# FUNCTION THEORY RELATED TO $H^\infty$ CONTROL


**ALAA ABDULWAHAB A. ABOUHAJAR**


A Thesis Submitted for the Degree of
Doctor of Philosophy


School of Mathematics and Statistics
Newcastle University
Newcastle upon Tyne
United Kingdom


16 May 2007

# Abstract

We define $\Gamma_E$, a subset of $\mathbb{C}^3$, related to the structured singular value $\mu$ of $2\times 2$ matrices. $\mu$ is used to analyse performance and robustness of linear feedback systems in control engineering. We find a characterisation for the elements of $\Gamma_E$ and establish a necessary and sufficient condition for the existence of an analytic function from the unit disc into $\Gamma_E$ satisfying an arbitrary finite number of interpolation conditions.

We prove a Schwarz Lemma for $\Gamma_E$ when one of the points in $\Gamma_E$ is $(0,0,0)$, then we show that in this case, the Carathéodory and Kobayashi distances between the two points in $\Gamma_E$ coincide.

We also give a characterisation of the interior, the topological boundary and the distinguished boundary of $\Gamma_E$, then we define $\Gamma_E$-inner functions and show that if there exists an analytic function from the unit disc into $\Gamma_E$ that satisfies the interpolating conditions, then there is a rational $\Gamma_E$-inner function that interpolates.

# Acknowledgement

I would like to thank both Dr. Michael White and Professor Nicholas Young for their patient guidance, encouragement and advice during the preparation of this thesis. I have been extremely fortunate to have had the opportunity to work with these two excellent supervisors who have provided me with all the assistance and attention to my work that I could have asked for.

Dr. Michael Dritschel for his support and help.

The School of Mathematics and Statistics at Newcastle University for providing invaluable support throughout.

Many thanks to Dr. Thomas Chadwick for his full support, continuous help and encouragement. His patience and understanding are most appreciated.

Thanks also should be given to the Ministry of Higher Education in Saudi Arabia for their kind help and guidance.

To my family; my father for providing me with financial support throughout my time as a student, especially for fully funding me during the years of study at Newcastle, and my mother for supporting me from the very beginning and for her unlimited help and encouragement, I owe her more than I could possibly express - thank you.

# Contents

# Chapter 1

# Interpolation Problems

We start by stating the Nevanlinna-Pick problem and explaining its importance, then present some of the results of Agler and Young for the symmetrised bidisc $\Gamma$. We prove later the analogue of these results of Agler and Young for our new set $\Gamma_E$ which we define in Section 2.1.

We also give some definitions and basic results in linear systems and explain how it relates to our project.

We use the following notations; $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{D}$ denotes the open unit disc, $\bar{\mathbb{D}}$ denotes the closed unit disc, $\mathbb{T}$ denotes the unit circle, $E$ denotes the space of diagonal $2 \times 2$ matrices, $M_2(\mathbb{C})$ denotes the space of $2 \times 2$ matrices, $\mathbb{C}^n$ is the set of complex $n$ vectors, $|.|$ is the absolute value of elements in $\mathbb{C}$ and $\|x\|$ is the Euclidean norm for $x \in \mathbb{C}^n$.

We write a $2 \times 2$ matrix $A$ as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

## 1.1 The Nevanlinna-Pick Problem

The classical Nevanlinna-Pick problem is as follows:

**The Nevanlinna-Pick Problem** *Given $n$ points $\lambda_1, \ldots, \lambda_n$ in the open unit disc $\mathbb{D}$ and $n$ complex numbers $w_1, \ldots, w_n$. Does there exist an analytic function $\varphi : \mathbb{D} \longrightarrow \mathbb{C}$ such that $\varphi(\lambda_j) = w_j$, for $j = 1, \ldots, n$* and $|\varphi(\lambda)| \leq 1$, *for all $\lambda \in \mathbb{D}$?*

This problem was solved by G. Pick in 1916 [2]. He showed that a necessary and sufficient condition is that the Pick matrix

$$\left[\frac{1 - \bar{w}_i w_j}{1 - \bar{\lambda}_i \lambda_j}\right]_{i,j=1}^{n}$$

is positive semi-definite.

The two-by-two spectral Nevanlinna-Pick problem is the following:

*Given distinct points $\lambda_1, \ldots, \lambda_n$ in the open unit disc $\mathbb{D}$ and $2 \times 2$ matrices $W_1, \ldots, W_n$, $n \geq 1$, find conditions for the existence of an analytic $2 \times 2$ matrix valued function $F$ on $\mathbb{D}$ such that*

$$F(\lambda_j) = W_j, \ j = 1, 2, \ldots, n,$$

*and*

$$\rho(F(\lambda)) \leq 1, \ \textit{for all } \lambda \in \mathbb{D}.$$

Here $\rho(.)$ denotes the spectral radius of a matrix. This problem, while being a special case of a classical topic, is also a test case of a fundamental question in $H^\infty$ control; the problem of $\mu$-synthesis. As such it has been the subject of a great deal of research during the last 25 years because a solution to the general problem would, among other applications, enable the design of automatic controllers which are robust with respect to structured uncertainty.

As yet there is no existing analytic solution to the problem of $\mu$-synthesis and therefore the standard approaches are computational; for example, the use of a Matlab toolbox [12]. Analysis of even special cases of the problem will therefore provide tests of the existing software and illuminate the difficulties associated with the more general problem.

Agler and Young [8] established a necessary and sufficient condition for the existence of a solution in the case of an arbitrary finite number of interpolation points, their result is as follows:

**Theorem 1.1.1** *Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$ for some $n \in \mathbb{N}$ and let $W_1, \ldots, W_n$ be $2 \times 2$ matrices, none of them a scalar multiple of the identity. The following two statements are equivalent:*

*(1) there exists an analytic $2\times 2$-matrix function $F$ on $\mathbb{D}$ such that $F(\lambda_j) = W_j$, $1 \le j \le n$ and $\rho(F(\lambda)) \le 1$ for all $\lambda \in \mathbb{D}$;*

*(2) there exists $b_1, \ldots, b_n, c_1, \ldots, c_n \in \mathbb{C}$ such that*

$$\left[\frac{I - \begin{bmatrix} \frac{1}{2}s_i & b_i \\ c_i & -\frac{1}{2}s_i \end{bmatrix}^* \begin{bmatrix} \frac{1}{2}s_j & b_j \\ c_j & -\frac{1}{2}s_j \end{bmatrix}}{1 - \bar{\lambda}_i \lambda_j}\right]_{i,j=1}^{n} \ge 0,$$

*where $s_j = \mathrm{tr} W_j$, $p_j = \det W_j$ and $b_j c_j = p_j - \frac{s_j^2}{4}$, $1 \le j \le n$.*

Agler and Young [8] studied the case where the target matrices $W_j$ are $2 \times 2$ hoping that a breakthrough in this case would show the way for the general problem. This led them to study the *symmetrised bidisc* which is defined to be the set

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1| \leq 1, |z_2| \leq 1\} \subset \mathbb{C}^2.$$

In other words, it is the set

$$\Gamma = \{(\mathrm{tr}A, \det A) : A \in M_2(\mathbb{C}),\ \rho(A) \leq 1\} \subset \mathbb{C}^2.$$

In this project, we study the Nevanlinna-Pick problem for $\Gamma_E$ and establish a necessary and sufficient condition for the existence of an analytic function from the unit disc into $\Gamma_E$ satisfying an arbitrary finite number of interpolation conditions.
In this project, we prove the analogue of the following results of Agler and Young for a different set, $\Gamma_E$, which we introduce in the next chapter.

The following characterisation of points of $G$, the interior of $\Gamma$, was given by Agler and Young [9].

**Theorem 1.1.2** *Let $s, p \in \mathbb{C}$. The following are equivalent:*

*(1)* $(s, p) \in G$*;*

*(2) the roots of the equation* $z^2 - sz + p = 0$ *lie in* $\mathbb{D}$*;*

*(3)* $|s - \bar{s}p| < 1 - |p|^2$*;*

*(4)* $|s| < 2$ *and, for all* $z \in \bar{\mathbb{D}}$, $\left|\dfrac{2zp - s}{2 - zs}\right| < 1$*;*

*(5)* $|p| < 1$ *and there exists* $\beta \in \mathbb{D}$ *such that* $s = \beta p + \bar{\beta}$*;*

*(6)* $2|s - \bar{s}p| + |s^2 - 4p| + |s|^2 < 4$*.*

A full proof of this lemma can be found in [9]. We present our characterisation for $\Gamma_E$ in Theorem 2.1.4

**Definition 1.1.3** *A function $f : \mathbb{D} \longrightarrow \mathbb{C}$ is a* Schur function *or belongs to the* Schur class $\mathcal{S}$ *(in the open unit disc $\mathbb{D}$) if $f$ is holomorphic in $\mathbb{D}$ and $\|f(z)\| \leq 1$ for all $z \in \mathbb{D}$.*

The next result of Agler and Young [9] relates the property of mapping the unit disc $\mathbb{D}$ analytically to $\Gamma$ and membership of the Schur class.

**Theorem 1.1.4** *For any function $\varphi = (s, p) : \mathbb{D} \longrightarrow \mathbb{C}^2$, the following are equivalent:*

*(1) $\varphi$ is analytic and maps $\mathbb{D}$ into $\Gamma$;*

*(2) there exists an analytic $2 \times 2$-matrix function $\psi = [\psi_{ij}]$ on $\mathbb{D}$ such that $\|\psi\|_\infty \leq 1$, $\mathrm{tr}\psi = 0$ identically on $\mathbb{D}$ and $\varphi = (2\psi_{11}, -\det \psi)$.*

The $2 \times 2$ matrix function $\psi$ appearing in condition (2) of Theorem 1.1.4 belongs to the Schur class. Agler and Young [8] found a realization formula for it. To present the realization formula for such an analytic $\Gamma$-valued function, we shall use the following notations; if $H, U$ and $Y$ are Hilbert spaces, and

$$A : H \longrightarrow H, \qquad B : U \longrightarrow H,$$
$$C : H \longrightarrow Y, \qquad D : U \longrightarrow Y$$

are bounded linear operators, then we define the operator

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](z) = D + Cz(1 - Az)^{-1}B : U \to Y$$

whenever $1 - Az$ is invertible.

**Corollary 1.1.5** *A function $\varphi = (s,p) : \mathbb{D} \longrightarrow \mathbb{C}^2$ maps $\mathbb{D}$ analytically into $\Gamma$ if and only if there exist a Hilbert space $H$ and a unitary operator*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : H \oplus \mathbb{C}^2 \longrightarrow H \oplus \mathbb{C}^2$$

*such that*

$$s = \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right] - \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right],$$

*and*

$$p = \left(\frac{1}{4}\mathrm{tr}^2 - \det\right) \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

*where* $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} : \mathbb{C}^2 \longrightarrow H,\ \ C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} : H \longrightarrow \mathbb{C}^2$ *and* $D = [D_{ij}]_{i,j=1}^2$.

Our analogue of these two result for $\Gamma_E$ is given in Theorem 2.2.5 and Corollary 2.2.6.

The following result of Agler and Young [8] reduces the problem of analytic interpolation from the unit disc $\mathbb{D}$ to $\Gamma$ to a standard classical matricial Nevanlinna-Pick problem.
In the following result, by *Nevanlinna-Pick data* we mean a finite set $\lambda_1, \dots, \lambda_n$ of finite distinct points in $\mathbb{D}$, where $n \in \mathbb{N}$, and an equal number of "target" matrices $W_1, \dots, W_n$ of type $m \times k$, say. We write these data as

$$\lambda_j \mapsto W_j,\ 1 \le j \le n. \tag{1.1}$$

We say that these data are *solvable*, if there exists a function $f$ in the Schur class such that

$$f(\lambda_j) = W_j,\ 1 \le j \le n.$$

Clearly, the Nevanlinna-Pick problem with data (1.1) is solvable if an only if

$$\left[\frac{I_k - W_i^* W_j}{1-\bar{\lambda}_i\lambda_j}\right]_{i,j=1}^n \geq 0.$$

In the next theorem, we use the following notations; $s_j = \mathrm{tr}(W_j)$, and $p_j = \det(W_j)$.

**Theorem 1.1.6** *Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$ for some $n \in \mathbb{N}$ and let $(s_j, p_j) \in \Gamma$ for $j = 1, \ldots, n$. There exists an analytic function $\varphi : \longrightarrow \Gamma$ such that*

$$\varphi(\lambda_j) = (s_j, p_j),\ 1 \leq 1 \leq n,$$

*if and only if there exist $b_j, c_j \in \mathbb{C}$ such that*

$$b_j c_j = p_j - \frac{s_j^2}{4},\ 1 \leq j \leq n,$$

*and the conditions*

$$\lambda_j \longmapsto \begin{bmatrix} \frac{1}{2}s_j & b_j \\ c_j & -\frac{1}{2}s_j \end{bmatrix},\ 1 \leq j \leq n,$$

*comprise solvable matricial Nevanlinna-Pick data.*

In Theorem 2.3.1, we present an analogue of Theorem 1.1.6 for $\Gamma_E$.

Agler and Young's Schwarz Lemma for the symmetrised bidisc [6] is as follows:

**Theorem 1.1.7** *Let $\lambda_0 \in \mathbb{D}$ and $(s_0, p_0) \in \Gamma$. The following are equivalent:*

*(1) There exists an analytic function $\varphi : \mathbb{D} \longrightarrow \Gamma$ such that $\varphi(0) = (0, 0)$ and $\varphi(\lambda_0) = (s_0, p_0)$;*

*(2)* $|s_0| < 2$ *and* $\dfrac{2|s_0 - \bar{s}_0 p_0| + |s_0^2 - 4p_0|}{4 - |s_0|^2} \leq |\lambda_0|.$

In chapter 3, we prove a Schwarz Lemma for $\Gamma_E$.

## 1.2 Linear Systems

In this section, we present some simple concepts in linear systems theory. In our study, we shall take all our linear systems to be finite dimensional.

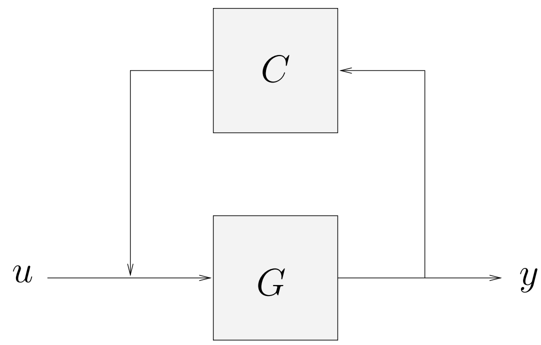



Figure 1.1: A feedback control block diagram

A *closed loop feedback system* is a system that can be described as in Figure 1.1 above. In this Figure, $G$ represents the *plant* and $C$ represents a *controller*. In such systems, we believe that the plant is what performs the main role of the system and the controller is what ensures that it behaves correctly. Mathematically, in a linear system, the plant and the controller can be considered as multiplication operators (by the Laplace transform, see [15]). Usually, there is no difference between the plant/controller and the multiplication operator it produces. In the case that $u$ is a $p$-dimensional vector input and $y$ is an $r$-dimensional vector output, the plant $G$ and the controller $C$ will be the matrices $r \times p$ and $p \times r$ respectively. It is clear to see that if $u$ and $y$ are scalar functions, then so are $G$ and $C$. The case that they are all scalars is called the SISO case, that is, single input, single output.

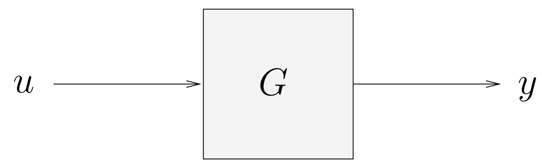



Figure 1.2: A simple block diagram

A simple block as in Figure 1.2, has $u$ as an input and $y$ as an output. It satisfies $y(s) = G(s)u(s)$. In this case, $u$, $y$ and $G$ are the Laplace transformation of the input, output and plant which are functions of time. The multiplication operators (which can be matricial) induced by the boxes in the relevant diagram are called *transfer functions*.

**Definition 1.2.1** *A system is* stable *if its transfer function is bounded and analytic in the right half-plane.*

This definition means that the system in Figure 1.2 is stable if and only if $|G(s)| < M$ for some $M \in \mathbb{R}$ and for all $s \in \mathbb{C}$. Note that $G(s)$ is in general only defined for Re $s > 0$.

**Definition 1.2.2** *A system is* internally stable *if the transfer function between each input and each branch of the system is stable.*

Note that systems with stable transfer functions can still have internal instabilities.

It is clear to see that the simple system given in Figure 1.2 is internally stable if and only if it is stable. Meanwhile, the system in Figure 1.1 is stable if and only if each of the following transfer functions is stable

$$(I + GC)^{-1}, \ (I + GC)^{-1}G, \ C(I + GC)^{-1}, \ C(I + GC)^{-1}G.$$

That is, the transfer function between each input and each branch in the system is stable.

It is a great interest to know which controllers $C$ stabilise the system in Figure 1.1 for a given $G$. To simplify this, assume that $G$ is rational and therefore has a co-prime factorisation. That is, there exist stable matrices $M$, $N$, $X$ and $Y$ such that $X$ and $Y$ are proper, real rational, and

$$G = NM^{-1} \text{ and } YN + XM = I.$$

The following result and proof in the scalar case can be found in [27].

**Theorem 1.2.3** *Let $G$ be a rational plant with co-prime factorisation $G = NM^{-1}$ as above. Then $C$ is a rational controller which internally stabilises the system given in Figure 1.1 if and only if*

$$C = (Y + MQ)(X - NQ)^{-1},$$

*for some stable proper, real rational function $Q$ for which $(X - NQ)^{-1}$ exists.*

In the scalar case, if $G = \dfrac{N}{M}$, then $C$ produces an internally stable system in Figure 1.1 if and only if

$$C = \frac{Y + MQ}{X - NQ},$$

for some $Q \in H^\infty$ with $X - NQ \neq 0$.

Observe that in the case of an internally stable single-input, single-output (SISO) system we have

$$\begin{aligned}
\frac{C}{1+GC} &= \frac{Y+MQ}{X-NQ}\frac{1}{1+\dfrac{N}{M}\dfrac{Y+MQ}{X-NQ}} \\
&= \frac{Y+MQ}{X-NQ}\frac{M(X-NQ)}{M(X-NQ)+N(Y+MQ)} \\
&= (Y+MQ)\frac{M}{NY+MX} \\
&= M(Y+MQ).
\end{aligned}$$

The Nevanlinna-Pick problem occurs in the context of robust stabilisation. The problem of robust stabilisation studies the possibility of constructing a controller which stabilises all feedback systems (as in Figure 1.1) with plants that are 'close' to $G$.

Let the system in Figure 1.1 be denoted by $(G, C)$ and the right half-plane by $\mathbb{H}$. We denote the set of functions that are analytic on the right half-plane with a unique limit at infinity by $A(\mathbb{H})$. The following result can be found in [27].

**Theorem 1.2.4** *Let $(G, C)$ be an internally stable SISO feedback system over $A(\mathbb{H})$ and suppose that*

$$\left\| \frac{C}{I+GC} \right\|_\infty = \varepsilon.$$

*Then $C$ stabilises $G + \Delta$ for all $\Delta \in A(\mathbb{H})$ with $\|\Delta\|_\infty < \dfrac{1}{\varepsilon}$.*

To see how the Nevanlinna-Pick problem and the robust stabilisation problem are closely related, suppose we seek a controller $C$ which stabilises the SISO system $(G+\Delta, C)$ whenever $\|\Delta\|_\infty < 1$. Moreover, suppose that $G$ is a real rational function and that $M$ and $N$ are also rational. Clearly by Theorem 1.2.4, it is enough to find $Q$ such that

$$\left\|\frac{C}{I+GC}\right\|_\infty = \|M(Y+MQ)\|_\infty = \|MY+M^2Q\|_\infty \stackrel{\text{def}}{=} \|T_1 - T_2Q\|_\infty \leq 1.$$

By changing the variables under the transform $\lambda = \dfrac{1-s}{1+s}$, we can work with functions on the unit disc rather than the right-half plane. Also, if $\varphi = T_1 - T_2Q$, we have $Q - T_1 = -T_2Q$. Thus,

$$\varphi(\lambda) = T_1(\lambda), \text{ for all } \lambda \in \bar{\mathbb{D}} \text{ with } T_2(\lambda) = 0.$$

Conversely, if $\varphi$ does interpolate $T_1$ at each of the zeros of $T_2$, then $\dfrac{T_1 - \varphi}{T_2}$ is analytic and bounded in $\mathbb{D}$ and thus it can be considered as $Q$. Therefore, our problem is to try to construct a function $\varphi$ on $\mathbb{D}$ such that $\|\varphi\|_\infty \leq 1$ and $\varphi(\lambda_j) = z_j$, for all $\lambda_j$ satisfying

$$T_1(\lambda_j) = z_j, \ \text{ and } T_2(\lambda_j) = 0.$$

Clearly, this describes the Nevanlinna-Pick problem, and therefore, this version of the robust stabilisation problem is exactly the same as the Nevanlinna-Pick problem.

Doyle [16] considered a slightly different robust stabilisation problem. He was the first to consider the *structured robust stabilisation problems.* His approach is based on the introduction of the *structured singular value*, which is defined relative to an underlying structure of operators which represent the permissible forms of the perturbation $\Delta$.

The structured singular value as defined in [16] is a function defined on matrices and denoted by $\mu(.)$. In the definition of $\mu(A)$, where $A \in M_2(\mathbb{C})$, there is an underlying structure $E$ (which is a subspace of $M_2(\mathbb{C})$) on which everything in the sequel depends.

**Definition 1.2.5** *Let $A \in M_2(\mathbb{C})$ and let $E$ be the space of $2 \times 2$ diagonal matrices. Then $\mu_E(A)$ is defined as follows:*

$$\mu_E(A) = \frac{1}{\inf\{\|X\| : \ X \in E, \ 1 - AX \ \textit{is singular}\}},$$

*where $\|X\|$ is the maximum singular value of $X$, unless no $X \in E$ makes $1 - AX$ singular, in which case, $\mu_E(A)$ is defined to be $0$.*

Note that, the operator norm of a square matrix $A$, denoted by $\|A\|$, is defined as the square root of the maximum eigenvalue of $A^*A$, that is, the maximum singular value of $A$. Meanwhile, the *spectral radius* $\rho(A)$ of an $n \times n$ matrix $A$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ is defined as follows:

$$\rho(A) = \max_{1 \leq i \leq n} |\lambda_i|.$$

The following remarks were given in [16].

**Remark 1.2.6** *Clearly from the definition of $\mu(A)$, we see that for any $\alpha \in \mathbb{C}$,*

$$\mu(\alpha A) = |\alpha|\mu(A).$$

**Remark 1.2.7** *A natural question is why we work with $\mu$ and not $1/\mu$. While it is clearly a matter of taste, there are important reasons; mathematically, $\mu$ is continuous, bounded and scales as indicated in the remark above. More importantly, it generalises the spectral radius and the maximum singular value. To see that, we state the results below which can be found along with full proofs in [16].*

**Lemma 1.2.8** *Let $A$ and $E$ be defined as in Definition 1.2.5. Then*

$$\mu_E(A) = \max_{\{X \in E : \|X\| \le 1\}} \rho(XA).$$

This lemma implies continuity of $\mu$ is based on continuity of the spectral radius and max functions and the compactness of $\{X \in E : \|X\| \le 1\}$.

**Remarks 1.2.9** *Let $A$ be defined as before. Then*

*(1) If $E = \{\delta I : \delta \in \mathbb{C}\}$, then $\mu_E(A) = \rho(A)$, the spectral radius of $A$.*

*(2) If $E = M_n(\mathbb{C})$, then $\mu_E(A) = \|A\|$.*

*(3) From the definition of $\mu$ and the two remarks above we have $\rho(A) \le \mu_E(A) \le \|A\|$.*

# Chapter 2

# Interpolation into $\Gamma_E$

In this chapter, we define a set $\Gamma_E$ related to the structured singular value of $2 \times 2$ matrices and find a characterisation of its elements. We establish a necessary and sufficient condition for the existence of an analytic function from the unit disc to $\Gamma_E$ satisfying an arbitrary finite number of interpolation conditions, then we find a realization formula for these interpolating functions.

## 2.1 Definitions and Characterisation of $\Gamma_E$

In this section, we define $\Gamma_E$ and give a characterisation of its elements. Let

$$E = \begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix},$$

where $z, w \in \mathbb{C}$.

**Definition 2.1.1** *The set $\Gamma_E$ is defined as follows:*

$$\Gamma_E = \{(a_{11}, a_{22}, \det(A)) : \; A \in M_2(\mathbb{C}), \mu_E(A) \leq 1\},$$

We denote the interior of $\Gamma_E$ by $G_E$ so that

$$G_E = \{(a_{11}, a_{22}, \det(A)) : \ A \in M_2(\mathbb{C}), \mu_E(A) < 1\}.$$

Observe that, the set

$$X = \{(a_{11}, a_{22}, \det(A)) : \ A \in M_2(\mathbb{C}), \mu_E(A) = 1\}$$

is not in $G_E$ because in this case we have $\mu_E(X) = 1$. Hence, for all $\varepsilon > 0$, where $\varepsilon$ is sufficiently small,

$$\mu_E\left((1+\varepsilon)A\right) = 1 + \varepsilon > 1.$$

Therefore, $(1+\varepsilon)X$ is not in $G_E$. Thus, $X \notin G_E$.

It is clear from the definition of $\Gamma_E$ that if $E$ consists of scalar multiples of the $2 \times 2$ identity matrix, that is $E = \{zI_2 : z \in \mathbb{C}\}$, then $\Gamma_E$ is the symmetrised bidisc $\Gamma$, for in this case we have

$$\begin{aligned} 1 - AX \text{ is singular} \quad &\Leftrightarrow \quad 1 - (\mathrm{tr}A)z + (\det A)z^2 \neq 0, \ \text{for all } z \in \mathbb{D} \\ &\Leftrightarrow \quad 1 - sz + pz^2 \neq 0, \ \text{for all } z \in \mathbb{D}, \end{aligned}$$

where $s = \mathrm{tr}A$ and $p = \det A$.

**Remark 2.1.2** *Let* $\Sigma_E = \{A \in M_2(\mathbb{C}) : \ \mu_E(A) \leq 1\}$ *and* $\Sigma = \{A \in M_2(\mathbb{C}) : \ \rho(A) \leq 1\}$, *where* $\rho(A)$ *is the spectral radius of the matrix* $A$. *We observe that the following diagram commutes:*

$$\begin{array}{ccc} \Sigma_E & \hookrightarrow & \Sigma \\ \downarrow & & \downarrow \varphi \\ \Gamma_E & \underset{\tau}{\longrightarrow} & \Gamma \end{array}$$

*where* $\varphi = (\mathrm{tr}, \det)$ *and* $\tau : \Gamma_E \longrightarrow \Gamma$ *is defined by*

$$\tau(a_{11}, a_{22}, \det A) = (a_{11} + a_{22}, \det A)\,.$$

$\mu_E$ is found to be be very useful for analysing the performance and robustness properties of linear feedback systems. A very important and interesting mathematical problem is to find a necessary and sufficient condition for the existence of analytic functions that interpolate from the unit disc $\mathbb{D}$ into $\Gamma_E$.

Our results include a theorem that reduces the problem of analytic interpolation from $\mathbb{D}$ to $\Gamma_E$ to a family of standard classical matricial Nevanlinna-Pick problems.

**Definition 2.1.3** *For* $z \in \mathbb{C}$ *and* $x = (x_1, x_2, x_3) \in \mathbb{C}^3$*, we define* $\Psi_z$ *and* $\Upsilon_z$ *as follows:*

$$\begin{aligned}
\Psi_z(x) &= \begin{cases} \dfrac{x_1 - zx_3}{1 - zx_2}, & \text{if } zx_2 \neq 1; \\ x_1 \text{ if } zx_2 = 1 \text{ and } x_1x_2 = x_3, \end{cases} \\
\Upsilon_z(x) &= \begin{cases} \dfrac{x_2 - zx_3}{1 - zx_1}, & \text{if } zx_1 \neq 1; \\ x_2 \text{ if } zx_1 = 1 \text{ and } x_1x_2 = x_3. \end{cases}
\end{aligned}$$

Note that, $\Psi_z$ is undefined if $x_1x_2 \neq x_3$ and $zx_2 = 1$. Also, $\Upsilon_z$ is undefined if $x_1x_2 \neq x_3$ and $zx_1 = 1$. We shall on occasions write $\Psi(z, x)$ and $\Upsilon(z, x)$ for $\Psi_z(x)$ and $\Upsilon_z(x)$, for $z \in \mathbb{C}$ and $x \in \mathbb{C}^3$.

In our first theorem we prove that $x = (x_1, x_2, x_3) \in \Gamma_E$ is equivalent to 11 different conditions.

**Theorem 2.1.4** *Let* $x = (x_1, x_2, x_3) \in \mathbb{C}^3$*. Then the following are equivalent:*

*(1)* $x \in \Gamma_E$.

*(2)* $1 - x_1 z - x_2 w + x_3 zw \neq 0$, *for all* $(z, w) \in \mathbb{D}^2$.

*(3)* $\begin{cases} |x_2|^2 + |x_1 - \bar{x}_2 x_3| + |x_1 x_2 - x_3| \leq 1, \ \textit{and} \\ |x_1| \leq 1. \end{cases}$

*(4)* $\begin{cases} |x_1|^2 + |x_2 - \bar{x}_1 x_3| + |x_1 x_2 - x_3| \leq 1, \ \textit{and} \\ |x_2| \leq 1. \end{cases}$

*(5)* $\begin{cases} \textit{Either } \Psi(., x) \textit{ is in the Schur class, or} \\ \textit{if } x_1 x_2 = x_3, \ |x_2| \leq 1. \end{cases}$

*(6)* $\begin{cases} \textit{Either } \Upsilon(., x) \textit{ is in the Schur class, or} \\ \textit{if } x_1 x_2 = x_3, \ |x_1| \leq 1. \end{cases}$

*(7)* *There exist* $b, c \in \mathbb{C}$ *such that* $bc = x_1 x_2 - x_3$ *and* $\left\| \begin{pmatrix} x_1 & b \\ c & x_2 \end{pmatrix} \right\| \leq 1.$

*(8)* *There exist* $b, c \in \mathbb{C}$ *such that* $|b| = |c| = |x_1 x_2 - x_3|^{1/2}$, $bc = x_1 x_2 - x_3$ *and* $\left\| \begin{pmatrix} x_1 & b \\ c & x_2 \end{pmatrix} \right\| \leq 1.$

*(9)* $\begin{cases} 1 - |x_1|^2 - |x_2|^2 + |x_3|^2 - 2|x_1 x_2 - x_3| \geq 0, \ \textit{and} \\ |x_1| \leq 1, \ |x_2| \leq 1, \ |x_3| \leq 1. \end{cases}$

*(10)* $\begin{cases} 1 - |x_1|^2 + |x_2|^2 - |x_3|^2 - 2|x_1 \bar{x}_3 - \bar{x}_2| \geq 0, \ \textit{and} \\ |x_1| \leq 1, \ |x_2| \leq 1. \end{cases}$

*(11)* $\begin{cases} 1 + |x_1|^2 - |x_2|^2 - |x_3|^2 - 2|x_2 \bar{x}_3 - \bar{x}_1| \geq 0, \ \textit{and} \\ |x_1| \leq 1, \ |x_2| \leq 1. \end{cases}$

**Proof** Our proof has the following structure:

$$
\begin{array}{ccccccccccccccc}
 & & & & (1) & & & & & & & & & & \\
 & & & & \Updownarrow & & & & & & & & & & \\
(11) & \Longleftrightarrow & (6) & \Longleftrightarrow & (2) & \Longleftrightarrow & (5) & \Longleftrightarrow & (10) & \Longleftrightarrow & (9) & \Longleftrightarrow & (8) & \Longleftrightarrow & (7) \\
 & & & & \Updownarrow & & & & & & & & & & \\
 & & & & (3) & & & & & & & & & & \\
 & & & & \Updownarrow & & & & & & & & & & \\
 & & & & (4) & & & & & & & & & &
\end{array}
$$

The implications from the proof can be found on the following pages:


$$
\begin{aligned}
(1) &\quad \Longleftrightarrow \quad (2):\ \text{P. } 24-26.\\
(2) &\quad \Longleftrightarrow \quad (3):\ \text{P. } 26-35.\\
(3) &\quad \Longleftrightarrow \quad (4):\ \text{P. } 35-36.\\
(2) &\quad \Longleftrightarrow \quad (5):\ \text{P. } 36.\\
(2) &\quad \Longleftrightarrow \quad (6):\ \text{P. } 37.\\
(7) &\quad \Longleftrightarrow \quad (8):\ \text{P. } 37-40.\\
(8) &\quad \Longleftrightarrow \quad (9):\ \text{P. } 40-43.\\
(5) &\quad \Longleftrightarrow \quad (10):\ \text{P. } 44.\\
(6) &\quad \Longleftrightarrow \quad (11):\ \text{P. } 45.\\
(9) &\quad \Longleftrightarrow \quad (10):\ \text{P. } 45-46.
\end{aligned}
$$

**(1)⇔(2)** First, we show that (1) $\Rightarrow$ (2). Let $E = \{\operatorname{diag}(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 \in \mathbb{C}\}$ and let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_2(\mathbb{C}).$$

When $\mu_E(A) \neq 0$, we have

$$\begin{aligned}
\mu_E(A) \leq 1 \;&\Rightarrow\; \inf\{\|X\| :\; X \in E,\; 1 - AX \text{ is singular}\} \geq 1 \\
&\Rightarrow\; \left\| \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} \right\| \geq 1 \text{ for all } X = \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} \text{ such that } 1 - AX \text{ is singular} \\
&\Rightarrow\; \max\{|z|, |w|\} \geq 1 \text{ for all } z, w \in \mathbb{C} \text{ such that} \\
&\qquad \det \begin{pmatrix} 1 - a_{11}z & -a_{12}w \\ -a_{21}z & 1 - a_{22}w \end{pmatrix} = 0 \\
&\Rightarrow\; [(1 - a_{11}z)(1 - a_{22}w) - a_{12}a_{21}zw = 0 \Rightarrow \max\{|z|, |w|\} \geq 1] \\
&\Rightarrow\; [z, w \in \mathbb{D} \Rightarrow 1 - a_{11}z - a_{22}w + \det(A)zw \neq 0] \\
&\Rightarrow\; 1 - x_1z - x_2w + x_3zw \neq 0, \text{ for all } (z, w) \in \mathbb{D}^2
\end{aligned}$$

where $x_1 = a_{11},\ x_2 = a_{22}$, and $x_3 = \det(A)$.

Conversely, when $\mu_E(A) \neq 0$, the proof of (2) $\Rightarrow$ (1) is as follows:

$$\begin{aligned}
\text{(2) holds} \;&\Rightarrow\; 1 - x_1z - x_2w + x_3zw \neq 0, \text{ for all } (z, w) \in \mathbb{D}^2 \\
&\Rightarrow\; \text{we can find a } 2 \times 2 \text{ matrix } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\
&\qquad \text{so that } x_1 = a_{11},\ x_2 = a_{22},\ x_3 = \det(A), \text{ and} \\
&\qquad 1 - x_1z - x_2w + x_3zw \neq 0, \text{ for all } (z, w) \in \mathbb{D}^2 \\
&\Rightarrow\; 1 - a_{11}z - a_{22}w + \det(A)zw \neq 0, \text{ for all } z, w \in \mathbb{D} \\
&\Rightarrow\; [(1 - a_{11}z)(1 - a_{22}w) - a_{12}a_{21}zw = 0 \Rightarrow \max\{|z|, |w|\} \geq 1]
\end{aligned}$$

$$
\begin{aligned}
(2) \text{ holds } &\Rightarrow \max\{|z|,|w|\} \geq 1 \text{ for all } z,w \in \mathbb{C} \text{ such that} \\
&\quad \det \begin{pmatrix} 1-a_{11}z & -a_{12}w \\ -a_{21}z & 1-a_{22}w \end{pmatrix} = 0 \\
&\Rightarrow \left\| \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} \right\| \geq 1 \text{ for all } X = \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} \text{ such that } 1 - AX \text{ is singular} \\
&\Rightarrow \inf\{\|X\| : \ X \in E, \ 1 - AX \text{ is singular}\} \geq 1 \\
&\Rightarrow \mu_E(A) \leq 1 \\
&\Rightarrow (1) \text{ holds.}
\end{aligned}
$$

That is, (1) $\Leftrightarrow$ (2) in the case that $\mu_E(A) \neq 0$.

The case that $\mu_E(A) = 0$: Let

$$
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{C}).
$$

We claim that $\mu_E(A) = 0 \iff a_{11} = 0, a_{22} = 0$ and at least one of $a_{12}$ and $a_{21}$ equals zero.

($\Leftarrow$) Suppose $a_{11} = 0$, $a_{22} = 0$ and at least one of $a_{12}$ and $a_{21}$ is zero, then $\det(1 - AX) = 1$ for all $X$ in $E$.

Hence, $1 - AX$ is non-singular for all $X$ in $E$, and therefore, $\mu_E(A) = 0$.

($\Rightarrow$) Now, suppose that $\mu_E(A) = 0$. This means that there is no $X$ in $E$ that makes $1 - AX$ singular. That is, $1 - AX$ is non-singular for all $X$ in $E$, which is the same as saying $\det(1 - AX) \neq 0, \forall X \in E$.

$$
\det(1 - AX) = 1 - a_{11}z - a_{22}w + (a_{11}a_{22} - a_{12}a_{21})zw \neq 0.
$$

Since this polynomial in $z, w$ has no zeros in $\mathbb{C}^2$, it is a non-zero constant, and hence $a_{11} = 0$, $a_{22} = 0$, and at least one of $a_{12}$ and $a_{21}$ is zero.

Therefore, for $x_1 = a_{11},\ x_2 = a_{22}$, and $x_3 = \det(A)$,

$$\mu_E(A) = 0 \iff 1 - x_1 z - x_2 w + x_3 zw \neq 0.$$

That is, (1) $\Leftrightarrow$ (2) in the case that $\mu_E(A) = 0$. This completes the proof that (1) $\Leftrightarrow$ (2).

We shall now show that **(2)$\Leftrightarrow$(3)**. We have

$$\begin{aligned}
\text{(2) holds} &\iff \forall z, w \in \mathbb{D},\ 1 - x_1 z - x_2 w + x_3 zw \neq 0,\\
&\iff \forall z, w \in \mathbb{D},\ z(x_3 w - x_1) \neq x_2 w - 1,\\
&\iff \begin{cases} \forall z, w \in \mathbb{D} \text{ such that } x_3 w \neq x_1,\ z \neq \dfrac{x_2 w - 1}{x_3 w - x_1}, \text{ and} \\ \text{if } \dfrac{x_1}{x_3} \in \mathbb{D},\ 0 \neq x_2 \dfrac{x_1}{x_3} - 1, \end{cases}\\
&\iff \begin{cases} f(\mathbb{D}) \cap \mathbb{D} = \emptyset, \text{ where } f(z) = \dfrac{x_2 z - 1}{x_3 z - x_1}, \text{ and} \\ \text{if } |x_1| < |x_3|,\ x_3 \neq x_1 x_2. \end{cases} \qquad (2.1)
\end{aligned}$$

We need to find $f(\mathbb{D})$. We consider the following cases:

(i) $|x_1| < |x_3|$,

(ii) $|x_1| > |x_3| \neq 0$,

(iii) $|x_3| = 0$,

(iv) $|x_1| = |x_3| \neq 0$ and $x_1 x_2 \neq x_3$,

(v) $|x_1| = |x_3| \neq 0$ and $x_1 x_2 = x_3$.

When $|x_3| \neq |x_1|$, $f$ maps

$$\begin{aligned}
\frac{x_1}{x_3} &\longmapsto \infty,\\
\frac{\bar{x}_3}{\bar{x}_1} &\longmapsto \frac{x_2 \bar{x}_3 - \bar{x}_1}{|x_3|^2 - |x_1|^2}.
\end{aligned}$$

Thus $f(\mathbb{T})$ is a circle of centre $\dfrac{x_2\bar{x}_3 - \bar{x}_1}{|x_3|^2 - |x_1|^2}$, where $\mathbb{T}$ is the unit circle.
$f(1) = \dfrac{x_2 - 1}{x_3 - x_1} \in f(\mathbb{T})$.
Therefore, the radius of $f(\mathbb{T})$ is

$$\left|\frac{x_2\bar{x}_3 - \bar{x}_1}{|x_3|^2 - |x_1|^2} - \frac{x_2 - 1}{x_3 - x_1}\right| = \left|\frac{x_3 - x_1x_2}{|x_3|^2 - |x_1|^2}\right|.$$

$f(\mathbb{D})$ is either the bounded component or the unbounded component of the circle of centre $\dfrac{x_2\bar{x}_3 - \bar{x}_1}{|x_3|^2 - |x_1|^2}$ and radius $\left|\dfrac{x_3 - x_1x_2}{|x_3|^2 - |x_1|^2}\right|$.
If $\left|\dfrac{x_1}{x_3}\right| < 1$, then $\infty = f(\dfrac{x_1}{x_3}) \in f(\mathbb{D})$ and so $f(\mathbb{D})$ is the unbounded component of $f(\mathbb{T})$. Likewise, if $\left|\dfrac{x_1}{x_3}\right| > 1$, then $f(\mathbb{D})$ is the bounded component of $f(\mathbb{T})$.

Case (i): The case that $|x_1| < |x_3|$.

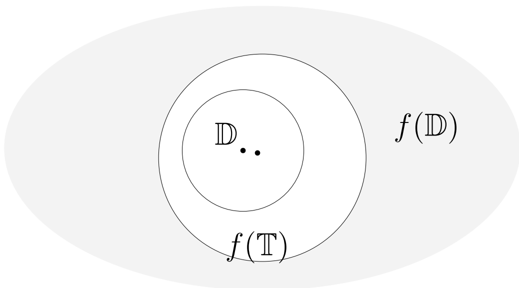



Figure 2.1: $f(\mathbb{D})$ is the unbounded component of $f(\mathbb{T})$

$$\text{(2) holds} \iff \begin{cases} f(\mathbb{D}) \cap \mathbb{D} = \emptyset, \\ x_3 \neq x_1x_2, \end{cases}$$

$$\iff \begin{cases} \text{distance between centres} + 1 \leq \text{radius of } f(\mathbb{T}), \\ x_3 \neq x_1x_2, \end{cases}$$

$$\iff \begin{cases} \left|\dfrac{x_2\bar{x}_3 - \bar{x}_1}{|x_3|^2 - |x_1|^2}\right| - \left|\dfrac{x_3 - x_1x_2}{|x_3|^2 - |x_1|^2}\right| \leq -1, \\ x_3 \neq x_1x_2, \end{cases}$$

$$(2) \text{ holds} \quad \Longleftrightarrow \quad \frac{|x_2\bar{x}_3 - \bar{x}_1| - |x_3 - x_1x_2|}{|x_3|^2 - |x_1|^2} \leq -1. \tag{2.2}$$

Automatically, $x_3 \neq x_1x_2$ if this holds.

