CLASSICAL AND APPROXIMATE METHODS IN THE DYNAMIC RESPONSE ANALYSIS OF A TRUSS SPAR IN WAVES

By

Keyvan Sadeghi, B.Sc., M.Sc.

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School of Marine Science and Technology
UNIVERSITY OF NEWCASTLE UPON TYNE

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Abstract

It is shown that, in the context of a linear theory, all radiation actions of fluid on a floating body can solely be represented by the fluid kinetic and potential energy associated with the wetted surface of the body. In this regard, it is indicated that the linear radiation damping can be expressed by a part of the fluid kinetic energy which has a bilinear form. The linear problem of a floating body motion is then studied in the context of a general linear dynamical system with such form of kinetic energy. From the Lagrange's equations of motion, an equation of motion is derived which generates the linear damping force directly from the bilinear kinetic energy without using any dissipation function. A variant of Hamilton's principle is introduced as the variational generator of this equation of motion.

It has been shown that in the context of a linear theory for a floating body with six degrees of freedom each of the 6×6 added mass and damping matrices contains three distinct Cartesian second-order tensors in regard to translational, rotational and interaction between translational and rotational oscillations. As a result of this, a new technique based on the transformation law of second order tensors is introduced for motion analysis of offshore platforms that can be used as an alternative to the common methods of motion analysis in offshore engineering.

Consistent with the transformation method, a viscous-radiation–diffraction model is proposed to include viscosity effects in the linear equations of motion derived from a potential radiation–diffraction analysis. This model is developed for both first- and second-order dynamic response analysis of a truss spar platform. The results obtained from this analysis are compared with experi-

mental data and the results of more conventional numerical approach. In case of the first-order uncoupled heave, the equation of motion with a nonlinear drag term is solved without any iteration in the frequency domain. For the slowly varying drift motion, the model yields a simple equation of motion which can be solved in the frequency domain easily and with fairly good accuracy.

Also in the first-order diffraction problem an approximate theory is proposed for the prediction of surge and pitch loads acting on a truncated vertical cylinder. The results of this theory are compared with the numerical results reported in the literature.

Dedication

To my best teacher Dr. Asghar Nosier.

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Nomenclature

a	Cylinder radius
a, b, c	Components of position vector, coefficients of the quadratic equation
$a_{\alpha\beta}$	Added mass coefficients
a_{ij}	Transformation symbol (direction cosines)
$b_{\alpha\beta}$	Damping coefficients
$c_{\alpha\beta}, \tilde{c}_{ij}$	Hydrostatic restoring coefficients
d	Draft, diameter
d_k	Components of position vector
e	Natural base of logarithms
e_i	Unit basis vectors
f_1	Surge force intensity of McCamy & Fuchs
f_{FK}	Froude–Krylov force per unit length
f_S	Scattering force per unit length
f_{SI}, f_{SD}	Inertia and damping component of Scattering force
f_{CI}, f_{CD}	Inertia and damping component of correction force
g	Gravitational acceleration
h	Water depth
h_d	Dynamic depth
i	Imaginary number, $\sqrt{-1}$
k	Wave number
k_{ij}, k_x	Mooring restoring coefficients
l	Arc length, cylinder length
m	Mass
$m_{\alpha\beta}$	Added mass coefficients of an immersed body
m_{ij}	Added mass tensor Components of an immersed body
n	Unit normal direction, counter

n_i	Components of unit normal vector
$\mathbf{n}$	Unit normal vector
p	Fluid pressure
q_α	Generalized coordinates
$\dot{q}_\alpha, \ddot{q}_\alpha$	Generalized velocities and accelerations
r	Cylinder radius
t	Time
u_i	Water particle's velocity component
x	Amplitude of the slowly varying surge response
x_{sig}	Significant response
x_i, x'_i, x, y, z	Cartesian coordinates

Capital letters

$A(ka)$	$1/H(ka)$
A_{ij}	Added mass tensor Components of a floating body
$B_d^{(2)}, \bar{B}_d^{(2)}$	Slow and mean wave drift damping coefficients
$B_{\alpha\beta}$	Radiation damping coefficients
B_{ij}	Radiation tensor, viscous damping matrix
C_d, C_D	Viscous damping coefficients
C_M	Inertia coefficient
C_S	Scattering coefficient
C_{SI}, C_{SD}	Inertia and damping components of C_S
D	Cylinder diameter
D_{ij}, E_{ij}	Radiation damping tensor components
D_{ij}^d	Quadratic transfer function of wave drift damping
D_i^d, D_i^q	Mean value of D_{ij}^d
E	Mechanical energy
F	Force
F_1, F_1^t	McCamy & Fuchs Surge force

F_C	Surge correction force
F_{CD}, F_{CI}	Damping and inertia components of F_C
F_{FK}	Froude–Krylov force
F_S	Scattering force
F_{SD}, F_{SI}	Damping and inertia component of F_S
H, H_w	Wave height
H_i	Magnitude of the linear transfer function
H_{ij}	$-\epsilon_{ijk}d_k$
H_S	Significant wave height
$H(\omega)$	Mechanical admittance
$H(ka)$	First kind Hankel function of order one
I_i	Invariants of a second-order tensor
I_{ij}	Added moment of inertia tensor component
$J_i(ka)$	Bessel function of the first kind and i-th order
L	Lagrangian function
M	Moment
$M_{\alpha\beta}$	Generalized mass coefficients
M_C	Pitch correction moment
P_α	Generalized momentum
P_{ij}	Component of a second-order pseudo-tensor
Q_α	Generalized force
$Q^d_{ij}, Q^p_{ij}, Q^q_{ij}$	Slow drift quadratic transfer functions
$Q^d_{ij}(0)$	Q^d_{ij} at zero forward speed
$Q^p_{ij}(0), Q^q_{ij}(0)$	Q^p_{ij}, Q^q_{ij} at zero forward speed
$Q^d_i(U), Q^q_i(U)$	Mean drift quadratic transfer functions
$Q^d_i(0), Q^q_i(0)$	$Q^d_i(U), Q^q_i(U)$ at zero forward speed
R	Rayliegh's dissipation function, radius
R_{ij}	Component of radiation tensor
$\mathbf{R}^0, \mathbf{R}^1, \mathbf{R}^2$	Zeroth, first and second moment radiation tensors

$S(\omega)$	Wave spectrum
$S_F(\mu)$	Slow-drift force spectrum
S_{ij}	Added product of inertia tensor component
T	Kinetic energy
T_{ij}	Component of a second-order tensor
U	velocity of the slow-drift motion
U_i	Translational velocity of rigid body
V	Potential energy
$\mathbf{V}$	Fluid velocity vector
W^{nc}	Work due to non-conservative forces
$\mathcal{W}$	Generalized work
X_{ij}	Added moment of inertia tensor component
$Y_i(ka)$	Bessel function of the second kind and order i
Z, Z_1, Z_2	Complex number

Greek letters

α, β	Rotation angles, phase angles
α_i, β_i	Phase angles
$\bar{\gamma}$	Spectral weighted average of γ
δ	Distance from centre of gravity to fair leads
δ	Surge force ratio, amplitude of slow-surge motion
$\bar{\delta}$	Spectral weighted average of surge force ratio
δ_{ij}	Kronecker delta
δ_{ij}^d	Difference frequency
ϵ_{ijk}	Components of alternator tensor
ϵ_i	Phase of the i-th regular wave component
$\zeta^{(0)}, \zeta^{(1)}, \zeta^{(2)}$	Perturbed surface elevations
η, η^R	Surface elevation
η_i	Amplitude of the i-th regular wave component

θ_i	Angular rotation vector component
μ	Difference wave frequency
ν	Water kinematic viscosity
ρ	Water density
τ_n	Integer coefficient ($\tau_0 = 1, \tau_n = 2, n \geq 1$)
ϕ	Velocity potential
$\varphi_\alpha, \psi_\alpha$	Unit- velocity and displacement potentials
ξ_α, ζ_α	Unit surface elevations
ω	Wave circular frequency
ω_c	Cut-off frequency
ω_e	Encounter frequency
ω_i	Frequency of i-th regular wave component
ω_n, ω_{in}	Natural frequencies
$\Delta\omega$	Difference wave frequency
Ω_i	Component of angular velocity vector

Operators

d	Differential operator
δ	First variation operator
$\bar{\delta}$	Conjugate first variation operator
∇	Gradient vector
∇^2	Laplacian

Subscripts

$i, j, \ldots$	Latin indices (range 1 to 3), counter
$\alpha, \beta, \ldots$	Greek indices (range 1 to 6)
B	Related to rigid body
D	Related to the diffraction problem, diffracted wave
F	Related to fluid
I	Incident wave

R	Related to the radiation problem
S	Scattering wave
S_B	Related to the wetted surface of the body

Superscripts

d	Difference frequency
(n)	Order n
q	Quadratic interaction of linear effects
B	Bilinear part
D	Related to the diffraction problem
Q	Quadratic part
R	Related to the radiation problem
T	Transpose

Abbreviations

FPSO	Floating production and off-loading
TLP	Tension leg platform
RAO	Response amplitude operator
PRAO	Pseudo response amplitude operator

Where a variable is used more than once to denote different quantities the context of the chapter should indicate the intended meaning.

DECLARATION

Except where reference is made to the work of others,

this thesis is believed to be original

Chapter 1

Introduction

The ongoing worldwide demand for oil and gas and the discovery of oil in deep water has pushed the offshore oil companies to deeper and deeper water. Because the production of oil in deep water is more expensive, an offshore platform designed to work in deep ocean environment must be a reliable low cost facility. Offshore platforms can be classified as being either fixed or compliant. Compliant platforms are not fixed at least in some degrees of freedom. A compliant platform may be a floating platform.

In shallow water, fixed structures are the most economical option. As the water depth increases compliant structures become more economical than conventional fixed structures. Among compliant offshore platforms suitable for deep water one can refer to floating production, storage and offloading tankers (FPSO's), Tension leg platforms (TLP's) and Spar platforms. Depending on the location, the well system, production rates and possible relocation, one type of compliant platform may becomes more economical than the others. As water depth further increases Spars become one of the most viable options.

1.1 Spar platforms

The spar platform has a number of features that make it a feasible option for deep water drilling and oil production. A conventional spar platform known as

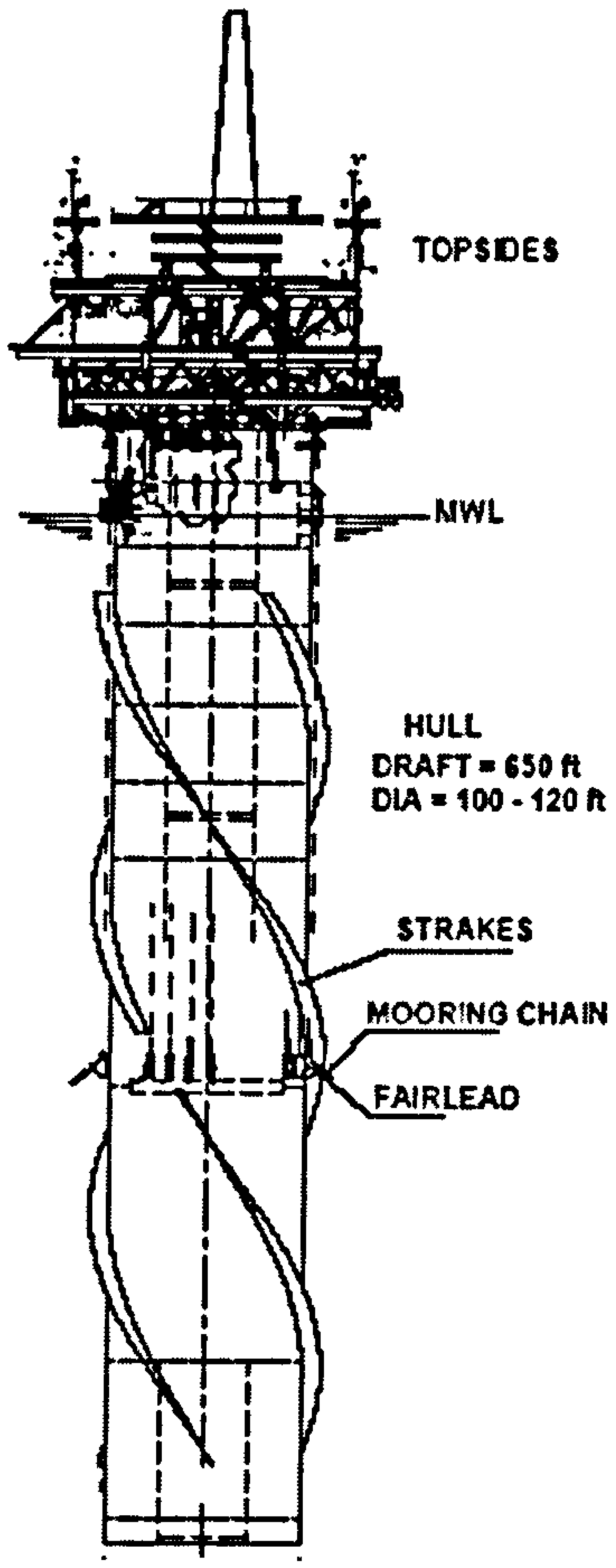

Figure 1.1: Schematic of a Classic Spar (Irani & Finn 2004)

classic spar is basically a floating large deep draft truncated vertical circular cylindrical structure (see Figure 1.1). The upper part of the hull provides the buoyancy and the lower part of the hull is flooded and therefore is pressure equalized to the sea and does not require a high strength shell structure. A part of this structure can also be used for storage of oil which can then be directly offloaded. The lowest compartment of the upper hull holds the ballast, which serves to control the draft and trim of the platform. Rigid steel risers are configured in a centrewell which runs through the hull. Spars are usually connected to the sea floor with mooring lines which are attached to fairleads at a point close to the structure's centre of gravity. Spiral strakes may have to be fitted to the hull surface of a classic spar to suppress vortex-induced vibration. A classic spar may be designed to serve as a storage and offloading, production or drilling platform, depending on the specific requirements. The hull of the structure may be of the order of 40m in diameter and 200m deep, depending on its application and the environment in which it works. The idea behind this concept is that due to the large draft, the motion responses of the platform to the wave loads should be correspondingly low. This has been proved through model tests and field observations.

The concept of a spar platform as an offshore structure is not new. Spar platforms now have a history of developement spanning several decades. The Floating Instrument Platform (FLIP) was built in 1961 to perform oceanographic research. Its favourable motion properties are well documented (Rudnick 1967). Developement of spar platforms for the offshore oil industry has been ongoing for several years. Over this period, extensive model tests have been performed to verify the motions, loads and other design characteristics. The Brent Spar was the first spar used by oil industry from 1976 to 1991 as an offshore oil storage/offloading terminal in the North Sea. The concept of a spar specifically as a production platform is relatively recent. The world's first production spar in the Gulf of Mexico is the Neptune Spar, which was installed in 588m water and started operation in 1997 (Glansville 1997). In later designs, such as the Chevron Spar, the concept was extended to include drilling capabilities.

Spar is a relatively inexpensive structure. Its simple hull can be built in most shipyards at low cost. The low motion responses of the spar configuration to the sea loads permit the installation of rigid risers, which are significantly less expensive than flexible risers. Also the low dynamic motions at the mooring fairleads allows the use of an array of taut or catenary mooring lines for station keeping of the platform that is easy to install, operate and relocate. The taut mooring system is economically more efficient than the catenary mooring system due to capital cost reduction and a smaller watch circle during operation. However, a short mooring can lead to resonance in the vertical plane motions, principally in heave (Lake et al. 2000). A spar platform can generally be operated in water depths up to 3000m. Due to its configuration, the centre of gravity of the spar platform is always below its centre of buoyancy, therefore, it is always stable and can support large topside loads. Another merit of the spar platform is the good protection of risers provided by the centrewell.

