Low Energy Quantum Gravity

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The gravitational force is the oldest force known to man and the least understood.(90)

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Abstract

This thesis investigates two very different aspects of quantum gravity.

In the first - and main - section, we examine the question of quantum gravitational contributions to the running of a coupling parameter alongside the various problems and issues that this raises. We treat quantum gravity as an effective field theory and use perturbative methods to address issues. Specifically, we look at a $\lambda\varphi^4$-type scalar coupling. In a gauge-invariant way, we consider a non-minimally coupled, massive scalar field, with non-constant background, in the presence of a cosmological constant and contrary to most of the literature, we also calculate all derivative terms. An effective action is constructed, renormalization counterterms calculated, and we find that, within certain bounds, gravity leads to asymptotic freedom of scalar field theory.

Furthermore, we investigate whether considering quadratic divergences in gravitational calculations can tell us anything useful. In this case we find non-vanishing quadratic divergences. However, we also recognise the possibility that quadratic divergences are somewhat of a red herring and that by suitable field redefinitions, we can eliminate these from our calculations.

The second section of the thesis addresses the possibility of superfluidity in a quark gluon plasma. We use the framework of AdS/CFT, with knowledge of black hole thermodynamics, to consider the duality between a black hole in anti-de Sitter space and a fluid existing on the boundary. Initially, we look at a simple case of a black hole possessing only mass and charge in AdS spacetime and calculate such properties as the entropy, temperature and specific heat capacity, identifying a telltale sign of a phase change (specific heat capacity tending to infinity) and of points of vanishing viscosity (corresponding with a zero entropy). After confirming that such a boundary exists, we take a different approach where we calculate and interpret the solutions to a relativistic Gross-Pitaevskii equation on a sphere. On projection back to R^3, the solutions are seen to be tori, which we choose

to interpret as vortex rings in analogy to the expected feature of those which are known to appear in a real superfluids.

Contents

List of Figures

Part I

Quantum Gravitational Corrections to Scalar Field Theory

Chapter 1

Introduction

1.1 Preface

A complete theory of quantum gravity is currently unknown. Although we have a good understanding of small length scales provided to us by the Standard Model of particle physics, and similarly at large length scales an understanding courtesy of Einstein's general relativity, efforts to find a single theory of quantum gravity are met abruptly with the problem of non-renormalizability.

The first analysis was performed in the classic paper of 't Hooft and Veltman (1) in which they considered a scalar field coupled to gravity at one-loop order in a traditional background field method and encountered the non-renormalizability of gravity. Later work by Deser and van Nieuwenhuizen (2), (3), (4) examined the coupling of gravity to an electromagnetic field and to fermions and the same problem appeared. Although higher derivative gravity was found to be renormalizable (8), it was discarded as a proper description of gravity because it was not found to be unitary.

Whilst a theory such as M-theory or loop quantum gravity may yet provide hope of progress towards a complete theory, one may take the viewpoint that the high energies where these theories should start to be uniquely testable, likely the order of the Planck mass, M_P ($\approx 10^{19}$ GeV), may never be attainable (cf. the highest energies available to us at the Large Hadron Collider $\approx 10^4$ GeV) and as such study of the combination of quantum mechanics and gravity in a low energy limit may be more insightful.

That said, it was Donoghue (5)(6) who first realised that it was possible to treat gravity as an effective field theory. Here, using a perturbative approach, increasing order terms are tamed by increasing powers of the Planck mass (a comprehensive review can be found in (7)).

The flaw is obvious of course: as we approach energies of the order of the Planck mass, the effective theory will break down. However, as long as we restrict ourselves to energies $E << M_P$, we find a theory that allows us to make quantitative predictions on the quantum gravitational corrections to whichever field theory we require. Indeed, it has been suggested that quantum gravitational effects could be found at energies on the order of those achievable at the LHC (12), (16), (13).

Moreover, we can take a more Popperian viewpoint (51). We suggest that in light of the falsafiability of the effective field theory approach juxtaposed with a model such as string theory which is not accessible to testing (or, more accurately, very hard to test as discussed above), we might suggest that effective field theory is the more complete model.

Building on the framework of Donoghue, an important result by Robinson and Wilczek (9) came to light which suggested that by coupling gravity to a Yang-Mills field and allowing the gauge coupling to run, the effect of gravity is such that we are led to asymptotic freedom (the shape of the running was given by them in Figure 1.1; Gogoladze et al (12), gave a slightly different graph in their work (see Figure 1.2)). That is to say that the gauge coupling vanishes for sufficiently high (but crucially sub-Planck scale) energy. The analagous situation of asymptotic freedom in the absence of gravity is, of course, well known and celebrated (10; 11). The result is most significant due to the completely different behaviour for theories which are not normally asymptotically free, such as QED and φ^4-theory.

[0]We shall be liberal when referring to *couplings*, *coupling constants* and *coupling parameters*, which will all be taken to mean the same idea of the strength of the interaction under investigation.

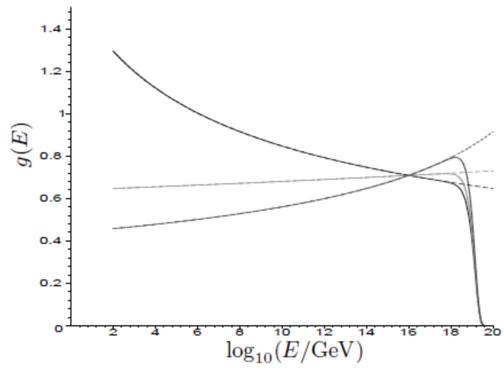

Figure 1.1: Robinson and Wilczek showed how the various couplings of the standard model would all become asymptotically free at an energy a few orders of magnitude below the Planck scale.

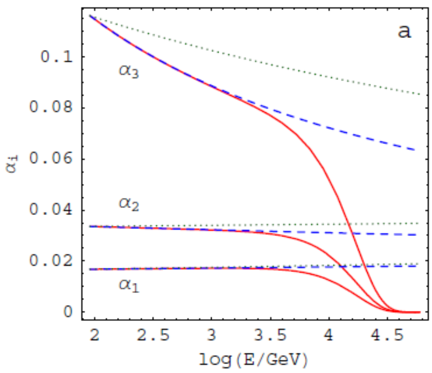

Figure 1.2: Gogoladze and Leung predicted the form of the running of the standard model couplings.

The result of (9) is very significant for numerous phenomenological reasons. Naturally, there is interest in any result which offers us an alternative to the typically unwieldy methods for many particle interactions at high energy. The ability to treat any gauge theory as a free theory in the presence of gravity would allow huge simplifications.

There are also potential implications for cosmology, particularly in the very early universe where one might imagine gravity to be comparative in strength to the other forces and where high particle densities are very difficult to deal with numerically.

Therefore, if we are able to grasp a better knowledge of the techniques employed in the field of running couplings and develop an understanding of the pitfalls which we must be wary of, then we could feasibly have a model with widespread application in particle physics phenomenology.

Alas, subsequent work by Pietrykowski (18) cast doubt on this result. By performing the calculation in a different choice of gauge, then arriving at a different result, he clearly demonstrated that the result of (9) was gauge-dependent. Several other papers using different approaches followed which all led to different conclusions, and also confirmed this gauge-dependency. Amongst these was the work of Ebert et al (20) who performed a traditional Feynman diagram calculation considering those diagrams which involved a coupling between the graviton and gauge fields, and thus would account for a difference with the known non-gravitational result. A paper by Tang and Wu (21) used a new technique known as loop regularization which first took into account quadratic divergences concluding that quadratic divergences were non-zero and should not be discarded when considering running of gauge parameters. Work by Toms (19) tried to clarify the situation by adopting a different approach which was inherently gauge independent and gauge condition independent. For an Einstein-Maxwell case, he showed that the result was certainly dependent on the gauge condition. The error in the original calculation (9) was that they used an off-shell flat background without an appropriate connection, the advantage of working off shell being that the gravitational metric can be as trivial as one wishes and the calculations simplified tremendously; the method of (19) allows for such a connection to be introduced.

Other work in the subject looked at the effect of a cosmological constant in (26) where it was noted that the presence of such a cosmological constant could dominate the running behaviour of a gauge theory.

A particularly interesting application of the theory was in extra-dimensional theories (22; 23; 24).

Yet further work looked at different types of particles, with various literature investigating the effect of gravity on Yukawa couplings (25; 31; 32) and scalars (47). More obscurely, the methods were used to predict the running of spacetime dimension itself (14)

Two distinct problems are apparent in this work in light of contrasting results.

The first problem lies in the observation that the choice of method can lead to various gauge problems. Traditional Feynman diagram methods yield different answers to the works of e.g. (19), while the methods in the asymptotic safety regime also have gauge dependence problems but offer hope of a complete theory of quantum gravity. Yet gauge condition dependence can affect the values of measureable quantities so this is clearly a difficulty that should be overcome.

The second problem was that the choice of regularisation scheme could be important, since the popular choice of dimensional regularisation retains only logarithmic divergences, whereas (21) used methods which retained the quadratic divergences.

A solution to both these problems was presented in (29). In that paper, the Vilkovisky-DeWitt method was again used to avoid any gauge problems, but rather than resorting to dimensional regularisation to compute the integrals, a heat kernel method with a proper time method was employed which kept the quadratic divergences intact while retaining gauge invariance and gauge-condition independence.

At around the same time, work by He et al (33) also addressed the problems using a technique which was free of gauge issues and which treated the quadratic divergences by using dimensional reduction in two dimensions. Such a dimensional reduction must be careful of the pathological nature of gravity in two dimensions; however, it appeared that the authors were careful to avoid such issues.

The interpretation of the results was called into question by (54) and (55).

Donoghue et al (54) questioned whether the idea of a running coupling was a useful concept, rather than a more traditional S-matrix approach, and whether the omission of higher derivative terms in the preceding work led to unphysical results. As the authors pointed out in (54), it is perfectly acceptable in theories such as pure QED or pure QCD to examine the running coupling by focussing only on those diagrams which contribute to e.g. the vacuum polarization. In (19) for example, a constant background field was chosen to simplify the calculations with the reasoning that the term proportional to the charge parameter varied quadratically in the background field and did not depend on any derivatives of the field, validating the calculations. When this approach is naively applied to a theory including gravity, Donoghue et al claim the technique to be inappropriate. That said, it is unclear how this does carry through to gravity and what sense one can make of a running coupling in gravitational theories; although it is suggested that one should simply include all possible terms which could contribute to the running coupling, including derivative terms.

Ellis (55) meanwhile argued that the results were unphysical on the grounds of the necessity of using the S-matrix to perform a physical calculation; though having made this argument, they do not offer a way to proceed (as noted in (30)). They also suggested, perhaps more correctly, that one must be careful to make sure the conclusions are still intact after field redefinitions.

Further criticism of the general method was raised by Nielsen (56) who suggested that the proper time method may need to be adjusted to ensure gauge independence of the quadratically divergent terms.

This thesis continues to examine the quantum gravitational contributions to the running of coupling constants and hopefully addresses some of the concerns raised by other authors, particularly by including all possible terms which could contribute to the running coupling.

1.1.1 Asymptotic Safety

The main calculation of this thesis will be focussed on the claim that gravity can introduce asymptotic freedom into theories which may or may not already exhibit such behaviour. In this scheme, the coupling parameter of the theory under investigation falls to zero at higher energies due to some interaction with the gravitational field. We have highlighted that this is an attractive feature for the theory to possess since it avoids the problem of many theories that the coupling strength approaches infinity as we go to ever smaller distances which would otherwise invalidate perturbation theory.

However, having the coupling parameter fall away to zero is not the only solution to this problem. Another possibility is that at short distances the coupling strength approaches a fixed, non-zero value. The field theory does not become free - the coupling strength does not vanish - but it is still *safe* from high-energy catastrophes; hence this is termed *asymptotic safety* and it is, in a way, the non-perturbative equivalent of our work. Crucially, this is still a UV complete theory. Such a model for gravity was first promoted by Weinberg (60). See (74; 75) for useful reviews.

The major difference between our work and the tools of asymptotic safety is that they use a non-perturbative approach and their central object of study is the functional renormalization group equation (FRGE). A wealth of literature has built up in this field (69; 70; 71; 72; 73; 76).

Of particular recent interest in the field, was a paper (78) which calculated the expectation value of the Higgs in a region hinted at by experiment. Donoghue (54) still notes problems for the ideas of running couplings in asymptotic safety.

1.2 Outline of Chapters

We begin in the first section by recalling the traditional background field method and the need to be careful when working off-shell. We then discuss a gauge invariant and gauge condition independent approach which will allow us to work with whichever background metric we desire. This approach is the Vilkovisky-DeWitt formalism. We will also introduce some of the tools that we will require such as the results for some dimensionally regularised integrals.

In the second section, we apply the Vilkovisky-DeWitt method to a scalar field model coupled to gravity. We will seek to keep the model fairly general, allowing the scalar field to be massive, contain a self interaction and some non-minimal coupling between the scalar field and the metric field. We will also work in the presence of a non-zero cosmological constant to keep the result more general. There will be no assumption that the background scalar field is constant and hence all derivative terms will be calculated. Furthermore, the choice of gauge will not be fixed until the end of the calculation so that it is always clear how any gauge dependent terms could affect the result. The work in this section will utilise dimensional regularisation. As such, the second section only provides the logarithmic divergences.

Following on naturally from the calculation in the second section, the goal of the third section is to find the quadratic divergences while still remaining free of gauge issues. We present a different method which relies on a heat kernel description and a normal coordinate expansion which are outlined in turn. It will then be important to discuss why this treatment of quadratic divergences has been found to be flawed, and how we may demonstrate that the quadratic divergences may be a significant feature of the model.

The fourth section will contain the discussion of these results, with attempts at interpreting their meaning and a comparison to the various literature on the subject and to demonstrate that with appropriate limits we can recover the well-known results in the absence of gravity.

A final section at first glance perhaps stands laterally to the rest of the work. A calculation using an interesting result of string theory, the duality of AdS/CFT, is presented in this section. We describe how a black hole existing in an n-dimensional anti-de Sitter spacetime can be used, at least qualitatively, to explain features of a fluid existing on the $n-1$-dimension boundary. We will discuss the consequences of this for heavy ion collisions. Some discussion of black hole thermodynamics will also be included here. Of course, if we look at the calculation holistically, we recognise that the calculation is simply another demonstration of performing high energy calculations via a simpler approach, and again of the interplay between quantum mechanics and gravity.

The appendices at the end will contain some details on the computer algebra employed throughout, in particular an explanation of the most important sections of the FORM code. It will also include numerous derivations not made explicit in the main body of the text.

1.3 Conventions

We use the Einstein convention that a repeated index signifies summation over that index.

We use a flat Euclidean background metric with signature $(1,1,1,1)$.

We use natural units, $c = \hbar = 1$.

Round and square brackets will indicate symmetrization and anti-symmetrization respectively:

$$T_{(\mu\nu)} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right), \tag{1.1}$$

$$T_{[\mu\nu]} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right). \tag{1.2}$$

We shall be using the Riemann tensor defined as

$$R^{\mu}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\nu\alpha} \tag{1.3}$$

with the Ricci tensor contracted as

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}. \tag{1.4}$$

The shorthand $h = h^{\alpha}_{\alpha}$ will be employed.

We will use the notation that a comma and a semicolon represent an ordinary derivative and a covariant derivative respectively.

Chapter 2

The Background Field Method

2.1 Vilkovisky-DeWitt method

In quantizing a gauge theory, there are two problems which must be overcome. First, we require invariance in the fundamental gauge transformations of the theory. In a background field theory, this is an easy problem to solve. In a traditional Feynman diagram approach the Slavnov-Taylor-Ward-Takahashi identities must be satisfied.

Second, we require that our calculation does not depend on the choice of gauge condition. This is an important feature since gauge condition dependence should not alter the values of physical observables. We introduce the gauge condition to avoid double-counting field configurations which are related to others by gauge transformations when integrating out the fields in the functional integral. One would usually choose a gauge condition and also their associated Faddeev-Popov ghost fields. How this gauge condition permeates through to our final result will be made explicit by keeping the gauge condition arbitrary until the end of the calculation.

In the background field method, it will be necessary to expand about a background field that is not a solution of the classical equations of motion. This is one possible source of gauge condition dependence. Therefore, it would be much better to modify the background field method from the start to ensure gauge condition independence in the effective action.

This was the approach of Vilkovisky and DeWitt and is the technique which we will follow in this work. It is helpful to outline some of the details of the method to refer to later. In what follows, we will follow the notation employed in (39).

When talking about arbitrary fields, it is invaluable to use the DeWitt notation. We will consider bosonic gauge fields denoted φ^i where the single index i contains all the labels of the fields relating to spacetime or to the gauge and also includes the spacetime coordinate. Summation will follow the usual Einstein convention except that we also integrate over any repeated coordinate.

We shall denote the classical action functional of our theory as $S[\varphi]$. We will assume that our theory is gauge invariant for some underlying infinitesimal parameters $\delta\epsilon^\alpha$. Then we can write our infinitesimal gauge transformations as

$$\delta\varphi^i = K^i_\alpha \delta\epsilon^\alpha \tag{2.1}$$

where the K^i_α are identified as generators of the gauge transformations. If we are requiring gauge invariance then the condition is

$$S[\varphi + \delta\phi] = S[\varphi]. \tag{2.2}$$

Now we expand this equation and require it hold to first order in the parameters $\delta\epsilon^\alpha$ and we are led to the statement of gauge invariance

$$K^i_\alpha[\varphi] S_{,i}[\varphi] = 0 \tag{2.3}$$

where the comma notation $S_{,i}[\varphi]$ denotes a derivative of $S[\varphi]$ with respect to the field φ^i. $S_{,i}[\varphi] = 0$ is simply Hamilton's principle of least action.

Now we address the problem of double-counting fields. We noted that quantizing gauge theories using the usual Feynman path integral, integrating over the space of all fields, will distinguish some fields from other fields to which they are related by a gauge transformation, when they are in fact physically equivalent. We wish to eliminate these redundant degrees of freedom. Geometrically speaking, if we have a space of fields $\mathcal{F}$ with some metric tensor g_{ij} then we require only the physical configuration space given by $\mathcal{F}/\mathcal{G}$ where $\mathcal{G}$, the set of gauge transformations, is factored out. It is worth pointing out at this stage that the choice of metric is not unique. The gravitational part may generally take the form

$$G^{\rho\sigma\lambda\tau} = \frac{1}{2}\left(\delta^{\rho\lambda}\delta^{\sigma\tau} + \delta^{\rho\tau}\delta^{\sigma\lambda} - a\delta^{\rho\sigma}\delta^{\lambda\tau}\right) \tag{2.4}$$

where a is a free parameter which can be determined from consideration of the higher derivative term in the classical action. It can be shown (57) that the Vilkovisky-DeWitt effective action which we will encounter shortly can depend on a. Therefore, while the Vilkovisky-DeWitt effective action will provide a gauge-independent tool, the price to pay is a loss of generality in the choice of metric. However, we will work with $a = \frac{1}{2}$ which corresponds to Einstein gravitation and say no more on this.

To specify $\mathcal{F}/\mathcal{G}$ choose (fix) a gauge,

$$\chi^{\alpha}[\varphi] = 0. \tag{2.5}$$

If we expand this as

$$\chi^{\alpha}[\varphi + \delta\varphi] = 0 \tag{2.6}$$

as $\delta\epsilon^{\alpha} \to 0$ then we note that

$$Q^{\alpha}_{\beta}[\varphi]\delta\epsilon^{\beta} = 0 \tag{2.7}$$

has the solution $\delta\epsilon^{\beta} = 0$ where we have defined

$$Q^{\alpha}_{\beta}[\varphi] = \chi^{\alpha}_{,i}K^{i}_{\beta}[\varphi] \tag{2.8}$$

and $\det Q^{\alpha}_{\beta} \neq 0$.

Our result does not depend on the choice of gauge condition, so we can choose whatever expediates the calculations. Before continuing, let us add a bit more clarity. An important point to highlight is the work of Fradkin and Tseytlin (42), where it was shown that the full Vilkovisky-DeWitt calculation is equivalent to a background field approach with the inclusion of a connection term, when working in the Landau-DeWitt gauge. As this is easier to work with, we will work in this simpler regime, rather than perform a full Vilkovisky-DeWitt calculation. Therefore, we may expect to see a gauge parameter (which will later be called ω) appear in the results but we should note that this is not a sign of gauge dependence when we select a particular value for this parameter in the final step. We opt then for the Landau-DeWitt gauge (also referred to as the background field gauge) here, where

$$\chi_\alpha = K_{\alpha i}[\bar{\varphi}]\eta^i = 0. \tag{2.9}$$

We begin by expanding the fields (still general at this stage) as some perturbations about some background fields

$$\varphi^i = \bar{\varphi}^i + \eta^i \tag{2.10}$$

The problem with the traditional background field method arises by expanding about a metric which is not a solution of the classical equations of motion which was the problem in (9).

Now we arrive at the crux of the method, the choice of connection on the space of fields.

We can calculate the connection by first considering a most general displacement in the space of fields which we write

$$d\varphi^i = \omega^i_\perp + \omega^i_\parallel \tag{2.11}$$

where

$$\omega^i_\parallel = K^i_\alpha d\epsilon^\alpha \tag{2.12}$$

and

$$\omega^i_\perp = P^i{}_j d\varphi^i \tag{2.13}$$

which satisfies

$$g_{ij}\omega_{\perp}^{i}\omega_{\parallel}^{j} = 0. \tag{2.14}$$

In (2.13) we introduced the projection operator

$$P^{i}{}_{j} = \delta_{j}^{i} - K_{\alpha}^{i}\gamma^{\alpha\beta}K_{\beta j} \tag{2.15}$$

with $K_{\beta j} = g_{ij}K_{\beta}^{i}$ as usual and $\gamma^{\alpha\beta}$ being the inverse of

$$\gamma_{\alpha\beta} = K_{\alpha}^{i}g_{ij}K_{\beta}^{j}. \tag{2.16}$$

It follows simply that

$$P^{i}{}_{j}K_{\alpha}^{j} = 0 \tag{2.17}$$

and

$$P^{i}{}_{j}P^{j}{}_{k} = P^{i}{}_{k}. \tag{2.18}$$

We interpret (2.17) as the projection operator $P^{i}{}_{j}$ having the property of projecting vectors perpendicular to the generators of gauge transformation. Now if we form the line element, we have

$$\begin{aligned} ds^2 &= g_{ij}d\varphi^{i}d\varphi^{j} \\ &= g_{ij}\left(\omega_{\perp}^{i} + \omega_{\parallel}^{i}\right)\left(\omega_{\perp}^{j} + \omega_{\parallel}^{j}\right) \\ &= g_{ij}\left(\omega_{\perp}^{i}\omega_{\perp}^{j} + \omega_{\parallel}^{i}\omega_{\parallel}^{j}\right) \\ &= g_{ij}^{\perp}\omega_{\perp}^{i}\omega_{\perp}^{j} + \gamma_{\alpha\beta}d\epsilon^{\alpha}d\epsilon^{\beta} \end{aligned} \tag{2.19}$$

with the second line following from (2.11), the third following from (2.14) and the final line following from (2.12), (2.13) and (2.16). The first term is then the line element on the space of fields and the second term is the line element on the gauge group.

This exhibits the local product structure $\mathcal{F} = (\mathcal{F}/\mathcal{G}) \times \mathcal{G}$ of the space of fields.

In the last line of (2.19) we have introduced the metric on the space of distinct gauge orbits

$$g^{\perp}_{ij} = P^k{}_i P^l{}_j g_{kl}. \tag{2.20}$$

In the Feynman functional integral, it is the space of distinct gauge orbits that is integrated over. Therefore the natural choice of connection comes from the requirement

$$0 = \bar{\nabla}_i g^{\perp}_{jk} = g^{\perp}_{jk,i} - \bar{\Gamma}^l_{ij} g^{\perp}_{lk} - \bar{\Gamma}^l_{ik} g^{\perp}_{jl} \tag{2.21}$$

with $\bar{\nabla}$ the covariant derivative with respect to the connection. It follows simply that

$$\bar{\Gamma}^l_{ij} g^{\perp}_{lk} = \frac{1}{2}\left(g^{\perp}_{jk,i} + g^{\perp}_{ki,j} - g^{\perp}_{ij,k}\right). \tag{2.22}$$

It would be usual to introduce the metric inverse to $g^{\perp}_{lk}$ and multiply both sides by this to arrive at an expression for our connection. However, $g^{\perp}_{ij} K^j_\alpha = 0$ so $g^{\perp}_{lk}$ is not invertible. Therefore, $\bar{\Gamma}^k_{ij}$ is only determined up to some arbitrary multiple of K^k_α that vanishes when it is contracted with $g^{\perp}_{lk}$. The form of $\bar{\Gamma}^k_{ij}$ can be shown to be (39)

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij} + T^k_{ij} + K^k_\alpha A^\alpha_{ij}. \tag{2.23}$$

Here Γ^k_{ij} is the usual Christoffel symbol for the metric g_{ij}; T^k_{ij} is an expression involving g_{ij}, K^i_α and its first derivatives $K^i_{\alpha},_\beta$; and A^α_{ij} is entirely arbitrary. We will further address this connection later.

Now, when the integration over the space of fields is carried out, the measure which follows formally from (2.19) is

$$d\mu\left[\varphi\right] = \left(\prod_i \omega^i_\perp\right)\left(\prod_\alpha d\epsilon^\alpha\right)\left(\det g^{\perp}_{ij}\right)^{1/2}\left(\det \gamma_{\alpha\beta}\right)^{1/2}. \tag{2.24}$$

In the present paper, we intend to calculate quantum corrections at one-loop order. We can simplify the measure using the identity for the delta function,

$$\delta\left[\chi^{\beta}\right] = \lim_{\alpha\to 0}(4\pi i\alpha)^{-1/2}\exp\left(\frac{i}{2\alpha}\chi^{\beta}\chi_{\beta}\right). \tag{2.25}$$

At one loop order then, the Vilkovisky-DeWitt effective action is

$$\Gamma[\bar{\varphi}] = S[\bar{\varphi}] - \ln\det Q_{\alpha\beta}[\bar{\varphi}] + \frac{1}{2}\lim_{\alpha\to 0}\ln\det\left(\nabla^{i}\nabla_{j}S[\bar{\varphi}] + \frac{1}{2\alpha}K^{i}_{\alpha}[\bar{\varphi}]K^{\alpha}_{j}[\bar{\varphi}]\right). \tag{2.26}$$

where $S[\bar{\varphi}]$ is the classical action, $Q_{\alpha\beta}$ is the ghost term, K^{i}_{α} are the generators of gauge transformations, and the covariant derivative is

$$\nabla_{i}\nabla_{j}S[\bar{\varphi}] = S_{,ij}[\bar{\varphi}] - \bar{\Gamma}^{k}_{ij}S_{,k}[\bar{\varphi}] \tag{2.27}$$

with $\bar{\Gamma}^{k}_{ij}$ being the connection term that is crucial for obtaining a gauge condition independent result.