Case (ii): The case that $|x_1| > |x_3|$.

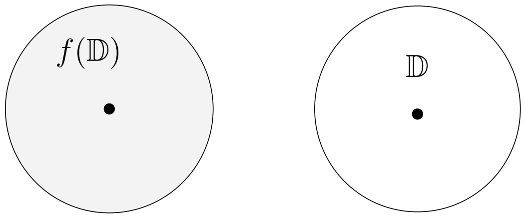



Figure 2.2: $f(\mathbb{D})$ is the bounded component of $f(\mathbb{T})$

We have

$$\begin{aligned}
(2) \text{ holds} \quad &\Longleftrightarrow \quad f(\mathbb{D}) \cap \mathbb{D} = \emptyset, \\
&\Longleftrightarrow \quad |\text{centre of } f(\mathbb{D})| \geq 1 + \text{ radius of } f(\mathbb{D}), \\
&\Longleftrightarrow \quad \left| \frac{x_2\bar{x}_3 - \bar{x}_1}{|x_3|^2 - |x_1|^2} \right| - \left| \frac{x_3 - x_1x_2}{|x_3|^2 - |x_1|^2} \right| \geq 1, \\
&\Longleftrightarrow \quad \frac{|x_2\bar{x}_3 - \bar{x}_1| - |x_3 - x_1x_2|}{|x_1|^2 - |x_3|^2} \geq 1. \qquad (2.3)
\end{aligned}$$

Both inequalities (2.2) and (2.3) can be written as follows:

$$(2) \text{ holds} \quad \Longleftrightarrow \quad \frac{|x_2\bar{x}_3 - \bar{x}_1| - |x_3 - x_1x_2|}{|x_1|^2 - |x_3|^2} \geq 1. \tag{2.4}$$

We multiply both sides of the equivalence (2.4) by

$$|x_2\bar{x}_3 - \bar{x}_1| + |x_3 - x_1x_2|,$$

which is strictly positive, then we factorise the left hand side and that will give

$$
\begin{aligned}
\text{(2) holds} \quad &\iff \quad \frac{|x_2\bar{x}_3 - \bar{x}_1| - |x_3 - x_1x_2|}{|x_1|^2 - |x_3|^2} \geq 1, \\
&\iff \quad \frac{|x_2\bar{x}_3 - \bar{x}_1|^2 - |x_3 - x_1x_2|^2}{|x_1|^2 - |x_3|^2} \geq |x_2\bar{x}_3 - \bar{x}_1| + |x_3 - x_1x_2|, \\
&\iff \quad \frac{(|x_2|^2 - 1)(|x_3|^2 - |x_1|^2)}{|x_1|^2 - |x_3|^2} \geq |x_2\bar{x}_3 - \bar{x}_1| + |x_3 - x_1x_2|, \\
&\iff \quad 1 - |x_2|^2 \geq |x_2\bar{x}_3 - \bar{x}_1| + |x_3 - x_1x_2|.
\end{aligned}
$$

Moreover, we know that (1) and (2) are equivalent, that is,

$$
\begin{aligned}
\text{(2) holds} \quad \iff \quad & \text{there exists a } 2 \times 2 \text{ matrix } A \text{ with } a_{11} = x_1, a_{22} = x_2, \det(A) = x_3, \\
& \text{and } \mu_E(A) \leq 1,
\end{aligned}
$$

Hence, $|x_1| \leq 1$, $|x_2| \leq 1$ and $|x_3| \leq 1$.

Therefore, in the cases that $|x_3| \neq |x_1|$ and $x_3 \neq 0$, that is, cases (i) and (ii),

$$
\text{(2) holds} \iff \begin{cases} |x_2|^2 + |x_1 - \bar{x}_2x_3| + |x_1x_2 - x_3| \leq 1, \text{ and} \\ |x_1| \leq 1. \end{cases} \tag{2.5}
$$

We shall show that this equivalence remains true when $|x_3| = |x_1|$ and when $x_3 = 0$.

Case (iii): First consider the case that $x_3 = 0$. If we take $x_3$ to be zero in equivalence (2.5) above, we get

$$
\begin{aligned}
\text{(2) holds} \quad &\iff \quad \begin{cases} |x_2|^2 + |x_1| + |x_1x_2| \leq 1, \text{ and} \\ |x_1| \leq 1 \end{cases} \\
&\iff \quad \begin{cases} |x_1|(1 + |x_2|) \leq 1 - |x_2|^2, \text{ and} \\ |x_1| \leq 1 \end{cases} \\
&\iff \quad \begin{cases} |x_1| \leq 1 - |x_2|, \text{ and} \\ |x_1| \leq 1 \end{cases} \\
&\iff \quad |x_1| + |x_2| \leq 1. \qquad (2.6)
\end{aligned}
$$

We claim that the equivalence (2.6) is true, that is, if $x_3 = 0$ then

$$(2) \text{ holds} \iff |x_1| + |x_2| \leq 1.$$

We shall show that

$$\forall z, w \in \mathbb{D},\ x_1 z + x_2 w \neq 1 \iff |x_1| + |x_2| \leq 1.$$

($\Leftarrow$) Suppose that there exist $z$ and $w$ in $\mathbb{D}$, such that $x_1 z + x_2 w = 1$. We show that $|x_1| + |x_2| > 1$.

Since $z$ and $w$ are in $\mathbb{D}$, $|z| < 1$ and $|w| < 1$. Therefore, $|x_1 z| < |x_1|$ and $|x_2 w| < |x_2|$.

Thus

$$\begin{aligned} 1 &= |x_1 z + x_2 w|, \\ &\leq |x_1 z| + |x_2 w|, \\ &< |x_1| + |x_2|. \end{aligned}$$

Hence,

$$\forall z, w \in \mathbb{D},\ x_1 z + x_2 w \neq 1 \Leftarrow |x_1| + |x_2| \leq 1.$$

($\Rightarrow$) Conversely, suppose that $|x_1| + |x_2| > 1$. We show that there exist $z$ and $w$ in $\mathbb{D}$, such that $x_1 z + x_2 w = 1$.

First case: If $|x_1| \neq 0, |x_2| \neq 0$. Let

$$z = \frac{|x_1|}{x_1(|x_1| + |x_2|)} \quad \text{and} \quad w = \frac{|x_2|}{x_2(|x_1| + |x_2|)},$$

$$|z| = |w| = \frac{1}{|x_1| + |x_2|} < 1.$$

Then

$$\begin{aligned} x_1 z + x_2 w &= x_1 \frac{|x_1|}{x_1(|x_1| + |x_2|)} + x_2 \frac{|x_2|}{x_2(|x_1| + |x_2|)}, \\ &= 1. \end{aligned}$$

Second case: At least one of $x_1, x_2$ is zero. We can suppose that
$x_1 = 0, |x_2| > 1$. Let $w = \dfrac{1}{x_2}, z = \dfrac{1}{2}$.
Then $z, w \in \mathbb{D}$ and $x_1 z + x_2 w = 1$.

Therefore,

$$|x_1| + |x_2| \leq 1 \Leftarrow \forall z, w \in \mathbb{D},\ x_1 z + x_2 w \neq 1.$$

Therefore, in the case that $x_3 = 0$,

$$(2) \text{ holds } \iff |x_1| + |x_2| \leq 1.$$

Case (iv): We claim also that the equivalence (2.5) is true when $|x_3| = |x_1|$.

As before,

$$(2) \text{ holds } \iff f(\mathbb{D}) \cap \mathbb{D} = \emptyset, \text{ where } f(z) = \frac{x_2 z - 1}{x_3 z - x_1}.$$

We have to find $f(\mathbb{D})$.

$f$ maps

$$\begin{aligned} \frac{x_1}{x_3} &\longmapsto \infty, \\ 0 &\longmapsto \frac{1}{x_1}, \\ \infty &\longmapsto \frac{x_2}{x_3}. \end{aligned}$$

$$f(\mathbb{D}) = \left\{ z \in \mathbb{C} : \left| z - \frac{1}{x_1} \right| < \left| z - \frac{x_2}{x_3} \right| \right\}.$$

$$\begin{aligned} f(\mathbb{D}) \cap \mathbb{D} = \emptyset \quad &\iff \quad 0 \notin f(\mathbb{D}), \text{ and} \\ &\quad\quad \text{the distance from 0 to the perpendicular bisector of } \frac{1}{x_1} \text{ and } \frac{x_2}{x_3} \geq 1, \\ &\iff \quad \left| \frac{1}{x_1} \right| \geq \left| \frac{x_2}{x_3} \right|, \text{ and} \\ &\quad\quad \text{the distance from 0 to the perpendicular bisector of } \frac{1}{x_1} \text{ and } \frac{x_2}{x_3} \geq 1. \end{aligned}$$

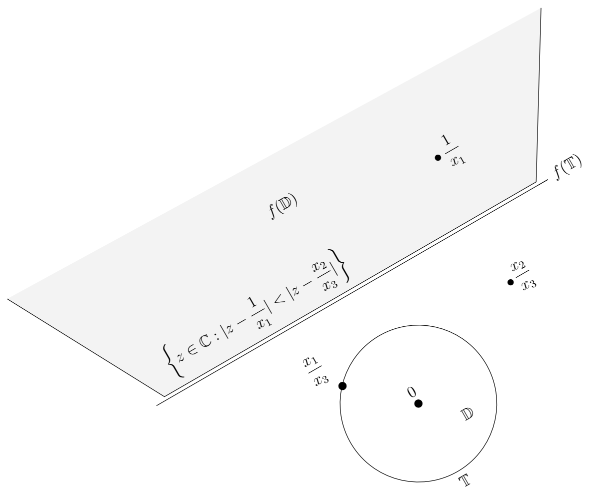



Figure 2.3: Finding $f(\mathbb{D})$

Choose $t$ so that

$$
\begin{aligned}
& \left| t\left(\frac{1}{x_1} - \frac{x_2}{x_3}\right) - \frac{1}{x_1} \right|^2 = \left| t\left(\frac{1}{x_1} - \frac{x_2}{x_3}\right) - \frac{x_2}{x_3} \right|^2 \\
\iff \quad & t^2 \left| \frac{1}{x_1} - \frac{x_2}{x_3} \right|^2 + \frac{1}{|x_1|^2} - 2t\mathrm{Re}\left\{ \left(\frac{1}{x_1} - \frac{x_2}{x_3}\right) \frac{1}{\bar{x}_1} \right\} \\
= \quad & t^2 \left| \frac{1}{x_1} - \frac{x_2}{x_3} \right|^2 + \left| \frac{x_2}{x_3} \right|^2 - 2t\mathrm{Re}\left\{ \left(\frac{1}{x_1} - \frac{x_2}{x_3}\right) \frac{\bar{x}_2}{\bar{x}_3} \right\} \\
\iff \quad & 2t\mathrm{Re}\left\{ \left(\frac{1}{x_1} - \frac{x_2}{x_3}\right) \left(\frac{\bar{x}_2}{\bar{x}_3} - \frac{1}{\bar{x}_1}\right) \right\} = \left| \frac{x_2}{x_3} \right|^2 - \frac{1}{|x_1|^2} \\
\iff \quad & -2t \left| \frac{1}{x_1} - \frac{x_2}{x_3} \right|^2 = \left| \frac{x_2}{x_3} \right|^2 - \frac{1}{|x_1|^2} \\
\iff \quad & t = \frac{\frac{1}{|x_1|^2} - \left| \frac{x_2}{x_3} \right|^2}{2\left| \frac{1}{x_1} - \frac{x_2}{x_3} \right|^2}, \ t > 0.
\end{aligned}
$$

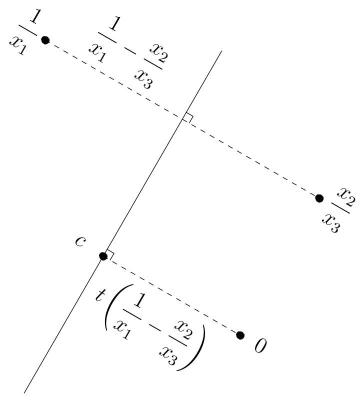



Figure 2.4: Distance between 0 and $c$

Hence, $c = t\left(\dfrac{1}{x_1} - \dfrac{x_2}{x_3}\right)$ can be easily calculated as follows:

$$
\begin{aligned}
c &= \frac{\dfrac{1}{|x_1|^2} - \left|\dfrac{x_2}{x_3}\right|^2}{2\left(\dfrac{1}{\bar{x}_1} - \dfrac{\bar{x}_2}{\bar{x}_3}\right)}, \\
&= \frac{|x_3|^2 - |x_1x_2|^2}{|x_1x_3|^2}\frac{\bar{x}_1\bar{x}_3}{2(\bar{x}_3 - \bar{x}_1\bar{x}_2)}, \\
&= \frac{|x_3|^2 - |x_1x_2|^2}{2x_1x_3(\bar{x}_3 - \bar{x}_1\bar{x}_2)}.
\end{aligned}
$$

Therefore, the distance from 0 to the perpendicular bisector of $\dfrac{1}{x_1}$ and $\dfrac{x_2}{x_3}$ is

$$|c| = \left|\frac{|x_3|^2 - |x_1x_2|^2}{2x_1x_3(\bar{x}_3 - \bar{x}_1\bar{x}_2)}\right| = \left|\frac{1 - |x_2|^2}{2|x_3 - x_1x_2|}\right|.$$

Hence, in the case that $|x_3| = |x_1| \neq 0, x_3 \neq x_1x_2$,

$$\text{(2) holds} \iff \begin{cases} |x_2| \leq 1, \ \dfrac{1 - |x_2|^2}{2|x_3 - x_1x_2|} \geq 1, \ \text{and} \\ |x_1| \leq 1 \end{cases}$$

$$
\text{(2) holds} \iff \begin{cases} 1-|x_2|^2 \geq 2|x_3-x_1x_2|, \text{ and} \\ |x_1| \leq 1 \end{cases}
$$

$$
\iff \begin{cases} |x_2|^2 + 2|x_3-x_1x_2| \leq 1, \text{ and} \\ |x_1| \leq 1, \end{cases}
$$

which is equivalent to the right hand side of (2.5), that is, equivalent to

$$
\begin{cases} |x_2|^2 + |x_1-\bar{x}_2x_3| + |x_1x_2-x_3| \leq 1, \text{ and} \\ |x_1| \leq 1, \end{cases}
$$

for when $|x_3| = |x_1|$, it is clear to see that

$$
\begin{aligned}
|x_2\bar{x}_3-\bar{x}_1| = |x_3-x_1x_2| \quad &\iff \quad |x_2\bar{x}_3-\bar{x}_1|^2 = |x_3-x_1x_2|^2 \\
&\iff \quad |x_2x_3|^2 + |x_1|^2 - 2\mathrm{Re}\{x_2\bar{x}_3x_1\} \\
&\qquad = |x_3|^2 + |x_1x_2|^2 - 2\mathrm{Re}\{x_1x_2\bar{x}_3\},
\end{aligned}
$$

which is always true for $|x_3| = |x_1|$.

Case (v): The case that $|x_3| = |x_1| \neq 0, x_3 = x_1x_2$. If $|x_2| < 1$, we have

$$
\begin{aligned}
|x_2|^2 + |x_2\bar{x}_3-\bar{x}_1| + |x_3-x_1x_2| \leq 1 \quad &\iff \quad |x_2|^2 + |x_2\bar{x}_1\bar{x}_2-\bar{x}_1| \leq 1 \\
&\iff \quad |x_2|^2 + |x_1|\,\big||x_2|^2-1\big| \leq 1 \\
&\iff \quad |x_1|(1-|x_2|^2) \leq 1-|x_2|^2 \\
&\iff \quad |x_1| \leq 1.
\end{aligned}
$$

If $|x_2| = 1, |x_3| = |x_1| \neq 0, x_3 = x_1x_2$, equivalence (2.5) holds. Also, if (2) holds, then $|x_1| \leq 1$ and $|x_2| \leq 1$.

Now, if $|x_1| \leq 1$ and $|x_2| \leq 1$,

$$\begin{aligned}
(2) \text{ holds} \quad &\iff \quad x_3zw - x_1z - x_2w + 1 \neq 0,\ \forall z, w \in \mathbb{D} \\
&\iff \quad x_1x_2zw - x_1z - x_2w + 1 \neq 0,\ \forall z, w \in \mathbb{D} \\
&\iff \quad x_1z(x_2w - 1) - (x_2w - 1) \neq 0,\ \forall z, w \in \mathbb{D} \\
&\iff \quad (x_1z - 1)(x_2w - 1) \neq 0,\ \forall z, w \in \mathbb{D} \\
&\iff \quad x_1z - 1 \neq 0 \text{ and } x_2w - 1 \neq 0,\ \forall z, w \in \mathbb{D} \\
&\iff \quad z \neq \frac{1}{x_1} \text{ and } w \neq \frac{1}{x_2},\ \forall z, w \in \mathbb{D},
\end{aligned}$$

which is true for $|x_1| \leq 1$ and $|x_2| \leq 1$.

This concludes the proof that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

**(3)$\Leftrightarrow$(4)** Since (1) $\Leftrightarrow$ (3), there exists a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_2(\mathbb{C}),$$

such that $(a_{11}, a_{22}, \det(A)) \in \Gamma_E$. Let

$$\widetilde{A} = \begin{bmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{bmatrix} = JAJ,$$

where $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Also, by the equivalence (1) $\Leftrightarrow$ (3) and by the definition of $\Gamma_E$, if $\mu_E(A) \neq 0$,

$$\begin{aligned}
\frac{1}{\mu_E(A)} \quad &= \quad \inf\{\|X\| : X \in E, 1 - AX \text{ is singular}\} \\
&= \quad \inf\{\|X\| : X \in E, 1 - J\widetilde{A}JX \text{ is singular}\}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\mu_E(A)} &= \inf\{\|X\| : X \in E, 1 - \widetilde{A}JXJ \text{ is singular}\} \\
&= \inf\{\|Y\| : Y = JXJ \in E, 1 - \widetilde{A}Y \text{ is singular}\} \\
&= \frac{1}{\mu_E(\widetilde{A})}.
\end{aligned}$$

Hence, if $\widetilde{x_1} = a_{22} = x_2$, $\widetilde{x_2} = a_{11} = x_1$, $\widetilde{x_3} = x_3$, we get

$$\begin{aligned}
\begin{cases} |x_2|^2 + |x_1 - \bar{x}_2 x_3| + |x_1x_2 - x_3| \le 1 \\ |x_1| \le 1 \end{cases} &\iff \mu_E(A) \le 1 \\
&\iff \mu_E(\widetilde{A}) \le 1 \\
&\iff \begin{cases} |\widetilde{x_2}|^2 + |\widetilde{x_1} - \bar{\widetilde{x_2}}\widetilde{x_3}| + |\widetilde{x_1}\widetilde{x_2} - \widetilde{x_3}| \le 1, \\ |\widetilde{x_1}| \le 1 \end{cases} \\
&\iff \begin{cases} |x_1|^2 + |x_2 - \bar{x}_1 x_3| + |x_1x_2 - x_3| \le 1, \\ |x_2| \le 1. \end{cases}
\end{aligned}$$

Moreover, in the case that $\mu_E(A) = 0$, the equivalence (3) $\Leftrightarrow$ (4) clearly holds since in this case $x = (0, 0, 0)$.

Hence, (3) $\Leftrightarrow$ (4).

**(2)$\Leftrightarrow$(5)** From the equivalence (2.1) of the proof of (2) $\Leftrightarrow$ (3), we find that

$$\begin{aligned}
\text{(2) holds} &\iff \begin{cases} f(\mathbb{D}) \cap \mathbb{D} = \emptyset, \text{ where } f(z) = \dfrac{x_2z - 1}{x_3z - x_1}, \\ \text{if } |x_1| < |x_3|,\ x_1x_2 \ne x_3. \end{cases} \\
&\iff \begin{cases} \dfrac{1}{f} \text{ is in the Schur class, where } \dfrac{1}{f(z)} = \dfrac{x_1 - zx_3}{1 - zx_2}, \text{ and} \\ \text{if } x_1x_2 = x_3,\ |x_2| \le 1. \end{cases}
\end{aligned}$$

Note that if $|x_1| < |x_3|$, $x_1x_2 \ne x_3$ always holds, and if $x_1x_2 = x_3$, $|x_1| \le 1,\ |x_2| \le 1$ (see case (v) of the proof of (2) $\Leftrightarrow$ (3) above). Thus

$$\begin{aligned}
\text{(2) holds} &\iff \begin{cases} \Psi(., x) \text{ is in the Schur class, and} \\ \text{if } x_1x_2 = x_3,\ |x_2| \le 1. \end{cases} \\
&\iff \text{(5) holds.}
\end{aligned}$$

$(\mathbf{2}) \Leftrightarrow (\mathbf{6})$ Similarly, let $f(z) = \dfrac{1 - zx_1}{x_2 - zx_3}$, by using the same method as in our proof of $(2) \Leftrightarrow (5)$, we find that

$$\text{(2) holds} \quad \Longleftrightarrow \quad \begin{cases} \dfrac{1}{f} \text{ is in the Schur class, where } \dfrac{1}{f(z)} = \dfrac{x_2 - zx_3}{1 - zx_1}, \text{ and} \\ \text{if } x_1x_2 = x_3, \ |x_1| \leq 1. \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} \Upsilon(., x) \text{ is in the Schur class, and} \\ \text{if } x_1x_2 = x_3, \ |x_1| \leq 1. \end{cases}$$

$$\Longleftrightarrow \quad \text{(6) holds.}$$

Now we show that $\mathbf{(7)} \Leftrightarrow \mathbf{(8)}$. Trivially $(8) \Rightarrow (7)$. Suppose (7) holds. Consider the analytic function $F : \mathbb{C}\backslash\{0\} \longrightarrow M_2(\mathbb{C})$ defined by

$$F(z) = \begin{bmatrix} x_1 & uz \\ \dfrac{v}{z} & x_2 \end{bmatrix}.$$

We show first that $\|F(z)\|$ is constant on $|z| = R$, where $R$ is the radius of a disc. For $|\omega| = 1$, we have

$$\begin{aligned} \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix} \begin{bmatrix} x_1 & uz \\ \dfrac{v}{z} & x_2 \end{bmatrix} \begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix} &= \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix} \begin{bmatrix} x_1\bar{\omega} & uz\omega \\ \dfrac{v}{z}\bar{\omega} & x_2\omega \end{bmatrix} \\ &= \begin{bmatrix} x_1 & uz\omega^2 \\ \dfrac{v}{z}\bar{\omega}^2 & x_2 \end{bmatrix}. \end{aligned}$$

Hence,

$$\|F(z)\| = \|F(\omega^2 z)\|.$$

Therefore, $\|F(z)\|$ is constant on $|z| = R$.

Moreover,

$$F(1) = \begin{bmatrix} x_1 & u \\ v & x_2 \end{bmatrix}, \tag{2.7}$$

and, for $u \neq 0$,

$$F\left(\frac{v}{u}\right) = \begin{bmatrix} x_1 & v \\ u & x_2 \end{bmatrix}. \tag{2.8}$$

Observe that the matrix in (2.8) is the transpose of the matrix in (2.7). Hence, they have the same norm. Therefore, in the case that $u \neq 0$, we have

$$\|F(1)\| = \left\|F\left(\frac{v}{u}\right)\right\|. \tag{2.9}$$

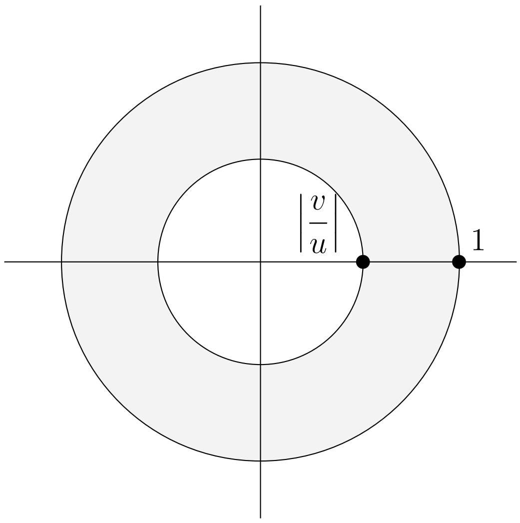



Figure 2.5: $\|F(z)\|$ is constant on $|z| = 1$

We have three cases; case (i) when $|v| < |u|$, case (ii) when $|v| > |u|$ and case (iii) when $|v| = |u|$.

Case (i) when $|v| < |u|$: By the maximum modulus principle (at $z^2 = \frac{v}{u}$), we find that

$$\|F(z)\| \leq |\text{values on the boundary}|.$$

Note that for all boundary values we have

$$\|F(w)\| = \|F(1)\|,$$

where $w$ is on the boundary. Therefore,

$$\|F(z)\| \leq \|F(1)\|.$$

Case (ii) when $|v| > |u|$: By the maximum modulus principle, we find that

$$\|F(z)\| \leq \left\| F\left(\frac{v}{u}\right) \right\|,$$

and by (2.9), we find that

$$\|F(z)\| \leq \|F(1)\|.$$

Therefore, in case (i) and case (ii), we find that $\|F(z)\| \leq \|F(1)\|$.

Case (iii) when $|v| = |u|$, we have

$$F(z) = \begin{bmatrix} x_1 & uz \\ \frac{v}{z} & x_2 \end{bmatrix},$$

where $z^2 = \dfrac{v}{u}$. Hence,

$$|uz| = |u| \left| \frac{v}{u} \right|^{1/2} = |uv|^{1/2},$$

and

$$\left| \frac{v}{z} \right| = |v| \left| \frac{u}{v} \right|^{1/2} = |uv|^{1/2}.$$

Hence,

$$|uz| = \left| \frac{v}{z} \right|.$$

Therefore, for a $2 \times 2$ matrix

$$\begin{bmatrix} x_1 & b \\ c & x_2 \end{bmatrix},$$

where $x_3 = x_1x_2 - bc$, that is, $bc = x_1x_2 - x_3$, we have

$$|b| = |c| = |bc|^{1/2}.$$

That is,

$$|b| = |c| = |x_1x_2 - x_3|^{1/2}.$$

Note that, in the case $uv = 0$, we suppose that $v = 0$. Define

$$G(z) = \begin{bmatrix} x_1 & uz \\ 0 & x_2 \end{bmatrix}.$$

Then we have

$$\|G(1)\| \geq \|G(0)\|,$$

where

$$G(1) = \begin{bmatrix} x_1 & u \\ 0 & x_2 \end{bmatrix}, \text{ and } G(0) = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}.$$

Therefore, $uv = 0 = x_1x_2 - x_3$, hence, $|u| = |v| = 0 = |x_1x_2 - x_3|^{1/2}$.

Thus, (7) $\Rightarrow$ (8), and therefore (7) $\Leftrightarrow$ (8).

To show that **(8)$\Leftrightarrow$(9)**, we need the following lemma.

**Lemma 2.1.5** *Let*

$$X = \begin{bmatrix} x_1 & b \\ c & x_2 \end{bmatrix} \in M_2(\mathbb{C}),$$

$bc = x_1x_2 - x_3$, *and* $|b| = |c| = |x_1x_2 - x_3|^{1/2}$, *where* $x_3 = \det(X)$. *Then the following hold:*

*(1)* $\|X\| \leq 1 \Leftrightarrow \begin{bmatrix} 1 - |x_1|^2 - |x_1x_2 - x_3| & -\bar{x}_1 b - x_2\bar{c} \\ -x_1\bar{b} - \bar{x}_2 c & 1 - |x_2|^2 - |x_1x_2 - x_3| \end{bmatrix} \geq 0,$

*(2)* $\det(1 - X^*X) = 1 - |x_1|^2 - |x_2|^2 + |x_3|^2 - 2|x_1x_2 - x_3|.$

**Proof** First we show that (1) holds. We know that

$$\|X\| \leq 1 \Leftrightarrow 1 - X^*X \geq 0.$$

Now we calculate $1 - X^*X$.

$$\begin{aligned} 1 - X^*X &= 1 - \begin{bmatrix} \bar{x}_1 & \bar{c} \\ \bar{b} & \bar{x}_2 \end{bmatrix} \begin{bmatrix} x_1 & b \\ c & x_2 \end{bmatrix}, \\ &= \begin{bmatrix} 1 - |x_1|^2 - |c|^2 & -\bar{x}_1 b - x_2 \bar{c} \\ -x_1 \bar{b} - \bar{x}_2 c & 1 - |x_2|^2 - |b|^2 \end{bmatrix}, \\ &= \begin{bmatrix} 1 - |x_1|^2 - |x_1 x_2 - x_3| & -\bar{x}_1 b - x_2 \bar{c} \\ -x_1 \bar{b} - \bar{x}_2 c & 1 - |x_2|^2 - |x_1 x_2 - x_3| \end{bmatrix}. \end{aligned}$$

Therefore,

$$\|X\| \leq 1 \iff \begin{bmatrix} 1 - |x_1|^2 - |x_1 x_2 - x_3| & -\bar{x}_1 b - x_2 \bar{c} \\ -x_1 \bar{b} - \bar{x}_2 c & 1 - |x_2|^2 - |x_1 x_2 - x_3| \end{bmatrix} \geq 0. \tag{2.10}$$

That is, (1) holds.

We shall show now that (2) holds. Observe that

$$\begin{aligned} \det(1 - X^*X) &= \det\left( \begin{bmatrix} 1 - |x_1|^2 - |x_1 x_2 - x_3| & -\bar{x}_1 b - x_2 \bar{c} \\ -x_1 \bar{b} - \bar{x}_2 c & 1 - |x_2|^2 - |x_1 x_2 - x_3| \end{bmatrix} \right) \\ &= (1 - |x_1|^2 - |x_1 x_2 - x_3|)(1 - |x_2|^2 - |x_1 x_2 - x_3|) \\ &\quad -(-\bar{x}_1 b - x_2 \bar{c})(-x_1 \bar{b} - \bar{x}_2 c), \end{aligned}$$

and moreover that

$$(-\bar{x}_1 b - x_2 \bar{c})(-x_1 \bar{b} - \bar{x}_2 c) = |x_1 x_2 - x_3|(|x_1|^2 + |x_2|^2) + 2\mathrm{Re}(\bar{x}_1 \bar{x}_2 (x_1 x_2 - x_3)),$$

and

$$\begin{aligned} (1-|x_1|^2-|x_1x_2-x_3|)(1-|x_2|^2-|x_1x_2-x_3|) &= 1-|x_1|^2-|x_2|^2+|x_1x_2-x_3|(|x_1|^2+|x_2|^2) \\ &\quad -2|x_1x_2-x_3|+|x_1x_2-x_3|^2+|x_1|^2|x_2|^2. \end{aligned}$$

Note that

$$\begin{aligned} |x_1|^2|x_2|^2 + |x_1x_2 - x_3|^2 - 2\mathrm{Re}(\bar{x}_1\bar{x}_2(x_1x_2 - x_3)) &= |x_1x_2 - (x_1x_2 - x_3)|^2 \\ &= |x_3|^2. \end{aligned}$$

Hence,

$$\begin{aligned} \det(1 - X^*X) &= (1 - |x_1|^2 - |x_1x_2 - x_3|)(1 - |x_2|^2 - |x_1x_2 - x_3|) \\ &\quad -(-\bar{x}_1 b - x_2\bar{c})(-x_1\bar{b} - \bar{x}_2 c) \\ &= 1 - |x_1|^2 - |x_2|^2 + |x_3|^2 - 2|x_1x_2 - x_3|, \qquad (2.11) \end{aligned}$$

That is (2) holds.

$\square$

We shall now prove (8) $\Rightarrow$ (9) Suppose that (8) holds. Let

$$X = \begin{bmatrix} x_1 & b \\ c & x_2 \end{bmatrix} \in M_2(\mathbb{C}).$$

Then $1 - X^*X \geq 0$, and so by Lemma 2.1.5,

$$\det(1 - X^*X) = 1 - |x_1|^2 - |x_2|^2 + |x_3|^2 - 2|x_1x_2 - x_3| \geq 0.$$

From (8), $\left\| \begin{bmatrix} x_1 & b \\ c & x_2 \end{bmatrix} \right\| \leq 1$, which implies that $|x_1| \leq 1$, $|x_2| \leq 1$ and $|x_3| \leq 1$. Thus (9) holds and hence, (8) $\Rightarrow$ (9).

(8) $\Leftarrow$ (9) Suppose (9) holds. Then $|x_1| \leq 1$, $|x_2| \leq 1$ and $|x_3| \leq 1$.
Since $1 - |x_1|^2 - |x_2|^2 + |x_3|^2 - 2|x_1x_2 - x_3| \geq 0$, then by Lemma 2.1.5,

$$(1-|x_1|^2-|x_1x_2-x_3|)(1-|x_2|^2-|x_1x_2-x_3|)-(-\bar{x}_1x_{12}-x_2\bar{x}_{21})(-x_1\bar{x}_{12}-\bar{x}_2x_{21}) \geq 0,$$

That is,

$$\det\left(\begin{bmatrix} 1-|x_1|^2-|x_1x_2-x_3| & -\bar{x}_1x_{12}-x_2\bar{x}_{21} \\ -x_1\bar{x}_{12}-\bar{x}_2x_{21} & 1-|x_2|^2-|x_1x_2-x_3| \end{bmatrix}\right) \geq 0,$$

We claim that $1-|x_1|^2-|x_1x_2-x_3| \geq 0$.

Since if

$$(1-|x_1|^2-|x_1x_2-x_3|)(1-|x_2|^2-|x_1x_2-x_3|) \geq 0,$$

then $1-|x_1|^2-|x_1x_2-x_3|$ and $1-|x_2|^2-|x_1x_2-x_3|$ are either both non-positive or both non-negative. Since (9) holds, and since $|x_3| \leq 1$ then they are both non-negative, for

$$\begin{aligned} 1-|x_1|^2-|x_2|^2+|x_3|^2-2|x_1x_2-x_3| \geq 0 \;&\Rightarrow\; 2-|x_1|^2-|x_2|^2-2|x_1x_2-x_3| \geq 1-|x_3|^2 \\ &\Rightarrow\; 2-|x_1|^2-|x_2|^2-2|x_1x_2-x_3| \geq 0 \\ &\Rightarrow\; 1-|x_1|^2-|x_1x_2-x_3| \geq 0, \text{ and} \\ &\phantom{\Rightarrow}\;\; 1-|x_2|^2-|x_1x_2-x_3| \geq 0. \end{aligned}$$

Therefore,

$$1-|x_1|^2-|x_1x_2-x_3| \geq 0, \text{ and } 1-|x_2|^2-|x_1x_2-x_3| \geq 0.$$

Hence,

$$\begin{aligned} (9)\text{ holds} \;&\Longrightarrow\; \begin{bmatrix} 1-|x_1|^2-|x_1x_2-x_3| & -\bar{x}_1x_{12}-x_2\bar{x}_{21} \\ -x_1\bar{x}_{12}-\bar{x}_2x_{21} & 1-|x_2|^2-|x_1x_2-x_3| \end{bmatrix} \geq 0 \\ &\Longrightarrow\; (8)\text{ holds.} \end{aligned}$$

That is, (8) $\Leftarrow$ (9). This completes the proof of (8) $\Leftrightarrow$ (9).

**(5)⇔(10)** Let $x_1, x_2, x_3 \in \mathbb{C}$. When $x_1x_2 = x_3$, (5) is the statement that $z \mapsto x_1$ is in the Schur class and $|x_2| \le 1$. Therefore in this case, $|x_1| \le 1$ and $|x_2| \le 1$.

$$
\begin{aligned}
\text{(5) holds} \quad & \Longleftrightarrow \quad \begin{cases} \text{either } x_1x_2 = x_3,\ |x_1| \le 1 \text{ and } |x_2| \le 1, \text{ or} \\ \left|\dfrac{x_1 - zx_3}{1 - zx_2}\right| \le 1,\ \forall z \in \mathbb{T},\ \text{and } |x_2| < 1, \end{cases} \\
& \Longleftrightarrow \quad \begin{cases} \text{either } x_1x_2 = x_3\ , |x_1| \le 1 \text{ and } |x_2| \le 1, \text{ or} \\ |x_1 - zx_3|^2 \le |1 - zx_2|^2,\ \forall z \in \mathbb{T},\ \text{and } |x_2| < 1 \end{cases} \\
& \Longleftrightarrow \quad \begin{cases} \text{either } x_1x_2 = x_3,\ |x_1| \le 1 \text{ and } |x_2| \le 1, \text{ or} \\ (x_1 - zx_3)(\bar{x}_1 - \bar{z}\bar{x}_3) \le (1 - zx_2)(1 - \bar{z}\bar{x}_2),\ \forall z \in \mathbb{T},\ \text{and } |x_2| < 1 \end{cases} \\
& \Longleftrightarrow \quad \begin{cases} \text{either } x_1x_2 = x_3,\ |x_1| \le 1 \text{ and } |x_2| \le 1, \text{ or} \\ |x_1|^2 - \bar{z}x_1\bar{x}_3 - z\bar{x}_1x_3 + |x_3|^2 \le 1 - \bar{z}\bar{x}_2 - zx_2 + |x_2|^2,\ \forall z \in \mathbb{T},\ |x_2| < 1 \end{cases} \\
& \Longleftrightarrow \quad \begin{cases} \text{either } x_1x_2 = x_3,\ |x_1| \le 1 \text{ and } |x_2| \le 1, \text{ or} \\ z(x_2 - \bar{x}_1x_3) + \bar{z}(\bar{x}_2 - x_1\bar{x}_3) \le 1 - |x_1|^2 + |x_2|^2 - |x_3|^2,\ \forall z \in \mathbb{T},\ |x_2| < 1 \end{cases} \\
& \Longleftrightarrow \quad \begin{cases} \text{either } x_1x_2 = x_3,\ |x_1| \le 1 \text{ and } |x_2| \le 1, \text{ or} \\ 2\mathrm{Re}[\bar{z}(\bar{x}_2 - x_1\bar{x}_3)] \le 1 - |x_1|^2 + |x_2|^2 - |x_3|^2,\ \forall z \in \mathbb{T},\ \text{and } |x_2| < 1 \end{cases} \\
& \Longleftrightarrow \quad \begin{cases} \text{either } x_1x_2 = x_3,\ |x_1| \le 1 \text{ and } |x_2| \le 1, \text{ or} \\ 2|\bar{x}_2 - x_1\bar{x}_3| \le 1 - |x_1|^2 + |x_2|^2 - |x_3|^2,\ \text{and } |x_2| < 1 \end{cases} \\
& \Longleftrightarrow \quad \begin{cases} \text{either } x_1x_2 = x_3,\ |x_1| \le 1 \text{ and } |x_2| \le 1, \text{ or} \\ 1 - |x_1|^2 + |x_2|^2 - |x_3|^2 - 2|\bar{x}_2 - x_1\bar{x}_3| \ge 0,\ \text{and } |x_2| < 1 \end{cases} \\
& \Longleftrightarrow \quad \begin{cases} 1 - |x_1|^2 + |x_2|^2 - |x_3|^2 - 2|\bar{x}_2 - x_1\bar{x}_3| \ge 0,\ \text{and} \\ |x_1| \le 1,\ |x_2| \le 1. \end{cases} \\
& \Longleftrightarrow \quad \text{(10) holds.}
\end{aligned}
$$

**(6)⇔(11)** Similarly, we find that

$$\text{(6) holds} \quad \Longleftrightarrow \quad \begin{cases} \text{either } x_1x_2 = x_3,\ |x_1| \le 1 \text{ and } |x_2| \le 1, \quad \text{or} \\ \left|\dfrac{x_2 - zx_3}{1 - zx_1}\right| \le 1,\ \forall z \in \mathbb{T},\ \text{ and } |x_1| < 1, \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} 1 + |x_1|^2 - |x_2|^2 - |x_3|^2 - 2|\bar{x}_1 - x_2\bar{x}_3| \ge 0, \\ |x_1| \le 1,\ |x_2| \le 1. \end{cases}$$

**(9)⇔(10)** By the equivalence (1) ⇔ (5), we find that

$$\begin{aligned} (x_1, x_2, x_3) \in \Gamma_E \quad &\Leftrightarrow \quad \begin{cases} \left|\dfrac{x_2 - zx_3}{1 - zx_1}\right| \le 1,\ \forall z \in \mathbb{T},\ \text{ and } |x_1| < 1,\ \text{ or} \\ x_1x_2 = x_3,\ \text{ and } |x_2| \le 1 \end{cases} \\ &\Leftrightarrow \quad \begin{cases} |x_2 - zx_3| \le |1 - zx_1|,\ \forall z \in \mathbb{T},\ \text{ and } |x_1| < 1,\ \text{ or} \\ x_1x_2 = x_3,\ \text{ and } |x_2| \le 1 \end{cases} \\ &\Leftrightarrow \quad \begin{cases} |\bar{x}_2 - \bar{z}\bar{x}_3| \le |1 - zx_1|,\ \forall z \in \mathbb{T},\ \text{ and } |x_1| < 1,\ \text{ or} \\ x_1x_2 = x_3,\ \text{ and } |x_2| \le 1 \end{cases} \\ &\Leftrightarrow \quad \begin{cases} |\bar{x}_2 z - \bar{x}_3| \le |1 - zx_1|,\ \forall z \in \mathbb{T},\ \text{ and } |x_1| < 1,\ \text{ or} \\ x_1x_2 = x_3,\ \text{ and } |x_2| \le 1 \end{cases} \\ &\Leftrightarrow \quad \begin{cases} \left|\dfrac{\bar{x}_3 - z\bar{x}_2}{1 - zx_1}\right|,\ \forall z \in \mathbb{T},\ \text{ and } |x_1| < 1,\ \text{ or} \\ x_1x_2 = x_3,\ \text{ and } |x_2| \le 1 \end{cases} \\ &\Leftrightarrow \quad (x_1, \bar{x}_3, \bar{x}_2) \in \Gamma_E. \end{aligned}$$

That is,

$$(x_1, x_2, x_3) \in \Gamma_E \Leftrightarrow (x_1, \bar{x}_3, \bar{x}_2) \in \Gamma_E.$$

Moreover, since

$$(x_1, x_2, x_3) \in \Gamma_E \Leftrightarrow (x_2, x_1, x_3) \in \Gamma_E,$$

we find that

$$
\begin{aligned}
(10) \text{ holds } &\Leftrightarrow \begin{cases} 1-|x_1|^2+|x_2|^2-|x_3|^2-2|x_1\bar{x}_3-\bar{x}_2| \geq 0, \\ |x_1| \leq 1, \ |x_2| \leq 1 \end{cases} \\
&\Leftrightarrow \begin{cases} 1-|x_1|^2-|x_2|^2+|x_3|^2-2|x_1x_2-x_3| \geq 0, \\ |x_1| \leq 1, \ |x_2| \leq 1, \ |x_3| \leq 1. \end{cases} \\
&\Leftrightarrow (9) \text{ holds.}
\end{aligned}
$$

□

The next result follows from equivalences (9) $\Leftrightarrow$ (10) and (3) $\Leftrightarrow$ (4) of Theorem 2.1.4.

