(y,SA)^{**}"] & (A,B)^* \arrow[dd, "(A,y)^*"] \\
(SY,SA) \arrow[r, "\eta"] \arrow[d, "(S(-,-))^{-1}"'] & (SY,SA)^{**} \arrow[dr, "\varphi^*_{A,SY}"] & \\
(Y,A) \arrow[rr, "\varphi_{Y,A}"'] & & (A,SY)^*
\end{tikzcd}$$

The upper left square commutes by equation (A.2.4). The square on the right commutes by equation (A.2.8*). The square on the bottom commutes by Lemma A.2.8. $\square$

## A.3 Mapping cone construction

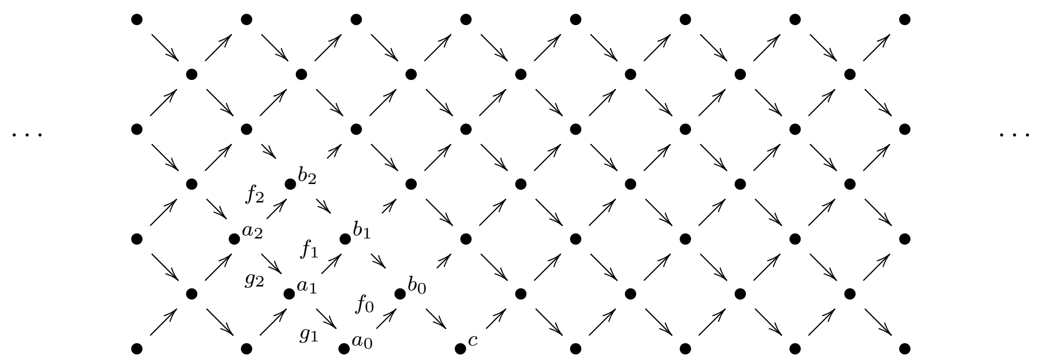
The author would like to thank Yann Palu for suggesting the constructions of mapping cones of (compositions of) downward morphisms in $D^b(\operatorname{mod} kA_7)$ and in the cluster category of Dynkin type $A_\infty$.

### A.3.1 The finite derived category $D^b(\operatorname{mod} kA_7)$

This section considers only the finite derived category $D^b(\operatorname{mod} kA_7)$, nevertheless the lemmas can be readily generalized in the finite derived category $D^b(\operatorname{mod} kA_n)$. As usual, let $\Sigma$ be the translation functor of $D^b(\operatorname{mod} kA_7)$.

Lemma A.3.1 and Lemma A.3.2 in this section have been mentioned in Remark 3.3.21.

Consider the Auslander-Reiten quiver of the finite derived category $D^b(\operatorname{mod} kA_7)$ in the following diagram.




The coordinates of some of the objects are as follows: $a_0 = (i,j), b_0 = (i,j+1), a_1 = (i-1,j), b_1 = (i-1,j+1), a_2 = (i-2,j), b_2 = (i-2,j+1)$ and $c = (i+1,j+1)$. As usual, the coordinates of objects on the bottom line satisfy the equation $y - x = 2$.

**Lemma A.3.1.** *(c.f. Lemma 3.3.16) The mapping cones of the maps $f_n : (i-n,j) \to (i-n,j+1)$ are all isomorphic to $c = (i+1,j+1)$.*

*Proof.* This is similar to Lemma 3.3.16. The two categories, the finite derived category $D^b(\operatorname{mod} kA_n)$ and the cluster category of Dynkin type $A_\infty$, are distinct, though their Auslander-Reiten quivers have some similarity. The action of the translation functor $\Sigma$ is not relevant in this situation. Nevertheless, a proof is given below. Since $a_0 \to b_0 \to c \to$ is an Auslander-Reiten triangle, the statement is true for $n = 0$. Suppose the statement is true for $n = p$, $p \geq 0$, i.e. the mapping cone of the map $f_p : a_p \to b_p$ is $c = (i+1, j+1)$. To see that the statement is true for $n = p+1$, consider the following commutative diagram,

$$\begin{tikzcd} a_{p+1} \arrow[r, "\mu_{p+1}"] \arrow[d, equal] & a_p \oplus b_{p+1} \arrow[d, "\jmath"] \\ a_{p+1} \arrow[r, "f_{p+1}"'] & b_{p+1}, \end{tikzcd}$$

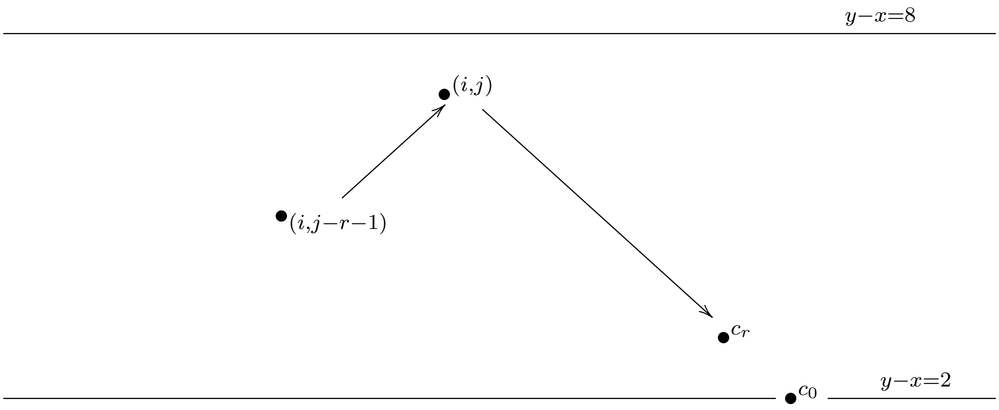
where $\mu_{p+1} = \begin{pmatrix} g_{p+1} \\ f_{p+1} \end{pmatrix}$ and the map $\jmath$ is the canonical surjection. By the octahedral axiom, it may be extended to the following commutative diagram,