Let us note a few things at this point. The arbitrary third term in (2.23) does not matter at one-loop order because we will have a term proportional to $K^{k}_{\alpha}S_{,k} = 0$ which is simply the expression of gauge invariance which we required from the start. It can be shown that at one loop order, T^{k}_{ij} takes the form (39)

$$T^{k}_{ij} = \frac{1}{2}\gamma^{\alpha\epsilon}\gamma^{\beta\sigma}K_{\alpha i}K_{\beta j}\left(K^{n}_{\epsilon}K^{k}_{\sigma;n} + K^{n}_{\sigma}K^{k}_{\epsilon;n}\right) - \gamma^{\alpha\beta}\left(K_{\alpha i}K^{k}_{\beta;j} + K_{\alpha j}K^{k}_{\beta;i}\right) \tag{2.28}$$

so that by repeated use of the Landau-DeWitt gauge condition, we find that

$$T^{k}_{ij}\eta^{i}\eta^{j} = 0 \tag{2.29}$$

and hence T^{k}_{ij} makes no contribution to the effective action for our particular choice of gauge. Of course, at higher loop order and in different gauges, there will be a contribution from T^{k}_{ij}. As a result of these two observations, we can simply replace the connection $\bar{\Gamma}^{k}_{ij}$ with the Christoffel connection Γ^{k}_{ij}.

Another important point should be made at this stage. If we are expanding the fields around such background fields that are solutions to the classical equations of motion then we have $S_{,i} = 0$. In this case, terms in the effective action arising from the connection also vanish. Conversely, if the background field is not a solution to the classical equations of motion (for example, expanding the gravitational field about a flat metric) then the connection must be included.

We can write (2.26) as an integration over the quantum fields η defined in (2.10),

$$\mathit{\Gamma}[\bar{\varphi}] = -\ln \int [d\eta] e^{-S_q} \tag{2.30}$$

where

$$S_q = \lim_{\alpha \to 0} \frac{1}{2} \eta^i \eta^j \left(S_{,ij} - \mathit{\Gamma}^k_{ij} S_{,k} + \frac{1}{2\alpha} K_{\alpha i} K^{\alpha}_j \right). \tag{2.31}$$

We now have a framework where we can choose a traditional action S and construct the Vilkovisky-DeWitt action $\mathit{\Gamma}[\bar{\varphi}]$.

For the sake of clarity, lets us reiterate that η here is a quantum field and that

$$\mathit{\Gamma}_G = -\ln \int [d\eta] e^{-S_q} \frac{1}{2} \ln \det \left\{ \nabla^i \nabla_j S[\bar{\varphi}] + \frac{1}{2\alpha} K^i_{\lambda}[\bar{\varphi}] K^{\alpha}_j[\bar{\varphi}] \right\} \tag{2.32}$$

We will later calculate the ghost action,

$$\mathit{\Gamma}_{GH} = -\ln \det Q_{\alpha\beta} = -\ln \int [d\bar{\eta} d\eta] e^{-\bar{\eta} Q^{\alpha}_{\beta} \eta^{\beta}} \tag{2.33}$$

and it will be understood when we do so that η (and $\bar{\eta}$) in this case represents a ghost field, with $Q_{\alpha\beta}$ defined in (2.8).

Chapter 3

φ^4-gravity

3.1 Logarithmic divergences of Einstein-φ^4 theory

At this stage, we can stop using the DeWitt notation that was so useful for the formal framework and begin to be more specific. We will choose an action then proceed to calculate the three terms of equation (2.31).

$$S = S_M + S_G, \tag{3.1}$$

where

$$S_M = \int d^n x |g(x)|^{1/2} \left\{ \frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi + \frac{1}{2} m^2 \varphi^2 + \frac{1}{2}\xi R \varphi^2 + U(\varphi) \right\}. \tag{3.2}$$

is the scalar field action, and

$$S_G = -\frac{2}{\kappa^2} \int d^n x |g(x)|^{1/2} (R - 2\Lambda), \tag{3.3}$$

is the gravitational Einstein-Hilbert action with the inclusion of a cosmological constant Λ. We have defined

$$\kappa^2 = 32\pi G, \tag{3.4}$$

with G Newton's gravitational constant. Here ξ represents a possible non-minimal coupling to the curvature that we include for generality, and $U(\varphi)$ is a potential term that we will take to be

$$U(\varphi) = \frac{\lambda}{4!}\varphi^4. \tag{3.5}$$

We write the fields (from equation (2.10)) as

$$\varphi^i = (g_{\mu\nu}(x), \varphi(x)), \tag{3.6}$$

and the quantum fields

$$\eta^i = (\kappa h_{\mu\nu}(x), \psi(x)), \tag{3.7}$$

with

$$\begin{aligned} g_{\mu\nu} &= \delta_{\mu\nu} + \kappa h_{\mu\nu}, & (3.8)\\ \varphi(x) &= \bar{\varphi}(x) + \psi(x). & (3.9) \end{aligned}$$

We have specified the background metric to be the flat metric which, we emphasise, is not a solution of the classical equation of motion. The background scalar field $\bar{\varphi}$ is kept general at this stage. It is useful to also list two results that follow from (3.8). The inverse metric is

$$g^{\mu\nu} = \delta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu}_{\lambda} h^{\lambda\nu} + O(h^3) \tag{3.10}$$

and the measure is

$$|g(x)|^{1/2} = \left(1 + \frac{\kappa}{2}h - \frac{\kappa^2}{4}h^{\alpha}_{\beta}h^{\beta}_{\alpha} + \frac{\kappa^2}{8}h^2\right) \tag{3.11}$$

where g is the determinant of $g_{\mu\nu}$.

We now calculate the generators K^i_{α} by considering the gauge transformations. For the scalar field part, it is easy to show that for a change in coordinates,

$$x^{'\mu} = x^{\mu} + \epsilon^{\mu}, \tag{3.12}$$

an infinitesimal change in the field is given by

$$\delta\varphi = -\delta\epsilon^{\mu}\partial_{\mu}\varphi \tag{3.13}$$

whilst for gravity a consideration of the metric transformation under an infinitesimal change of coordinates leads to

$$\delta g_{\mu\nu} = -\delta\epsilon^{\lambda}g_{\mu\nu,\lambda} - \delta\epsilon^{\lambda}{}_{,\mu}g_{\lambda\nu} - \delta\epsilon^{\lambda}{}_{,\nu}g_{\lambda\mu}. \tag{3.14}$$

Writing now equation (2.1) in uncondensed notation and making the index associations $i \to \varphi(x)$, $\alpha \to \lambda$, we have

$$\delta\varphi = \int d^n x' \left\{ K^{\varphi(x)}_{\lambda}(x,x')\delta\epsilon^{\lambda}(x') \right\} \tag{3.15}$$

and

$$\delta g_{\mu\nu} = \int d^n x' \left\{ K^{g_{\mu\nu}(x)}_{\lambda}(x,x')\delta\epsilon^{\lambda}(x') \right\} \tag{3.16}$$

with the index associations $i \to g_{\mu\nu}(x)$, $\alpha \to \lambda$. By comparison of (3.14) and (3.13) with (3.15) and (3.16) we deduce

$$\begin{aligned} K^{\varphi(x)}_{\lambda}(x,x') &= -\varphi(x)_{,\lambda}\delta(x,x') && (3.17)\\ K^{g_{\mu\nu}(x)}_{\lambda}(x,x') &= -g^{\mu\nu,\lambda}(x)\delta(x,x') - g_{\mu\lambda}(x)\partial_{\nu}\delta(x,x') - g_{\lambda\nu}(x)\partial_{\mu}\delta(x,x') && (3.18) \end{aligned}$$

where $\delta(x,x')$ is the symmetric Dirac delta function.

The gauge condition is then

$$\chi_{\lambda}(x) = \frac{2}{\kappa}(\partial^{\mu}h_{\mu\lambda} - \frac{1}{2}\partial_{\lambda}h) - \omega\partial_{\lambda}\bar{\varphi}\psi, \tag{3.19}$$

where we introduce a parameter ω that must be set equal to one for the gauge condition independent result (the Landau-DeWitt gauge). The first part of $\chi_\lambda(x)$ follows from combining (3.18) and the DeWitt metric (3.21). We keep it general at this stage to illustrate the gauge condition dependence of the standard formalism. In this way we can easily compare our results at the end with other choices of gauge condition such as the De Donder gauge ($\omega = 0$).

For the computation of the connection term we consider the metric of the space of fields. The natural line element is

$$ds^2 = \int d^n x d^n x' \left\{ g_{g_{\mu\nu}(x) g_{\lambda\sigma}(x')} dg_{\mu\nu}(x) dg_{\lambda\sigma}(x') + g_{\varphi(x)\varphi(x')} d\varphi(x) d\varphi(x') \right\} \tag{3.20}$$

where we have chosen the metric for the gravity fields to be the DeWitt metric

$$\frac{1}{2\kappa^2} |g(x)|^{1/2} \left(g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda} - g^{\mu\nu} g^{\lambda\sigma} \right) \tag{3.21}$$

and the scalar field metric is simply

$$g_{\varphi(x)\varphi(x')} = |g(x)|^{1/2} \delta(x, x'). \tag{3.22}$$

The extra factor of κ^{-2} in (3.21) is necessary to ensure that both terms have the same dimensions and that the line element has units of length squared.

Using the metric components from (3.20) we can calculate the Christoffel symbols. The non-zero Christoffel symbols are

$$\begin{aligned} \Gamma^{\varphi(x'')}_{\varphi(x) g_{\mu\nu}(x')} &= \frac{1}{2} \int d^n \bar{x} g^{\varphi\varphi}(x'', \bar{x}) \left\{ \frac{\delta g_{\varphi\varphi}(x, \bar{x})}{\delta g_{\mu\nu}(x')} + \frac{\delta g_{g_{\mu\nu}\varphi}(\bar{x}, x')}{\delta \varphi(x)} - \frac{\delta g_{\varphi g_{\mu\nu}}(x, x')}{\delta \varphi(\bar{x})} \right\} \\ &= \frac{1}{4} \sqrt{g(x'')} \sqrt{g(x)} g^{\mu\nu}(x) \delta(x'', x) \delta(x, x'), \end{aligned} \tag{3.23}$$

$$
\begin{aligned}
\Gamma^{g_{\alpha\beta}(x'')}_{\varphi(x)\varphi(x')} &= \frac{1}{2}\int d^n\bar{x} g^{g_{\alpha\beta}g_{\mu\nu}}(x'',\bar{x})\left\{\frac{\delta g_{\varphi g_{\mu\nu}}(x,\bar{x})}{\delta\varphi(x')}+\frac{\delta g_{g_{\mu\nu}\varphi}(\bar{x},x')}{\delta\varphi(x)}-\frac{\delta g_{\varphi\varphi}(x,x')}{\delta g_{\mu\nu}(\bar{x})}\right\} \\
&= -\frac{1}{4}\sqrt{g(x'')}\left\{g_{\alpha(\mu}(x'')g_{\nu)\beta}(x'')\right. \\
&\quad \left.+\frac{1}{2-n}g_{\mu\nu}(x'')g_{\alpha\beta}(x'')\right\}\sqrt{g(x)}g^{\mu\nu}(x)\delta(x,x'')\delta(x,x'), \qquad (3.24)
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma^{g_{\lambda\tau}(x'')}_{g_{\mu\nu}(x)g_{\rho\sigma}(x')} &= \frac{1}{2}\int d^n\bar{x} g^{g_{\lambda\tau}g_{\alpha\beta}}(x'',\bar{x})\left\{\frac{\delta g_{g_{\mu\nu}g_{\alpha\beta}}(x,\bar{x})}{\delta g_{\rho\sigma}(x')}+\frac{\delta g_{g_{\alpha\beta}g_{\rho\sigma}}(\bar{x},x')}{\delta g_{\mu\nu}(x)}-\frac{\delta g_{g_{\mu\nu}g_{\rho\sigma}}(x,x')}{\delta g_{\alpha\beta}(\bar{x})}\right\} \\
&= \delta(x'',x')\delta(x'',x)\left[-\delta^{(\mu}_{(\lambda}g^{\nu)(\rho}\delta^{\sigma)}_{\tau)}+\frac{1}{4}g^{\mu\nu}\delta^{\rho}_{(\lambda}\delta^{\sigma}_{\tau)}+\frac{1}{4}g^{\rho\sigma}\delta^{\mu}_{(\lambda}\delta^{\nu}_{\tau)}\right. \\
&\quad \left.-\frac{1}{2(2-n)}g_{\lambda\tau}g^{\mu(\rho}g^{\sigma)\nu}+\frac{1}{4(2-n)}g_{\lambda\tau}g^{\mu\nu}g^{\rho\sigma}\right]. \qquad (3.25)
\end{aligned}
$$

We need to multiply this by $S_{,i}$, which are functional derivatives of equation (3.1). If we consider once again the Taylor expansion of $S[\varphi]$ about the background field $\bar{\varphi}^i$, the required term can be deduced after some partial integration from the term linear in the quantum fields η^i.

After some calculation, we arrive at the action (2.31),

$$
S_q = S_0 + S_1 + S_2 + S_3 + S_4, \qquad (3.26)
$$

where the subscripts count the order of the background scalar field with

$$
\begin{aligned}
S_0 &= \int d^n x\left\{-\frac{1}{2}h^{\mu\nu}\Box h_{\mu\nu}+\frac{1}{4}h\Box h+\left(\frac{1}{\alpha}-1\right)\left(\partial^\mu h_{\mu\nu}-\frac{1}{2}\partial_\nu h\right)^2\right. \\
&\quad -\Lambda\left(h^{\mu\nu}h_{\mu\nu}-\frac{1}{2}h^2\right)\left[1+\frac{v}{2}\left(\frac{n-4}{2-n}\right)\right] \\
&\quad \left.+\frac{1}{2}\partial^\mu\psi\partial_\mu\psi+\frac{1}{2}m^2\psi^2+\frac{vn\Lambda}{4-2n}\psi^2\right\}, \qquad (3.27)
\end{aligned}
$$

$$S_1 = \kappa \int d^n x \left\{ \frac{1}{2}(h\delta^{\mu\nu} - 2h^{\mu\nu})\partial_\mu\bar{\varphi}\partial_\nu\psi + \frac{1}{2}m^2\bar{\varphi}h\psi + \xi\bar{\varphi}(h^{\mu\nu}{}_{,\mu\nu} - \Box h)\psi \right.$$
$$\left. -\frac{\omega}{\alpha}(\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h)\partial^\nu\bar{\varphi}\psi - \frac{v}{4}(-\Box\bar{\varphi} + m^2\bar{\varphi})h\psi \right\}, \tag{3.28}$$

$$S_2 = \kappa^2 \int d^n x \left\{ \frac{1}{2}\left(h^{\mu\lambda}h_\lambda{}^\nu - \frac{1}{2}hh^{\mu\nu} - \frac{1}{4}\delta^{\mu\nu}h^{\alpha\beta}h_{\alpha\beta} + \frac{1}{8}h^2\delta^{\mu\nu}\right)\partial_\mu\bar{\varphi}\partial_\nu\bar{\varphi} \right.$$
$$+\frac{1}{2}\left(\frac{1}{8}h^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}\right)m^2\bar{\varphi}^2 + \frac{1}{2}\xi(R_2 + \frac{1}{2}hR_1)\bar{\varphi}^2 + \frac{\lambda}{4\kappa^2}\bar{\varphi}^2\psi^2 \tag{3.29}$$
$$+\frac{v}{4}h_{\mu\nu}h_{\lambda\sigma}\left[\frac{1}{2}\delta^{\mu\nu}T_2^{\lambda\sigma} - \delta^{\mu\lambda}T_2^{\nu\sigma} + \frac{1}{4(n-2)}T_2\left(\delta^{\mu\sigma}\delta^{\nu\lambda} + \delta^{\mu\lambda}\delta^{\nu\sigma} - \delta^{\mu\nu}\delta^{\lambda\sigma}\right)\right]$$
$$\left. -\frac{v}{8(2-n)}T_2\psi^2 + \frac{\omega^2}{4\alpha}(\partial^\mu\bar{\varphi}\partial_\mu\bar{\varphi})\psi^2 \right\}, \tag{3.30}$$

$$S_3 = \kappa \int d^n x \frac{\lambda}{12}\left(1 - \frac{v}{2}\right)\bar{\varphi}^3 h\psi, \tag{3.31}$$

$$S_4 = \kappa^2 \int d^n x \left\{ \frac{\lambda}{24}\bar{\varphi}^4\left(\frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu}\right) - \frac{v}{8(2-n)}T_4\psi^2 \right.$$
$$\left. +\frac{v}{4}h_{\mu\nu}h_{\lambda\sigma}\left[\frac{1}{2}\delta^{\mu\nu}T_4^{\lambda\sigma} - \delta^{\mu\lambda}T_4^{\nu\sigma} + \frac{1}{4(n-2)}T_4\left(\delta^{\mu\sigma}\delta^{\nu\lambda} + \delta^{\mu\lambda}\delta^{\nu\sigma} - \delta^{\mu\nu}\delta^{\lambda\sigma}\right)\right]\right\}. \tag{3.32}$$

Here v is the parameter that counts the connection contribution. It should be set to one for the correct gauge condition independent result and to zero to compare with the (incorrect) traditional result. $T_{n\,\mu\nu}$ for $n = 2, 4$ represents the energy-momentum tensor terms of order $\bar{\varphi}^2$ and $\bar{\varphi}^4$ given by

$$T_{2\,\mu\nu} = \partial_\mu\bar{\varphi}\partial_\nu\bar{\varphi} - \frac{1}{2}\delta_{\mu\nu}\partial^\alpha\bar{\varphi}\partial_\alpha\bar{\varphi} - \frac{1}{2}\delta_{\mu\nu}m^2\bar{\varphi}^2 + \xi\delta_{\mu\nu}(\Box\bar{\varphi}^2) - \xi\partial_\mu\partial_\nu\bar{\varphi}^2, \tag{3.33}$$
$$T_{4\mu\nu} = -\delta_{\mu\nu}\frac{\lambda}{4!}\bar{\varphi}^4, \tag{3.34}$$
$$T_2 = \left(1 - \frac{n}{2}\right)\partial^\mu\bar{\varphi}\partial_\mu\bar{\varphi} - \frac{n}{2}m^2\bar{\varphi}^2 + (n-1)\xi\Box\bar{\varphi}^2, \tag{3.35}$$
$$T_4 = -\frac{n\lambda}{4!}\bar{\varphi}^4. \tag{3.36}$$

$R_2 + \frac{1}{2}hR_1$ is the quadratic part of $|g|^{1/2}R$ given by

$$\begin{aligned} R_2 + \frac{1}{2}hR_1 &= h^{\mu\nu}\Box h_{\mu\nu} - 2h_{\mu\nu}\partial^\mu\partial^\lambda h_\lambda{}^\nu - \partial^\lambda h_\lambda{}^\mu\partial^\nu h_{\mu\nu} + \partial^\lambda h_\lambda{}^\mu\partial_\mu h + h^{\mu\nu}\partial_\mu\partial_\nu h \\ &+\frac{3}{4}\partial^\lambda h^{\mu\nu}\partial_\lambda h_{\mu\nu} - \frac{1}{4}\partial^\lambda h\partial_\lambda h - \frac{1}{2}\partial^\lambda h^{\mu\nu}\partial_\mu h_{\lambda\nu} + \frac{1}{2}h\partial_\mu\partial_\nu h^{\mu\nu} - \frac{1}{2}h\Box h. \end{aligned} \tag{3.37}$$

The graviton and scalar propagators follow from S_0 in the usual way. The terms in S_1 and S_2 will be treated as interactions. We can write the scalar propagator as

$$G(x,x') = \int \frac{d^np}{(2\pi)^n}e^{ip\cdot(x-x')}G(p), \tag{3.38}$$

and the graviton propagator as

$$G_{\rho\sigma\lambda\tau}(x,x') = \int \frac{d^np}{(2\pi)^n}e^{ip\cdot(x-x')}G_{\rho\sigma\lambda\tau}(p). \tag{3.39}$$

Using the result for S_0 leads to

$$G(p) = \frac{1}{p^2+M^2}\ , \tag{3.40}$$

where

$$M^2 = m^2 + \frac{nv\Lambda}{2-n} = m^2 - 2v\Lambda \tag{3.41}$$

with $n \to 4$ in the second equality, and,

$$G_{\rho\sigma\lambda\tau}(p) = \frac{\delta_{\rho\lambda}\delta_{\sigma\tau} + \delta_{\rho\tau}\delta_{\sigma\lambda} - \frac{2}{n-2}\delta_{\rho\sigma}\delta_{\lambda\tau}}{2\left(p^2-2\lambda\right)} + \frac{1}{2}(\alpha-1)\frac{\delta_{\rho\lambda}p_\sigma p_\tau + \delta_{\rho\tau}p_\sigma p_\lambda + \delta_{\sigma\lambda}p_\rho p_\tau + \delta_{\sigma\tau}p_\rho p_\lambda}{\left(p^2-2\lambda\right)\left(p^2-2\alpha\lambda\right)}, \tag{3.42}$$

where we have defined

$$\lambda = \Lambda + v\Lambda\left(\frac{n-4}{4-2n}\right). \tag{3.43}$$

In our calculations of the pole terms, the Vilkovisky-DeWitt correction in (3.43) will make no contributions to the poles when $n \to 4$, and we may set $\lambda \to \Lambda$ in this limit. We use

Wick's theorem and the basic pairings

$$\langle\psi(x)\psi(x')\rangle = G(x,x'), \tag{3.44}$$

$$\langle h_{\rho\sigma}(x)h_{\lambda\tau}(x')\rangle = G_{\rho\sigma\lambda\tau}(x,x'), \tag{3.45}$$

to evaluate the effective action to order $\bar{\varphi}^2$.

With these tools we can now proceed to work out terms in the effective action. We will begin with the quadratic terms. These were calculated in this author's earlier paper (47) but we include the calculations here for completeness. Before we do so, let us discuss the Wick rules and calculation of Feynman integrals.

3.2 Wick Rules

Wick's theorem is a powerful tool that reduces a complicated expression containing many quantum fields into a combinatorics problem. A product of quantum fields must be combined in all possible combinations and to obtain the effective action we must remove any expressions which relate to diagrams which are not one particle irreducible (1PI).

In the absence of any derivative operators, we can recognise that the expressions only take on a handful of forms; when we reintroduce the derivative operators, these will appear as prefactors to the resulting Green's functions and will not affect the general structure of the equations. To be more verbose, S_1 essentially contains terms which look like $h\psi$, S_2 like hh or $\psi\psi$, S_3 again like $h\psi$ and S_4 again like hh or $\psi\psi$.

The simplest term is $\langle S_4\rangle$ with the reduction into Green's functions following immediately. We have either

$$h_A h_B = G_{AB}(x,x) \tag{3.46}$$

(where the upper case index on the gravitons represent whatever indices may actually be present) or

$$\psi\psi = \mathit{\Delta}(x,x). \tag{3.47}$$

Figure 3.1: The curly line represents a graviton while a dashed line represents a scalar.

Figure 3.2: The possible diagrams in S_4

Using the definitions giving in Fig 3.1, we can represent these possibilities by Fig 3.2.

Next, consider $\langle S_1 S_3 \rangle$. The basic structure of terms appearing in this expression are of the form

$$h_A h'_B \psi \psi' \tag{3.48}$$

where the primes indicate the coordinate, e.g. $h' \equiv h(x')$. In this case, the application of Wick's Theorem is trivial. The result is

$$h_A h'_B \psi \psi' = G_{AB}(x, x') \varDelta(x, x'). \tag{3.49}$$

The relevant diagrams for this interaction are given in Fig 3.3.

Continuing on, consider $\left\langle S_2^2 \right\rangle$. This time, the basic structure of the terms will be of the form

$$h_A h_B h'_C h'_D \tag{3.50}$$

Figure 3.3: The possible diagrams in $S_1 S_3$

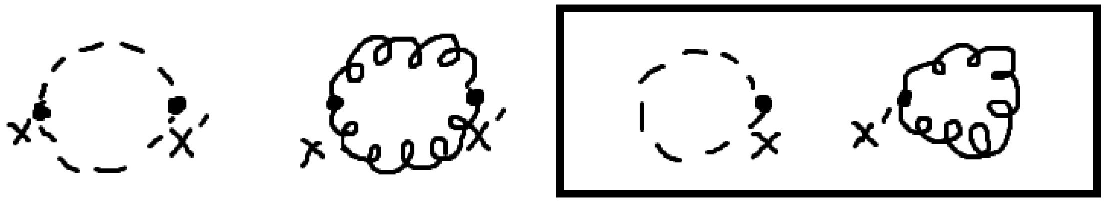

Figure 3.4: The possible diagrams in S_2^2. The boxed interactions are not 1-PI and so are not counted.

and

$$\psi\psi\psi'\psi'. \tag{3.51}$$

This time application of Wick's theorem to (3.50) is not so simple. Recall that we seek only to retain 1PI diagrams at one loop order, so that a term in (3.50) where we pair h_A with h_B and h_C with h_D would be both two loop and disconnected; hence such terms are discarded. The same argument applies to any term of the form $hh\psi'\psi'$. The result for the graviton part is

$$h_A h_B h'_C h'_D = G_{AC}(x,x')G_{BD}(x,x') + G_{AD}(x,x')G_{BC}(x,x'). \tag{3.52}$$

Since the scalar fields carry no extra indices, the analogous expression for scalars is simpler. The result is

$$\psi\psi\psi'\psi' = 2\Delta(x,x')\Delta(x,x'). \tag{3.53}$$

The corresponding diagrams are given in Fig 3.4.