**Corollary 2.1.6** *We have*

$$
\begin{aligned}
(x_1,x_2,x_3) \in \Gamma_E &\iff (x_1,\bar{x}_3,\bar{x}_2) \in \Gamma_E, \text{ and} \\
(x_1,x_2,x_3) \in \Gamma_E &\iff (x_2,x_1,x_3) \in \Gamma_E.
\end{aligned}
$$

**Corollary 2.1.7** *The following holds:*

$$
(s,p) \in \Gamma \Leftrightarrow \left(\frac{s}{2},\frac{s}{2},p\right) \in \Gamma_E.
$$

**Proof** Recall that, from Theorem 1.1.2 and Theorem 2.1.4, we have

$$
\begin{aligned}
(s,p) \in \Gamma &\Leftrightarrow |s-\bar{s}p| \leq 1-|p|^2, \text{ and} \\
(x_1,x_2,x_3) \in \Gamma_E &\Leftrightarrow 1-|x_1|^2+|x_2|^2-|x_3|^2-2|x_1\bar{x}_3-\bar{x}_2| \geq 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left(\frac{s}{2},\frac{s}{2},p\right) \in \Gamma_E &\Leftrightarrow 1-|p|^2-2\left|\frac{s}{2}\bar{p}-\frac{\bar{s}}{2}\right| \geq 0 \\
&\Leftrightarrow 1-|p|^2 \geq |s-\bar{s}p| \\
&\Leftrightarrow (s,p) \in \Gamma.
\end{aligned}
$$

□

## 2.2 A Necessary Condition for Interpolation into $\Gamma_E$

By the *Nevanlinna-Pick data* we mean a finite set $\lambda_1, \ldots, \lambda_n$ of distinct points in $\mathbb{D}$, where $n \in \mathbb{N}$, and an equal number of "target points" $w_1, \ldots, w_n$ in $\mathbb{C}$. We write these data

$$\lambda_j \mapsto w_j,\ 1 \le j \le n. \tag{2.12}$$

We say that these data are *solvable* if there exists a function $F$ in the Schur class such that $F(\lambda_j) = w_j$, $1 \le j \le n$. By the classical theorem of Pick, the Nevanlinna-Pick problem (2.12) is solvable if and only if the "Pick matrix"

$$\left[\frac{1 - \bar{w}_i w_j}{1 - \bar{\lambda}_i \lambda_j}\right]_{i,j=1}^{n}$$

is positive semi-definite.

The next result follows immediately by (1) $\Leftrightarrow$ (5) of Theorem 2.1.4.

**Corollary 2.2.1** *Let $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$, $x^{(j)} = (x_1^{(j)}, x_2^{(j)}, x_3^{(j)}) \in \Gamma_E$ for $j = 1, 2, \cdots, n$. A necessary condition for the existence of an analytic function $f : \mathbb{D} \longmapsto \Gamma_E$ such that $f(\lambda_j) = x^{(j)}$, $1 \le j \le n$, is that for all $\omega \in \mathbb{T} \backslash \{\bar{x}_2^{(j)} :\ 1 \le j \le n\}$,*

$$\lambda_j \longmapsto \frac{x_1^{(j)} - \omega x_3^{(j)}}{1 - \omega x_2^{(j)}},\ 1 \le j \le n$$

*are solvable Nevanlinna-Pick data.*

**Corollary 2.2.2** *Let $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$, $x^{(j)} = (x_1^{(j)}, x_2^{(j)}, x_3^{(j)}) \in \Gamma_E$ for $j = 1, 2, \cdots, n$. If there exists an analytic function $f : \mathbb{D} \longmapsto \Gamma_E$ such that $f(\lambda_j) = x^{(j)}$, $1 \le j \le n$, then for all $\omega \in \mathbb{T} \backslash \{\bar{x}_2^{(j)} :\ 1 \le j \le n\}$,*

$$\left[\frac{1 - \overline{\Psi_\omega(x^{(i)})}\Psi_\omega(x^{(j)})}{1 - \bar{\lambda}_i \lambda_j}\right]_{i,j=1}^{n} \ge 0.$$

**Remark 2.2.3** *The converse of Corollary 2.2.2 is not true.*

**Example** We consider interpolating points of the form $x_3^{(j)} = x_1^{(j)} x_2^{(j)}$, $1 \leq j \leq n$. In this case, the converse of Corollary 2.2.2 suggests a sufficient condition would be that there exists an analytic function $f : \mathbb{D} \longmapsto \Gamma_E$ such that $f(\lambda_j) = x^{(j)},\ 1 \leq j \leq n$, but

$$\Psi_\omega\left(x^{(j)}\right) = \frac{x_1^{(j)} - \omega x_3^{(j)}}{1 - \omega x_2^{(j)}} = \frac{x_1^{(j)} - \omega x_1^{(j)} x_2^{(j)}}{1 - \omega x_2^{(j)}} = \frac{x_1^{(j)}\left(1 - \omega x_2^{(j)}\right)}{1 - \omega x_2^{(j)}} = x_1^{(j)}.$$

Hence, we can take $x_2^{(j)}$ to be any $n$ points in $\mathbb{D}$ which cannot be interpolated by $\lambda_j \mapsto x_2^{(j)}$. In this case, we certainly cannot solve the interpolation $\lambda_j \mapsto x^{(j)}$.

In the case of the symmetrised bidisc $\Gamma$, Agler and Young [3], [9] gave a necessary condition for interpolation into $\Gamma$. Their result is as follows: *Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$ and let $z_j = (s_j, p_j)$ be in $G$ for $j = 1, \ldots, n$, where $G$ is the interior of $\Gamma$. If there exists an analytic function $h : \mathbb{D} \longrightarrow G$ such that $h(\lambda_j) = z_j$, for $j = 1, \ldots, n$, then for all $\omega \in \mathbb{T}$,*

$$\left[\frac{1 - \overline{\Phi_\omega(z_i)}\Phi_\omega(z_j)}{1 - \bar{\lambda}_i \lambda_j}\right]_{i,j=1}^{n} \geq 0,$$

*where* $\Phi_\omega(z_j) = \dfrac{2\omega p_j - s_j}{2 - \omega s_j}$, $1 \leq j \leq n$.

They believe that the converse of their result fails to hold in general, however, in [9], they show that it does hold when $n = 2$. In the case of $\Gamma_E$, we state the following question for the converse of Corollary 2.2.2.

**Question 2.2.4** *Let $\lambda_1, \ldots, \lambda_n \in \mathbb{D},\ x^{(j)} = (x_1^{(j)}, x_2^{(j)}, x_3^{(j)}) \in \Gamma_E$ for*

$j = 1, 2, \cdots, n$. *If for all* $\omega \in \mathbb{T}\backslash\{\bar{x}_1^{(j)}, \bar{x}_2^{(j)} : 1 \le j \le n\}$,

$$\left[\frac{1 - \overline{\Psi_\omega(x^{(i)})}\Psi_\omega(x^{(j)})}{1 - \bar{\lambda}_i\lambda_j}\right]_{i,j=1}^n \ge 0 \quad \text{and} \quad \left[\frac{1 - \overline{\Upsilon_\omega(x^{(i)})}\Upsilon_\omega(x^{(j)})}{1 - \bar{\lambda}_i\lambda_j}\right]_{i,j=1}^n \ge 0,$$

*does there exist an analytic function* $f : \mathbb{D} \longrightarrow \Gamma_E$ *such that*

$$f(\lambda_j) = x^{(j)},\ 1 \le j \le n?$$

The next result relates the property of mapping $\mathbb{D}$ analytically to $\Gamma_E$ and membership of the Schur class.

**Theorem 2.2.5** *For any function* $\varphi = (\varphi_1, \varphi_2, \varphi_3) : \mathbb{D} \longmapsto \mathbb{C}^3$, *the following statements are equivalent:*

*(1)* $\varphi$ *is analytic and maps* $\mathbb{D}$ *into* $\Gamma_E$;

*(2) there exists an analytic* $2 \times 2$-*matrix valued function* $\psi = [\psi_{ij}]$ *on* $\mathbb{D}$ *such that* $\|\psi\|_\infty \le 1$ *and* $\varphi = (\psi_{11}, \psi_{22}, \det(\psi))$.

**Proof (1)⇒(2)** Let $\varphi : \mathbb{D} \longrightarrow \Gamma_E$ be analytic. We shall construct

$$\psi = \begin{bmatrix} \varphi_1 & \psi_{12} \\ \psi_{21} & \varphi_2 \end{bmatrix},$$

analytic in $\mathbb{D}$ such that $\varphi_1\varphi_2 - \psi_{12}\psi_{21} = \varphi_3$ and $\|\psi\|_\infty \le 1$. That is, $\psi_{12}\psi_{21}$ has to be $\varphi_1\varphi_2 - \varphi_3$.

Since $\varphi : \mathbb{D} \longrightarrow \Gamma_E$ is analytic, $\varphi_1\varphi_2 - \varphi_3 \in H^\infty$, and so by a theorem of F. Riesz, which follows easily from inner-outer factorisation [17], there exist functions $\psi_{12}, \psi_{21} \in H^\infty$ such that $\psi_{12}\psi_{21} = \varphi_1\varphi_2 - \varphi_3$ and importantly

$$|\psi_{12}| = |\psi_{21}| = |\varphi_1\varphi_2 - \varphi_3|^{1/2} \text{ on } \mathbb{T}.$$

Since $\varphi : \mathbb{D} \longmapsto \Gamma_E$, then by the equivalence (1) $\Leftrightarrow$ (9) of Theorem 2.1.4, we have

$$\begin{cases} 1 - |\varphi_1|^2 - |\varphi_2|^2 + |\varphi_3|^2 - 2|\varphi_1\varphi_2 - \varphi_3| \geq 0, \text{ and} \\ |\varphi_1| \leq 1, \ |\varphi_2| \leq 1, \ |\varphi_3| \leq 1. \end{cases}$$

Therefore, by (2) of Lemma 2.1.5, we find that

$$\det(1 - \psi^*\psi) = 1 - |\varphi_1|^2 - |\varphi_2|^2 + |\varphi_3|^2 - 2|\varphi_1\varphi_2 - \varphi_3| \geq 0.$$

We need to show that $1 - |\varphi_1|^2 - |\varphi_1\varphi_2 - \varphi_3| \geq 0$. Since $(\varphi_1, \varphi_2, \varphi_3) \in \Gamma_E$,

$$\begin{aligned} 1 - |\varphi_1|^2 - |\varphi_1\varphi_2 - \varphi_3| &\geq |\varphi_2 - \bar{\varphi}_1\varphi_3| \geq 0, \text{ and} \\ 1 - |\varphi_2|^2 - |\varphi_1\varphi_2 - \varphi_3| &\geq |\varphi_1 - \bar{\varphi}_2\varphi_3| \geq 0 \end{aligned}$$

Hence

$$\begin{aligned} \text{(1) holds} &\implies \begin{bmatrix} 1 - |\varphi_1|^2 - |\varphi_1\varphi_2 - \varphi_3| & -\bar{\varphi}_1\varphi_{12} - \varphi_2\bar{\varphi}_{21} \\ -\varphi_1\bar{\varphi}_{12} - \bar{\varphi}_2\varphi_{21} & 1 - |\varphi_2|^2 - |\varphi_1\varphi_2 - \varphi_3| \end{bmatrix} \geq 0, \\ &\implies \|\psi\|_\infty \leq 1. \end{aligned}$$

That is (1) $\Rightarrow$ (2).

**(2)$\Rightarrow$(1)** Suppose that $\|\psi\|_\infty \leq 1$. Then by Lemma 2.1.5,

$$\det(1 - \psi^*\psi) = 1 - |\varphi_1|^2 - |\varphi_2|^2 + |\varphi_3|^2 - 2|\varphi_1\varphi_2 - \varphi_3| \geq 0.$$

Also, our assumption that $\|\psi\|_\infty \leq 1$ implies that $|\varphi_1| \leq 1$, $|\varphi_2| \leq 1$ and $|\varphi_3| \leq 1$. Hence,

$$||\psi||_\infty \leq 1 \implies \begin{cases} 1 - |\varphi_1|^2 - |\varphi_2|^2 + |\varphi_3|^2 - 2|\varphi_1\varphi_2 - \varphi_3| \geq 0, \\ |\varphi_1| \leq 1, \ |\varphi_2| \leq 1, \ |\varphi_3| \leq 1. \end{cases}$$

That is, (2) $\Rightarrow$ (1). Thus, (1) $\Leftrightarrow$ (2). $\square$

The next result allows us to find a realization formula for analytic functions from $\mathbb{D}$ to $\Gamma_E$. The $2 \times 2$ matrix function $\psi$ appearing in condition (2) of Theorem 2.2.5 belongs to the Schur class, and can therefore be realised, as in the Realization Theorem.

**Corollary 2.2.6** *A function $\varphi = (\varphi_1, \varphi_2, \varphi_3) : \mathbb{D} \longrightarrow \mathbb{C}^3$ maps $\mathbb{D}$ analytically into $\Gamma_E$ if and only if there exist a Hilbert space $H$ and a unitary operator*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : H \oplus \mathbb{C}^2 \longrightarrow H \oplus \mathbb{C}^2$$

*such that*

$$\varphi_1 = \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right], \ \varphi_2 = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right] \ \textit{and}\ \varphi_3 = \det \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

*where $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} : \mathbb{C}^2 \longrightarrow H,\ C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} : H \longrightarrow \mathbb{C}^2$ and $D = [D_{ij}]_{i,j=1}^2$.*

**Proof** ($\Longrightarrow$) Given the analytic function $\varphi : \mathbb{D} \longrightarrow \Gamma_E$, choose $\psi$ as in Theorem 2.2.5, so that $\psi$ is in the Schur class and $\varphi = (\psi_{11}, \psi_{22}, \det(\psi))$. By the Realization Theorem, there exists a Hilbert space $H$ and a unitary operator

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ on } H \oplus \mathbb{C}^2$$

such that for all $\lambda \in \mathbb{D}$,

$$\begin{aligned} \psi(\lambda) &= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](\lambda) \\ &= D + C\lambda(1 - A\lambda)^{-1}B \end{aligned}$$

$$\psi(\lambda) \;=\; \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \lambda(1-A\lambda)^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix}.$$

Thus

$$\psi_{11} = \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right], \quad \psi_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$$

and so

$$\varphi_1 = \psi_{11} = \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right], \quad \varphi_2 = \psi_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right], \text{ and}$$

$$\varphi_3 = \det\psi = \det \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

($\Longleftarrow$) Conversely, if $H, A, B, C$ and $D$ are as described, then the function

$$\chi = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = [\chi_{ij}]$$

is analytic and $\|\chi\|_\infty \leq 1$ by the Realization Theorem. By hypothesis,

$$\varphi_j = \left[\begin{array}{c|c} A & B_j \\ \hline C_j & D_{jj} \end{array}\right], \; j = 1, 2$$

and

$$\varphi_3 = \det\chi.$$

Hence, by Theorem 2.2.5, $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ maps $\mathbb{D}$ analytically into $\Gamma_E$.

$\square$

## 2.3 The Nevanlinna-Pick Problem for $\Gamma_E$

In this section, we establish a necessary and sufficient condition for the existence of an analytic function $\mathbb{D} \longrightarrow \Gamma_E$ satisfying an arbitrary finite number of interpolation conditions.

**Theorem 2.3.1** *Let $\lambda_1, \cdots, \lambda_n$ be distinct points in $\mathbb{D}$ for some $n \in \mathbb{N}$ and let $(x_1^j, x_2^j, x_3^j) \in \Gamma_E$ for $j = 1, \cdots, n$. There exists an analytic function $\varphi : \mathbb{D} \longmapsto \Gamma_E$ such that*

$$\varphi(\lambda_j) = (x_1^j, x_2^j, x_3^j),\ 1 \le j \le n,$$

*if and only if there exist $b_j, c_j \in \mathbb{C}$ such that*

$$b_j c_j = x_1^j x_2^j - x_3^j,\ 1 \le j \le n, \tag{2.13}$$

*and the conditions*

$$\lambda_j \longmapsto \begin{bmatrix} x_1^j & b_j \\ c_j & x_2^j \end{bmatrix},\ 1 \le j \le n, \tag{2.14}$$

*comprise solvable Nevanlinna-Pick data.*

**Proof**($\Rightarrow$) Suppose $\varphi$ as described exists. By Theorem 2.2.5, there is an analytic $2 \times 2$ matrix-valued function $\psi$ on $\mathbb{D}$ such that $\|\psi\|_\infty \le 1$ and $\varphi = (\psi_{11}, \psi_{22}, \det(\psi))$. Choose

$$b_j = \psi_{12}(\lambda_j),\ c_j = \psi_{21}(\lambda_j),\ 1 \le j \le n.$$

Then

$$\psi(\lambda_j) = \begin{bmatrix} x_1^j & \psi_{12}(\lambda_j) \\ \psi_{21}(\lambda_j) & x_2^j \end{bmatrix} = \begin{bmatrix} x_1^j & b_j \\ c_j & x_2^j \end{bmatrix}$$

and

$$x_3^j = x_3(\lambda_j) = \det(\psi(\lambda_j)) = x_1^j x_2^j - b_j c_j.$$

Thus, the equations (2.13) are satisfied, and for this choice of $b_j, c_j$, the matricial Nevanlinna-Pick data with (2.14) is indeed solvable, since $\psi$ is a solution of it.

($\Leftarrow$) Suppose $b_j, c_j$ can be found such that the equations (2.13) hold and the matricial Nevanlinna-Pick data (2.14) are solvable, with solution $\chi = [\chi_{ij}]$. Thus $\chi$ is a $2 \times 2$ Schur function, and

$$\chi(\lambda_j) = \begin{bmatrix} x_1^j & b_j \\ c_j & x_2^j \end{bmatrix}.$$

Define functions $\varphi_1, \varphi_2, \varphi_3$ by

$$\begin{aligned} \varphi_1 &= \chi_{11}, \\ \varphi_2 &= \chi_{22}, \\ \varphi_3 &= \chi_{11}\chi_{22} - \det(\chi), \end{aligned}$$

and let $\varphi = (\varphi_1, \varphi_2, \varphi_3)$. By Theorem 2.2.5, $\varphi$ maps $\mathbb{D}$ analytically to $\Gamma_E$ and we have

$$\begin{aligned} \varphi_1(\lambda_j) &= \chi_{11}(\lambda_j) = x_1^j, \\ \varphi_2(\lambda_j) &= \chi_{22}(\lambda_j) = x_2^j, \\ \varphi_3(\lambda_j) &= x_1^j x_2^j - b_j c_j = x_3^j. \end{aligned}$$

□

The next corollary follows immediately from Theorem 2.3.1.

**Corollary 2.3.2** *Let $\lambda_1, \cdots, \lambda_n$ be distinct points in $\mathbb{D}$ for some $n \in \mathbb{N}$ and let $x^j = (x_1^j, x_2^j, x_3^j) \in \Gamma_E$, for $j = 1, 2, \cdots, n$. The following two statements are equivalent:*

*(1) there exists an analytic function $\varphi : \mathbb{D} \longrightarrow \Gamma_E$ such that $\varphi(\lambda_j) = x^j$, $1 \le j \le n$;*

*(2) there exists* $b_1, \cdots, b_n, c_1, \cdots, c_n \in \mathbb{C}$ *such that*

$b_j c_j = x_1^j x_2^j - x_3^j$, $1 \le j \le n$, *and*

$$\left[\frac{I - \begin{bmatrix} x_1^i & b_i \\ c_i & x_2^i \end{bmatrix}^* \begin{bmatrix} x_1^j & b_j \\ c_j & x_2^j \end{bmatrix}}{1 - \bar{\lambda}_i \lambda_j}\right]_{i,j=1}^n \ge 0.$$

**Theorem 2.3.3** *The following identity is valid for all* $z \in \mathbb{D}$, $x_1, x_2, x_3 \in \mathbb{C}$ *and* $r \in [0,1)$.

$$|1 - rzx_2|^2 - |rx_1 - rzx_3|^2 = r^2\{|1 - zx_2|^2 - |x_1 - zx_3|^2\} + (1-r)(1 + r - 2r\mathrm{Re}(zx_2)).$$

**Proof** For all $z \in \mathbb{D}$, $x_1, x_2, x_3 \in \mathbb{C}$ and $r \in [0,1)$,we have

$$\begin{aligned}
|1 - rzx_2|^2 - |rx_1 - rzx_3|^2 &= (1 - rzx_2)(1 - r\bar{z}\bar{x}_2) - r^2(x_1 - zx_3)(\bar{x}_1 - \bar{z}\bar{x}_3) \\
&= 1 - rzx_2 - r\bar{z}\bar{x}_2 + r^2|z|^2|x_2|^2 \\
&\quad - r^2(|x_1|^2 - \bar{z}x_1\bar{x}_3 - z\bar{x}_1x_3 + |z|^2|x_3|^2) \\
&= r^2(1 - zx_2 - \bar{z}\bar{x}_2 + |z|^2|x_2|^2) \\
&\quad - r^2(|x_1|^2 - \bar{z}x_1\bar{x}_3 - z\bar{x}_1x_3 + |z|^2|x_3|^2) \\
&\quad + 1 + r - r(zx_2 + \bar{z}\bar{x}_2) - r(1 + r - r(zx_2 + \bar{z}\bar{x}_2)) \\
&= r^2(1 - zx_2)(1 - \bar{z}\bar{x}_2) - r^2(x_1 - zx_3)(\bar{x}_1 - \bar{z}\bar{x}_3) \\
&\quad + (1 - r)(1 + r - r(zx_2 + \bar{z}\bar{x}_2)) \\
&= r^2\{|1 - zx_2|^2 - |x_1 - zx_3|^2\} + (1 - r)(1 + r - 2r\mathrm{Re}(zx_2)).
\end{aligned}$$

□

**Theorem 2.3.4** *$\Gamma_E$ is not convex, though it is starlike about the point $(0,0,0)$.*

**Proof** To show that $\Gamma_E$ is not convex, we give the following example: The points $(1,i,i)$ and $(-i,1,-i)$ are in $\Gamma_E$, but the mid-point of these points is $\left(\frac{1-i}{2}, \frac{1+i}{2}, 0\right) \notin \Gamma_E$, for

$$|x_1|^2 + |x_2 - \bar{x}_1 x_3| + |x_1 x_2 - x_3| = \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{1}{2} = 1 + \frac{\sqrt{2}}{2} \nleq 1.$$

To prove that $\Gamma_E$ is starlike about the point $(0,0,0)$, we need to show that if $x = (x_1, x_2, x_3) \in \Gamma_E$ and $0 \leq r < 1$ then $(rx_1, rx_2, rx_3) \in \Gamma_E$. Fix $(x_1, x_2, x_3) \in \Gamma_E$ and $r \in [0,1)$. By Theorem 2.1.4 we have, for all $z \in \mathbb{D}$,

$$\left| \frac{x_1 - zx_3}{1 - zx_2} \right| \leq 1,$$

that is,

$$|1 - zx_2|^2 - |x_1 - zx_3|^2 \geq 0.$$

Therefore, by Theorem 2.3.3, we have, for $r \in [0,1)$ and for all $z \in \mathbb{D}$,

$$\begin{aligned} |1 - rzx_2|^2 - |rx_1 - rzx_3|^2 &= r^2\{|1 - zx_2|^2 - |x_1 - zx_3|^2\} + (1-r)(1 + r - 2r\mathrm{Re}(zx_2)) \\ &\geq 0. \end{aligned}$$

Therefore,

$$\left| \frac{rx_1 - rzx_3}{1 - rzx_2} \right| \leq 1,$$

for all $z \in \mathbb{D}$. Hence, by Theorem 2.1.4, $(rx_1, rx_2, rx_3) \in \Gamma_E$. Thus, $\Gamma_E$ is a starlike about the point $(0,0,0)$.

□

# Chapter 3

# A Schwarz Lemma for $\Gamma_E$

The classical Schwarz Lemma gives a necessary and sufficient condition for the solvability of a two-point interpolation problem for analytic functions from the open unit disc $\mathbb{D}$ into itself. This lemma has many generalisations in which the two copies of $\mathbb{D}$ are replaced by other domains. One of our goals is to prove a Schwarz Lemma for $\Gamma_E$.

**The Classical Schwarz Lemma** *Given* $\lambda_0 \in \mathbb{D}\backslash\{0\}$ *and* $z_0 \in \mathbb{D}$, *there exists an analytic function* $f : \mathbb{D} \to \mathbb{D}$ *such that* $f(0) = 0$ *and* $f(\lambda_0) = z_0$ *if and only if* $|z_0| \leq |\lambda_0|$.

In section 3.4, we describe a large group of holomorphic automorphisms of $G_E$, which we conjecture to be all the automorphisms of $G_E$. We do this using automorphisms induced by Möbius automorphisms and the natural involution $(a, b, p) \in \Gamma_E \mapsto (b, a, p)$ of $\Gamma_E$.

We also use the Möbius automorphisms in Chapter 4 to find the distinguished boundary of $\Gamma_E$ and to prove some other results.

## 3.1 A special case of the Schwarz Lemma for $\Gamma_E$

In this section we prove the Schwarz Lemma for $\Gamma_E$ in the case that the two points in $\Gamma_E$ are $(0,0,0)$ and $(a,b,0)$.

**Theorem 3.1.1** *Let $0 \leq b_0 < a_0 < 1 - b_0$ and let $\lambda_0 \in \mathbb{D}$. The following are equivalent:*

*(1) There exists $h : \mathbb{D} \longrightarrow G_E$ such that $h(0) = (0,0,0)$ and $h(\lambda_0) = (a_0, b_0, 0)$;*

*(2) $|\lambda_0| \geq \dfrac{a_0}{1-b_0}$;*

*(3) there exists an analytic function $F : \mathbb{D} \longrightarrow M_2(\mathbb{C})$ such that for all $\lambda \in \mathbb{D}$, $\|F(\lambda)\| < 1$ and*

$$F(0) = \begin{bmatrix} 0 & \zeta \\ 0 & 0 \end{bmatrix}, \quad F(\lambda_0) = \begin{bmatrix} a_0 & \tau\sqrt{a_0 b_0} \\ \tau^{-1}\sqrt{a_0 b_0} & b_0 \end{bmatrix},$$

*for some $\zeta \in (0,1)$ and $\tau > 0$.*

**Proof (1)⇒(2)** Suppose that there exists $h : \mathbb{D} \longrightarrow G_E$ such that $h(0) = (0,0,0)$ and $h(\lambda_0) = (a_0, b_0, 0)$.

We know that for any $\omega \in \mathbb{T}$, the function $\Psi_\omega$ defined by

$$\Psi_\omega(a,b,p) = \frac{a - \omega p}{1 - \omega b}$$

maps $G_E$ to $\mathbb{D}$, and the same is true for

$$\Upsilon_\omega(a,b,p) = \frac{b - \omega p}{1 - \omega a}.$$

Therefore, $\Psi_\omega \circ h$ maps $\mathbb{D}$ to $\mathbb{D}$,

$$\begin{aligned} \Psi_\omega \circ h(0) &= \Psi_\omega(0,0,0) \ = \ 0, \\ \Psi_\omega \circ h(\lambda_0) &= \Psi_\omega(a_0, b_0, 0) = \frac{a_0}{1-\omega b_0}. \end{aligned}$$

Also,

$$\begin{aligned} \Upsilon_\omega \circ h(0) &= \Upsilon_\omega(0,0,0) \ = \ 0, \\ \Upsilon_\omega \circ h(\lambda_0) &= \Upsilon_\omega(a_0, b_0, 0) = \frac{b_0}{1-\omega a_0}. \end{aligned}$$

By the Schwarz Lemma, for any $\omega \in \mathbb{T}$,

$$|\Psi_\omega \circ h(\lambda_0)| \le |\lambda_0| \ \text{ and } \ |\Upsilon_\omega \circ h(\lambda_0)| \le |\lambda_0|.$$

Hence,

$$\sup_\omega |\Psi_\omega \circ h(\lambda_0)| \le |\lambda_0| \ \text{ and } \ \sup_\omega |\Upsilon_\omega \circ h(\lambda_0)| \le |\lambda_0|.$$

Hence,

$$\frac{a_0}{1-b_0} \le |\lambda_0| \ \text{ and } \ \frac{b_0}{1-a_0} \le |\lambda_0|.$$

Therefore, (1) $\Rightarrow$ (2).

**(2)$\Rightarrow$(3)** Suppose that (2) holds. We can suppose that $\lambda_0 = \dfrac{a_0}{1-b_0}$ so that $0 < \lambda_0 < 1$.

Let $\zeta = \sqrt{\dfrac{b_0}{1-b_0}}$, $\tau = \sqrt{\dfrac{1-b_0}{a_0}}$ and let

$$X_1 = \begin{bmatrix} 0 & \zeta \\ 0 & 0 \end{bmatrix}, \ X_2 = \begin{bmatrix} a_0 & \tau\sqrt{a_0 b_0} \\ \tau^{-1}\sqrt{a_0 b_0} & b_0 \end{bmatrix} = \begin{bmatrix} \sqrt{a_0} \\ \tau^{-1}\sqrt{b_0} \end{bmatrix} \begin{bmatrix} \sqrt{a_0} & \tau\sqrt{b_0} \end{bmatrix}.$$

Note that $\zeta \in (0,1)$ and $\|X_2\| < 1$, for

$$\begin{aligned}
\|X_2\| &= \sqrt{\left(a_0 + \frac{b_0}{\tau^2}\right)(a_0 + \tau^2 b_0)} \\
&= \sqrt{a_0^2 + b_0^2 + \tau^2 a_0 b_0 + \frac{a_0 b_0}{\tau^2}} \\
&= \sqrt{a_0^2 + b_0^2 + b_0(1-b_0) + \frac{a_0^2 b_0}{1-b_0}} \\
&= \sqrt{\frac{a_0^2(1-b_0) + b_0(1-b_0) + a_0^2 b_0}{1-b_0}} \\
&= \sqrt{\frac{a_0^2 + b_0(1-b_0)}{1-b_0}} \\
&= \sqrt{\frac{a_0^2}{1-b_0} + b_0},
\end{aligned}$$

and since $a_0 < 1 - b_0$, then $\frac{a_0}{1-b_0} < 1$, which implies that $\frac{a_0^2}{1-b_0} < a_0$. Therefore,

$$\begin{aligned}
\|X_2\| &= \sqrt{\frac{a_0^2}{1-b_0} + b_0} \\
&< \sqrt{a_0 + b_0} \\
&< 1.
\end{aligned}$$

Consider the Möbius transformation $\mathcal{M}_{X_1}(X_2)$ defined by

$$\mathcal{M}_{X_1}(X_2) = (1 - X_1 X_1^*)^{-1/2}(X_1 - X_2)(1 - X_1^* X_2)^{-1}(1 - X_1^* X_1)^{1/2}.$$

Note that $\mathcal{M}_{X_1}(X_2)$ is defined and is a contraction, for

$$1 - \mathcal{M}_{X_1}(X_2)^* \mathcal{M}_{X_1}(X_2) = (1 - X_1^* X_1)^{1/2}(1 - X_2^* X_1)^{-1}(1 - X_2^* X_2)(1 - X_1^* X_2)^{-1}(1 - X_1^* X_1)^{1/2},$$

and since $\|X_1\| < 1$, $\|X_2\| < 1$ and $(1 - X_1^* X_2)$ is invertible, $\mathcal{M}_{X_1}(X_2)$ is defined and $\|\mathcal{M}_{X_1}(X_2)\| < 1$, see Chapter 12 of [29].

Now we have

$$
\begin{aligned}
\mathcal{M}_{X_1}(X_2) &= \begin{bmatrix} (1-\zeta^2)^{-1/2} & 0 \\ \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -a_0 & \zeta - \tau\sqrt{a_0 b_0} \\ \\ -\tau^{-1}\sqrt{a_0 b_0} & -b_0 \end{bmatrix} \\
&\quad \begin{bmatrix} 1 & 0 \\ \\ -a_0\zeta & 1-\zeta\tau\sqrt{a_0 b_0} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ \\ 0 & (1-\zeta^2)^{1/2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{-a_0}{\sqrt{1-\zeta^2}} & \frac{\zeta-\tau\sqrt{a_0b_0}}{\sqrt{1-\zeta^2}} \\ \\ -\tau^{-1}\sqrt{a_0b_0} & -b_0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \\ \frac{a_0\zeta}{1-\zeta\tau\sqrt{a_0b_0}} & \frac{1}{1-\zeta\tau\sqrt{a_0b_0}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \\ 0 & \sqrt{1-\zeta^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{-a_0}{\sqrt{1-\zeta^2}} + \frac{a_0\zeta(\zeta-\tau\sqrt{a_0b_0})}{\sqrt{1-\zeta^2}(1-\zeta\tau\sqrt{a_0b_0})} & \frac{\zeta-\tau\sqrt{a_0b_0}}{\sqrt{1-\zeta^2}(1-\zeta\tau\sqrt{a_0b_0})} \\ \\ -\tau^{-1}\sqrt{a_0b_0} - \frac{\zeta a_0 b_0}{1-\zeta\tau\sqrt{a_0b_0}} & \frac{-b_0}{1-\zeta\tau\sqrt{a_0b_0}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \\ 0 & \sqrt{1-\zeta^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{-a_0}{\sqrt{1-\zeta^2}} + \frac{a_0\zeta(\zeta-\tau\sqrt{a_0b_0})}{\sqrt{1-\zeta^2}(1-\zeta\tau\sqrt{a_0b_0})} & \frac{\zeta-\tau\sqrt{a_0b_0}}{1-\zeta\tau\sqrt{a_0b_0}} \\ \\ \frac{-\sqrt{a_0b_0}}{\tau} - \frac{\zeta a_0 b_0}{1-\zeta\tau\sqrt{a_0b_0}} & \frac{-b_0\sqrt{1-\zeta^2}}{1-\zeta\tau\sqrt{a_0b_0}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{-a_0\sqrt{1-\zeta}}{1-\zeta\tau\sqrt{a_0b_0}} & \frac{\zeta-\tau\sqrt{a_0b_0}}{1-\zeta\tau\sqrt{a_0b_0}} \\ \\ \frac{-\sqrt{a_0b_0}}{\tau(1-\zeta\tau\sqrt{a_0b_0})} & \frac{-b_0\sqrt{1-\zeta}}{1-\zeta\tau\sqrt{a_0b_0}} \end{bmatrix}
\end{aligned}
$$

$$\mathcal{M}_{X_1}(X_2) = \begin{bmatrix} \dfrac{-a_0\sqrt{1-2b_0}}{(1-b_0)\sqrt{1-b_0}} & \dfrac{b_0\sqrt{b_0}}{(1-b_0)\sqrt{1-b_0}} \\ \\ \dfrac{-a_0\sqrt{b_0}}{(1-b_0)\sqrt{1-b_0}} & \dfrac{-b_0\sqrt{1-2b_0}}{(1-b_0)\sqrt{1-b_0}} \end{bmatrix} =: X_3.$$

Let $F : \mathbb{D} \longrightarrow M_2(\mathbb{C})$ be given by $F(\lambda) = \mathcal{M}_{-X_1}\left(\dfrac{\lambda}{\lambda_0}X_3\right)$, $\lambda \in \mathbb{D}$.

That is,

$$F(\lambda) = \mathcal{M}_{\begin{bmatrix} 0 & -\sqrt{\dfrac{b_0}{1-b_0}} \\ 0 & 0 \end{bmatrix}}\left(\dfrac{\lambda}{\lambda_0}X_3\right)$$

Note that $F$ is analytic in $\mathbb{D}$,

$$\begin{aligned} F(0) &= X_1 = \begin{bmatrix} 0 & \zeta \\ 0 & 0 \end{bmatrix}, \\ F(\lambda_0) &= X_2 = \begin{bmatrix} a_0 & \tau\sqrt{a_0 b_0} \\ \tau^{-1}\sqrt{a_0 b_0} & b_0 \end{bmatrix}, \end{aligned}$$

where $\zeta = \sqrt{\dfrac{b_0}{1-b_0}} \in (0,1)$, $\tau = \sqrt{\dfrac{1-b_0}{a_0}} > 0$.
We must show that $\|F(\lambda)\| < 1$, for all $\lambda \in \mathbb{D}$.

First, we shall show that $\|X_3\| = \lambda_0$. i.e., that $\|X_3^* X_3\| = \lambda_0^2$.

$$\det(X_3^* X_3) = \left(\frac{a_0 b_0(1-2b_0) + a_0 b_0^2}{(1-b_0)^3}\right)^2 = \left(\frac{a_0 b_0}{(1-b_0)^2}\right)^2 = \frac{a_0^2 b_0^2}{(1-b_0)^4},$$

$$\operatorname{tr}(X_3^* X_3) = \frac{(a_0^2+b_0^2)(1-2b_0) + (a_0^2+b_0^2)b_0}{(1-b_0)^3} = \frac{a_0^2+b_0^2}{(1-b_0)^2}.$$

The squares $s_0^2, s_1^2$ of the singular values $s_0, s_1$ of $X_3$ are the roots of the equation

$$y^2 - \operatorname{tr}(X_3^* X_3)y + \det(X_3^* X_3) = 0 \text{ in } y,$$

that is, of the equation

$$y^2 - \frac{a_0^2 + b_0^2}{(1-b_0)^2} y + \frac{a_0^2 b_0^2}{(1-b_0)^4} = 0.$$

Hence,

$$s_0^2 = \frac{a_0^2}{(1-b_0)^2} \quad \text{and} \quad s_1^2 = \frac{b_0^2}{(1-b_0)^2}.$$

Note that $b_0 < a_0$, and so $\dfrac{a_0^2}{(1-b_0)^2} > \dfrac{b_0^2}{(1-b_0)^2}$.

Therefore, $\|X_3\|^2 = \dfrac{a_0^2}{(1-b_0)^2}$, and hence, $\|X_3\| = \lambda_0$. Now we show that $\|F(\lambda)\| < 1$ for all $\lambda \in \mathbb{D}$.

Since $\left\| \dfrac{\lambda}{\lambda_0} X_3 \right\| = |\lambda| \left\| \dfrac{X_3}{\lambda_0} \right\| = |\lambda| < 1$, we have $\|F(\lambda)\| < 1$, for all $\lambda \in \mathbb{D}$.

Thus, $F$ has the required properties and so (2) $\Rightarrow$ (3).

**(3)$\Rightarrow$(1)** Let $F$ satisfy condition (3) and let $h = (F_{11}, F_{22}, \det F)$. Clearly, $h$ is analytic. We know that

$$F(0) = \begin{bmatrix} 0 & \zeta \\ 0 & 0 \end{bmatrix} \text{ and } F(\lambda_0) = \begin{bmatrix} a_0 & \tau\sqrt{a_0 b_0} \\ \tau^{-1}\sqrt{a_0 b_0} & b_0 \end{bmatrix},$$

for some $\zeta \in (0,1)$ and $\tau > 0$.

Therefore,

$$\begin{aligned} h(0) &= (F_{11}, F_{22}, \det F)(0) &= (0,0,0), \\ h(\lambda_0) &= (F_{11}, F_{22}, \det F)(\lambda_0) = (a_0, b_0, 0). \end{aligned}$$

For any $\lambda \in \mathbb{D}$, we have $\|F(\lambda)\| < 1$, and so by (3) of Remark 1.2.9, $\mu_E(F(\lambda)) \leq \|F(\lambda)\| < 1$. Hence $(F_{11}(\lambda), F_{22}(\lambda), \det F(\lambda)) \in G_E$. Thus $h$ maps $\mathbb{D}$ into $G_E$.

$\square$

In the next result we give an explicit formula for an analytic function $f : \mathbb{D} \longrightarrow G_E$ such that $f(0) = (0, 0, 0)$ and $f(\lambda_0) = (a_0, b_0, 0)$, where $\lambda_0 \in \mathbb{D}$.