Classic spars have low damping and relatively low heave natural period. A combination of these two characteristics and swell may lead to linearly excited heave resonant motion of the spar. In addition, when the ambient deep current is a major factor, the drag on the long cylindrical hull can be significant (Haslum & Faltinsen 1999, Tao et al. 2001, Datta et al. 1999). To reduce the effect of these factors, an alternative design known as truss spar is considered which received considerable attention as a more economical design (see Figure 1.2). Truss spar is also structurally more efficient when oil storage is not required. The truss spar is a structure composed of three sections. The upper part is a hard tank similar to the classic spar, the middle part is a trusslike framework constructed of slender members and a number of horizontal heave plates and the lower part is a soft tank at the keel. The soft tank mainly contains solid ballast to provide stability, whereas the hard tank provides buoyancy and contains trim ballast. The multiple horizontal heave plates greatly increase the added mass and viscous damping of the structure in heave motion. Thus the effective vertical mass of the platform, and hence its natural period in heave, increases to similar values of a comparable classic spar and is well above the dominant wave periods. In addition, the increased added mass contributes to a reduction in the heave motion despite the increase in the heave wave exciting force relative to that of a similar classic spar due to a shallower cylinder draft. Results of numerical simulations and model tests indicate that the motion behavior of the truss spar is better than the classic spar mainly due to increased added mass and viscous damping and the truss spar is known to be more stable than the conventional one. A truss spar has a much lower drag area than a classic spar so that the current and associated mooring loads are reduced. It is also less susceptible to the vortex-induced

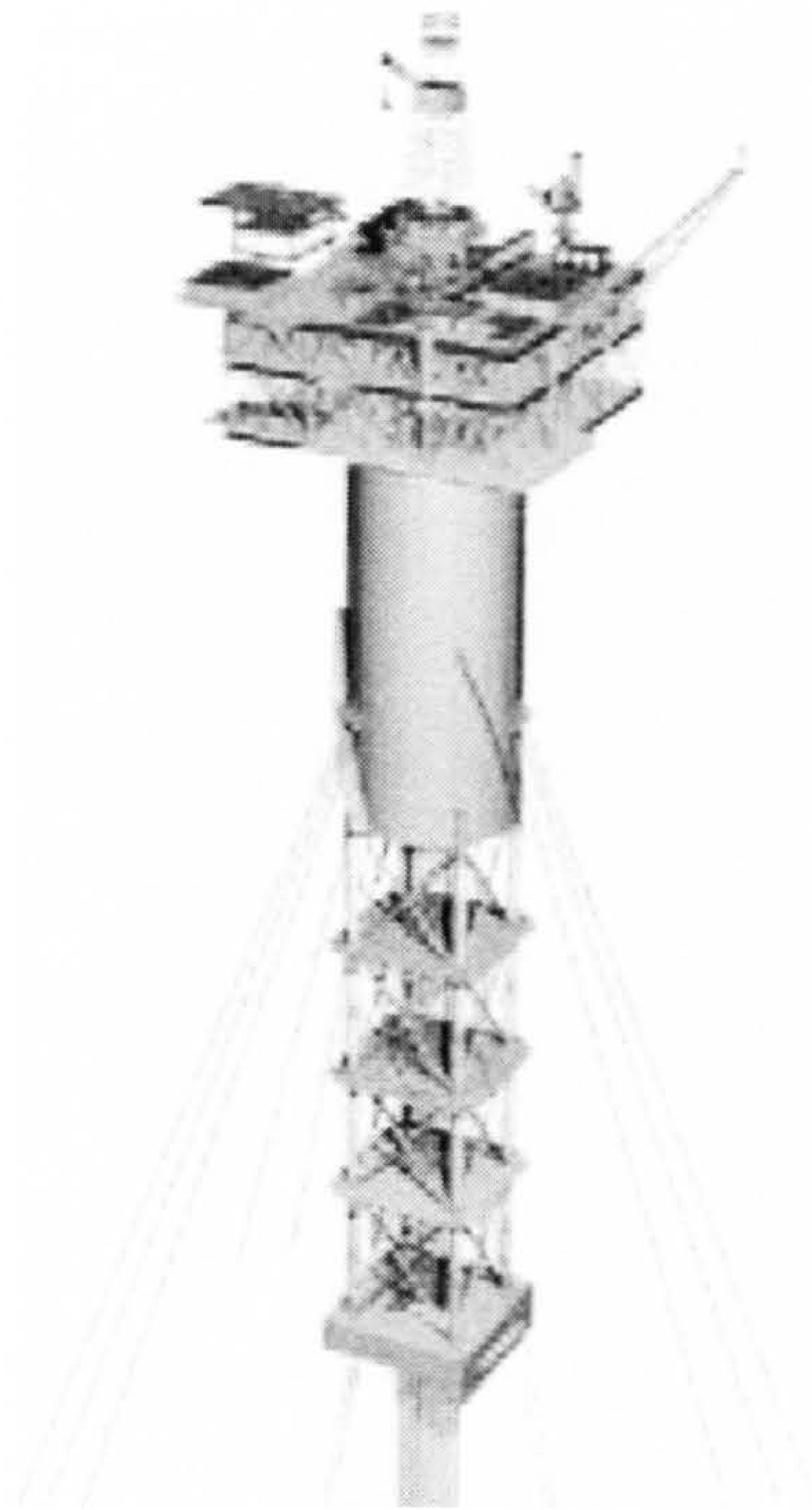

Figure 1.2: Schematic of a Truss Spar (Irani & Finn 2004)

vibration because the buoyant cylindrical section is short compared to the full length of the classic spar (Downie et al. 2000). As of today, three classic spars and eight truss spars have been installed in the Gulf of Mexico. In addition to the prevalent concepts of the classic spar and truss spar, other types of spar platforms have been considered such as minispars, shoe-box spars and cell spars. The cell spar (Finn & Maher 2003) consists of several smaller diameter tubes joined together in contrast to a single diameter tube forming the hull (see Figure 1.3). Some of these designs are in advanced construction or planning phases (Irani & Finn 2004).

1.2 Study objectives

The prediction of the dynamic behavior of spar platforms is essential to the design of their structure and subsystems such as mooring lines, risers and tethers. One approach to this is the use of scaled model tests. As was mentioned, spar platforms are designed to operate in deep and ultra deep water. For such

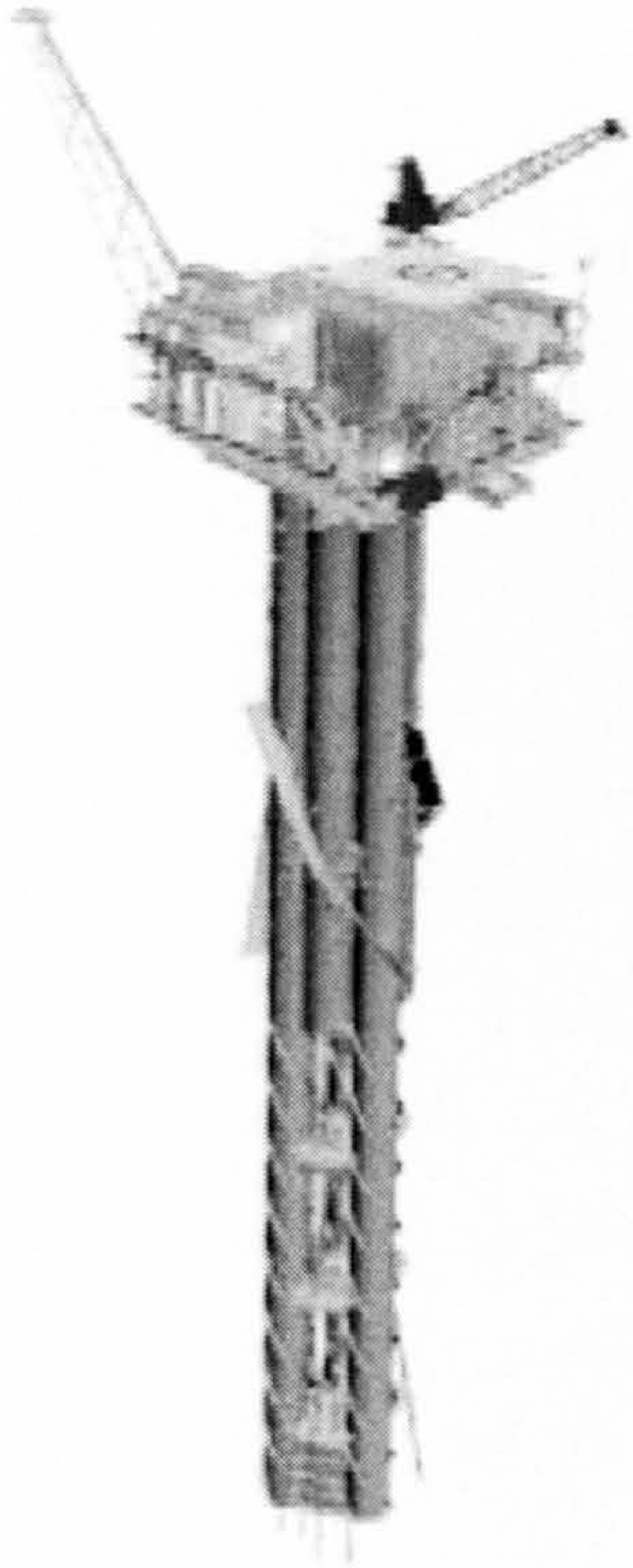

Figure 1.3: Schematic of a Cell Spar (Irani & Finn 2004)

water depths the model testing can be applied in various levels of complexity. The most expensive setup is when the whole system including the platform and the full-depth mooring lines and risers are modelled. In this case, there are several experimental difficulties. As Stansberg et al. (2002) mentioned, one difficulty is that no tank facility in the world is sufficiently deep to perform the testing of the complete system within reasonable limits of model scale in 1500–3000 m depth. Therefore, the size of the model must be excessively reduced. In these circumstances, the reliability of the experimental results will be reduced due to the scale effect that occurs from the incompatibility between Reynolds and Froude numbers (Nishimoto et al. 2002). To reduce the uncertainty of results, very accurate models and measuring instruments together with the highest level of expertise must be employed that increase the cost of the model testing. A less expensive setup is the use of the so-called hybrid method where a combination of model tests at reduced depths with computer simulations will be applied (Stansberg et al. 2002). In the simplest setup, a reasonably large model is used for the hull and the mooring system will be modelled only approximately by means of springs with linear elastic characteristics. Therefore, the test can be down in many testing basins with less cost and if the accurate measurement of the dynamic effects of mooring lines, such as mooring line damping, is not required, valuable information can still be gathered from the test results.

Another method for the prediction of the behavior of a spar platform is by numerical simulation. Numerical techniques for the prediction of wave effects have achieved an important role in offshore engineering, and have assured an importance comparable to physical experiments. For a deep water compliant structure like a spar platform the numerical program should be capable of including the nonlinear quadratic terms for the prediction of the second-order forces which are crucial for the design of the mooring system of the structure. In complex procedures, the conventional linear panel methods will be extended to include all second order effects. The program may also employ higher-order panel methods where the solution and/or the geometry are represented by polynomial or B-spline functions. Even if relatively simpler linear panel methods are used, valuable information can still be obtained for the calculation of mean and slowly varying drift responses of the platform. In any case, using a numerical simulation like the experimental model testing is not free of challenge. Difficult numerical problems, such as the evaluation of the nonlinear second-order boundary conditions on the free surface or the removal of irregular frequencies may be encountered that must be tackled. These problems as Lee et al. (1991) mentioned are rich and varied as the spectrum of the second-order results.

Experimental and fully developed numerical methods tend to be expensive and need special facilities and expertise to be applied. Both methods are problem specific and are suitable for later or final stages of design. In addition to experimental and numerical methods, approximate methods are another important tool for dynamic response analysis of offshore platforms.

These methods are not expensive and their application does not usually need highly specialised expertise. They can economically be applied in repetitive analysis and their output usually conveys sufficient insight. This makes them more suitable for the early stages of design. More importantly, simplified approximate methods are essential to form a basis for the physical understanding that is required for evaluation of model test results or for sound judgement of outputs of sophisticated computer programs.

The methodology selected for the research presented in this thesis was neither experimental nor numerical. This was firstly, due to the size limilations of the available model basin in the department and secondly because a practical method of prediction was being sought that does not require extensive computer resources, nor the application of highly specialised and complex numerical techniques.

The object of the research presented in this thesis was to derive and develop simplified approximate methods, based on theoretical methods, that can be applied for the dynamic response analysis of spar platforms. Some of the approximate models and numerical schemes presented are not limited to spar platforms and can also be applied to other types of offshore structures.

1.3 Outline of the Thesis

The study starts with the application of energy methods to the formulation of the motion problem of a floating body. Traditional marine hydrodynamics is dominated by Newtonian mechanics. The Newtonian and Lagrangian approaches each have their own merits and complement each other. In complicated problems, like the problems of fluid-structure interaction, energy methods can be used to derive alternative methods that are more suited to some problems. Therefore, Chapter 2 is devoted to a Lagrangian treatment of the linear theory of the motion of a floating body. The mathematical expressions of the kinetic and potential energy associated with the wetted surface of the body derived in Chapter 2 are then used to express tensor properties of added mass and damping coefficients in Chapter 3. Chapter 4 presents an approximate theory for prediction of the surge and pitch loads acting on truncated vertical cylinders floating in deep water. Some parametric studies regarding the diameter to draft ratio of the truncated cylinder are presented and the results of the developed approximate method are compared with the available numerical results. Based on the findings of Chapter 3, a transformation method for the response analysis of a compliant floating structure is proposed which is used for the dynamic response analysis of a truss spar platform in Chapter 5. Also, in this chapter, a viscous–radiation–diffraction model is introduced to include viscous effects consistently with the transformation method in the equations of motion. In Chapter 6 the slow-drift surge motion of a truss spar platform is studied by using approximations based on the viscous–radiation–diffraction model and the results are compared with the available experimental data. Finally, overall conclusions and recommendations for future work are presented in Chapter 7. Some of the theoretical material related to the early chapters of the thesis are presented in the appendices in order to make the thesis more readable.

Chapter 2

On the classical linear theory of motion of a floating body

2.1 Introduction

Methods of analytical mechanics were introduced to the marine hydrodynamics at the late nineteenth century. Lord Kelvin (1879) applied these methods in the study of motion problem of a submerged body. Since then different analytical methods have been applied to the fluid-body interaction problems. In some applications, following the work of Lord Kelvin, the Lagrangian function is replaced by the kinetic energy (Lamb 1932, Wang 1976). Similarly and mostly in fluid problems without the presence of a rigid body the fluid pressure is taken in place of the Lagrangian function (see for instace Luke 1967). In other applications, the principle of virtual velocity is the basis of the analytical method (Milne-Thomson 1968, Athanassoulis & Loukakis 1985). In this work, the problem of fluid–structure interaction is studied in the context of analytical mechanics by a new approach. As a point of departure and following Miloh (1984) the original Lagrangian function, as the difference between the kinetic and the potential energy, is chosen to start the analytical representation. Our method is, however, different from that of Miloh (1984) as he generalized Luke's (1967) variational principle whereas we shall use a variant of Lagrange's equations of motion.