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & a_p \arrow[r, equal] \arrow[d, "\imath"] & a_p \arrow[r] \arrow[d, "f_p"] & 0 \arrow[d] \\ a_{p+1} \arrow[r, "\mu_{p+1}"] \arrow[d, equal] & a_p \oplus b_{p+1} \arrow[r] \arrow[d, "\jmath"] & b_p \arrow[r] \arrow[d] & \Sigma a_{p+1} \arrow[d, equal] \\ a_{p+1} \arrow[r, "f_{p+1}"'] \arrow[d] & b_{p+1} \arrow[r] \arrow[d] & c \arrow[r] \arrow[d] & \Sigma a_{p+1} \arrow[d] \\ 0 \arrow[r] & \Sigma a_p \arrow[r, equal] & \Sigma a_p \arrow[r] & 0, \end{tikzcd}$$

where the map $\imath$ is the canonical injection. The distinguished triangle on the second row is an Auslander-Reiten triangle. Hence the mapping cone of the map $f_{p+1} : a_{p+1} \to b_{p+1}$ is the same as the mapping cone of the map $f_p : a_p \to b_p$, which is $c = (i+1, j+1)$ by the induction hypothesis. $\square$

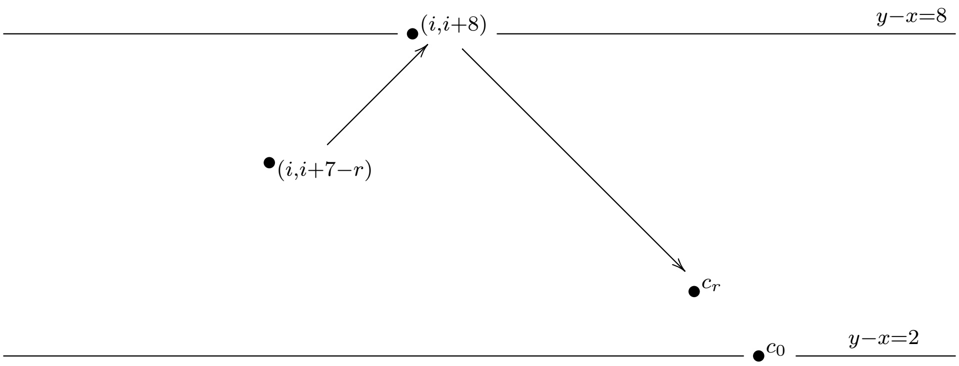
The following is similar to Lemma 3.3.18, and the proof is left to the reader. The action of $\Sigma$ is implicitly alluded and is not entirely irrelevant in this situation (c.f. Lemma A.3.1). This will be illustrated in Example A.3.3.

**Lemma A.3.2.** *(c.f. Lemma 3.3.18) Let $3 \leq j - i \leq 8$ and consider the following sketch where the Auslander-Reiten quiver of $D^b(\mathrm{mod}\, kA_7)$ lies in the region bounded by $y - x = 2$ and $y - x = 8$.*




*For* $0 \le r \le j-i-2$, *let* $a_r = (i, j-r)$. *Let* $f_r$ *be the morphism* $f_r : (i, j-r-1) \to (i, j-r)$, *where* $0 \le r \le j-i-3$. *Then for* $0 \le r \le j-i-3$, *the composition* $f_0 \dots f_{r-1} f_r : (i, j-r-1) \to (i, j)$ *has mapping cone* $c_r = (j-r-2, j)$.

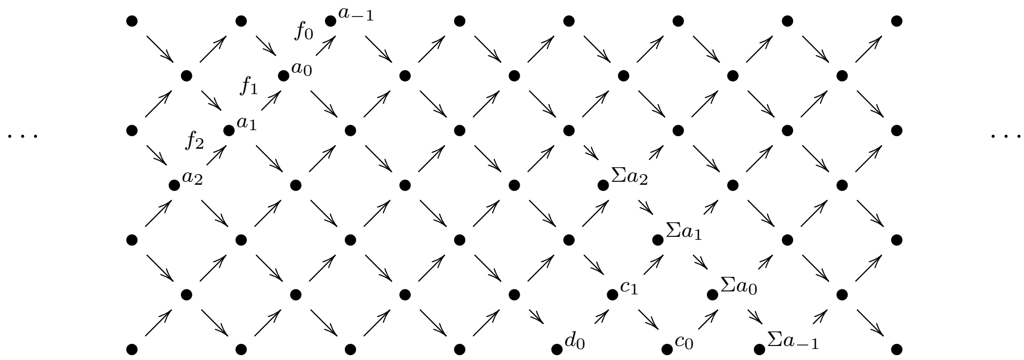
*Example* A.3.3. Consider the following sketch.