**Corollary 3.1.2** *Let $0 \leq b_0 < a_0 < 1 - b_0$ and let $\lambda_0 \in \mathbb{D}$. If there exists an analytic function $f : \mathbb{D} \longrightarrow G_E$ such that $f(0) = (0, 0, 0)$ and $f(\lambda_0) = (a_0, b_0, 0)$, then $f = (f_1, f_2, f_3) : \mathbb{D} \longrightarrow G_E$ can be given by*

$$f(\lambda) = \left( \frac{\lambda a_0(1 - 2b_0)}{a_0(1 - b_0) - \lambda b_0^2}, \frac{\lambda b_0(1 - 2b_0)}{a_0(1 - b_0) - \lambda b_0^2}, \frac{\lambda b_0 \left(\lambda(1 - b_0) - a_0\right)}{a_0(1 - b_0) - \lambda b_0^2} \right),$$

*for all $\lambda \in \mathbb{D}$.*

**Proof** Suppose that there exists an analytic function $f : \mathbb{D} \longrightarrow G_E$ such that $f(0) = (0, 0, 0)$ and $f(\lambda_0) = (a_0, b_0, 0)$. By Theorem 3.1.1, there exists an analytic function $F : \mathbb{D} \longrightarrow M_2(\mathbb{C})$ such that for all $\lambda \in \mathbb{D}$, $\|F(\lambda)\| < 1$ and

$$F(0) = \begin{bmatrix} 0 & \zeta \\ 0 & 0 \end{bmatrix} := X_1, \; F(\lambda_0) = \begin{bmatrix} a_0 & \tau\sqrt{a_0 b_0} \\ \tau^{-1}\sqrt{a_0 b_0} & b_0 \end{bmatrix} := X_2,$$

where $\lambda_0 = \dfrac{a_0}{1 - b_0}$, $\zeta = \sqrt{\dfrac{b_0}{1 - b_0}} \in (0, 1)$ and $\tau = \sqrt{\dfrac{1 - b_0}{a_0}} > 0$.
As in the proof of Theorem 3.1.1, we may take

$$F(\lambda) = \mathcal{M}_{-X_1}\left(\frac{\lambda}{\lambda_0} X_3\right),$$

and

$$X_3 = \begin{bmatrix} \dfrac{-a_0\sqrt{1 - 2b_0}}{(1 - b_0)\sqrt{1 - b_0}} & \dfrac{b_0\sqrt{b_0}}{(1 - b_0)\sqrt{1 - b_0}} \\ \\ \dfrac{-a_0\sqrt{b_0}}{(1 - b_0)\sqrt{1 - b_0}} & \dfrac{-b_0\sqrt{1 - 2b_0}}{(1 - b_0)\sqrt{1 - b_0}} \end{bmatrix}.$$

Hence,

$$\mathcal{M}_{-X_1}\left(\frac{\lambda}{\lambda_0}X_3\right) = (1-X_1X_1^*)^{-1/2}\left(-X_1-\frac{\lambda}{\lambda_0}X_3\right)\left(1+X_1^*\frac{\lambda}{\lambda_0}X_3\right)^{-1}(1-X_1^*X_1)^{1/2}$$

$$= \begin{bmatrix} \sqrt{\frac{1-b_0}{1-2b_0}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\lambda\sqrt{1-2b_0}}{\sqrt{1-b_0}} & \frac{-\sqrt{b_0}(a_0-\lambda b_0)}{a_0\sqrt{1-b_0}} \\ \frac{-\lambda\sqrt{b_0}}{\sqrt{1-b_0}} & \frac{\lambda b_0\sqrt{1-2b_0}}{a_0\sqrt{1-b_0}} \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ \frac{\lambda a_0\sqrt{b_0(1-2b_0)}}{a_0(1-b_0)-\lambda b_0^2} & \frac{a_0(1-b_0)}{a_0(1-b_0)-\lambda b_0^2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\frac{1-2b_0}{1-b_0}} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & \frac{-\sqrt{b_0}(a_0-\lambda b_0)}{a_0\sqrt{1-2b_0}} \\ \frac{-\lambda\sqrt{b_0}}{\sqrt{1-b_0}} & \frac{\lambda b_0\sqrt{1-2b_0}}{a_0\sqrt{1-b_0}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\lambda a_0\sqrt{b_0(1-2b_0)}}{a_0(1-b_0)-\lambda b_0^2} & \frac{a_0(1-b_0)}{a_0(1-b_0)-\lambda b_0^2} \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\frac{1-2b_0}{1-b_0}} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-\frac{\lambda b_0(a_0-\lambda b_0)}{a_0(1-b_0)-\lambda b_0^2} & \frac{-\sqrt{b_0}(1-b_0)(a_0-\lambda b_0)}{\sqrt{1-2b_0}(a_0(1-b_0)-\lambda b_0^2)} \\ \frac{-\lambda\sqrt{b_0}}{\sqrt{1-b_0}}+\frac{\lambda^2 b_0\sqrt{b_0}(1-2b_0)}{\sqrt{1-b_0}(a_0(1-b_0)-\lambda b_0^2)} & \frac{\lambda b_0\sqrt{1-b_0}\sqrt{1-2b_0}}{a_0(1-b_0)-\lambda b_0^2} \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\frac{1-2b_0}{1-b_0}} \end{bmatrix}$$

$$\mathcal{M}_{-X_1}\left(\frac{\lambda}{\lambda_0}X_3\right) = \begin{bmatrix} \dfrac{\lambda a_0(1-2b_0)}{a_0(1-b_0)-\lambda b_0^2} & \dfrac{-\sqrt{b_0(1-b_0)}(a_0-\lambda b_0)}{a_0(1-b_0)-\lambda b_0^2} \\ \\ \dfrac{-\lambda\sqrt{b_0(1-b_0)}(a_0-\lambda b_0)}{a_0(1-b_0)-\lambda b_0^2} & \dfrac{\lambda b_0(1-2b_0)}{a_0(1-b_0)-\lambda b_0^2} \end{bmatrix} := F(\lambda).$$

Hence,

$$f_1(\lambda) = \frac{\lambda a_0(1-2b_0)}{a_0(1-b_0)-\lambda b_0^2},$$

$$f_2(\lambda) = \frac{\lambda b_0(1-2b_0)}{a_0(1-b_0)-\lambda b_0^2},$$

$$\begin{aligned} f_3(\lambda) &= \det(f(\lambda)) = \frac{\lambda^2 a_0 b_0(1-2b_0)^2 - \lambda b_0(1-b_0)(a_0-\lambda b_0)^2}{\left(a_0(1-b_0)-\lambda b_0^2\right)^2} \\ &= \frac{\lambda b_0\left(\lambda(1-b_0)-a_0\right)}{a_0(1-b_0)-\lambda b_0^2}. \end{aligned}$$

Thus, for all $\lambda \in \mathbb{D}$,

$$f(\lambda) = \left(\frac{\lambda a_0(1-2b_0)}{a_0(1-b_0)-\lambda b_0^2}, \frac{\lambda b_0(1-2b_0)}{a_0(1-b_0)-\lambda b_0^2}, \frac{\lambda b_0\left(\lambda(1-b_0)-a_0\right)}{a_0(1-b_0)-\lambda b_0^2}\right).$$

□

## 3.2 A More General Schwarz Lemma for $\Gamma_E$

In this section, we prove a more general case of the Schwarz Lemma for $\Gamma_E$. Given $(a,b,p) \in G_E$, we find necessary and sufficient conditions for the existence of an analytic function $F : \mathbb{D} \longrightarrow G_E$ such that $F(0) = (0,0,0)$ and $F(\lambda_0) = (a,b,p)$, where $\lambda_0 \in \mathbb{D}\backslash\{0\}$.

**Theorem 3.2.1** *Let $U \in M_2(\mathbb{C})$, $\|U\| < 1$ and let $\lambda_0 \in \mathbb{D}\backslash\{0\}$. Then there exists an analytic function $G \in \mathcal{S}_{2\times 2}$ such that*

$$G(0) = \begin{bmatrix} * & * \\ * & 0 \end{bmatrix} \text{ and } G(\lambda_0) = U,$$

*if and only if* $\det(M) \leq 0$, *where*

$$M = \begin{bmatrix} [(1-\rho^2 U^*U)(1-U^*U)^{-1}]_{11} & [(1-\rho^2)(1-UU^*)^{-1}U]_{21} \\ [(1-\rho^2)U^*(1-UU^*)^{-1})]_{12} & [(UU^*-\rho^2)(1-UU^*)^{-1}]_{22} \end{bmatrix} \tag{3.1}$$

*and $\rho = |\lambda_0|$.*

**Proof** Suppose that $\|U\| < 1$ and that there exists an analytic function $G \in \mathcal{S}_{2\times 2}$ such that

$$G(0) = \begin{bmatrix} * & * \\ * & 0 \end{bmatrix} \text{ and } G(\lambda_0) = U.$$

Moreover, assume that

$$\mathcal{M}_U(G(\lambda_0)) = 0 \text{ and } \mathcal{M}_U(G(\lambda)) = B_{\lambda_0}(\lambda)H(\lambda),$$

where $H \in \mathcal{S}_{2\times 2}$ and $B_{\lambda_0}(\lambda) = \dfrac{\lambda - \lambda_0}{1 - \bar{\lambda}_0\lambda}$.

Therefore, there exists $G \in \mathcal{S}_{2\times 2}$ such that $G(0) = \begin{bmatrix} * & * \\ * & 0 \end{bmatrix}$ and $G(\lambda_0) = U$

$\Leftrightarrow$ $\exists\, H \in \mathcal{S}_{2\times 2}$ such that $G(0) = \mathcal{M}_{-U}(B_{\lambda_0}(0)H(0)) = \mathcal{M}_{-U}(-\lambda_0 H(0))$ has $2 \times 2$ entry 0

Let $X = -\lambda_0 H(0)$, where $H(0)$ is a constant of norm 1. Then

Such a $G$ exists $\Leftrightarrow \exists\, X \in M_2(\mathbb{C})$ such that $\|X\| \leq |\lambda_0| < 1$ and $[\mathcal{M}_{-U}(X)]_{22} = 0$.

We have

$$\mathcal{M}_{-U}(X) = (AX+B)(CX+D)^{-1}, \quad \text{where} \quad A = (1-UU^*)^{-1/2},\ B = (1-UU^*)^{-1/2}U,$$
$$C = U^*(1-UU^*)^{-1/2},\ D = (1-U^*U)^{-1/2}.$$

Therefore,

$$\begin{aligned}
\text{Such a } G \text{ exists} \quad &\Leftrightarrow \quad \exists\, X \in M_2(\mathbb{C}) \text{ such that } \|X\| \le \rho, \text{ and} \\
&\qquad \langle (AX+B)(CX+D)^{-1}e_2, e_2\rangle = 0 \\
&\Leftrightarrow \quad \exists\, X \in M_2(\mathbb{C}) \text{ such that } \|X\| \le \rho, \xi \in \mathbb{C}^2, \xi \neq 0, \text{ and} \\
&\qquad \begin{cases} \langle (AX+B)\xi, e_2\rangle = 0, \text{ and} \\ \langle (CX+D)\xi, e_1\rangle = 0 \end{cases} \\
&\Leftrightarrow \quad \exists\, X \in M_2(\mathbb{C}) \text{ such that } \|X\| \le \rho, \xi \in \mathbb{C}^2, \xi \neq 0, \text{ and} \\
&\qquad \xi \in (X^*A^* + B^*)e_2^{\perp} \cap (X^*C^* + D^*)e_1^{\perp} \\
&\Leftrightarrow \quad \exists\, X \in M_2(\mathbb{C}) \text{ such that } \|X\| \le \rho \text{ and} \\
&\qquad \operatorname{span}\{X^*A^*e_2 + B^*e_2, X^*C^*e_1 + D^*e_1\} \neq \mathbb{C}^2 \\
&\Leftrightarrow \quad \exists\, X \in M_2(\mathbb{C}) \text{ such that } \|X\| \le \rho \text{ and} \\
&\qquad X^*A^*e_2 + B^*e_2 \text{ and } X^*C^*e_1 + D^*e_1 \text{ are linearly dependent} \\
&\Leftrightarrow \quad \exists\, X \in M_2(\mathbb{C}) \text{ such that } \|X\| \le \rho \text{ and } \alpha_1, \alpha_2 \in \mathbb{C} \text{ not both zero} \\
&\qquad \text{such that } \alpha_2(X^*A^*e_2 + B^*e_2) + \alpha_1(X^*C^*e_1 + D^*e_1) = 0 \\
&\Leftrightarrow \quad \exists\, X \in M_2(\mathbb{C}) \text{ such that } \|X\| \le \rho \text{ and } \alpha_1, \alpha_2 \in \mathbb{C} \text{ not both zero} \\
&\qquad \text{such that } X^*(\alpha_1 C^*e_1 + \alpha_2 A^*e_2) = -(\alpha_1 D^*e_1 + \alpha_2 B^*e_2) \\
&\Leftrightarrow \quad \exists\, \alpha_1, \alpha_2 \in \mathbb{C} \text{ not both zero, such that} \\
&\qquad \|\alpha_1 D^*e_1 + \alpha_2 B^*e_2\|^2 \le \rho^2 \|\alpha_1 C^*e_1 + \alpha_2 A^*e_2\|^2
\end{aligned}$$

$$
\begin{aligned}
\text{Such a } G \text{ exists} \quad \Leftrightarrow \quad & \exists\, \alpha_1, \alpha_2 \in \mathbb{C} \text{ not both zero, such that} \\
& \langle D^*e_1, D^*e_1\rangle \alpha_1\bar{\alpha}_1 + \langle D^*e_1, B^*e_2\rangle \alpha_1\bar{\alpha}_2 + \langle B^*e_2, D^*e_1\rangle \bar{\alpha}_1\alpha_2 \\
& + \langle B^*e_2, B^*e_2\rangle \alpha_2\bar{\alpha}_2 - \rho^2[\langle C^*e_1, C^*e_1\rangle \alpha_1\bar{\alpha}_1 + \langle C^*e_1, A^*e_2\rangle \alpha_1\bar{\alpha}_2 \\
& + \langle A^*e_2, C^*e_1\rangle \bar{\alpha}_1\alpha_2 + \langle A^*e_2, A^*e_2\rangle \alpha_2\bar{\alpha}_2] \le 0 \\
\Leftrightarrow \quad & M := \begin{bmatrix} \langle D^*e_1, D^*e_1\rangle - \rho^2\langle C^*e_1, C^*e_1\rangle & \langle D^*e_1, B^*e_2\rangle - \rho^2\langle C^*e_1, A^*e_2\rangle \\ \langle B^*e_2, D^*e_1\rangle - \rho^2\langle A^*e_2, C^*e_1\rangle & \langle B^*e_2, B^*e_2\rangle - \rho^2\langle A^*e_2, A^*e_2\rangle \end{bmatrix} \\
& \text{is not positive definite} \\
\Leftrightarrow \quad & \text{ either } \langle D^*e_1, D^*e_1\rangle - \rho^2\langle C^*e_1, C^*e_1\rangle \le 0, \text{ or } \det(M) \le 0.
\end{aligned}
$$

Hence, we can write $M$ as follows:

$$
\begin{aligned}
M &= \begin{bmatrix} [DD^* - \rho^2 CC^*]_{11} & [BD^* - \rho^2 AC^*]_{21} \\ [DB^* - \rho^2 CA^*]_{12} & [BB^* - \rho^2 AA^*]_{22} \end{bmatrix} \\
&= \begin{bmatrix} [(1-UU^*)^{-1} - \rho^2 U^*U(1-U^*U)^{-1}]_{11} & [(1-UU^*)^{-1}U - \rho^2(1-UU^*)^{-1}U]_{21} \\ [U^*(1-UU^*)^{-1} - \rho^2 U^*(1-UU^*)^{-1}]_{12} & [UU^*(1-UU^*)^{-1} - \rho^2(1-UU^*)^{-1}]_{22} \end{bmatrix} \\
&= \begin{bmatrix} [(1-\rho^2U^*U)(1-U^*U)^{-1}]_{11} & [(1-\rho^2)(1-UU^*)^{-1}U]_{21} \\ [(1-\rho^2)U^*(1-UU^*)^{-1})]_{12} & [(UU^*-\rho^2)(1-UU^*)^{-1}]_{22} \end{bmatrix}.
\end{aligned}
$$

Note that, since $\|U\| < 1$ and $\rho \in (0,1)$, then

$$[(1-\rho^2U^*U)(1-U^*U)^{-1}]_{11} > 0,$$

for, let $Q = U^*U$, then $Q$ is a positive matrix and strictly contractive. Hence, $(1-\rho^2 Q)(1-Q)^{-1}$ is a positive function of a positive variable on the spectrum of $Q$, therefore, by the Spectral Theorem, $(1-\rho^2Q)(1-Q)^{-1}$ is positive.

Therefore, there exists an analytic function $G \in \mathcal{S}_{2\times 2}$ such that

$$G(0) = \begin{bmatrix} * & * \\ * & 0 \end{bmatrix}, \quad \text{and} \quad G(\lambda_0) = U,$$

if and only if $\det(M) \leq 0$, where $M$ is given as in (3.1).

$\square$

The proof of the next result in the case that $ab = p$ follows immediately by applying the classical Schwarz Lemma, in this case we have

$$|\lambda_0| \geq \max\{|a|, |b|\}.$$

**Theorem 3.2.2** *Let $(a, b, p) \in G_E$, $ab \neq p$, $\lambda_0 \in \mathbb{D}\backslash\{0\}$ and let $w \in \mathbb{C}$ satisfy $w^2 = ab - p$. Then there exist $\tau > 0$ and $F \in \mathcal{S}_{2\times 2}$ such that*

$$F(0) = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad F(\lambda_0) = \begin{bmatrix} a & \tau w \\ \tau^{-1} w & b \end{bmatrix},$$

*if and only if*

$$\text{either} \begin{cases} |b| \leq |a|, \text{ and} \\ |\lambda_0| \geq \dfrac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2} \end{cases} \text{or} \begin{cases} |a| < |b|, \text{ and} \\ |\lambda_0| \geq \dfrac{|b - \bar{a}p| + |ab - p|}{1 - |a|^2} \end{cases}.$$

To prove this theorem, we need the following Lemma.

**Lemma 3.2.3** *Let $a, b, p$ be defined as in Theorem 3.2.2, $0 < \lambda_0 < 1$ and let*

$$U = \begin{bmatrix} \dfrac{a}{\lambda_0} & \tau w \\ \\ \dfrac{\tau^{-1} w}{\lambda_0} & b \end{bmatrix}$$

*and*

$$M = \begin{bmatrix} [(1 - \rho^2 U^* U)(1 - U^* U)^{-1}]_{11} & [(1 - \rho^2)(1 - UU^*)^{-1} U]_{21} \\ [(1 - \rho^2) U^* (1 - UU^*)^{-1})]_{12} & [(UU^* - \rho^2)(1 - UU^*)^{-1}]_{22} \end{bmatrix},$$

*where* $\rho = |\lambda_0| = \lambda_0$. *Then*

$$\det\left(M \det(1-U^*U)\right) = C_{-4}\tau^{-4} + C_{-2}\tau^{-2} + C_0 + C_2\tau^2 + C_4\tau^4,$$

*where*

$$\begin{aligned}
C_{-4} &= \frac{-|ab-p|^2}{\lambda_0^2},\\
C_4 &= -|ab-p|^2\lambda_0^2,\\
C_{-2} &= -|ab-p|\left(-\lambda_0^2 + |a|^2 + |b|^2 - \frac{|p|^2}{\lambda_0^2}\right) + \left(1-|a|^2-|b|^2+|p|^2\right)\frac{|ab-p|}{\lambda_0^2}\\
&\quad - \left(\frac{1-\lambda_0^2}{\lambda_0}\right)^2 |ab-p|,\\
C_2 &= -|ab-p|\left(-\lambda_0^2 + |a|^2 + |b|^2 - \frac{|p|^2}{\lambda_0^2}\right) + \left(1-|a|^2-|b|^2+|p|^2\right)|ab-p|\lambda_0^2\\
&\quad - \left(\frac{1-\lambda_0^2}{\lambda_0}\right)^2 |p|^2|ab-p|,\\
C_0 &= -2|ab-p|^2 + (|a|^2+|b|^2)(1+|p|^2) - (|a|^4+|b|^4+2|p|^2) + \lambda_0^2(1-|a|^2)(|b|^2-1)\\
&\quad + \frac{1}{\lambda_0^2}\left(|p|^2(|a|^2+|b|^2-|p|^2) - |a|^2|b|^2\right).
\end{aligned}$$

**Proof** First, we calculate $\det(1-U^*U)$.

$$\begin{aligned}
U^*U &= \begin{bmatrix} \dfrac{\bar a}{\lambda_0} & \dfrac{\tau^{-1}\bar w}{\lambda_0} \\ \\ \tau\bar w & \bar b \end{bmatrix} \begin{bmatrix} \dfrac{a}{\lambda_0} & \tau w \\ \\ \dfrac{\tau^{-1} w}{\lambda_0} & b \end{bmatrix}\\
&= \begin{bmatrix} \dfrac{|a|^2}{\lambda_0^2} + \dfrac{|w|^2}{\lambda_0^2\tau^2} & \dfrac{\tau w\bar a}{\lambda_0} + \dfrac{b\bar w}{\tau\lambda_0} \\ \\ \dfrac{a\tau\bar w}{\lambda_0} + \dfrac{\bar b w}{\lambda_0\tau} & \tau^2|w|^2 + |b|^2 \end{bmatrix}.
\end{aligned}$$

Hence,

$$1 - U^*U = \begin{bmatrix} 1 - \dfrac{|a|^2}{\lambda_0^2} - \dfrac{|w|^2}{\lambda_0^2\tau^2} & -\dfrac{\tau w\bar{a}}{\lambda_0} - \dfrac{b\bar{w}}{\tau\lambda_0} \\ \\ -\dfrac{a\tau\bar{w}}{\lambda_0} - \dfrac{\bar{b}w}{\lambda_0\tau} & 1 - \tau^2|w|^2 - |b|^2 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \det(1 - U^*U) &= \left(1 - \frac{|a|^2}{\lambda_0^2} - \frac{|w|^2}{\lambda_0^2\tau^2}\right)\left(1 - \tau^2|w|^2 - |b|^2\right) - \left(\frac{\tau w\bar{a}}{\lambda_0} + \frac{b\bar{w}}{\tau\lambda_0}\right)\left(\frac{a\tau\bar{w}}{\lambda_0} + \frac{\bar{b}w}{\lambda_0\tau}\right) \\ &= 1 - \frac{|a|^2}{\lambda_0^2} - |b|^2 + \frac{|p|^2}{\lambda_0^2} - |ab - p|\left(\tau^2 + \frac{1}{\lambda_0^2\tau^2}\right). \end{aligned}$$

Now, we calculate $M$. We start by calculating $[(1 - \rho^2U^*U)(1 - U^*U)^{-1}]_{11}$.

Clearly,

$$1 - \rho^2U^*U = \begin{bmatrix} 1 - |a|^2 - \dfrac{|w|^2}{\tau^2} & -\lambda_0\left(\tau w\bar{a} + \dfrac{b\bar{w}}{\tau}\right) \\ \\ -\lambda_0\left(a\tau\bar{w} + \dfrac{\bar{b}w}{\tau}\right) & 1 - \lambda_0^2\left(\tau^2|w|^2 + |b|^2\right) \end{bmatrix},$$

and

$$\begin{aligned} (1 - U^*U)^{-1} &= \frac{1}{1 - \dfrac{|a|^2}{\lambda_0^2} - |b|^2 + \dfrac{|p|^2}{\lambda_0^2} - |ab - p|\left(\tau^2 + \dfrac{1}{\lambda_0^2\tau^2}\right)} \times \\ & \begin{bmatrix} 1 - \tau^2|ab - p|^2 - |b|^2 & \dfrac{\tau w\bar{a}}{\lambda_0} + \dfrac{b\bar{w}}{\lambda_0\tau} \\ \\ \dfrac{a\tau\bar{w}}{\lambda_0} + \dfrac{\bar{b}w}{\tau\lambda_0} & 1 - \dfrac{|a|^2}{\lambda_2^2} - \dfrac{|ab - p|}{\tau^2\lambda_0^2} \end{bmatrix}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
\left[(1-\rho^2U^*U)(1-U^*U)^{-1}\right]_{11} &= \frac{\left(1-|a|^2-\dfrac{|ab-p|}{\tau^2}\right)(1-|b|^2-\tau^2|ab-p|)}{1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}-|ab-p|\left(\tau^2+\dfrac{1}{\lambda_0^2\tau^2}\right)} \\
&\quad -\frac{\left(\tau w\bar{a}+\dfrac{b\bar{w}}{\tau}\right)\left(a\tau\bar{w}+\dfrac{\bar{b}w}{\tau}\right)}{1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}-|ab-p|\left(\tau^2+\dfrac{1}{\lambda_0^2\tau^2}\right)} \\
&= \frac{1-|a|^2-|b|^2+|p|^2-|ab-p|\left(\tau^2+\dfrac{1}{\tau^2}\right)}{1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}-|ab-p|\left(\tau^2+\dfrac{1}{\lambda_0^2\tau^2}\right)}.
\end{aligned}
$$

Now we shall calculate $[(UU^*-\rho^2)(1-UU^*)^{-1}]_{22}$.

$$
\begin{aligned}
UU^* &= \begin{bmatrix} \dfrac{a}{\lambda_0} & \tau w \\ \\ \dfrac{\tau^{-1}w}{\lambda_0} & b \end{bmatrix} \begin{bmatrix} \dfrac{\bar{a}}{\lambda_0} & \dfrac{\tau^{-1}\bar{w}}{\lambda_0} \\ \\ \tau\bar{w} & \bar{b} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{|a|^2}{\lambda_0^2}+\tau^2|ab-p| & \dfrac{a\bar{w}}{\tau\lambda_0^2}+\bar{b}\tau w \\ \\ \dfrac{\bar{a}w}{\tau\lambda_0^2}+b\tau\bar{w} & \dfrac{|ab-p|}{\tau^2\lambda_0^2}+|b|^2 \end{bmatrix}.
\end{aligned}
$$

Hence,

$$
UU^*-\rho^2 = \begin{bmatrix} \dfrac{|a|^2}{\lambda_0^2}+\tau^2|ab-p|-\lambda_0^2 & \dfrac{a\bar{w}}{\tau\lambda_0^2}+\bar{b}\tau w \\ \\ \dfrac{\bar{a}w}{\tau\lambda_0^2}+b\tau\bar{w} & \dfrac{|ab-p|}{\tau^2\lambda_0^2}+|b|^2-\lambda_0^2 \end{bmatrix},
$$

and

$$(1-UU^*)^{-1} = \frac{1}{1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}-|ab-p|\left(\tau^2+\dfrac{1}{\lambda_0^2\tau^2}\right)} \times \begin{bmatrix} 1-\dfrac{|ab-p|}{\tau^2\lambda_0^2}-|b|^2 & \dfrac{a\bar{w}}{\tau\lambda_0^2}+\bar{b}\tau w \\ \\ \dfrac{\bar{a}w}{\tau\lambda_0^2}+b\tau\bar{w} & 1-\dfrac{|a|^2}{\lambda_0^2}-\tau^2|ab-p| \end{bmatrix}.$$

Therefore,

$$\begin{aligned}\left[(UU^*-\rho^2)(1-UU^*)^{-1}\right]_{22} &= \frac{\left(-\lambda_0^2+\dfrac{|ab-p|}{\tau^2\lambda_0^2}+|b|^2\right)\left(1-\dfrac{|a|^2}{\lambda_0^2}-\tau^2|ab-p|\right)}{1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}-|ab-p|\left(\tau^2+\dfrac{1}{\lambda_0^2\tau^2}\right)} \\ &\quad+\frac{\left(\dfrac{a\bar{w}}{\tau\lambda_0^2}+\bar{b}\tau w\right)\left(\dfrac{\bar{a}w}{\tau\lambda_0^2}+b\tau\bar{w}\right)}{1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}-|ab-p|\left(\tau^2+\dfrac{1}{\lambda_0^2\tau^2}\right)} \\ &= \frac{-\lambda_0^2+|a|^2+|b|^2-\dfrac{|p|^2}{\lambda_0^2}+|ab-p|\left(\tau^2\lambda_0^2+\dfrac{1}{\tau^2\lambda_0^2}\right)}{1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}-|ab-p|\left(\tau^2+\dfrac{1}{\lambda_0^2\tau^2}\right)}.\end{aligned}$$

Moreover, we calculate the following:

$$\left[(1-\rho^2)(1-UU^*)^{-1}U\right]_{21} = \frac{(1-\lambda_0^2)\left(w+ab\tau^2\bar{w}-\tau^2w|ab-p|\right)}{\lambda_0\tau\left(1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}-|ab-p|\left(\tau^2+\dfrac{1}{\lambda_0^2\tau^2}\right)\right)},$$

and

$$\left[(1-\rho^2)U^*(1-UU^*)^{-1}\right]_{12} = \frac{(1-\lambda_0^2)\left(\bar{w}+\bar{a}\bar{b}\tau^2w-\tau^2\bar{w}|ab-p|\right)}{\lambda_0\tau\left(1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}-|ab-p|\left(\tau^2+\dfrac{1}{\lambda_0^2\tau^2}\right)\right)}.$$

Therefore,

$$M = \frac{1}{1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}-|ab-p|\left(\tau^2+\dfrac{1}{\lambda_0^2\tau^2}\right)} \times$$
$$\begin{bmatrix} 1-|a|^2-|b|^2+|p|^2-|ab-p|\left(\tau^2+\frac{1}{\tau^2}\right) & \frac{(1-\lambda_0^2)}{\lambda_0}\left(\tau\bar{a}\bar{b}w-\tau\bar{w}|ab-p|+\frac{\bar{w}}{\tau}\right) \\ \frac{(1-\lambda_0^2)}{\lambda_0}\left(\tau ab\bar{w}-\tau w|ab-p|+\frac{w}{\tau}\right) & -\lambda_0^2+|a|^2+|b|^2-\frac{|p|^2}{\lambda_0^2}+|ab-p|\left(\tau^2\lambda_0^2+\frac{1}{\tau^2\lambda_0^2}\right) \end{bmatrix}.$$

Note that

$$\begin{aligned} \|U\|<1 &\Leftrightarrow \begin{bmatrix} 1-\dfrac{|a|^2}{\lambda_0^2}-\dfrac{|w|^2}{\lambda_0^2\tau^2} & -\dfrac{\bar{a}\tau w}{\lambda_0}-\dfrac{b\bar{w}}{\tau\lambda_0} \\ -\dfrac{a\tau\bar{w}}{\lambda_0}-\dfrac{\bar{b}w}{\lambda_0\tau} & 1-\tau^2|w|^2-|b|^2 \end{bmatrix}>0 \\ &\Leftrightarrow \begin{cases} \tau^2|ab-p|<1-|b|^2, \text{ and} \\ \det(1-U^*U)=1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}-|ab-p|\left(\tau^2+\dfrac{1}{\tau^2\lambda_0^2}\right)>0 \end{cases} \\ &\Leftrightarrow \begin{cases} \tau^2<\dfrac{1-|b|^2}{|ab-p|}, \text{ and} \\ \tau^2+\dfrac{1}{\tau^2\lambda_0^2}<\dfrac{1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}}{|ab-p|}. \end{cases} \end{aligned}$$

Therefore,

$$M\det(1-U^*U)=\begin{bmatrix} 1-|a|^2-|b|^2+|p|^2-|ab-p|\left(\tau^2+\frac{1}{\tau^2}\right) & \frac{(1-\lambda_0^2)}{\lambda_0}\left(\tau\bar{a}\bar{b}w-\tau\bar{w}|ab-p|+\frac{\bar{w}}{\tau}\right) \\ \frac{(1-\lambda_0^2)}{\lambda_0}\left(\tau ab\bar{w}-\tau w|ab-p|+\frac{w}{\tau}\right) & -\lambda_0^2+|a|^2+|b|^2-\frac{|p|^2}{\lambda_0^2}+|ab-p|\left(\tau^2\lambda_0^2+\frac{1}{\tau^2\lambda_0^2}\right) \end{bmatrix}.$$

Hence,

$$\det\left(M\det(1-U^*U)\right)=C_{-4}\tau^{-4}+C_{-2}\tau^{-2}+C_0+C_2\tau^2+C_4\tau^4,$$

where

$$
\begin{aligned}
C_{-4} &= -\frac{|ab-p|^2}{\lambda_0^2}, \\
C_4 &= -\lambda_0^2|ab-p|^2, \\
C_{-2} &= -|ab-p|\left(-\lambda_0^2+|a|^2+|b|^2-\frac{|p|^2}{\lambda_0^2}\right)+(1-|a|^2-|b|^2+|p|^2)\frac{|ab-p|}{\lambda_0^2} \\
&\quad -\left(\frac{1-\lambda_0^2}{\lambda_0}\right)^2|ab-p|, \\
C_2 &= -|ab-p|\left(-\lambda_0^2+|a|^2+|b|^2-\frac{|p|^2}{\lambda_0^2}\right)+(1-|a|^2-|b|^2+|p|^2)|ab-p|\lambda_0^2 \\
&\quad -\left(\frac{1-\lambda_0^2}{\lambda_0}\right)^2(ab\bar{w}-w|ab-p|)(\bar{a}\bar{b}w-\bar{w}|ab-p|) \\
&= -|ab-p|\left(-\lambda_0^2+|a|^2+|b|^2-\frac{|p|^2}{\lambda_0^2}\right)+(1-|a|^2-|b|^2+|p|^2)|ab-p|\lambda_0^2 \\
&\quad -\left(\frac{1-\lambda_0^2}{\lambda_0}\right)^2|p|^2|ab-p|, \\
C_0 &= -\lambda_0^2|ab-p|^2+(1-|a|^2-|b|^2+|p|^2)|\left(-\lambda_0^2+|a|^2+|b|^2-\frac{|p|^2}{\lambda_0^2}\right)-\frac{|ab-p|^2}{\lambda_0^2} \\
&\quad -\left(\frac{1-\lambda_0^2}{\lambda_0}\right)^2\big(w(\bar{a}\bar{b}w-\bar{w}|ab-p|)+\bar{w}(ab\bar{w}-w|ab-p|)\big) \\
&= -2|ab-p|^2+(|a|^2+|b|^2)(1+|p|^2)-(|a|^4+|b|^4+2|p|^2)+\lambda_0^2(1-|a|^2)(|b|^2-1) \\
&\quad +\frac{1}{\lambda_0^2}\left(|p|^2(|a|^2+|b|^2-|p|^2)-|a|^2|b|^2\right).
\end{aligned}
$$

$\square$

**Proof of Theorem 3.2.2** We shall show that there exist $\tau > 0$ and $F \in \mathcal{S}_{2\times 2}$ such that $\|F\| < 1$ and

$$
F(0) = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad F(\lambda_0) = \begin{bmatrix} a & \tau w \\ \tau^{-1}w & b \end{bmatrix},
$$

if and only if

$$\text{either } \begin{cases} |b| \le |a|, \text{ and} \\ |\lambda_0| > \dfrac{|a-\bar{b}p| + |ab-p|}{1-|b|^2} \end{cases} \text{ or } \begin{cases} |a| < |b|, \text{ and} \\ |\lambda_0| > \dfrac{|b-\bar{a}p| + |ab-p|}{1-|a|^2} \end{cases}.$$

Note that if $F$ exists, then $F = G \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}$, $z \in \mathbb{D}$, for some $G \in \mathcal{S}_{2\times 2}$.

Hence, there exists an analytic function $F \in \mathcal{S}_{2\times 2}$ and $\tau > 0$ such that

$$F(0) = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad F(\lambda_0) = \begin{bmatrix} a & \tau w \\ \tau^{-1} w & b \end{bmatrix}$$

if and only if there exists an analytic function $G \in \mathcal{S}_{2\times 2}$ such that

$$G(0) = \begin{bmatrix} * & * \\ * & 0 \end{bmatrix}, \quad \text{and} \quad G(\lambda_0) = U,$$

where $U \in M_2(\mathbb{C})$ and $\|U\| < 1$.

That is, by Theorem 3.2.1, if and only if $\det(M) \le 0$, where $M$ is given by (3.1).

That is, if and only if $\det\left(M \det(1 - U^*U)\right) \le 0$, where $U \in M_2(\mathbb{C})$ and $\|U\| < 1$.

Let $x = \lambda_0 \tau^2 + \dfrac{1}{\lambda_0 \tau^2}$ so that $x^2 = \lambda_0^2 \tau^4 + \dfrac{1}{\lambda_0^2 \tau^4} + 2$. Hence, we can write $\det\left(M \det(1 - U^*U)\right)$ as follows:

$$\begin{aligned} \det\left(M \det(1 - U^*U)\right) &= -|ab-p|^2 x^2 + |ab-p| \left( \frac{2|p|^2 - |a|^2 - |b|^2}{\lambda_0} + \lambda_0 (2 - |a|^2 - |b|^2) \right) x \\ &\quad + 2|ab-p|^2 - 2|ab-p|^2 + (|a|^2 + |b|^2)(1 + |p|^2) \\ &\quad - (|a|^4 + |b|^4 + 2|p|^2) + \lambda_0^2 (1 - |a|^2)(|b|^2 - 1) \\ &\quad + \frac{1}{\lambda_0^2} \left( |p|^2 (|a|^2 + |b|^2 - |p|^2) - |a|^2 |b|^2 \right). \end{aligned}$$

Therefore,

$$\exists \text{ such an } F \quad \Leftrightarrow \quad \begin{cases} \exists\, \tau > 0 \text{ such that } \tau^2 < \dfrac{1-|b|^2}{|ab-p|}, \\[2ex] \tau^2\lambda_0 + \dfrac{1}{\tau^2\lambda_0} < \lambda_0 \dfrac{\dfrac{|a|^2}{\lambda_0^2} - |b|^2 + \dfrac{|p|^2}{\lambda_0^2}}{|ab-p|}, \\[2ex] -|ab-p|^2x^2 + |ab-p|\left(\dfrac{2|p|^2-|a|^2-|b|^2}{\lambda_0} + \lambda_0(2-|a|^2-|b|^2)\right)x \\ +(|a|^2+|b|^2)(1+|p|^2) - (|a|^4+|b|^4+2|p|^2) + \lambda_0^2(1-|a|^2)(|b|^2-1) \\ +\dfrac{1}{\lambda_0^2}\left(|p|^2(|a|^2+|b|^2-|p|^2) - |a|^2|b|^2\right) \le 0, \\ \text{where } x = \lambda_0\tau^2 + \dfrac{1}{\lambda_0\tau^2}. \end{cases} \tag{3.2}$$

Let

$$\begin{aligned} f(x) \;=\; & -|ab-p|^2x^2 + |ab-p|\left(\frac{2|p|^2-|a|^2-|b|^2}{\lambda_0} + \lambda_0(2-|a|^2-|b|^2)\right)x \\ & +(|a|^2+|b|^2)(1+|p|^2) - (|a|^4+|b|^4+2|p|^2) + \lambda_0^2(1-|a|^2)(|b|^2-1) \\ & +\frac{1}{\lambda_0^2}\left(|p|^2(|a|^2+|b|^2-|p|^2) - |a|^2|b|^2\right), \end{aligned}$$

and let

$$Y = \lambda_0 \frac{1 - \dfrac{|a|^2}{\lambda_0^2} - |b|^2 + \dfrac{|p|^2}{\lambda_0^2}}{|ab-p|}.$$

We shall find under what conditions it is true that there exists $x \in [2, Y)$ such that $f(x) \le 0$.

Note that

$$\nexists \text{ such an } x \iff f(2) > 0 \text{ and } f(Y) \ge 0.$$

Next, we calculate $f(Y)$.

$$
\begin{aligned}
f(Y) &= -|ab-p|^2\lambda_0^2\frac{\left(1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}\right)^2}{|ab-p|^2} \\
&\quad +|ab-p|\lambda_0\frac{\left(1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}\right)}{|ab-p|}\left(\frac{2|p|^2-|a|^2-|b|^2}{\lambda_0}+\lambda_0(2-|a|^2-|b|^2)\right) \\
&\quad +(|a|^2+|b|^2)(1+|p|^2)-(|a|^4+|b|^4+2|p|^2) \\
&\quad +\lambda_0(1-|a|^2)(|b|^2-1)+\frac{1}{\lambda_0}\left(|p|^2(|a|^2+|b|^2-|p|^2)-|a|^2|b|^2\right) \\
&= -\lambda_0^2\left(1-\frac{|a|^2}{\lambda_0^2}-|b|^2+\frac{|p|^2}{\lambda_0^2}\right)^2+\left(1-\frac{|a|^2}{\lambda_0^2}-|b|^2+\frac{|p|^2}{\lambda_0^2}\right)\left(2|p|^2-|a|^2-|b|^2\right) \\
&\quad +\lambda_0^2\left(1-\frac{|a|^2}{\lambda_0^2}-|b|^2+\frac{|p|^2}{\lambda_0^2}\right)\left(2-|a|^2-|b|^2\right)+(|a|^2+|b|^2)(1+|p|^2) \\
&\quad -(|a|^4+|b|^4+2|p|^2)+\lambda_0^2(1-|a|^2)(|b|^2-1)+\frac{1}{\lambda_0^2}\left(|a|^2|p|^2+|b|^2|p|^2-|p|^4-|a|^2|b|^2\right) \\
&= 0.
\end{aligned}
$$

That is, $f(Y)=0$. We also have

$$
f(2)=-\lambda_0^2\left(1-|a|^2-\frac{|b|^2}{\lambda_0^2}+\frac{|p|^2}{\lambda_0^2}-\frac{2|ab-p|}{\lambda_0}\right)\left(1-\frac{|a|^2}{\lambda_0^2}-|b|^2+\frac{|p|^2}{\lambda_0^2}-\frac{2|ab-p|}{\lambda_0}\right).
$$

Moreover, $f(x)$ has the following roots:

$$
\begin{aligned}
Y_1 &= \lambda_0\frac{1-|a|^2-\dfrac{|b|^2}{\lambda_0^2}+\dfrac{|p|^2}{\lambda_0^2}}{|ab-p|}, \text{ and} \\
Y_2 &= \lambda_0\frac{1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}}{|ab-p|}=Y.
\end{aligned}
$$

We shall find conditions for when $f(2)\le 0$. Let

$$
\begin{aligned}
X_1 &= 1-|a|^2-\frac{|b|^2}{\lambda_0^2}+\frac{|p|^2}{\lambda_0^2}-\frac{2|ab-p|}{\lambda_0}, \\
X_2 &= 1-\frac{|a|^2}{\lambda_0^2}-|b|^2+\frac{|p|^2}{\lambda_0^2}-\frac{2|ab-p|}{\lambda_0}.
\end{aligned}
$$

First, we show that $X_2 > 0$. From (3.2) we have

$$
\begin{aligned}
2 < \lambda_0 \frac{1 - \frac{|a|^2}{\lambda_0^2} - |b|^2 + \frac{|p|^2}{\lambda_0^2}}{|ab - p|} \quad &\Leftrightarrow \quad 2 < \frac{1 - \frac{|a|^2}{\lambda_0^2} - |b|^2 + \frac{|p|^2}{\lambda_0^2}}{\frac{|ab-p|}{\lambda_0}} \\
&\Leftrightarrow \quad \left(1 - \frac{|a|^2}{\lambda_0^2} - |b|^2 + \frac{|p|^2}{\lambda_0^2} - \frac{2|ab-p|}{\lambda_0}\right) > 0 \\
&\Leftrightarrow \quad X_2 > 0 \\
&\Leftrightarrow \quad \left(\frac{a}{\lambda_0}, b, \frac{p}{\lambda_0}\right) \in G_E.
\end{aligned}
$$

Therefore, $X_2 > 0$.

Now we shall find under what conditions it is true that $X_1 \geq 0$ so that $f(2) \leq 0$. There are two cases; when $0 < X_2 \leq X_1$ and when $0 \leq X_1 < X_2$.