In § 2.2 Lagrange's equations of motion are used to derive equations of motion for a submerged body moving in an unbounded fluid. In § 2.3 the linear radiation problem of a floating body is studied by an energy method. Rather than confining the method to a Lagrangian one and comparing the results at the end with Newtonian results, a combined application of both Lagrangian and Newtonian approaches is used. In addition, the consistency requirement of both approaches is applied throughout. This leads to the following equation in § 2.3,

$$T_{S_B} = \frac{1}{2}\,\dot{q}_\alpha\, a_{\alpha\beta}\, \dot{q}_\beta + \frac{1}{2}\, q_\alpha b_{\alpha\beta}\, \dot{q}_\beta \tag{2.1}$$

in which T_{S_B} is the kinetic energy of the fluid associated with the wetted surface of the body, q_α and $\dot{q}_\alpha$ are generalized displacements and velocities of the body and $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are added mass and damping coefficients. Then Lagrange's equations of motion of a floating body are derived in § 2.4. In § 2.5, a variant of Lagrange's equations of motion is derived which generates the radiation damping force from (2.1) without using a dissipation function. This equation in the absence of non-conservative forces can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E}{\partial \dot{q}_\gamma} + \frac{\partial E}{\partial q_\gamma} = 0. \tag{2.2}$$

in which E is the total mechanical energy. In § 2.6 a variant of Hamilton's principle is derived which can be stated as

$$\bar{\delta}\int_{t_1}^{t_2} E\,\mathrm{d}t = 0 \tag{2.3}$$

where $\bar{\delta}$ is an antisymmetric variational operator introduced in § 2.6. This variational equation generates (2.2) directly.

Throughout this chapter, the fluid flow is assumed to be incompressible and irrotational. For the same quantity, subscripts F and B are used to distinguish between the fluid and rigid body contributions. Unless it is explicitly specified, Greek indices range 1 to 6, Latin indices range 1 to 3 and summation convention is implied on repeated indices.

2.2 Hydrodynamic problem of a submerged body

Consider a rigid body immersed in an otherwise unbounded fluid where the rigid body and the fluid are initially at rest. If the rigid body suddenly be moved in an arbitrary manner, the surrounding fluid will be set into motion. Taking the rigid body and the fluid as a single dynamical system, the Lagrange's equations of motion for the system can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial(L_F + L_B)}{\partial \dot{q}_\alpha} - \frac{\partial(L_F + L_B)}{\partial q_\alpha} = Q_\alpha, \qquad \alpha = 1, \ldots, 6 \tag{2.4}$$

in which q_α, the generalized coordinates of the system, are displacement components of the rigid body, that is,

$$\dot{q}_\alpha = U_i : \quad \alpha = i, \qquad \dot{q}_\alpha = \Omega_i : \quad \alpha = i + 3, \tag{2.5}$$

where U_i is the translational velocity of the rigid body at an origin fixed to the body, Ω_i is the angular velocity of the body, Q_α are generalized external forces acting on the body and L_F and L_B are, respectively, fluid and rigid body Lagrangians, i.e., $L_F = T_F - V_F$ and $L_B = T_B - V_B$. The kinetic energy of the fluid can be written as

$$T_F = \frac{1}{2}\rho \int_{\mathcal{V}} \mathbf{V} \cdot \mathbf{V} \, \mathrm{d}\mathcal{V}, \tag{2.6}$$

where $\mathbf{V}$ is the fluid particle velocity and $\mathcal{V}$ is the 3-dimensional fluid domain bounded between the body surface S_B and a surface S_∞ far from the body enclosing it. Because the fluid flow is assumed to be incompressible and irrotational, the velocity field will be potential and the velocity potential ϕ satisfies the Laplac's equation. Therefore by using the divergence theorem it follows from (2.6) that

$$T_F = \frac{1}{2}\rho \int_S \phi \frac{\partial \phi}{\partial n} \mathrm{d}S \tag{2.7}$$

in which $S = S_B \cup S_\infty$. Now following Kelvin's approximation that the potential energy is invariant (Miloh & Hauptman 1980), the force due to the potential energy can be neglected and therefore (2.4) will be simplified as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial (T_F + T_B)}{\partial \dot{q}_\alpha} - \frac{\partial (T_F + T_B)}{\partial q_\alpha} = Q_\alpha, \qquad \alpha = 1, \ldots, 6. \tag{2.8}$$

Taking the fluid terms to the right-hand side of the equation, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_B}{\partial \dot{q}_\alpha} - \frac{\partial T_B}{\partial q_\alpha} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_F}{\partial \dot{q}_\alpha} + \frac{\partial T_F}{\partial q_\alpha} + Q_\alpha. \tag{2.9}$$

Therefore, the fluid inetia force Q_{F_α} can be written as

$$Q_{F_\alpha} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_F}{\partial \dot{q}_\alpha} + \frac{\partial T_F}{\partial q_\alpha}. \tag{2.10}$$

Lord Kelvin (1879) derived (2.10) by using Hamilton's principle and taking the fluid kinetic energy in place of fluid Lagrangian. In the literature, equations (2.10) are known as Kelvin–Kirchhoff hydrodynamical equations (Athanassoulis and Loukakis 1985). If one proceeds one step further and expand T_F in (2.7) one obtains

$$T_F = \frac{1}{2}\rho \int_{S_B} \phi \frac{\partial \phi}{\partial n} \mathrm{d}S + \frac{1}{2}\rho \int_{S_\infty} \phi \frac{\partial \phi}{\partial n} \mathrm{d}S. \tag{2.11}$$

However, it is well known that the integrand of the second integral on the right-hand side of (2.11) vanishes at infinity (Newman 1977, p.135), therefore it follows that

$$T_F = \frac{1}{2}\rho \int_{S_B} \phi \frac{\partial \phi}{\partial n} \mathrm{d}S. \tag{2.12}$$

Now if T_F in (2.12) is denoted by T_{S_B}, equation (2.8) takes the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\,(T_{S_B}+T_B)}{\partial\dot{q}_\alpha}-\frac{\partial\,(T_{S_B}+T_B)}{\partial q_\alpha}=F_\alpha,\qquad \alpha=1,\ldots,6. \tag{2.13}$$

This equation indicates that to derive equations of motion of a submerged body it is not necessary to consider the fluid and the body as one dynamical system. The fluid kinetic energy associated with the surface of a submerged body contains all inertia effects of fluid on the rigid body. In the next section we shall obtain a similar result for a floating body.

2.3 A combined Newtonian-Lagrangian approach to the linear radiation problem

Consider a floating body which is oscillating sinusoidally with small amplitudes in linear incident waves. The floating body is assumed to be rigid with six degrees of freedom. The motion of the rigid body and the fluid particles are referred to an inertial Cartesian coordinate system x_i, with its origin fixed on the undisturbed free surface and its x_3–axis vertically upwards. A second coordinate system x_i' is parallel to x_i and is fixed to the floating body at the center of gravity of the body with its x_3'–axis coincident with the x_3–axis in the reference configuration. We define three translational motions parallel to x_i'–axis by q_i and three rotational motions about the same axes by q_{i+3}. Therefore, sinusoidal oscillations of the body can be denoted by $q_\alpha = |q_\alpha|\,\sin(\omega t + \theta_\alpha)$, where $|q_\alpha|$ is a real small amplitude. Provided that $|q_\alpha|$ is a small first-order quantity, the distinction between the inertial coordinate system and the body fixed coordinate system will be a source of second-order effects that can be neglected (Newman 1977, p.287).

In a linear theory the wave–body interaction problem can be decomposed into a radiation prblem and a diffraction problem. In both problems, the fluid flow is assumed to be incompressible and irrotational and therefore potential.

In the radiation problem, the displacements q_α and velocities $\dot{q}_\alpha$ of the rigid body can be regarded as the generalized coordinates of the fluid which is a nonholonomic mechanical system.

The boundary value problem for the radiation velocity potential ϕ^R can be expressed as follows (Sarpkaya & Isaacson 1981, pp.435–440):

$$\nabla^2\phi^R = 0 \qquad \text{for } -d < z < 0, \tag{2.14a}$$

$$\phi^R_{,tt} + g\,\phi^R_{,z} = 0 \qquad \text{at } z = 0, \tag{2.14b}$$

$$\eta^R = -1/g\,\phi^R_{,t} \qquad \text{at } z = 0, \tag{2.14c}$$

$$\partial\phi^R/\partial n = U_n = \dot{q}_\alpha n_\alpha \qquad \text{on } S_B, \tag{2.14d}$$

$$\phi^R_{,z} = 0 \qquad \text{at } z = -d, \tag{2.14e}$$

$$\text{Radiated waves are outgoing.} \tag{2.14f}$$

A comma in (2.14) means partial differentiation ($\phi^R_{,z} = \partial\phi^R/\partial z$), η^R is the radiated wave elevation, U_n is the velocity of the body surface in the direction normal to itself and n_α is the component of the generalized unit normal vector of S_B. The last boundary condition in (2.14) can be stated by Sommerfield radiation condition in terms of complex potential ϕ^R_c as follows (Mei 1989)

$$\sqrt{kr}\left(\frac{\partial\phi^R_c}{\partial r} - \mathrm{i}k\,\phi^R_c\right) \to 0, \qquad kr \to \infty. \tag{2.15}$$

By using the divergence theorem the fluid kinetic energy, T_F, can be represented over the boundary of the fluid domain, which is composed of the submerged body surface S_B, free surface S_F, seabed $S_{B'}$ and a surface S_∞ at the farfield. In other words, $T_F = T_{S_B} + T_{S_F} + T_{S_{B'}} + T_{S_\infty}$, where for instance

$$T_{S_B} = \frac{1}{2}\rho\int_{S_B}\phi^R\frac{\partial\phi^R}{\partial n}\,\mathrm{d}S. \tag{2.16}$$

$T_{S_{B'}}$ is zero, T_{S_F} do not contribute to the Lagrangian function and T_{S_∞} can be employed to represent the fluid radiation damping (see Miloh 1984). However, we shall presently show that T_{S_B} can be used in place of T_{S_∞} to derive the fluid damping effects on the floating body. It should be emphasized that ϕ^R here is a real quantity. Now we shall prove the following theorem.

Theorem 1 *The kinetic energy of fluid associated with the wetted surface of the body, T_{S_B}, contains all the inertia and damping effects of fluid on the rigid floating body.*

Proof. For complex potential ϕ_c^R and complex amplitude q_α^c it is possible to use the complex form of Kirchhoff's decomposition, $\phi_c^R = q_\alpha^c\, \varphi_\alpha^c$ (see Newman 1977), however, we prefer to work with real potential ϕ^R and consider the following decomposition

$$\phi^R = \varphi_\alpha\, \dot{q}_\alpha + \psi_\alpha\, q_\alpha. \tag{2.17}$$

In (2.17), φ_α and ψ_α are real velocity potentials and have dimensions of m and m/s for translatory motions, respectively, and dimensions of m^2 and m^2/s for rotatory motions, respectively. Substituting (2.17) into (2.14) and using (2.15) yields two coupled boundary value problems for steady state potentials φ_α and ψ_α as follows

$$\nabla^2\varphi_\alpha = 0 \qquad \text{for } -d < z < 0, \tag{2.18a}$$

$$\omega^2\varphi_\alpha - g\,\varphi_{\alpha,_z} = 0, \qquad \text{at } z = 0, \tag{2.18b}$$

$$\zeta_\alpha = 1/\tau\; \varphi_\alpha\big|_{z=0} \qquad \text{at } z = 0, \tag{2.18c}$$

$$\partial\varphi_\alpha/\partial n = n_\alpha \qquad \text{on } S_B, \tag{2.18d}$$

$$\partial\varphi_\alpha/\partial z = 0 \qquad \text{at } z = -d, \tag{2.18e}$$

$$\left(\frac{\partial\varphi_\alpha}{\partial r} - \frac{k}{\omega}\psi_\alpha\right) \to 0, \qquad kr \to \infty. \tag{2.18f}$$

and

$$\nabla^2\psi_\alpha = 0 \qquad \text{for } -d < z < 0, \tag{2.19a}$$

$$\omega^2\psi_\alpha - g\,\psi_{\alpha,_z} = 0, \qquad \text{at } z = 0, \tag{2.19b}$$

$$\xi_\alpha = 1/(\omega\tau)\; \psi_\alpha\big|_{z=0} \qquad \text{at } z = 0, \tag{2.19c}$$

$$\partial\psi_\alpha/\partial n = 0 \qquad \text{on } S_B, \tag{2.19d}$$

$$\partial\psi_\alpha/\partial z = 0 \qquad \text{at } z = -d, \tag{2.19e}$$

$$\left(\frac{\partial\psi_\alpha}{\partial r} + \omega k\,\varphi_\alpha\right) \to 0, \qquad kr \to \infty. \tag{2.19f}$$

where

$$\tau = \begin{cases} 1 & : \quad \alpha = 1,2,3, \\ 1/k & : \quad \alpha = 4,5,6. \end{cases},$$

and the following linear superposition is assumed among the radiated wave elevations η^R, ζ_α and ξ_α

$$\eta^R = -1/g\,\tau\,\zeta_\alpha\,\ddot{q}_\alpha - \omega/g\,\tau\,\xi_\alpha\,\dot{q}_\alpha. \tag{2.20}$$

Two boundary value problems for φ_α and ψ_α in (2.18) and (2.19) are coupled through (2.20) and radiation conditions (2.18f) and (2.19f). Radiation conditions (2.18f) and (2.19f) are derived from complex radiation condition (2.15), where steady state potentials φ_α and ψ_α are taken to be proportional with steady state potentials Re $\left[\phi_c^R\right]$ and Im $\left[\phi_c^R\right]$, respectively. Now substitution of (2.17) into the linear form of the Bernoulli's equation, $p^R = -\rho\,\partial\phi^R/\partial t$, gives the radiation dynamic pressure as

$$p^R = -\rho\,\varphi_\alpha\,\ddot{q}_\alpha - \rho\,\psi_\alpha\,\dot{q}_\alpha\,. \tag{2.21}$$

Using the Newtonian approach, the integration of p^R over the body wetted surface yields

$$Q_\alpha^R = -\rho\int_{S_B}\varphi_\alpha\,n_\beta\,\mathrm{d}S\;\ddot{q}_\beta - \rho\int_{S_B}\psi_\alpha\,n_\beta\,\mathrm{d}S\;\dot{q}_\beta, \tag{2.22}$$

where Q_α^R are generalized radiation forces and, following the common assumption of the linear theory of motion of a floating body, S_B is the mean wetted surface of the body. Using the boundary conditions (2.18d) it follows that

$$Q_\alpha^R = -\left(\rho\int_{S_B}\varphi_\alpha\frac{\partial\varphi_\beta}{\partial n}\,\mathrm{d}S\right)\ddot{q}_\beta - \left(\rho\int_{S_B}\psi_\alpha\frac{\partial\varphi_\beta}{\partial n}\,\mathrm{d}S\right)\dot{q}_\beta. \tag{2.23}$$

Now defining

$$a_{\alpha\beta} = \rho\int_{S_B}\varphi_\alpha\frac{\partial\varphi_\beta}{\partial n}\,\mathrm{d}S\,, \tag{2.24a}$$

$$b_{\alpha\beta} = \rho\int_{S_B}\psi_\alpha\frac{\partial\varphi_\beta}{\partial n}\,\mathrm{d}S\,, \tag{2.24b}$$

yields

$$Q_\alpha^R = -a_{\alpha\beta}\,\ddot{q}_\beta - b_{\alpha\beta}\,\dot{q}_\beta. \tag{2.25}$$

In (2.25), $a_{\alpha\beta}$ which are coefficients of accelerations $\ddot{q}_\beta$ are added mass coefficients of the floating body. Also, following Mei (1989), by calculating the average rate of work done by the force Q_α^R to the fluid over one period it can be shown that $b_{\alpha\beta}$ are damping coefficients. Having obtained the required knowledge from a Newtonian approach we shall now return to the Lagrangian formulation and consider the kinetic energy of the fluid corresponding to the wetted surface of the floating body, T_{S_B}, given in (2.16). Using the linear decomposition of (2.17) the integrand in (2.16) can be written as

$$\phi^R\frac{\partial\phi^R}{\partial n} = (\dot{q}_\alpha\,\varphi_\alpha + q_\alpha\,\psi_\alpha)\left(\dot{q}_\beta\,\frac{\partial\varphi_\beta}{\partial n} + q_\beta\,\frac{\partial\psi_\beta}{\partial n}\right). \tag{2.26}$$

However, from boundary condition (2.19d) we have $\partial\psi_\beta/\partial n = 0$, therefore,

$$\phi^R\frac{\partial\phi^R}{\partial n} = \dot{q}_\alpha\,\varphi_\alpha\frac{\partial\varphi_\beta}{\partial n}\,\dot{q}_\beta + q_\alpha\,\psi_\alpha\frac{\partial\varphi_\beta}{\partial n}\,\dot{q}_\beta. \tag{2.27}$$

Introducing (2.27) into (2.16) yields

$$T_{S_B} = \frac{1}{2}\,\dot{q}_\alpha\left(\rho\int_{S_B}\varphi_\alpha\frac{\partial\varphi_\beta}{\partial n}\,\mathrm{d}S\right)\dot{q}_\beta + \frac{1}{2}\,q_\alpha\left(\rho\int_{S_B}\psi_\alpha\frac{\partial\varphi_\beta}{\partial n}\,\mathrm{d}S\right)\dot{q}_\beta. \tag{2.28}$$

Now from the Newtonian treatment of the problem, equation (2.24), we know that the first and second brackets on the right-hand side of (2.28) are added mass and damping coefficients, respectively. Thus

$$T_{S_B} = \frac{1}{2}\dot{q}_\alpha\, a_{\alpha\beta}\,\dot{q}_\beta + \frac{1}{2}q_\alpha b_{\alpha\beta}\,\dot{q}_\beta. \tag{2.29}$$

This equation shows that the kinetic energy of fluid associated with the wetted surface of the body contains all the linear inertia and damping effects of the fluid on the floating body. Therefore, the proof is completed.