Moving onto $\langle S_1^2 S_2 \rangle$ requires us to be more careful. Now we either have terms

$$h_A h'_B \psi\psi'\psi''\psi'' \tag{3.54}$$

or

$$h_A h'_B h''_C h''_D \psi\psi'. \tag{3.55}$$

Aside from a technical problem [1], we can apply Wick's theorem again. (3.54) becomes

$$h_A h'_B \psi\psi'\psi''\psi'' = 2G_{AB}(x,x')\Delta(x,x'')\Delta(x',x'') \tag{3.56}$$

and (3.55) becomes

$$h_A h'_B h''_C h''_D \psi\psi' = \left(G_{AC}(x,x'')G_{BD}(x',x'') + G_{AD}(x,x'')G_{BC}(x',x'')\right)\Delta(x,x'). \tag{3.57}$$

Again, we have been careful to ignore terms of order above one-loop or terms that are non-1PI [2]. The relevant diagrams are in Fig 3.5.

$\langle S_1^4 \rangle$ is simpler in some regard than $\langle S_1^2 S_2 \rangle$, in that all the terms again have the same structure,

$$h_A h'_B h''_C h'''_D \psi\psi'\psi''\psi'''. \tag{3.58}$$

Applying Wick's theorem to (3.58), we have

$$h_A h'_B h''_C h'''_D \psi\psi'\psi''\psi''' = G_{AB}(x,x')G_{CD}(x'',x''')\Delta(x,x'')\Delta(x',x''')+5\,more\,terms, \tag{3.59}$$

where the extra terms not shown explicitly are those formed by matching up all the pairs which do not lead to disconnected diagrams. Finally, the relevant diagrams here are in Fig 3.6.

[1] In terms of coding, $\langle S_1^2 S_2 \rangle$ presents a unique difficulty in that we can not blindly apply Wick rules without first identifying whether we have a term of type (3.54) or (3.55).

[2] In fact, because of the nature of our calculation - that the parts of our action we are considering are all quadratic in the quantum part of the fields - it is not possible to have a term that is of two-loop order whilst being 1PI

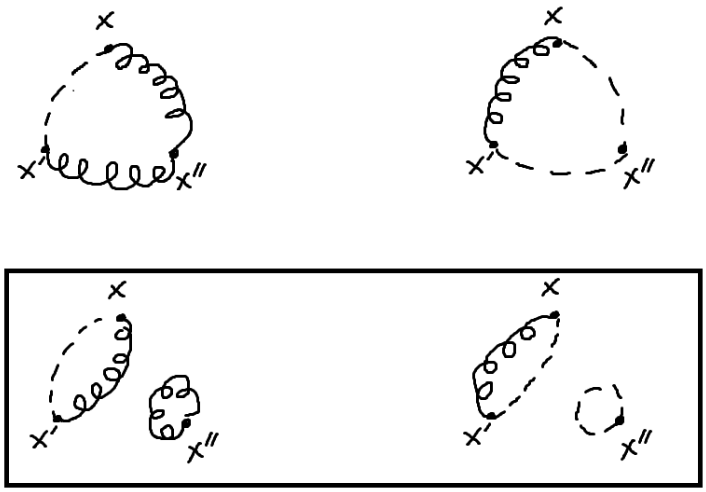

Figure 3.5: The possible diagrams in $S_1^2 S_2$. The boxed interactions are not 1-PI and so are not counted.

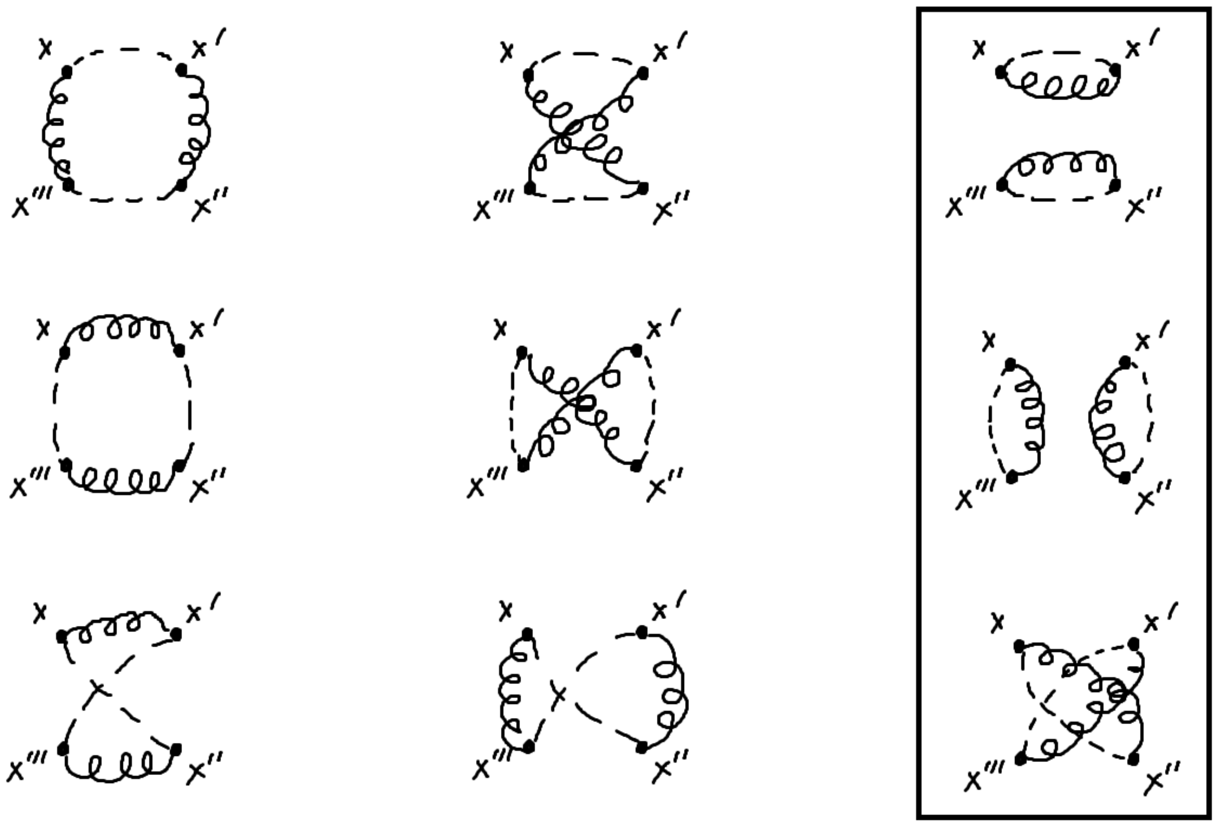

Figure 3.6: The possible diagrams in S_1^4. The boxed interactions are not 1-PI and so are not counted.

3.2.1 Momentum space transformation

An important step in the calculation is to transform from coordinate space to momentum space. For each term in the expansion of e^{-S_q} , we need to be able to eliminate a momentum from the exponential factor. This will be equivalent to selecting the momentum that we want to integrate over, such that the remaining *un-integrated* momenta will multiply an exponential containing those same momenta and will simply integrate to introduce Dirac δ-functions and their derivatives. In doing so, we will reduce our expression to a single variable.

Again, let us investigate each term in the expansion in turn.

For $\langle S_4 \rangle$, there is no need for any momentum shift. The term (3.46) transforms as

$$G_{AB}(x,x) = \int \frac{d^n k}{(2\pi)^n} G_{AB}(k). \tag{3.60}$$

Likewise, (3.47) transforms as

$$\Delta(x,x) = \int \frac{d^n k}{(2\pi)^n} \Delta(k). \tag{3.61}$$

Introducing our expressions for $G_{AB}(k)$ and $\Delta(k)$ from (3.60) and (3.61) promptly yields a result for $\langle S_4 \rangle$.

For $\langle S_1 S_3 \rangle$, (3.49) transforms to

$$G_{AB}(x,x')\Delta(x,x') = \int\int \frac{d^n p}{(2\pi)^n} \frac{d^n k}{(2\pi)^n} G_{AB}(p)\Delta(k) e^{i[(k+p)\cdot(x-x')]} \tag{3.62}$$

and it is easily observed that either $k \to k - p$ or $p \to p - k$ eliminates one of the momenta from the exponential as we claimed was required. However, we will introduce an algorithm for deciding which of these substitutions to make. To wit, the terms are ordered such that the graviton terms are on the left; scalars to the right. Our convention is such that we assign momenta from the right side of the expression with the labels k, p (then q then r for the more complicated terms): to elucidate, in (3.62), reading from the right, we first need to assign a momentum space transformation to the $\Delta(x, x')$ term, which we call k, then assign a momentum to the $G_{AB}(x, x')$ term which we call p. We then eliminate from the exponential factor the momentum which appears leftmost in the expression (p in this case) and integrate over this momentum. Therefore, we choose $k \to k - p$ so that

$$G_{AB}(x, x')\Delta(x, x') = \int\int \frac{d^n p}{(2\pi)^n} \frac{d^n k}{(2\pi)^n} G_{AB}(p)\Delta(k - p)e^{i[(k)\cdot(x-x')]} \tag{3.63}$$

A similar analysis applies to $\langle S_2^2 \rangle$ and, for example,

$$G_{AC}(x, x')G_{BD}(x, x') = \int\int \frac{d^n p}{(2\pi)^n} \frac{d^n k}{(2\pi)^n} G_{AC}(p)G_{BD}(k - p)e^{i[(k)\cdot(x-x')]}. \tag{3.64}$$

3.2.2 Summary of results

Taking altogether the steps of:

- applying Wick rules;
- switching to momentum space;
- applying an appropriate momentum shift;

it is possible to write rules which take us straight from an expression involving the quantum parts h and ψ to a momentum expression which we can integrate to find our final answer.

3.3 Feynman integrals

It is useful to list some results which appear often in the calculation. They can all be derived from the first integral

$$\int \frac{d^n p}{(2\pi)^n} \frac{1}{(p^2 + 2p \cdot q + m^2)^\alpha} = (4\pi)^{-n/2} \frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)} (-q^2 + m^2)^{n/2-\alpha} \tag{3.65}$$

by repeated differentiation with respect to the momentum q_μ and then setting $q_\mu = 0$ at the final step. The derivation of the first integral follows, for example, (50), with the exception that we have a Euclidean metric rather than the Minkowski metric used therein. We start with the integral

$$I(q) = \int \frac{d^n p}{(p^2 + 2pq + m^2)^\alpha} \tag{3.66}$$

We choose polar coordinates for the spatial part so we have $(p_0, r, \phi, \theta_1, \theta_2, ..., \theta_{n-3})$ and so

$$\begin{aligned} d^n p &= dp_0 r^{n-2} dr d\phi \sin\theta_1 d\theta_1 \sin^2\theta_2 d\theta_2 \dots \sin^{n-3}\theta_{n-3} d\theta_{n-3} && (3.67) \\ &= dp_0 r^{n-2} dr d\phi \prod_{k=1}^{n-3} \sin^k \theta_k d\theta_k && (3.68) \end{aligned}$$

for

$$-\infty < p_0 < \infty$$

$$0 < r < \infty$$

$$0 < \phi < 2\pi$$

$$0 < \theta_i < \pi.$$

We then have (after performing the simple integral over ϕ)

$$I_n(q) = 2\pi \int_{-\infty}^{\infty} dp_0 \int_0^\infty r^{n-2} dr \int_0^\pi \frac{\prod_{k=1}^{n-3} \sin^k \theta_k d\theta_k}{(p^2 + 2pq + m^2)^\alpha} \tag{3.69}$$

To do the integral over the θ_i we can use the formula

$$\int_0^{\pi/2} (\sin\theta)^{2n-1} (\cos\theta)^{2m-1} d\theta = \frac{1}{2} \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} \tag{3.70}$$

which with $m = \frac{1}{2}$ leads to

$$\int_0^{\pi} (\sin\theta)^k d\theta = \pi^{1/2} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)} \tag{3.71}$$

hence we have

$$I_n(q) = 2\pi^{(n-1)/2} \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \int_{-\infty}^{\infty} dp_0 \int_0^{\infty} \frac{r^{n-2} dr}{(p_0^2 + r^2 + 2pq + m^2)^{\alpha}} \tag{3.72}$$

We now switch momentum variables to $p'_\mu = p_\mu + q_\mu$ and choose to evaluate in the frame $q_\mu = (\mu, \mathbf{0})$. Then we have

$$I_n(q) = 2\pi^{(n-1)/2} \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \int_{-\infty}^{\infty} dp'_0 \int_0^{\infty} \frac{r^{n-2} dr}{(p_0'^2 + r^2 - q^2 + m^2)^{\alpha}} \tag{3.73}$$

For convenience, we define $M^2 = p_0'^2 - q^2 + m^2$ and use the definition of the Euler beta function,

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^{\infty} dt t^{2x-1} (1+t^2)^{-x-y} \tag{3.74}$$

with

$$\begin{aligned} x &= \frac{1+\beta}{2} \\ y &= \alpha - \frac{1+\beta}{2} \\ t &= \frac{s}{M} \end{aligned} \tag{3.75}$$

to get the integral

$$\int_0^{\infty} ds \frac{s^{\beta}}{(s^2 + M^2)^{\alpha}} = \frac{\Gamma\left(\frac{1+\beta}{2}\right)\Gamma\left(\alpha - \frac{1+\beta}{2}\right)}{2(M^2)^{\alpha-(1+\beta)/2}\Gamma(\alpha)}. \tag{3.76}$$

We use this first to perform the radial integral, then the time part (noting an extra factor of 2 due to the domain of p_0 compared with r). We arrive eventually at the formula

$$I_n(q) = (\pi)^n/2\Gamma\left(\alpha - n/2\right)\left(-q^2 + m^2\right)^{n/2-\alpha} \tag{3.77}$$

where we can insert the extra factor of $1/(2\pi)$ to get the formula (3.65).

Altogether, we have the useful integrals:

$$\int \frac{d^n p}{(2\pi)^n} \frac{1}{(p^2+m^2)^\alpha} = (4\pi)^{-n/2} \frac{\Gamma\left(\alpha - n/2\right)}{\Gamma\left(\alpha\right)} \left(m^2\right)^{n/2-\alpha} \tag{3.78}$$

$$\int \frac{d^n p}{(2\pi)^n} \frac{p_\mu}{(p^2+m^2)^\alpha} = 0 \tag{3.79}$$

$$\int \frac{d^n p}{(2\pi)^n} \frac{p_\mu p_\nu}{(p^2+m^2)^\alpha} = \frac{1}{2}\left(4\pi\right)^{-n/2} \delta_{\mu\nu} \frac{\Gamma\left(\alpha - 1 - n/2\right)}{\Gamma\left(\alpha\right)} \left(m^2\right)^{1+n/2-\alpha} \tag{3.80}$$

$$\int \frac{d^n p}{(2\pi)^n} \frac{p^2}{(p^2+m^2)^\alpha} = \frac{1}{2}\left(4\pi\right)^{-n/2} n \frac{\Gamma\left(\alpha - 1 - n/2\right)}{\Gamma\left(\alpha\right)} \left(m^2\right)^{1+n/2-\alpha} \tag{3.81}$$

$$\int \frac{d^n p}{(2\pi)^n} \frac{p_\mu p_\nu p_\lambda}{(p^2+m^2)^\alpha} = \int \frac{d^n p}{(2\pi)^n} \frac{p^2 p_\lambda}{(p^2+m^2)^\alpha} = 0 \tag{3.82}$$

$$\begin{aligned} \int \frac{d^n p}{(2\pi)^n} \frac{p_\mu p_\nu p_\gamma p_\delta}{(p^2+m^2)^\alpha} &= \frac{1}{4}\left(4\pi\right)^{-n/2} \frac{\Gamma\left(\alpha - 2 - n/2\right)}{\Gamma\left(\alpha\right)} \left(m^2\right)^{2+n/2-\alpha} \\ &\quad \times (\delta_{\mu\nu}\delta_{\gamma\delta} + \delta_{\nu\gamma}\delta_{\mu\delta} + \delta_{\mu\gamma}\delta_{\nu\delta}) \end{aligned} \tag{3.83}$$

Clearly, any odd power of the momentum p in the numerator results in an overall odd integrand and thus leads to a zero integral.

3.3.1 Feynman Parametrization

Now we need a technique that will allow us to solve such an integral as, for example, (3.78); fortunately, such techniques exist. Feynman built upon the work of Schwinger when he noted that the product of any two functions, say A and B, in a denominator can be combined by observing that

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[Ax + B(1-x)]^2}. \tag{3.84}$$

For example, consider the integral

$$I = \int \frac{d^n p}{(2\pi)^n} \frac{1}{(p^2 - 2\lambda)} \frac{1}{[(k-p)^2 + m^2]}. \tag{3.85}$$

If we invoke the Feynman parametrization, then this can be re-expressed as

$$I = \int \frac{d^n p}{(2\pi)^n} \int_0^1 \left((p^2 - 2\lambda)x + ((k-p)^2 + m^2)(1-x)\right)^{-2}. \tag{3.86}$$

If we rewrite

$$(p^2 - 2\lambda)x + ((k-p)^2 + m^2)(1-x) = (p - k(1-x))^2 + k^2 x(1-x) - 2\lambda x + m^2(1-x) \tag{3.87}$$

then by performing a momentum shift $p \to p + k(1-x)$, we end up with

$$I = \int \frac{d^n p}{(2\pi)^n} \int_0^1 \left(p^2 + M^2\right)^{-2}. \tag{3.88}$$

where $M^2 = (k^2 x + m^2)(1-x) - 2\lambda x$. We can now use our standard results from the previous section to perform the momentum integral. In dimensional regularization we would take $n \to 4 + \epsilon$ and use the expansions of the gamma function, etc.

We can generate similar integrals to 3.84 by simply differentiating, for example

$$\frac{1}{A^2 B} = -\frac{\partial}{\partial A} \int_0^1 dx \frac{1}{(Ax + B(1-x))^2} = 2\int_0^1 dx\, x \frac{1}{(Ax + B(1-x))^3}. \tag{3.89}$$

By repeated differentiation we simply find

$$\frac{1}{A^n B} = n \int_0^1 dx \frac{1}{(Ax + B(1-x))^{n+1}}. \tag{3.90}$$

Similar equations hold for a higher number of functions. When we consider the full φ^4 calculation, we will encounter terms with far more functions in the denominator (in fact, as many as eight).[3]

However, its should be pointed out at this stage that this is an increasingly unwieldy way to perform the calculations. While this method will give us exact results for both the finite and pole parts, we are only interested in the pole parts. If we are able to ignore the finite parts, then there exists a simpler (and more attractive to computer modelling) technique which we may employ.

3.3.2 Retaining only pole parts

As we claimed in the previous section, there is a easier way to calculate the Feynman integrals if we are only interest in the pole part. Since we are integrating over all momentum (call it p), $p^2 \gg m^2$ and we can expand the integral in powers of p. For example,

$$\frac{1}{(k-p)^2 + m^2} = \frac{1}{p^2} + \frac{2k \cdot p}{p^4} + \frac{3k^2 - m^2}{p^4} + \dots \tag{3.92}$$

and, using dimensional regularization, we will only retain the logarithmically divergent terms, defining the basic logarithmically divergent integral to be (in four dimensions)

$$L = \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^4}. \tag{3.93}$$

[3]For completeness, let us state the expression for n such terms:

$$\int d^n p \frac{1}{A_1(p) \dots A_n(p)} = (n-1)! \int_0^1 dz_1 \dots \int_0^1 dz_n \int \frac{\delta(z_1 + \dots + z_n - 1)}{(z_1 A_1 \dots z_n A_n)^n} \tag{3.91}$$

For example, the pole part (PP) of the integral relating to 3.92 expression is

$$PP\left(\int \frac{d^4p}{(2\pi)^4}\frac{1}{(k-p)^2+m^2}\right) = (3k^2-m^2)L. \tag{3.94}$$

Hereafter, we will proceed to calculate integrals in this fashion. However, by way of examples, let us consider the integrals for S_1^2 which appear in the quadratic calculation in the case using Feynman parametrisation, retaining only the pole part. We have terms such as

$$\big\langle h(x)\psi(x)h(x')\psi(x')\big\rangle = \frac{3-\alpha}{16\pi^2\epsilon}\delta(x,x'), \tag{3.95}$$

$$\big\langle h(x)\psi(x)\partial'^{\alpha}\partial'_{\alpha}h(x')\psi(x')\big\rangle = -\frac{3-\alpha}{24\pi^2\epsilon}m^2\delta(x,x'), \tag{3.96}$$

$$\Big\langle h(x)\psi(x)\partial'^{\alpha}\partial'_{\beta}h^{\beta}_{\alpha}(x')\psi(x')\Big\rangle = \frac{-3+2\alpha}{48\pi^2\epsilon}m^2\delta(x,x'), \tag{3.97}$$

$$\big\langle h(x)\psi(x)h(x')\partial'_{\mu}\psi(x')\big\rangle = -\frac{3(3-\alpha)}{8\pi^2\epsilon}\partial_{\mu}\delta(x,x'), \tag{3.98}$$

$$\big\langle h(x)\psi(x)h^{\mu\nu}(x')\partial'_{\mu}\psi(x')\big\rangle = -\frac{3\alpha-2}{16\pi^2\epsilon}\partial_{\mu}\delta(x,x'), \tag{3.99}$$

$$\Big\langle h(x)\psi(x)\partial^{\alpha'}h(x')\psi(x')\Big\rangle = -\frac{3-\alpha}{8\pi^2\epsilon}\partial^{\alpha}\delta(x,x'), \tag{3.100}$$

$$\big\langle h(x)\psi(x)\partial_{\mu'}h^{\mu\alpha}(x')\psi(x')\big\rangle = \frac{2\alpha-3}{16\pi^2\epsilon}\partial^{\alpha}\delta(x,x'), \tag{3.101}$$

$$\big\langle \partial^{\mu}\partial_{\mu}h(x)\psi(x)\partial^{\nu}\partial'_{\nu}h(x')\psi(x')\big\rangle = \frac{3-\alpha}{48\pi^2\epsilon}m^2(m^2-2\partial^{\mu}\partial_{\mu})\delta(x,x'), \tag{3.102}$$

$$\Big\langle \partial^{\mu}\partial_{\mu}h(x)\psi(x)\partial^{\alpha'}\partial_{\beta'}h(x')\psi(x')\Big\rangle = \frac{3-2\alpha}{96\pi^2\epsilon}m^2(m^2-2\partial^{\mu}\partial_{\mu})\delta(x,x'), \tag{3.103}$$

$$\big\langle \partial^{\mu}\partial_{\mu}h(x)\psi(x)\partial'_{\nu}h(x')\psi(x')\big\rangle = \frac{3-\alpha}{8\pi^2\epsilon}m^2\left(\frac{17}{6}m^2\partial_{\mu}+\frac{7}{3}\partial_{\mu}\partial^{\nu}\partial_{\nu}\right)\delta(x,x'), \tag{3.104}$$

$$\Big\langle \partial^{\mu}\partial_{\mu}h(x)\psi(x)h^{\alpha\beta}(x')\partial'_{\beta}\psi(x')\Big\rangle = \frac{\alpha-1}{8\pi^2\epsilon}\left(\frac{1}{6}m^2\partial^{\mu}-\frac{5}{6}\partial^{\mu}\partial^{\nu}\partial_{\nu}\right)\delta(x,x'), \tag{3.105}$$

$$\Big\langle \partial^{\mu}\partial_{\mu}h(x)\psi(x)\partial^{\alpha'}h(x')\psi(x')\Big\rangle = -\frac{3-\alpha}{8\pi^2\epsilon}m^2(3m^2\partial^{\alpha}+2\partial^{\alpha}\partial^{\mu}\partial_{\mu})\delta(x,x'), \tag{3.106}$$

$$\Big\langle \partial^{\mu}\partial_{\mu}h(x)\psi(x)\partial'_{\alpha}h^{\alpha\beta}(x')\psi(x')\Big\rangle = \frac{2\alpha-3}{48\pi^2\epsilon}m^2\partial^{\mu}\delta(x,x'), \tag{3.107}$$

$$\Big\langle \partial^{\alpha}\partial_{\beta}h^{\beta}_{\alpha}(x)\psi(x)\partial^{\rho'}\partial'_{\sigma}h^{\sigma}_{\rho}(x')\psi(x')\Big\rangle = \frac{3-4\alpha}{192\pi^2\epsilon}m^2(m^2-2\partial^{\mu}\partial_{\mu})\delta(x,x'), \tag{3.108}$$

$$\left\langle \partial^\alpha \partial_\beta h^\beta_\alpha(x)\psi(x)h(x')\partial'_\mu\psi(x')\right\rangle = \frac{3-2\alpha}{8\pi^2\epsilon}\partial_\mu\delta(x,x'), \tag{3.109}$$

$$\left\langle \partial^\alpha \partial_\beta h^\beta_\alpha(x)\psi(x)h^{\mu\nu}(x')\partial'_\nu\psi(x')\right\rangle = \frac{2\alpha-1}{96\pi^2\epsilon}(5\partial^\mu\partial^\nu\partial_\nu + m^2\partial^\mu)\delta(x,x'), \tag{3.110}$$

$$\left\langle \partial^\alpha \partial_\beta h^\beta_\alpha(x)\psi(x)\partial^{\lambda'}h(x')\psi(x')\right\rangle = \frac{3-2\alpha}{48\pi^2\epsilon}\partial^\lambda\delta(x,x'), \tag{3.111}$$

$$\left\langle \partial^\alpha \partial_\beta h^\beta_\alpha(x)\psi(x)\partial'_\mu h^{\mu\nu}(x')\psi(x')\right\rangle = -\frac{m^2}{48\pi^2\epsilon}\partial^\nu\delta(x,x'), \tag{3.112}$$

$$\langle h(x)\partial_\mu\psi(x)h(x')\partial'_\nu\psi(x')\rangle = \frac{3-\alpha}{8\pi^2\epsilon}\left(\frac{1}{3}\partial_\mu\partial_\nu + \frac{1}{36}\delta_{\mu\nu} + \frac{m^2}{12}\delta_{\mu\nu}\right)\delta(x,x'), \tag{3.113}$$

$$\begin{aligned}\left\langle h(x)\partial_\mu\psi(x)h^{\alpha\beta}(x')\partial'_\beta\psi(x')\right\rangle &= \frac{1}{16\pi^2\epsilon}\left[\frac{1}{12}(-38\alpha+42)\partial^\alpha\partial_\mu + \frac{1}{36}(20\alpha-21)\delta^\alpha_\mu\partial^\nu\partial_\nu\right.\\ &\left.+\frac{1}{12}(2\alpha-1)\delta^\alpha_\mu m^2\right]\delta(x,x'),\end{aligned} \tag{3.114}$$

$$\left\langle h(x)\partial_\mu\psi(x)\partial^{\alpha'}h(x')\psi(x')\right\rangle = \frac{\alpha-3}{192\pi^2\epsilon}(m^2)^2\delta^\alpha_\mu\delta(x,x'), \tag{3.115}$$

$$\langle h(x)\partial_\mu\psi(x)\partial'_\nu h^\nu\alpha(x')\psi(x')\rangle = \frac{2\alpha-3}{16\pi^2\epsilon}\left(\frac{1}{36}\delta^\alpha_\mu\partial^\nu\partial_\nu - \frac{1}{12}\delta^\alpha_\nu m^2 + \frac{4}{3}\partial^\alpha\partial_\mu\right)\delta(x,x'), \tag{3.116}$$

$$\begin{aligned}\left\langle h^{\mu\nu}(x)\partial_\nu\psi(x)h^{\lambda\tau}(x')\partial'_\tau\psi(x')\right\rangle &= \frac{1}{16\pi^2\epsilon}\left[\left(-\frac{17}{144}-\frac{10}{144}\alpha\right)\delta^{\mu\lambda}m^2 + \left(\frac{80}{36}-\frac{83}{36}\alpha\right)\partial^\mu\partial^\lambda\right.\\ &\left.+\left(\frac{515}{432}-\frac{518}{432}\alpha\right)\delta^{\mu\lambda}\partial^\nu\partial_\nu\right]\delta(x,x')\end{aligned} \tag{3.117}$$

which are the terms which contribute to just the quadratic term. This is clearly a cumbersome approach, hence the earlier introduction of the technique of identifying only the pole parts.