In the case that $0 < X_2 \leq X_1$, we have, for $0 < \lambda_0 < 1$,

$$
\begin{aligned}
X_2 \leq X_1 \quad &\Leftrightarrow \quad 1 - \frac{|a|^2}{\lambda_0^2} - |b|^2 + \frac{|p|^2}{\lambda_0^2} - \frac{2|ab-p|}{\lambda_0} \leq 1 - |a|^2 - \frac{|b|^2}{\lambda_0^2} + \frac{|p|^2}{\lambda_0^2} - \frac{2|ab-p|}{\lambda_0} \\
&\Leftrightarrow \quad |a|^2 - \frac{|a|^2}{\lambda_0^2} \leq |b|^2 - \frac{|b|^2}{\lambda_0^2} \\
&\Leftrightarrow \quad |a|^2\left(1 - \frac{1}{\lambda_0^2}\right) \leq |b|^2\left(1 - \frac{1}{\lambda_0^2}\right) \\
&\Leftrightarrow \quad |a|^2 \geq |b|^2 \\
&\Leftrightarrow \quad |a| \geq |b|.
\end{aligned}
$$

Similarly, in the case that $0 \leq X_1 < X_2$, we have, for $0 < \lambda_0 < 1$,

$$
X_1 < X_2 \Leftrightarrow |a| < |b|.
$$

Therefore, there exists an $x \in [0, Y)$ such that

$f(x) = \det\left(M \det(1 - U^*U)\right) \leq 0$ if and only if

$$
\begin{cases}
|b| \leq |a|, \text{ and} \\
\lambda_0 > \dfrac{2|ab-p|}{1 - \frac{|a|^2}{\lambda_0^2} - |b|^2 + \frac{|p|^2}{\lambda_0^2}},
\end{cases}
$$

or

$$\begin{cases} |a| < |b|, \text{ and} \\ \lambda_0 > \dfrac{2|ab-p|}{1-|a|^2-\dfrac{|b|^2}{\lambda_0^2}+\dfrac{|p|^2}{\lambda_0^2}}. \end{cases}$$

Thus,

$$\exists \text{ such an } F \in \mathcal{S}_{2\times 2} \Leftrightarrow \begin{cases} \exists\, \tau > 0 \text{ such that } \tau^2 < \dfrac{1-|b|^2}{|ab-p|}, \\ |b| \leq |a|, \text{ and} \\ \lambda_0 > \dfrac{2|ab-p|}{1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}}. \end{cases}$$

From the proof above, we observe that, when $|b| \leq |a|$,

$$\begin{aligned} \lambda_0 > \frac{2|ab-p|}{1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}} \quad &\Leftrightarrow \quad 1-\frac{|a|^2}{\lambda_0^2}-|b|^2+\frac{|p|^2}{\lambda_0^2}-2\frac{|ab-p|}{\lambda_0} > 0 \\ &\Leftrightarrow \quad \left(\frac{a}{\lambda_0}, b, \frac{p}{\lambda_0}\right) \in G_E \\ &\Leftrightarrow \quad |b|^2 + \left|\frac{a}{\lambda_0}+\frac{\bar{b}p}{\lambda_0}\right| + \left|\frac{ab-p}{\lambda_0}\right| < 1,\ |a| \leq |\lambda_0| \\ &\Leftrightarrow \quad \lambda_0 > \frac{|a-\bar{b}p|+|ab-p|}{1-|b|^2}. \end{aligned}$$

Similarly, when $|a| < |b|$,

$$\begin{aligned} \lambda_0 > \frac{2|ab-p|}{1-|a|^2-\dfrac{|b|^2}{\lambda_0^2}+\dfrac{|p|^2}{\lambda_0^2}} \quad &\Leftrightarrow \quad \left(a, \frac{b}{\lambda_0}, \frac{p}{\lambda_0}\right) \in G_E \\ &\Leftrightarrow \quad \lambda_0 > \frac{|b-\bar{a}p|+|ab-p|}{1-|a|^2}. \end{aligned}$$

Hence, there exists $\tau > 0$ and $F \in \mathcal{S}_{2\times 2}$ such that $\|F\| < 1$ and

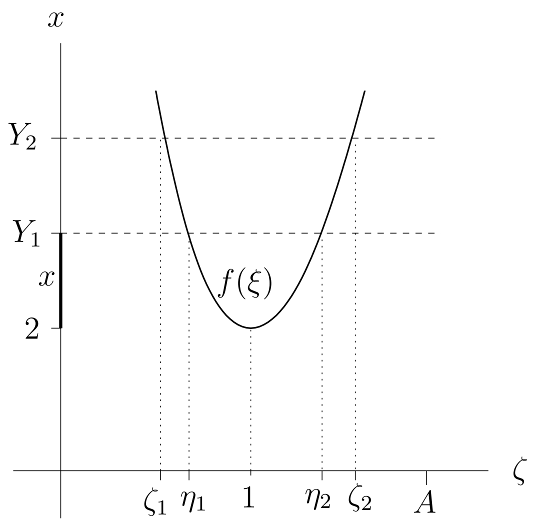
$$F(0) = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad F(\lambda_0) = \begin{bmatrix} a & \tau w \\ \tau^{-1} w & b \end{bmatrix},$$

if and only if

$$\text{either}\begin{cases}|b|\le|a|, \text{ and}\\ |\lambda_0|>\dfrac{|a-\bar{b}p|+|ab-p|}{1-|b|^2}\end{cases}\text{or}\begin{cases}|a|<|b|, \text{ and}\\ |\lambda_0|>\dfrac{|b-\bar{a}p|+|ab-p|}{1-|a|^2}\end{cases}.$$

Finally, we shall find under what $a, b, p$ does there exist $\tau > 0$ such that

$$\begin{cases}\lambda_0\tau^2<\lambda_0\dfrac{1-|b|^2}{|ab-p|}=:A, \text{ and}\\ \\ \lambda_0\tau^2+\dfrac{1}{\lambda_0\tau^2}<\lambda_0\dfrac{1-\dfrac{|a|^2}{\lambda_0^2}-|b|^2+\dfrac{|p|^2}{\lambda_0^2}}{|ab-p|}=:Y_2\end{cases}. \tag{3.3}$$




Let $\zeta = \lambda_0\tau^2$. We shall find under what conditions on $A$ and $Y_2$ there exists $\zeta < A$ such that $\zeta + \dfrac{1}{\zeta} < Y_2$.

Let $f(\zeta) = \zeta + \dfrac{1}{\zeta}$. Note that $\zeta + \dfrac{1}{\zeta} \ge 2$.

If $A \ge 1$, then there exists $\zeta < A$ such that $\zeta + \dfrac{1}{\zeta} < Y_2$. If $A < 1$, then there exists $\zeta < A$ such that $\zeta + \dfrac{1}{\zeta} < Y_2$ if and only if $A > \zeta_1$, where $\zeta_1 \le 1$ is a root of $\zeta + \dfrac{1}{\zeta} = Y_2$ and $\zeta_2 \ge 1$ is the other root.

Therefore, there exists $\zeta < A$ such that $f(\zeta) < Y_2$ if and only if $A > \zeta_1$.
When $\zeta_1 < A \leq \zeta_2$, there exists $\zeta < A$ such that $f(\zeta) < Y_2$ if and only if $\zeta \in [\zeta_1, A)$, and when $A > \zeta_2$ there exists $\zeta < A$ such that $f(\zeta) < Y_2$ if and only if $\zeta \in [\zeta_1, \zeta_2]$.
Note that

$$\begin{aligned}
A + \frac{1}{A} - Y_2 &= \lambda_0 \frac{1-|b|^2}{|ab-p|} + \frac{|ab-p|}{\lambda_0(1-|b|^2)} - \lambda_0 \frac{1 - \frac{|a|^2}{\lambda_0^2} - |b|^2 + \frac{|p|^2}{\lambda_0^2}}{|ab-p|} \\
&= \frac{1}{\lambda_0(1-|b|^2)|ab-p|}\left(|ab-p|^2 - (1-|b|^2)(-|a|^2+|p|^2)\right) \\
&= \frac{1}{\lambda_0(1-|b|^2)|ab-p|}\left(-2\mathrm{Re}\{ab\bar{p}\} + |a|^2 + |b|^2|p|^2\right) \\
&= \frac{|a-\bar{b}p|}{\lambda_0(1-|b|^2)|ab-p|} \geq 0.
\end{aligned}$$

Hence, $A > 1$. That is, there exists $\zeta < A$ such that $\zeta + \dfrac{1}{\zeta} < Y_2$.
Thus,

$$\begin{aligned}
\exists\, \tau > 0 \text{ such that (3.3) holds} &\Leftrightarrow A > 1 \\
&\Leftrightarrow \lambda_0 > \frac{|ab-p|}{1-|b|^2}.
\end{aligned}$$

Note that, $f(Y_1) = 0 = f(Y_2)$ and $f(2) \leq 0$, thus, $x < Y_1$ or $x > Y_2$, therefore, $x \in [2, Y_1]$.
Therefore, when $\lambda_0 > \dfrac{|ab-p|}{1-|b|^2}$, there exists $\tau > 0$ such that $\tau^2 < \dfrac{1-|b|^2}{|ab-p|}$ and the range of values of $\tau$ is given by $\zeta_1 \leq \lambda_0\tau^2 \leq \zeta_2$, where $\zeta_1$ and $\zeta_2$ are the roots of $\zeta + \dfrac{1}{\zeta} = Y_2$, that is, the roots of $\lambda_0\tau^2 + \dfrac{1}{\lambda_0\tau^2} = Y_2$.
Therefore, there exists $\tau > 0$ such that $\tau^2 < \dfrac{1-|b|^2}{|ab-p|}$ where the range of values of $\tau$ is as given above and $F \in \mathcal{S}_{2\times 2}$ such that $\|F\| < 1$ and

$$F(0) = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad F(\lambda_0) = \begin{bmatrix} a & \tau w \\ \tau^{-1} w & b \end{bmatrix}, \tag{3.4}$$

if and only if

$$\text{either} \begin{cases} |b| \leq |a|, \text{ and} \\ |\lambda_0| > \dfrac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2} \end{cases} \text{or} \begin{cases} |a| < |b|, \text{ and} \\ |\lambda_0| > \dfrac{|b - \bar{a}p| + |ab - p|}{1 - |a|^2} \end{cases}.$$

$\square$

In Theorem 3.2.2, if $\|U\| \to 1$, we pick $\lambda_\varepsilon = |\lambda_0|(1 + \varepsilon)$, where $\varepsilon > 0$ is small enough so that $0 < |\lambda_0| < |\lambda_\varepsilon| < 1$ and $\|U_\varepsilon\| < 1$, where

$$U_\varepsilon = \begin{bmatrix} \dfrac{a}{\lambda_\varepsilon} & \tau w \\ \\ \dfrac{\tau^{-1} w}{\lambda_\varepsilon} & b \end{bmatrix}.$$

We proceed in the proof exactly as we did above but using $\lambda_\varepsilon$ instead of $\lambda_0$. Then we apply Montel's Theorem which states: *Any locally bounded sequence of holomorphic functions $f_n$ defined on an open subset of $\mathbb{C}$ has a subsequence which converges uniformly to a holomorphic function $f$ on compact subsets.*

In this case, we find that, since $F_\varepsilon : \mathbb{D} \longrightarrow \Gamma_E$, where $\Gamma_E$ is bounded, $F_\varepsilon$ has a subsequence that converges to a holomorphic function $F : \mathbb{D} \longrightarrow \Gamma_E$. Moreover, since $\lambda_\varepsilon = \lambda_0(1 + \varepsilon)$, then for $\varepsilon < \delta$, $\lambda_\varepsilon \in I = [1, \delta]$ is a compact line-segment. Hence, $\lambda_\varepsilon \longrightarrow \lambda_0$. Therefore, $F_\varepsilon(\lambda_\varepsilon) \longrightarrow F(\lambda_0)$.

**Corollary 3.2.4** *Let $(a, b, p) \in G_E$ and $\lambda_0 \in \mathbb{D} \backslash \{0\}$. Then there exists an analytic function $h : \mathbb{D} \longrightarrow G_E$ such that $h(0) = (0, 0, 0)$ and $h(\lambda_0) = (a, b, p)$ if and only if*

$$\textit{either} \begin{cases} |b| \leq |a|, \textit{ and} \\ |\lambda_0| \geq \dfrac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2} \end{cases} \textit{or} \begin{cases} |a| \leq |b|, \textit{ and} \\ |\lambda_0| \geq \dfrac{|b - \bar{a}p| + |ab - p|}{1 - |a|^2} \end{cases}.$$

Observe that, when $|a| = |b|$,

$$\begin{aligned} |a - \bar{b}p|^2 - |b - \bar{a}p|^2 &= |a|^2 + |b|^2|p|^2 - 2\mathrm{Re}(ab\bar{p}) - \big(|b|^2 + |a|^2|p|^2 - 2\mathrm{Re}(ab\bar{p})\big) \\ &= (|a|^2 - |b|^2)(1 - |p|^2) \\ &= 0. \end{aligned}$$

Therefore, if $|a| = |b|$, then $|a - \bar{b}p| = |b - \bar{a}p|$.

## 3.3 The Carathéodory and Kobayashi Distances

In this section, we prove that the Carathéodory and the Kobayashi distances between two points in $\Gamma_E$ are equal, where one point is $(0, 0, 0)$.

We write $d$ for the pseudo-hyperbolic distance on $\mathbb{D}$ which is defined as follows:

$$d(z, w) = \left| \frac{z - w}{1 - \bar{w}z} \right|, \ z, w \in \mathbb{D}.$$

The *Carathéodory extremal problem* for a domain $\Omega$ and for a given pair of points $z_1, z_2 \in \Omega$ is to find the quantity

$$C_\Omega(z_1, z_2) \stackrel{def}{=} \sup \left\{ d\left(F(z_1), F(z_2)\right) : \ F \text{ maps } \Omega \text{ analytically into } \mathbb{D} \right\}.$$

Any $F$ for which the supremum on the right hand side is attained is called *Carathéodory extremal function* for $\Omega$ and the points $z_1, z_2$, and $C_\Omega$ is called the *Carathéodory distance* on $\Omega$.

The *Kobayashi extremal problem* for a pair of points $z_1, z_2 \in \Omega$ is to find the quantity

$$\delta_\Omega(z_1, z_2) = \inf\{d(\lambda_1, \lambda_2)\},$$

over all pairs $\lambda_1, \lambda_2 \in \mathbb{D}$ such that there exists an analytic function $h : \mathbb{D} \longrightarrow \Omega$ such that $h(\lambda_1) = z_1$ and $h(\lambda_2) = z_2$. Any such function $h$ for which the infimum is attained is called a *Kobayashi extremal function* for $\Omega$ and the points $z_1, z_2$. The *Kobayashi distance* $K_\Omega$ on $\Omega$ is defined to be the largest pseudo-distance on $\Omega$ dominated by $\delta_\Omega$, [14].

It is standard that

$$C_\Omega \leq K_\Omega \leq \delta_\Omega. \tag{3.5}$$

Lempert's theorem [14] asserts that $C_{\mathcal{D}} = K_{\mathcal{D}}$ for any convex domain $\mathcal{D}$. Although the symmetrised bidisc $\Gamma$ is not convex, Agler and Young [9] proved the equality of the Carathéodory and Kobayashi distances on $G$, the interior of $\Gamma$.

Note that, the Carathéodory and the Kobayashi distances are metrics on bounded domains in $\mathbb{C}^n$, which is the case for $G_E$ since it is bounded.

In this section, we show that the Carathéodory and the Kobayashi distances between the points $(0,0,0)$ and $(a,b,p)$ are equal in $G_E$.

**Theorem 3.3.1** *Let $a, b, p \in \mathbb{C}$ and $ab \neq p$. If $z_1 = (0,0,0)$ and $z_2 = (a,b,p)$ are in $G_E$ then*

$$C_{G_E}(z_1, z_2) = K_{\Gamma_E}(z_1, z_2) = \begin{cases} \dfrac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2}, & \textit{if } |b| \leq |a|, \textit{ or} \\ \\ \dfrac{|b - \bar{a}p| + |ab - p|}{1 - |a|^2} & \textit{if } |a| \leq |b|. \end{cases}$$

**Proof** By the Schwarz Lemma for $\Gamma_E$, i.e., Corollary 3.2.4, we have

$$\delta_{G_E}(z_1, z_2) = \begin{cases} \dfrac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2}, & \text{if } |b| \leq |a|, \text{ or} \\ \\ \dfrac{|b - \bar{a}p| + |ab - p|}{1 - |a|^2}, & \text{if } |a| \leq |b|. \end{cases}$$

For $\omega \in \mathbb{T}$, define $F_\omega : G_E \longrightarrow \mathbb{C}$ by

$$F_\omega(a', b', p') = \begin{cases} \Psi_\omega(a', b', p'), & \text{if } |b| \leq |a|, \text{ or} \\ \\ \Upsilon_\omega(a', b', p'), & \text{if } |a| \leq |b|. \end{cases}$$

By Theorem 2.1.4, $\Psi_\omega$ and $\Upsilon_\omega$ map $G_E$ to $\mathbb{D}$. That is, $F_\omega : G_E \longrightarrow \mathbb{D}$. Moreover, $F_\omega(z_1) = 0$ and hence by the definition of $G_E$,

$$C_{G_E} \geq \max_{\omega \in \mathbb{T}} |F_\omega(z_2)|.$$

Now

$$|F_\omega(z_2)| = \begin{cases} \left| \dfrac{a - \omega p}{1 - \omega b} \right|, & \text{if } |b| \leq |a|, \text{ or} \\ \\ \left| \dfrac{b - \omega p}{1 - \omega a} \right|, & \text{if } |a| \leq |b|. \end{cases}$$

As we have shown in the proof of Theorem 2.1.4, the two linear rational transformations

$$z \mapsto \begin{cases} \dfrac{a - zp}{1 - zb}, & \text{if } |b| \leq |a|, \text{ or} \\ \\ \dfrac{b - zp}{1 - za}, & \text{if } |a| \leq |b|, \end{cases}$$

map $\mathbb{T}$ to circles of centre $c_1, c_2$ and radius $R_1, R_2$, respectively, where

$$\begin{cases} c_1 = \dfrac{a - \bar{b}p}{1 - |b|^2} \text{ and } R_1 = \dfrac{|ab - p|}{1 - |b|^2}, & \text{if } |b| \leq |a|, \text{ or} \\ \\ c_2 = \dfrac{b - \bar{a}p}{1 - |a|^2} \text{ and } R_2 = \dfrac{|ab - p|}{1 - |a|^2}, & \text{if } |a| \leq |b|. \end{cases}$$

Thus,

$$C_{G_E}(z_1,z_2) \geq \begin{cases} |c_1| + R_1 = \dfrac{|a-\bar{b}p| + |ab-p|}{1-|b|^2} = \delta_{\Gamma_E}(z_1,z_2), \text{ if } |b| \leq |a|, \text{ or} \\ \\ |c_2| + R_2 = \dfrac{|b-\bar{a}p| + |ab-p|}{1-|a|^2} = \delta_{\Gamma_E}(z_1,z_2), \text{ if } |a| \leq |b|. \end{cases} \tag{3.6}$$

Hence, by (3.5) and (3.6), we find that

$$C_{G_E}(z_1,z_2) = \delta_{G_E}(z_1,z_2) = K_{G_E}(z_1,z_2) = \begin{cases} \dfrac{|a-\bar{b}p| + |ab-p|}{1-|b|^2}, \text{ if } |b| \leq |a|, \text{ or} \\ \\ \dfrac{|b-\bar{a}p| + |ab-p|}{1-|a|^2}, \text{ if } |a| \leq |b|. \end{cases}$$

□

## 3.4 Automorphisms of $G_E$

An *endomorphism* is a homomorphism from a domain to itself. An endomorphism that is also an isomorphism is called an *automorphism*.
In this section, we study the automorphisms of $G_E$. We conjecture that we have found all the automorphisms of $G_E$, but we have not obtained a proof of this thus far.

**Definition 3.4.1** *Let $a, b, c, d$ be complex numbers, where $ad - bc \neq 0$. A* Möbius automorphism *from $\mathbb{C}^\infty$ to $\mathbb{C}^\infty$, where $\mathbb{C}^\infty = \mathbb{C} \cup \{\infty\}$, is an analytic function $\mu$ such that*

$$\mu(z) = \frac{az+b}{cz+d}.$$

Note that the composition of Möbius automorphisms is a Möbius automorphism, and the inverse of a Möbius automorphism is a Möbius automorphism.

Let $x \in \Gamma_E$. We define $\tilde{\Upsilon}_x(z) : \mathbb{D} \longrightarrow \mathbb{D}$ as follows: For all $z \in \mathbb{D}$,

$$\tilde{\Upsilon}_x(z) = \frac{x_3 z - x_2}{x_1 z - 1} = \Upsilon_z(x).$$

In order to define the action of the Möbius automorphisms $\mu$ on the left and the right of $G_E$, we consider the following:

For each $x = (x_1, x_2, x_3) \in G_E$, define a matrix

$$X = \begin{bmatrix} x_3 & -x_2 \\ x_1 & -1 \end{bmatrix}.$$

We pick a matrix

$$M_\mu = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

such that $\det(M) \neq 0$, which induces a Möbius automorphism $\mu : \mathbb{D} \longrightarrow \mathbb{D}$,

$$\mu(z) = \frac{az + b}{cz + d}, \forall z \in \mathbb{D}.$$

In the case of the action of Möbius automorphisms on the left hand side of $G_E$, we observe that

$$\begin{aligned} M_\mu X &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_3 & -x_2 \\ x_1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} ax_3 + bx_1 & -(ax_2 + b) \\ cx_3 + dx_1 & -(cx_2 + d) \end{bmatrix}. \end{aligned}$$

Clearly, since $z \in \mathbb{D}$, $cz + d \neq 0$ as $\mu$ has no pole in $\mathbb{D}$, and since $x \in G_E$, $|x_i| < 1$, $1 \leq i \leq 3$. Hence, $cx_2 + d \neq 0$. Therefore, $M_\mu X$ induces the same

Möbius map as

$$\begin{bmatrix} x_3' & -x_2' \\ x_1' & -1 \end{bmatrix} := \begin{bmatrix} \dfrac{ax_3+bx_1}{cx_2+d} & -\dfrac{ax_2+b}{cx_2+d} \\ \\ \dfrac{cx_3+dx_1}{cx_2+d} & -1 \end{bmatrix}.$$

Observe that the choice of $M_\mu$ used to represent the Möbius map above does not change the resulting $(x_1', x_2', x_3')$.

We use the notation $A \equiv B$ if $A$ and $B$ induce the same Möbius map.

Define $\mu_x^l = (x_1', x_2', x_3')$. We have

$$\mu_x^l := \left( \frac{cx_3+dx_1}{cx_2+d}, \frac{ax_2+b}{cx_2+d}, \frac{ax_3+bx_1}{cx_2+d} \right) = x'.$$

We shall show that $x' \in G_E$. There are two cases; (i) when $x_1x_2 \neq x_3$ and (ii) when $x_1x_2 = x_3$.

Case (i): In the case that $x_1x_2 \neq x_3$, we have

$$x' = (x_1', x_2', x_3') \in G_E \Leftrightarrow \tilde{\Upsilon}_{x'}(\bar{\mathbb{D}}) \subset \mathbb{D}. \tag{3.7}$$

The equivalence (3.7) follows from Theorem 4.2.8 that we include later. A full proof of the theorem can be found in Section 4.2 where we study $G_E$ and present a characterisation for its elements.

We shall show that $\tilde{\Upsilon}_{x'}(\bar{\mathbb{D}}) \subset \mathbb{D}$. Clearly, since $\mu : \mathbb{D} \longrightarrow \mathbb{D}$ and by Theorem 2.1.4, $x \in \Gamma_E \Leftrightarrow \Upsilon_x(z) : \bar{\mathbb{D}} \longrightarrow \bar{\mathbb{D}}$, we have $\mu\left(\tilde{\Upsilon}_x(z)\right) : \bar{\mathbb{D}} \longrightarrow \mathbb{D}$.

Observe that

$$\mu\left(\tilde{\Upsilon}_x(z)\right) \quad = \quad \tilde{\Upsilon}_{x'}(z).$$

Thus, $\tilde{\Upsilon}_{x'}(z) : \bar{\mathbb{D}} \longrightarrow \mathbb{D}$.

Case (ii): When $x_1x_2 = x_3$, we have

$$\begin{aligned}
x' = (x_1', x_2', x_3') &= \left(\frac{cx_1x_2 + dx_1}{cx_2 + d}, \frac{ax_2 + b}{cx_2 + d}, \frac{ax_1x_2 + bx_1}{cx_2 + d}\right) \\
&= \left(x_1, \frac{ax_2 + b}{cx_2 + d}, x_1\frac{ax_2 + b}{cx_2 + d}\right) \\
&= (x_1, \mu(x_2), x_1\mu(x_2)).
\end{aligned}$$

For all $z \in \bar{\mathbb{D}}$,

$$\begin{aligned}
\tilde{\Upsilon}_{x'}(z) &= \frac{x_3'z - x_2'}{x_1'z - 1} \\
&= \frac{x_1\mu(x_2)z - \mu(x_2)}{x_1z - 1} \\
&= \mu(x_2)\frac{x_1z - 1}{x_1z - 1} \\
&= \mu(x_2).
\end{aligned}$$

Since $\mu : \mathbb{D} \longrightarrow \mathbb{D}$, $\tilde{\Upsilon}_{x'}(z) \in \mathbb{D}$, for all $z \in \bar{\mathbb{D}}$. That is, $\tilde{\Upsilon}_{x'}(\bar{\mathbb{D}}) \subset \mathbb{D}$. Thus, $x' \in G_E$.

Similarly, we define $XM_\mu$, the action of Möbius automorphisms on the right hand side of $G_E$. In this case, we find that

$$XM_\mu \equiv \begin{bmatrix} \dfrac{x_3a - x_2c}{-x_1b + d} & \dfrac{x_3b - x_2d}{-x_1b + d} \\ & \\ \dfrac{x_1a - c}{-x_1b + d} & -1 \end{bmatrix} := \begin{bmatrix} x_3' & -x_2' \\ x_1' & -1 \end{bmatrix}$$

and

$$\mu_x^r := \left(\frac{x_1a - c}{-x_1b + d}, \frac{-x_3b + x_2d}{-x_1b + d}, \frac{x_3a - x_2c}{-x_1b + d}\right) = x'' \in G_E.$$

**Lemma 3.4.2** *Let*

$$X = \begin{bmatrix} x_3 & -x_2 \\ x_1 & -1 \end{bmatrix},$$

*and let* $\mu_1$ *and* $\mu_2$ *be Möbius automorphisms of* $\mathbb{D}$ *defined by*

$$\begin{aligned}\mu_1(z) &= \frac{a_1 z + b_1}{c_1 z + d_1}, \ \forall z \in \mathbb{D},\\ \mu_2(z) &= \frac{a_2 z + b_2}{c_2 z + d_2}, \ \forall z \in \mathbb{D},\end{aligned}$$

*such that* $a_1 d_1 - b_1 c_1 \neq 0$ *and* $a_2 d_2 - b_2 c_2 \neq 0$. *Pick the matrices* $M_{\mu_1}$ *and* $M_{\mu_2}$ *which induce* $\mu_1$ *and* $\mu_2$ *respectively,*

$$M_{\mu_1} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \ \textit{and}\ M_{\mu_2} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix},$$

*such that* $\det(M_{\mu_1}) \neq 0$ *and* $\det(M_{\mu_2}) \neq 0$. *Then the following hold:*

*(1)* $M_{\mu_2}(M_{\mu_1} X) = M_{\mu_2 \circ \mu_1} X$.

*(2)* $(X M_{\mu_2}) M_{\mu_1} = X M_{\mu_2 \circ \mu_1}$.

*(3)* $(M_{\mu_1} X) M_{\mu_2} = M_{\mu_1}(X M_{\mu_2})$.

**Proof** There are two cases; (i) $x_3 \neq x_1 x_2$, and (ii) $x_3 = x_1 x_2$.

**Case (i): (1)** When $x_3 \neq x_1 x_2$. Recall that

$$M_{\mu_1} X \equiv \begin{bmatrix} \dfrac{a_1 x_3 + b_1 x_1}{c_1 x_2 + d_1} & -\dfrac{a_1 x_2 + b_1}{c_1 x_2 + d_1} \\ \\ \dfrac{c_1 x_3 + d_1 x_1}{c_1 x_2 + d_1} & -1 \end{bmatrix},$$

where $c_1 x_2 + d_1 \neq 0$.

We shall now calculate $M_{\mu_2}(M_{\mu_1} X)$.

$$M_{\mu_2}(M_{\mu_1} X) = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} \dfrac{a_1 x_3 + b_1 x_1}{c_1 x_2 + d_1} & -\dfrac{a_1 x_2 + b_1}{c_1 x_2 + d_1} \\ \\ \dfrac{c_1 x_3 + d_1 x_1}{c_1 x_2 + d_1} & -1 \end{bmatrix}$$

$$M_{\mu_2}(M_{\mu_1}X) \quad = \quad \begin{bmatrix} a_2\dfrac{a_1x_3+b_1x_1}{c_1x_2+d_1}+b_2\dfrac{c_1x_3+d_1x_1}{c_1x_2+d_1} & -a_2\dfrac{a_1x_2+b_1}{c_1x_2+d_1}-b_2 \\ \\ c_2\dfrac{a_1x_3+b_1x_1}{c_1x_2+d_1}+d_2\dfrac{c_1x_3+d_1x_1}{c_1x_2+d_1} & -c_2\dfrac{a_1x_2+b_1}{c_1x_2+d_1}-d_2 \end{bmatrix}.$$

Since $\mu_1$ is a Möbius endomorphism of $\mathbb{D}$ and $|x_2| < 1$ as $x \in G_E$, we have $\left|\dfrac{a_1x_2+b_1}{c_1x_2+d_1}\right| < 1$ and since $-\dfrac{d_2}{c_2} \notin \mathbb{D}$, $\left|\dfrac{d_2}{c_2}\right| > 1$. Hence, $-\dfrac{c_2}{d_2}\dfrac{a_1x_2+b_1}{c_1x_2+d_1} \neq 1$. Thus, $-c_2(a_1x_2+b_1) - d_2(c_1x_2+d_1) \neq 0$.

Therefore,

$$M_{\mu_2}(M_{\mu_1}X) \equiv \begin{bmatrix} \dfrac{a_2(a_1x_3+b_1x_1)+b_2(c_1x_3+d_1x_1)}{c_2(a_1x_2+b_1)+d_2(c_1x_2+d_1)} & -\dfrac{a_2(a_1x_2+b_1)+b_2(c_1x_2+d_1)}{c_2(a_1x_2+b_1)+d_2(c_1x_2+d_1)} \\ \\ \dfrac{c_2(a_1x_3+b_1x_1)+d_2(c_1x_3+d_1x_1)}{c_2(a_1x_2+b_1)+d_2(c_1x_2+d_1)} & -1 \end{bmatrix}.$$

We shall now calculate $(M_{\mu_2}M_{\mu_1})(X)$. We have

$$\mu_2 \circ \mu_1 = \mu_2(\mu_1(z)) = \frac{(a_1a_2+b_2c_1)z+(a_2b_1+b_2d_1)}{(c_2a_1+d_2c_1)z+(c_2b_1+d_2d_1)}.$$

Hence, for the Möbius automorphism $\mu_2 \circ \mu_1$, we define $M_{\mu_2\circ\mu_1}$ by

$$M_{\mu_2\circ\mu_1} = \begin{bmatrix} a_1a_2+b_2c_1 & a_2b_1+b_2d_1 \\ \\ c_2a_1+d_2c_1 & c_2b_1+d_2d_1 \end{bmatrix},$$

such that $\det(M_{\mu_2\circ\mu_1}) \neq 0$. Since $\mu_2 \circ \mu_1$ is an automorphism of $\mathbb{D}$,

$c_2b_1 + d_1d_2 \neq 0$. Therefore,

$$M_{\mu_2\circ\mu_1} \equiv \begin{bmatrix} -\dfrac{a_1a_2+b_2c_1}{c_2b_1+d_1d_2} & -\dfrac{a_2b_1+b_2d_1}{c_2b_1+d_1d_2} \\ \\ -\dfrac{c_2a_1+d_2c_1}{c_2b_1+d_1d_2} & -1 \end{bmatrix}.$$

Hence,

$$M_{\mu_2\circ\mu_1}X \quad = \quad \begin{bmatrix} -\dfrac{a_1a_2+b_2c_1}{c_2b_1+d_1d_2} & -\dfrac{a_2b_1+b_2d_1}{c_2b_1+d_1d_2} \\ \\ -\dfrac{c_2a_1+d_2c_1}{c_2b_1+d_1d_2} & -1 \end{bmatrix} \begin{bmatrix} x_3 & -x_2 \\ x_1 & -1 \end{bmatrix}$$

$$M_{\mu_2\circ\mu_1}X = \begin{bmatrix} -\dfrac{x_3(a_1a_2+b_2c_1)+x_1(a_2b_1+b_2d_1)}{c_2b_1+d_1d_2} & \dfrac{x_2(a_1a_2+b_2c_1)+a_2b_1+b_2d_1}{c_2b_1+d_1d_2} \\ \\ -\dfrac{x_3(c_2a_1+d_2c_1)+x_1(c_2b_1+d_1d_2)}{c_2b_1+d_1d_2} & \dfrac{x_2(c_2a_1+d_2c_1)+c_2b_1+d_1d_2}{c_2b_1+d_1d_2} \end{bmatrix}.$$

As before, $x_2(c_2a_1+d_2c_1)+c_2b_1+d_1d_2 \neq 0$. Hence,

$$M_{\mu_2\circ\mu_1}X \equiv \begin{bmatrix} \dfrac{x_3(a_1a_2+b_2c_1)+x_1(a_2b_1+b_2d_1)}{x_2(c_2a_1+d_2c_1)+c_2b_1+d_1d_2} & -\dfrac{x_2(a_1a_2+b_2c_1)+a_2b_1+b_2d_1}{x_2(c_2a_1+d_2c_1)+c_2b_1+d_1d_2} \\ \\ \dfrac{x_3(c_2a_1+d_2c_1)+x_1(c_2b_1+d_1d_2)}{x_2(c_2a_1+d_2c_1)+c_2b_1+d_1d_2} & -1 \end{bmatrix}.$$

Therefore, (1) holds when $x_1x_2 \neq x_3$.

**Case (ii): (1)** In the case that $x_3 = x_1x_2$, we have

$$\begin{aligned} M_{\mu_1}X &= \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} x_1x_2 & -x_2 \\ x_1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} a_1x_1x_2+b_1x_1 & -(a_1x_2+b_1) \\ c_1x_1x_2+d_1x_1 & -(c_1x_2+d_1) \end{bmatrix} \\ &\equiv \begin{bmatrix} x_1\dfrac{a_1x_2+b_1}{c_1x_2+d_1} & -\dfrac{a_1x_2+b_1}{c_1x_2+d_1} \\ x_1 & -1 \end{bmatrix} \\ &\equiv \begin{bmatrix} x_1\mu_1(x_2) & -\mu_1(x_2) \\ x_1 & -1 \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} M_{\mu_2}(M_{\mu_1}X) &= \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} x_1\mu_1(x_2) & -\mu_1(x_2) \\ x_1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} a_2x_1\mu_1(x_2)+b_2x_1 & -(a_2\mu_1(x_2)+b_2) \\ c_2x_1\mu_1(x_2)+d_2x_1 & -(c_2\mu_1(x_2)+d_2) \end{bmatrix}. \end{aligned}$$

As in case (i), since $\mu_{\mu_1}$ is an automorphism of $\mathbb{D}$, then $|\mu_1(x_2)| < 1$. We also have $\frac{d_2}{c_2} \notin \mathbb{D}$, which implies that $\left|\frac{d_2}{c_2}\right| \geq 1$.
If $c_2 = 0$ and $c_2\mu_1(x_2) + d_2 = 0$ then $d_2 = 0$, but $a_2d_2 - b_2c_2 \neq 0$, therefore, $c_2\mu_1(x_2) + d_2 \neq 0$.

Therefore,

$$\begin{aligned} M_{\mu_2}(M_{\mu_1}X) &\equiv \begin{bmatrix} \dfrac{a_2x_1\mu_1(x_2) + b_2x_1}{c_2\mu_1(x_2) + d_2} & -\dfrac{a_2\mu_1(x_2) + b_1}{c_2\mu_1(x_2) + d_2} \\ x_1 & -1 \end{bmatrix} \\ &\equiv \begin{bmatrix} x_1\mu_2\left(\mu_1(x_2)\right) & -\mu_2\left(\mu_1(x_2)\right) \\ x_1 & -1 \end{bmatrix}. \end{aligned}$$

Now we calculate the right hand side of (1). We have

$$\begin{aligned} M_{\mu_2\circ\mu_1}X &= \begin{bmatrix} -\dfrac{a_2a_1 + b_2c_1}{c_2b_1 + d_2d_1} & -\dfrac{a_2b_1 + b_2d_1}{c_2b_1 + d_2d_1} \\ & \\ -\dfrac{c_2a_1 + d_2c_1}{c_2b_1 + d_2d_1} & -1 \end{bmatrix} \begin{bmatrix} x_1x_2 & -x_2 \\ x_1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -\dfrac{(a_2a_1 + b_2c_1)x_1x_2 + x_1(a_2b_1 + b_2d_1)}{c_2b_1 + d_2d_1} & \dfrac{x_2(a_2a_1 + b_2c_1) + a_2b_1 + b_2d_1}{c_2b_1 + d_2d_1} \\ & \\ -\dfrac{(c_2a_1 + d_2c_1)x_1x_2 + x_1(c_2b_1 + d_2d_1)}{c_2b_1 + d_2d_1} & \dfrac{x_2(c_2a_1 + d_2c_1) + c_2b_1 + d_2d_1}{c_2b_1 + d_2d_1} \end{bmatrix}. \end{aligned}$$

Note that

$$\begin{aligned} c_2\mu_1(x_2) + d_2 \neq 0 \quad &\Rightarrow \quad c_2\frac{a_1x_2 + b_1}{c_1x_2 + d_1} + d_2 \neq 0 \\ &\Rightarrow \quad c_2(a_1x_2 + b_1) + d_2(c_1x_2 + d_1) \neq 0 \\ &\Rightarrow \quad x_2(c_2a_1 + d_2c_1) + c_2b_1 + d_2d_1 \neq 0. \end{aligned}$$

Therefore,

$$
\begin{aligned}
M_{\mu_2\circ\mu_1}X &\equiv \begin{bmatrix} \dfrac{x_1\left(a_2(a_1x_2+b_1)+b_2(c_1x_2+d_1)\right)}{c_2(a_1x_2+b_1)+d_2(c_1x_2+d_1)} & -\dfrac{a_2(a_1x_2+b_1)+b_2(c_1x_2+d_1)}{c_2(a_1x_2+b_1)+d_2(c_1x_2+d_1)} \\ \\ x_1 & -1 \end{bmatrix} \\
&\equiv \begin{bmatrix} x_1\dfrac{a_2\dfrac{a_1x_2+b_1}{c_1x_2+d_1}+b_2}{c_2\dfrac{a_1x_2+b_1}{c_1x_2+d_1}+d_2} & -\dfrac{a_2\dfrac{a_1x_2+b_1}{c_1x_2+d_1}+b_2}{c_2\dfrac{a_1x_2+b_1}{c_1x_2+d_1}+d_2} \\ \\ x_1 & -1 \end{bmatrix} \\
&\equiv \begin{bmatrix} x_1\dfrac{a_2\mu_1(x_2)+b_2}{c_2\mu_1(x_2)+d_2} & -\dfrac{a_2\mu_1(x_2)+b_2}{c_2\mu_1(x_2)+d_2} \\ x_1 & -1 \end{bmatrix} \\
&\equiv \begin{bmatrix} x_1\mu_2(\mu_1(x_2)) & -\mu_2(\mu_1(x_2)) \\ x_1 & -1 \end{bmatrix}.
\end{aligned}
$$

Therefore, (1) holds when $x_1x_2 = x_3$.

The proof of (2) and (3) follows by the associativity of matrix multiplication.

□

The next result follows from the previous lemma.

**Corollary 3.4.3** *Let $x \in G_E$ and let $\mu_1$, $\mu_2$ be Möbius automorphisms of $\mathbb{D}$. Then the following hold:*

*(1)* $\mu_2(\mu_1 x) = (\mu_2\mu_1)x.$

*(2)* $(x\mu_2)\mu_1 = x(\mu_2\mu_1).$

*(3)* $(\mu_1 x)\mu_2 = \mu_1(x\mu_2).$

**Remarks 3.4.4** *Let $x = (a, b, p) \in G_E$ and let $\mu_1$ and $\mu_2$ be Möbius automorphisms of $\mathbb{D}$. Then*

*(1) The following are holomorphic automorphisms of $G_E$:*

*(1) $x \mapsto (\mu_1 x)\mu_2 = \mu_1(x\mu_2)$.*

*(2) $(a, b, p) \mapsto (b, a, p)$.*

*(2) The the following are non-holomorphic automorphisms of $G_E$:*

*(1) $(a, b, p) \mapsto (a, \bar{p}, \bar{b})$.*

*(2) $(a, b, p) \mapsto (\bar{p}, b, \bar{a})$.*

**Theorem 3.4.5** *Let $x = (x_1, x_2, x_3) \in G_E$, $\tilde{\Upsilon}_x(z) = \dfrac{x_3 z - x_2}{x_1 z - 1} : \mathbb{D} \longrightarrow \mathbb{D}$. Then there exist Möbius automorphisms $\mu_1, \mu_2$ of $\mathbb{D}$ such that*

$$\mu_1 \tilde{\Upsilon}_x \mu_2 = rz,$$

*where $0 \leq r < 1$.*

**Proof** When $x_1 x_2 = x_3$, our result follows easily since in this case we have, $\tilde{\Upsilon}_x(z) = x_2$. Hence, we can choose $\mu_2$ to be any automorphism of $\mathbb{D}$ and $\mu_1$ to be an automorphism of $\mathbb{D}$ such that $\mu_1(x_2) = 0$.

When $x_1 x_2 \neq x_3$, there are two cases; case (i) when $0 < r < 1$, and case (ii) when r=0.

**Case (i):** When $0 < r < 1$, let $x \in G_E$, and let $|x_1| \neq 1$.

For $\mu_1, \mu_2 : \mathbb{D} \longrightarrow \mathbb{D}$, automorphisms of $\mathbb{D}$, defined by

$$\mu_1(z) = \frac{a_1 z + b_1}{\bar{b}_1 z + \bar{a}_1}, \ \forall z \in \mathbb{D},$$

$$\mu_2(z) = \frac{a_2 z + b_2}{\bar{b}_2 z + \bar{a}_2}, \ \forall z \in \mathbb{D}.$$

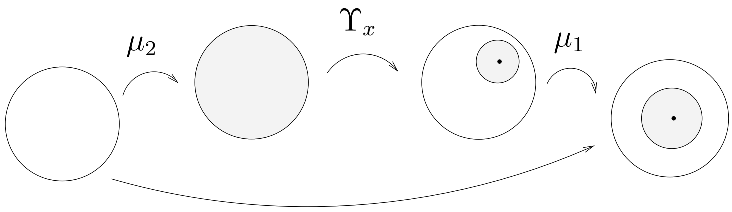



Figure 3.1: $\mu_1 \circ \Upsilon_x \circ \mu_2(z)$

Define $M_{\mu_1}$ and $M_{\mu_2}$ by

$$M_{\mu_1} = \begin{bmatrix} a_1 & b_1 \\ \bar{b}_1 & \bar{a}_1 \end{bmatrix} \text{ and } M_{\mu_2} = \begin{bmatrix} a_2 & b_2 \\ \bar{b}_2 & \bar{a}_2 \end{bmatrix},$$

such that $\det(M_{\mu_1})$ and $\det(M_{\mu_2})$. Recall that $x \in \Gamma_E \Leftrightarrow \tilde{\Upsilon}_x(z) : \mathbb{D} \longrightarrow \mathbb{D}$.

Hence, we define

$$X = \begin{bmatrix} x_3 & -x_2 \\ x_1 & -1 \end{bmatrix}$$

that induces $\tilde{\Upsilon}_x(z) = \dfrac{x_3 z - x_2}{x_1 z - 1}$.