It should be noted that it is not extraordinary to consider the second term on the right-hand side of (2.29) as a kinetic energy. Equation (2.29) is a particular case of the general form of the kinetic energy when it is written in terms of generalized coordinates (e.g., see Goldstein et al. 2002 p.25, Rosenberg 1977 p.202, Greenwood 1977 p.49).

Now we shall consider the fluid potential energy corresponding to the wetted surface of the floating body, i.e., V_{S_B}. This potential energy can be decomposed into two parts. The first part corresponds to the mean wetted surface of the body and is associated with the generalized static buoyancy forces. The second part is related to the change in the wetted surface of the body due to its motions and can be written as a positive definite quadratic form in terms of hydrostatic restoring coefficients $c_{\alpha\beta}$ (Miloh 1984). Taking the static equilibrium position of the floating body as the reference point of the displacements q_α, the generalized body weight forces and the generalized zeroth-order buoyancy forces can be disregarded and V_{S_B} can be written as

$$V_{S_B} = \frac{1}{2}q_\alpha\, c_{\alpha\beta}\, q_\beta. \tag{2.30}$$

Equation (2.30) states that V_{S_B} contains all the linear restoring effects of fluid on the floating body. The result that the kinetic and potential energy may be expressed only in terms of integrals over the submerged portion of the body to represent the fluid inertia and restoring effects on a floating body has been known in the classical marine hydrodynamics (e.g., see Bessho 1970, Miloh 1984). However, the result that the fluid damping effects can also be expressed by T_{S_B} is new. In addition, the formulation of a damping phenomenon by an energy rather than a dissipation function is believed to be novel in the analytical mechanics. It is worth to mention that (2.25), (2.29) and (2.30) can be used to derive tensor properties of the radiation coefficients (see Chapter 2 or Sadeghi & Incecik 2005*b*). Now from (2.29) and (2.30) we can deduce that the kinetic and potential energy of the fluid associated with the submerged surface of the body, contain all the inertia, damping and restoring effects of the fluid on the rigid body. An immediate consequence of this result is:

Corollary 1 *The fluid Lagrangian function of the linear radiation problem can be defined as follows*

$$L_F = L_{S_B} = T_{S_B} - V_{S_B} = \frac{1}{2}\,\dot{q}_\alpha\, a_{\alpha\beta}\,\dot{q}_\beta + \frac{1}{2}\,q_\alpha b_{\alpha\beta}\,\dot{q}_\beta - \frac{1}{2}\,q_\alpha\, c_{\alpha\beta}\,q_\beta. \qquad (2.31)$$

Equations (2.29) and (2.30) represent all radiation effects of the fluid on the floating body. The only remaining effect of fluid is the non-conservative excitation forces due to dynamic diffracrion pressure p^D acting on S_B,

$$Q_\alpha^D = \int_{S_B} p^D\, n_\alpha\, \mathrm{d}S. \qquad (2.32)$$

In (2.32), $p^D = -\rho\,\partial\phi^D/\partial t$ and ϕ^D is the diffraction potential. In summary, in the Lagrangian formulation, like the newtonian formulation, all fluid actions on the floating body can be stated by dynamic quantities associated with the wetted surface of the floating body. It means that, as far as equations of motion of a floating body are concerned, it is not necessary to consider the fluid kinetic energy associated with the free surface, seabed or an enclosing surface at infinity. Therefore for a floating body, similar to an immersed body, it is not necessary to consider the rigid body and the whole fluid as one dynamical system.

2.4 Lagrange's equations of motion for a floating body

We shall first start by a Newtonian approach. If the momentum principle of Newton and the linear form of angular momentum principle of Euler are combined, the linear Newton–Euler equations of motion,

$$\frac{\mathrm{d}P_\alpha}{\mathrm{d}t} = Q_\alpha, \tag{2.33}$$

can be used to derive equations of motion of a floating body. In (2.33) P_i are components of linear momentum vector, P_{i+3} are components of angular momentum vector and Q_α are generalized external forces. Because in Newtonian mechanics the effect of every conservative force or inertia force can be obtained when it is assumed as a non-conservative force, it is not required to distinguish between inertia, potential and non-conservative forces of fluid. All fluid forces can be assumed as non-conservative forces which through total fluid pressure p acts on the wetted surface of the body. Therefore equation (2.33) can be rewritten as follows

$$\frac{\mathrm{d}P_{B\alpha}}{\mathrm{d}t} = Q_{F\alpha} + Q_{B\alpha}, \tag{2.34}$$

where

$$Q_{F\alpha} = Q_\alpha^R + Q_\alpha^D + Q_\alpha^S$$

in which Q_α^R and Q_α^D are generalized radiation and diffraction fluid forces given by (2.25) and (2.32), respectively, and Q_α^S is the generalized hyrostatic force acting on the surface of the body. In the context of a linear theory, the hydrostatic pressure acting on the wetted surface of the body gives rise to the generalized zeroth-order static buoyancy forces and the linear first-order restoring forces, the latter can be written as $c_{\alpha\beta}\, q_\beta$ in which $c_{\alpha\beta}$ are hydrostatic restoring coefficients given in (2.30). Also from rigid body dynamics we have

$$\frac{\mathrm{d}P_{B\alpha}}{\mathrm{d}t} = M_{\alpha\beta}\, \ddot{q}_\beta, \tag{2.35}$$

where $M_{\alpha\beta}$ are generalized mass coefficients of the rigid floating body. Substitution of body inertia forces of (2.35), the fluid radiation forces of (2.25) and the fluid restoring forces into linear Newton–Euler equations of motion (2.34) and noting that the generalized zeroth–order static buoyancy forces cancel the generalized body weight forces Q_{B_α}, the linear equations of motion of a floating body becomes

$$(M_{\alpha\beta} + a_{\alpha\beta})\, \ddot{q}_\beta + b_{\alpha\beta}\, \dot{q}_\beta + c_{\alpha\beta}\, q_\beta = Q^D_\alpha. \tag{2.36}$$

where Q^D_α are generalized diffraction forces. Now we shall turn back to the Lagrangian approach. By using the fluid Lagrangian as defined in (2.31) and the rigid body Lagrangian as $L_B = T_B - V_B$, the Lagrange's equations of motion for a floating body can be written as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\,(L_{S_B} + L_B)}{\partial \dot{q}_\alpha} - \frac{\partial\,(L_{S_B} + L_B)}{\partial q_\alpha} = Q^D_\alpha, \qquad \alpha = 1, \ldots, 6, \tag{2.37}$$

where the kinetic energy of the rigid body is defined as $T_B = \frac{1}{2}\, \dot{q}_\alpha\, M_{\alpha\beta}\, \dot{q}_\beta$. Now taking the equilibrium position of the floating body as the reference point of the generalized coordinates q_α and substituting (2.31) into (2.37) yield the equations of motion of the floating body as follows

$$(M_{\alpha\beta} + a_{\alpha\beta})\, \ddot{q}_\beta + c_{\alpha\beta}\, q_\beta = Q^D_\alpha. \tag{2.38}$$

Unlike (2.36), obtained from a Newtonian approach, (2.38) does not have a damping term. The kinetic energy T_{S_B} defined in (2.29) provides the required information about the damping but (2.37) does not generate a damping force from T_{S_B}. To see why this happens we shall consider the kinetic energy in (2.29) more carefully,

$$T_{S_B} = \frac{1}{2}\dot{q}_\alpha\, a_{\alpha\beta}\,\dot{q}_\beta + \frac{1}{2}q_\alpha b_{\alpha\beta}\,\dot{q}_\beta. \tag{2.39}$$

The first term on the right-hand side of this equation is a symmetric positive-definite quadractic form and the second term is a bilinear form. Hence, one may write

$$T_{S_B} = T^Q_{S_B} + T^B_{S_B} \tag{2.40}$$

in which

$$T^Q_{S_B} = \frac{1}{2}\dot{q}_\alpha\, a_{\alpha\beta}\,\dot{q}_\beta, \tag{2.41a}$$

$$T^B_{S_B} = \frac{1}{2}q_\alpha b_{\alpha\beta}\,\dot{q}_\beta. \tag{2.41b}$$

Now substituting $T^B_{S_B}$ in the operators of the Lagrange's equations of motion gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^B_{S_B}}{\partial \dot{q}_\alpha} = \frac{1}{2}b_{\alpha\beta}\,\dot{q}_\beta, \tag{2.42a}$$

$$\frac{\partial T^B_{S_B}}{\partial q_\alpha} = \frac{1}{2}b_{\alpha\beta}\,\dot{q}_\beta. \tag{2.42b}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^B_{S_B}}{\partial \dot{q}_\alpha} - \frac{\partial T^B_{S_B}}{\partial q_\alpha} = 0. \tag{2.43}$$

In other words the damping force produced by the first term of (2.37) cancels the damping force produced by the second term of these equations. As a result (2.37) does not generate any radiation damping force.

It must be mentioned that, this should not be counted as a shortcoming of the Lagrange's equation of motion as Lagrange (1788) derived his equation based on the assumption that non-conservative forces are not present in the system. Therefore, it is not surprising that Lagrange's equation does not predict a radiation damping force.

Before dealing with this problem more fundamentally, we shall follow the usual approach in analytical mechanics in dealing with a damping force and add the damping force to the left-hand side of equations of motion (2.37) by means of Rayliegh's dissipation function, which in the problem in hand can be defined based on $T^B_{S_B}$ in (2.41b) as follows

$$R_{S_B} = \frac{1}{2}\,\dot{q}_\alpha\, b_{\alpha\beta}\,\dot{q}_\beta. \tag{2.44}$$

Therefore equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\,(L_{S_B}+L_B)}{\partial\dot{q}_\alpha} - \frac{\partial\,(L_{S_B}+L_B)}{\partial q_\alpha} + \frac{\partial R_{S_B}}{\partial\dot{q}_\alpha} = Q^D_\alpha, \qquad \alpha = 1,\ldots,6, \tag{2.45}$$

obtained from (2.37) will give equations of motion the same as (2.36). Following the common convention of analytical mechanics in naming equations, (2.45) can be called Lagrange's equatios of motion for a floating body.

2.5 A variant of Lagrange's equations of motion

The fact that the kinetic energy defined in (2.29) contains the complete dynamics of the damping together with the fact that the force–momentum formulation of a problem must be consistent with its energy formulation and both formulations must result in the same differential equations of motion, is sufficient to believe that there must be a variant of Lagrange's equations of motion which without use of a dissipation function generates the linear damping force from the kinetic energy and delivers the differential equations of motion (2.36). Now turning back to (2.45), we argue that the bilinear kinetic energy $T^B_{S_B}$ contains the required information about the linear damping force and is a natural part of the problem and therefore adding a new scalar function, R_{S_B}, to the problem is unnecessary. Hence, we prefer to use $T^B_{S_B}$ rather than R_{S_B}. In addition, using $T^B_{S_B}$ over R_{S_B} has two advantages. First, unlike R_{S_B} which has dimensions of power, $T^B_{S_B}$ has the dimension of energy and in this regard is consistent with $T^Q_{S_B}$ and V_{S_B}. Second, adding the force $\partial R_{S_B}/\partial \dot{q}_\alpha$ to the Lagrange's equations of motion is a feature of Newtonian mechanics used in the Lagrangian mechanics while by using $T^B_{S_B}$, as (2.42) shows, the damping force can be expressed in a Lagrangian way through either of the operators of the Lagrange's equations of motion. Therefore, rather than (2.45) we shall consider the following equations as the linear equations of motion of a floating body

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\left(L_{S_B}+L_B\right)}{\partial \dot{q}_\alpha}-\frac{\partial\left(L_{S_B}+L_B\right)}{\partial q_\alpha}+2\frac{\partial T^B_{S_B}}{\partial q_\alpha}=Q^D_\alpha, \tag{2.46}$$

If one excludes $T^B_{S_B}$ and defines a new quadratic Lagrangian as

$$L^Q_{S_B}=T^Q_{S_B}-V_{S_B}, \tag{2.47}$$

then because according to (2.43) $T^B_{S_B}$ has no effect, (2.46) can be written as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\left(L^Q_{S_B}+L_B\right)}{\partial \dot{q}_\alpha}-\frac{\partial\left(L^Q_{S_B}+L_B\right)}{\partial q_\alpha}+2\frac{\partial T^B_{S_B}}{\partial q_\alpha}=Q^D_\alpha. \tag{2.48}$$

At this point, in order to put our problem in a broader context we consider a general dynamical system with constitutive relations equivalent with those of the linear radiation problem of a floating body and study that equivalent system. The results of our study are then applicable to the linear radiation problem of a floating body as well as any other dynamical system with the same constitutive laws. As our equivalent system we define a dynamical system with kinetic energy T and potential energy V where constitutive relations governing T and V are as follows

$$T=T^Q+T^B, \tag{2.49a}$$

$$T^Q=T^Q\left(\dot{q}_\alpha\right)=\frac{1}{2}\,\dot{q}_\alpha\,a_{\alpha\beta}\,\dot{q}_\beta, \tag{2.49b}$$

$$T^B = T^B(\dot{q}_\alpha, q_\alpha) = \frac{1}{2} q_\alpha b_{\alpha\beta} \dot{q}_\beta, \tag{2.49c}$$

$$V = V(q_\alpha) = \frac{1}{2} q_\alpha c_{\alpha\beta} q_\beta. \tag{2.49d}$$

in which $a_{\alpha\beta}$, $b_{\alpha\beta}$ and $c_{\alpha\beta}$ are symmetric steady state inertia, damping and stiffness coefficients of the system, T^Q and V are positive definite quadratic forms and Greek subscripts range 1 to the number of generalized coordinates of the system. An important consequence of (2.49c) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^B}{\partial \dot{q}_\gamma} = \frac{\partial T^B}{\partial q_\gamma}. \tag{2.50}$$