3.4 Quadratic part from scalars and gravitons

Now we are in a position to be able to put all the previous sections together and calculate the various parts of the effective action. The quadratic gauge field contribution is

$$\Gamma_{G2} = \langle S_2 \rangle - \frac{1}{2}\langle S_1^2 \rangle. \tag{3.118}$$

After some calculation we find

$$\langle S_1^2 \rangle = \kappa^2 L \int d^4x \left\{ A_2(\Box\bar{\varphi})^2 + B_2\bar{\varphi}\Box\bar{\varphi} + C_2\bar{\varphi}^2 \right\}, \tag{3.119}$$

where

$$\begin{aligned}
A_2 &= -\frac{3}{8}v^2 + \frac{1}{4}\omega v + \frac{1}{8}\alpha v^2 + \frac{1}{2}\xi + \frac{3}{4}v + \frac{1}{2}\omega - \frac{1}{2}\alpha v, && (3.120)\\
B_2 &= \frac{3}{2}\alpha\xi^2 m^2 - \frac{3}{2}\xi m^2 + \frac{3}{2}\xi m^2 v - \frac{9}{4}m^2 v + \frac{3}{4}m^2 v^2 - \frac{1}{2}\omega m^2 - \frac{1}{4}\omega m^2 v + \alpha m^2 v \\
&\quad -\frac{1}{4}\alpha m^2 v^2 + \frac{3}{2}m^2 - \Lambda\omega^2 - 3\Lambda\xi v^2 - 2\alpha m^2 + 2\Lambda\alpha\omega - \Lambda\alpha^2 \\
&\quad -\frac{\Lambda\omega^2 v}{\alpha} + 2\Lambda\omega v - \Lambda\alpha v + \frac{\omega^2 m^2}{2\alpha} + 3\Lambda\xi, && (3.121)\\
C_2 &= -\frac{3}{2}m^4 + \frac{3}{2}m^4 v - \frac{3}{8}m^4 v^2 + \frac{1}{2}\alpha m^4 - \frac{1}{2}\alpha m^4 v + \frac{1}{8}\alpha m^4 v^2 - 6\Lambda\xi m^2 \\
&\quad -3\Lambda\xi m^2 v + 3\Lambda\xi m^2 v^2 + 3\xi m^4 - \frac{3}{2}\xi m^4 v - 6\Lambda^2\xi^2 + 3\Lambda\xi^2 m^2 - 6\Lambda^2\xi^2 v \\
&\quad -\frac{3}{2}\xi^2 m^4 + 6\Lambda\xi^2 m^2 v - 6\Lambda^2\xi^2 v^2. && (3.122)
\end{aligned}$$

Note that we cannot take $\alpha \to 0$ in this expression.

For $\langle S_2 \rangle$ we find the form of (3.119) but with

$$\begin{aligned}
A_2 &= 0, && (3.123)\\
B_2 &= \frac{1}{8}v^2\Lambda - \frac{1}{16}vm^2 - \frac{\omega^2 v\Lambda}{2\alpha} + \frac{\omega^2 m^2}{4\alpha}, && (3.124)\\
C_2 &= -\frac{3}{2}\Lambda m^2 - \Lambda\alpha^2 m^2 - 3\xi\Lambda^2 + \frac{\lambda v\Lambda}{2\kappa^2} - \frac{\lambda m^2}{4\kappa^2} - \frac{1}{4}\Lambda m^2 v^2 + \frac{1}{8}m^4 v. && (3.125)
\end{aligned}$$

Now we form Γ_{G2} in (3.118) and note that the $1/\alpha$ terms cancel out as they must so that we can take the $\alpha \to 0$ limit in the expression for the effective action. This leaves the form of (3.119) but with

$$\begin{aligned}
A_2 &= \frac{3}{16}v^2 - \frac{1}{8}\omega v - \frac{1}{16}\alpha v^2 - \frac{1}{4}\xi - \frac{3}{8}v - \frac{1}{4}\omega + \frac{1}{4}\alpha v, && (3.126)\\
B_2 &= \frac{1}{8}v^2\Lambda + \frac{17}{16}vm^2 - \frac{3}{4}\alpha\xi^2 m^2 + \frac{3}{4}\xi m^2 - \frac{3}{4}\xi m^2 v - \frac{3}{8}v^2 m^2 + \frac{1}{4}\omega m^2 + \frac{1}{8}\omega m^2 v \\
&\quad -\frac{1}{2}\alpha m^2 v + \frac{1}{8}\alpha m^2 v^2 - \frac{3}{4}m^2 + \frac{1}{2}\Lambda\omega^2 + \frac{3}{2}\Lambda\xi v^2 + \alpha m^2 - \Lambda\alpha\omega + \frac{1}{2}\Lambda\alpha^2 \\
&\quad -\Lambda\omega v + \frac{1}{2}\Lambda\alpha v - \frac{3}{2}\Lambda\xi, && (3.127)\\
C_2 &= -\frac{3}{2}\Lambda m^2 - \Lambda\alpha^2 m^2 - 3\xi\Lambda^2 + \frac{\lambda v\Lambda}{2\kappa^2} - \frac{\lambda m^2}{4\kappa^2} - \frac{1}{4}\Lambda m^2 v^2 - \frac{5}{8}m^4 v + \frac{3}{4}m^4 + \frac{3}{16}m^4 v^2 \\
&\quad -\frac{1}{4}\alpha m^4 + \frac{1}{4}\alpha m^4 v - \frac{1}{16}\alpha m^4 v^2 + 3\Lambda\xi m^2 + \frac{3}{2}\Lambda\xi m^2 v - \frac{3}{2}\Lambda\xi m^2 v^2 - \frac{3}{2}\xi m^4 \\
&\quad +\frac{3}{4}\xi m^4 v + 3\Lambda^2\xi^2 - \frac{3}{2}\Lambda\xi^2 m^2 + 3\Lambda^2\xi^2 v + \frac{3}{4}\xi^2 m^4 - 3\Lambda\xi^2 m^2 v + 3\Lambda^2\xi^2 v^2. && (3.128)
\end{aligned}$$

The ghost action is also

$$S_{GH} = \int d^n x \left\{ -\frac{2}{\kappa^2}\bar{\eta}^\lambda \Box \eta_\lambda + \omega\bar{\eta}^\lambda \eta_\mu \partial^\mu \bar{\varphi}\partial_\lambda\bar{\varphi} \right\}. \tag{3.129}$$

There is no contribution to the quadratic part of the effective action. There is a contribution to the quartic part that is

$$\Gamma_{GH4} = \frac{1}{2}\langle (S_{GH2})^2 \rangle = -\frac{1}{8}\omega^2\kappa^4 L \int d^4x \left(\partial^\mu\bar{\varphi}\partial_\mu\bar{\varphi}\right)^2. \tag{3.130}$$

It remains to calculate the scalar and graviton contributions to the quartic effective action.

3.5 Quartic part from scalars and gravitons

The quartic expression for the effective action (apart from the ghost terms) is

$$\Gamma_{G4} = \left\langle S_4 - S_1 S_3 - \frac{1}{2}S_2^2 + \frac{1}{2}S_2 S_1^2 - \frac{1}{24}S_1^4 \right\rangle. \tag{3.131}$$

For $\langle S_4 \rangle$ we find

$$\langle S_4 \rangle = \frac{\kappa^2 \lambda}{24} L \int d^4x \bar{\varphi}^4 \left\{ \frac{v}{4} m^2 - \Lambda(2\alpha^2 + 3 + \frac{v^2}{2}) \right\}. \tag{3.132}$$

For $\langle S_1 S_3 \rangle$ we find

$$\begin{aligned} \langle S_1 S_3 \rangle \quad = \quad & \frac{\kappa^2 \lambda L}{12} \left(1 - \frac{v}{2}\right) \int d^4x \Big\{ \bar{\varphi}^4 \Big[-3m^2 + \frac{3}{2} v m^2 + \alpha m^2 - \frac{1}{2} \alpha v m^2 \\ & \qquad -6\xi\Lambda - 6v\xi\Lambda + 3\xi m^2 \Big] \\ & + \bar{\varphi}^3 \Box \bar{\varphi} \Big[6\xi - \frac{3}{2} v + \frac{1}{2}\omega + \frac{3}{2}\alpha\xi + \frac{1}{2}\alpha v - 6\xi + \frac{3}{2} - \alpha - \frac{3}{2}\alpha\xi \Big] \Big\}. \end{aligned} \tag{3.133}$$

3.5.1 Calculation of the λ divergence

To simplify the calculations we first assume that $\bar{\varphi}$ is constant. This is sufficient to consider the renormalization of λ since the relevant pole is proportional to $\bar{\varphi}^4$.

For $\langle S_2^2 \rangle$ we find (with $\bar{\varphi}$ constant)

$$\begin{aligned} \langle S_2^2 \rangle \quad = \quad & \kappa^4 L \int d^4x \bar{\varphi}^4 \Big\{ \left(\frac{3}{16} + \frac{1}{8}\alpha^2 \right) m^4 + \frac{3}{2}\xi m^2 \Lambda + \frac{\lambda^2}{8\kappa^4} \\ & - \frac{\lambda v m^2}{16\kappa^2} + \frac{v^2 m^4}{32} + \frac{9}{2}\xi^2 \Lambda^2 \Big\}. \end{aligned} \tag{3.134}$$

For $\langle S_2 S_1^2 \rangle$ we find (with $\bar{\varphi}$ constant)

$$\begin{aligned} \langle S_2 S_1^2 \rangle \quad = \quad & \kappa^4 L \int d^4x \bar{\varphi}^4 \Big\{ \frac{3}{2}\xi \left(\frac{1}{2} m^2 - \frac{v}{4} m^2 \right)^2 + \frac{3}{2}\xi m^2 \left(\frac{1}{2} m^2 - \frac{v}{4} m^2 \right) \\ & + \xi^2 \left(1 - \frac{v}{2}\right) \left(3\Lambda m^2 - \frac{3}{4} m^4 - \frac{3}{2}\Lambda m^2 \alpha^2 - \frac{3}{2}\alpha\Lambda m^2 + \frac{3}{4}\alpha m^4 + \frac{3}{2} v\Lambda m^2 - \frac{3}{2} v\alpha\Lambda m^2 \right) \\ & + \xi \left(\frac{\lambda}{\kappa^2} - \frac{v}{2} m^2 \right) \left(\frac{3}{4} v m^2 - \frac{3}{2} m^2 \right) + \xi^2 m^2 \left(-\frac{3}{2}\Lambda + \frac{3}{8} m^2 - \frac{3}{4}\Lambda v \right) \\ & + \xi^3 \left(\frac{3}{16} m^4 - \frac{3}{4} v\Lambda m^2 + \frac{3}{4} v^2 \Lambda^2 - \frac{3}{4}\Lambda m^2 + \frac{3}{2} v\Lambda^2 + \frac{9}{4}\Lambda^2 \right) \\ & + \xi^2 \left(\frac{\lambda}{4\kappa^2} - \frac{v}{8} m^2 \right) \left(-6\Lambda + 6m^2 - 12v\Lambda \right) \Big\}. \end{aligned} \tag{3.135}$$

For $\langle S_1^4 \rangle$ we find (with $\bar{\varphi}$ constant)

$$
\begin{aligned}
\langle S_1^4 \rangle \quad = \quad & 6\kappa^4 L \int d^4x \bar{\varphi}^4 \Big\{ \xi^2 m^4 \left(\frac{27}{2} - \frac{3}{2}\alpha - \frac{27}{2}v + \frac{3}{2}\alpha v + \frac{27}{8}v^2 - \frac{3}{8}\alpha v^2 \right) \\
& + \xi^3 \Lambda m^2 (36 + 18v - 18v^2) + \xi^3 m^4 (9v - 18) + \frac{27}{4}\xi^4 m^4 \\
& + \xi^4 \Lambda^2 (27 + 36v + 27v^2) + \xi^4 \Lambda m^2 (-18 - 27v) \Big\}. \qquad (3.136)
\end{aligned}
$$

Putting these all together, and dropping the derivative terms to keep only the $\bar{\varphi}^4$ term gives

$$
\Gamma_{G4} = \kappa^4 L \int d^4x \bar{\varphi}^4 A, \qquad (3.137)
$$

where A is given by

$$
\begin{aligned}
A \quad = \quad & -\frac{\lambda^2}{16\kappa^4} + \frac{\lambda}{\kappa^2} \left(\frac{\xi \Lambda}{2} + \frac{m^2}{4} + \frac{v^2 m^2}{16} - \frac{\alpha v^2 m^2}{48} - \frac{\xi \Lambda v^2}{4\kappa^2} + \frac{\alpha v m^2}{12} + \frac{\xi \Lambda v}{4} - \frac{3\xi^2 \Lambda}{4\kappa^2} \right. \\
& \left. + \frac{3\xi^2 m^2}{4} + \frac{v \xi m^2}{2} - \frac{3\xi^2 \Lambda v}{2} - \frac{5 v m^2}{24} - \frac{\Lambda}{8} - \xi m^2 - \frac{\alpha m^2}{12} - \frac{\Lambda v^2}{48} - \frac{\Lambda \alpha^2}{12} \right) \\
& - \frac{75}{8} \xi^3 \Lambda m^2 + \frac{3}{4} \xi^3 v \Lambda^2 + \frac{3}{8} \xi^3 v^2 \Lambda^2 + \frac{3}{4} \xi^2 m^2 \Lambda - \frac{1}{16} m^4 \alpha^2 + \frac{9}{16} \xi m^4 - \frac{27}{4} \xi^4 \Lambda^2 \\
& - \frac{57}{16} \xi^2 m^4 + \frac{147}{32} \xi^3 m^4 + \frac{9}{8} \xi^3 \Lambda^2 + \frac{51}{16} \xi^2 m^4 v + \frac{3}{4} \xi^2 m^4 \alpha - \frac{9}{64} \xi m^4 v^2 - \frac{27}{32} \xi^2 m^4 v^2 \\
& - \frac{9}{4} \xi^3 m^4 v - 9 \xi^4 \Lambda^2 v - \frac{27}{4} \xi^4 \Lambda^2 v^2 + \frac{9}{2} \xi^4 \Lambda m^2 - \frac{3}{4} \xi^2 m^2 \Lambda \alpha^2 - \frac{3}{4} \xi^2 m^2 \alpha \Lambda - \frac{3}{8} \xi^2 m^2 \Lambda v \alpha \\
& + \frac{3}{8} \xi^2 m^2 v \Lambda \alpha^2 - \frac{9}{16} \xi^2 m^4 \alpha v + \frac{3}{8} \xi^2 m^2 \Lambda v^2 + \frac{3}{8} \xi^2 m^2 \Lambda v^2 \alpha - \frac{39}{8} \xi^3 \Lambda v m^2 + \frac{3}{32} \xi^2 m^4 \alpha v^2 \\
& + \frac{9}{2} \xi^3 \Lambda m^2 v^2 + \frac{27}{4} \xi^4 \Lambda m^2 v - \frac{1}{64} v^2 m^4 - \frac{9}{4} \xi^2 \Lambda^2 - \frac{3}{32} m^4 - \frac{3}{4} \xi m^2 \Lambda - \frac{27}{16} \xi^4 m^4. \quad (3.138)
\end{aligned}
$$

For the correct case $\alpha = 0, v = 1$ we have

$$
\begin{aligned}
A \quad = \quad & -\frac{\lambda^2}{16\kappa^4} + \frac{\lambda}{\kappa^2} \left(-\frac{\xi m^2}{2} - \frac{7\Lambda}{48} + \frac{3\xi^2 m^2}{4} + \frac{\xi \Lambda}{2} + \frac{5m^2}{48} - \frac{9\xi^2 \Lambda}{4} \right) \\
& - \frac{45}{2} \xi^4 \Lambda^2 - \frac{3}{4} \xi m^2 \Lambda + \frac{27}{64} \xi m^4 - \frac{9}{4} \xi^2 \Lambda^2 - \frac{27}{16} \xi^4 m^4 - \frac{7}{64} m^4 + \frac{9}{4} \xi^3 \Lambda^2 \\
& + \frac{9}{8} \xi^2 m^2 \Lambda - \frac{39}{4} \xi^3 \Lambda m^2 + \frac{45}{4} \xi^4 \Lambda m^2 - \frac{39}{32} \xi^2 m^4 + \frac{75}{32} \xi^3 m^4. \qquad (3.139)
\end{aligned}
$$

[3] We refer, of course, to our technique as correct. $\alpha = 0$ is the limit that must be taken in our expression for the effective action whilst $v = 1$ is the case where we include the connection term, which we explained was crucial, being off-shell.

In the minimally coupled case ($\xi = 0$) this becomes

$$A = -\frac{\lambda^2}{16\kappa^4} + \frac{\lambda}{\kappa^2}\left(-\frac{7\Lambda}{48} + \frac{5m^2}{48}\right) - \frac{7}{64}m^4. \tag{3.140}$$

For the traditional case ($v = 0$), for pure gravity ($\Lambda = 0$), and with minimal coupling ($\xi = 0$) we find

$$A = -\frac{\lambda^2}{16\kappa^4} + \frac{\lambda m^2}{4\kappa^2}\left(1 - \frac{\alpha}{3}\right) - \frac{3}{32}m^4 - \frac{1}{16}m^4\alpha^2. \tag{3.141}$$

This demonstrates the gauge condition dependence of the traditional result. Note that in the absence of gravity we find the correct result (46) of

$$\Gamma = \int d^4x\bar{\varphi}^4\ \frac{\lambda^2}{128\pi^2(n-4)}. \tag{3.142}$$

3.5.2 Non-constant $\bar{\varphi}$

Whilst considering $\bar{\varphi}$ to be constant may be sufficient for considering the running of the scalar coupling constant, we should complete the calculation for one loop order by considering divergences resulting from the derivatives of $\bar{\varphi}$. The same technique is applied to significantly more terms and we find, *in addition to the parts calculated in the previous section*, the non-constant parts

$$\begin{aligned}\left\langle S_2^2\right\rangle \quad = \quad & \kappa^4 L\int\frac{d^4x}{(2\pi)^4}\bar{\varphi}^3\Box\bar{\varphi}(m^2(-3/16+(1/4)\alpha^2+\xi-(3/2)v)+\Lambda(1/2+\xi(3/4)) \\ & +\frac{\lambda}{4\kappa^2}(-1/4+(1/2)v)) \\ & +\kappa^4 L\int\frac{d^4x}{(2\pi)^4}\bar{\varphi}^2(\Box\bar{\varphi})^2(3/2+(1/8)\alpha^2-(1/16)v+3\xi(1/2) \\ & -(1/16)v^2)\end{aligned} \tag{3.143}$$

$$\begin{aligned}
\langle S_1^2 S_2\rangle &= \kappa^4 L \int \frac{d^4x}{(2\pi)^4}\bar{\varphi}^3\Box\bar{\varphi}m^2((3/2)\xi - (1/4)v + (3/4)\alpha + (1/4)\alpha^2 + (1/2)\alpha v) \\
&\quad + \Lambda(1-(1/2)v)(3/8 - \alpha - v + (1/4)\alpha^2 + \xi^2) + \frac{\lambda}{4\kappa^2}((1/2)v - 3/4 - (1/4)\alpha) \\
&\quad + \frac{\omega}{\alpha}(vm^2 + \xi v) + \Lambda(1-(1/2)v)(1 + v - (3/4)\xi^2)) \\
&\quad + \kappa^4 L \int \frac{d^4x}{(2\pi)^4}\bar{\varphi}^2(\Box\bar{\varphi})^2((3/2)\alpha - 2\xi + (1/2)v - (1/4)v^2 - (3/4)\xi + 1/4 + \frac{\omega^2}{\alpha^2}((1/2)v \\
&\quad + (3/4)\alpha - 1/2))
\end{aligned} \tag{3.144}$$

$$\begin{aligned}
\langle S_1^4\rangle &= \kappa^4 L \int \frac{d^4x}{(2\pi)^4}\bar{\varphi}^3\Box\bar{\varphi}(m^2\xi^2((3/2)\alpha - (3/4)\alpha v + (27/4)v^2 + (3/4)\alpha v^2) \\
&\quad + \Lambda(\xi^4(9-18v) + \xi^3(18 + 9v - 9v^2)) + \frac{\lambda}{4\kappa^2}((1/2)\xi + (3/4)\alpha + (1/2)v + (1/4)\alpha v) \\
&\quad + \frac{\omega}{\alpha}(12m^2 v + 12m^2\xi + \Lambda(12 + 6v - 6v^2 - 9\xi^2 + (9/2)\xi v))) \\
&\quad + \kappa^4 L \int \frac{d^4x}{(2\pi)^4}\bar{\varphi}^2(\Box\bar{\varphi})^2(-(3/2)\alpha(1 - v + v^2) + \xi^4(18 - 36v) + \xi^3(18 + 9v - (27/2)v^2) \\
&\quad + \xi^2((3/2)\alpha - (3/4)v + (27/2)v^2) + \xi((1/2)\alpha v + (27/4)v^2 + (3/4)\alpha v^2) \\
&\quad + 12\frac{\omega^2}{\alpha^2}((1/2)v - 1/2 + (3/4)\alpha))
\end{aligned} \tag{3.145}$$

Putting these results together with those previously calculated, we find the term quartic in the background field (recall that we have already calculated the corresponding ghost term in 3.130) to be

$$\Gamma_{G4} = \kappa^4 L \int d^4x \left(\bar{\varphi}^4 A + \bar{\varphi}^3\Box\bar{\varphi}B + \bar{\varphi}^2(\Box\bar{\varphi})^2 C\right), \tag{3.146}$$

where

$$
\begin{aligned}
A &= -\frac{\lambda^2}{16\kappa^4}+\frac{\lambda}{\kappa^2}\left(\frac{\xi\,\Lambda}{2}+\frac{m^2}{4}+\frac{v^2m^2}{16}-\frac{\alpha\,v^2m^2}{48}-\frac{\xi\,\Lambda\,v^2}{4\kappa^2}+\frac{\alpha\,vm^2}{12}+\frac{\xi\,\Lambda\,v}{4}-\frac{3\xi^2\Lambda}{4\kappa^2}\right.\\
&\left.+\frac{3\xi^2m^2}{4}+\frac{v\xi\,m^2}{2}-\frac{3\xi^2\Lambda\,v}{2}-\frac{5vm^2}{24}-\frac{\Lambda}{8}-\xi\,m^2-\frac{\alpha\,m^2}{12}-\frac{\Lambda\,v^2}{48}-\frac{\Lambda\,\alpha^2}{12}\right)\\
&-\frac{75}{8}\,\xi^3\Lambda\,m^2+\frac{3}{4}\,\xi^3v\Lambda^2+\frac{3}{8}\,\xi^3v^2\Lambda^2+\frac{3}{4}\,\xi^2m^2\Lambda-\frac{1}{16}\,m^4\alpha^2+\frac{9}{16}\,\xi\,m^4-\frac{27}{4}\,\xi^4\Lambda^2\\
&-\frac{57}{16}\,\xi^2m^4+\frac{147}{32}\,\xi^3m^4+\frac{9}{8}\,\xi^3\Lambda^2+\frac{51}{16}\,\xi^2m^4v+\frac{3}{4}\,\xi^2m^4\alpha-\frac{9}{64}\,\xi\,m^4v^2-\frac{27}{32}\,\xi^2m^4v^2\\
&-\frac{9}{4}\,\xi^3m^4v-9\,\xi^4\Lambda^2v-\frac{27}{4}\,\xi^4\Lambda^2v^2+\frac{9}{2}\,\xi^4\Lambda\,m^2-\frac{3}{4}\,\xi^2m^2\Lambda\,\alpha^2-\frac{3}{4}\,\xi^2m^2\alpha\,\Lambda-\frac{3}{8}\,\xi^2m^2\Lambda\,v\alpha\\
&+\frac{3}{8}\,\xi^2m^2v\Lambda\,\alpha^2-\frac{9}{16}\,\xi^2m^4\alpha\,v+\frac{3}{8}\,\xi^2m^2\Lambda\,v^2+\frac{3}{8}\,\xi^2m^2\Lambda\,v^2\alpha-\frac{39}{8}\,\xi^3\Lambda\,vm^2+\frac{3}{32}\,\xi^2m^4\alpha\,v^2\\
&+\frac{9}{2}\,\xi^3\Lambda\,m^2v^2+\frac{27}{4}\,\xi^4\Lambda\,m^2v-\frac{1}{64}\,v^2m^4-\frac{9}{4}\,\xi^2\Lambda^2-\frac{3}{32}\,m^4-\frac{3}{4}\,\xi\,m^2\Lambda-\frac{27}{16}\,\xi^4m^4. \quad (3.147)
\end{aligned}
$$

$$
\begin{aligned}
B &= -\frac{1}{384}(-24\frac{\lambda}{\kappa^2}+47\alpha\frac{\lambda}{\kappa^2}+74\frac{\lambda}{\kappa^2}v-288\Lambda\xi^4v+144\Lambda\xi^3v-144\Lambda\xi^3v^2-96\alpha vm^2\\
&+108\xi^2v^2m^2+24\xi^2\alpha m^2-24\frac{\lambda}{\kappa^2}v^2+2\frac{\lambda}{\kappa^2}\xi-31\frac{\lambda}{\kappa^2}\alpha v+8\frac{\lambda}{\kappa^2}v\omega+8\frac{\lambda}{\kappa^2}\alpha v^2-96\xi m^2\\
&-240vm^2+144\xi\Lambda-144\alpha m^2+144\xi^4\Lambda+288\xi^3\Lambda+96\Lambda v(3/8-\alpha-v+(1/4)\alpha^2+\xi^2)\\
&-36m^2-96\Lambda-12\xi^2m^2\alpha v+12\xi^2m^2\alpha v^2-16\omega\frac{\lambda}{\kappa^2}) \qquad (3.148)
\end{aligned}
$$

$$
\begin{aligned}
C &= -\frac{1}{96}(60+2\xi\alpha v+3\xi\alpha v^2-72\alpha-6\alpha v^2-144\xi^4v+36\xi^3v-54\xi^3v^2+6\xi^2\alpha\\
&-3v\xi^2+54\xi^2v^2+27\xi v^2+72\xi^4+72\xi^3+6\alpha^2-27v+204\xi+9v^2) \qquad (3.149)
\end{aligned}
$$

Note that there are no terms proportional to $\bar{\varphi}(\Box\bar{\varphi})^3$ or $(\Box\bar{\varphi})^4$ as such terms, by virtue of their high numbers of derivatives, are not able to be tamed with any dimensionful quantities such that they can contribute to logarithmic divergences. Also note that, as expected in the Vilkovisky-DeWitt method, although terms proportional to $1/\alpha$, $1/\alpha^2$ appear in the individual expressions for the parts of the action, they vanish in our overall effective action, and we can take the correct limit $\alpha \to 0$.