We have

$$\begin{aligned} M_{\mu_2} X &= \begin{bmatrix} a_2 & b_2 \\ \bar{b}_2 & \bar{a}_2 \end{bmatrix} \begin{bmatrix} x_3 & -x_2 \\ x_1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} a_2 x_3 + b_2 x_1 & -(a_2 x_2 + b_2) \\ \bar{b}_2 x_3 + \bar{a}_2 x_1 & -(\bar{b}_2 x_2 + \bar{a}_2) \end{bmatrix} \\ &\equiv \begin{bmatrix} \dfrac{a_2 x_3 + b_2 x_1}{\bar{b}_2 x_2 + \bar{a}_2} & -\dfrac{a_2 x_2 + b_2}{\bar{b}_2 x_2 + \bar{a}_2} \\ \\ \dfrac{\bar{b}_2 x_3 + \bar{a}_2 x_1}{\bar{b}_2 x_2 + \bar{a}_2} & -1 \end{bmatrix}. \end{aligned}$$

Let

$$
\begin{aligned}
x'_1 &= \frac{\bar{b}_2 x_3 + \bar{a}_2 x_1}{\bar{b}_2 x_2 + \bar{a}_2}, \\
x'_2 &= \frac{a_2 x_2 + b_2}{\bar{b}_2 x_2 + \bar{a}_2}, \\
x'_3 &= \frac{a_2 x_3 + b_2 x_1}{\bar{b}_2 x_2 + \bar{a}_2}.
\end{aligned}
$$

We are seeking $x'_2 = \bar{x'}_1 x'_3$. That is,

$$
\frac{a_2 x_2 + b_2}{\bar{b}_2 x_2 + \bar{a}_2} = \left( \frac{b_2 \bar{x}_3 + a_2 \bar{x}_1}{b_2 \bar{x}_2 + a_2} \right) \left( \frac{a_2 x_3 + b_2 x_1}{\bar{b}_2 x_2 + \bar{a}_2} \right).
$$

That is,

$$
(a_2 x_2 + b_2)(b_2 \bar{x}_2 + a_2) = (b_2 \bar{x}_3 + a_2 \bar{x}_1)(a_2 x_3 + b_2 x_1). \tag{3.8}
$$

We shall now find values of $a_2$ and $b_2$, such that (3.8) holds and $|b_2| < |a_2|$. Note that, since $\mu_2$ maps $\mathbb{T} \longrightarrow \mathbb{T}$ and the interior to the interior (of the unit disc), then $|a_2|^2 - |b_2|^2 > 0$, therefore, $|a_2| > |b_2|$.

Thus, we have

$$
\begin{aligned}
& (a_2 x_2 + b_2)(b_2 \bar{x}_2 + a_2) = (b_2 \bar{x}_3 + a_2 \bar{x}_1)(a_2 x_3 + b_2 x_1) \\
\Rightarrow \quad & a_2 b_2 - a_2 b_2 |x_1|^2 + a_2 b_2 |x_2|^2 - a_2 b_2 |x_3|^2 + a_2^2 (x_2 - \bar{x}_1 x_3) + b_2^2 (\bar{x}_2 - x_1 \bar{x}_3) = 0 \\
\Rightarrow \quad & a_2 b_2 (1 - |x_1|^2 + |x_2|^2 - |x_3|^2) + a_2^2 (x_2 - \bar{x}_1 x_3) + b_2^2 (\bar{x}_2 - x_1 \bar{x}_3) = 0.
\end{aligned}
$$

Hence, we can take

$$
\begin{aligned}
a_2 &= 1 - |x_1|^2 + |x_2|^2 - |x_3|^2 + \sqrt{(1 - |x_1|^2 + |x_2|^2 - |x_3|^2)^2 - 4|x_2 - \bar{x}_1 x_3|^2}, \\
b_2 &= 2(\bar{x}_1 x_3 - x_2).
\end{aligned}
$$

Since $x = (x_1, x_2, x_3) \in G_E$, then

$(1 - |x_1|^2 + |x_2|^2 - |x_3|^2)^2 - 4|x_2 - \bar{x}_1 x_3|^2 \geq 0$, and clearly, $|b_2| < |a_2|$.

Now we shall find an $r$ such that $\mu_1 \Upsilon_x \mu_2 = rz$.

$$\begin{aligned}
M_{\mu_2} X M_{\mu_1} &= \begin{bmatrix} x'_3 & -x'_2 \\ x'_1 & -1 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ \bar{b}_1 & \bar{a}_1 \end{bmatrix} \\
&= \begin{bmatrix} a_1 x'_3 - \bar{b}_1 x'_2 & b_1 x'_3 - \bar{a}_1 x'_2 \\ a_1 x'_1 - \bar{b}_1 & b_1 x'_1 - \bar{a}_1 \end{bmatrix} \\
&\equiv \begin{bmatrix} \dfrac{a_1 x'_3 - \bar{b}_1 x'_2}{-b_1 x'_1 + \bar{a}_1} & \dfrac{b_1 x'_3 - \bar{a}_1 x'_2}{-b_1 x'_1 + \bar{a}_1} \\[2ex] \dfrac{a_1 x'_1 - \bar{b}_1}{-b_1 x'_1 + \bar{a}_1} & -1 \end{bmatrix}.
\end{aligned}$$

We seek an $r$ such that

$$\begin{bmatrix} \dfrac{a_1 x'_3 - \bar{b}_1 x'_2}{-b_1 x'_1 + \bar{a}_1} & \dfrac{b_1 x'_3 - \bar{a}_1 x'_2}{-b_1 x'_1 + \bar{a}_1} \\[2ex] \dfrac{a_1 x'_1 - \bar{b}_1}{-b_1 x'_1 + \bar{a}_1} & -1 \end{bmatrix} = \begin{bmatrix} -r & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.9}$$

Hence,

$$\begin{aligned}
\frac{a_1 x'_1 - \bar{b}_1}{-b_1 x'_1 + \bar{a}_1} &= 0 \Rightarrow b_1 = \bar{a}_1 \bar{x}'_1, \\
\frac{b_1 x'_3 - \bar{a}_1 x'_2}{-b_1 x'_1 + \bar{a}_1} &= 0 \Rightarrow x'_2 = \bar{x}'_1 x'_3, \\
\frac{-a_1 x'_3 + \bar{b}_1 x'_2}{-b_1 x'_1 + \bar{a}_1} &= r.
\end{aligned}$$

Therefore, we can take $a_1 = 1$, and $b_1 = \bar{x}'_1$, so that $|b_1| < |a_1|$, hence,

$$r = \frac{x'_1 x'_2 - x'_3}{1 - |x'_1|^2}, \ |x'_1| \neq 1.$$

Now we substitute the values of $x_1', x_2', x_3'$ in $r$.

$$\begin{aligned}
x_1'x_2' - x_3' &= \left(\frac{\bar{b}_2x_3 + \bar{a}_2x_1}{\bar{b}_2x_2 + \bar{a}_2}\right)\left(\frac{a_2x_2 + b_2}{\bar{b}_2x_2 + \bar{a}_2}\right) - \frac{a_2x_3 + b_2x_1}{\bar{b}_2x_2 + \bar{a}_2} \\
&= \frac{(\bar{b}_2x_3 + \bar{a}_2x_1)(a_2x_2 + b_2) - (a_2x_3 + b_2x_1)(\bar{b}_2x_2 + \bar{a}_2)}{(\bar{b}_2x_2 + \bar{a}_2)^2} \\
&= \frac{(x_1x_2 - x_3)(|a_2|^2 - |b_2|^2)}{(\bar{b}_2x_2 + \bar{a}_2)^2},
\end{aligned}$$

and

$$\begin{aligned}
1 - |x_1'|^2 &= 1 - \left(\frac{\bar{b}_2x_3 + \bar{a}_2x_1}{\bar{b}_2x_2 + \bar{a}_2}\right)\left(\frac{b_2\bar{x}_3 + a_2\bar{x}_1}{b_2\bar{x}_2 + a_2}\right) \\
&= \frac{|a_2|^2(1 - |x_1|^2) + |b_2|^2(|x_2|^2 - |x_3|^2) + a_2\bar{b}_2(x_2 - \bar{x}_1x_3) + \bar{a}_2b_2(\bar{x}_2 - x_1\bar{x}_3)}{(\bar{b}_2x_2 + \bar{a}_2)(b_2\bar{x}_2 + a_2)}
\end{aligned}$$

Therefore,

$$r = \frac{(x_1x_2 - x_3)(|a_2|^2 - |b_2|^2)(b_2\bar{x}_2 + a_2)}{(\bar{b}_2x_2 + \bar{a}_2)(|a_2|^2(1 - |x_1|^2) + |b_2|^2(|x_2|^2 - |x_3|^2) + a_2\bar{b}_2(x_2 - \bar{x}_1x_3) + \bar{a}_2b_2(\bar{x}_2 - x_1\bar{x}_3))}.$$

**Case (ii):** when r=0, we have $\tilde{\Upsilon}_x(z) = \dfrac{x_3z - x_2}{x_1z - 1} = 0, \ \forall z \in \mathbb{D}$. Therefore,

$$\begin{aligned}
x_3z - x_2 = 0, \text{ for all } z \in \mathbb{D} \ &\Rightarrow \ x_3z = x_2, \text{ for all } z \in \mathbb{D} \\
&\Rightarrow \ x_2 = 0 \text{ and } x_3 = 0.
\end{aligned}$$

Therefore, we can find a Möbius automorphism $\mu$ of the $\mathbb{D}$, $\mu(z) = \dfrac{az + b}{\bar{b}z + \bar{a}}$, $\forall z \in \mathbb{D}$, $|b| < |a|$, then define the matrix $M_\mu$ by

$$M_\mu = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}, \ |b| < |a|,$$

so that

$$\begin{bmatrix} 0 & 0 \\ x_1 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

In this case, we have

$$\begin{bmatrix} 0 & 0 \\ x_1 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ ax_1 - \bar{b} & bx_1 - \bar{a} \end{bmatrix}$$
$$\equiv \begin{bmatrix} 0 & 0 \\ \dfrac{ax_1 - \bar{b}}{\bar{a} - bx_1} & -1 \end{bmatrix}.$$

Hence, $\bar{b} = ax_1$, which implies that $|b| < |a|$. Therefore, we can take $a = 1$ and $b = \bar{x}_1$ so that

$$\begin{bmatrix} 0 & 0 \\ x_1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \bar{x}_1 \\ x_1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & |x_1|^2 + 1 \end{bmatrix}$$
$$\equiv \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus, in the case that $r = 0$, we have $x = (x_1, 0, 0) \in G_E$.

Finally, we shall show that $r$ is unique. For $\tilde{\Upsilon}_r$ and $\tilde{\Upsilon}_s$, define $X_r$ and $X_s$, respectively by

$$X_r = \begin{bmatrix} -r & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad X_s = \begin{bmatrix} -s & 0 \\ 0 & -1 \end{bmatrix},$$

where $r, s \geq 0$. Hence, for $|b_j| < |a_j|$, $j = 1, 2$,

$$M_{\mu_2} X_r = X_s M_{\mu_1} \quad \Rightarrow \quad \begin{bmatrix} a_2 & b_2 \\ \bar{b}_2 & \bar{a}_2 \end{bmatrix} \begin{bmatrix} -r & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -s & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ \bar{b}_1 & \bar{a}_1 \end{bmatrix}$$
$$\Rightarrow \quad \begin{bmatrix} -a_2 r & -b_2 \\ -\bar{b}_2 r & -\bar{a}_2 \end{bmatrix} = \begin{bmatrix} -a_1 s & -b_1 s \\ -\bar{b}_1 & -\bar{a}_1 \end{bmatrix}$$
$$\Rightarrow \quad \begin{bmatrix} \dfrac{-a_2 r}{\bar{a}_2} & \dfrac{-b_2}{\bar{a}_2} \\ \dfrac{-\bar{b}_2 r}{\bar{a}_2} & -1 \end{bmatrix} = \begin{bmatrix} \dfrac{-a_1 s}{\bar{a}_1} & \dfrac{-b_1 s}{\bar{a}_1} \\ \dfrac{-\bar{b}_1}{\bar{a}_1} & -1 \end{bmatrix}, \tag{3.10}$$

where $a_1, a_2 \neq 0$ because $\mu_1, \mu_2$ are Möbius automorphisms.

There are three cases; case (i) when $r \neq 0$ and $b_1, b_2 \neq 0$, case (ii) when $r \neq 0$ and either $b_1 = 0$ or $b_2 = 0$, and case (iii) is when $r = 0$.

Case (i): If $r \neq 0$, then it is easy to see from (3.10) that $|r| = |s|$, which implies that $r = s$ since $r, s \geq 0$. Therefore, $r$ is unique.

Case (ii): This is the case that $r \neq 0$ and either $b_1$, or $b_2$ is zero, which is the scalar case. This case also follows easily from (3.10). Hence, $r = s$ and thus $r$ is unique.

Case (iii): If $r = 0$, then from (3.10), we find that $a_1 s = 0$, which implies that $s = 0$ (since $a_1 \neq 0$), therefore, $r = s = 0$, and hence, $r$ is unique.

□

The next result is a generalisation of Theorem 3.3.1 when $ab = p$. In this result, we show that the Carathéodory and the Kobayashi distances between two points $x_1 = (a_1, b_1, p_1)$ and $x_2 = (a_2, b_2, p_2)$ in $G_E$ are equal in the case that $a_1 b_1 = p_1$.

**Lemma 3.4.6** *Let $a_i, b_i, p_i \in \mathbb{C}$, $i = 1, 2$. If $x_1 = (a_1, b_1, p_1)$ and $x_2 = (a_2, b_2, p_2)$ are in $G_E$ such that $a_1 b_1 = p_1$, then*

$$C_{G_E}(x_1, x_2) = K_{G_E}(x_1, x_2).$$

**Proof** Observe that, since $x_1 = (a_1, b_1, a_1 b_1) \in G_E$,

$$\Upsilon_z(x_1) = \frac{a_1 b_1 z - b_1}{a_1 z - 1} = \frac{b_1(a_1 z - 1)}{a_1 z - 1} = b_1.$$

As before, since

$$\Upsilon_z(x_1) = \frac{a_1 b_1 z - b_1}{a_1 z - 1} : \mathbb{D} \longrightarrow \mathbb{D},$$

then for each $x_1 = (a_1, b_1, a_1 b_1) \in G_E$, we define the matrix

$$X_1 = \begin{bmatrix} a_1 b_1 & -b_1 \\ a_1 & -1 \end{bmatrix}. \tag{3.11}$$

Consider the Möbius automorphisms of the $\mathbb{D}$, defined as follows:

$$\begin{aligned} \mu_1 &= z \mapsto \frac{z + \bar{a}_1}{a_1 z + 1}, \ \forall z \in \mathbb{D}, \text{ and} \\ \mu_2 &= z \mapsto \frac{z - b_1}{\bar{b}_1 z - 1}, \ \forall z \in \mathbb{D}. \end{aligned}$$

For $\mu_1$ and $\mu_2$ as above, define $M_{\mu_1}$ and $M_{\mu_2}$ by

$$M_{\mu_1} = \begin{bmatrix} 1 & \bar{a}_1 \\ a_1 & 1 \end{bmatrix}, \ |a_1| < 1 \text{ and } M_{\mu_2} = \begin{bmatrix} 1 & -b_1 \\ \bar{b}_1 & -1 \end{bmatrix}, \ |b_1| < 1.$$

We shall multiply $X_1$ by $M_{\mu_1}$ and $M_{\mu_2}$ to move the point $x_1$ to $(0, 0, 0)$. Let $|a_1| < 1$, then

$$\begin{aligned} X_1 M_{\mu_1} &= \begin{bmatrix} a_1 b_1 & -b_1 \\ a_1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \bar{a}_1 \\ a_1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & b_1(|a_1|^2 - 1) \\ 0 & |a_1|^1 - 1 \end{bmatrix} \\ &\equiv \begin{bmatrix} 0 & -b_1 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

Let $|b_1| < 1$, then

$$\begin{aligned} M_{\mu_2} X_1 M_{\mu_1} &= \begin{bmatrix} 1 & -b_1 \\ \bar{b}_1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -b_1 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 1 - |b_1|^2 \end{bmatrix}, \end{aligned}$$

which corresponds to the point $(0,0,0)$. Since $C_{G_E}$ and $K_{G_E}$ are invariant distances, the holomorphic automorphisms of $G_E$ are isometric with respect to $C_{G_E}$ and $K_{G_E}$. Therefore, the Möbius automorphisms on the left and the right hand sides are isometric with respect to both distances, $C_{G_E}$ and $K_{G_E}$. Therefore, we can take $x_1 = (0,0,0)$ and $x_2 = (a_2, b_2, p_2)$ in $G_E$. Thus, by Theorem 3.3.1,

$$C_{G_E}(x_1, x_2) = K_{G_E}(x_1, x_2).$$

$\square$

**Theorem 3.4.7** *Let $x = (x_1, x_2, x_3) \in \mathbb{C}^3$. Then the following hold:*

*(1) If $|x_1| < |x_3|$ then $(x \in \Gamma_E \Leftrightarrow |x_1|^2 + |x_1 x_2 - x_3| \geq |x_3|^2 + |x_1 - \bar{x}_2 x_3|)$.*

*(2) If $|x_2| < |x_3|$ then $(x \in \Gamma_E \Leftrightarrow |x_2|^2 + |x_1 x_2 - x_3| \geq |x_3|^2 + |x_2 - \bar{x}_1 x_3|)$.*

*(3) If $|x_1| < |x_2|$ then $(x \in \Gamma_E \Leftrightarrow |x_1|^2 + |x_2 - \bar{x}_1 x_3| \geq |x_2|^2 + |x_1 - \bar{x}_2 x_3|)$.*

*(4) If $|x_2| < |x_1|$ then $(x \in \Gamma_E \Leftrightarrow |x_2|^2 + |x_1 - \bar{x}_2 x_3| \geq |x_1|^2 + |x_2 - \bar{x}_1 x_3|)$.*

**Proof (1)** Let $\left|\frac{x_1}{x_3}\right| < 1$. That is, $|x_1| < |x_3|$. Define a Möbius automorphism $\mu$ of $\mathbb{D}$ by

$$\mu = \frac{x_3(\bar{x}_3 z - \bar{x}_1)}{\bar{x}_3(x_1 z - 1)}.$$

For this $\mu$ define $M_\mu$ by

$$M_\mu = \begin{bmatrix} 1 & -\frac{\bar{x}_1}{\bar{x}_3} \\ \frac{x_1}{x_3} & -1 \end{bmatrix}$$

so that $M_\mu$ induces $\mu$. As before, for each $x \in \Gamma_E$, we define

$$X = \begin{bmatrix} x_3 & -x_2 \\ x_1 & -1 \end{bmatrix}.$$

Then

$$
\begin{aligned}
M_\mu X &= \begin{bmatrix} 1 & -\dfrac{\bar{x}_1}{\bar{x}_3} \\ \dfrac{x_1}{x_3} & -1 \end{bmatrix} \begin{bmatrix} x_3 & -x_2 \\ x_1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} x_3 - \dfrac{|x_1|^2}{\bar{x}_3} & -x_2 + \dfrac{\bar{x}_1}{\bar{x}_3} \\ 0 & \dfrac{-x_1 x_2}{x_3} + 1 \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{|x_3|^2 - |x_1|^2}{\bar{x}_3} & \dfrac{-x_2\bar{x}_3 + \bar{x}_1}{\bar{x}_3} \\ 0 & \dfrac{-x_1 x_2 + x_3}{x_3} \end{bmatrix} \\
&\equiv \begin{bmatrix} \dfrac{x_3(|x_3|^2 - |x_1|^2)}{\bar{x}_3(x_1 x_2 - x_3)} & \dfrac{x_3(x_2\bar{x}_3 - \bar{x}_1)}{\bar{x}_3(x_1 x_2 - x_3)} \\ 0 & -1 \end{bmatrix} := \begin{bmatrix} x_3' & -x_2' \\ x_1' & -1 \end{bmatrix}.
\end{aligned}
$$

Since $x' = (x_1', x_2', x_3') \in \Gamma_E$, then by (1)$\Leftrightarrow$(4) of Theorem 2.1.4, we have

$$
\begin{aligned}
x \in \Gamma_E &\Leftrightarrow \left|\frac{x_2\bar{x}_3 - \bar{x}_1}{x_1 x_2 - x_3}\right| + \left|\frac{|x_1|^2 - |x_3|^2}{x_1 x_2 - x_3}\right| \le 1 \\
&\Leftrightarrow \begin{cases} |x_1 - \bar{x}_2 x_3| + \big||x_1|^2 - |x_3|^2\big| \le |x_1 x_2 - x_3|, \text{ and} \\ |x_1 - \bar{x}_2 x_3| \le |x_1 x_2 - x_3| \end{cases}
\end{aligned}
$$

Therefore, if $|x_1| < |x_3|$, then

$$x \in \Gamma_E \Leftrightarrow |x_1|^2 + |x_1 x_2 - x_3| \ge |x_3|^2 + |x_1 - \bar{x}_2 x_3|.$$

That is, (1) holds.

**(2)** This holds immediately since $(x_1, x_2, x_3) \in \Gamma_E \Leftrightarrow (x_2, x_1, x_3) \in \Gamma_E$.

**(2)** This holds immediately since $(x_1, x_2, x_3) \in \Gamma_E \Leftrightarrow (x_1, \bar{x}_3, \bar{x}_2) \in \Gamma_E$.

**(4)** This holds immediately since $(x_1, x_2, x_3) \in \Gamma_E \Leftrightarrow (x_2, x_1, x_3) \in \Gamma_E$.

$\square$

# Chapter 4

# The Topological and the Distinguished Boundaries of $\Gamma_E$ and $\Gamma_E$-Inner Functions

In this chapter, we define $\Gamma_E^{(r)}$ and give a characterisation for its elements. We also define the topological boundary and the distinguished boundary of $\Gamma_E$. In section 4.5 we define $\Gamma_E$-inner functions and present some results concerning this type of functions. We also give a general formula for rational $\Gamma_E$-inner functions.
We shall use the following notations; $\partial\Gamma_E$ denotes the *topological boundary* of $\Gamma_E$, $G_E$ denotes the interior of $\Gamma_E$ and $b\Gamma_E$ denotes the *distinguished boundary* of $\Gamma_E$.

**Definition 4.0.1** *Let $K$ be a compact subset of $\mathbb{C}^n$. The distinguished boundary of $K$ is defined to be the Šilov boundary of the algebra $A(K)$ of functions continuous on $K$ and analytic on the interior of $K$, that is, the* *smallest closed subset of $K$ on which every function in $A(K)$ attains its maximum modulus.*

## 4.1 Characterisation of $\Gamma_E^{(r)}$

Let $\mathbb{B}(0;1/r)$ denote the open ball of centre 0 and radius $1/r$ in $\mathbb{C}^2$, and $\mathbb{D}_{1/r}$ denote the disc of radius $1/r$ in $\mathbb{C}$, where $0<r<1$.
We define $\Gamma_E^{(r)}$ as follows:

$$\Gamma_E^{(r)} = \{(a_{11}, a_{22}, \det(A)) : \ A \in M_2(\mathbb{C}), \mu_E(A) \leq r\},$$

where $0<r<1$.

**Lemma 4.1.1** $(x_1, x_2, x_3) \in \Gamma_E^{(r)} \Leftrightarrow \left(\frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r^2}\right) \in \Gamma_E$, *where* $0<r<1$.

**Proof** From the definitions of $\Gamma_E$ and $\Gamma_E^{(r)}$ and by Remark 1.2.6, which states:

$$\mu_E(\lambda A) = |\lambda| \mu_E(A),$$

where $\lambda \in \mathbb{C}$ and $A \in M_2(\mathbb{C})$, we find that

$$\begin{aligned}
(x_1, x_2, x_3) \in \Gamma_E^{(r)} \quad &\Leftrightarrow \quad \mu_E\left(\begin{bmatrix} x_1 & b \\ c & x_2 \end{bmatrix}\right) \leq r, \text{ for some } b, c \in \mathbb{C} \text{ such that } bc = x_1x_2 - x_3, \\
&\Leftrightarrow \quad \frac{1}{r}\mu_E\left(\begin{bmatrix} x_1 & b \\ c & x_2 \end{bmatrix}\right) \leq 1, \text{ for some } b, c \in \mathbb{C} \text{ such that } bc = x_1x_2 - x_3 \\
&\Leftrightarrow \quad \mu_E\left(\begin{bmatrix} \frac{x_1}{r} & \frac{b}{r} \\ \frac{c}{r} & \frac{x_2}{r} \end{bmatrix}\right) \leq 1, \text{ for some } b, c \in \mathbb{C} \text{ such that } \frac{b}{r}\frac{c}{r} = \frac{x_1}{r}\frac{x_2}{r} - \frac{x_3}{r^2} \\
&\Leftrightarrow \quad \left(\frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r^2}\right) \in \Gamma_E,
\end{aligned}$$

$\square$

**Corollary 4.1.2** *Let $z \in \mathbb{C}$ and $x \in \mathbb{C}^3$. We define $\Psi_z^{(r)}$ and $\Upsilon_z^{(r)}$ as follows:*

$$\Psi_z^{(r)}(x) = \begin{cases} \dfrac{1}{r}\dfrac{rx_1 - zx_3}{r - zx_2}, & \text{if } zx_2 \neq r, \text{ and} \\ \dfrac{x_1}{r}, & \text{if } zx_2 = r \text{ and } x_1x_2 = x_3, \end{cases}$$

$$\Upsilon_z^{(r)}(x) = \begin{cases} \dfrac{1}{r}\dfrac{rx_2 - zx_3}{r - zx_1}, & \text{if } zx_1 \neq r, \text{ and} \\ \dfrac{x_2}{r}, & \text{if } zx_1 = r \text{ and } x_1x_2 = x_3. \end{cases}$$

Note that $\Psi_z^{(r)}$ is undefined when $x_1x_2 \neq x_3$ and $zx_2 = r$. Also, $\Upsilon_z^{(r)}$ is undefined when $x_1x_2 \neq x_3$ and $zx_1 = r$.

The next result follows from Theorem 2.1.4 and Lemma 4.1.1.

**Theorem 4.1.3** *Let $x \in \mathbb{C}^3$ and $0 < r < 1$. Then the following are equivalent:*

*(1)* $x \in \Gamma_E^{(r)}$.

*(2)* $1 - x_1z - x_2w + x_3zw \neq 0$, *for all* $(z, w) \in \mathbb{B}(0; 1/r)$.

*(3)* $$\begin{cases} \left|\dfrac{x_2}{r}\right|^2 + \left|\dfrac{x_1}{r} - \dfrac{\bar{x}_2}{r}\dfrac{x_3}{r^2}\right| + \left|\dfrac{x_1}{r}\dfrac{x_2}{r} - \dfrac{x_3}{r^2}\right| \leq 1, \text{ and} \\ \left|\dfrac{x_1}{r}\right| \leq 1. \end{cases}$$

*(4)* $$\begin{cases} \left|\dfrac{x_1}{r}\right|^2 + \left|\dfrac{x_2}{r} - \dfrac{\bar{x}_1}{r}\dfrac{x_3}{r^2}\right| + \left|\dfrac{x_1}{r}\dfrac{x_2}{r} - \dfrac{x_3}{r^2}\right| \leq 1, \text{ and} \\ \left|\dfrac{x_2}{r}\right| \leq 1. \end{cases}$$

*(5)* $$\begin{cases} \Psi_z^{(r)}(x) \text{ is in the Schur class,} \\ \text{if } x_1x_2 = x_3, \left|\dfrac{x_2}{r}\right| \leq 1. \end{cases}$$

*(6)* $$\begin{cases} \Upsilon_z^{(r)}(x) \text{ is in the Schur class,} \\ \text{if } x_1x_2 = x_3, \left|\dfrac{x_1}{r}\right| \leq 1. \end{cases}$$

*(7) There exist* $b, c \in \mathbb{C}$ *such that* $bc = x_1x_2 - x_3$ *and* $\left\| \begin{pmatrix} x_1 & b \\ c & x_2 \end{pmatrix} \right\| \le r$.

*(8) There exist* $b, c \in \mathbb{C}$ *such that* $|b| = |c| = |x_1x_2 - x_3|^{1/2}$, $bc = x_1x_2 - x_3$ *and* $\left\| \begin{pmatrix} x_1 & b \\ c & x_2 \end{pmatrix} \right\| \le r$.

*(9)* $\begin{cases} 1 - \left|\dfrac{x_1}{r}\right|^2 - \left|\dfrac{x_2}{r}\right|^2 + \left|\dfrac{x_3}{r^3}\right|^2 - 2\left|\dfrac{x_1}{r}\dfrac{x_2}{r} - \dfrac{x_3}{r^2}\right| \ge 0, \ \textit{and} \\ \left|\dfrac{x_1}{r}\right| \le 1, \ \left|\dfrac{x_2}{r}\right| \le 1, \ \left|\dfrac{x_3}{r}\right| \le 1. \end{cases}$

*(10)* $\begin{cases} 1 - \left|\dfrac{x_1}{r}\right|^2 + \left|\dfrac{x_2}{r}\right|^2 - \left|\dfrac{x_3}{r^2}\right|^2 - 2\left|\dfrac{x_1}{r}\dfrac{\bar{x}_3}{r^2} - \dfrac{\bar{x}_2}{r}\right| \ge 0, \ \textit{and} \\ \left|\dfrac{x_1}{r}\right| \le 1, \ \left|\dfrac{x_2}{r}\right| \le 1. \end{cases}$

*(11)* $\begin{cases} 1 + \left|\dfrac{x_1}{r}\right|^2 - \left|\dfrac{x_2}{r}\right|^2 - \left|\dfrac{x_3}{r^2}\right|^2 - 2\left|\dfrac{x_2}{r}\dfrac{\bar{x}_3}{r^2} - \dfrac{\bar{x}_1}{r}\right| \ge 0, \ \textit{and} \\ \left|\dfrac{x_1}{r}\right| \le 1, \ \left|\dfrac{x_2}{r}\right| \le 1. \end{cases}$

The next result follows from Theorem 4.1.3.

**Corollary 4.1.4** *We have*

$$G_E = \bigcup_{0<r<1} \Gamma_E^{(r)}.$$

## 4.2 The Topological Boundary of $\Gamma_E$

In this section, we study the topological boundary and the interior of $\Gamma_E$. Our results include a characterisation of points in the topological boundary and in the interior of $\Gamma_E$.

Recall that

$$x \in \Gamma_E \Leftrightarrow \begin{cases} |x_2|^2 + |x_1 - \bar{x}_2x_3| + |x_1x_2 - x_3| \le 1, \ \text{and} \\ |x_1| \le 1, \end{cases} \tag{4.1}$$

**Lemma 4.2.1** *Let $x \in \mathbb{C}^3$. Then*

$$x \in \partial\Gamma_E \Leftrightarrow \begin{cases} |x_2|^2 + |x_1 - \bar{x}_2 x_3| + |x_1 x_2 - x_3| = 1, \ \textit{and} \\ |x_1| \leq 1, \end{cases} \tag{4.2}$$

**Proof** If the equality in (4.2) holds, then $f(\bar{\mathbb{D}}) \cap \bar{\mathbb{D}} \in \mathbb{T}$, where $f(z) = \dfrac{x_2 z - 1}{x_3 z - x_1}$ for all $z \in \bar{\mathbb{D}}$, because for $x$ to be in the topological boundary of $\Gamma_E$, we must have that the image of the disc touches the unit circle.

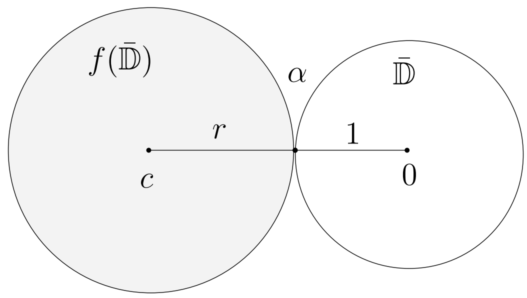



Figure 4.1: $f(\bar{\mathbb{D}}) \cap \bar{\mathbb{D}} = \{\alpha\} \in \mathbb{T}$

The proof of this result is similar to that of (2) $\Leftrightarrow$ (3) of Theorem 2.1.4, where in this case since $x$ is on the topological boundary of $\Gamma_E$, we have

$$x \in \partial\Gamma_E \Leftrightarrow |\text{centre of } f(\bar{\mathbb{D}})| = 1 + \text{radius of } f(\bar{\mathbb{D}}).$$

$\square$

The result above is not what one might expect. One might think that boundary points can also arise from having equality in the second inequality of (4.1). Below, we show that there are no such points.

**Remark 4.2.2** *There are no points* $x \in \partial\Gamma_E$ *such that*

$$\begin{cases} |x_2|^2 + |x_1 - \bar{x}_2 x_3| + |x_1 x_2 - x_3| < 1, \text{ and} \\ |x_1| = 1, \end{cases} \tag{4.3}$$

Suppose that (4.3) holds. Let $x_1 = e^{i\theta}$ as $|x_1| = 1$ and let $\varepsilon = e^{i\theta}x_2 - x_3$.

$$\begin{aligned} |x_2|^2 + |e^{i\theta} - \bar{x}_2 x_3| + |e^{i\theta}x_2 - x_3| < 1 \ &\Rightarrow \ |x_2|^2 + |e^{i\theta} - \bar{x}_2(e^{i\theta}x_2 - \varepsilon)| + |\varepsilon| < 1 \\ &\Rightarrow \ |x_2|^2 + |e^{i\theta} - e^{i\theta}x_2\bar{x}_2 + \bar{x}_2\varepsilon| + |\varepsilon| < 1 \\ &\Rightarrow \ t^2 + |(1 - t^2) + e^{-i\theta}\bar{x}_2\varepsilon| + |\varepsilon| < 1, \end{aligned}$$

where $t = |x_2|$, but since $|x_2| < 1$ by (4.3), we have

$$t^2 + |(1 - x_2\bar{x}_2) + e^{-i\theta}\bar{x}_2\varepsilon| + |\varepsilon| \geq t^2 + 1 - t^2 - |\bar{x}_2\varepsilon| + |\varepsilon| = 1 - |x_2\varepsilon| + |\varepsilon| \geq 1,$$

which a contradiction. Therefore, if $x \in \partial\Gamma_E$, then (4.3) does not hold.

**Remark 4.2.3** *Lemma 4.2.1 shows that for all* $x = (x_1, x_2, x_3)$, *where* $|x_j| \leq 1,\ j = 1, 2, 3$ *such that*

$$|x_2|^2 + |x_1 - \bar{x}_2 x_3| + |x_1 x_2 - x_3| = 1, \tag{4.4}$$

*there exists* $z = \dfrac{1 - \alpha x_1}{x_2 - \alpha x_3} \in \mathbb{T}$ *such that* $\Psi(z, x) \in \mathbb{T}$.

**Example** If $x_3 = 0$, we have $z = \dfrac{1 - \alpha x_1}{x_2}$. In this case, equation (4.4) is equivalent to $|x_1| + |x_2| = 1$ and so $\Psi(z, x) = \dfrac{1}{\alpha} \in \mathbb{T}$. Hence the points $x = (|x_1|, 1 - |x_1|, 0)$ are on the boundary of $\Gamma_E$.

**Remark 4.2.4** *For all* $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ *such that*

$$|x_2|^2 + |x_1 - \bar{x}_2 x_3| + |x_1 x_2 - x_3| = 1,$$

*there exists* $\alpha$ *such that* $\dfrac{1}{\Upsilon_\alpha(x)} \in \mathbb{T}$*, for some* $x$*.*

*Thus*

$$\begin{aligned}\Psi\left(\frac{1-\alpha x_1}{x_2-\alpha x_3}, x\right) &= \frac{x_1(x_2-\alpha x_3)-(1-\alpha x_1)x_3}{x_2-\alpha x_3-(1-\alpha x_1)x_2} \\ &= \frac{x_1x_2-x_3}{\alpha(x_1x_2-x_3)} \\ &= \frac{1}{\alpha}.\end{aligned}$$

*Therefore, for* $z = \dfrac{1-\alpha x_1}{x_2-\alpha x_2} \in \bar{\mathbb{D}},\ \alpha \in \mathbb{T}$*, we have* $|\Psi(z,x)| = \left|\dfrac{1}{\alpha}\right| = 1.$

The table below is a guide to some of our results concerning the characterisation of the topological boundary and the interior of $\Gamma_E$.

| Case | $x \in \partial\Gamma_E$ | $x \in G_E$ |
|---|---|---|
| $x_1x_2 \neq x_3$ | Theorem 4.2.5 | Theorem 4.2.8 |
| $x_1x_2 = x_3$ | Theorem 4.2.6 | Theorem 4.2.9 |

**Theorem 4.2.5** *For* $x \in \Gamma_E$*, when* $x_1x_2 \neq x_3$*,*

$$x \in \partial\Gamma_E \Leftrightarrow \tilde{\Upsilon}_x(\bar{\mathbb{D}}) \cap \mathbb{T} \neq \emptyset,$$

*where* $\tilde{\Upsilon}_x(z) = \dfrac{x_3z-x_2}{x_1z-1}$*, for all* $z \in \bar{\mathbb{D}}$*.*

**Proof** Note that in the case that $x_1x_2 \neq x_3$, we have $|x_1| \neq 1$.

Consider the Möbius automorphism $\mu$ of $\bar{\mathbb{D}}$ given by:

$$\mu(z) = \frac{z+\bar{x}_1}{x_1z+1},\ \forall z \in \bar{\mathbb{D}}.$$

Since $\mu$ is an automorphism of $\bar{\mathbb{D}}$, this is the same as showing that

$$x \in \partial\Gamma_E \Leftrightarrow \tilde{\Upsilon}_x\left(\mu\left(\bar{\mathbb{D}}\right)\right) \cap \mathbb{T} \neq \emptyset.$$

The map $\tilde{\Upsilon}_x \circ \mu$ is linear taking

$$\mathbb{D} \ni z \longmapsto \frac{x_1x_2 - x_3}{1-|x_1|^2} z + \frac{x_2 - \bar{x}_1 x_3}{1-|x_1|^2},$$

and so maps $\mathbb{D}$ to a disc with centre $C = \dfrac{x_2 - \bar{x}_1 x_3}{1-|x_1|^2}$ and radius $R = \dfrac{|x_1x_2 - x_3|}{1-|x_1|^2}$. Thus,

$$\begin{aligned}
\tilde{\Upsilon}_x(\bar{\mathbb{D}}) \cap \mathbb{T} \neq \emptyset \quad &\Leftrightarrow \quad |C| + R = 1 \\
&\Leftrightarrow \quad \frac{|x_2 - \bar{x}_1 x_3|}{1-|x_1|^2} + \frac{|x_1x_2 - x_3|}{1-|x_1|^2} = 1 \\
&\Leftrightarrow \quad \begin{cases} |x_1|^2 + |x_2 - \bar{x}_1 x_3| + |x_1x_2 - x_3| = 1, \text{ and} \\ |x_2| \leq 1. \end{cases} \\
&\Leftrightarrow \quad x \in \partial\Gamma_E.
\end{aligned}$$

□

**Theorem 4.2.6** *For $x \in \Gamma_E$, when $x_1x_2 = x_3$,*

$$x \in \partial\Gamma_E \Leftrightarrow |x_1| \leq 1 \textit{ and } |x_2| = 1, \textit{ or } |x_1| = 1 \textit{ and } |x_2| \leq 1.$$

**Proof** By Lemma 4.2.1, we have

$$\begin{aligned}
x \in \partial\Gamma_E \quad &\Leftrightarrow \quad \begin{cases} |x_2|^2 + |x_1 - \bar{x}_2 x_3| + |x_1x_2 - x_3| = 1, \text{ and} \\ |x_1| \leq 1. \end{cases} \\
&\Leftrightarrow \quad \begin{cases} |x_2|^2 + |x_1|\,|1 - |x_2|^2| = 1, \text{ and} \\ |x_1| \leq 1. \end{cases} \\
&\Leftrightarrow \quad \begin{cases} |x_1|\,|1 - |x_2|^2| = 1 - |x_2|^2, \text{ and} \\ |x_1| \leq 1. \end{cases} \\
&\Leftrightarrow \quad |x_1| \leq 1 \text{ and } |x_2| = 1, \text{ or } |x_1| = 1 \text{ and } |x_2| \leq 1.
\end{aligned}$$

□

The next characterisation of points on the topological boundary of $\Gamma_E$ follows from Lemma 4.2.1 and Remark 4.2.3.

**Corollary 4.2.7** *Let $x = (x_1, x_2, x_3) \in \mathbb{C}^3$. Then the following are equivalent.*

*(1)* $x \in \partial\Gamma_E$.

*(2)* $$\begin{cases} |x_2|^2 + |x_1 - \bar{x}_2 x_3| + |x_1 x_2 - x_3| = 1, \text{ and} \\ |x_1| \le 1. \end{cases}$$

*(3)* $$\begin{cases} |x_1|^2 + |x_2 - \bar{x}_1 x_3| + |x_1 x_2 - x_3| = 1, \text{ and} \\ |x_2| \le 1. \end{cases}$$

*(4) There exist $b, c \in \mathbb{C}$ such that $bc = x_1 x_2 - x_3$ and* $\left\| \begin{pmatrix} x_1 & b \\ c & x_2 \end{pmatrix} \right\| = 1.$

*(5) There exist $b, c \in \mathbb{C}$ such that $|b| = |c| = |x_1 x_2 - x_3|^{1/2}$, $bc = x_1 x_2 - x_3$ and* $\left\| \begin{pmatrix} x_1 & b \\ c & x_2 \end{pmatrix} \right\| = 1.$

*(6)* $$\begin{cases} 1 - |x_1|^2 - |x_2|^2 + |x_3|^2 - 2|x_1 x_2 - x_3| = 0, \text{ and} \\ |x_1| \le 1, \ |x_2| \le 1, \ |x_3| \le 1. \end{cases}$$

*(7)* $$\begin{cases} 1 - |x_1|^2 + |x_2|^2 - |x_3|^2 - 2|x_1 \bar{x}_3 - \bar{x}_2| = 0, \text{ and} \\ |x_1| \le 1, \ |x_2| \le 1. \end{cases}$$

*(8)* $$\begin{cases} 1 + |x_1|^2 - |x_2|^2 - |x_3|^2 - 2|x_2 \bar{x}_3 - \bar{x}_1| = 0, \text{ and} \\ |x_1| \le 1, \ |x_2| \le 1. \end{cases}$$

Next, we shall find conditions for when $x$ is in the interior of $\Gamma_E$.