Moreover, from constitutive relations (2.49b) and (2.49d) it follows that

$$\frac{\partial T^Q}{\partial q_\gamma} = 0, \tag{2.51a}$$

$$\frac{\partial V}{\partial \dot{q}_\gamma} = 0. \tag{2.51b}$$

By analogy with (2.48), equations of motion of the system can be expressed as,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^Q}{\partial \dot{q}_\gamma} - \frac{\partial L^Q}{\partial q_\gamma} + 2\frac{\partial T^B}{\partial q_\gamma} = Q_\gamma. \tag{2.52}$$

Now by expanding L^Q and then using (2.51), equation (2.52) takes the simplified form of

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^Q}{\partial \dot{q}_\gamma} + \frac{\partial V}{\partial q_\gamma} + 2\frac{\partial T^B}{\partial q_\gamma} = Q_\gamma. \tag{2.53}$$

For a dynamical system governed by (2.49) a work–energy relation can be obtained from (2.53) by multiplying both sides of this equation by $\dot{q}_\gamma$. Details of the derivation are given in the Appendix A and the result is

$$\frac{\mathrm{d}E^Q}{\mathrm{d}t} + 2R = \frac{\mathrm{d}W^{nc}}{\mathrm{d}t}, \tag{2.54}$$

where $E^Q = T^Q + V$ is the quadratic total mechanical energy, $R = \dot{q}_\gamma\, \partial T^B / \partial q_\gamma$ is the Rayleigh's dissipation function and W^{nc} is the work done by non-conservative forces Q_γ. When the damping is zero, T^B and R will vanish and $T = T^Q$ and $E = E^Q$ and we obtain the familiar work–energy relation,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}W^{nc}}{\mathrm{d}t}.$$

By using (2.50) and (2.51) the equations of motion (2.53) can be recast in a more compact and meaningful form. In order to find that form, we shall rewrite (2.53) as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_\gamma}\left(T^Q + 0\right) + 0 + \frac{\partial V}{\partial q_\gamma} + \frac{\partial T^B}{\partial q_\gamma} + \frac{\partial T^B}{\partial q_\gamma} = Q_\gamma. \tag{2.55}$$

Now substituting for the first and second zeros from (2.51b) and (2.51a), respectively, and using (2.50) for the last term on the left–hand side of (2.55), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_\gamma}\left(T^Q + V\right) + \frac{\partial}{\partial q_\gamma}\left(V + T^B + T^Q\right) + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^B}{\partial \dot{q}_\gamma} = Q_\gamma \tag{2.56}$$

but $T^Q + T^B + V = T + V = E$ is the total mechanical energy of the system, therefore, the following equations of motion will be obtained from (2.56),

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E}{\partial \dot{q}_\gamma} + \frac{\partial E}{\partial q_\gamma} = Q_\gamma, \tag{2.57}$$

Equations (2.57) are our equations of motion for a mechanical system governed by constitutive relations (2.49). For the same system, in the absence of a dissipation function, the Lagrange's equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\gamma} - \frac{\partial L}{\partial q_\gamma} = Q_\gamma. \tag{2.58}$$

One may call (2.57) as conjugate Lagrange's equations of motion. As it is evident (2.57) is expressed only in terms of T and V without using any dissipation function and is naturally generating a linear damping force. As Lanczos (1970) rightly asserted, T and V indeed contain the complete dynamics of a problem when used in the context of a suitable principle and in the case of a dynamical system described by constitutive relations (2.49) this principle may be stated by (2.57).

Now consider a damping free kinetic energy. In this case $T^B = 0$ and constitutive relations (2.49) reduce to the common constitutive relations of

$$T = T^Q = T^Q(\dot{q}_\alpha) = \frac{1}{2}\dot{q}_\alpha\, a_{\alpha\beta}\, \dot{q}_\beta, \tag{2.59a}$$

$$V = V(q_\alpha) = \frac{1}{2} q_\alpha c_{\alpha\beta}\, q_\beta. \tag{2.59b}$$

Because for (2.59), relations (2.51) are still valid, one can show that in this case

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E}{\partial \dot{q}_\gamma} + \frac{\partial E}{\partial q_\gamma} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\gamma} - \frac{\partial L}{\partial q_\gamma}. \tag{2.60}$$

This means that for a dynamical system with a damping-free kinetic energy conjugate Lagrange's equations of motion (2.57) can be used in place of Lagrange's equations of motion (2.58). Now turning back to the linear radiation problem of a floating body, the constitutive equations of the rigid body alone regardless of fluid effects are the same as (2.59), therefore from (2.57) and (2.60), corresponding equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_B}{\partial \dot{q}_\gamma} + \frac{\partial E_B}{\partial q_\gamma} = Q^R_\gamma + Q^D_\gamma, \tag{2.61}$$

where Q^R_γ and Q^D_γ are fluid radiation and diffraction forces and $E_B = T_B + V_B$. Furthermore, since the fluid kinetic and potential energy associated with the body wetted surface given by (2.29) and (2.30) satisfy the constitutive equations (2.49) the fluid radiation force can be written from (2.57) as follows

$$Q^R_\gamma = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_{S_B}}{\partial \dot{q}_\gamma} - \frac{\partial E_{S_B}}{\partial q_\gamma}, \tag{2.62}$$

where

$$E_{S_B} = T_{S_B} + V_{S_B} = \frac{1}{2}\dot{q}_\alpha\, a_{\alpha\beta}\,\dot{q}_\beta + \frac{1}{2}q_\alpha b_{\alpha\beta}\,\dot{q}_\beta + \frac{1}{2}q_\alpha\, c_{\alpha\beta}\, q_\beta \tag{2.63}$$

is the total mechanical energy of the fluid associated with the wetted surface of the floating body. It is easy to show that the Kelvin-Kirchhoff hyrodynamical equations (2.10) are a special form of the new energy–force equations (2.62). Now introducing for radiation forces from (2.62) into (2.61) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_\gamma}(E_{S_B} + E_B) + \frac{\partial}{\partial q_\gamma}(E_{S_B} + E_B) = Q_\gamma^D, \tag{2.64}$$

or in an operator form,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_\gamma} + \frac{\partial}{\partial q_\gamma}\right)(E_{S_B} + E_B) = Q_\gamma^D. \tag{2.65}$$

Substituting (2.63) into (2.65) and using $T_B = \frac{1}{2}\dot{q}_\alpha\, M_{\alpha\beta}\,\dot{q}_\beta$ yields

$$(M_{\alpha\beta} + a_{\alpha\beta})\,\ddot{q}_\beta + b_{\alpha\beta}\,\dot{q}_\beta + c_{\alpha\beta}\,q_\beta = Q_\alpha^D \tag{2.66}$$

which is consistent with the Newtonian result. Equation (2.65) is the conjugate Lagrange's equation of motion of a floating body which with one less scalar function with respect to the Lagrange's equations of motion (2.45) delivers the same differential equations of motion.

2.6 A variant of Hamilton's principle

Now we shall return to our mechanical system governed by (2.49) and write (2.57) and (2.58) in an operator form,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_\gamma} + \frac{\partial}{\partial q_\gamma}\right)E = Q_\gamma, \tag{2.67}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_\gamma} - \frac{\partial}{\partial q_\gamma}\right) L = Q_\gamma. \tag{2.68}$$

In (2.67) both the operator and the energy functional E are symmetric while in (2.68) both the operator and the energy functional L are antisymmetric. Now consider Hamilton's principle

$$\delta \int_{t_1}^{t_2} L\,\mathrm{d}t = \int_{t_1}^{t_2} \delta L\ \mathrm{d}t = 0 \tag{2.69}$$

in which δ is the first variation operator and $L = L\,(q_\alpha,\, \dot{q}_\alpha)$ is the Lagrangian. If one proceeds step by step and derives the left-hand side of (2.68) from (2.69), one finds that the antisymmetry of the operator in (2.68) is a result of the symmetry of the first variation operator δ. To show that δ is symmetric, consider an arbitrary functional $F(q, \dot{q})$, chosen such to be consistent with $L\,(q_\alpha,\, \dot{q}_\alpha)$, then we have

$$\delta F = \frac{\partial F}{\partial q}\,\delta q + \frac{\partial F}{\partial \dot{q}}\,\delta \dot{q} = \left(\delta q\,\frac{\partial}{\partial q} + \delta \dot{q}\frac{\partial}{\partial \dot{q}}\right) F. \tag{2.70}$$

Introducing partial variations $\delta_q F$ and $\delta_{\dot{q}} F$, where δ_q and $\delta_{\dot{q}}$ are partial variation operators defined by $\delta_q = \frac{\partial}{\partial q}\,\delta q$ and $\delta_{\dot{q}} = \frac{\partial}{\partial \dot{q}}\,\delta \dot{q}$, resprectively, and denoting δ with $\delta_{q,\dot{q}}$ it follows that

$$\delta = \delta_{q,\dot{q}} = \delta_q + \delta_{\dot{q}} = \delta_{\dot{q}} + \delta_q = \delta_{\dot{q},q}\,. \tag{2.71}$$

which indicates that the operator δ is a symmetric operator. As was mentioned, the symmetry of δ is the reason of the antisymmetry of the Lagrange's operator in (2.68). This encourages us to introduce an antisymmetric first variation operator $\bar{\delta}$ as follows

$$\bar{\delta} = \bar{\delta}_{q,\dot{q}} = \delta_q - \delta_{\dot{q}} = \frac{\partial}{\partial q}\,\delta q - \frac{\partial}{\partial \dot{q}}\,\delta \dot{q}, \tag{2.72}$$

We shall call $\bar{\delta}$ the *conjugate* first variation operator. Now rather than Hamilton's principle consider the following variational equation

$$\bar{\delta}\int_{t_1}^{t_2} E\,\mathrm{d}t = \int_{t_1}^{t_2}\bar{\delta}E\,\mathrm{d}t = 0. \tag{2.73}$$

Using (2.72) it follows from (2.73) that

$$\int_{t_1}^{t_2}\frac{\partial E}{\partial q}\,\delta q\,\mathrm{d}t - \int_{t_1}^{t_2}\frac{\partial E}{\partial \dot{q}}\frac{\mathrm{d}}{\mathrm{d}t}(\delta q)\,\mathrm{d}t = 0. \tag{2.74}$$

Using integration by parts technique for the second integral on the left-hand side of (2.74) yields

$$\int_{t_1}^{t_2}\left(\frac{\partial E}{\partial q} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E}{\partial \dot{q}}\right)\delta q\,\mathrm{d}t - \left(\frac{\partial E}{\partial \dot{q}}\delta q\right)\Bigg|_{t_1}^{t_2} = 0. \tag{2.75}$$

For admissible variation δq, which vanishes at t_1 and t_2, (2.75) yields the essential and natural boundary conditions of the problem and the following equations of motion

$$\frac{\partial E}{\partial q} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E}{\partial \dot{q}} = 0. \tag{2.76}$$

Therefore, by transfering the antisymmetry from the integrand of the Hamilton's principle to its operator we have obtained (2.73) which in the limits of the considered mechanical system is more powerful than (2.69) in the sense that it generates the damping force naturally. Note that the variational equation stated by (2.73) is not an extremum principle unless $T^B = 0$.

So far it is assumed that non-conservative forces are not present in the system. For a dynamical system with q_α as the generalized displacements and Q_α as generalized non-conservative forces, the inner product of q_α and Q_α can be defined as *generalized work*, $\mathcal{W} = Q_\alpha q_\alpha$. $\mathcal{W}$ is used to denote the generalized work to distinguish it from W^{nc} which is the work done by non-conservative forces Q_α from time t_1 to t_2. The virtual work done by Q_α through virtual displacements δq_α is then

$$\delta\mathcal{W} = Q_\alpha \delta q_\alpha. \tag{2.77}$$

On the other hand, definitions (2.71) and (2.72) imply that

$$\bar{\delta} q = \delta q, \tag{2.78a}$$

$$\bar{\delta} \dot{q} = -\delta \dot{q}. \tag{2.78b}$$

Therefore, from (2.77) and (2.78a) it can be deduced that

$$\delta\mathcal{W} = \bar{\delta}\mathcal{W}. \tag{2.79}$$

Finally, our extended variational equation, which is applicable to a dynamical system governed by (2.49) and subjected to non-conservative forces, can be written as

$$\bar{\delta}\int_{t_1}^{t_2} (E - \mathcal{W})\,\mathrm{d}t = 0, \tag{2.80}$$

which corresponds to the extended Hamilton's principle

$$\delta\int_{t_1}^{t_2} (L + \mathcal{W})\,\mathrm{d}t = 0. \tag{2.81}$$

Equation (2.80) may be called conjugate Hamilton's equation. These equations generate the conjugate Lagrange's equations of motion (2.67). In (2.80) contrary to the extended Hamilton's principle given by (2.81), both the operator and the integrand are antisymmetric.

Chapter 3

Tensor Properties of Added-mass and Damping Coefficients

3.1 Introduction

In marine hydrodynamics like other branches of continuum mechanics it is customary to use index notation and summation convention when writing equations in a compact form but since a marine vehicle is usually assumed as a rigid body and a rigid body in a three-dimensional space generally has six degrees of freedom, the range of indices in marine hydrodynamics is assumed to be 1 to 6 rather than the usual range of 1 to 3. This range convention helps to write equations in a very compact form but sometimes the resulted compactness hides some valuable information. One of those important information which is hidden and ignored due to the traditional range convention of marine hydrodynamics is the tensor character of added mass and damping coefficients of immersed and floating bodies. If $m_{\alpha\beta}$ denotes added mass coefficients of an immersed body where α and β as usual range 1 to 6, it is shown in § 3.2 that $m_{\alpha\beta}$ contain three distinct Cartesian second-order tensors in three-dimensional space.

In the study of tensor properties of suspension particles, Happel and Brenner (1965) obtained similar tensors. Their study is limited to the case of a rigid particle immersed in an unbounded fluid, whose results can be used for an immersed marine structure. Here the theory is extended to the case of a floating body. As a result of this extention, powerful tools of the tensor analysis which have been used in other branches of mechanics since long time ago can now be applied in offshore engineering. An application of this method in the response analysis of a truss spar platform is shown in chapter 5.

Throughout this chapter by an immersed body we refer to a body moving either in an unbounded fluid or in deep water and far from the free surface, seabed and all other boundaries. By a floating body we refer to a body oscillating near or on the free surface of the fluid. Greek indices range 1 to 6, Latin indices range 1 to 3 and summation convention is implied on repeated indices. In addition, the word "tensor" is used to refer to tensors, pseudo-tensors and some quantities which obey the transformation law of a tensor, and unless it is explicitly specified by a "tensor" is meant a tensor in that broad inexact sense. The mathematical background of the stated material can be found in Borisenko & Taparov (1968), Reddy & Rasmussen (1982), Malvern (1969) and Arfken & Weber (2001).