3.6 Renormalization

Now we proceed to calculate the counterterms. By comparing 3.142 with the original action 3.1 we can read off the renormalized counterterms. This is the goal of this section. We follow the standard renormalization techniques developed by 't Hooft (45).

3.6.1 Beta functions

Now that we have calculated the counterterms, we are in a position to be able to calculate the beta functions. In general, we have

$$\beta(g) = \mu \frac{\partial g}{\partial \mu} \tag{3.150}$$

which relates the coupling strength, g, with the energy scale, μ. Inserting our renormalized couplings, we have, for an energy scale E and a coupling parameter λ, the β-function

$$\beta(\lambda) = E \frac{\partial}{\partial E} \lambda. \tag{3.151}$$

We should take an aside to develop some physical understanding of the β-function and of the running of coupling parameters. It is perhaps instructive to consider a few illustrative examples.

First, consider QED in the absence of gravity. It was found in (59) that the coupling parameter for electromagnetism increases at shorter length scales. The energy of a single central electron polarises the vacuum such that virtual pairs are created. If we consider the virtual e^+e^- pair to form a dipole, then they align such that the positronic part shields the charge of the central electron. Far from the electron then, the effective coupling strength that is observed is lower than the actual coupling strength of the electron alone, due to the *screening* of the positrons. As we go to shorter scales, we gradually see more of the force, so that, qualitatively, the coupling constant increases with decreasing length scale.

Now consider instead the case of QCD. In this case, the nature of the interaction between the central particle and the virtual quark-antiquark pairs it induces should this time screen the colour charge. However, the force carriers themselves, the gluons, carry a colour charge so in fact the overall effect is such that there exists *anti-screening* whereby the force field is augmented. In this case, as we move to ever shorter scales we will see less of this augmentation and as such, qualitatively, the force lowers - in fact, vanishes - with decreasing length scale. This is known as asymptotic freedom and corresponds to a negative value of the β-function.

But how might one interpret gravity in this (anti-)screening regime? In pure quantum gravity, it is known that gravity has an anti-screening effect. In (68), it was shown that the behaviour of the β-function depended on the choice of gauge parameter, and that in a physical gauge, antiscreening was calculated.

Of course, gravitational charges are masses, so any virtual pairs fluctuating into existence will surely increase the mass. Therefore, an antiscreening effect due to gravity is to be expected. Morever, when approaching the central mass, we see less of the surrounding mass associated with the virtual pairs and therefore the gravitational force appears weaker at shorter lengths; we can say gravity is asymptotically free. However, for any other field polarising the vacuum, aside from electric charge, colour charge or whatever more exotic charges they might carry, in all cases they share a fundamental property, mass. In some sense then, it is not surprising that gravity is claimed to lead to the asymptotic freedom of any gauge theory.

Having calculated the pole part of the effective action, we now seek to renormalize our original action. We will use a counterterm procedure where our counterterms must compensate for the pole parts. The bare masses are replaced by dimensionless renormalized masses by the introduction of a counterterm factor and a length scale [4].

[4] Originally in (45), t'Hooft used a mass scale, typically denoted μ in most texts. Our length scale, l, here is simply the reciprocal of that.

Since the action itself should be dimensionless, we find first that the bare field should be replaced by

$$\bar{\varphi}_B = l^{2-n/2} Z_\varphi \bar{\varphi}_R \tag{3.152}$$

where we have introduced a length l, the counterterm factor is $Z_\varphi = 1 + \delta\varphi$ and $\bar{\varphi}_R$ is our renormalized field.

Next, we look at the mass. We do not require any extra factor involving l in fron of the mass parameter to correct the dimensionality of the mass term and so we simply have

$$m_B^2 = m_R^2 + \delta m^2. \tag{3.153}$$

Finally, consider the coupling to the φ^4. Dimensional considerations lead to

$$\lambda_B = l^{2(4-n)}(\lambda_R + \delta\lambda). \tag{3.154}$$

As a convenience, having introduced these expressions for the bare quantities, hereafter we shall drop the subscript R for renormalized quantities.

3.6.2 Renormalization of the mass, m^2

Using 3.152 and 3.153, we can see that

$$\frac{1}{2}m^2\bar{\varphi}_B^2 \to \frac{1}{2}(m_B^2 + \delta m_B^2 + m_B^2 \delta Z_\varphi) l^{4-n} \bar{\varphi}^2. \tag{3.155}$$

We find

$$\delta Z_\varphi = -\frac{\kappa^2 B}{4\pi^2(n-4)} \tag{3.156}$$

$$\delta m^2 = -\frac{\kappa^2(C + m^2 B)}{4\pi^2(n-4)} \tag{3.157}$$

with the values of B and C from the Γ_2 calculation.

3.6.3 Renormalization of the scalar coupling, λ

Using 3.152 and 3.154, we can see that

$$\frac{1}{4!}\lambda\bar{\varphi}_B^4 \to \frac{1}{24}(\lambda + \delta\lambda + 2\lambda\delta Z_\varphi)l^{2(4-n)}\bar{\varphi}^4. \tag{3.158}$$

and with

$$Z_\varphi = 1 + \frac{e_1}{(n-4)} + \frac{e_2}{(n-4)^2} + \dots \tag{3.159}$$

and following (27), we eventually are led to (with $\alpha = 0$ and $\omega = 1$, per our method)

$$\beta_{gravity} = \frac{\lambda\kappa^2}{64\pi^2}(5m^2\lambda + 2m^2 - 21m^4\kappa^2) \tag{3.160}$$

Chapter 4

Quadratic Divergences

Following the work of several authors (29; 33; 21), we now turn our attention to the issue of quadratic divergences.

4.1 Heat Kernel Method

To deal with expressions like

$$L_{\Delta} = \frac{1}{2} \ln \det \Delta^i_j \tag{4.1}$$

we will use the heat kernel. The method we will employ is based on the proper time method of Schwinger (79) which was developed by DeWitt (40). In this section we will follow the framework of (27).

If we have an operator Δ^i_j, then its heat kernel $K^i_j(x, x'; \tau)$ is defined to be the solution to the heat-type equation [1]

$$-\frac{\partial}{\partial \tau} K^i_j(x, x'; \tau) = \Delta^i_k K^k_j(x, x'; \tau) \tag{4.2}$$

with the boundary condition $K^i_j(x, x'; \tau) = \delta^i_j \delta(x, x')$.

[1] We diverge slightly from (79) here. Schwinger referred to the parameter τ as the proper time and solved a Schrodinger-type equation. We have Wick rotated to imaginary τ and hence solve a heat-type equation.

The Green function for Δ^i_j is defined in the usual way by

$$\Delta^i_k G^k_j(x,x') = \delta^i_j \delta(x,x') \tag{4.3}$$

and related to the heat kernel by

$$G^i_j(x,x') = \int_0^\infty d\tau K^i_j(x,x';\tau). \tag{4.4}$$

$$L_\Delta = -\frac{1}{2}\int d^nx \int_0^\infty \frac{d\tau}{\tau} trK^i_j(x,x;\tau). \tag{4.5}$$

We can relate L_Δ to the heat kernel by use of the asymptotic expansion as $\tau \to 0$,

$$K^i_j(x,x;\tau) \sim (4\pi\tau)^{-n/2} \sum_{r=0}^{\infty} \tau^r E_r{}^i_j(x) \tag{4.6}$$

where n is the spacetime dimension and $E_r{}^i_j(x)$ are the heat kernel coefficients.

If we denote the divergent part of an expression as **divp**, then we can express the divergent part of L_Δ by

$$\mathbf{divp}(L_\Delta) = -\frac{1}{2}\int d^nx \mathbf{divp}\left\{\int_{\tau_c}^\infty \frac{d\tau}{\tau} trK^i_j(x,x;\tau)\right\} \tag{4.7}$$

with the proper time cut-off τ_c. If we use the asymptotic expansion (4.6) then we can perform the integration. Recalling that the asymptotic expansion is only valid for $\tau \to 0$, the upper limit of the integral is irrelevant. For $n = 4$ we find the divergent part to be

$$\mathbf{divp}(L_\Delta) = -\frac{1}{32\pi^2}\int d^4x \left\{\frac{1}{2\tau_c^2} trE_0 + \frac{1}{\tau_c} trE_1 - (\ln \tau_c) trE_2\right\} \tag{4.8}$$

Now we will make a connection with a conventional energy cut-off. If we note that the operator Δ^i_j has units of $(length)^{-2}$, then from (4.2), τ_c must have units of $(length)^2$, or units of $(energy)^{-2}$ in $\hbar = c = 1$ units. So we can replace the proper time cut-off τ_c by an equivalent energy cut-off E_c. We then have

$$\mathbf{divp}(L_\Delta) = -\frac{1}{32\pi^2}\int d^4x \left\{\frac{1}{2}E_c^4 trE_0 + E_c^2 trE_1 - (\ln E_c^2) trE_2\right\}. \tag{4.9}$$

Before we proceed further, let us compare this technique with dimensional regularization which we employed earlier. Applying dimensional regularization to (4.5), we find (for $n = 4 + \epsilon$) a simple pole at

$$\mathbf{divp}(L_\Delta) = \frac{1}{16\pi^2\epsilon}\int d^4x trE_2. \tag{4.10}$$

We were able to take the lower limit of the proper time integration at $\tau = 0$ here since we regulated the integral by the usual analytic continuation in ϵ.

Comparing this result with (4.9), it is clear that the coefficient of the ϵ^{-1} term from dimensional regularization matches that of the term multiplying the $\ln E_c^{-1}$ term in the proper time cut-off method. Significantly, the quartic and quadratic divergences are seen not to appear in dimensional regularization; that is to say, they have been regulated to zero. If the cut-off energy E_c is considered to be a fundamental scale in the effective field theory then it may be folly to ignore the quadratic and quartic terms.

4.2 Normal Coordinate Expansion

If we want the quadratic divergences, then we need to find E_1. This can be found by working out the pole part of the Green function G^i_j in dimensional regularization, a result found in (85), namely

$$\begin{aligned}\mathbf{divp}G^i_j(x,x) &= \mathbf{divp}\left\{\int_0^1 d\tau K^i_j(x,x;\tau)\right\} \\ &= -\frac{1}{8\pi^2\epsilon}E_1{}^i_j(x).\end{aligned} \tag{4.11}$$

For clarity, the utility of dimensional regularization here is simply to identify the coefficient

E_1; the use of the cut-off described earlier stands alone from this. (Another, more general, way to calculate the E_1 coefficient without resorting to dimensional regularization is known which leads to the same result).

The first step will be to adopt the local momentum space approach of (86). The operator Δ^i_j is taken to have the general form

$$\Delta^i_j = (A^{\alpha\beta})^i_j \partial_\alpha \partial_\beta + (B^\alpha)^i_j \partial_\alpha + (C)^i_j \tag{4.12}$$

where the coefficients $(A^{\alpha\beta})^i_j$, $(B^\alpha)^i_j$ and $(C)^i_j$ can all depend on the background field.

Normal coordinates are introduced at the point x' with $x^\mu = x'^\mu + y^\mu$. Expanding the coefficients of (4.12) about $y^\mu = 0$, we find

$$(A^{\alpha\beta})^i{}_j = (A_0^{\alpha\beta})^i{}_j + \sum_{n=1}^{\infty} (A^{\alpha\beta}{}_{\mu_1 \ldots \mu_n})^i{}_j y^{\mu_1} \ldots y^{\mu_n} \tag{4.13}$$

$$(B^\alpha)^i{}_j = (B_0^\alpha)^i{}_j + \sum_{n=1}^{\infty} (B^\alpha{}_{\mu_1 \ldots \mu_n})^i{}_j y^{\mu_1} \ldots y^{\mu_n} \tag{4.14}$$

$$(C)^i{}_j = (C_0)^i{}_j + \sum_{n=1}^{\infty} (C_{\mu_1 \ldots \mu_n})^i{}_j y^{\mu_1} \ldots y^{\mu_n}. \tag{4.15}$$

Recall (4.3) and let the operator (4.12) act on the Green function,

$$G^i_j(x, x') = \int \frac{d^n p}{(2\pi)^n} e^{ip\cdot y} G^i_j(p). \tag{4.16}$$

A combination of the expansions (4.13), the realisation

$$y^{\mu_1} \ldots y^{\mu_n} e^{ip\cdot y} = (-i)^n \frac{\partial^n}{\partial p_{\mu_1} \ldots \partial p_{\mu_n}} e^{ip\cdot y} \tag{4.17}$$

and some partial integration to shift the momentum integrals from the exponential lead to the result

$$\begin{aligned}\delta_i^k &= -\sum_{n=0}^{\infty} i^n (A^{\alpha\beta}{}_{\mu_1\ldots\mu_n})_{ij} \frac{\partial^n}{\partial p_{\mu_1}\ldots\partial p_{\mu_n}}[p_\alpha p_\beta G^{jk}(p)] \\ &\quad + i\sum_{n=0}^{\infty} i^n (B^{\alpha}{}_{\mu_1\ldots\mu_n})_{ij} \frac{\partial^n}{\partial p_{\mu_1}\ldots\partial p_{\mu_n}}[p_\alpha G^{jk}(p)] \\ &\quad + \sum_{n=0}^{\infty} i^n (C_{\mu_1\ldots\mu_n})_{ij} \frac{\partial^n}{\partial p_{\mu_1}\ldots\partial p_{\mu_n}}[G^{jk}(p)] \end{aligned} \tag{4.18}$$

where the $n = 0$ terms in the sum are defined to be the first terms in the expansions (4.13).

If we write the Green function as

$$G^{jk}(p) = G_0^{jk}(p) + G_1^{jk}(p) + G_2^{jk}(p) + \ldots \tag{4.19}$$

where $G_n^{jk}(p) \sim p^{-2-n}$ for large p, then we can work order by order to deduce $G^{jk}(p)$. Since we only need to find the pole part of the Green function in (4.11), then for $n \to 4$ we only need to keep terms up to G_2^{jk}.

First, consider the p^0 term in (4.18). We find

$$(A_0^{\alpha\beta})_{ij} p_\alpha p_\beta G_0^{jk}(p) = -\delta_i^k \tag{4.20}$$

to be the leading asymptotic behaviour of $G^{jk}(p)$.

Now look at the term of order p^{-1}. We find,

$$0 = -i(A_\mu^{\alpha\beta})_{ij} \frac{\partial}{\partial p_\mu}(p_\alpha p_\beta G_0^{jk}] + i(B_0^\alpha)_{ij} p_\alpha G_0^{jk} - (A_0^{\alpha\beta})_{ij} p_\alpha p_\beta G_1^{jk}. \tag{4.21}$$

If we let $G_1^{jk} = G_0^{jl}\tilde{G}_l^k$ we have

$$0 = -i(A_\mu^{\alpha\beta})_{ij} \frac{\partial}{\partial p_\mu}[p_\alpha p_\beta G_0^{jk}] + i(B_0^\alpha)_{ij} p_\alpha G_0^{jk} - (A_0^{\alpha\beta})_{ij} p_\alpha p_\beta G_0^{jl}\tilde{G}_l^k \tag{4.22}$$

and we can use (4.20) to simplify the final term, so that

$$\tilde{G}_i^k = i(A_\mu^{\alpha\beta})_{ij}\frac{\partial}{\partial p_\mu}[p_\alpha p_\beta G_0^{jk}] - i(B_0^\alpha)_{ij}p_\alpha G_0^{jk} \tag{4.23}$$

and so

$$G_1^{ij}(p) = G_0^{il}i(A_\mu^{\alpha\beta})_{lk}\frac{\partial}{\partial p_\mu}[p_\alpha p_\beta G_0^{kj}] - G_0^{il}(B_0^\alpha)_{lk}p_\alpha G0^{kj}. \tag{4.24}$$

A similar analysis for the term of order p^{-2} leads to

$$\begin{aligned} G_2^{ij}(p) \quad = \quad & iG_0^{il}(A^{\alpha\beta}{}_\mu)_{lk}\frac{\partial}{\partial p_\mu}\left[p_\alpha p_\beta G_1^{kj}\right] - iG_0^{il}(A^{\alpha\beta}{}_{\mu\nu})_{lk}\frac{\partial^2}{\partial p_\mu \partial p_\nu}\left[p_\alpha p_\beta G_0^{kj}\right] \\ & -iG_0^{il}(B_0^\alpha)_{lk}p_\alpha G_1^{kj} + iG_0^{il}(B_\mu^\alpha)_{lk}\frac{\partial}{\partial p_\mu}\left[p_\alpha G_0^{kj}\right] - iG_0^{il}(C_0)_{lk}G_0^{kj} \end{aligned} \tag{4.25}$$

4.3 Application to ϕ^4-gravity

We follow the method of (29) for ϕ^4 theory rather than electromagnetism with the difference that here we calculate all derivative terms. To begin, we will rewrite S in the more convenient form

$$\begin{aligned} S_q = \int d^n x \Big\{ & A^{\alpha\beta\rho\sigma\gamma\delta}h_{\alpha\beta}\partial_\rho\partial_\sigma h_{\gamma\delta} + B^{\rho\alpha\beta\sigma\gamma\delta}\partial_\rho h_{\alpha\beta}\partial_\sigma h_{\gamma\delta} + C^{\alpha\beta\gamma\delta}h_{\alpha\beta}h_{\gamma\delta} \\ & + D^{\gamma\delta\alpha\beta}\partial_\gamma\partial_\delta h_{\alpha\beta}\psi + E^{\gamma\alpha\beta}\partial_\gamma h_{\alpha\beta}\psi + F^{\alpha\beta\gamma}h_{\alpha\beta}\partial_\gamma\psi \\ & + G^{\alpha\beta}h_{\alpha\beta}\psi + H^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi + I\psi^2 \Big\} \end{aligned} \tag{4.26}$$

where

$$\begin{aligned}
A^{\alpha\beta\rho\sigma\gamma\delta} &= P_0^{\alpha\beta\rho\sigma\gamma\delta} + \kappa^2 P_2^{\alpha\beta\rho\sigma\gamma\delta} && (4.27)\\
B^{\rho\alpha\beta\sigma\gamma\delta} &= Q_0^{\rho\alpha\beta\sigma\gamma\delta} + \kappa^2 Q_2^{\rho\alpha\beta\sigma\gamma\delta} && (4.28)\\
C^{\alpha\beta\gamma\delta} &= P_0^{\alpha\beta\gamma\delta} + \kappa^2 P_2^{\alpha\beta\gamma\delta} + \kappa^2 P_4^{\alpha\beta\gamma\delta} && (4.29)\\
D^{\gamma\delta\alpha\beta} &= \kappa P_1^{\gamma\delta\alpha\beta} && (4.30)\\
E^{\gamma\alpha\beta} &= \kappa Q_1^{\gamma\alpha\beta} && (4.31)\\
F^{\alpha\beta\gamma} &= \kappa P_1^{\alpha\beta\gamma} && (4.32)\\
G^{\alpha\beta} &= \kappa P_1^{\alpha\beta} + \kappa P_3^{\alpha\beta} && (4.33)\\
H^{\alpha\beta} &= P_0^{\alpha\beta} && (4.34)\\
I &= P_0 + \kappa^2 P_2 + \kappa^2 P_4 && (4.35)
\end{aligned}$$

where P_0, P_1, etc. are given in the next section.

4.3.1 Redefined terms

We can rewrite S_0 as

$$\begin{aligned}
S_0 = \int d^n x \qquad &\Big\{ h_{\alpha\beta}\partial_\rho\partial_\sigma h_{\gamma\delta} P_0^{\alpha\beta\rho\sigma\gamma\delta}(x) + \partial_\rho h_{\alpha\beta}\partial_\sigma h_{\gamma\delta} Q_0^{\rho\alpha\beta\sigma\gamma\delta}(x) \\
&+ h_{\alpha\beta}h_{\gamma\delta}P_0^{\alpha\beta\gamma\delta}(x) + \partial_\alpha\psi\partial_\beta\psi P_0^{\alpha\beta}(x) + \psi^2 P_0(x) \Big\} \qquad (4.36)
\end{aligned}$$

where

$$P_0^{\alpha\beta\rho\sigma\gamma\delta} = -\frac{1}{4}\delta^{\rho\sigma}\left(\delta^{\alpha\gamma}\delta^{\delta\beta}+\delta^{\alpha\delta}\delta^{\gamma\beta}-\delta^{\alpha\beta}\delta^{\gamma\delta}\right) \tag{4.37}$$

$$Q_0^{\rho\alpha\beta\sigma\gamma\delta} = \left(\frac{1}{\alpha}-1\right)\left[\delta^{\rho(\alpha}\delta^{\beta)(\delta}\delta^{\gamma)\alpha}-\frac{1}{2}\left(\delta^{\alpha(\rho}\delta^{\sigma)\beta}\delta^{\gamma\delta}+\delta^{\gamma(\rho}\delta^{\sigma)\delta}\delta^{\alpha\beta}\right)\right.$$
$$\left.+\frac{1}{4}\delta^{\alpha\beta}\delta^{\rho\sigma}\delta^{\gamma\delta}\right] \tag{4.38}$$

$$P_0^{\alpha\beta\gamma\delta} = -\Lambda\left(1+\frac{v}{2}\left(\frac{n-4}{2-n}\right)\right)\left(\delta^{\alpha(\gamma}\delta^{\delta)\beta}-\frac{1}{2}\delta^{\alpha\beta}\delta^{\gamma\delta}\right) \tag{4.39}$$

$$P_0^{\alpha\beta} = \frac{1}{2}\delta^{\alpha\beta} \tag{4.40}$$

$$P_0 = \frac{1}{2}m^2+\frac{vn\Lambda}{4-2n} \tag{4.41}$$

We can rewrite S_1 as

$$S_1 = \kappa\int d^nx\left\{h_{\alpha\beta}\partial_\gamma\psi P_1^{\alpha\beta\gamma}(x)+h_{\alpha\beta}\psi P_1^{\alpha\beta}(x)+\partial_\gamma\partial_\delta h_{\alpha\beta}\psi P_1^{\gamma\delta\alpha\beta}(x)+\partial_\gamma h_{\alpha\beta}\psi Q_1^{\gamma\alpha\beta}(x)\right\} \tag{4.42}$$

where

$$P_1^{\alpha\beta\gamma}(x) = \frac{1}{2}\left(\delta^{\alpha\beta}\delta^{\mu\gamma}-\delta^{\alpha\mu}\delta^{\beta\gamma}-\delta^{\alpha\gamma}\delta^{\mu\beta}\right)\partial_\mu\bar{\varphi} \tag{4.43}$$

$$P_1^{\alpha\beta}(x) = \left[\left(\frac{1}{2}-\frac{v}{4}\right)m^2\bar{\varphi}+\frac{v}{4}\Box\bar{\varphi}\right]\delta^{\alpha\beta} \tag{4.44}$$

$$P_1^{\gamma\delta\alpha\beta}(x) = \frac{1}{2}\xi\bar{\varphi}\left(\delta^{\alpha\gamma}\delta^{\beta\delta}+\delta^{\alpha\delta}\delta^{\beta\gamma}-2\delta^{\gamma\delta}\delta^{\alpha\beta}\right) \tag{4.45}$$

$$Q_1^{\gamma\alpha\beta}(x) = -\frac{\omega}{2\alpha}\left(\delta^{\alpha\gamma}\delta^{\beta}_{\nu}+\delta^{\beta\gamma}\delta^{\alpha}_{\nu}-\delta^{\gamma}_{\nu}\delta^{\alpha\beta}\right)\partial^\nu\bar{\varphi} \tag{4.46}$$