For $-1 \le n \le 5$, let $a_n = (i, i+7-n)$. Let $f_n$ be the morphism $f_n : (i, i+7-n) \to (i, i+8-n)$, where $0 \le n \le 5$. Then for $0 \le r \le 5$, the composition $f_0 \dots f_{r-1} f_r : (i, i+7-r) \to (i, i+8)$ has mapping cone $c_r = (i+6-r, i+8)$.

*Proof.* Consider the object $c_0 = (i+6, i+8)$ on the bottom line $y-x = 2$. It lies in the same descending line as $(i, i+8)$, where the composition $f_0 \dots f_{r-1} f_r$ maps to. The mapping cone $c_r = (i+6-r, i+8)$ lies in the same descending line as $c_0$, but it is $r$ steps up the line. This is how we understand the location of the mapping cone $c_r$.

Consider the Auslander-Reiten quiver of the finite derived category $D^b(\operatorname{mod} kA_7)$.




The statement is true for $r = 0$ by Lemma A.3.2. Suppose the statement is true for $r = p$, $p \geq 0$, and then the statement is also required to be true for $r = p + 1$.

Consider the following commutative diagram,

$$\begin{tikzcd}
a_{p+1} \arrow[r, "f_{p+1}"] \arrow[d, equal] & a_p \arrow[d, "g"] \\
a_{p+1} \arrow[r, "gf_{p+1}"'] & a_{-1},
\end{tikzcd}$$

where $g$ is the morphism $f_0 \dots f_p : (i, i + 7 - p) \to (i, i + 8)$.

By the octahedral axiom, it may be extended to the following commutative diagram,

$$\begin{tikzcd}
a_{p+1} \arrow[r, "f_{p+1}"] \arrow[d, equal] & a_p \arrow[r, "\rho"] \arrow[d, "g"] & d_p \arrow[r] \arrow[d] & \Sigma a_{p+1} \arrow[d, equal] \\
a_{p+1} \arrow[r, "gf_{p+1}"] \arrow[d] & a_{-1} \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & \Sigma a_{p+1} \arrow[d] \\
0 \arrow[r] \arrow[d] & c_p \arrow[r, equal] \arrow[d, "\varrho"] & c_p \arrow[r] \arrow[d, "(\Sigma\rho)\varrho"] & 0 \arrow[d] \\
\Sigma a_{p+1} \arrow[r] & \Sigma a_p \arrow[r, "\Sigma\rho"] & \Sigma d_p \arrow[r] & \Sigma^2 a_{p+1}.
\end{tikzcd}$$

By the induction hypothesis, the mapping cone of the morphism $g$ is $c_p = (i+6-p, i+8)$, which gives the distinguished triangle in the second column. By Lemma A.3.1, the mapping cone of the morphism $f_{p+1}$ is the object $d_p = (i + 5 - p, i + 7 - p)$. This gives the distinguished triangle on the first row.

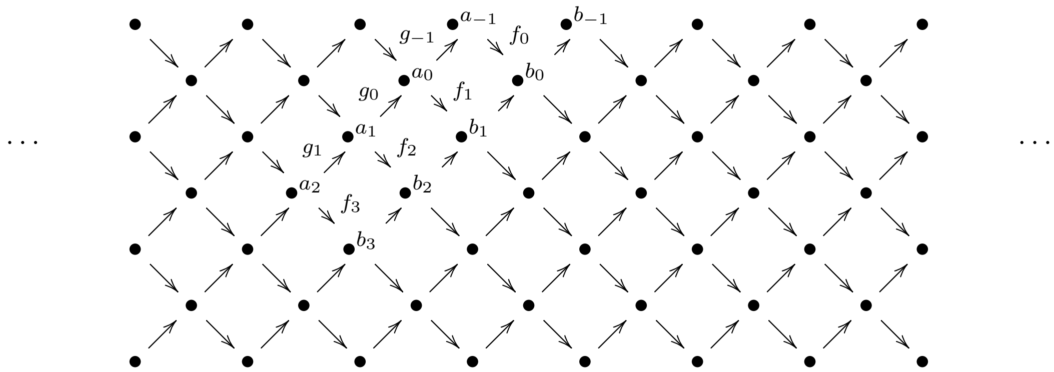
The map $\rho : a_p \to d_p$ is non-zero, as otherwise $a_{p+1} \cong \Sigma^{-1} d_p \oplus a_p$ by Lemma 0.2.2(v), which is not possible since $a_{p+1}$ is indecomposable. Therefore the map $\Sigma\rho : \Sigma a_p \to \Sigma d_p$ is non-zero as well. Similarly, the map $\varrho : c_p \to \Sigma a_p$ is non-zero. For $-1 \leq p \leq 4$, $d_p$ lies on the line $y = i + 7 - p$ and hence $\Sigma d_p$ lies on the line $x = i + 6 - p$. This gives $\Sigma d_p$ in $\mathcal{R}(c_p)$. Therefore the composition $(\Sigma\rho)\varrho : c_p \to \Sigma d_p$ is non-zero by Lemma 3.2.3, and the distinguished triangle $d_p \to * \to c_p \to \Sigma d_p$ is non-split.