3.2 Second-order tensors of radiation problem

3.2.1 Motion in unbounded fluid

We shall first consider the added mass coefficients of an immersed body and begin with (2.12), i.e.,

$$T_{S_B} = \frac{1}{2}\rho \int_{S_B} \phi \frac{\partial \phi}{\partial n} \mathrm{d}S. \tag{3.1}$$

For a body which moves with generalized velocity $\dot{q}_\alpha$ if Kirchhoff decomposition is used, fluid velocity potential ϕ can be expressed as the following linear superposition

$$\phi = \varphi_\alpha \dot{q}_\alpha, \tag{3.2}$$

Using (3.2) in (3.1) and noting that velocity potentials φ_α are steady state quantities, it follows that

$$T_{S_B} = \frac{1}{2}\rho\,\dot{q}_\alpha \left(\int_{S_B} \varphi_\alpha \frac{\partial \varphi_\beta}{\partial n}\,\mathrm{d}S \right) \dot{q}_\beta = \frac{1}{2}\dot{q}_\alpha\, m_{\alpha\beta}\,\dot{q}_\beta\,, \tag{3.3}$$

where coefficients $m_{\alpha\beta}$ defined in the equation above are steady state added mass coefficients of the immersed body. From (3.3) the following equation can be introduced as an alternative definition for added mass coefficients

$$m_{\alpha\beta} = \frac{\partial^2 T_{S_B}}{\partial \dot{q}_\alpha\,\partial \dot{q}_\beta}\,. \tag{3.4}$$

It is of interest to note that because in (3.3) α and β are dummy indices, it follows that $m_{\alpha\beta}$ are symmetric coefficients, that is, $m_{\alpha\beta} = m_{\beta\alpha}$. Now, replacing Greek indices with Latin indices in (3.3) and expanding the right-hand side of this equation, it follows that

$$T = \frac{1}{2}(\dot{q}_i\, m_{ij}\, \dot{q}_j + \dot{q}_{i+3}\, m_{i+3,j}\, \dot{q}_j + \dot{q}_i\, m_{i,j+3}\, \dot{q}_{j+3} + \dot{q}_{i+3}\, m_{i+3,j+3}\, \dot{q}_{j+3}) \tag{3.5}$$

If one defines,

$$m_{i+3,j+3} = I_{ij}, \qquad m_{i,j+3} = J_{ij}, \tag{3.6}$$

because added mass coefficients $m_{\alpha\beta}$ are symmetric, it follows from the second equation in (3.6) that

$$m_{i+3,j} = J_{ji}. \tag{3.7}$$

Therefore using (3.6) and (3.7), and denoting $\dot{q}_i$ and $\dot{q}_{i+3}$, respectively, with U_i and Ω_i, equation (3.5) takes the following form

$$T = \frac{1}{2} U_i m_{ij} U_j + \frac{1}{2} \Omega_i J_{ji} U_j + \frac{1}{2} U_i J_{ij} \Omega_j + \frac{1}{2} \Omega_i I_{ij} \Omega_j, \tag{3.8}$$

or in matrix form

$$T = \frac{1}{2} \langle U \rangle [m] \{U\} + \frac{1}{2} \langle \Omega \rangle [J]^T \{U\} + \frac{1}{2} \langle U \rangle [J] \{\Omega\} + \langle \Omega \rangle [I] \{\Omega\}, \tag{3.9}$$

or equivalently

$$T = \left\langle \begin{array}{cc} \langle U \rangle & \langle \Omega \rangle \end{array} \right\rangle \begin{bmatrix} [m] & [J] \\ [J]^T & [I] \end{bmatrix}, \left\{ \begin{array}{c} \{U\} \\ \{\Omega\} \end{array} \right\} \tag{3.10}$$

where the partitioned square matrix in (3.10) is equivalent to $m_{\alpha\beta}$ in (3.1) and $\langle \cdot \rangle$ and $\{ \cdot \}$ are used to denote a row and a column vector, respectively. Now since the kinetic energy T on the left–hand side of (3.8) is a zeroth–order tensor (a scalar) and U_i and Ω_i in each term on the right–hand side of (3.8) are components of two first–order tensors (two vectors), it follows from the quotient rule in tensor algebra that m_{ij}, J_{ij} and I_{ij} must be components of three distinct Cartesian second–order tensors. One can call m_{ij}, J_{ij} and I_{ij} as the components of the added mass, added-product of inertia and added-moment of inertia tensors, respectively. Alternatively, we prefer to call them the components of the zeroth–moment, first–moment and second–moment–added mass tensors, respectively. For each tensor, we shall use both names interchangeably.

3.2.2 Effect of a free surface

In order to obtain second-order tensors related to the linear radiation problem of a floating body, we shall consider (2.39) and decompose it as we did in (2.41), that is,

$$T^Q = \frac{1}{2}\,\dot{q}_\alpha\, A_{\alpha\beta}\, \dot{q}_\beta, \qquad T^B = \frac{1}{2}\, q_\alpha\, B_{\alpha\beta}\, \dot{q}_\beta. \tag{3.11}$$

where $A_{\alpha\beta}$ and $B_{\alpha\beta}$ are used in place of $a_{\alpha\beta}$ and $b_{\alpha\beta}$, respectively. Because in (3.11) α and β are dummy indices, it follows that added mass coefficients of a floating body are symmetric. Similarly, from (2.44) it follows that damping coefficients $B_{\alpha\beta}$ are symmetric. Expanding (3.11) in a fashion similar to the one used in (3.5) results in the following equations

$$T^Q = \frac{1}{2}(\dot{q}_i\, A_{ij}\, \dot{q}_j + \dot{q}_{i+3}\, A_{i+3,j}\, \dot{q}_j + \dot{q}_i\, A_{i,j+3}\, \dot{q}_{j+3} + \dot{q}_{i+3}\, A_{i+3,j+3}\, \dot{q}_{j+3}) \tag{3.12a}$$
$$T^B = \frac{1}{2}(q_i\, B_{ij}\, \dot{q}_j + q_{i+3}\, B_{i+3,j}\, \dot{q}_j + q_i\, B_{i,j+3}\, \dot{q}_{j+3} + q_{i+3}\, B_{i+3,j+3}\, \dot{q}_{j+3}) \tag{3.12b}$$

Using U_i, Ω_i and θ_i, respectively, to denote $\dot{q}_i$, $\dot{q}_{i+3}$ and q_{i+3} together with the following definitions

$$A_{i,j+3} = S_{ij}, \tag{3.13a}$$
$$A_{i+3,j+3} = X_{ij}, \tag{3.13b}$$
$$B_{i,j+3} = D_{ij}, \tag{3.13c}$$
$$B_{i+3,j+3} = E_{ij}. \tag{3.13d}$$

and also taking into account the consequence of the symmetry of $A_{\alpha\beta}$ and $B_{\alpha\beta}$, that is

$$\begin{aligned} A_{i+3,j} &= S_{ji}, \\ B_{i+3,j} &= D_{ji}, \end{aligned} \tag{3.14}$$

results in the following two equations:

$$T^Q = \frac{1}{2}\, U_i A_{ij}\, U_j + \frac{1}{2}\,\Omega_i S_{ji}\, U_j + \frac{1}{2}\, U_i S_{ij}\, \Omega_j + \frac{1}{2}\,\Omega_i\, X_{ij}\, \Omega_j, \tag{3.15a}$$

$$T^B = \frac{1}{2}\, q_i\, B_{ij}\, U_j + \frac{1}{2}\, \theta_i\, D_{ji}\, U_j + \frac{1}{2}\, q_i\, D_{ij}\, \Omega_j + \frac{1}{2}\, \theta_i\, E_{ij}\, \Omega_j. \quad (3.15b)$$

Equation (3.15) can be written in the forms similar to (3.9) and (3.10). Because on the left-hand side of (3.15) T^Q and T^B are two scalars (two zeroth-order tensors) and on the right-hand side of (3.15) U_i, Ω_i, q_i and θ_i are components of four first-order tensors, it follows from the quotient rule that A_{ij}, S_{ij}, X_{ij}, B_{ij}, D_{ij}, and E_{ij}, must be components of six distinct Cartesian second-order tensors. In (3.15) A_{ij}, S_{ij} and X_{ij} are components of the added mass, the added-product of inertia and the added-moment of inertia tensors of a floating body, respectively. Components B_{ij} are damping tensor components corresponding to the translational oscillations of the floating body; E_{ij} are components of the damping tensor corresponding to the rotational oscillations and D_{ij} are components of the damping tensor corresponding to the interaction between translational and rotational oscillations. In analogy with m_{ij}, J_{ij} and I_{ij} we shall call B_{ij}, D_{ij} and E_{ij} respectively, the components of the zeroth-moment, first-moment and second-moment damping tensors. One may refer to the nine second order tensors m_{ij}, J_{ij}, I_{ij}, A_{ij}, S_{ij}, X_{ij}, B_{ij}, D_{ij} and E_{ij} as radiation tensors. We shall refer to three tensors m_{ij}, A_{ij} and B_{ij} as zeroth-moment radiation tensors; to three tensors J_{ij}, S_{ij} and D_{ij} as first-moment radiation tensors and to I_{ij}, X_{ij} and E_{ij} as second-moment radiation tensors.

Now consider (3.8) since in this equation i and j are dummy indices it follows that

$$m_{ij} = m_{ji} \quad \text{and} \quad I_{ij} = I_{ji}. \quad (3.16)$$

In addition, the symmetry of $A_{\alpha\beta}$ and $B_{\alpha\beta}$ implies that

$$A_{ij} = A_{ji} \quad \text{and} \quad X_{ij} = X_{ji}, \quad (3.17a)$$

$$B_{ij} = B_{ji} \quad \text{and} \quad E_{ij} = E_{ji}. \quad (3.17b)$$

Therefore, zeroth-moment and second-moment radiation tensors are symmetric tensors.

By using a similar approach as used in this section it can be shown that for the linear radiation problem of a floating body, in addition to the added mass and damping matrices the 6×6 hydrostatic restoring matrix introduced in (2.30) also contains three distinct Cartesian second–order tensors in regard to translational, rotational and interaction between translational and rotational degrees of freedom.

3.3 Tensor properties of radiation coefficients

Having shown that radiation coefficients $m_{\alpha\beta}$, $A_{\alpha\beta}$ and $B_{\alpha\beta}$ contain nine distinct second–order tensors, all powerful tools of tensor analysis become available for the radiation problem of an immersed or a floating body. Some of these tools are related to the problem of obtaining the components of a tensor in one coordinate system when these components are known in another coordinate system. We shall express the rotation, reflection and translation transformation laws for radiation coefficients in the following sub-sections.

3.3.1 The transformation law of radiation tensors

Consider two right–handed rectangular Cartesian coordinate systems x_i and x'_i with the same origin where primed coordinate system x'_i is obtained by rotating unprimed coordinate system x_i about the common origin. It is known that if R_{ij} are components of a second–order tensor in x_i coordinate system, they transform to components R'_{ij} in x'_i coordinate system by the following transformation law

$$R'_{ij} = a_{ik}\, a_{jl}\, R_{kl}, \tag{3.18}$$

in which a_{ij} is the direction cosines or the transformation symbol. For $\mathbf{e}'_i$ and $\mathbf{e}_i$ respectively, as unit basis vectors of primed– and unprimed–coordinate systems, a_{ij} is defined as

$$a_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j = \cos(\widehat{x'_i,\, x_j}). \tag{3.19}$$

Since it is shown that each of the nine radiation tensors related to the radiation problem is a second-order tensor, therefore (3.18) will be valid for each of them. Hereafter we use R_{ij} to refer to any of the nine radiation tensors in general. Using (3.18), the tranformation law for m_{ij}, J_{ij} and I_{ij} in matrix form can be written as

$$[m'] = [a]\,[m]\,[a]^T, \tag{3.20a}$$

$$[J'] = [a]\,[J]\,[a]^T, \tag{3.20b}$$

$$[I'] = [a]\,[I]\,[a]^T. \tag{3.20c}$$

Equation (3.20) shows that although m_{ij}, J_{ij} and I_{ij} are related to the same body and to each other through $m_{\alpha\beta}$ but they can be obtained in a rotated coordinate system independently.

For an arbitrary three-dimensional body the transformation law (3.18) in component form is derived in Appendix B. As a consequence of the transformation law (3.18), the component of any radiation tensor R_{ij} in the direction of an arbitrary unit vector $\mathbf{n} = n_i \mathbf{e}_i$ is $R_{ij}\, n_i\, n_j$. Also the first, second and third invariants of radiation tensor R_{ij} can be found from the following relations

$$\begin{aligned}
I_1 &= R_{ii}, \\
I_2 &= \frac{1}{2}(R_{ij}\,R_{ij} - R_{ii}\,R_{jj}), \\
I_3 &= \frac{1}{6}(R_{ii}\,R_{jj}\,R_{kk} - 3R_{ii}\,R_{jk}\,R_{jk} + 2R_{ij}\,R_{jk}\,R_{ki}).
\end{aligned} \tag{3.21}$$

In addition, since zeroth–moment and second–moment radiation tensors are symmetric tensors, their principal values are all real and the corresponding principal directions are mutually orthogonal to each other. Furthermore, the maximum (minimum) value of a tensor component is equal to the maximum (minimum) principal value. Moreover, if the immersed or floating body has a plane of symmetry, the direction perpendicular to that plane is a principal direction of radiation tensors and the other two principal directions lie in the plane of symmetry.

It should be mentioned that, in practice, it is more convenient to obtain radiation coefficients of a rotated body rather than those coefficients for a body in a rotated coordinate system. This can be done by considering x_i and x'_i coordinate systems as inertia and body-fixed coordinate systems, respectively, such that before rotation x'_i coincides x_i coordinate system. If the radiation coefficients are known in the body-fixed coordinate system, then for the rotated body we have

$$R_{ij} = a_{ki}\, a_{lj}\, R'_{kl}, \tag{3.22}$$

where a_{ki} can be assumed as the object rotation matrix.

3.3.2 Radiation tensors and improper orthogonal transformations

An orthogonal transformation, like the one governed by (3.19), is defined to be a proper transformation when $\det(a_{ij})$ is equal to $+1$ and an improper transformation when $\det(a_{ij})$ is equal to -1. A proper transformation preserves the handedness of the coordinate system whereas an improper transformation changes the handedness. A second-order tensor is a quantity that obeys the transformation law (3.18), whether the transformation is proper or improper. On the other hand, a psuedo-tensor is a quantity whose transformation law is similar to that of a tensor but has $\det(a_{ij})$ as a coefficient on its right-hand side. Therefore a pseudo-tensor differs from a tensor when the transformation is an improper one.