We rewrite S_2 as

$$S_2 = \kappa^2\int d^nx\left\{h_{\alpha\beta}h_{\gamma\delta}P_2^{\alpha\beta\gamma\delta}(x)+h_{\alpha\beta}\partial_\rho\partial_\sigma h_{\gamma\delta}P_2^{\alpha\beta\rho\sigma\gamma\delta}(x)+\partial_\rho h_{\alpha\beta}\partial_\sigma h_{\gamma\delta}Q_2^{\rho\alpha\beta\sigma\gamma\delta}(x)+\psi^2P_2(x)\right\} \tag{4.47}$$

where

$$
\begin{aligned}
P_2^{\alpha\beta\gamma\delta}(x) &= \frac{1}{2}\Bigg[\frac{1}{2}\left(\delta^{\alpha(\mu}\delta^{\lambda)\beta}\delta^{\gamma}_{(\lambda}\delta^{\nu)\delta} + \delta^{\gamma(\mu}\delta^{\lambda)\delta}\delta^{\alpha}_{(\lambda}\beta^{\nu)\delta}\right) \\
&\quad -\frac{1}{4}\left(\delta^{\alpha\beta}\delta^{\gamma(\mu}\delta^{\nu)\delta} + \delta^{\gamma\delta}\delta^{\alpha(\mu}\delta^{\nu)\beta}\right) - \frac{1}{4}\delta^{\mu\nu}\delta^{\alpha(\gamma}\delta^{\delta)\beta} + \frac{1}{8}\delta^{\alpha\beta}\delta^{\gamma\delta}\delta^{\mu\nu}\Bigg]\partial_\mu\bar{\varphi}\partial^\nu\bar{\varphi} \\
&\quad +\frac{1}{2}m^2\bar{\varphi}^2\left(\frac{1}{8}\delta^{\alpha\beta}\delta^{\gamma\delta} - \frac{1}{4}\delta^{\alpha(\gamma}\delta^{\delta)\beta}\right) + \frac{v}{4}\left[T_2\right]^{\mu\nu\lambda\sigma}\frac{1}{2}\left(\delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)}\delta^{\gamma}_{(\lambda}\delta^{\delta}_{\sigma)} + \delta^{\alpha}_{(\lambda}\delta^{\beta}_{\sigma)}\delta^{\gamma}_{(\mu}\delta^{\delta}_{\nu)}\right) \\
P_2^{\alpha\beta\rho\sigma\gamma\delta}(x) &= \frac{1}{2}\xi\bar{\varphi}^2\Bigg\{\delta^{\rho\sigma}\delta^{\alpha(\gamma}\delta^{\delta)\beta} - \left(\delta^{\rho(\alpha}\delta^{\beta)(\delta}\delta^{\gamma)\sigma} + \delta^{\sigma(\alpha}\delta^{\beta)(\delta}\delta^{\gamma)\rho}\right) + \delta^{\gamma\delta}\delta^{\alpha(\rho}\delta^{\sigma)\beta} + \frac{1}{2}\delta^{\alpha\beta}\delta^{\rho(\gamma}\delta^{\delta)\sigma} \\
&\quad -\frac{1}{2}\delta^{\alpha\beta}\delta^{\rho\sigma}\delta^{\gamma\delta}\Bigg\} \qquad (4.48) \\
Q_2^{\rho\alpha\beta\sigma\gamma\delta}(x) &= \frac{1}{2}\xi\bar{\varphi}^2\Bigg\{-\delta^{\rho(\alpha}\delta^{\beta)(\gamma}\delta^{\delta)\sigma} + \frac{1}{2}\left(\delta^{\gamma\delta}\delta^{\alpha(\rho}\delta^{\sigma)\beta} + \delta^{\alpha\beta}\delta^{\gamma(\rho}\delta^{\sigma)\delta}\right) + \frac{3}{4}\delta^{\rho\sigma}\delta^{\alpha(\gamma}\delta^{\delta)\beta} - \frac{1}{4}\delta^{\rho\sigma}\delta^{\alpha\beta}\delta^{\gamma\delta} \\
&\quad -\frac{1}{2}\delta^{\sigma(\alpha}\delta^{\beta)(\delta}\delta^{\gamma)\rho}\Bigg\} \qquad (4.49) \\
P_2(x) &= \frac{\lambda}{4\kappa^2}\bar{\varphi}^2 - \frac{v}{8(2-n)}T_2 + \frac{\omega^2}{4\alpha}\left(\partial^\mu\bar{\varphi}\partial_\mu\bar{\varphi}\right) \qquad (4.50)
\end{aligned}
$$

We can rewrite S_3 as

$$S_3 = \kappa\int d^n x h_{\alpha\beta}\psi P_3^{\alpha\beta}(x) \qquad (4.51)$$

where

$$P_3^{\alpha\beta}(x) = \frac{\lambda}{12}\left(1 - \frac{v}{2}\right)\bar{\varphi}^3\delta^{\alpha\beta} \qquad (4.52)$$

We can rewrite S_4 as

$$S_4 = \kappa^2\int d^n x\left\{h_{\alpha\beta}h_{\mu\nu}P_4^{\alpha\beta\mu\nu}(x) + \psi^2 P_4(x)\right\} \qquad (4.53)$$

where

$$
\begin{aligned}
P_4^{\alpha\beta\gamma\delta}(x) &= \frac{\lambda\bar{\varphi}^4}{192}\left(\delta^{\alpha\beta}\delta^{\gamma\delta} - 2\delta^{\alpha(\gamma}\delta^{\delta)\beta}\right) \\
&\quad +\frac{v}{8}\left[T_4\right]^{\mu\nu\lambda\sigma}\left(\delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)}\delta^{\gamma}_{(\lambda}\delta^{\delta}_{\sigma)} + \delta^{\alpha}_{(\lambda}\delta^{\beta}_{\sigma)}\delta^{\gamma}_{(\mu}\delta^{\delta}_{\nu)}\right) \qquad (4.54) \\
P_4(x) &= -\frac{v}{8(2-n)}T_4. \qquad (4.55)
\end{aligned}
$$

It should be noted that many of these terms may be made markedly simpler, however we have not considered the graviton $h_{\mu\nu}$ to be symmetric. (for technical reasons in the FORM implementation), therefore any terms multiplying $h_{\mu\nu}$ can be simplified by just removing any symmetry over the indices (α, β) in the above expressions.

4.3.2 A, B, Cs

Using the fields themselves as the index labels for these coefficients, we have

$$(A^{\alpha\beta})_{h_{\mu\nu}h_{\lambda\tau}} = -G^{\lambda\tau\mu\nu}\delta^{\alpha\beta} + \left\{1 - \frac{1}{\alpha}\right\} G_{\sigma}{}^{\alpha\mu\nu}G^{\sigma\beta\lambda\tau} + \left\{1 - \frac{1}{\alpha}\right\} G_{\sigma}{}^{\beta\mu\nu}G^{\sigma\alpha\lambda\tau} \tag{4.56}$$

with the DeWitt metric

$$G^{\rho\sigma\lambda\tau} = \frac{1}{2}\left(\delta^{\rho\lambda}\delta^{\sigma\tau} + \delta^{\rho\tau}\delta^{\sigma\lambda} - \delta^{\rho\sigma}\delta^{\lambda\tau}\right). \tag{4.57}$$

$$(A^{\alpha\beta})_{\psi h_{\lambda\tau}} = A^{\alpha\beta}_{h_{\lambda\tau}\psi} = D^{\alpha\beta\lambda\tau} = \frac{1}{2}\kappa\xi((\delta^{\alpha\lambda}\delta^{\beta\tau} + \delta^{\beta\lambda}\delta^{\alpha\tau}) - 2\delta^{\alpha\beta}\delta^{\lambda\tau})\bar{\varphi} \tag{4.58}$$

$$(A^{\alpha\beta})_{\psi\psi} = -H^{\alpha\beta} \tag{4.59}$$

$$\begin{aligned}
(B^{\alpha})_{h_{\mu\nu}h_{\lambda\tau}} \quad = \quad & \kappa^2\xi\bar{\varphi}\partial^{\alpha}\bar{\varphi}(\frac{3}{4}\delta^{\mu\nu}\delta^{\lambda\tau} - \frac{7}{8}\delta^{\mu\lambda}\delta^{\nu\tau} - \frac{7}{8}\delta^{\mu\tau}\delta^{\nu\lambda}) \\
& -\kappa^2\xi\bar{\varphi}\partial^{\mu}\bar{\varphi}(\frac{3}{4}\delta^{\alpha\nu}\delta^{\lambda\tau} - \frac{7}{16}\delta^{\alpha\lambda}\delta^{\nu\tau} - \frac{7}{16}\delta^{\alpha\tau}\delta^{\nu\lambda}) \\
& -\kappa^2\xi\bar{\varphi}\partial^{\nu}\bar{\varphi}(\frac{3}{4}\delta^{\alpha\mu}\delta^{\lambda\tau} - \frac{7}{16}\delta^{\alpha\lambda}\delta^{\mu\tau} - \frac{7}{16}\delta^{\alpha\tau}\delta^{\mu\lambda}) \\
& -\kappa^2\xi\bar{\varphi}\partial^{\lambda}\bar{\varphi}(\frac{3}{4}\delta^{\alpha\tau}\delta^{\mu\nu} - \frac{7}{16}\delta^{\alpha\mu}\delta^{\nu\tau} - \frac{7}{16}\delta^{\alpha\nu}\delta^{\mu\tau}) \\
& -\kappa^2\xi\bar{\varphi}\partial^{\tau}\bar{\varphi}(\frac{3}{4}\delta^{\alpha\lambda}\delta^{\mu\nu} - \frac{7}{16}\delta^{\alpha\mu}\delta^{\nu\lambda} - \frac{7}{16}\delta^{\alpha\nu}\delta^{\mu\lambda})
\end{aligned} \tag{4.60}$$

$$
\begin{aligned}
(B^\alpha)_{h_{\gamma\delta}\psi} &= \kappa\partial^\alpha\bar{\varphi}\delta^{\gamma\delta}(\xi + \frac{1}{2} - \frac{\omega}{2\alpha}) \\
&\quad +\kappa\partial^\gamma\bar{\varphi}\delta^{\alpha\delta}(-\frac{\xi}{2} - \frac{1}{2} + \frac{\omega}{2\alpha}) \\
&\quad +\kappa\partial^\delta\bar{\varphi}\delta^{\alpha\gamma}(-\frac{\xi}{2} - \frac{1}{2} + \frac{\omega}{2\alpha})
\end{aligned}
\tag{4.61}
$$

$$
\begin{aligned}
(B^\alpha)_{\psi h_{\gamma\delta}} &= \kappa\partial^\alpha\bar{\varphi}\delta^{\gamma\delta}(\xi - \frac{1}{2} + \frac{\omega}{2\alpha}) \\
&\quad +\kappa\partial^\gamma\bar{\varphi}\delta^{\alpha\delta}(-\frac{\xi}{2} + \frac{1}{2} - \frac{\omega}{2\alpha}) \\
&\quad +\kappa\partial^\delta\bar{\varphi}\delta^{\alpha\gamma}(-\frac{\xi}{2} + \frac{1}{2} - \frac{\omega}{2\alpha})
\end{aligned}
\tag{4.62}
$$

$$
(B^\alpha)_{\psi\psi} = -\partial_\beta H^{\alpha\beta} = 0 \tag{4.63}
$$

$$
\begin{aligned}
(C)_{\psi h_{\gamma\delta}} = (C)_{h_{\gamma\delta}\psi} &= \frac{\kappa}{2}\left[\bar{\varphi}\delta^{\gamma\delta}m^2(1-v/2) + \frac{\bar{\varphi}^3}{6}\lambda(1-v/2) - \Box\bar{\varphi}\delta^{\gamma\delta}(1-v/2)\right] \\
&\quad \frac{\kappa}{2}\left[-2\xi\Box\bar{\varphi}\delta^{\gamma\delta} + 2\partial^\gamma\partial^\delta\bar{\varphi}(1+\xi+\omega/\alpha) - \Box\bar{\varphi}\delta^{\gamma\delta}\alpha/\omega\right]
\end{aligned}
\tag{4.64}
$$

$$
(C)_{\psi\psi}\kappa^2 = \kappa^2\left(\frac{\bar{\varphi}^2\lambda}{4\kappa^2} + \frac{1}{2}m^2 + (\partial\bar{\varphi})^2 + \frac{\omega^2}{\alpha}\right) \tag{4.65}
$$

Now we proceed to calculate the required normal coordinate expansions of $(A^{\alpha\beta})^i_j$, $(B^\alpha)^i_j$ and $(C)^i_j$, following the approach of (28), and we use these to calculate the terms in the Green function.

We start with G_0^{ij}. From (4.20) we find, rather expectedly,

$$G_0^{\psi\psi} = \frac{1}{p^2} \tag{4.66}$$

and

$$G_0^{h_{\rho\sigma}h_{\lambda\tau}} = \frac{\mathcal{G}^{\rho\sigma\lambda\tau}}{p^2} + \frac{1}{2}(\alpha - 1)\frac{\delta^{\rho\lambda}p^\sigma p^\tau + \delta^{\rho\tau}p^\sigma p^\lambda + \delta^{\sigma\lambda}p^\rho p^\tau + \delta^{\sigma\tau}p^\rho p^\lambda}{p^4}. \tag{4.67}$$

The leading order behaviour of our propagators is as expected.

We can then use (4.66) and (4.67) to calculate G_1^{ij}. We find

$$G_1^{\psi\psi} = iG_0^{\psi\psi}(A_\mu^{\alpha\beta})_{\psi\psi}\frac{\partial}{\partial p_\mu}[p_\alpha p_\beta G_0^{\psi\psi}] - iG_0^{\psi\psi}(B_0^\alpha)_{\psi\psi}p_\alpha G_0^{\psi\psi} \tag{4.68}$$

$$G_1^{\psi h_{\lambda\tau}} = iG_0^{\psi\psi}(A_\mu^{\alpha\beta})_{\psi h_{\rho\sigma}}\frac{\partial}{\partial p_\mu}[p_\alpha p_\beta G_0^{h_{\rho\sigma}h_{\lambda\tau}}] - iG_0^{\psi\psi}(B_0^\alpha)_{\psi h_{\rho\sigma}}p_\alpha G_0^{h_{\rho\sigma}h_{\lambda\tau}} \tag{4.69}$$

$$G_1^{h_{\rho\sigma}\psi} = iG_0^{h_{\rho\sigma}h_{\lambda\tau}}(A_\mu^{\alpha\beta})_{h_{\lambda\tau}\psi}\frac{\partial}{\partial p_\mu}[p_\alpha p_\beta G_0^{\psi\psi}] - iG_0^{h_{\rho\sigma}h_{\lambda\tau}}(B_0^\alpha)_{h_{\lambda\tau}\psi}p_\alpha G_0^{\psi\psi} \tag{4.70}$$

and

$$G_1^{h_{\rho\sigma}h_{\lambda\tau}} = iG_0^{h_{\rho\sigma}h_{\gamma\delta}}(A_\mu^{\alpha\beta})_{h_{\gamma\delta}h_{\pi\omega}}\frac{\partial}{\partial p_\mu}[p_\alpha p_\beta G_0^{h_{\pi\omega}h_{\lambda\tau}}] - iG_0^{h_{\rho\sigma}h_{\gamma\delta}}(B_0^\alpha)_{h_{\gamma\delta}h_{\pi\omega}}p_\alpha G_0^{h_{\pi\omega}h_{\lambda\tau}} \tag{4.71}$$

In turn, we can use these results to calculate G_2^{ij}. Since we only want the trace, we need only consider the diagonal terms.

Having calculated these terms, we suspected that this approach may be somewhat of a red herring so we don't dwell too much on the details. However, our final result for the term that could contribute towards the running coupling, that is the constant background field term, is (with $\xi = \varLambda = 0$)

$$Tr(E_1) = \bar{\varphi}^4(\frac{3}{4} - \frac{3\omega}{2} + \frac{3\omega\alpha}{4} - \frac{\alpha}{4} + \frac{3\omega^2}{16}). \tag{4.72}$$

4.3.3 Proof of equation (4.18)

Starting with [2]

$$\Delta^i_k G^k_j(x,x') = \delta^i_j \delta(x,x') = \int \frac{d^n p}{(2\pi)^n} \delta^i_j e^{ip\cdot y} \tag{4.73}$$

with

$$\Delta^i_j = (A^{\alpha\beta})^i_j \partial_\alpha \partial_\beta + (B^\alpha)^i_j \partial_\alpha + (C)^i_j \tag{4.74}$$

we find after some rearrangement and a bit of partial integration of the RHS that we have

$$\begin{aligned}
\int \frac{d^n p}{(2\pi)^n} \delta^i_j e^{ip\cdot y} \quad = \quad & -\sum_{n=0}^{\infty} (A^{\alpha\beta}{}_{\mu_1 \ldots \mu_n}) \int \frac{d^n p}{(2\pi)^n} \frac{\partial^n}{\partial p_{\mu_1} \ldots \partial p_{\mu_n}} [p_\alpha p_\beta G^k_j(p)] \\
& + i \sum_{n=0}^{\infty} (B^{\alpha}{}_{\mu_1 \ldots \mu_n}) \int \frac{d^n p}{(2\pi)^n} \frac{\partial^n}{\partial p_{\mu_1} \ldots \partial p_{\mu_n}} [p_\alpha G^k_j(p)] \\
& + \sum_{n=0}^{\infty} (C_{\mu_1 \ldots \mu_n}) \int \frac{d^n p}{(2\pi)^n} \frac{\partial^n}{\partial p_{\mu_1} \ldots \partial p_{\mu_n}} [G^k_j(p)]
\end{aligned} \tag{4.75}$$

and hence (4.18).

[2]The expression to the right obviously following from the definition of the δ-function, taking the δ^i_j term inside the integral for later comparison.

Chapter 5

Conclusions, Discussion and Outlook

We now have the ability to compare our results with other literature. First, consider the mass term. (31) worked in the Feynman gauge (with $\alpha = 1$) and found

$$Z_{\varphi^4} - 1 = \frac{\kappa^2}{16\pi^2} 4m^2 \frac{2}{n-4} \tag{5.1}$$

whereas we found

$$\delta Z_\varphi = -\frac{\kappa^2 B}{4\pi^2(n-4)} \tag{5.2}$$

$$\delta m^2 = -\frac{\kappa^2(C + m^2 B)}{4\pi^2(n-4)} \tag{5.3}$$

For the choices of the parameters used in (31), we find agreement for our expression for δm^2.

Next consider the λ coupling. We find ($\alpha = \xi = \Lambda = 0$)

$$\lambda_B = \frac{1}{2\pi^2(n-4)} \left[-3\lambda^2 + \kappa^2 \lambda m^2 (\frac{3}{8}\omega - 16)) + \frac{5}{8} m^2 \kappa^2 - \frac{21}{32} m^4 \kappa^4 \right] \tag{5.4}$$

We found an expression for the gravitational part of the beta function for the coupling parameter λ. If we ignore any dependence on the derivative terms, we are led to (with $\Lambda = \xi = 0$ and with our correct choice of the other parameters)

$$\beta_{gravity} = \frac{\lambda\kappa^2}{64\pi^2}(5m^2\lambda + 2m^2 - 21m^4\kappa^2). \tag{5.5}$$

Therefore, gravity makes $\lambda\varphi^4$ theory asymptotically free only if $|\lambda| < \frac{1}{5}(2 - 21m^2\kappa^2)$. Otherwise, the coupling will run to increasingly high values as we approach the Planck scale. (For the special case when $\lambda = \frac{1}{5}(2-21m^2\kappa^2)$, there is no gravitational contribution to the λ running). We do not agree therefore with the general result of (9) that gravity always leads to asymptotic freedom.

However, as we have non-zero terms in our results proportional to e.g. $\bar{\varphi}^3\Box\bar{\varphi}$, we can use the field equation

$$\Box\bar{\varphi} = -m^2\bar{\varphi} - \frac{\lambda}{6}\bar{\varphi}^3 \tag{5.6}$$

to remove all terms containing $\Box\bar{\varphi}$ but this will change the term multiplying $\bar{\varphi}^4$. Therefore, it is clear that the derivative terms can affect the running of coupling constant as discussed in (55). How the derivatives terms actually affect the renormalization counter terms is not clear though and would be an important area of investigation in the future. However, by naively using the field equation however, we will arrive at (relabelling the quadratic mass term, also previously called B, to B_2)

$$\delta\lambda = \frac{\kappa^4}{2\pi^2(n-4)}\left(\frac{1}{4}m^4C - m^2\frac{1}{2}B + A + \frac{\lambda}{\kappa^2}B_2\right) \tag{5.7}$$

and we can proceed to work out β_λ from here following the renormalization techniques above.

Using our previously calculated expression for A, B, C and B_2 (from (3.148), (3.149), (3.150) and (3.121)), and choosing $\Lambda = \xi = \alpha = 0$, $v = \omega = 1$, we find

$$\delta\lambda = \frac{1}{2\pi^2(n-4)}\left\{\frac{9}{64}\kappa^4 m^4 + \frac{223}{384}\kappa^2\lambda m^2 - \frac{1}{16}\lambda^2\right\} \tag{5.8}$$

and by the usual beta function procedure, we find

$$\beta_\lambda = \frac{1}{2\pi^2}\left\{\frac{223}{384}\kappa^2\lambda^2 m^2 - \frac{1}{8}\lambda^3\right\}. \tag{5.9}$$

5.0.4 More work

Having made our conclusions, let us finish by suggesting some ways in which the methods outlined in this work may be applied to more situations and the phenomena we might expect to see.

The most obvious extension is to apply the work here to another matter field coupled to gravity. A fermion model coupled to gravity, already examined by a few authors in simpler regimes, would be worth examining in more details using the Vilkovisky-DeWitt technique but retaining all derivative terms. Then, in a composite model with fermions and the scalars discussed here, we could examine the quantum gravitational contributions to the running of the Yukawa coupling. And looking at the full Yang-Mills case coupled to gravity is also an important extension (See Appendix for preliminary work on this).

A more involved calculation would involve a general curved metric tensor, which would allow us to calculate the running of the gravitational parameter, κ. This would leave us in a position to perform a full renormalization group analysis. This is the logical extension of the work of this thesis. However, it has been noted that running of κ might not be meaningful (89).

We could also extend the calculation to look at the case of two (or higher) loop diagrams. However, as noted previously, the number of terms involved in such a calculation might mean this is currently not technologically feasible to attempt and this author believes a more thorough approach would be required to avoid any errors.

5.0.5 Software Acknowledgements

The computationally intensive calculations of this thesis were performed with the assistance of FORM (48), and some were crosschecked against results worked out independently via Cadabra (49).

Part II

AdS/CFT and the Quark Gluon Plasma

Chapter 6

Black Hole Thermodynamics and AdS/CFT

6.1 Black Hole Thermodynamics

There are striking analogies between the thermodynamics of fluids and the thermodynamics of black holes. This connection is an important one in that it provides a clear link between gravity and quantum mechanics.

In particular, we are able to write down a set of thermodynamic laws for black holes which correspond closely to the well known laws for classical fluids. We consider these in the first section.

6.1.1 Zeroth Law

First, consider the zeroth law for classical thermodynamics. This is a statement of constant temperature at equilibrium throughout a classical fluid. Equally, we can search for a constant for black holes - indeed, it is known that the surface gravity, κ of a black hole is constant over its horizon.

6.1.2 First Law

The first law is summed up in the mass differential of a black hole,

$$dM = \frac{\kappa}{8\pi} dA + \varOmega dJ + \varPhi dQ \tag{6.1}$$

for mass M, surface gravity κ, horizon area A, angular velocity $\varOmega$, angular momentum J, electrostatic potential $\varPhi$ and electrostatic charge Q (108). A similar law for the energy differential of a fluid is well known and is

$$dE = TdS + \varOmega dJ + \varPhi dQ \tag{6.2}$$

for energy E, temperature T and entropy S. If we match up the second and third terms to their obvious counterparts - i.e. rotation and charge of the black hole correspond to rotation and charge of a fluid - and use the result from the zeroth law, that the temperature is equivalent to the surface gravity then we can surmise that there is a relationship between the entropy and area. Indeed, the relationship between entropy *of the black hole* and its area is well known (88) and is expressed by

$$S = \frac{1}{4} A. \tag{6.3}$$

6.1.3 Second Law

The second law in classical thermodynamics is that entropy always increases. The same holds true for black holes, or if we consider the analogous area instead, then the area of the black hole must always increase, i.e. black holes must always increase in size. However, if Hawking radiation (88) is to hold true, then a black hole can evaporate which leads to a decrease in the black hole size. Instead there has been hypothesised the existance of a generalized second law wherein the *total* entropy of the system is always increasing.

6.1.4 Third Law

The third and final classical law is the statement that temperature can never reach absolute zero. Likewise, the corresponding third law for black holes states that a black hole cannot have vanishing surface gravity, κ. However, in extremal black holes - those with the minimum mass possible given a charge and rotation - this is not the case, but no evidence of such black holes have ever been observed.

6.2 AdS-CFT Correspondence

Maldacena (92) realised that there existed a duality between an n-dimensional anti-de Sitter spacetime and a conformal field theory on its $(n-1)$ dimensional boundary, now termed the AdS/CFT correspondence.

In particular, if we consider a black hole in such a spacetime then we can qualitatively describe a thermal state of Super-Yang-Mills theory on the boundary (94; 95; 96; 98; 99; 100). Although this is a massless theory, if we are considering energies close to a phase transition then quarks may behave as though they were massless (101; 102; 103). Such a calculation then can provide insight into the dynamics of heavy ion collisions such as those which occur at the Relativistic Heavy Ion Collider (RHIC) or in the ALICE experiment at the LHC.

In such experiments, two heavy ions of Au (RHIC) (91) or Pb (ALICE) are collided with centre of mass energies around the TeV scale. The energies are enough to overcome confinement and the resulting soup of quarks and gluons is termed the quark gluon plasma (QGP).

Experiments have suggested that the QGP formed from the collisions of heavy ions exhibits a liquid state with a very small viscosity and that the viscosity observed may be close to the AdS-CFT predictions (107). A relationship exist which relates the entropy (density), s, to the viscosity, η given by

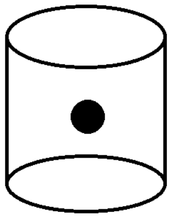

Figure 6.1: Here, a black hole exists alone in anti de-Sitter spacetime. The radius of the cylinder represents the length scale associated with anti-de Sitter spacetime.