Let an object $e$ have coordinates $(i + 5 - p, i + 8)$. By Lemma A.3.2, the mapping cone of $d_p \to e$ is $c_p$. Since $(c_p, \Sigma d_p)$ is one-dimensional, the object $*$ is indeed equal to $e$. Therefore the mapping cone of the morphism $gf_{p+1}$ is $e = (i + 5 - p, i + 8) = (i + 6 - (p + 1), i + 8) = c_{p+1}$ as desired. $\square$

The following two lemmas consider mapping cones of (compositions of) downward morphisms, and they are delivered entirely out of leisure. Their proofs are the mirror versions of the proofs of Lemma A.3.1 and Lemma A.3.2 respectively, since the action of the translation functor $\Sigma$ corresponds to the glide reflection. Nevertheless, they are given below, so that the reader can be guided formally and rigorously with good intuition.

They also engender yet another significance in the construction of distinguished triangles in the quotient categories. This is similar to the forthcoming section, and the reader can refer to the description given there.

Consider again the Auslander-Reiten quiver in the following diagram.




For $-1 \leq n \leq 5$, let $a_n = (i, i + 7 - n)$, $b_n = (i + 1, i + 8 - n)$.

**Lemma A.3.4.** *(c.f. Lemma A.3.1)*

*The mapping cones of the maps* $f_n : (i, i + 8 - n) \to (i + 1, i + 8 - n)$ *are all isomorphic to* $b_{-1} = (i + 1, i + 9)$.

*Proof.* Since $a_{-1} \to b_0 \to b_{-1} \to$ is an Auslander-Reiten triangle, the statement is true for $n = 0$. Suppose the statement is true for $n = p$, $p \geq 0$, i.e. the mapping cone of the map $f_p : a_{p-1} \to b_p$ is $b_{-1} = (i+1, i+9)$. To see that the statement is true for $n = p+1$, consider the following commutative diagram,

$$\begin{tikzcd}
a_p \arrow[r, "\mu_{p-1}"] \arrow[d, equal] & a_{p-1} \oplus b_{p+1} \arrow[d, "\jmath"] \\
a_p \arrow[r, "f_{p+1}"'] & b_{p+1},
\end{tikzcd}$$

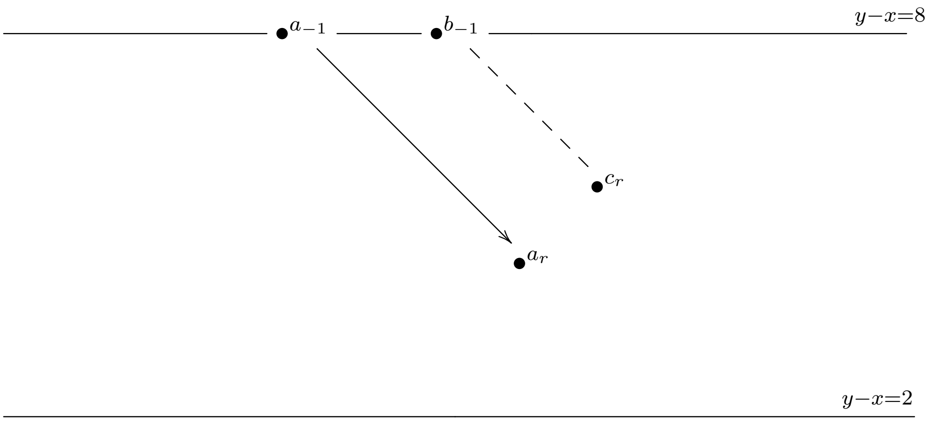
where $\mu_{p-1} = \begin{pmatrix} g_{p-1} \\ f_{p+1} \end{pmatrix}$ and the map $\jmath$ is the canonical surjection. By the octahedral axiom, it may be extended to the following commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & a_{p-1} \arrow[r, equal] \arrow[d, "\imath"] & a_{p-1} \arrow[r] \arrow[d, "f_p"] & 0 \arrow[d] \\
a_p \arrow[r, "\mu_{p-1}"] \arrow[d, equal] & a_{p-1} \oplus b_{p+1} \arrow[r] \arrow[d, "\jmath"] & b_p \arrow[r] \arrow[d] & \Sigma a_p \arrow[d, equal] \\
a_p \arrow[r, "f_{p+1}"'] \arrow[d] & b_{p+1} \arrow[r] \arrow[d] & c \arrow[r] \arrow[d] & \Sigma a_p \arrow[d] \\
0 \arrow[r] & \Sigma a_{p-1} \arrow[r, equal] & \Sigma a_{p-1} \arrow[r] & 0,
\end{tikzcd}$$

where the map $\imath$ is the canonical injection. The distinguished triangle on the second row is an Auslander-Reiten triangle. Hence the mapping cone of the map $f_{p+1} : a_p \to b_{p+1}$ is the same as the mapping cone of the map $f_p : a_{p-1} \to b_p$, which is $b_{-1} = (i+1, i+9)$ by the induction hypothesis. □