Now consider two rectangular Cartesian coordinate systems x_i and x'_i with the same origin where the primed coordinate system x'_i is obtained from unprimed coordinate system x_i by an improper orthogonal transformation (let say by a reflection in a coordinate plane). To study the behaviour of radiation tensors of an immersed body under improper orthogonal transformations, recall (3.8)

$$T = \frac{1}{2}\,U_k\,m_{kl}\,U_l + \frac{1}{2}\,\Omega_k\,J_{lk}\,U_l + \frac{1}{2}\,U_k\,J_{kl}\,\Omega_l + \frac{1}{2}\,\Omega_k\,I_{kl}\,\Omega_l. \tag{3.23}$$

Because on the right-hand side of this equation U_k is the component of a first-order tensor (a polar vector) and Ω_k is the component of a first-order pseudo-tensor (an axial vector), under an improper orthogonal transformation we have

$$U_k = a_{ik}\,U'_i, \qquad \Omega_k = -a_{ik}\,\Omega'_i. \tag{3.24}$$

A similar equation can be written by replacing free index k in (3.24) with l and changing the dummy index i with j. Using these equations on the right-hand side of (3.23) gives us

$$\begin{aligned} T = &\frac{1}{2}\,U'_i\,(a_{ik}\,a_{jl}\,m_{kl})\,U'_j + \frac{1}{2}\,\Omega'_i(-a_{ik}\,a_{jl}\,J_{lk})\,U'_j + \\ &\frac{1}{2}\,U'_i\,(-a_{ik}\,a_{jl}\,J_{kl})\,\Omega'_j + \frac{1}{2}\,\Omega'_i\,(a_{ik}\,a_{jl}\,I_{kl})\,\Omega'_j. \end{aligned} \tag{3.25}$$

On the other hand, since T is an invariant, in the primed coordinate system we have

$$T = \frac{1}{2}\,U'_i\,m'_{ij}\,U'_j + \frac{1}{2}\,\Omega'_i\,J'_{ji}\,U'_j + \frac{1}{2}\,U'_i\,J'_{ij}\,\Omega'_j + \frac{1}{2}\,\Omega'_i\,I'_{ij}\,\Omega'_j. \tag{3.26}$$

Equating the right-hand sides of (3.25) and (3.26), since components U'_i and Ω'_i are independent of each other and in general non-zero, we find that

$$\begin{aligned} m'_{ij} &= a_{ik}\,a_{jl}\,m_{kl}, \\ J'_{ij} &= -a_{ik}\,a_{jl}\,J_{kl}, \\ I'_{ij} &= a_{ik}\,a_{jl}\,I_{kl}. \end{aligned} \tag{3.27}$$

Therefore m_{ij} and I_{ij} are components of two tensors while J_{ij} is the component of a pseudo-tensor. Now turn back to (3.23) and note that U_k is the component of a polar vector and Ω_k is the component of an axial vector. In addition, tensor components m_{ij} and I_{ij} are multiplied on both sides by either polar or axial vectors while pseudo–tensor component J_{ij} is multiplied on one side by a polar vector and on the other side by an axial vector. We shall apply this rule to study the behaviour of added mass and damping tensors of a floating body under improper orthogonal transformations. To this end, consider (3.15)

$$T^Q = \frac{1}{2}\,U_i A_{ij}\,U_j + \frac{1}{2}\,\Omega_i S_{ji}\,U_j + \frac{1}{2}\,U_i S_{ij}\,\Omega_j + \frac{1}{2}\,\Omega_i\,X_{ij}\,\Omega_j, \quad (3.28a)$$

$$T^B = \frac{1}{2} q_i\,B_{ij}\,U_j + \frac{1}{2}\Theta_i\,D_{ji}\,U_j + \frac{1}{2}\,q_i\,D_{ij}\,\Omega_j + \frac{1}{2}\,\Theta_i\,E_{ij}\,\Omega_j,. \quad (3.28b)$$

In this equation U_i and q_i are components of two polar vectors and Ω_i and Θ_i are components of two axial vectors. Therefore S_{ij} and D_{ij} which are multipled by both polar and axial vectors and produce scalar values are components of two second-order pseudo-tensors; and A_{ij}, X_{ij}, B_{ij} and E_{ij} which are only multiplied by either polar or axial vectors and give scalar values are components of four tensors. In other words, we can write

$$S'_{ij} = -a_{ik}\,a_{jl}\,S_{kl}, \qquad D'_{ij} = -a_{ik}\,a_{jl}\,D_{kl}. \quad (3.29)$$

and

$$\begin{aligned} A'_{ij} &= a_{ik}\,a_{jl}\,A_{kl}, \\ X'_{ij} &= a_{ik}\,a_{jl}\,X_{kl}, \\ B'_{ij} &= a_{ik}\,a_{jl}\,B_{kl}, \\ E'_{ij} &= a_{ik}\,a_{jl}\,E_{kl}. \end{aligned} \quad (3.30)$$

Equations (3.27), (3.29) and (3.30) show that the zeroth–moment and second–moment radiation tensors obey the transformation law of a tensor while the first–moment radiation tensors obey the transformation law of a pseudo–tensor. If the components T_{ij} are used to refer to the components of one of the zeroth–

moment or second–moment radiation tensors and the components P_{ij} are used to refer to the components of one of the first–moment radiation tensors, then (3.27) to (3.30) can be summarized as follows

$$T'_{ij} = a_{ik}\, a_{jl}\, T_{kl}, \tag{3.31a}$$

$$P'_{ij} = -a_{ik}\, a_{jl}\, P_{kl}. \tag{3.31b}$$

The effect of body symmetries on radiation tensors

Equation (3.31) can be used to study the effect of body symmetries on the radiation tensors of an immersed or a floating body. Assume that a body has a plane of symmetry and the x_k-axis of the unprimed coordinate system is perpendicular to that plane. Now consider a primed coordinate system such that it is the reflection of unprimed coordinte system in the symmetry plane. Then the transformation symbol will be

$$a^k_{ij} = \begin{cases} -1 & : \quad i = j = k, \\ \delta_{ij} & : \quad \text{otherwise.} \end{cases} \qquad : \quad k = 1, 2, 3, \tag{3.32}$$

where δ_{ij} is the Kronecker delta. It means that apart from one of the leading diagonal elements which is -1 the rest of matrix $[a]$ is the same as identity matrix. Therefore it follows that $\det(a_{ij}) = -1$ so the transformation defined by (3.32) is an improper one. Hence, equation (3.31) governs the transformation. On the other hand, due to the symmetry of the body there must be no difference between the components of tensor $\mathbf{T}$ and pseudo–tensor $\mathbf{P}$ in primed– and unprimed–coordinate systems. In other words due to the body symmetry T_{ij} and P_{ij} remain invariant under the transformation (3.32). Consequently, equation (3.31) in matrix form takes the following form

$$[T] = [a]\,[T]\,[a]^T, \qquad [P] = -[a]\,[P]\,[a]^T. \tag{3.33}$$

Now if (3.32) is substituted into (3.33), because only zero is equal to its additive inverse, it follows that some of the T_{ij} and P_{ij} components corresponding to a symmetric body must be zero. Observing the results of substitution of (3.32) into (3.33) for three cases of $k = 1, 2, 3$ reveals the following symmetry rules: If x_k-axis, ($k = 1, 2, 3$), be perpendicular to the symmetry plane of the body then:

1. *For matrix* $[T]$, *corresponding to a tensor* $\mathbf{T}$, *all off-diagonal elements in the kth–row and kth–column are zero.*

2. *For matrix* $[P]$, *corresponding to a pseudo–tensor* $\mathbf{P}$, *only off-diagonal elements in the kth–row and kth–column are non–zero.*

These rules can be used to find the zero components of tensors and pseudo–tensors corresponding to a body with 1, 2 or 3 orthogonal symmetry planes without doing the matrix multiplications of (3.33). For other kinds of symmetry a_{ij} will be defined by an equation different from (3.31) but (3.33) is still valid. The final results are given in the literature (see for instance Happel & Brenner (1965) pp.183-192).

3.3.3 Parallel–axes–theorem for radiation tensors

Consider two coordinate systems x_i and x'_i both fixed with respect to a rigid body at O and O' such that the corresponding axes of two coordinate systems are parallel. If U_i and U'_i denote the components of the translational velocity of the body at O and O' respectively, and Ω_i is the component of the angular velocity of the body, then because O and O' can be assumed as two points of the rigid body and because $\mathbf{e}'_i = \mathbf{e}_i$, one can write

$$U_i = U'_i - \epsilon_{ijk}\Omega_j d_k \,, \tag{3.34}$$

where ϵ_{ijk} is the component of the alternator tensor and d_k is the component of the position vector of O' with respect to O. Introducing the anti-symmetric tensor components $H_{ij} = -\epsilon_{ijk}d_k$, equation (3.34) takes the following forms

$$U_i = U_i' + H_{ik}\Omega_k \tag{3.35a}$$

$$U_j = U_j' + H_{jl}\Omega_l\,. \tag{3.35b}$$

Substituting from the right-hand side of (3.35) for U_i and U_j the second-order tensors U_iU_j, Ω_iU_j and $U_i\Omega_j$ in (3.8) can be written as

$$U_iU_j = U_i'U_j' + U_i'H_{jl}\Omega_l + H_{ik}\Omega_kU_j' + H_{ik}H_{jl}\Omega_k\Omega_l \tag{3.36a}$$

$$\Omega_iU_j = \Omega_iU_j' + \Omega_iH_{jl}\Omega_l \tag{3.36b}$$

$$U_i\Omega_j = U_i'\Omega_j + H_{ik}\,\Omega_k\Omega_j\,. \tag{3.36c}$$

Now introducing from (3.36) into (3.8) and changing the dummy indices i and j, respectively, with dummy indices k and l yields

$$\begin{aligned} T = {} & \frac{1}{2}U_i'm_{ij}U_j' + \frac{1}{2}\Omega_i\left\{J_{ij}^T + H_{ik}^Tm_{kj}\right\}U_j' + \frac{1}{2}U_i'\left\{J_{ij} + m_{il}H_{lj}\right\}\Omega_j \\ & + \frac{1}{2}\Omega_i\left\{H_{ik}^Tm_{kl}H_{lj} + H_{ik}^TJ_{kj} + J_{il}^TH_{lj} + I_{ij}\right\}\Omega_j. \end{aligned} \tag{3.37}$$

On the other hand, the kinetic energy given by (3.8) in the primed coordinate system takes the following form

$$T' = \frac{1}{2}U_i'm_{ij}'U_j' + \frac{1}{2}\Omega_i'J_{ij}'^TU_j' + \frac{1}{2}U_i'J_{ij}'\Omega_j' + \frac{1}{2}\Omega_i'I_{ij}'\Omega_j'. \tag{3.38}$$

But the kinetic energy is a scalar quantity and therefore is an invariant and the angular velocity of the rigid body is a free vector. Therefore (3.38) can be written as follows

$$T = \frac{1}{2}U_i'm_{ij}'U_j' + \frac{1}{2}\Omega_iJ_{ij}'^TU_j' + \frac{1}{2}U_i'J_{ij}'\Omega_j + \frac{1}{2}\Omega_iI_{ij}'\Omega_j. \tag{3.39}$$

Now equating the right-hand sides of (3.39) and (3.37) and taking into account that U_i' and Ω_i are independent, arbitrary and generally non-zero, it follows that

$$m'_{ij} = m_{ij}, \tag{3.40a}$$

$$J'^T_{ij} = J^T_{ij} + H^T_{ik} m_{kj}, \tag{3.40b}$$

$$J'_{ij} = J_{ij} + m_{ik} H_{kj}, \tag{3.40c}$$

$$I'_{ij} = H^T_{ik} m_{kl} H_{lj} + H^T_{ik} J_{kj} + J^T_{ik} H_{kj} + I_{ij}. \tag{3.40d}$$

Equations (3.40) are the translation law from unprimed coordinate system to the primed coordinate system. If added mass coefficients in the unprimed coordinate system are known, (3.40) can be used to find those coefficients in the primed coordinate system. Equation (3.40) can also be written as follows

$$m_{ij} = m'_{ij}, \tag{3.41a}$$

$$J^T_{ij} = J'^T_{ij} + H_{ik} m'_{kj}, \tag{3.41b}$$

$$J_{ij} = J'_{ij} + m'_{ik} H^T_{kj}, \tag{3.41c}$$

$$I_{ij} = H_{ik} m'_{kl} H^T_{lj} + H_{ik} J'_{kj} + J'^T_{ik} H^T_{kj} + I'_{ij}. \tag{3.41d}$$

This equation can be used to derive added mass coefficients in the unprimed coordinate system when those coefficients are known in the primed coordinate system. Because m_{ij} and m'_{ij} are symmetric, (3.40b) and (3.41b) are the transpose of (3.40c) and (3.41c), respectively. Therefore there are only three independent equations in (3.40) and (3.41). Equation (3.40) in matrix form can be written as follows

$$[m'] = [m], \tag{3.42a}$$

$$[J'] = [J] + [m]\,[H], \tag{3.42b}$$

$$[I'] = [H]\,[m]\,[H]^T + [H]^T [J] + [J]^T [H] + [I]. \tag{3.42c}$$

where

$$[H]^T\,[m]\,[H] = [H]\,[m]\,[H]^T$$

is used.

For a floating body, equations similar to (3.40) for added mass and radiation damping coefficients can be obtained by substituting from (3.35) for U_i and from a similar equation for q_i into (3.15). The details of derivations are given in the Appendix C and the results are

$$[A'] = [A], \tag{3.43a}$$

$$[S'] = [S] + [A]\,[H], \tag{3.43b}$$

$$[X'] = [H]\,[A]\,[H]^T + [H]^T [S] + [S]^T [H] + [X], \tag{3.43c}$$

for added mass matrices, and

$$[B'] = [B], \tag{3.44a}$$

$$[D'] = [D] + [B]\,[H], \tag{3.44b}$$

$$[E'] = [H]\,[B]\,[H]^T + [H]^T [D] + [D]^T [H] + [E], \tag{3.44c}$$

for radiation damping matrices. In summary, If $\mathbf{R}^0$, $\mathbf{R}^1$ and $\mathbf{R}^2$ are used to denote zeroth–, first– and second–moment radiation tensors in general, then the translation law for radiation tensors of a rigid body, immersed or floating, can be written as

$$\begin{aligned} \mathbf{R}'^1 &= \mathbf{R}^0 \cdot \mathbf{H} + \mathbf{R}^1, \\ \mathbf{R}'^2 &= \mathbf{H} \cdot \mathbf{R}^0 \cdot \mathbf{H}^T + \mathbf{R}^{1^T} \cdot \mathbf{H} + \mathbf{H}^T \cdot \mathbf{R}^1 + \mathbf{R}^2, \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} \mathbf{R}^1 &= \mathbf{R}'^0 \cdot \mathbf{H}^T + \mathbf{R}'^1, \\ \mathbf{R}^2 &= \mathbf{H} \cdot \mathbf{R}'^0 \cdot \mathbf{H}^T + \mathbf{R}'^{1^T} \cdot \mathbf{H}^T + \mathbf{H} \cdot \mathbf{R}'^1 + \mathbf{R}'^2, \end{aligned} \tag{3.46}$$

or in matrix form

$$\begin{aligned}
\left[R'^1\right] &= [R^0]\,[H]^T + [R^1], \\
\left[R'^2\right] &= [H]\,[R^0]\,[H]^T + [R^1]^T\,[H] + [H]^T\,[R^1] + [R^2].
\end{aligned} \tag{3.47}$$

and

$$\begin{aligned}
\left[R^1\right] &= [R'^0]\,[H]^T + [R'^1], \\
\left[R^2\right] &= [H]\,[R'^0]\,[H]^T + [R'^1]^T\,[H]^T + [H]\,[R'^1] + [R'^2].
\end{aligned} \tag{3.48}$$

Equation (3.42) to (3.48) show that only zeroth–moment radiation tensors, which are independent of the choice of coordinate system are truly second–order tensors. The first–moment and second–moment radiation tensors are not precisely tensor quantities since they are dependent on the position of the origin of coordinate system. One may call $\mathbf{R}^1$ and $\mathbf{R}^2$ generalized tensors.

3.4 Application of transformation method

Tensor properties of radiation coefficients in regard to different coordinate transformations can be used in the hydrodynamic analysis of offshore structures. The rotation law, known as the transformation law, can be used to obtain radiation coefficients of inclined or rotated members of a structure. The reflection law can be used to quickly determine the zero radiation coefficients of symmetric bodies. The translation law, known as parallel-axes-theorem, can be used to calculate the radiation coefficients of members or substructures of a platform about the global coordinate system of the platform. Therefore, total added mass coefficients of an offshore structure can be obtained by the transformation method.

In the absence of these tensor tools the common method of calculation of added mass coefficients of offshore structures made of slender members has been based on the Morison equation (1950) and the normal component approach. In that method, in order to compute added mass coefficients of each element of the structure it is required to calculate forces acting on that element. One advantage of the transformation method, equations (3.18) and (3.47), is that there is no need for calculation of any force. It is only required to know the added mass coefficients of an element in an arbitrary coordinate system. Those coefficients in another coordinate system can be obtained by using the transformation method. In other words, it is not necessary to apply the transformation method to each and every element of the structure, i.e., when sub–structure B can be produced by rotating, reflecting or translating sub–structure A, the radiation coefficients of B can be obtained from those of A by using the appropriate transformation law, rather than from direct calculations. Considering that offshore structures are usually made of identical sub-structures, this reduces the amount of calculations greatly. This advantage could become more significant if the wave excitation and viscous forces acting on the slender members of a platform can also be obtained by a method consistent with the transformation method. With this purpose Sadeghi et al. (2004) proposed a model for the response analysis of a truss spar platform. In addition to the transformation method, a viscous-radiation-diffraction model was introduced which together generates a new technique for response analysis of offshore platforms. This model with more details is introduced in chapter 5.