**Theorem 4.2.8** *For $x \in \Gamma_E$, when $x_1x_2 \neq x_3$,*

$$x \in G_E \Leftrightarrow \tilde{\Upsilon}_x(\bar{\mathbb{D}}) \subset \mathbb{D},$$

*where* $\tilde{\Upsilon}_x(z) = \dfrac{x_3z - x_2}{x_1z - 1}$, *for all $z \in \bar{\mathbb{D}}$.*

**Proof** Since $\tilde{\Upsilon}_x$ is in the Schur class, we have

$$\begin{aligned} x \in G_E \quad &\Leftrightarrow \quad x \notin \partial\Gamma_E \\ &\Leftrightarrow \quad \nexists z \in \bar{\mathbb{D}} \text{ such that } \tilde{\Upsilon}_x(z) \in \mathbb{T} \\ &\Leftrightarrow \quad \tilde{\Upsilon}_x(\bar{\mathbb{D}}) \subset \mathbb{D}. \end{aligned}$$

□

**Theorem 4.2.9** *For $x \in \Gamma_E$, when $x_1x_2 = x_3$,*

$$x \in G_E \Leftrightarrow |x_1| < 1 \text{ and } |x_2| < 1.$$

**Proof** Clearly, by Theorem 4.2.8,

$$x \in G_E \Leftrightarrow \begin{cases} |x_2|^2 + |x_1 - \bar{x}_2x_3| + |x_1x_2 - x_3| < 1, \text{ and} \\ |x_1| < 1. \end{cases}$$

Hence, if $x_1x_2 = x_3$, then

$$\begin{aligned} x \in G_E \quad &\Leftrightarrow \quad \begin{cases} |x_2|^2 + |x_1|\,(1 - |x_2|^2) < 1, \text{ and} \\ |x_1| < 1. \end{cases} \\ &\Leftrightarrow \quad \begin{cases} |x_1|\,(1 - |x_2|^2) < 1 - |x_2|^2, \text{ and} \\ |x_1| < 1. \end{cases} \\ &\Leftrightarrow \quad |x_1| < 1 \text{ and } |x_2| < 1. \end{aligned}$$

□

The next characterisation of points in the interior of $\Gamma_E$ follows from Theorem 4.1.3, Corollary 4.1.4 and Lemma 4.2.1.

**Corollary 4.2.10** *Let $x \in \mathbb{C}^3$ and $G_E = \{(a_{11}, a_{22}, \det(A)) : \mu_E(A) < 1\}$. Then the following are equivalent.*

*(1)* $x \in G_E$.

*(2)* $\begin{cases} |x_2|^2 + |x_1 - \bar{x}_2 x_3| + |x_1 x_2 - x_3| < 1, \ \textit{and} \\ |x_1| < 1. \end{cases}$

*(3)* $\begin{cases} |x_1|^2 + |x_2 - \bar{x}_1 x_3| + |x_1 x_2 - x_3| < 1, \ \textit{and} \\ |x_2| < 1. \end{cases}$

*(4)* $\begin{cases} \Psi(., x) \ \textit{is analytic in} \ \mathbb{D} \ \textit{and} \ |\Psi(., x)| < 1 \\ \textit{if} \ x_1 x_2 = x_3, \ |x_2| < 1. \end{cases}$

*(5)* $\begin{cases} \Upsilon(., x) \ \textit{is analytic in} \ \mathbb{D} \ \textit{and} \ |\Upsilon(., x)| < 1, \\ \textit{if} \ x_1 x_2 = x_3, \ |x_1| < 1. \end{cases}$

*(6) There exist $b, c \in \mathbb{C}$ such that $bc = x_1 x_2 - x_3$ and* $\left\| \begin{pmatrix} x_1 & b \\ c & x_2 \end{pmatrix} \right\| < 1.$

*(7) There exist $b, c \in \mathbb{C}$ such that $|b| = |c| = |x_1 x_2 - x_3|^{1/2}$, $bc = x_1 x_2 - x_3$ and* $\left\| \begin{pmatrix} x_1 & b \\ c & x_2 \end{pmatrix} \right\| < 1.$

*(8)* $\begin{cases} 1 - |x_1|^2 - |x_2|^2 + |x_3|^2 - 2|x_1 x_2 - x_3| > 0, \ \textit{and} \\ |x_1| < 1, \ |x_2| < 1, \ |x_3| < 1. \end{cases}$

*(9)* $\begin{cases} 1 - |x_1|^2 + |x_2|^2 - |x_3|^2 - 2|x_1 \bar{x}_3 - \bar{x}_2| > 0, \ \textit{and} \\ |x_1| < 1, \ |x_2| < 1. \end{cases}$

*(10)* $\begin{cases} 1 + |x_1|^2 - |x_2|^2 - |x_3|^2 - 2|x_2 \bar{x}_3 - \bar{x}_1| > 0, \ \textit{and} \\ |x_1| < 1, \ |x_2| < 1. \end{cases}$

**Remark 4.2.11** *If $x = (x_1, x_2, x_3) \in G_E$, then $|x_3| \neq 1$.*

To see this, let $|x_3| = 1$. We may assume that $x_3 = 1$. Then

$$\begin{aligned} x \in G_E \quad &\Leftrightarrow \quad |x_2|^2 + |x_1 - \bar{x}_2 x_3| + |x_1 x_2 - x_3| < 1 \qquad (4.5) \\ &\Leftrightarrow \quad |x_2|^2 + |x_1 - \bar{x}_2| + |x_1 x_2 - 1| < 1. \end{aligned}$$

Let $\varepsilon = x_1 - \bar{x}_2$ so that $x_1 = \bar{x}_2 + \varepsilon$. Then

$$\begin{aligned} x \in G_E \quad &\Leftrightarrow \quad |x_2|^2 + |\varepsilon| + |x_2(\bar{x}_2 + \varepsilon) - 1| < 1 \\ &\Leftrightarrow \quad |x_2|^2 + |\varepsilon| + \big||x_2|^2 + \varepsilon x_2 - 1\big| < 1, \end{aligned}$$

but

$$|x_2|^2 + |\varepsilon| + \big||x_2|^2 + \varepsilon x_2 - 1\big| \geq |x_2|^2 + |\varepsilon| - |x_2|^2 - |\varepsilon x_2| + 1 \geq 1,$$

which is a contradiction, unless $|x_2| = 1$, which cannot happen since $|x_2| < 1$ by (4.5). Therefore, if $x \in G_E$, then $|x_3| < 1$.

## 4.3 Peak Sets

We need the following definitions and results from T. Gamelin [21] in order to find the distinguished boundary of $\Gamma_E$. Throughout this section, let $K$ be a compact metric space and $A$ be a uniform algebra on $K$.

**Definition 4.3.1** *A point $x \in K$ is a* peak point *if there is a function $f \in A$ such that $f(x) = 1$ while $|f(y)| < 1$ for $y \in K$, $y \neq x$. Any function $f$ which satisfies this condition is said to* peak at *$x$ relative to $K$.*

**Definition 4.3.2** *A closed subset $H$ of $K$ is a* peak set *if there is a function $f \in A$ such that $f(x) = 1$ for $x \in H$, and $|f(y)| < 1$ for $y \in K \backslash H$. Any function $f$ satisfying this condition is said to be* peak on $H$.

**Definition 4.3.3** *A closed subset $H$ of $K$ is a* p-set, *or* generalised peak set *if it is the intersection of peak sets.*

We state the following result from Gamelin's book without a proof. For a full proof see Theorem 12.5 of [21].

**Theorem 4.3.4** *Let $B$ be a closed subspace of $C(K)$ and $B^\perp$ be the space of all measures orthogonal to $B$. Let $H$ be a closed subset of $K$ such that $m_H \in B^\perp$ for all measures $m \in B^\perp$. Let $f \in B|_H$, and let $p$ be a positive continuous function on $K$ such that $|f(y)| \leq p(y)$ for $y \in H$. Then there is $g \in B$ such that $g|_H = f$ and $|g(x)| \leq p(x)$ for all $x \in K$.*

**Corollary 4.3.5** *If $H$ is a p-set of $A$, and $F \subseteq H$ is a p-set of $A|_H$, then $F$ is a p-set of $A$.*

**Example** In Definition 4.3.2, if we take $K$ to be $\Gamma_E$ and $H = \{(x_1, \bar{x}_1, 1) : x_1 \in \bar{\mathbb{D}}\}$, then $p = \dfrac{1 + x_3}{2}$ has a peak set $H$.

## 4.4 The Distinguished Boundary of $\Gamma_E$

Agler and Young [9] have shown that the distinguished boundary of the symmetrised bidisc $\Gamma$ is the set

$$\begin{aligned} b\Gamma &= \{(z_1 + z_2, z_1 z_2) : \; z_1, z_2 \in \mathbb{T}\} \\ &= \{(s, p) \in \mathbb{C}^2 : \; s = \bar{s}p, |p| = 1, |s| \leq 2\}. \end{aligned}$$

In this section, we shall show that the distinguished boundary of $\Gamma_E$ is the set

$$B := \{x = (x_1, x_2, x_3) \in \Gamma_E : \; x_1 = \bar{x}_2 x_3, \; |x_3| = 1\}.$$

**Lemma 4.4.1** $\Gamma_E \cap \mathbb{R}^3$ *is a tetrahedron.*

**Proof** Let $x = (a, b, p) \in \mathbb{R}^3$. Recall that, if $|a| < 1$, then

$$x \in \Gamma_E \Leftrightarrow \left| \frac{b - zp}{1 - za} \right| \leq 1, \; \forall z \in \mathbb{T}.$$

We shall find

$$\sup_{z \in \mathbb{T}} \left| \frac{zp - b}{za - 1} \right|$$

when $a, b, p \in \mathbb{R}$. Observe that, when $|a| < 1$,

$$\begin{aligned} \begin{bmatrix} p & -b \\ a & -1 \end{bmatrix} \begin{bmatrix} 1 & -\bar{a} \\ a & -1 \end{bmatrix} &= \begin{bmatrix} p - ab & b - p\bar{a} \\ 0 & 1 - a\bar{a} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{ab - p}{1 - a\bar{a}} & -\dfrac{b - \bar{a}p}{1 - a\bar{a}} \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

Therefore,

$$\sup_{z \in \mathbb{T}} \left| \frac{zp - b}{za - 1} \right| = \sup_{w \in \mathbb{T}} \left| \frac{(p - ab)w + (b - \bar{a}p)}{1 - a\bar{a}} \right|.$$

Observe that

$$\begin{bmatrix} \dfrac{ab - p}{1 - a\bar{a}} & -\dfrac{b - \bar{a}p}{1 - a\bar{a}} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} w \\ 1 \end{bmatrix}$$

is biggest at $w = \pm 1$.

Hence, $\dfrac{pz - b}{az - 1}$ is biggest at

$$\begin{bmatrix} 1 & -\bar{a} \\ a & -1 \end{bmatrix} \begin{bmatrix} \pm 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \pm 1 & -\bar{a} \\ \pm a & -1 \end{bmatrix}.$$

That is, at

$$\begin{cases} \dfrac{1-\bar{a}}{a-1} = -1, \text{ or} \\ \\ \dfrac{-1-\bar{a}}{-a-1} = 1. \end{cases}$$

Thus, $\dfrac{pz-b}{az-1}$ is biggest at $z = \mp 1$. Therefore,

$$\begin{aligned}(a,b,p) \in \Gamma_E \cap \mathbb{R}^3,\ |a| < 1 \quad &\Leftrightarrow \quad \frac{|p-b|}{|a-1|} \le 1 \text{ and } \frac{|-p-b|}{|-a-1|} \le 1 \\ &\Leftrightarrow \quad -1 \le \frac{p-b}{a-1} \le 1 \text{ and } -1 \le \frac{-p-b}{-a-1} \le 1 \\ &\Leftrightarrow \quad -(a-1) \ge p-b \ge a-1 \text{ and } a+1 \ge -p-b \ge -a-1.\end{aligned}$$

Thus, $\Gamma_E \cap \mathbb{R}^3$ is a tetrahedron with four faces given by the inequalities:

$$\begin{aligned} -a+b-p+1 \ge 0 \quad &, \quad -a-b+p+1 \ge 0, \\ a+b+p+1 \ge 0 \quad &, \quad a-b-p+1 \ge 0, \end{aligned} \tag{4.6}$$

where $a, b, p \in [-1, 1]$.

Moreover, the vertices of this tetrahedron are:

$$(1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1).$$

In the case that $|a| = 1$, we have

$$\begin{aligned} x \in \Gamma_E \quad &\Leftrightarrow \quad \begin{cases} |a|^2 + |b - \bar{a}p| + |ab - p| \le 1, \text{ and} \\ |b| \le 1 \end{cases} \\ &\Leftrightarrow \quad \begin{cases} |b - \bar{a}p| + |ab - p| \le 0, \text{ and} \\ |b| \le 1 \end{cases} \\ &\Leftrightarrow \quad \begin{cases} b = \bar{a}p,\ ab = p, \text{ and} \\ |b| \le 1 \end{cases} \\ &\Leftrightarrow \quad |b| = |p| \text{ and } |b| \le 1. \end{aligned}$$

Clearly, points $(a, b, p) \in \Gamma_E \cap \mathbb{R}^3$ such that $|a| = 1$ and $|p| = |b| \leq 1$ are in the tetrahedron described above.

$\square$

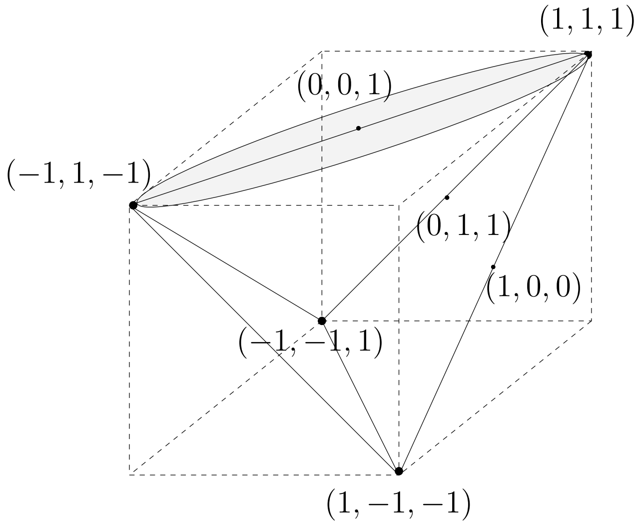



Figure 4.2: $\Gamma_E$ in the real case

**Lemma 4.4.2** *The set* $C = \{x \in \Gamma_E : \ x_1 = \bar{x}_2 x_3, \ |x_2| = 1$ *and* $|x_3| < 1\}$ *is disjoint from the distinguished boundary of* $\Gamma_E$.

**Proof** Let $x \in C$. We can fix $x_2$ with $|x_2| = 1$ and define a map $h$ from the open disc into $C$ as follows:

$$\begin{aligned} h : \mathbb{D} &\longrightarrow C, \\ w &\longmapsto (\bar{x}_2 w, x_2, w). \end{aligned}$$

Therefore, we have an analytic disc in $C$ that contains the point $x$, but no point in the interior of an analytic disc can be a peak point. Therefore, such an $x$ is not in the distinguished boundary of $\Gamma_E$.

$\square$

**Lemma 4.4.3** *A point $x$ such that $|x_1| = 1$, $|x_3| < 1$ and $x_1x_2 = x_3$ is not in the distinguished boundary of $\Gamma_E$.*

***Proof*** *Let $z \in \mathbb{D}$, i.e., $|z| < 1$. Then $(x_1, z, x_1z) \in \Gamma_E$, because*

$$\begin{aligned} |x_2|^2 + |x_1 - \bar{x}_2x_3| + |x_1x_2 - x_3| &= |z|^2 + |x_1 - \bar{z}x_1z| \\ &= |z|^2 + |x_1||1 - |z|^2| \\ &= |z|^2 + (1 - |z|^2) \leq 1. \end{aligned}$$

*Hence, there exist analytic discs that contain the points $x \in \Gamma_E$ with $|x_1| = 1$ and $x_1x_2 = x_3$ or with $|x_2| = 1$ and $x_1x_2 = x_3$. Therefore, such points $x$ are not in the distinguished boundary of $\Gamma_E$.*

□

**Theorem 4.4.4** *The point $(p, \bar{p}, 1)$ is a peak point relative to $H \subseteq \Gamma_E$, where $H = \{(x_1, \bar{x}_1, 1) : x \in \mathbb{D}\}$.*
*Moreover, $(p, \bar{p}, 1)$ is a peak point relative to $\Gamma_E$.*

**Proof** Let $x = (x_1, x_2, x_3) \in H \subseteq \Gamma_E$. If $x_3 = 1$ and $x_1 = \bar{x}_2$, then we can define a function $f_p$ that peaks at $(p, \bar{p}, 1)$ in $H$, where $0 \leq p \leq 1$, by

$$f_p(x_1, x_2, x_3) = 1 - \frac{(x_1 - p)(x_2 - \bar{p})}{4}.$$

Clearly, $f_p$ is an analytic function in $H$ such that when $x_1 \neq p$, we have

$$\begin{aligned} f_p(x_1, \bar{x}_1, 1) &= 1 - \frac{(x_1 - p)(\bar{x}_1 - \bar{p})}{4} \\ &= 1 - \frac{|x_1 - p|^2}{4} < 1. \end{aligned}$$

Hence, $|f_p(x_1, \bar{x}_1, 1)| < 1$. Moreover, $f(p, \bar{p}, 1) = 1$. Therefore, $f_p$ peaks at $(p, \bar{p}, 1)$ in $H$.

Let $d$ be a metric on $\mathbb{C}^3$ and let

$$g(x_1, x_2, x_3) = \frac{1}{1 + d(x, H)}.$$

Then $|f_p(y)| \leq g(y)$, for all $y \in H$. Hence, by Theorem 4.3.4, $f_p$ is analytic on $\Gamma_E$ with $|f_p(y)| \leq g(y)$, for all $y \in \Gamma_E$. Therefore, $(p, \bar{p}, 1)$ is a peak point relative to $\Gamma_E$.

$\square$

**Remark 4.4.5** *The function* $g(x) = \dfrac{(x_1x_2 - x_3) + 1}{2}$ *peaks at* $(0, 0, -1)$ *relative to* $\Gamma_E$*, for,* $g$ *is an analytic function in* $\mathbb{D}$ *such that* $|g(x)| < 1$*, because*

$$|g(x)| = \left|\frac{(x_1x_2 - x_3) + 1}{2}\right| \leq \left|\frac{x_1x_2 - x_3}{2}\right| + \left|\frac{1}{2}\right| < \frac{1}{2} + \frac{1}{2} = 1,$$

*and* $g(0, 0, -1) = 1$*. Moreover,* $|g(x)| = 1 \Rightarrow x = (0, 0, -1)$*, for,*

$$\begin{aligned} |g(x)| = 1 \quad &\Rightarrow \quad \left|\frac{(x_1x_2 - x_3) + 1}{2}\right| = 1 \\ &\Rightarrow \quad x_1x_2 - x_3 = 1, \ \text{by (3) of Theorem 2.1.4} \\ &\Rightarrow \quad |x_1| = 0 \ \text{and} \ |x_2| = 0, \end{aligned}$$

*Hence,* $x_1 = 0 = x_2$ *and* $x_3 = -1$*.*

Let $b\Gamma_E$ denote the distinguished boundary of $\Gamma_E$ and recall that $B$ is defined as follows:

$$B = \{x \in \Gamma_E : x_1 = \bar{x}_2x_3, |x_3| = 1\}.$$

Next, we find necessary and sufficient conditions for $x$ to be in $B$ and in $b\Gamma_E$, then we show that $b\Gamma_E = B$. The table below is a guide to these results.

| Case | $x \in B$ | $x \in b\Gamma_E$ | $b\Gamma_E = B$ |
|---|---|---|---|
| $x_1x_2 \neq x_3$ | Theorem 4.4.6 | Theorem 4.4.8 | Theorem 4.4.10 |
| $x_1x_2 = x_3$ | Theorem 4.4.7 | Theorem 4.4.9 | Theorem 4.4.10 |

**Theorem 4.4.6** *Let* $\tilde{\Upsilon}_x(z) = \dfrac{x_3z - x_2}{x_1z - 1}$*, for all* $z \in \bar{\mathbb{D}}$*. When* $x_1x_2 \neq x_3$*,*

$$x \in B \Leftrightarrow \tilde{\Upsilon}_x(\bar{\mathbb{D}}) = \bar{\mathbb{D}}.$$

**Proof** In the case that $x_1x_2 \neq x_3$, we have $|x_1| \neq 1$. Consider the Möbius automorphism $\mu$ of $\bar{\mathbb{D}}$,

$$\mu(z) = \frac{z + \bar{x}_1}{x_1z + 1}, \ \forall z \in \bar{\mathbb{D}},$$

Since $\mu$ is an automorphism of $\bar{\mathbb{D}}$, it suffices to prove that

$$x \in B \Leftrightarrow \tilde{\Upsilon}_x\left(\mu\left(\bar{\mathbb{D}}\right)\right) = \bar{\mathbb{D}}.$$

The map $\tilde{\Upsilon}_x \circ \mu$ is linear, taking

$$\mathbb{D} \ni z \longmapsto \frac{x_1x_2 - x_3}{1 - |x_1|^2} z + \frac{x_2 - \bar{x}_1x_3}{1 - |x_1|^2},$$

and so maps $\mathbb{D}$ to a disc with centre $C = \dfrac{x_2 - \bar{x}_1x_3}{1 - |x_1|^2}$ and radius $R = \dfrac{|x_1x_2 - x_3|}{1 - |x_1|^2}$. Therefore,

$$\begin{aligned}
\tilde{\Upsilon}_x(\bar{\mathbb{D}}) = \bar{\mathbb{D}} \ &\Leftrightarrow \ C = 0 \text{ and } R = 1 \\
&\Leftrightarrow \ x_2 = \bar{x}_1x_3 \text{ and } |x_1x_2 - x_3| = 1 - |x_1|^2 \\
&\Leftrightarrow \ x_2 = \bar{x}_1x_3 \text{ and } |x_1\bar{x}_1x_3 - x_3| = 1 - |x_1|^2 \\
&\Leftrightarrow \ x_2 = \bar{x}_1x_3 \text{ and } |x_3|\left|1 - |x_1|^2\right| = 1 - |x_1|^2 \\
&\Leftrightarrow \ x_1 = \bar{x}_2x_3 \text{ and } |x_3| = 1 \\
&\Leftrightarrow \ x \in B.
\end{aligned}$$

□

**Theorem 4.4.7** *When $x_1x_2 = x_3$,*

$$|x_1| = |x_2| = |x_3| = 1 \Leftrightarrow x \in B.$$

**Proof** ($\Rightarrow$) This implication is clear; if $x_1x_2 = x_3$ and $|x_1| = |x_2| = |x_3| = 1$, then

$$x_1x_2\bar{x}_2 = \bar{x}_2x_3 \Rightarrow x_1|x_2|^2 = \bar{x}_2x_3 \Rightarrow x_1 = \bar{x}_2x_3.$$

Hence, $x_1 = \bar{x}_2x_3$ and $|x_3| = 1$.

($\Leftarrow$) If $x_1x_2 = x_3$, $x_1 = \bar{x}_2x_3$ and $|x_3| = 1$, then

$$x_1 = \bar{x}_2x_3 \Rightarrow |x_1| = |\bar{x}_2x_3| \Rightarrow |x_1| = |x_2|,$$

Hence,

$$x_1x_2 = x_3 \Rightarrow \bar{x}_2x_3x_2 = x_3 \Rightarrow |x_2| = 1.$$

Therefore, $|x_1| = |x_2| = |x_3| = 1$.

$\square$

In the following results, we find conditions on $x$ so that it is in the distinguished boundary $b\Gamma_E$ of $\Gamma_E$.

**Theorem 4.4.8** *When $x_1x_2 \neq x_3$,*

$$x \in b\Gamma_E \Leftrightarrow \tilde{\Upsilon}_x(\bar{\mathbb{D}}) = \bar{\mathbb{D}},$$

*where* $\tilde{\Upsilon}_x(z) = \dfrac{x_3z - x_2}{x_1z - 1}$, *for all* $z \in \bar{\mathbb{D}}$.

**Proof** ($\Leftarrow$) If $\tilde{\Upsilon}_x(\bar{\mathbb{D}}) = \bar{\mathbb{D}}$, then $\tilde{\Upsilon}_x$ is a Möbius automorphism of $\mathbb{D}$. Hence, we can compose it with its inverse $\tilde{\Upsilon}_x^{-1}$ so that we get $\tilde{\Upsilon}_e$, where $e = (0, 0, -1)$.

Therefore, there exists an automorphism $\mu$ of $\Gamma_E$ that sends points of $\Gamma_E$ to $(0, 0, -1)$, which is a peak point by Remark 4.4.5. Hence, $x \in b\Gamma_E$.

($\Rightarrow$) If $\tilde{\Upsilon}_x(\bar{\mathbb{D}}) \neq \bar{\mathbb{D}}$, then we have two cases; (i) either $x$ is in the interior of $\Gamma_E$, or (ii) that $\tilde{\Upsilon}_x(\bar{\mathbb{D}})$ touches $\mathbb{T}$.

Case (i): Points in $G_E$ are not peak points because we can embed an analytic disc $w \longmapsto (x_1, x_2 + \varepsilon w, x_3) \in \Gamma_E$ for $|w| < 1$ and a small $\varepsilon$, this analytic disc contains $(x_1, x_2, x_3)$, hence $(x_1, x_2, x_3)$ cannot be in the distinguished boundary by the maximum modulus principle.

Case (ii): Let $x = (x_1, x_2, x_3)$. Recall that when $\tilde{\Upsilon}_x(z)$ touches $\mathbb{T}$, we have $\tilde{\Upsilon}_x(1) = \{\alpha\} \in \mathbb{T}$. We may compose $\tilde{\Upsilon}_x(1)$ with an automorphism so that we move $\{\alpha\}$ to 1. Therefore, we have $\tilde{\Upsilon}_x(1) = 1$. Hence,

$$\frac{x_3 - x_2}{x_1 - 1} = 1.$$

That is,

$$x_3 - x_2 - x_1 + 1 = 0. \tag{4.7}$$

For $x \in \Gamma_E$, define

$$X = \begin{bmatrix} x_3 & -x_2 \\ x_1 & -1 \end{bmatrix}.$$

Let $\mathbb{H}$ denote the right half-plane. Consider the map $h : \bar{\mathbb{H}} \longrightarrow \bar{\mathbb{D}}$, defined as

$$h(z) = \frac{-z + 1}{-z - 1},$$

which corresponds to

$$\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Let $L(z) : \bar{\mathbb{H}} \longrightarrow \bar{\mathbb{H}}$. Then

$$\begin{aligned}
\begin{bmatrix} L_3 & L_2 \\ L_1 & 1 \end{bmatrix} &= \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_3 & -x_2 \\ x_1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -x_3 + x_2 & x_3 + x_2 \\ -x_1 + 1 & x_1 + 1 \end{bmatrix} \\
&= \begin{bmatrix} x_3 - x_2 + x_1 - 1 & -x_3 - x_2 - x_1 - 1 \\ -x_3 + x_2 + x_1 - 1 & x_3 + x_2 - x_1 - 1 \end{bmatrix} \\
&= \begin{bmatrix} x_3 - x_2 + x_1 - 1 & -x_3 - x_2 - x_1 - 1 \\ 0 & x_3 + x_2 - x_1 - 1 \end{bmatrix} \\
&\equiv \begin{bmatrix} \dfrac{x_3 - x_2 + x_1 - 1}{x_3 + x_2 - x_1 - 1} & \dfrac{-x_3 - x_2 - x_1 - 1}{x_3 + x_2 - x_1 - 1} \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

From (4.7), we have $x_3 = x_1 + x_2 - 1$. Therefore, $L_1 = 0$ and

$$L_3 = \frac{x_3 - x_2 + x_1 - 1}{x_3 + x_2 - x_1 - 1} = \frac{x_1 + x_2 - 1 - x_2 + x_1 - 1}{x_1 + x_2 - 1 + x_2 - x_1 - 1} = \frac{1 - x_1}{1 - x_2},$$

$$L_2 = \frac{-x_3 - x_2 - x_1 - 1}{x_3 + x_2 - x_1 - 1} = \frac{-x_1 - x_2 + 1 - x_2 - x_1 - 1}{x_1 + x_2 - 1 + x_2 - x_1 - 1} = \frac{x_1 + x_2}{1 - x_2}.$$

Therefore, $L(z) = L_3 z + L_2$, where $L_3$ must clearly be a real non-negative constant. We can write $L(z)$ as follows

$$L(z) = \begin{bmatrix} \dfrac{1 - x_1}{1 - x_2} & \dfrac{x_1 + x_2}{1 - x_2} \\ 0 & 1 \end{bmatrix}.$$

Hence, we can define a family of analytic functions $L_\varepsilon$ as

$$L_\varepsilon(z) = \begin{bmatrix} L_3 & L_2 + \varepsilon\lambda \\ 0 & 1 \end{bmatrix},$$

where $\varepsilon > 0$ is sufficiently small and $\lambda \in \mathbb{D}$. Therefore, given a solution as $L(z) = L_3 z + L_2$, where $L_3 \geq 0$ and $L_2 : \bar{\mathbb{D}} \longrightarrow \bar{\mathbb{H}}$, we have

$$\begin{aligned}
\begin{bmatrix} x_3' & -x_2' \\ x_1' & -1 \end{bmatrix} &= \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} L_3 & L_2 + \varepsilon\lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -L_3 + L_2 + \varepsilon\lambda & -L_3 - L_2 - \varepsilon\lambda \\ 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} L_3 - L_2 - \varepsilon\lambda + 1 & L_3 + L_2 + \varepsilon\lambda - 1 \\ L_3 - L_2 - \varepsilon\lambda - 1 & L_3 + L_2 + \varepsilon\lambda + 1 \end{bmatrix} \\
&= \begin{bmatrix} -\dfrac{L_3 - L_2 - \varepsilon\lambda + 1}{L_3 + L_2 + \varepsilon\lambda + 1} & -\dfrac{L_3 + L_2 + \varepsilon\lambda - 1}{L_3 + L_2 + \varepsilon\lambda + 1} \\ \\ -\dfrac{L_3 - L_2 - \varepsilon\lambda - 1}{L_3 + L_2 + \varepsilon\lambda + 1} & -1 \end{bmatrix},
\end{aligned}$$

where

$$L_3 = \frac{1 - x_1}{1 - x_2}, \; L_2 = \frac{x_1 + x_2}{1 - x_2} \text{ and } x_3 = x_1 + x_2 - 1.$$

Observe that

$$\begin{aligned}
x_1' &= \frac{-2x_1 - \varepsilon\lambda(1 - x_2}{-2 - \varepsilon\lambda(1 - x_2)} \to x_1 \text{ as } \varepsilon \to 0, \\
x_2' &= \frac{-2x_2 - \varepsilon\lambda(1 - x_2)}{-2 - \varepsilon\lambda(1 - x_2)} \to x_2 \text{ as } \varepsilon \to 0, \\
x_3' &= \frac{2(1 - x_1 - x_2) - \varepsilon\lambda(1 - x_2)}{-2 - \varepsilon\lambda(1 - x_2)} \to -1 + x_1 + x_2 = x_3 \text{ as } \varepsilon \to 0.
\end{aligned}$$

Hence, as $\varepsilon \to 0$, $(x_1', x_2', x_3') \to (x_1, x_2, x_3) \in \Gamma_E$. Therefore, this family $L_\varepsilon$ of functions is analytic and contains $L(z)$. Therefore, $x = (x_1, x_2, x_3)$ is not a peak point and so it is not in the distinguished boundary of $\Gamma_E$. Thus, $\tilde{\Upsilon}_x(\bar{\mathbb{D}}) = \bar{\mathbb{D}}$. $\square$

**Theorem 4.4.9** *For $x \in \Gamma_E$, when $x_1x_2 = x_3$,*

$$x \in b\Gamma_E \Leftrightarrow |x_1| = |x_2| = |x_3| = 1.$$

**Proof** ($\Rightarrow$) If $x_1x_2 = x_3$ and $|x_1| < 1$ or $|x_2| < 1$, say $|x_2| < 1$, then there exists an analytic disc $(x_1, z, x_1z)$, $z \in \mathbb{D}$, this disc contains $(x_1, x_2, x_1x_2)$. Therefore, points $x \in \Gamma_E$ such that $x_1x_2 = x_3$ and $|x_1| < 1$ or $|x_2| < 1$ are not in the distinguished boundary of $\Gamma_E$. Hence, points $x$ such that $x_1x_2 = x_3$ and $|x_1| = 1 = |x_2|$ are in $b\Gamma_E$. Since we also have, $x_1 = \bar{x}_2x_3$ and $|x_3| = 1$, we have $|x_1| = |x_2| = |x_3| = 1$.
($\Leftarrow$) We have $|x_1| = |x_2| = |x_3| = 1$. Let $p = x_1$, hence, $\bar{p} = x_2\bar{x}_3$ and $p\bar{p} = x_1x_2\bar{x}_3 = x_3\bar{x}_3 = |x_3|^2 = 1$. Therefore, $x = (x_1, x_2\bar{x}_3, x_3\bar{x}_3) = (p, \bar{p}, 1)$, which is a peak point by Theorem 4.4.4. Thus, $x \in b\Gamma_E$.

$\square$

The next result follows immediately from Theorems 4.4.6, 4.4.8, 4.4.7 and 4.4.9. It shows that the distinguished boundary of $\Gamma_E$ is in fact $B$. That is,

$$B = b\Gamma_E = \{x \in \Gamma_E : \; x_1 = \bar{x}_2x_3, \; |x_3| = 1\}.$$

**Corollary 4.4.10** *We have*

$$x \in b\Gamma_E \Leftrightarrow x \in B.$$

**Theorem 4.4.11** *Let $x = (x_1, x_2, x_3) \in \mathbb{C}^3$. Then*

$$x \in b\Gamma_E \Leftrightarrow \begin{cases} |\Psi_\omega(x)| = 1, \; \forall\omega \in \mathbb{T}, \; \textit{and} \\ |\Upsilon_\omega(x)| = 1, \; \forall\omega \in \mathbb{T}. \end{cases}$$

**Proof** Since $\Psi_z$ and $\Upsilon_z$ are automorphisms of $\bar{\mathbb{D}}$, then by continuity we have that $\Psi_x(\mathbb{T}) = \mathbb{T}$ and $\Upsilon_z(\mathbb{T}) = \mathbb{T}$.

$\square$

## 4.5 $\Gamma_E$-Inner Functions

In this section, we define $\Gamma_E$-inner functions and present some results concerning this type of functions.

Fatou's Theorem [29] states that a bounded analytic function on the unit disc has radial limits at almost every boundary point.

An $H^\infty$ function on $\mathbb{D}$ that has unit modulus almost everywhere on $\mathbb{T}$ is called an *inner* function [27]. Blaschke factors are inner and therefore so are finite Blaschke products. Infinite Blaschke products are also inner, a full proof of this can be found in [2].

The following definition of $\Gamma$-inner functions can be found in [8].

**Definition 4.5.1** *A* $\Gamma$-inner function *is an analytic function* $\varphi : \mathbb{D} \longrightarrow \Gamma$ *for which almost all radial limits* $\varphi\left(e^{i\theta}\right)$, $\theta \in \mathbb{R}$, *lie in the distinguished boundary* $b\Gamma$ *of* $\Gamma$ *(defined as the Šilov boundary of the algebra of continuous functions on* $\Gamma$ *which are analytic on the interior of* $\Gamma$*).*

We define $\Gamma_E$-inner functions as follows:

**Definition 4.5.2** *A* $\Gamma_E$-inner function *is an analytic function* $\varphi : \mathbb{D} \longrightarrow \Gamma_E$ *such that* $\varphi(\lambda) \in b\Gamma_E$ *for almost all* $\lambda \in \mathbb{T}$.

An example for a $\Gamma_E$-inner function is $\varphi(z) = (z, z^2, z^3),\ z \in \mathbb{D}$.

**Definition 4.5.3** *A matrix valued function* $F$ *on* $\mathbb{D}$ *is said to be a* matricial inner function *if it is unitary almost everywhere on* $\mathbb{T}$.

**Theorem 4.5.4** *A function* $\varphi : \mathbb{D} \longrightarrow \Gamma_E$ *is a* $\Gamma_E$*-inner function if and only if there exists a* $2 \times 2$ *matricial inner function* $\psi : \mathbb{D} \longrightarrow M_2(\mathbb{C})$ *such* *that for all $\lambda \in \mathbb{D}$,*

$$\varphi(\lambda) = (\psi_{11}(\lambda), \psi_{22}(\lambda), \det \psi(\lambda)) .$$

*Moreover, if $\psi$ is rational, so is $\varphi$.*

**Proof** $(\Leftarrow)$ Suppose that there exists such a matricial inner function $\psi : \mathbb{D} \longrightarrow M_2(\mathbb{C})$. Then

$$\|\psi(\lambda)\| \leq 1, \text{ for all } \lambda \in \mathbb{D}.$$

Therefore, by Theorem 2.2.5, $\varphi$ is analytic and maps $\mathbb{D} \longrightarrow \Gamma_E$.
Since $\psi$ is an inner function, for almost all $\lambda \in \mathbb{T}$,

$$\psi(\lambda) = \begin{bmatrix} \psi_{11}(\lambda) & \psi_{12}(\lambda) \\ \psi_{21}(\lambda) & \psi_{22}(\lambda) \end{bmatrix} \text{ is unitary,}$$

which means that

$$\begin{aligned} \|\psi_{11}\|^2 + \|\psi_{21}\|^2 &= 1, \\ \|\psi_{12}\|^2 + \|\psi_{22}\|^2 &= 1, \\ \psi_{11}\bar{\psi}_{12} + \psi_{21}\bar{\psi}_{22} &= 0. \end{aligned}$$

We shall show that $\varphi(\lambda) \in b\Gamma_E$, for almost all $\lambda \in \mathbb{T}$. That is, for almost all $\lambda \in \mathbb{T}$,

$$(\psi_{11}(\lambda), \psi_{22}(\lambda), \det \psi(\lambda)) \in b\Gamma_E.$$

That is, we shall show that for almost all $\lambda \in \mathbb{T}$,

$$\| \det \ \psi(\lambda)\| = 1 \text{ and } \psi_{11}(\lambda) = \overline{\psi_{22}(\lambda)} \det \psi(\lambda).$$

Since $\psi$ is unitary, $\|\psi\| = 1$, therefore, $\| \det \psi(\lambda)\| = 1$, for almost all $\lambda \in \mathbb{T}$.

Moreover, since $\|\psi_{22}\|^2 + \|\psi_{12}\|^2 = 1$ and $\psi_{11}\|\psi_{12}\|^2 = -\bar{\psi}_{22}\psi_{12}\psi_{21}$, then

$$\begin{aligned}
\psi_{11} &= \psi_{11}(\|\psi_{22}\|^2 + \|\psi_{12}\|^2) \\
&= \psi_{11}\|\psi_{22}\|^2 + \psi_{11}\|\psi_{12}\|^2 \\
&= \psi_{11}\|\psi_{22}\|^2 - \bar{\psi}_{22}\psi_{12}\psi_{21} \\
&= \bar{\psi}_{22}(\psi_{11}\psi_{22} - \psi_{12}\psi_{21}) \\
&= \bar{\psi}_{22} \det \psi.
\end{aligned}$$

Therefore,

$$\varphi(\lambda) = (\psi_{11}(\lambda), \psi_{22}(\lambda), \det \psi(\lambda)) \in b\Gamma_E.$$

Thus, $\varphi$ is a $\Gamma_E$-inner function.

It is clear from the definition of $\varphi$ that if $\psi$ is rational then so is $\varphi$.

($\Rightarrow$) Conversely, suppose that $\varphi$ is a $\Gamma_E$-inner function. Construct the $2 \times 2$ Schur function $\psi$ exactly as in the proof of Theorem 2.2.5. Thus, we have $[\psi_{ij}]$ analytic such that $\|\psi\|_\infty \leq 1$, and $\varphi = (\psi_{11}, \psi_{22}, \det \psi)$.

Hence, there exists functions $\psi_{12}, \psi_{21} \in H^\infty$ such that

$$\|\psi_{12}\|^2 = \|\psi_{21}\|^2 = \|\psi_{11}\psi_{22} - \det \psi\| \text{ on } \mathbb{T},$$

where $\det \psi = \psi_{11}\psi_{22} - \psi_{12}\psi_{21}$.

Since $\varphi = (\psi_{11}, \psi_{22}, \det \psi)$ is a $\Gamma_E$-inner function, then for almost all $\lambda \in \mathbb{T}$,

$$\psi_{11}(\lambda) = \overline{\psi_{22}(\lambda)} \det \psi(\lambda) \text{ and } \|\det \psi(\lambda)\| = 1.$$

We shall show that $\psi$ is unitary almost everywhere. That is, we shall how that

$$\begin{aligned}
\|\psi_{11}\|^2 + \|\psi_{21}\|^2 &= 1, \\
\psi_{11}\bar{\psi}_{12} + \psi_{21}\bar{\psi}_{22} &= 0.
\end{aligned}$$

There are two cases; case (i): when $\psi_{11} \neq 0$, $\psi_{22} \neq 0$ and case (ii): when either $\psi_{11} = 0$ or $\psi_{22} = 0$.

**Case (i):** In the case that $\psi_{11} \neq 0$ and $\psi_{22} \neq 0$, we have $\varphi = (\psi_{11}, \psi_{22}, \det\psi)$ is a $\Gamma_E$-inner. Hence

$$\psi_{11} = \bar{\psi}_{22} \det\psi \Rightarrow \|\psi_{11}\|^2 = \|\psi_{22}\|^2.$$

We also have

$$\|\psi_{12}\|^2 = \|\psi_{21}\|^2 \text{ on } \mathbb{T}.$$

Therefore,

$$\|\psi_{11}\|^2 + \|\psi_{21}\|^2 = \|\psi_{12}\|^2 + \|\psi_{22}\|^2.$$

To show that $\psi$ is unitary, it is enough to show that $\|\psi_{11}\|^2 + \|\psi_{21}\|^2 = 1$, or that $\psi_{11}\bar{\psi}_{12} + \psi_{21}\bar{\psi}_{22} = 0$, because

$$\begin{aligned}
\psi_{11}\bar{\psi}_{12} + \psi_{21}\bar{\psi}_{22} = 0 \quad &\Leftrightarrow \quad \psi_{11}\|\psi_{12}\|^2 + \psi_{12}\psi_{21}\bar{\psi}_{22} = 0 \\
&\Leftrightarrow \quad \psi_{11}\|\psi_{12}\|^2 + \bar{\psi}_{22}(\psi_{11}\psi_{22} - \det\psi) = 0 \\
&\Leftrightarrow \quad \psi_{11}\|\psi_{12}\|^2 + \psi_{11}\|\psi_{22}\|^2 - \bar{\psi}_{22}\det\psi = 0 \\
&\Leftrightarrow \quad \psi_{11}\|\psi_{12}\|^2 + \psi_{11}\|\psi_{22}\|^2 - \psi_{11} = 0 \\
&\Leftrightarrow \quad \psi_{11}(\|\psi_{12}\|^2 + \|\psi_{22}\|^2 - 1) = 0 \\
&\Leftrightarrow \quad \|\psi_{12}\|^2 + \|\psi_{22}\|^2 = 1.
\end{aligned}$$

Since $\|\psi_{12}\|^2 = \|\psi_{21}\|^2 = \|\psi_{11}\psi_{22} - \det\psi\|$ on $\mathbb{T}$, $\psi_{11} = \bar{\psi}_{22}\det\psi$ and $\|\det\psi\| = 1$, then

$$\begin{aligned}
\|\psi_{22}\|^2 + \|\psi_{12}\|^2 \quad &= \quad \|\psi_{22}\|^2 + \|\psi_{11}\psi_{22} - \det\psi\| \\
&= \quad \|\psi_{22}\|^2 + \big\|\|\psi_{22}\|^2\det\psi - \det\psi\big\|
\end{aligned}$$

$$\begin{aligned}
\|\psi_{22}\|^2 + \|\psi_{12}\|^2 &= \|\psi_{22}\|^2 + \|\det\psi\| \left\|1 - \|\psi_{22}\|^2\right\| \\
&= \|\psi_{22}\|^2 + 1 - \|\psi_{22}\|^2 \\
&= 1.
\end{aligned}$$

Hence,

$$\|\psi_{11}\|^2 + \|\psi_{21}\|^2 = 1 = \|\psi_{12}\|^2 + \|\psi_{22}\|^2.$$

Therefore, $\psi$ is unitary, and hence, $\psi$ is an inner function.