**Lemma A.3.5.** *(c.f. Lemma A.3.2, Example A.3.3)*

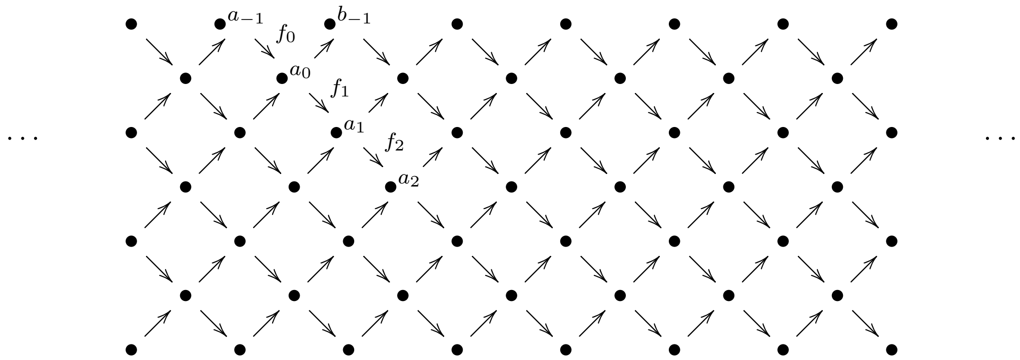
*Consider the following sketch.*




*For* $-1 \leq n \leq 5$*, let* $a_n = (j-7+n, j)$ *and* $b_n = (j-6+n, j+1)$*. Let* $f_n$ *be the morphism* $f_n : (j-8+n, j) \to (j-7+n, j)$*, where* $0 \leq n \leq 5$*. Then for* $0 \leq r \leq 5$*, the composition* $f_r \dots f_1 f_0 : (j-8, j) \to (j-7+r, j)$ *has mapping cone* $c_r = b_{r-1} = (j-7+r, j+1)$*.*

*Proof.* Consider the object $b_{-1} = (j-7, j+1)$ on the top line $y - x = 8$. The mapping cone $c_r = (j-7+r, j+1)$ lies in the same descending line as $b_{-1}$, but it is $r$ steps down the line. The object $b_{-1}$ lies immediately to the right of $a_{-1}$, where $a_{-1}$ is whence the composition $f_r \dots f_1 f_0$ maps from. This is how we understand the location of the mapping cone $c_r$.

Consider again the Auslander-Reiten quiver in the following diagram.




The statement is true for $r = 0$ by Lemma A.3.4. Suppose the statement is true for $r = p$, $p \geq 0$, and then the statement is also required to be true for $r = p + 1$.

Consider the following commutative diagram,

$$
\begin{tikzcd}
a_{-1} \arrow[r, "g"] \arrow[d, equal] & a_p \arrow[d, "f_{p+1}"] \\
a_{-1} \arrow[r, "f_{p+1}g"'] & a_{p+1},
\end{tikzcd}
$$

where $g$ is the morphism $f_p \dots f_0 : (j-8, j) \to (j-7+p, j)$.

By the octahedral axiom, it may be extended to the following commutative diagram,

$$
\begin{tikzcd}
a_{-1} \arrow[r, "g"] \arrow[d, equal] & a_p \arrow[r, "\rho"] \arrow[d, "f_{p+1}"] & c_p \arrow[r] \arrow[d] & \Sigma a_{-1} \arrow[d, equal] \\
a_{-1} \arrow[r, "f_{p+1}g"] \arrow[d] & a_{p+1} \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & \Sigma a_{-1} \arrow[d] \\
0 \arrow[r] \arrow[d] & d_p \arrow[r, equal] \arrow[d, "\varrho"] & d_p \arrow[r] \arrow[d, "(\Sigma\rho)\varrho"] & 0 \arrow[d] \\
\Sigma a_{-1} \arrow[r] & \Sigma a_p \arrow[r, "\Sigma\rho"] & \Sigma c_p \arrow[r] & \Sigma^2 a_{-1}.
\end{tikzcd}
$$

The distinguished triangle on the first row is given by the induction step. The mapping cone of the morphism $f_{p+1}$ is $d_p = (j-6+p, j+2+p)$ by Lemma A.3.4, which gives the distinguished triangle in the second column.

The map $\rho : a_p \to c_p$ is non-zero, as otherwise $a_{-1} \cong \Sigma^{-1}c_p \oplus a_p$ by Lemma 0.2.2(v), which is not possible since $a_{-1}$ is indecomposable. Therefore the map $\Sigma\rho : \Sigma a_p \to \Sigma c_p$ is non-zero as well. Similarly, the map $\varrho : d_p \to \Sigma a_p$ is non-zero. Since $\Sigma c_p = (j, j+2+p)$ lies on the line $y = j+2+p$, this gives $\Sigma c_p$ in $\mathcal{R}(d_p)$. Therefore the composition $(\Sigma\rho)\varrho : d_p \to \Sigma c_p$ is non-zero by Lemma 3.2.3, and the distinguished triangle $c_p \to * \to d_p \to \Sigma c_p$ is non-split.