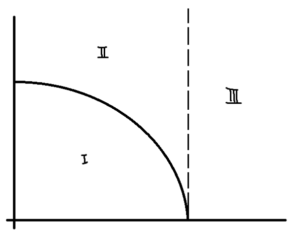

Figure 6.2: We recognise 3 different regions on the graph with zero, one or two black hole solutions. The vertical axis represents the temperature whilst the horizontal axis represents chemical potential. The graph may also represent a baryon in a confined state in the lower left segment becoming a sea of quarks and gluons given enough energy. The dashed line represents a critical potential, μ_C, for which superfluidity may occur.

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{6.4}$$

Therefore a state with zero entropy should be expected to have vanishing viscosity, which would also be expected in a superfluid.

Using the AdS/CFT correspondence, we would encounter two black hole solutions. Typically, one would discard the smaller black hole. However, we could also consider the two solutions to correspond separately to two components of a Landau fluid, with one component existing in a condensate and the other existing as thermal excitations.

Consider a Reissner-Nordström black hole in $d = n+1$ dimensional AdS spacetime. The line element is (104)

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_{n-1} \tag{6.5}$$

where

$$V(r) = 1 - \frac{m}{r^{n-1}} + \frac{q^2}{r^{2n-4}} + \frac{r^2}{l^2} \tag{6.6}$$

for some factors q and m proportional to charge and mass and l the radius of curvature of the AdS spacetime.

We can introduce the horizon r_+ by $g^{rr} = 0$ or $V(r) = 0$ and use this to eliminate the mass. We also have the surface gravity given by $\kappa = \frac{1}{2}V'(r_+)$. Also, we had $T = \frac{\kappa}{2\pi}$ so that

$$T = \frac{nr_+^{2n-2} + (n-2)l^2 r_+^{2n-4} - (n-2)q^2 l^2}{4\pi l^2 r_+^{2n-3}}. \tag{6.7}$$

We can also introduce the electrostatic potential (which can be equated with the chemical potential in the fluid case)

$$\mu = \frac{1}{c}\frac{q}{r_+^{n-2}} \tag{6.8}$$

with

$$c = \sqrt{\frac{2(n-2)}{n-1}} \tag{6.9}$$

to simplify further. We then have

$$nr_+^2 - 4\pi^2 l^2 r_+ + l^2(n-2)(1-\mu^2c^2) = 0 \tag{6.10}$$

with solutions

$$r_+ = \frac{2\pi l^2}{n}\left(T \pm \sqrt{T^2 - \frac{1}{4l^2}n(n-2)(1-\mu^2c^2)}\right). \tag{6.11}$$

Clearly the sign of the expression under the square root dictates the number of solutions with some boundary. If $T^2 - \frac{1}{4l^2}n(n-2)(1-\mu^2c^2)$ is negative then we have no black hole solutions and if it positive then two exist. When it is identically zero, then there is a single solution, and this will represent a phase transition.

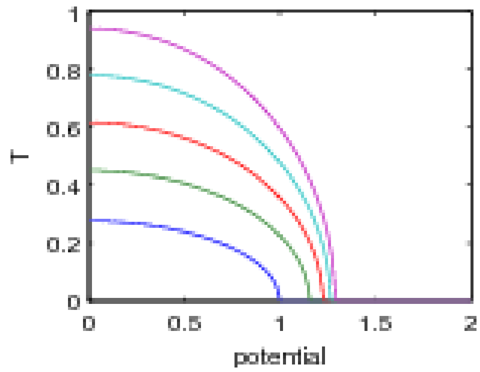

Figure 6.3: Plot of the different regions for varying dimension n.

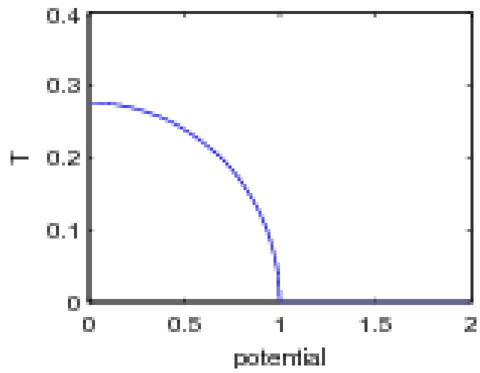

Figure 6.4: Fig 6.3, identifying just the dimensionality of interest, $n = 3$.

Therefore, we can construct a phase diagram with two clearly different regions (see Fig 6.2, Fig 6.3, Fig 6.4).

Another useful quantity to calculate is the entropy. Recalling that the entropy is equivalent to the area of the event horizon, we can write

$$S = \frac{\omega_{n-1}}{4} r_+^{n-1} \tag{6.12}$$

$$= \frac{\omega_{n-1}}{4} \left(\frac{2\pi l^2}{n}\right)^{n-1} \left(T \pm \sqrt{T^2 - \frac{1}{4l^2} n(n-2)(1-\mu^2 c^2)}\right)^{n-1} \tag{6.13}$$

where

$$\omega_n = \frac{16\pi}{(n-2)Vol(S^{n-2})}. \tag{6.14}$$

There is again a region in which two solutions exist which we may interpret as a two-fluid system. In particular, there is a choice which can be made for the potential, $\mu = 1/c$, for which there exists both a zero solution $S = 0$ and a non-zero solution $S = \omega_{n-1}\left(\frac{\pi l^2 T}{n}\right)^{n-1}$ - the fluid-superfluid mixture we claimed.

6.2.1 Phase Change

We should now check that a phase boundary does indeed exist. The signature of a phase transition is that the heat capacity changes abruptly. In classical thermodynamics, we have the specific heat capacity given by

$$C = T\left(\frac{\partial S}{\partial T}\right)_{condition} \tag{6.15}$$

where we can choose a set of conditions for our system, e.g. constant volume or pressure. It is also well known that close to a phase boundary, the specific heat capacity behaves as

$$C \sim |T - T_C|^{-\alpha} \tag{6.16}$$

for some critical temperature T_C and with $\alpha > 0$.

Hence, if we search for the value of the black hole parameters for which $C \to \infty$, which will occur for $T = T_C$, then we should be led to an equation containing such parameters for a phase boundary.

Let us look again at the Reissner-Nordström black hole where we will want to examine the behaviour of the heat capacity

$$C_\mu = T\left(\frac{\partial S}{\partial T}\right)_\mu . \tag{6.17}$$

We could write this is terms of Jacobians, as

$$C_\mu = T\frac{\partial(S,\mu)}{\partial(T,\mu)} = T\frac{\partial(S,\mu)}{\partial(r_+,Q)}\frac{\partial(r_+,Q)}{\partial(T,\mu)} \tag{6.18}$$

and then examine

$$\left(\frac{\partial(r_+,Q)}{\partial(T,\mu)}\right)^{-1} = 0. \tag{6.19}$$

If we were interested in a more general black hole with more properties, this would be a useful technique, but since we are only interested in a fairly simple Reissner-Nordström black hole, we can just perform the derivatives directly. Since $T = \kappa/2\pi$, we can write

$$\begin{aligned} C_\mu &= \kappa\left(\frac{\partial S}{\partial \kappa}\right)_\mu \\ &= \kappa\frac{\omega_{n-1}}{4}\left(\frac{1}{n\alpha^2}\right)^{n-1}(n-1)\left[\kappa \pm \left(\kappa^2 - n(n-2)\alpha^2(1-\mu^2c^2)\right)^{1/2}\right]^{n-2} \\ &\quad \times \left[1 \pm \kappa\left(\kappa^2 - n(n-2)\alpha^2(1-\mu^2c^2)\right)^{-1/2}\right]. \end{aligned} \tag{6.20}$$

Thus,

$$\kappa^2 - n(n-2)\alpha^2(1-\mu^2c^2) \to 0 \Rightarrow C_\mu \to \infty \tag{6.21}$$

which occurs for $T_C = \frac{1}{2\pi}\sqrt{n(n-2)\alpha^2(1-\mu^2c^2)}$, where we have labelled the critical temperature T_C.

6.3 A Super-Yang-Mills Superfluid

Let us consider a system containing massless quarks with a typical self-interaction term. At high energies, we have a relativistic version of the Gross-Pitaevskii equation, or equivalently the Klein-Gordon equation with an interaction term. We have a stationary condensate, so the usual time derivatives vanish. We have (see for example (110))

$$-\nabla^2\Psi + g^2\left|\Psi^2\right|\Psi + m_B^2\Psi = \mu^2\Psi \tag{6.22}$$

where g and m_B are some coupling constants.

Let us make a brief aside to consider the geometry we will be working in. Here, we wish to work on a 3-sphere so we have

$$W^2 + X^2 + Y^2 + Z^2 = l^2 \tag{6.23}$$

where we identify the radius of the sphere, l, with the characteristic length scale of anti-de Sitter space and W, X, Y, Z are the four coordinates mapping the 3-sphere. A useful choice is to consider a Hopf fibration, which defines a point $CP^1 = S^2$ and allows us to choose

$$z^0 = X + iY \tag{6.24}$$

and

$$z^1 = Z + iW \tag{6.25}$$

and introduce the Euler angles (θ, ϕ, ψ) by

$$z^0 = l \sin\frac{\theta}{2} e^{i(\psi+\phi)} \tag{6.26}$$

and

$$z^1 = l \cos\frac{\theta}{2} e^{i(\psi-\phi)}. \tag{6.27}$$

The line element on the 3-sphere can then be written down as

$$\begin{aligned} ds^2 &= \left|dz^0\right|^2 + \left|dz^1\right|^2 \\ &= \frac{l^2}{4}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{l^2}{4}(d\psi + \cos\theta d\phi)^2. \end{aligned} \tag{6.28}$$

The Laplacian acting on scalars in general is

$$\nabla^2 = \frac{1}{g^{1/2}}\left(\frac{\partial}{\partial x^\mu} g^{\mu\nu} g^{1/2} \frac{\partial}{\partial x^\nu}\right). \tag{6.29}$$

If we choose an ansatz

$$\Psi = R(\theta) e^{in_a(\phi+\psi)/2 + in_b(\phi-\psi)/2} \tag{6.30}$$

for some integer wavenumbers n_a and n_b then the equation (6.22) reduces to an ordinary differential equation

$$(1-z^2)\frac{d^2R}{dz^2} - 2z\frac{dR}{dz} - \frac{n_a^2}{2(1-z)}R - \frac{n_b^2}{2(1+z)}R + \nu(\nu+1)R - \frac{1}{4}g^2l^2R^3 = 0 \tag{6.31}$$

where we have simplified somewhat by introducing $z = -\cos\theta$ and introduced $\nu(\nu+1) = \frac{1}{4}(\mu^2 - m_b^2)l^2$. As ν is related to the chemical potential, we shall hereafter refer to it as the *energy parameter*. The singular nature of this equation for $R(z)$ dictates the boundary conditions to be $R(-1) = 0$ for $n_b \neq 0$ and $R(1) = 0$ for $n_a \neq 0$.

Finally, we can make the substitution

$$w = \frac{1}{2}Rgl \tag{6.32}$$

to further simplify (6.31) so that

$$(1-z^2)w'' - 2zw' - \frac{n_a^2}{2(1-z)}w - \frac{n_b^2}{2(1+z)}w + \nu(\nu+1)w - \frac{w^3}{2} = 0. \tag{6.33}$$

Before looking at solving this equation, let us make some important comments.

The Gross-Pitaevskii equation (6.22) can also be solved using relativistic fluid mechanics (110) with a fluid-flux covector (n_B, $n_B\mathbf{u}$). This is related to the wavefunction by

$$n_B = 2\mu\varPsi\varPsi^*, \quad n_B\mathbf{u} = -i(\varPsi^* d\varPsi - \varPsi d\varPsi^*). \tag{6.34}$$

If we are expecting a superfluid, then one key property to search for is the appearance of quantum vortices. If a vortex is present, then that solution will possess a circulation around curves $\varGamma$ given by

$$C = \int_{\varGamma} \mathbf{u}. \tag{6.35}$$

For our choice of ansatz, the circulation is constant outside the vortex cores, and given by

$$C_a = \frac{2\pi n_a}{\mu}, \quad C_b = \frac{2\pi n_b}{\mu} \tag{6.36}$$

for curves around the two possible axes of rotation. In the non-relativistic limit, $\mu = m_B$ and we recover the typical quanta of circulation. The quantised parts of our solution are our wavenumbers which correspond directly to the angular momenta of some vortices within the fluid,

$$J_a = n_a, \quad J_b = n_b. \tag{6.37}$$

Most importantly, (6.37) exhibits the appearance of quantum gravity in our work where the black holes dual to our fluid necessarily have quantised angular momenta. It has been shown that black hole angular momentum may be quantised; AdS black holes in $n = 3$ can be expressed in terms of a conformal algebra suggesting states of quantised mass and angular momentum (112).

Whilst we could now attempt to solve (6.31) numerically, we will instead examine two particular cases for which analytic solutions exist.

6.3.1 $n_a = n_b$

First, let us examine the situation where both the integers n_a and n_b are equal. In this case (6.31) becomes

$$(1-z^2)\frac{d^2R}{dz^2} - 2z\frac{dR}{dz} - \frac{n_a^2}{(1-z^2)}R + \nu(\nu+1)R - \frac{1}{4}g^2l^2R^3 = 0 \tag{6.38}$$

Making the substitution for w as above, it is possible to find an analytic solution here by choosing w to be given by an expansion in terms of Jacobi polynomials, specifically

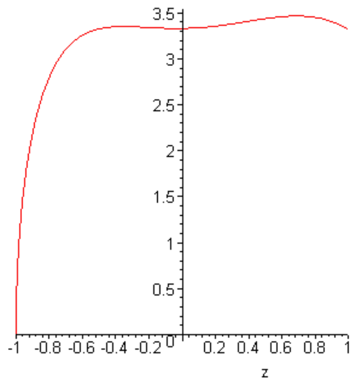

Figure 6.5: A typical plot of the value of w against z for n_b=n_a case, where the rotation and circulation in each direction are equal.

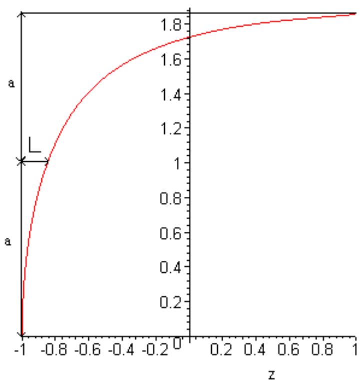

Figure 6.6: The vortex width was defined as full width at half height. The vertical height represents the value of the function R.

$$w = \sum_{l=0}^{\infty} (1+z)^{1/2} a_l P_l^{(n_b,0)}(-z) \tag{6.39}$$

with

$$P_l^{(n_b,0)}(-z) = \sum_{m=0}^{l} \binom{l}{m} \frac{\Gamma(n_b+l+m+1)}{\Gamma(n_b+m+1)} \left(\frac{z-1}{z}\right)^m \tag{6.40}$$

and a_l some coefficients to be determined.

An example of the output is Fig 6.5. Also, Fig 6.6 shows how we defined the width of a vortex. A check on the results for the $l = 0$ mode is included in the appendix.

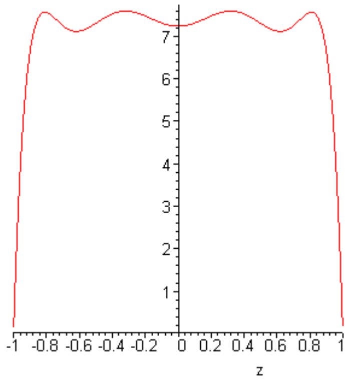

Figure 6.7: A typical plot of the value of w against z for the n_b=0 case.

6.3.2 $n_b = 0$

Now let us turn to the situation where one of the integers is set equal to zero, e.g. let us choose $n_b = 0$. Now (6.31) is

$$(1-z^2)\frac{d^2R}{dz^2} - 2z\frac{dR}{dz} - \frac{n_a^2}{2(1-z)}R + \nu(\nu+1)R - \frac{1}{4}g^2l^2R^3 = 0 \tag{6.41}$$

and again we can substitute for w. This time, w can be given by an expansion in terms of associated Legendre functions:

$$w = (1-z^2)\sum_{l=0}^{\infty} a_l P_l^{n_a}(z), \tag{6.42}$$

with the associated Legendre function given explicitly by

$$P_l^{n_a}(z) = \frac{(-1)^{n_a}}{2^l l!}(1-z^2)^{n_a/2}\frac{d^{n_a+l}}{dz^{n_a+l}}(z^2-1)^l \tag{6.43}$$

An example of the shape of this graph is Fig 6.7.

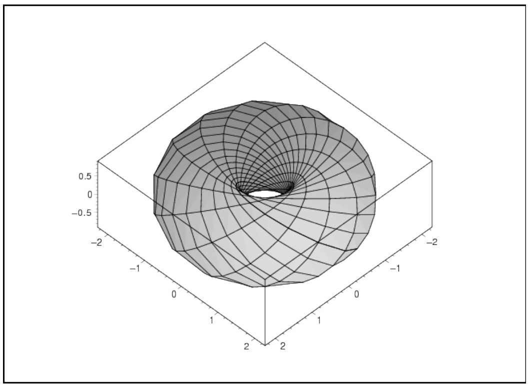

Figure 6.8: The solutions for w, after projection back to R^3, are tori which we interpret as vortex rings.

6.3.3 Stereographic projection $S^3 \to R^3$

To interpret the solution, it is useful to now project the solutions back into Euclidean space. We use a change of coordinates

$$x = \frac{\sin\frac{\theta}{2}\cos\frac{\psi+\phi}{2}}{1-\cos\frac{\theta}{2}\sin\frac{\psi-\phi}{2}} \tag{6.44}$$

$$y = \frac{\sin\frac{\theta}{2}\sin\frac{\psi-\phi}{2}}{1-\cos\frac{\theta}{2}\sin\frac{\psi-\phi}{2}} \tag{6.45}$$

$$z = \frac{\cos\frac{\theta}{2}\cos\frac{\psi-\phi}{2}}{1-\cos\frac{\theta}{2}\sin\frac{\psi-\phi}{2}}. \tag{6.46}$$

When we do so, the solution is seen to be a series of tori (as a function of the potential μ). We then suggest that these solutions may be vortex rings Fig 6.8. The relationship between the energy of the vortices and their width is shown in Fig 6.10.

6.3.4 Calculation of bound on g

If we start with an expression for the conserved charge

$$N_B = 2\mu \int \Psi^* \Psi d\Omega_3. \tag{6.47}$$

The ground state from the Gross-Pitaevskii equation is

$$\Psi_0 = \frac{(\mu^2 - m_B^2)^1/2}{g}. \tag{6.48}$$

We can write

$$N_B(groundstate) = \frac{4\mu(\mu^2 - m_B^2)\pi^2 l^3}{g^2}, \tag{6.49}$$

or

$$g^2 = \frac{4\mu(\mu^2 - m_B^2)\pi^2 l^3}{N_B}. \tag{6.50}$$

It is clear that (6.50) is maximal for fixed μ and $m_B = 0$, i.e.

$$g^2 < \frac{4\pi^2 \mu^3 l^3}{N_B}. \tag{6.51}$$

Now, for an $SU(N)$ gauge theory, (92)

$$4\pi^2 l^3 = N^2, \tag{6.52}$$

and so

$$g^2 \leq \frac{\mu_C^3 N^2}{N_B}, \tag{6.53}$$

where we have replaced the potential by the critical potential, μ_C.

By choosing values for the various free parameters in our Gross-Pitaevskii equation, we can also show (see Fig 6.10) that there exists a power law relation between the energy parameter, ν, and the vortex length, L.

6.4 Conclusions and discussion for AdS-CFT

By using results from AdS/CFT correspondence, we were able to make comparisons between a black hole in 5-dimensional anti-de Sitter spacetime and a fluid existing on its 4-dimensional boundary. We found that there exists solutions to the Gross-Pitaevskii

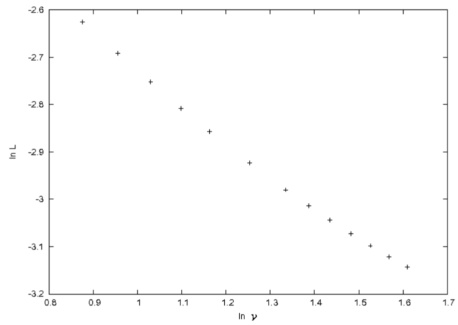

Figure 6.9: Plot of the energy related parameter ν against the vortex width.

equation which took the shape of vortex rings. These are a key feature of quantum fluids so this is good evidence of superfluidity of the quark-gluon plasma which is formed in heavy ion collisions. We also noted that are two different black hole phases and correspondingly there are two fluid phases. One phases corresponds to confined quarks (i.e. hadrons) while the other corresponds to the free quarks of the plasma. However, there is also a point in our phase space where entropy drops to zero and hence superfluidity occurs.

Chapter 7

Overall Conclusions

7.1 Overall conclusions

We have studied two aspects of quantum gravity in this thesis. In the first part we have found that there is a prescription for describing gravity as an effective field theory, which we suggest may be the true theory of quantum gravity, rather than a UV-complete description. We studied φ^4-theory and by trying to keep our action more general than other authors, we hope to have side-stepped a lot of the problems highlighted in the introduction (such as gauge-dependency) whilst picking out some interesting phenomenology. In particular, we have shown that there exist a set of bounds for the scalar coupling constant between which the theory is asymptotically free. Outside of these regions, scalar field theory has similar behaviour to that in the absence of gravity. More than this, we have hinted at how the work could be extended to other types of fields (section 5.0.4) and suggested some first steps in combining Yang-Mills and gravity (Appendix A.4). We exposed the utility (indeed, the necessity) of computer algebra packages such as FORM in this type of work. We also considered quadratic divergences and looked at a heat kernel method for calculating such terms. Yet we heeded the comments in the literature regarding field redefinitions where it was explained that quadratic divergences can not contribute to the running behaviour of a coupling parameter; moreover, in light of commentary in the literature, we have seen that gravity and its effect on other types of field is still a difficult problem and it is not entirely clear how one should proceed, with different techniques often leading to different answers.

The second part of the thesis looked at AdS-CFT correspondence and in particular the duality between a black hole and a fluid. We compared different properties of the fluid with those of the black hole and found that the black hole solution admitted different regions which could tell us qualitatively about the particles in the fluid. Of note, it told us that there were different phases which experiment tells us are the confined phase and the QGP phase. A particular property we looked at was the entropy of the fluid, corresponding with black hole area. We saw that this might lead to have a very low viscosity state, that could be superfluid. From this, we investigated a relativistic Gross-Pitaevskii equation under certain conditions. Our main result was that the solutions resembled vortex rings which is a key feature of superfluid behaviour.

Appendix A

Appendix

A.1 AdS/CFT calculation

A.1.1 Check for $l = 0$ Jacobi Polynomials

We do a manual check on the results, in the simple case where $l = 0$, hence $P_0 = 1$.

We then have

$$w = (1+z)^{1/2}a_0 \quad , w' = \frac{1}{2}(1+z)^{-1/2}a_0. \tag{A.1}$$

Substituting this into our action, we find

$$\begin{aligned} S &= \int_{-1}^{1} \left\{ -(1-z^2)(1+z)a_0^2 - \frac{n_a^2}{2}a_0^2 + \nu(\nu+1)(1+z)a_0^2 - \frac{(1+z)^2}{2}a_0^4 \right\} \\ &= a_0^2(\frac{-4}{3} - n_a^2 + 2\nu(\nu+1)) - a_0^4 \end{aligned} \tag{A.2}$$

which has a non-trivial minimum at

$$a_0 = \frac{1}{2}\sqrt{4\nu(\nu+1) - \frac{8}{3} - 2n_a^2}. \tag{A.3}$$

For particular values of ν and n_a, we confirm that a_0 here is the same as those calculated using Maple with predefined hypergeometric functions.

A.2 Scalar field-gravity calculation

This section of the appendix is intended as a manual to explain the function of various parts of the code used to calculate the required terms in the effective action.

Here, in turn, particular segments of the code will be displayed, followed by discussion of how it works and why such techniques were used. A few useful tricks will be outlined too.

First let us consider the code for scalar field theory coupled to gravity.

A.2.1 Declarations

As with many programming languages, in FORM we begin with some declarations. At the very top, we define the dimension:

```
Dimension n;
```

We keep a general symbol n for the dimension until the point where we introduce dimensional regularisation to calculate the integrals.

```
Symbols n, kappa, xi, m, ...
Indices x, alpha, beta, ...
Indices x1, alpha1, beta1, ...
Indices x2, alpha2, beta2, ...
Indices dummyS11, dummyS12, ...
```

Symbols are defined here. This includes our parameters v and ω that allow us to compare with a traditional Feynman diagram method in various choices of gauge, which can be set to whichever values we require at the last stage of the calculation.

Indices have also been defined here. It is easier to keep track of symbols if, for example, in calculating a 3 point function, we label our functions with numbered indices to match the corresponding coordinate (e.g. `alpha1` for `x1`).

We retain a number of dummy indices, which will be passed as arguments all the way back to the top level function. This helps to avoid a situation where we might encounter too many repeated (more than 2 of the same) indices.

Next up is our list of functions. A useful feature of FORM is that it allows us to declare the functions to be commutative or non-commutative. This allows us to perform derivatives by the use of a commutator relation. The prefactors which multiply the quantum fields can be chosen to be commutative functions since these will generally involve only symbols, delta functions and background scalar fields.

Another useful feature is the ability to allow a function to be (anti-)symmetric in its indices. We have

```
Functions h(symmetric), gb(symmetric);
```

representing the graviton and background gravitational field. We keep the background general, rather than choosing it to be flat, so that it is easier to write identities to set derivatives of the field to be zero (if we set it to be a delta straight away, then it would just raise or lower the index on the derivative operator).

A.2.2 Effective Action

The main task for FORM will be to calculate the expansion of the function

$$e^{-S} \approx < S^4 > - < S_3 S_1 > - < S_2^2 > + \frac{1}{2} < S_1^2 S_2 > + \frac{1}{24} < S_1^4 > \tag{A.4}$$

where the angle brackets indicate the typical calculation using Wick's theorem and we expand to fourth order in the background fields. Note that we do not include anything to represent an integral sign - as far as the FORM code is concerned, this is superfluous.