**Case (ii):** In the case that either $\psi_{11} = 0$ or $\psi_{22} = 0$, it is easy to show that $\psi$ is an inner function, for, suppose that $\psi_{11} = 0$.
Since $\psi_{11} = \bar{\psi}_{22} \det\psi$ and $\|\det\psi\| = 1$, then

$$\psi_{22} = \bar{\psi}_{11} \det\psi.$$

Hence, $\psi_{11} = \psi_{22} = 0$. Moreover, when $\psi_{11} = \psi_{22} = 0$, we have

$$\|\psi_{12}\|^2 = \|\psi_{21}\|^2 = \|\det\psi\| = 1.$$

Therefore,

$$\|\psi_{11}\|^2 + \|\psi_{21}\|^2 = 1 = \|\psi_{12}\|^2 + \|\psi_{22}\|^2 \text{ and } \psi_{11}\bar{\psi}_{12} + \psi_{21}\bar{\psi}_{22} = 0.$$

Thus, $\psi$ is unitary, and hence, $\psi$ is an inner function.

□

**Theorem 4.5.5** *Let $x^{(j)} = \left(x_1^{(j)}, x_2^{(j)}, x_3^{(j)}\right) \in \Gamma_E$ and $\lambda_j \in \mathbb{D}$, $1 \le j \le n$. If there exists an analytic function $f : \mathbb{D} \longrightarrow \Gamma_E$ such that $f(\lambda_j) = x^{(j)}$, $1 \le j \le n$, then there exists a rational $\Gamma_E$-inner function $\psi : \mathbb{D} \longrightarrow \Gamma_E$ that satisfies the same interpolating conditions.*

**Proof** Let $x^{(j)} = \left(x_1^{(j)}, x_2^{(j)}, x_3^{(j)}\right) \in \Gamma_E$ and $\lambda_j \in \mathbb{D}$, $1 \leq j \leq n$. Suppose that there exists an analytic function $f : \mathbb{D} \longrightarrow \Gamma_E$ such that $f(\lambda_j) = x^{(j)}$, $1 \leq j \leq n$. Then by Theorem 2.3.1, there exists $b_j, c_j \in \mathbb{C}$ such that

$$b_j c_j = x_1^{(j)} x_2^{(j)} - x_3^{(j)},\ 1 \leq j \leq n,$$

$$\begin{bmatrix} x_1^{(j)} & b_j \\ c_j & x_2^{(j)} \end{bmatrix} \text{ are contractions}$$

and

$$\lambda_j \mapsto \begin{bmatrix} x_1^{(j)} & b_j \\ c_j & x_2^{(j)} \end{bmatrix},\ 1 \leq j \leq n$$

are solvable matricial Nevanlinna-Pick data. That is, there exists a solution to the matrix interpolation above. Since there exists a solution to the matricial interpolating problem, then there exists a solution that is rational and inner [2].

Therefore, by Theorem 4.5.4, there exists a rational $\Gamma_E$-inner solution that solves the interpolation problem.

□

Later in Theorem 4.5.7, we give a general formula for rational $\Gamma_E$-inner functions from $\mathbb{D}$ to $\Gamma_E$.

The next result follows from Theorem 3.1.1 and the definition of $\Gamma_E$-inner functions. In this result, we find a formula for a rational $\Gamma_E$-inner function $F : \mathbb{D} \longrightarrow \Gamma_E$ such that $F(0) = (0, 0, 0)$ and $F(\lambda_0) = (a, b, 0)$.

**Theorem 4.5.6** *Let $0 \leq b_0 < a_0 < 1 - b_0$ and let $\lambda_0 \in \mathbb{D}$. If there exists an analytic function $h : \mathbb{D} \longrightarrow G_E$ such that $h(0) = (0, 0, 0)$ and $h(\lambda_0) = (a_0, b_0, 0)$, then there exists a rational $\Gamma_E$-inner function*

$F = (F_1, F_2, F_3) : \mathbb{D} \longrightarrow \Gamma_E$, *satisfies that* $F(0) = (0,0,0)$ *and* $F(\lambda_0) = (a_0, b_0, 0)$. *This* $F$ *can be given as follows:*

$$F(\lambda) = \left( \frac{\lambda(1-2b_0)}{1-b_0-\lambda b_0\varphi(\lambda)}, \frac{\lambda\varphi(\lambda)(1-2b_0)}{1-b_0-\lambda b_0\varphi(\lambda)}, \frac{\lambda\left(\lambda\varphi(\lambda)(1-b_0)-b_0\right)}{1-b_0-\lambda b_0\varphi(\lambda)} \right),$$

*for all* $\lambda \in \mathbb{D}$, *where* $\varphi$ *is a scalar inner function such that* $\varphi(\lambda_0) = \sigma$, $0 \le \sigma \le 1$.

**Proof** Let $\lambda_0 \in \mathbb{D}$. Suppose there exists an analytic function $h : \mathbb{D} \longrightarrow G_E$ such that $h(0) = (0,0,0)$ and $h(\lambda_0) = (a_0, b_0, 0)$. Therefore, by Theorem 3.1.1, there exists an analytic function $f : \mathbb{D} \longrightarrow M_2(\mathbb{C})$ such that

$$f(0) = \begin{bmatrix} 0 & \zeta \\ 0 & 0 \end{bmatrix} := X_1 \text{ and } f(\lambda_0) = \begin{bmatrix} a_0 & \tau\sqrt{a_0b_0} \\ \tau^{-1}\sqrt{a_0b_0} & b_0 \end{bmatrix} := X_2.$$

where $\lambda_0 = \dfrac{a_0}{1-b_0}$, $\zeta = \sqrt{\dfrac{b_0}{1-b_0}} \in (0,1)$, $\tau = \sqrt{\dfrac{1-b_0}{a_0}} > 0$.
Let

$$f(\lambda) = \mathcal{M}_{-X_1}\left(\frac{\lambda}{\lambda_0}X_3\right), \ \forall \lambda \in \mathbb{D},$$

where

$$X_3 = \begin{bmatrix} \dfrac{-a_0\sqrt{1-2b_0}}{(1-b_0)\sqrt{1-b_0}} & \dfrac{b_0\sqrt{b_0}}{(1-b_0)\sqrt{1-b_0}} \\ \\ \dfrac{-a_0\sqrt{b_0}}{(1-b_0)\sqrt{1-b_0}} & \dfrac{-b_0\sqrt{1-2b_0}}{(1-b_0)\sqrt{1-b_0}} \end{bmatrix} = \mathcal{M}_{X_1}(X_2).$$

Let

$$\lambda_0^{-1}X_3 = U \begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix} V,$$

where $0 \le \sigma \le 1$ and $U, V$ are unitaries.
If $\sigma = 1$, then $\lambda_0^{-1}X_3$ is inner. Suppose that $\sigma < 1$. We are seeking a rational $\Gamma_E$-inner function $F$ such that $F(0) = X_1$ and $F(\lambda_0) = X_2$. Hence,

$$\mathcal{M}_{X_1} \circ F(0) = 0 \text{ and } \mathcal{M}_{X_1} \circ F(\lambda_0) = X_3.$$

We write $\mathcal{M}_{X_1} \circ F = \lambda G$, where $G$ is a Schur function such that

$$G(\lambda) = \lambda_0^{-1} X_3 = U \begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix} V.$$

Let

$$U^* G V^* = \begin{bmatrix} 1 & 0 \\ 0 & \varphi \end{bmatrix},$$

where $\varphi$ is a scalar inner function such that $\varphi(\lambda_0) = \sigma$. Let

$$\varphi(\lambda) = \frac{(\lambda - \lambda_0) - \sigma(\lambda\bar{\lambda}_0 - 1)}{\bar{\sigma}(\lambda - \lambda_0) - (\lambda\bar{\lambda}_0 - 1)}.$$

Clearly, $\varphi$ is an inner function that maps $\lambda_0 \mapsto \sigma$.

Therefore, we can take

$$F(\lambda) = \mathcal{M}_{-X_1}\left(\lambda\, U \begin{bmatrix} 1 & 0 \\ 0 & \varphi(\lambda) \end{bmatrix} V\right).$$

This $F$ is inner and satisfies $F(0) = X_1$ and $F(\lambda_0) = X_2$.

We shall find unitaries $U, V$ such that $\lambda_0^{-1} X_3 = U \begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix} V$.

We have $\lambda_0^{-1} = \dfrac{1 - b_0}{a_0}$. Hence,

$$\lambda_0^{-1} X_3 = \begin{bmatrix} \dfrac{-\sqrt{1-2b_0}}{\sqrt{1-b_0}} & \dfrac{b_0\sqrt{b_0}}{a_0\sqrt{1-b_0}} \\ \\ \dfrac{-\sqrt{b_0}}{\sqrt{1-b_0}} & \dfrac{-b_0\sqrt{1-2b_0}}{a_0\sqrt{1-b_0}} \end{bmatrix} := X_3^0.$$

Let $X_3^0 = UP$, where $U$ us a unitary and $P$ is a positive definite Hermitian, i.e., $P^* = P$.

Therefore,

$$\begin{aligned}
P &= \left(X_3^{0*}X_3^0\right)^{1/2} \\
&= \left(\begin{bmatrix} \dfrac{-\sqrt{1-2b_0}}{\sqrt{1-b_0}} & \dfrac{-\sqrt{b_0}}{\sqrt{1-b_0}} \\ & \\ \dfrac{b_0\sqrt{b_0}}{a_0\sqrt{1-b_0}} & \dfrac{-b_0\sqrt{1-2b_0}}{a_0\sqrt{1-b_0}} \end{bmatrix} \begin{bmatrix} \dfrac{-\sqrt{1-2b_0}}{\sqrt{1-b_0}} & \dfrac{b_0\sqrt{b_0}}{a_0\sqrt{1-b_0}} \\ & \\ \dfrac{-\sqrt{b_0}}{\sqrt{1-b_0}} & \dfrac{-b_0\sqrt{1-2b_0}}{a_0\sqrt{1-b_0}} \end{bmatrix}\right)^{1/2} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{b_0}{a_0} \end{bmatrix}.
\end{aligned}$$

Since $X_3^0 = UP$, then

$$\begin{aligned}
U &= X_3^0 P^{-1} \\
&= \begin{bmatrix} \dfrac{-\sqrt{1-2b_0}}{\sqrt{1-b_0}} & \dfrac{b_0\sqrt{b_0}}{a_0\sqrt{1-b_0}} \\ & \\ \dfrac{-\sqrt{b_0}}{\sqrt{1-b_0}} & \dfrac{-b_0\sqrt{1-2b_0}}{a_0\sqrt{1-b_0}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{a_0}{b_0} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{-\sqrt{1-2b_0}}{\sqrt{1-b_0}} & \dfrac{\sqrt{b_0}}{\sqrt{1-b_0}} \\ & \\ \dfrac{-\sqrt{b_0}}{\sqrt{1-b_0}} & \dfrac{-\sqrt{1-2b_0}}{\sqrt{1-b_0}} \end{bmatrix}.
\end{aligned}$$

Therefore, $V$ is the identity $2\times 2$ matrix and so $\lambda_0^{-1}X_3 = UPV$ as required.

Next, we construct a $\Gamma_E$-inner function $F = (F_1, F_2, F_3) : \mathbb{D} \longrightarrow \Gamma_E$ such that $F(0) = (0,0,0)$ and $F(\lambda_0) = (a_0, b_0, 0)$.

We have

$$F(\lambda) = \left(\lambda\, U \begin{bmatrix} 1 & 0 \\ 0 & \varphi(\lambda) \end{bmatrix} V\right)$$

$$
\begin{aligned}
F(\lambda) &= \lambda \begin{bmatrix} \dfrac{-\sqrt{1-2b_0}}{\sqrt{1-b_0}} & \dfrac{\sqrt{b_0}}{\sqrt{1-b_0}} \\ & \\ \dfrac{-\sqrt{b_0}}{\sqrt{1-b_0}} & \dfrac{-\sqrt{1-2b_0}}{\sqrt{1-b_0}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \varphi(\lambda) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \lambda \begin{bmatrix} \dfrac{-\sqrt{1-2b_0}}{\sqrt{1-b_0}} & \dfrac{-\sqrt{b_0}\varphi(\lambda)}{\sqrt{1-b_0}} \\ & \\ \dfrac{\sqrt{b_0}}{\sqrt{1-b_0}} & \dfrac{-\sqrt{1-2b_0}\varphi(\lambda)}{\sqrt{1-b_0}} \end{bmatrix} := \lambda \tilde{X}_3
\end{aligned}
$$

Therefore, after calculations we have

$$
\begin{aligned}
F(\lambda) &= \mathcal{M}_{-X_1}\left(\lambda \tilde{X}_3\right) \\
&= \begin{bmatrix} \dfrac{\lambda(1-2b_0)}{1-b_0-\lambda b_0\varphi(\lambda)} & \dfrac{-(1-\lambda\varphi(\lambda))\sqrt{b_0(1-b_0)}}{1-b_0-\lambda b_0\varphi(\lambda)} \\ & \\ \dfrac{-\lambda(1-\lambda\varphi(\lambda))\sqrt{b_0(1-b_0)}}{1-b_0-\lambda b_0\varphi(\lambda)} & \dfrac{\lambda\varphi(\lambda)(1-2b_0)}{1-b_0-\lambda b_0\varphi(\lambda)} \end{bmatrix},
\end{aligned}
$$

where $\lambda_0 = \dfrac{a_0}{1-b_0} < 1$, and $\varphi(\lambda_0) = \sigma = \dfrac{b_0}{a_0} < 1$.

Clearly,

$$
\begin{aligned}
F_1(\lambda) &= \frac{\lambda(1-2b_0)}{1-b_0-\lambda b_0\varphi(\lambda)}, \\
F_2(\lambda) &= \frac{\lambda\varphi(\lambda)(1-2b_0)}{1-b_0-\lambda b_0\varphi(\lambda)},
\end{aligned}
$$

and

$$
\begin{aligned}
F_3(\lambda) = \det F(\lambda) &= \frac{\lambda^2\varphi(\lambda)(1-2b_0)^2 - \lambda b_0(1-b_0)(1-\lambda\varphi(\lambda))^2}{(1-b_0-\lambda b_0\varphi(\lambda))^2} \\
&= \frac{\lambda(\lambda\varphi(\lambda)(1-b_0)-b_0)}{1-b_0-\lambda b_0\varphi(\lambda)}.
\end{aligned}
$$

Therefore,

$$F(\lambda) = \left( \frac{\lambda(1-2b_0)}{1-b_0-\lambda b_0\varphi(\lambda)}, \frac{\lambda\varphi(\lambda)(1-2b_0)}{1-b_0-\lambda b_0\varphi(\lambda)}, \frac{\lambda(\lambda\varphi(\lambda)(1-b_0)-b_0)}{1-b_0-\lambda b_0\varphi(\lambda)} \right).$$

Clearly, $F_1(0) = F_2(0) = F_3(0) = 0$, and

$$\begin{aligned} F_1(\lambda_0) &= \frac{a_0(1-2b_0)}{(1-b_0)^2 - b_0^2} = a_0, \\ F_2(\lambda_0) &= \frac{b_0(1-2b_0)}{(1-b_0)^2 - b_0^2} = b_0, \\ F_3(\lambda_0) &= \frac{a_0(b_0-b_0)}{(1-b_0)^2 - b_0^2} = 0. \end{aligned}$$

Note that $F$ is a $\Gamma_E$-inner function, because, for almost all $\lambda \in \mathbb{T}$,

$$\begin{aligned} |F_3(\lambda)| &= \frac{|\lambda| \left| \lambda\varphi(\lambda) - \dfrac{b_0}{1-b_0} \right|}{\left| 1 - \lambda\varphi(\lambda)\dfrac{b_0}{1-b_0} \right|} \\ &= \left| \frac{\bar{\lambda}\bar{\varphi(\lambda)} - \dfrac{b_0}{1-b_0}}{1-\lambda\varphi(\lambda)\dfrac{b_0}{1-b_0}} \right| \\ &= \left| \frac{\dfrac{1}{\lambda}\dfrac{1}{\varphi(\lambda)} - \dfrac{b_0}{1-b_0}}{1-\lambda\varphi(\lambda)\dfrac{b_0}{1-b_0}} \right| \\ &= \left| \frac{1}{\lambda} \right| \left| \frac{1}{\varphi(\lambda)} \right| \left| \frac{1-\lambda\varphi(\lambda)\dfrac{b_0}{1-b_0}}{1-\lambda\varphi(\lambda)\dfrac{b_0}{1-b_0}} \right| \\ &= 1. \end{aligned}$$

Moreover, $F_1(\lambda) = \overline{F_2(\lambda)} F_3(\lambda)$ because

$$
\begin{aligned}
\overline{F_2(\lambda)}F_3(\lambda) &= \frac{\bar{\lambda}\overline{\varphi(\lambda)}(1-2b_0)}{1-b_0-\bar{\lambda}b_0\bar{\varphi(\lambda)}}\frac{\lambda(\lambda\varphi(\lambda)(1-b_0)-b_0)}{1-b_0-\lambda b_0\varphi(\lambda)} \\
&= \frac{|\lambda|^2\dfrac{1}{\varphi(\lambda)}(1-2b_0)\left(\lambda\varphi(\lambda)(1-b_0)-b_0\right)}{\left(1-b_0-\dfrac{b_0}{\lambda\varphi(\lambda)}\right)\left(1-b_0-\lambda b_0\varphi(\lambda)\right)} \\
&= \frac{\lambda(1-2b_0)}{1-b_0-\lambda b_0\varphi(\lambda)} \\
&= F_1(\lambda).
\end{aligned}
$$

Hence, $F$ is a rational $\Gamma_E$-inner function.

□

In the next result, we use the following notations; $\mu$ denotes a Möbius automorphism of $\mathbb{D}$ and $B$ denotes a Blaschke product, that is,

$$B_a(z) = \zeta\prod_{i=1}^{n}\frac{z-a_i}{\bar{a}_i z-1},$$

where $\zeta \in \mathbb{T}$ and $a_i \in \mathbb{D}$.

**Theorem 4.5.7** *If $x = (x_1, x_2, x_3) : \mathbb{D} \longrightarrow \Gamma_E$ is a rational $\Gamma_E$-inner function, then it is of the form:*

$$x(\lambda) = \left(B_1(\lambda)x_1(\lambda), B_2(\lambda)x_2(\lambda), B_1(\lambda)B_2(\lambda)B(\lambda)\right),$$

*for all $\lambda \in \mathbb{D}$, where $B_1, B_2, B$ are Blaschke products,*

$$
\begin{aligned}
x_1(\lambda) &= \alpha\frac{\prod_{i=1}^{m'}(\lambda-z_i)}{\prod_{j=1}^{n'}(\lambda-p_j)}, \quad \textit{and} \\
x_2(\lambda) &= \beta\frac{\prod_{i=1}^{m}(\lambda-w_i)}{\prod_{j=1}^{n}(\lambda-q_j)},
\end{aligned}
$$

*where $\alpha, \beta \in \mathbb{T}$, $|p_j|$, $|q_j| > 1$, $z_i, w_i \in \mathbb{C}$ and $m', m, n', n \in \mathbb{N}$.*

**Proof** We shall assume that $x_1(\lambda)$, $x_2(\lambda)$ and $x_3(\lambda)$ have no roots at zero by picking a Möbius automorphism $\mu$ so that $x_1(\tilde{\lambda})$, $x_2(\tilde{\lambda})$ and $x_3(\tilde{\lambda})$ have no root at zero, where $\tilde{\lambda} = \mu(\lambda)$. During the proof of this theorem, we shall write $\lambda$ instead of $\tilde{\lambda}$. Note that later in the proof, we shall use a Blaschke product $B_1(\lambda)$ that makes $x_1(\lambda)$ and $x_3(\lambda)$ have no common zero in $\mathbb{D}$ and a Blaschke product $B_2(\lambda)$ that makes $x_2(\lambda)$ and $x_3(\lambda)$ have no common zero in $\mathbb{D}$.

Now, we are seeking a rational $\Gamma_E$-inner function

$$x(\lambda) = (x_1(\lambda), x_2(\lambda), x_3(\lambda)) : \mathbb{D} \longrightarrow \Gamma_E.$$

Since $x$ is a $\Gamma_E$-inner function, then $x(\lambda) \in b\Gamma_E$ for almost all $\lambda \in \mathbb{T}$. Therefore, $x_1 = \bar{x}_2 x_3$ and $|x_3(\lambda)| = 1$, for almost all $\lambda \in \mathbb{T}$. Thus, $x_3$ has to be a Blaschke product. That is, $x_3(\lambda) = B(\lambda)$, where $\lambda \in \mathbb{D}$.

Let

$$\begin{aligned} x_1(\lambda) &= \alpha \frac{\prod_{i=1}^{m'}(\lambda - z_i)}{\prod_{j=1}^{n'}(\lambda - p_j)}, \text{ and} \\ x_2(\lambda) &= \beta \frac{\prod_{i=1}^{m}(\lambda - w_i)}{\prod_{j=1}^{n}(\lambda - q_j)}, \end{aligned}$$

where $|p_j|,\ |q_j| > 1$ and $z_i, w_i \in \mathbb{C}$. We have

$$\begin{aligned} x_1 = \bar{x}_2 x_3 \text{ on } \mathbb{T} \ &\Rightarrow\ x_1(\lambda) = \overline{x_2(\lambda)} x_3(\lambda),\ |\lambda| = 1 \\ &\Rightarrow\ x_1(\lambda) = \overline{x_2(\lambda)} B(\lambda),\ |\lambda| = 1 \\ &\Rightarrow\ \alpha \frac{\prod_{i=1}^{m'}(\lambda - z_i)}{\prod_{j=1}^{n'}(\lambda - p_j)} = \beta \overline{\frac{\prod_{i=1}^{m}(\lambda - w_i)}{\prod_{j=1}^{n}(\lambda - q_j)}} B_a(\lambda),\ |\lambda| = 1 \end{aligned}$$

$$
\begin{aligned}
x_1 = \bar{x}_2 x_3 \text{ on } \mathbb{T} \quad \Rightarrow \quad & \alpha \frac{\prod_{i=1}^{m'} (\lambda - z_i)}{\prod_{j=1}^{n'} (\lambda - p_j)} = \bar{\beta} \frac{\prod_{i=1}^{m} (\bar{\lambda} - \bar{w}_i)}{\prod_{j=1}^{n} (\bar{\lambda} - \bar{q}_j)} B_a(\lambda), \ \bar{\lambda} = \frac{1}{\lambda} \\
\Rightarrow \quad & \alpha \frac{\prod_{i=1}^{m'} (\lambda - z_i)}{\prod_{j=1}^{n'} (\lambda - p_j)} = \bar{\beta} \frac{\prod_{i=1}^{m} \left( \frac{1}{\lambda} - \bar{w}_i \right)}{\prod_{j=1}^{n} \left( \frac{1}{\lambda} - \bar{q}_j \right)} B_a(\lambda), \ \lambda \in \mathbb{T} \\
\Rightarrow \quad & \alpha \frac{\prod_{i=i}^{m'} (\lambda - z_i)}{\prod_{j=1}^{n'} (\lambda - p_j)} = \frac{\bar{\beta} \lambda^{-m} \prod_{i=1}^{m} (1 - \lambda \bar{w}_i)}{\lambda^{-n} \prod_{j=1}^{n} (1 - \lambda \bar{q}_j)} B_a(\lambda), \ \lambda \in \mathbb{T} \\
\Rightarrow \quad & \alpha \frac{\prod_{i=1}^{m'} (\lambda - z_i)}{\prod_{j=1}^{n'} (\lambda - p_j)} = \bar{\beta} \frac{\lambda^{-m}}{\lambda^{-n}} \frac{\prod_{i=1}^{m} (1 - \lambda \bar{w}_i)}{\prod_{j=1}^{n} (1 - \lambda \bar{q}_j)} \, \zeta \prod_{k=1}^{n''} \frac{\lambda - a_k}{1 - \bar{a}_k \lambda}, \ \lambda, \zeta \in \mathbb{T},
\end{aligned}
$$

where $|a_k| < 1$.

We may assume that $x_1(\lambda)$ and $x_3(\lambda)$ have no common zeros in $\mathbb{D}$, for all $\lambda \in \mathbb{D}$. Otherwise, we can take $x_1(w) = 0 = x_3(w)$, $w \in \mathbb{D}$.

Now consider

$$
x'(\lambda) = \left( x_1(\lambda) \left( \frac{z\bar{w} - 1}{z - w} \right), x_2(\lambda), x_3(\lambda) \left( \frac{z\bar{w} - 1}{z - w} \right) \right),
$$

where $z, w \in \mathbb{D}$. Clearly, $x'(\lambda) : \mathbb{D} \longrightarrow \Gamma_E$, for

$$
\begin{aligned}
& |x_2(\lambda)|^2 + \left| \frac{z\bar{w} - 1}{z - w} \right| |x_1(\lambda) - \overline{x_2(\lambda)} x_3(\lambda)| + \left| \frac{z\bar{w} - 1}{z - w} \right| |x_1(\lambda) x_2(\lambda) - x_3(\lambda)| \\
= \ & |x_2(\lambda)|^2 + |x_1(\lambda) - \overline{x_2(\lambda)} x_3(\lambda)| + |x_1(\lambda) x_2(\lambda) - x_3(\lambda)| \\
\leq \ & 1
\end{aligned}
$$

because $x(\lambda) = (x_1(\lambda), x_2(\lambda), x_3(\lambda)) : \mathbb{D} \longrightarrow \Gamma_E$.

Moreover, since $x(\lambda)$ is a $\Gamma_E$-inner function, then $x'(\lambda) : \mathbb{D} \longrightarrow \Gamma_E$ is also a $\Gamma_E$-inner function, for

$$
\left| x_3(\lambda) \left( \frac{z\bar{w} - 1}{z - w} \right) \right| = |x_3(\lambda)| = 1,
$$

and

$$x_1(\lambda)\left(\frac{z\bar{w}-1}{z-w}\right) = \overline{x_2(\lambda)}x_3(\lambda)\left(\frac{z\bar{w}-1}{z-w}\right) \Rightarrow x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda).$$

Therefore, we have

$$\alpha\frac{\prod_{i=1}^{m'}(\lambda - z_i)}{\prod_{j=1}^{n'}(\lambda - p_j)} = \bar{\beta}\frac{\lambda^{-m}}{\lambda^{-n}}\frac{\prod_{i=1}^{m}(1-\lambda\bar{w}_i)}{\prod_{j=1}^{n}(1-\lambda\bar{q}_j)}\,\zeta\prod_{i=1}^{n''}\frac{\lambda - a_i}{1-\bar{a}_i\lambda}, \tag{4.8}$$

where $|p_j| > 1$, $|q_j| > 1$ and $|a_i| < 1$.

Note that, unless $\lambda$ is a root of unity, $\dfrac{\lambda^{-m}}{\lambda^{-n}} = 1$ implies that $m = n$.

Observe the following:

(1) All $\lambda = a_i$ are not roots of the left hand side by reduction hypothesis (that $x_1$ and $x_3$ have no common roots), thus, $\lambda - a_i$ cancels $\dfrac{1}{1-\lambda\bar{q}_j}$, and therefore, $a_i = \dfrac{1}{\bar{q}_j}$.

(2) All $\dfrac{1}{1-\lambda\bar{q}_j}$ are poles on the right hand side in $\mathbb{D}$, so $\lambda - \dfrac{1}{q_j}$ must cancel with $\lambda - a_i$.

(3) All $\lambda = p_j$ are poles of the left hand side, so it must be $\dfrac{1}{1-\lambda\bar{a}_i}$. Hence, $p_i$'s are $\dfrac{1}{\bar{a}_i}$'s. Moreover, all $\lambda\bar{a}_i = 1$ are poles of the right hand side, so $\dfrac{1}{1-\lambda\bar{a}_i}$ must cancel $\dfrac{1}{\lambda - p_i}$.

(4) All $\lambda = z_i \in \mathbb{D}$ are zeros of the left hand side by reduction hypothesis (that $x_1$ and $x_3$ have no common root), so they are not $a_i$'s, thus, $z_i$'s are $\dfrac{1}{\bar{w}_i}$'s. If $z_i$ is outside $\mathbb{D}$ or on the unit circle $\mathbb{T}$, then $z_i$ has to be $\dfrac{1}{\bar{w}_i}$, because $B_a(\lambda)$ is a Blaschke product which means that it has its zeros in $\mathbb{D}$.

From (1) and (2), we have that all $\dfrac{1}{\bar{q}_i}$'s are $a_i$'s. Hence, (3) implies that all $p_j$'s are $q_j$'s.

Therefore, equation (4.8) can be written as follows:

$$\alpha \frac{\prod_{i=1}^{n}(\lambda \bar{w}_i - 1)}{\prod_{i=1}^{n}(\lambda - p_i)} = \bar{\beta} \frac{\prod_{i=1}^{n}(1 - \lambda \bar{w}_i)}{\prod_{i=1}^{n}(1 - \lambda \bar{p}_i)} \, \zeta \prod_{i=1}^{n} \left( \frac{\lambda \bar{p}_i - 1}{\lambda - \bar{p}_i} \right), \tag{4.9}$$

where $|p_i| > 1$ and $\zeta \in \mathbb{T}$.

Similarly, we assume that $x_2(\lambda)$ and $x_3(\lambda)$ have no common zero in $\mathbb{D}$, where $\lambda \in \mathbb{D}$ and find that

$$\beta \frac{\prod_{i=1}^{n}(\lambda \bar{z}_i - 1)}{\prod_{i=1}^{n}(\lambda - p_i)} = \bar{\alpha} \frac{\prod_{i=1}^{n}(1 - \lambda \bar{z}_i)}{\prod_{i=1}^{n}(1 - \lambda \bar{p}_i)} \, \zeta \prod_{i=1}^{n} \left( \frac{\lambda \bar{p}_i - 1}{\lambda - \bar{p}_i} \right), \tag{4.10}$$

where $|p_i| > 1$ and $\zeta \in \mathbb{T}$.

Therefore, from equations (4.9) and (4.10), we find that all $w_i$'s are $z_i$'s.

Therefore, a general $\Gamma_E$-inner function is $x = (x_1, x_2, x_3) : \mathbb{D} \longrightarrow \Gamma_E$, where

$$\begin{aligned}
x_1(\lambda) &= \alpha \frac{\prod_{i=1}^{n}(\lambda \bar{w}_i - 1)}{\prod_{i=1}^{n}(\lambda - p_i)}, \\
\overline{x_2(\lambda)} &= \bar{\beta} \frac{\prod_{i=1}^{n}(1 - \lambda \bar{w}_i)}{\prod_{i=1}^{n}(1 - \lambda \bar{p}_i)} \Rightarrow x_2(\lambda) = \beta \frac{\prod_{i=1}^{n}(\lambda - z_i)}{\prod_{i=1}^{n}(\lambda - q_i)}, \\
x_3(\lambda) &= \zeta \prod_{i=1}^{n} \left( \frac{\lambda \bar{p}_i - 1}{\lambda - \bar{p}_i} \right).
\end{aligned}$$

Thus, a general $\Gamma_E$-inner function $x : \mathbb{D} \longrightarrow \Gamma_E$ is given by:

$$x(\tilde{\lambda}) = \left( B_1(\tilde{\lambda}) x_1(\tilde{\lambda}), B_2(\tilde{\lambda}) x_2(\tilde{\lambda}), B_1(\tilde{\lambda}) B_2(\tilde{\lambda}) B(\tilde{\lambda}) \right),$$

where $\mu(\lambda) = \tilde{\lambda}$ is the Möbius automorphism so that $x_1(\lambda)$, $x_2(\lambda)$ and $x_3(\lambda)$ have no root at zero, $B_1(\tilde{\lambda})$ is a Blaschke product that makes $x_1(\lambda)$ and

$x_3(\lambda)$ have no common zero in $\mathbb{D}$ and $B_2(\tilde{\lambda})$ is a Blaschke product that makes $x_2(\lambda)$ and $x_3(\lambda)$ have no common zero in $\mathbb{D}$.

Note that, $B_1(\tilde{\lambda}) = B_1(\mu(\lambda))$ is a Blaschke product, for

$$\begin{aligned} B_1(\mu(\lambda)) &= \zeta \prod_{i=1}^{n} \mu_i(\mu(\lambda)) \\ &= \zeta \prod_{i=1}^{n} (\mu_i \circ \mu)(\lambda) \\ &= \zeta \prod_{i=1}^{n} \tilde{\mu_i}(\lambda), \end{aligned}$$

where $\tilde{\mu_i}$ is a Möbius automorphism of $\mathbb{D}$. Also, $B_2(\tilde{\lambda})$ is a Blaschke product.

Since $\mu(\lambda) = \tilde{\lambda}$, then $\dfrac{\tilde{\lambda} - a_i}{\tilde{\lambda} - q_i}$ has roots at

$$\tilde{\lambda} = \mu(\lambda) = a_i, \text{ that is, at } \lambda = \mu^{-1}(a_i)$$

and has poles at

$$\tilde{\lambda} = \mu(\lambda) = q_i, \text{ that is, at } \lambda = \mu^{-1}(q_i).$$

Therefore, we can write the general $\Gamma_E$-inner function $x = (x_1, x_2, x_3) : \mathbb{D} \longrightarrow \Gamma_E$ as follows:

$$x(\lambda) = \left(B_1(\lambda)x_1(\lambda), B_2(\lambda)x_2(\lambda), B_1(\lambda)B_2(\lambda)B(\lambda)\right),$$

where $\lambda \in \mathbb{D}$.

Note that in the case that $z_i = 0$ or $w_i = 0$, we have $m \neq n$ in equation (4.8), hence, $\lambda$ on both sides will cancel some of $\dfrac{\lambda^{-m}}{\lambda^{-n}}$.

$\square$

**Example** From Theorem 4.5.6 an example of a $\Gamma_E$-inner function, $f : \mathbb{D} \longrightarrow \Gamma_E$ such that $f(0) = (0,0,0)$ and $f(\lambda_0) = (a, b, 0)$, where $\lambda_0 \in \mathbb{D}$, is given by

$$f(\lambda) = \left( \frac{\lambda(1-2b)}{1-b-\lambda b\varphi(\lambda)}, \frac{\lambda\varphi(\lambda)(1-2b)}{1-b-\lambda b\varphi(\lambda)}, \frac{\lambda\left(\lambda\varphi(\lambda)(1-b)-b\right)}{1-b-\lambda b\varphi(\lambda)} \right),$$

for all $\lambda \in \mathbb{D}$, where $\varphi$ is a scalar inner function such that $\varphi(\lambda_0) = \sigma$, where $0 \leq \sigma \leq 1$.

# Chapter 5

# Areas for Further Study

This project has touched a number of different mathematical and engineering areas, including interpolation theory, complex geometry, linear systems and control engineering. There are many questions that arise naturally as a consequence of our work on this new set $\Gamma_E$.

We draw comparisons mainly with the work of Agler and Young because we adopted their approach to derive most of our results.

It seems a natural question to ask, when presented with a necessary condition for the existence of an analytic function that maps the disc into $\Gamma_E$, as in Corollary 2.2.2, if this condition is sufficient. We know that it is not, but what if we added more conditions, would that make them sufficient? We do not have an answer to this question. We stated Question 2.2.4 on what we believe are sufficient conditions for interpolation from the disc into $\Gamma_E$. In the case of the symmetrised bidisc $\Gamma$, Agler and Young know that an analogous sufficient condition fails to hold in general but it does hold when $n = 2$, they have provided a proof of this in [9].

After proving a Schwarz Lemma for $\Gamma_E$ in the case that one of the points is $(0,0,0)$, we show that in this case, the Carathéodory and the Kobayashi distances between two points in $G_E$ are equal. It would be interesting to know if the Carathéodory and the Kobayashi distances between any pair of points in $\Gamma_E$ are equal or not. By using Möbius automorphisms, we can send a point $(a,b,p) \in \Gamma_E$ to $(0,0,0)$, but we can only do this in the case that $ab = p$, which corresponds to $A$ being a triangular matrix. This way we show that the Carathéodory and the Kobayashi distances are equal between any two points in $G_E$ such that $ab = p$.

We believe that we have found all the automorphisms of $G_E$, but since we do not have a proof to support our claim, this remains open for study.

In this project, we concentrated on studying $\Gamma_E$. One might consider $\Sigma_E$, we expect that our results can be lifted to this domain. Another interesting set to consider is $\Gamma_E$, where $E$ is an upper triangular matrix of the form $\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$, where $\lambda \in \mathbb{D}$.

Our problem relates to robust stabilisation and interpolation, we hope that the results of this project will make a significant contribution to the field by throwing light on a hard, concrete special case.

# Bibliography


[1] J. Agler, *On the Representation of Certain Holomorphic Functions Defined on a Polydisc,* Oper. Theory Adv. Appl., Vol. 48 (1990), 47-66.

[2] Jim Agler and John E. McCarthy, *Pick Interpolation and Hilbert Function Spaces,* American Mathematical Society, (2000).

[3] J. Agler and N.J. Young, *A Commutant Lifting Theorem for a Domain in $\mathbb{C}^2$ and Spectral Interpolation,* J. Funct. Anal., Vol. 161, No. 2 (1999), 452-477.

[4] J. Agler and N.J. Young, *Operators Having the Symmetrized Bidisc as a Spectral Set,* Proc. Edinburgh Math. Soc., (2) Vol. 43, No.1 (2000), 195-210.

[5] J. Agler and N.J. Young, *The Two-Point Spectral Nevanlinna-Pick Problem,* Integral Equations Operator Theory, Vol. 37, No. 4 (2000), 375-385.

[6] J. Agler and N.J. Young, *A Schwarz Lemma for the Symmetrized Bidisc,* Bull. London Math. Soc., Vol. 33, No. 2 (2001), 175-186.

[7] J. Agler and N. J. Young, *A Model Theory for $\Gamma$-Contractions,* J. Operator Theory, Vol. 49, No. 1 (2003), 45-60.

[8] J. Agler and N. J. Young, *The Two-by-Two Spectral Nevanlinna-Pick Problem,* Trans. Amer. Math. Soc., Vol. 356, No. 2 (2004), 573-585.

[9] J. Agler and N. J. Young, *The Hyperbolic Geometry of the Symmetrized Bidisc,* J. Geom. Anal., Vol. 14, No. 3 (2004), 375-403.

[10] H. Bercovici, C. Foias and A. Tannenbaum, *A Spectral Commuting Lifting Theorem,* Trans. Amer. Math. Soc., Vol. 325, No. 2 (1991), 741-763.

[11] Hari Bercovici, *Spectral Versus Classical Nevanlinna-Pick Interpolation in Dimension Two,* Electron. J. Linear Algebra, Vol. 10 (2003), 60-64.

[12] R. Y. Chiang and M. G. Safonov, *Robust-Control Toolbox for Use with MATLAB: User's Guide,* Mathworks, Inc., (1992).

[13] Constantin Costara, *The $2\times 2$ Spectral Nevanlinna-Pick Problem,* J. London Math. Soc., Vol. 71, No. 2, (2005), 684-702.

[14] S. Dineen, *The Schwarz Lemma,* Oxford University Press, (1989).

[15] J. C. Doyle, B. A. Francis and A. R. Tannenbaum, *Feedback Control Theory,* Macmillan, New York, (1992).

[16] J. Doyle and A. Packard, *The Complex Structured Singular Value,* Automatica J. IFAC, Vol. 29, No. 1 (1993), 71-109.

[17] P. L. Duren, *Theory of $H^p$ Spaces,* Academic Press, New York, (1970).

[18] A. Edigarian and W. Zwonek *Geometry of the Symmetrized Polydisc,* Arch. Math. (Basel), Vol. 84, No. 4 (2005), 364-374.

[19] A. Feintuch and A. Markus, *The Structured Norm of a Hilbert Space Operator With Respect to a Given Algebra of Operators,* Oper. Theory Adv. Appl., Vol. 115 (2000), 163-183.

[20] C. Foias and A. E. Frazho, *The Commutant Lifting Approach to Interpolation Problems,* Birkhäuser Verlag, Basel, (1990).

[21] T. W. Gamelin *Uniform Algebras,* Prentice-Hall, Inc., Englewood Cliffs, N.J., (1969).

[22] R. A. Horn and C. R. Johnson, *Matrix Analysis,* Cambridge University Press, Cambridge, (1999).

[23] S. G. Krantz, *Function Theory of Several Complex Variables,* Wadsworth and Brooks/Cole Advances Books and Software, Pacific Grove, Calif., (1992).

[24] B. Sz-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space,* North-Holland Pub. Co., Amsterdam, (1970).

[25] David Ogle, *Operator and Function Theory of the Symmetrized Polydisc,* PhD Thesis, University of Newcastle upon Tyne, Newcastle upon Tyne, (1999).

[26] A. D. Osborne, *Complex Variables and their Applications,* Addison Wesley Longman, Harlow, England, (1999).

[27] J. R. Partington, *Interpolation, Identification, and Sampling,* New Series, Vol. 17 of LMS Monographs, Oxford University Press, Oxford, (1997).

[28] V. Paulsen, *Completely Bounded Maps and Operator Algebras,* Cambridge University Press, New York, (2002).

[29] N. Young, *An Introduction to Hilbert Space*, Cambridge University Press, Cambridge, (1988).