Let an object $e$ have coordinates $(j-6+p, j+1)$. By Lemma A.3.4, the mapping cone of $c_p \to e$ is $d_p$. Since $(d_p, \Sigma c_p)$ is one-dimensional, the object $*$ is indeed equal to $e$. Therefore the mapping cone of the morphism $f_{p+1}g$ is $e = (j-6+p, j+1) = (j-7+(p+1), j+1) = c_{p+1}$ as desired. $\square$

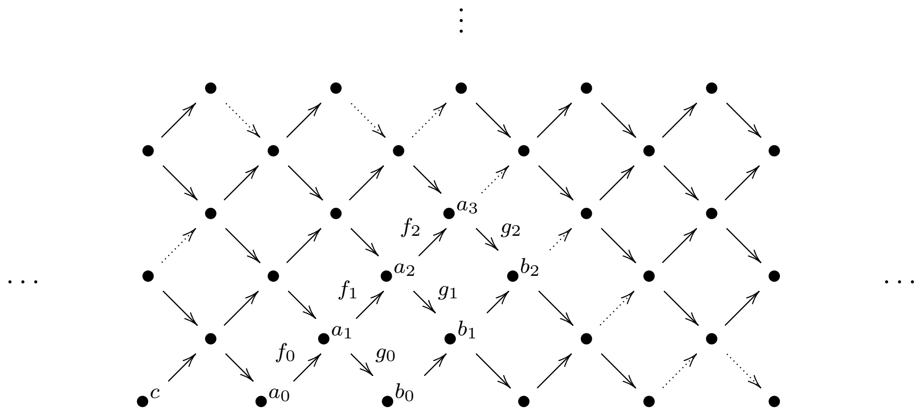
### A.3.2 The cluster category of Dynkin type $A_\infty$

This section considers mapping cones of (compositions of) downward (forward) morphisms in the cluster category $\mathcal{D}$ of Dynkin type $A_\infty$. As usual, let $\Sigma$ be the translation functor of $\mathcal{D}$.

Given a subcategory $\mathcal{X}$ of $\mathcal{D}$ which is both precovering and preenveloping, the quotient category $\mathcal{D}_{\mathcal{X}}$ is pretriangulated. $\mathcal{D}_{\mathcal{X}}$ is triangulated if and only if $\tau\mathcal{X} = \mathcal{X}$, where $\tau$ is the Auslander-Reiten translation. This is described in Section 3.1.

The distinguished triangles in $\mathcal{D}_{\mathcal{X}}$ are described in [23, Setup 1.1]. They are obtained either by apprehending $\mathcal{X}$ as preenveloping or as precovering. If $\mathcal{X}$ is perceived as preenveloping, then Lemma 3.3.16 and Lemma 3.3.18 are needed (Example 3.3.19). In a different manner, if $\mathcal{X}$ is perceived as precovering, then the following lemmas are needed (Example A.3.8).

Consider the Auslander-Reiten quiver of $\mathcal{D}$ in the following diagram.




Let us write down the coordinates of some of the objects shown: $a_0 = (i, j), b_0 = (i+1, j+1), a_1 = (i, j+1), b_1 = (i+1, j+2), a_2 = (i, j+2), b_2 = (i+1, j+3)$ and $c = (i-1, j-1)$. As usual, the coordinates of objects on the bottom line satisfy the equation $y - x = 2$.

**Lemma A.3.6.** *(c.f. Lemma 3.3.16, Lemma A.3.4) The mapping cones of the maps $g_n : (i, j+n+1) \to (i+1, j+n+1)$ are all isomorphic to $c = \Sigma a_0 = (i-1, j-1)$.*

*Proof.* Since $a_0 \to a_1 \to b_0 \to \Sigma a_0$ is an Auslander-Reiten triangle, the statement is true for $n = 0$. Suppose the statement is true for $n = p$, $p \geq 0$, i.e. the mapping cone of the map $g_p : a_{p+1} \to b_p$ is $c = (i-1, j-1)$. To see that the statement is true for $n = p+1$, consider the following commutative diagram,

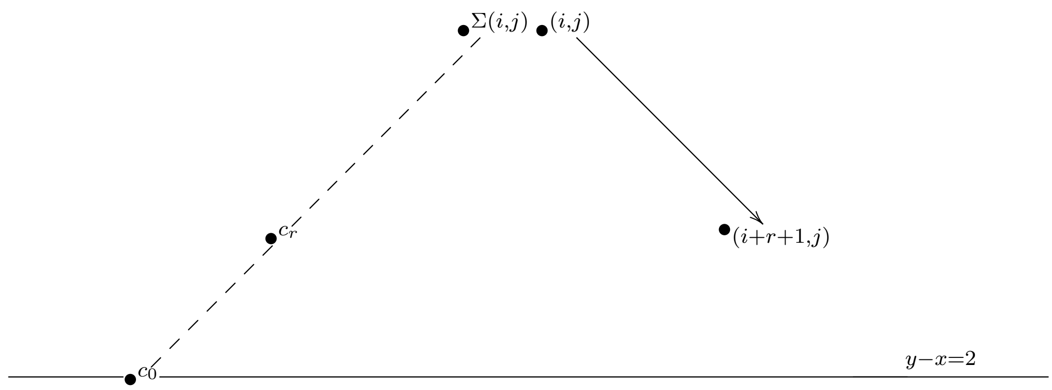
$$\begin{tikzcd} a_{p+1} \arrow[r, "\mu_{p+1}"] \arrow[d, equal] & a_{p+2} \oplus b_p \arrow[d, "\jmath"] \\ a_{p+1} \arrow[r, "g_p"'] & b_p, \end{tikzcd}$$