A.2.3 Checking the form of S_1, S_2, ...

We can also use FORM to check our expressions for the various parts of the action S_1, S_2, etc. Our top level local function is the sum of a function representing the classical part of the action, a function representing the connection term and a function representing the gauge-fixing term.

For the classical part, we can simply write down our classical action in the most obvious way in FORM with appropriate identities for the metric tensor, the square of its determinant, derivatives of the background field, and so on. We must be most careful where lots of indices are contracted, such as with the curvature terms, R and $\frac{1}{2}\xi R\phi^2$.

For the connection term, we use FORM to check that the Christoffel symbols (which can contain many delta functions) multiply the first functional derivatives of the action to give the correct terms.

For the gauge fixing term, FORM does not offer much advantage in checking the result, but we include it so that it is easier to sum the terms to get the complete one-loop order effective action.

A trick that is used in FORM is this: wherever a quantum field appears in our identities, we multiply by a factor q. Then, when we want to collect those terms that are quadratic in the quantum fields, we can simply set all other powers of q to be zero, and then set $q^2 = 1$. In this way, all that remains in our output is the term we are interested in. Of course, we must also be careful of the cosmological constant term, containing no powers of q and thus not vanishing, but this is easy to keep track of.

Similarly, we could multiply the background fields by such a factor, then order the terms in our output by powers of this factor. This would separate S_0, S_1 and so on.

A.2.4 Pre-sorting

Ignoring the prefactors, we are essentially left with gravitons and quantum scalar fields, and derivatives thereof. Since we defined the derivative operators and these fields as non-commutative, it is helpful to first write simple rules to place the gravitons on one side of the equation, and the scalar fields on the other. An identity

```
id psi(x1?)*h(mu?,nu?,x2?) = h(mu,nu,x2)*psi(x1);
```

moves a graviton (with no derivatives) to the left of a scalar (with no derivative). Similar terms exist to manipulate the derivatives. The equation is now more maleable but there are still many terms to calculate. Consider S_1^4. S_1 contains 4 terms; therefore, S_1^4 contains 256 terms. While we could write 256 separate rules, we can reduce the number of rules significantly. If we write the action in order of the derivatives of quantum fields (and at most, we find there may be two such derivatives, say $\partial_\mu \partial_\nu h_{\alpha\beta}$) then simple combinatronics tells us that terms with a similar derivative structure number just 28 (14 each for the gravitons and quantum scalar fields).

A.2.5 Propagators and Wick's Theorem

Now we write the rules to convert the various products of quantum fields and their derivatives into products of propagators with derivatives attached.

A.2.6 Transform to momentum space and performing derivatives

Each propagator in coordinate space can be transformed to momentum space by identities such as

```
id partial(mu1?, x1?)*Gx(alpha1?,beta1?,alpha2?,beta2?,x1?,x2?)
= Gp(alpha1,beta1,alpha2,beta2, p1) *partial(mu1,x1)* exp(p1,x1,x2);
```

which assigns a particular momentum to a pair of coordinates. The derivative operator contains a coordinate label to make sure it acts on the correct coordinate in the exponential `exp(p1,x1,x2)` .

A.2.7 Momenta integrals

Once we have products of momenta and propagators in momentum space, we resort to a more familiar computer algebra package, Maple, to calculate the various integrals. In fact, we need only calculate the pole part of each integral. Therefore, we simply Taylor expand the propagators and retain only the terms of the order p^{-4}. The results will be expressed in terms of a basic logarithmic integral L.

A.2.8 Simplifying the answer

Our expressions at this stage will consist of numerical factors, the basic logarithmic integral L and various powers and derivatives of the background scalar field.

A.2.9 Using the field equation

The final step is to note that the solution can be reduced using the field equation. We have

$$\Box\phi = -m^2\phi - \frac{\lambda}{6}\phi^3 \tag{A.5}$$

which in FORM is described by a rule such as

```
id Box(x)*phi(x) = -m^2*phi(x) - lambd/6*phi(x)^3
```

A.3 Quadratic divergences

In a similar vein to the final steps in the calculation of the logarithmic divergences, the final steps of the calculation of the quadratic divergences are also computationally intensive. Therefore, we once again seek the assistance of FORM in performing the lengthy manipulations and calculations.

Once we have evaluated expressions for all the components of $(A^{\alpha\beta})_{ij}$, $(B^{\alpha})_{ij}$ and $(C)_{ij}$, we can calculate, first G_1^{ij} and then G_2^{ij}.

A.3.1 Calculating derivatives

Again, we use one of FORM's most useful features - the ability to define functions as non-commuting or commuting and to be able to switch between properties at different stages of a calculation.

To perform derivatives we introduce a non-commuting function which we call *delp* which represents the derivative with respect to a momentum p. *delp* carries an index which corresponds to the index on the momentum we are differentiating with respect to, e.g. $delp(theta)$ would correspond to the operator $\frac{\partial}{\partial p^\theta}$.

We implement the following rules for derivatives

$$\frac{\partial}{\partial p^\mu} p_\nu = \delta_{\mu\nu}, \tag{A.6}$$

$$\frac{\partial}{\partial p^\mu} p^{-n} = n p^\mu p^{-n-2}. \tag{A.7}$$

In the final step of our calculations, with the derivative operators commuted all the way to the right, we let $delp = 0$.

To sort the results, we employ the following trick. We let the momentum p change from a non-commuting variable to a commuting variable. Doing so gathers all factors of p and p^{-1} together within a term so that we can manipulate them. Of course, we only do this after derivatives have been calculated otherwise correct order would not be maintained between momenta and derivative operators.

Any other uses of FORM in this calculation echo those that we have already employed in the logarithmic calculation, such as contracting equations with multiple Kronecker deltas, and require no further discussion here.

A.4 Yang-Mills-gravity calculation

Having considered scalar field theory in the previous section, let us now examine the effect of gravity on another important class of fields - the Yang-Mills field. Yang-Mills gauge theory is the study of fields which are related to an underlying $SU(N)$ symmetry. For example, in the standard model, $SU(2) \times U(1)$ describes electroweak interactions and $SU(3)$ describes quantum chromodynamics.

Here, we will consider Yang-Mills, in the sense of quantum chromodynamics, coupled to gravity. The result is well known (indeed, Nobel prize-worthy (10; 11)) in the absence of gravity.

Let us begin, as before, by writing down an action. We have

$$S = S_M + S_G \tag{A.8}$$

where

$$S_M = \frac{1}{4} \int |g(x)|^{1/2} F_{\alpha\beta a} F^{\alpha\beta a} \tag{A.9}$$

with the Yang-Mills field strength tensor given by

$$F_{\alpha\beta a} = \partial_\alpha A_{\beta a} - \partial_\beta A_{\alpha a} + g f_a{}^{bc} A_{\alpha b} A_{\beta c} \tag{A.10}$$

for some vector potential $A_{\alpha a}$, Yang-Mills parameter g (not to be confused with the metric tensor $g(x)$) and Yang-Mills structure constants $f_a{}^{bc}$.

As before,

$$S_G = -\frac{2}{\kappa^2} \int d^n x |g(x)|^{1/2} (R - 2\Lambda). \tag{A.11}$$

Introduce the quantum fields $\eta = (a_{\alpha a}, h_{\mu\nu})$ by

$$\begin{aligned} g_{\mu\nu} &= \delta_{\mu\nu} + \kappa h_{\mu\nu} && \text{(A.12)} \\ A_{\alpha a} &= \bar{A}_{\alpha a} + a_{\alpha a} && \text{(A.13)} \end{aligned}$$

so that the second functional derivative $S_{,ij}$ can be read off from the term quadratic in the quantum fields.

Considering infinitesimal gauge transformations of the field we find

$$\delta A_{\alpha a} = \int d^n x' K_{\alpha ab}(x,x')\delta\epsilon^b(x') \tag{A.14}$$

.

allowing us to read off the generators K.

The only non-zero components of the field-space metric this time around are again the diagonal components g_{AA} and g_{gg}.

We use the field space metric to construct the field space Christoffel symbols. The non-zero components are

$$\begin{aligned}\varGamma^{g_{\lambda\tau}(x)}_{g_{\mu\nu}(x')g_{\rho\sigma}(x'')} \quad = \quad & \Big[-\delta^{(\mu}_{(\lambda}g^{\nu)(\rho}\delta^{\sigma)}_{\tau)} + \frac{1}{4}g^{\mu\nu}\delta^{\rho}_{(\lambda}\delta^{\sigma}_{\tau)} + \frac{1}{4}g^{\rho\sigma}\delta^{\mu}_{(\lambda}\delta^{\nu}_{\tau)} - \frac{1}{2(2+nc)}g_{\lambda\tau}g^{\mu(\rho}g^{\sigma)\nu} \\ & -\frac{c}{4(2+nc)}g_{\lambda\tau}g^{\mu\nu}g^{\rho\sigma}\Big]\delta(x'',x')\delta(x'',x),\end{aligned} \tag{A.15}$$

$$\varGamma^{A^a_\mu(x)}_{A^b_\nu(x')g_{\lambda\tau}(x'')} = \frac{1}{4}\delta^a_b\delta(x',x)\delta(x',x'')(g^{\lambda\tau}\delta^\nu_\mu - g^{\nu\lambda}\delta^\tau_\mu - g^{\nu\tau}\delta^\lambda_\mu) \tag{A.16}$$

and

$$\varGamma^{g_{\mu\nu}(x)}_{A^a_\lambda(x')A^b_\tau(x'')} = \frac{1}{2}\delta(x',x)\delta(x',x'')\delta_{ab}\delta^{(\lambda}_\mu\delta^{\tau)}_\nu \tag{A.17}$$

Functional derivatives of (A.8) gives us

$$\frac{\partial S}{\partial A_{\lambda b}} = \partial_\mu \bar{F}^{\mu\lambda b} + g f^{bca}\bar{F}^{\lambda}{}_{\nu a}\bar{A}^\nu_c \tag{A.18}$$

and

$$\frac{\partial S}{\partial g_{\lambda\tau}} = \frac{2}{\kappa^2}\Lambda\delta^{\lambda\tau} + \frac{1}{8}\delta^{\lambda\tau}\bar{F}_{\alpha\beta a}\bar{F}^{\alpha\beta a} - \frac{1}{2}\bar{F}^{\lambda}{}_{\alpha a}\bar{F}^{\tau\alpha a} \tag{A.19}$$

The connection term is then

$$\begin{aligned}
\frac{1}{2}\eta^i\eta^j\Gamma^k_{ij}S_{,k} &= \\
&= \frac{1}{2}\eta^i\eta^j\left[\Gamma^{g_{\lambda\tau}(x)}_{g_{\mu\nu}(x')g_{\rho\sigma}(x'')}\frac{\partial S}{\partial g_{\lambda\tau}(x)} + \Gamma^{g_{\mu\nu}(x)}_{A^a_\lambda(x')A^b_\tau(x'')}\frac{\partial S}{\partial g_{\mu\nu}(x)} + \Gamma^{A^a_\mu(x)}_{A^b_\nu(x')g_{\lambda\tau}(x'')}\frac{\partial S}{\partial A^a_\mu(x)}\right] \\
&= \bar{F}_{\alpha\beta a}\bar{F}^{\alpha\beta a}\kappa^2\left(\frac{1}{16}h^2 - \frac{1}{8}h_{\gamma\delta}h^{\gamma\delta} + \frac{1}{16\kappa^2}a_{\gamma b}a^{\gamma b}\right) \\
&\quad + \bar{F}_{\alpha\beta a}\bar{F}_\gamma{}^{\beta a}\kappa^2\left(-\frac{1}{4}hh^{\alpha\delta} + \frac{1}{2}h^{\alpha\mu}h^\gamma_\mu - \frac{1}{4}a^\alpha_b a^{\gamma b}\right) \\
&\quad + \partial_\alpha\bar{F}^\alpha{}_{\beta a}\kappa\left(\frac{1}{4}ha^{\beta a} - \frac{1}{2}h^{\beta\gamma}a^b_\gamma\right) \\
&\quad + \bar{F}_{\alpha\beta a}f^{bca}\bar{A}^\beta_c\kappa g\left(\frac{1}{4}ha^\alpha_b - \frac{1}{2}h^\alpha_\gamma a^\gamma_b\right) \\
&\quad - \Lambda\left(h^2 + 2h_{\alpha\beta}h^{\alpha\beta} + \frac{1}{\kappa^2}a_{\alpha a}a^{\alpha a}\right) \qquad \text{(A.20)}
\end{aligned}$$

Now combine the above equations, as with scalar-gravity, as

$$\begin{aligned}
S_q &= \eta^i\eta^j\left(S_{,ij} + \Gamma^k_{ij}S_{,k} + \frac{1}{2\xi}K^\alpha_i K_{j\alpha}\right) \\
&= S_0 + S_1 + S_2 + S_3 + S_4. \qquad \text{(A.21)}
\end{aligned}$$

From S_0, we can determine the propagators, then proceed as with the scalar case. With the above calculations as a starting point, we could repeat the main calculation of this thesis for Einstein-Yang-Mills.

Bibliography

[1] G. ‘t Hooft and M. Veltman, Annales de L’Institute Henri Poincaré **20**, 69 (1974).

[2] S. Deser and P. van Nieuwenhuizen, Phys. Rev. Lett. **32**, 245 (1974).

[3] S. Deser and P. van Nieuwenhuizen, Phys. Rev. D **10**, 401 (1974).

[4] S Deser, H. Tsao, and P. van Nieuwenhuizen, Phys. Rev. D **10**, 3337 (1974).

[5] J. F. Donoghue, Phys. Rev. Lett. **72**, 2996 (1994).

[6] J. F. Donoghue, Phys. Rev. D **50**, 3874 (1994).

[7] C. P. Burgess. Quantum gravity in everyday life: General relativity as an effective field theory. *Living Rev. Relativity*, 7, 2004.

[8] K.S. Stelle, Phys. Rev. D, **16**, 953969 (1977).

[9] S. P. Robinson and F. Wilczek, Phys. Rev. Lett. **96**, 231601 (2006).

[10] D. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973).

[11] D. Politzer, Phys. Rev. Lett. **30**, 1346 (1973).

[12] I. Gogoladze and C. N. Leung, Phys. Letts. B **645**, 451 (2007).

[13] J.R. Bhatt, S. Petra and U. Sarkar, Mod. Phys. Lett A **25** 283 (2010).

[14] M. Maziashvili, Int.J.Mod.Phys., **D18** 2209-2213, (2009).

[15] Q. Huang, J. High Energy Phys. **03**, 053 (2007).

[16] A. Nicolis, N. Arkani-Hamed, L. Motl and C. Vafa, J. High Energy Phys. **06**, 060 (2007).

[17] M. Johnson, T. Banks and A. Shomer, J. High Energy Phys. **09**, 049 (2006).

[18] A. R. Pietrykowski, Phys. Rev. Lett. **98**, 061801 (2007).

[19] D. J. Toms, Phys. Rev. D **76**, 045015 (2007).

[20] D. Ebert, J. Plefka, and A. Rodigast, Phys. Lett. B **660**, 579 (2008).

[21] Y. Tang and Y. Wu, Gravitational contributions to the running of gauge couplings, 2008. arXiv:0807.0331v2 [hep-ph].

[22] D. Ebert, J. Plefka, and A. Rodigast, J. High Energy Phys. **02**, 028 (2008).

[23] F. Wu and M. Zhong. Correction to the running of the gauge couplings out of extra dimensional gravity. (2008). arXiv:0809.1913v1 [hep-ph].

[24] F. Wu and M. Zhong, Phys. Lett. B **659**, 694 (2008).

[25] A. Rodigast and T. Schuster, Phys. Rev. D **79**, 125017 (2009).

[26] D. J. Toms, Phys. Rev. Lett. **101**, 131301 (2008).

[27] D. J. Toms, Phys. Rev. D **80**, 064040 (2009).

[28] D. J. Toms, Phys. Rev. D **84**, 084016 (2011).

[29] D. J. Toms, Nature, **468**, 5659 (2010).

[30] D. J. Toms, Nature, **479**, E6 ,(2011).

[31] A. Rodigast and T. Schuster. Gravitational corrections to Yukawa and ϕ^4 interactions. 2009. arXiv:0908.2422v1 [hep-th].

[32] O. Zanusso, L. Zambelli, G. P. Vacca, and R. Percacci, Phys. Lett. B **689** 90 (2010).

[33] H.-J. He, X.-F. Wang and Z.-Z. Xianyu, Phys. Rev. D **83**, 125014 (2011).

[34] I. Antoniadis, J. Iliopoulos, and T. N. Tomaras, Nucl. Phys. B **267**, 497 (1986).

[35] B. de Wit and N. D. Hari Dass, Nucl. Phys. B **374**, 99 (1992).

[36] G. A. Vilkovisky, Nucl. Phys. B **234**, 125 (1984).

[37] G. A. Vilkovisky, in S. M. Christensen, editor, *The Quantum Theory of Gravity* (Adam Hilger, Bristol, 1984).

[38] B. S. DeWitt, in C. J. Isham I. A. Batalin and G. A. Vilkovisky, editors, *Quantum Field Theory and Quantum Statistics*, volume 1, (Adam Hilger, Bristol, 1987).

[39] L. Parker and D. J. Toms, *Quantum Field Theory in Curved Spacetime: Quantized fields and Gravity* (Cambridge University Press, Cambridge, 2009).

[40] B. S. DeWitt, *The Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).

[41] S. Falkenberg, S.D. Odintsov, *Gauge Dependence of the Effective Average Action in Einstein Gravity, Recent Developments in Theoretical and Experimental General Relativity, Gravitation, and Relativistic Field Theories* (1999)

[42] E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. B **234**, 509 (1984).

[43] A. Rebhan, Nucl. Phys. B **288**, 832 (1987).

[44] G. ‘t Hooft and M. Veltman, Nucl. Phys. B **44**, 189 (1972).

[45] G. ‘t Hooft, Nucl. Phys. B **61**, 455 (1973).

[46] J. C. Collins, Phys. Rev. D **10**, 1213 (1974).

[47] P. T. Mackay and D. J. Toms, Phys. Lett. B. **684**, 4-5 (2010).

[48] J. A. M. Vermaseren. New features of FORM. arXiv:math-ph/0010025.

[49] K. Peeters, Comp. Phys. Commun. **41**, 16 (2007);Introducing Cadabra: a symbolic computer algebra system for field theory problems. 2007. arXiv:hep-th/0701238.

[50] L.H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1996).

[51] K. Popper, *The Logic of Scientific Discovery* (Routledge, 2002).

[52] D.J. Gross, F. Wilczek, Phys. Rev. Lett. **30**, 13431346 (1973).

[53] D. Politzer, Phys.Rev.Lett. **30**, 1346-1349, (1973).

[54] M.M. Anber, J.F. Donoghue and M.E.Houssieny, Phys. Rev. D **83**, 124003 (2011).

[55] J. Ellis, N.E. Mavromatos, On the interpretation of gravitational corrections to gauge couplings. arXiv:1012.4353 [hep-th].

[56] N.K. Nielsen. The Einstein-Maxwell system, Ward identities, and the Vilkovisky construction arXiv:1109.2699v1 [hep-th].

[57] S. D. Odintsov, Pis'ma Zh. Eksp. Teor. Fiz. **53** 4 (1991).

[58] J.C.C. Felipe, L.C.T. Brito, M. Sampaio and M.C. Nemes, Phys. Lett. , **B700**, 86-89 (2011).

[59] L. D. Landau, A. A. Abrikosov, and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR **95** 1177-80 (1954).

[60] S. Weinberg, Erice Subnucl. Phys. **1** (1976).

[61] S.M. Christensen and M.J. Duff, Phys. Lett. B, **79**, 213, (1978).

[62] R. Gastmans, R. Kallosh and C. Truffin, Nucl. Phys. B **133**, 417, (1978).

[63] S. Weinberg Ultraviolet divergences in quantum theories of gravitation. In "General Relativity: An Einstein centenary survey", ed. S. W. Hawking and W. Israel, chapter 16, 790-831 (Cambridge University Press, 1979).

[64] L. Smolin, Nucl. Phys. B **208**, 439-466, (1982).

[65] H. Kawai and M. Ninomiya, Nuclear Physics B **336**, 115-145, (1990).

[66] I. Jack, D.R.T. Jones, Nucl.Phys.B **358**, 695-712 (1991).

[67] D. Dou and R. Percacci, Class. Quant. Grav. **15**, 3449, (1998).

[68] S. Falkenberg, S.D. Odintsov, Int. J. Mod. Phys. **A13**, 607-623 (1998).

[69] R. Percacci and D. Perini, Phys. Rev. **D68**, 044018 (2003).

[70] M. Niedermaier, Nucl. Phys. B 673, 131-169. (2003).

[71] Martin Reuter, Frank Saueressig Fortsch. Phys. 52, 650-654 (2004).

[72] Daniel F. Litim, Phys. Rev. Lett. **92**, 201301, (2005).

[73] R. Percacci,Phys. Rev. **D73**, 041501 (2006).

[74] M. Niedermaier, M. Reuter, The Asymptotic Safety Scenario in Quantum Gravity, (2006) http://relativity.livingreviews.org/Articles/lrr-2006-5/.

[75] R. Percacci, A short introduction to asymptotic safety., (2011) arXiv:1110.6389 [hep-th].

[76] Martin Reuter and Frank Saueressig, Quantum Einstein Gravity, (2012) arXiv:1202.2274 [hep-th].

[77] J.E. Daum, U. Harst, M. Reuter, JHEP, **10** (2010).

[78] M. Shaposhnikov and C. Wetterich, Phys. Lett. B **683**, 196 (2010).

[79] J.S. Schwinger, Phys. Rev. **82** 664 (1951).

[80] K. Kirsten, *Spectral Functions in Mathematics and Physics* (Chapman and Hall/CRC,London, 2002).

[81] I.G. Avramandi, *Heat Kernel and Quantum Gravity* (Lecture Notes in Physics Monographs) (Springer, New York, 2000).

[82] P.B. Gilkey, J. Diff. Geom., **10** 601 (1975).

[83] A.O. Barvinsky and G.A. Vilkovisky, Phys. Rep. **119** 1 (1985).

[84] L. Abbott, Nucl. Phys. B **185** 189 (1981).

[85] D.J. Toms, Phys. Rev. D **26** 2713 (1982).

[86] T.S. Bunch, L. Parker, Phys. Rev. D **20** 2499 (1979).

[87] A. Rebhan, Nucl. Phys. B **298**, 726 (1988).

[88] S.W. Hawking, Comm. in Math. Phys., **46**, 2, (1976).

[89] M.M. Anber,J.F. Donoghue, On the running of the gravitational constant, arXiv:1111.2875v2 [hep-th]

[90] P. van Nieuwenhuizen, Phys. Rep., **68** 4 (1981).

[91] B. Muller, J.L. Nagle, Ann.Rev.Nucl.Part.Sci. **56**, 93-135,(2006).

[92] J.M. Maldacena., Adv.Theor.Math.Phys. **2**, 231-252,(1998).

[93] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, Phys.Lett.B **428:105-114**, (1998).

[94] E. Witten, Adv.Theor.Math.Phys. **2:253-291**, (1998).

[95] A. Chamblin, R. Emparan, C.V. Johnson, R.C. Myers, Phys.Rev. D60 (1999).

[96] S.W. Hawking, C.J. Hunter, M.M. Taylor-Robinson, Phys.Rev.D59:064005,(1999).

[97] S.W. Hawking, G.F.R. Ellis, *Large Scale Structure of Spacetime* (Cambridge University Press, 1973).

[98] M.M. Caldarelli, G. Cognola, D. Klemm, Class.Quant.Grav. 17 (2000).

[99] A. Chamblin, R. Emparan, C.V. Johnson, R.C. Myers, Phys.Rev.D60:104026,(1999).

[100] G.W. Gibbons, M.J. Perry, C.N. Pope, Class.Quant.Grav.22:1503-1526,(2005).

[101] H. Nastase The RHIC fireball as a dual black hole (2005) arXiv:hep-th/0501068.

[102] E. Shuryak, S.-J. Sin, I. Zahed J.Korean Phys.Soc. **50** 384-397,(2007).

[103] R.A. Janik, R. Peschanski Phys.Rev.D73 **045013** (2006).

[104] S. Ross, Black Hole Thermodynamics, arXiv:hep-th/0502195

[105] P. Kovtun, D.T. Son, A.O. Starinets Phys.Rev.Lett. **94** (2005).

[106] R. Nouicer, for the PHOBOS Collaboration, *Elliptic Flow, Initial Eccentricity and Elliptic Flow fluctuations in Heavy Ion Collisions at RHIC* (2007) arXiv:0707.4424.

[107] P. Romatschke, U. Romatschke, Phys.Rev.Lett. **99** 172301 (2007).

[108] S.W. Hawking, Commun.Math.Phys. **87** 577 (1983).

[109] E. Witten, Adv.Theor.Math.Phys. **2** 505-532 (1998).

[110] P. Roberts, N.G. Berloff, J. Phys. A. **33**, 4025 (2000).

[111] Z.-W. Chong, M. Cvetic, H. Lu, C.N. Pope, Phys.Rev.Lett. **95** 161301 (2005).

[112] D. Birmingham, I. Sachs, S. Sen, Phys.Lett. **B424** (1998).

[113] M. Cvetic, F. Larsen, Nucl.Phys.B **531:239-255**,(1998).

[114] A. Strominger, C. Vafa, Phys.Lett.B **379:99-104**,(1996).

[115] J.C. Breckenridge , R.C. Myers , A.W. Peet , C. Vafa, Phys.Lett. B **391:93-98** (1997).

[116] G.T. Horowitz, Donald Marolf, JHEP, **9807:014**,(1998).

[117] E. Shuryak, Prog.Part.Nucl.Phys. **53** (2004).

[118] I. Arsene et al. BRAHMS collaboration, Nucl.Phys.A **757:1-27** (2005).

[119] PHENIX Collaboration, K. Adcox, et al, Nucl.Phys.A **757: 184-283** (2005).

[120] STAR Collaboration: J. Adams, et al, Nucl.Phys.A **757:102-183**,(2005).

[121] B.B. Back, et al (PHOBOS), Nucl.Phys.A **757:28-101**,(2005).