where $\mu_{p+1} = \begin{pmatrix} f_{p+1} \\ g_p \end{pmatrix}$ and the map $\jmath$ is the canonical surjection. By the octahedral axiom, it may be extended to the following commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & a_{p+2} \arrow[r, equal] \arrow[d, "\imath"] & a_{p+2} \arrow[r] \arrow[d, "g_{p+1}"] & 0 \arrow[d] \\
a_{p+1} \arrow[r, "\mu_{p+1}"] \arrow[d, equal] & a_{p+2} \oplus b_p \arrow[r] \arrow[d, "\jmath"] & b_{p+1} \arrow[r] \arrow[d] & \Sigma a_{p+1} \arrow[d, equal] \\
a_{p+1} \arrow[r, "g_p"'] \arrow[d] & b_p \arrow[r] \arrow[d] & c \arrow[r] \arrow[d] & \Sigma a_{p+1} \arrow[d] \\
0 \arrow[r] & \Sigma a_{p+2} \arrow[r, equal] & \Sigma a_{p+2} \arrow[r] & 0,
\end{tikzcd}$$

where the map $\imath$ is the canonical injection. The distinguished triangle on the second row is an Auslander-Reiten triangle. Hence the mapping cone of the map $g_{p+1} : a_{p+2} \to b_{p+1}$ is the same as the mapping cone of the map $g_p : a_{p+1} \to b_p$, which is $c = (i-1, j-1)$ by the induction hypothesis. $\square$

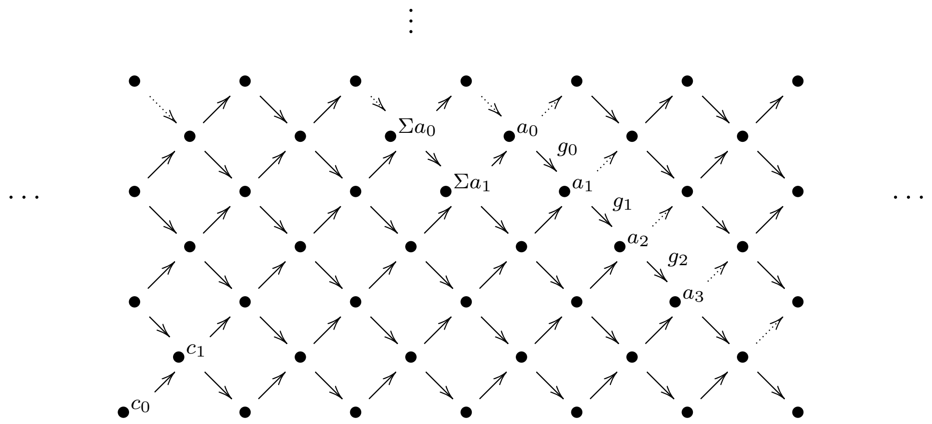
**Lemma A.3.7.** *(c.f. Lemma 3.3.18, Lemma A.3.5) Consider the following sketch.*




*For* $0 \leq n$, *let* $a_n = (i+n, j)$. *Let* $g_n$ *be the morphism* $g_n : (i+n, j) \to (i+n+1, j)$. *Then for* $0 \leq r$, *the composition* $g_r g_{r-1} \ldots g_0 : (i,j) \to (i+r+1, j)$ *has mapping cone* $c_r = (i-1, i+1+r)$.

*Proof.* Consider the object $c_0 = (i-1, i+1)$ on the bottom line $y - x = 2$. It lies in the same ascending line as $\Sigma(i,j)$. The mapping cone $c_r = (i-1, i+1+r)$ lies in the same ascending line as $c_0$, but it is $r$ steps up the line. This is how we understand the location of the mapping cone $c_r$.

Consider again the Auslander-Reiten quiver of $\mathcal{D}$.




The statement is true for $r = 0$ by Lemma A.3.6. Suppose the statement is true for $r = p$, $p \geq 0$, and then the statement is also required to be true for $r = p + 1$.

Consider the following commutative diagram,

$$\begin{tikzcd}
a_0 \arrow[r, "g"] \arrow[d, equal] & a_{p+1} \arrow[d, "g_{p+1}"] \\
a_0 \arrow[r, "g_{p+1}g"'] & a_{p+2},
\end{tikzcd}$$

where $g$ is the morphism $g_p g_{p-1} \dots g_0 : (i, j) \to (i + p + 1, j)$.

By the octahedral axiom, it may be extended to the following commutative diagram,

$$\begin{tikzcd}
a_0 \arrow[r, "g"] \arrow[d, equal] & a_{p+1} \arrow[r, "\rho"] \arrow[d, "g_{p+1}"] & d_p \arrow[r] \arrow[d] & \Sigma a_0 \arrow[d, equal] \\
a_0 \arrow[r, "g_{p+1}g"'] \arrow[d] & a_{p+2} \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & \Sigma a_0 \arrow[d] \\
0 \arrow[r] \arrow[d] & e_p \arrow[r, equal] \arrow[d, "\varrho"] & e_p \arrow[r] \arrow[d, "(\Sigma\rho)\varrho"] & 0 \arrow[d] \\
\Sigma a_0 \arrow[r] & \Sigma a_{p+1} \arrow[r, "\Sigma\rho"] & \Sigma d_p \arrow[r] & \Sigma^2 a_0.
\end{tikzcd}$$

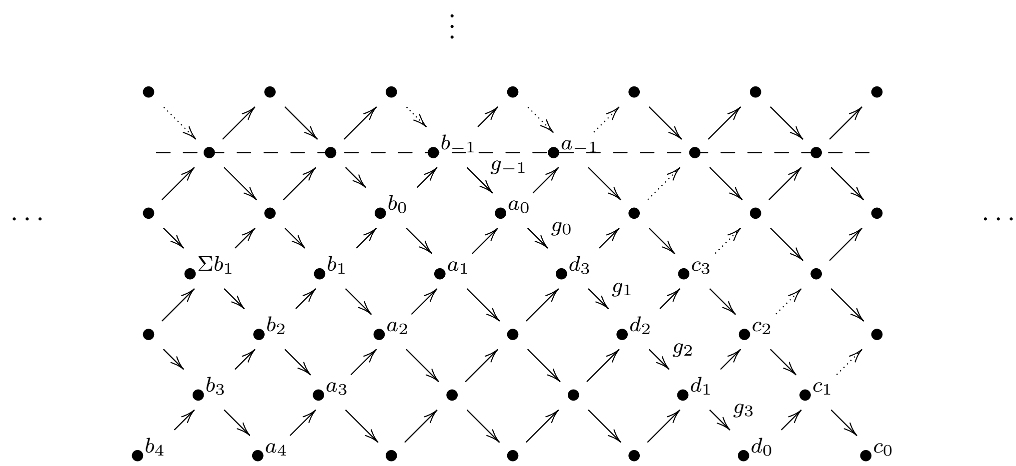
Again consider the object $c_0 = (i-1, i+1)$ on the bottom line $y - x = 2$. It lies in the same ascending line as $\Sigma a_0$, where $a_0$ is the place from where the morphism $g$ maps. By the induction hypothesis, the mapping cone of the morphism $g$ lies in the same ascending line as $c_0$, but it is $p$ steps up the line, i.e. $d_p = c_p = (i-1, i+1+p)$. This gives the distinguished triangle on the first row. The mapping cone of the morphism $g_{p+1}$ is $e_p = (i+p, i+p+2)$ by Lemma A.3.6, which gives the distinguished triangle in the second column.

The map $\rho : a_{p+1} \to d_p$ is non-zero, as otherwise $a_0 \cong \Sigma^{-1} d_p \oplus a_{p+1}$ by Lemma 0.2.2(v), which is not possible since $a_0$ is indecomposable. Therefore the map $\Sigma\rho : \Sigma a_{p+1} \to \Sigma d_p$ is non-zero as well. Similarly, the map $\varrho : e_p \to \Sigma a_{p+1}$ is non-zero. Therefore the composition $(\Sigma\rho)\varrho : e_p \to \Sigma d_p$ is non-zero by Lemma 3.3.8, and the distinguished triangle $d_p \to * \to e_p \to \Sigma d_p$ is non-split.

Let an object $s$ have coordinates $(i-1, i+2+p)$. By Lemma 3.3.16, the mapping cone of $d_p \to s$ is $e_p$. Since $(e_p, \Sigma d_p)$ is one-dimensional, the object $*$ is indeed equal to $s$. Therefore the mapping cone of the morphism $g_{p+1}g$ is $* = s = (i-1, i+2+p) = (i-1, i+1+(p+1)) = c_{p+1}$ as desired. $\square$

The section finishes with the following example which illustrates the obtaining of the distinguished triangles in the quotient category $\mathcal{D}_{\mathcal{X}}$ in virtue of $\mathcal{X}$-precovers. The reader can refer to [23, Setup 1.1] for more details.

*Example* A.3.8. *(c.f. Example 3.3.19)* Consider again the Auslander-Reiten quiver and the subcategory $\mathcal{X}$ in Example 3.3.19. As usual, let $\sigma$ be the translation functor of $\mathcal{D}_{\mathcal{X}}$.




For example, the morphism $g = g_2 g_1 g_0 : a_0 \to d_1$ is an $\mathcal{X}$*-epimorphism* in $\mathcal{D}$, i.e. a morphism such that each morphism $x \to d_1$ with $x$ in $\mathcal{X}$ factors through $g$ (c.f. Lemma 3.3.11). Extend $g : a_0 \to d_1$ to the distinguished triangle $a_0 \xrightarrow{g} d_1 \to b_2 \to \Sigma a_0$ in $\mathcal{D}$ by Lemma A.3.7, i.e. $b_2$ is the mapping cone of the morphism $g$. On the other hand, the mapping cone of the morphism $gg_{-1}$ is $\Sigma b_1$ by Lemma A.3.7, where $gg_{-1} : b_{-1} \to d_1$ is an $\mathcal{X}$-precover of

$d_1$ by Lemma 3.3.11. Then the diagram $\Sigma^{-1}\Sigma b_1 \to \Sigma^{-1} b_2 \to a_0 \to d_1$, which is the diagram $b_1 \to a_2 \to a_0 \to d_1$, considered in $\mathcal{D}_{\mathcal{X}}$, is defined to be a distinguished triangle in $\mathcal{D}_{\mathcal{X}}$, so that $\sigma^{-1}(d_1) = b_1$. Compare in Example 3.3.19(iii) the distinguished triangle $a_2 \to a_0 \to d_1 \to c_2$ in $\mathcal{D}_{\mathcal{X}}$.

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