Chapter 4

Approximation of Surge and Pitch Loads on Truncated Vertical Cylinders

4.1 Introduction

Truncated vertical circular cylinders are used to make deep water floating offshore platforms like truss spars. In deep water, when the cylinder is not sufficiently deep, no simple exact analytical experssion exists for the calculation of surge and pitch loads. In this case numerical methods can be used for prediction of loads but expensive software and expertise are required. An alternative approach is the use of the analytical and semi-analytical methods. For examples in this regard, one can refer to the works of Miles & Gilbert (1968) and Garrett (1971) who studied the diffraction problem of a circular dock. Black et al. (1971) calculated the loads on a truncated cylinder which is either floating or sitting on the seabed. Yeung (1981) derived a set of theoretical added masses and damping coefficients for a floating circular cylinder in water of finite depth. Bhatta & Rahman (1995, 2003) presented analytical and numerical results for the diffraction loads and radiation coefficients of floating circular cylinders in finite-depth water. An approximate but less elaborate

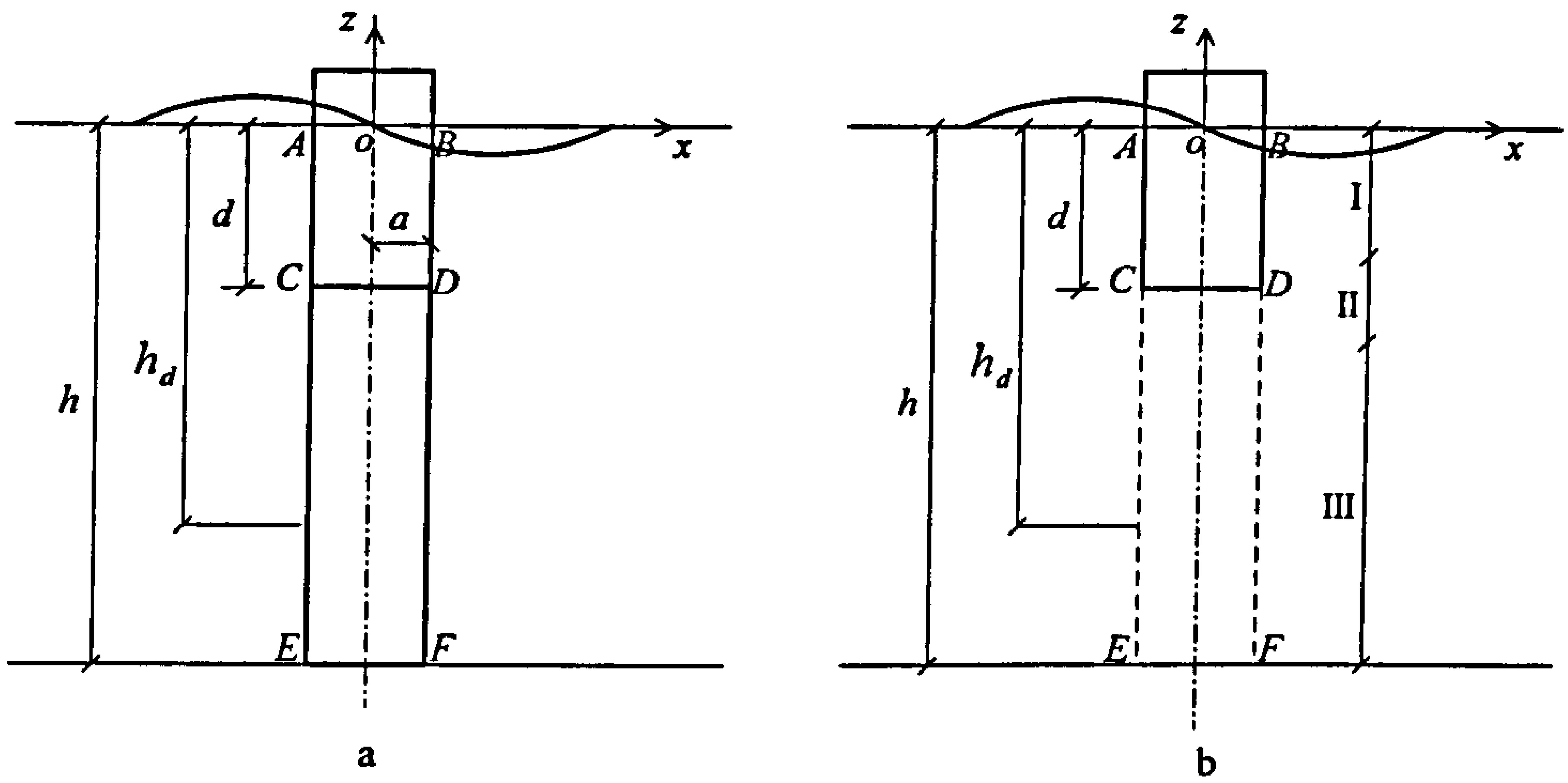

Figure 4.1: (a) Bottom mounted cylinder; (b) Truncated cylinder

method for prediction of loads acting on a truncated cylinder is the use of McCamy & Fuchs (1954) diffraction theory through the Niedzwecki & Duggal (1992) approximation. In this chapter, we shall first study the accuracy of this method in the following then a more accurate approximation will be introduced in §§ 4.3 and 4.4.

4.2 Approximation of surge and pitch loads

Let us consider a surface piercing bottom mounted circular cylinder and a surface piercing truncated circular cylinder as shown in figures 4.1. The cylinder in figure 4.1a is divided into two parts, ABCD and CDEF. The ABCD part is geometrically equivalent with the truncated cylinder in Figure 4.1b. Using the diffraction theory of McCamy & Fuchs (1954) the inline force per unit length acting on the cylinder in figure 4.1a in deep water is,

$$f_1 = \frac{2\rho g H}{k} A(ka) e^{kz} \cos(\alpha_1 - \omega t) \tag{4.1}$$

where

$$A(ka) = \frac{1}{|H_1'(ka)|} \tag{4.2a}$$

$$\alpha_1 = \tan^{-1}\left[\frac{J_1'(ka)}{Y_1'(ka)}\right] \tag{4.2b}$$

where k is the wave number, a is the cylinder radius, H_1 is the first order Hankel function of the first kind (Abramowitz & Stegun 1965) and J_1 and Y_1 are the first order Bessel functions of the first and second kind, respectively. Mogridge & Jamieson (1976) integrated (4.1) over the water depth to obtain the total surge force acting on a cylinder like the one in figure 4.1a, obtaining,

$$F_1^T = \frac{2\rho g H}{k^2} A(ka)[1 - \mathrm{e}^{kh}]\cos(\alpha_1 - \omega t) \tag{4.3}$$

McCamy & Fuchs (1954) formulated the moment due to force f_1, which can be written about the origin o as follows,

$$M_2^T = \frac{2\rho g H}{k^2} A(ka)\left[-\frac{1}{k} + \mathrm{e}^{kh}\left(h + \frac{1}{k}\right)\right]\cos(\alpha_5 - \omega t) \tag{4.4}$$

Niedzwecki & Duggal (1992) used the experssion of force per unit length of McCamy & Fuchs (1954) to estimate the surge force acting on a truncated cylinder by limiting the integration in vertical direction to the submerged length of the cylinder. Therefore, for a truncated cylinder with draft d the surge force per unit wave amplitude is,

$$\frac{F_1}{H/2} = \frac{4\rho g}{k^2} A(ka)[1 - \mathrm{e}^{kd}]\cos(\alpha_1 - \omega t), \tag{4.5}$$

and the pitch moment per unit wave amplitude about o can be written as

$$\frac{M_2}{H/2} = \frac{4\rho g}{k^2} A(ka)\left[-\frac{1}{k} + \mathrm{e}^{kd}\left(d + \frac{1}{k}\right)\right]\cos(\alpha_5 - \omega t). \tag{4.6}$$

From linear wave theory it is well known that in deep water the wave motion decays rapidly in depth and usually by neglecting the higher order nonlinear effects the water beneath a certain depth is almost at rest. We shall refer to the water between the free surface and the upper boundary of the quiescent water as the *dynamic* depth. Below the dynamic depth the kinetic energy of water particles is negligible and as far as the kinetic energy is concerned there is no difference between a mass of solid and a mass of fluid when the particles of the matter are at rest. Therefore the solid cylinder in this region can be replaced by a mass of water without a considerable change in the horizontal force. Based on this argument the total horizontal force acting on a truncated cylinder in deep water can be approximated by limiting the integration of force intensity of McCamy & Fuchs (1954) to the cylinder draft when the cylinder draft is sufficiently deep. This is basically the idea behind the Niedzwecki & Duggal (1992) approximation. This approximation was verified by experiments (Niedzwecki & Duggal 1992). The accuracy of (4.5) can be assessed by defining the ratio of F_1/F_1^T by δ, then

$$\delta = \frac{1-\mathrm{e}^{-kd}}{1-\mathrm{e}^{-kh}}, \qquad (0 < \delta < 1). \tag{4.7}$$

Assuming $d = \gamma h$, $(0 < \gamma < 1)$, then for a known δ, solving (4.7) for γ yields

$$\gamma(\omega) = \frac{-\ln\left[1-\delta(1-\mathrm{e}^{-kh})\right]}{kh}. \tag{4.8}$$

In analogy with the centre of mass, the weighted average of γ in the random sea can be written as,

$$\bar{\gamma} = \frac{\int_0^{\omega_c} \gamma(\omega)S(\omega)\mathrm{d}\omega}{\int_0^{\infty} S(\omega)\mathrm{d}\omega}. \tag{4.9}$$

In which $S(\omega)$ is the sea spectrum and ω_c is a cut-off frequency such that for $\omega > \omega_c$, $S(\omega)$ is close to zero. For instance, using $h = 500$m and $a = 15.75$m given for the truss spar in Chapter 5, then for $\delta = 0.99$, $\omega_c = 1.5$rad/sec and sea spectrum as JONSWAP spectrum with specifications given in page 97, it is found that $\bar{\gamma} = 0.434$ and $\bar{\gamma}h = 217$m. It means that the mean surge force associated with the first 217 meters of the cylinder in figure 4.1a with parameters given above is 99 percent of the total mean surge force acting on the whole length of the cylinder. Therefore the mean error of (4.5) for the

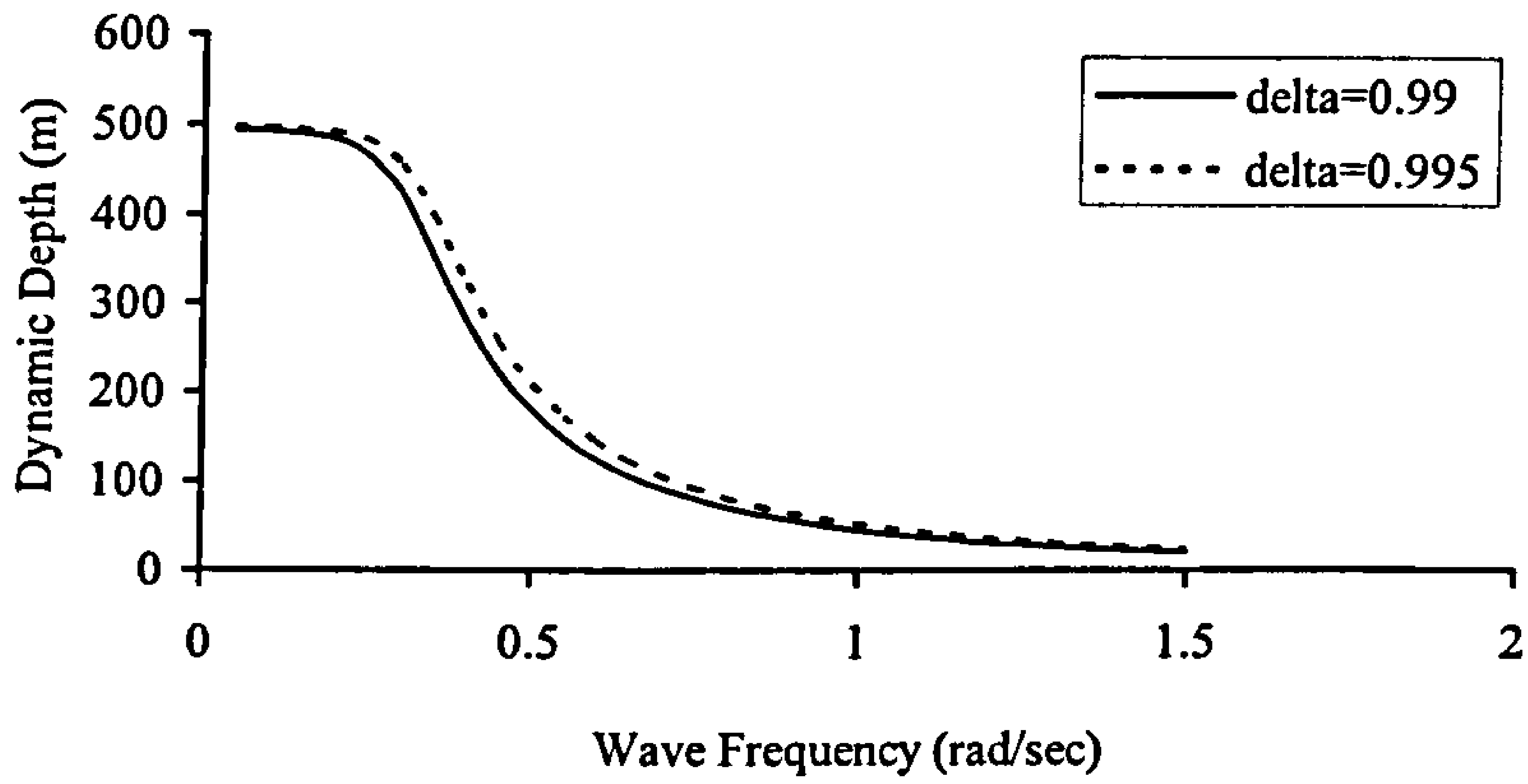

Figure 4.2: Dynamic depth in 500m water depth

truncated cylinder with 217m draft in 500m water depth is almost zero. On the other hand, if the draft of a truncated cylinder is known, then from (4.7) the random weighted average of δ can be obtained from

$$\bar{\delta} = \frac{\int_0^{\omega_c} \delta(\omega) S(\omega) \mathrm{d}\omega}{\int_0^{\infty} S(\omega) \mathrm{d}\omega}.$$

Again for a cylinder with the same parameters as above, if the draft of cylinder ABCD in figure 4.1a and its equivalent truncated cylinder in figure (4.1b) be 59.2m (as the draft of the truss spar hull in Chapter 5), the mean difference of 27.4 percent will be found. This difference does not represent the error of (4.5) but it can be used as a measure to show that as the draft of truncated cylinder decreases the error of (4.5) increases.

The accuracy of the surge force given by (4.5) in each wave frequency may be related to the dynamic depth, h_d,

$$h_d = \tilde{\gamma}(\omega) h. \tag{4.10}$$

As pointed out h_d is a water depth beneath it the water particles are almost at rest and have negligible kinetic energy, so $\tilde{\gamma}$ in (4.10) corresponds to a δ value close to 1. For the considered example, the dynamic depth, obtained from (4.8) and (4.10), is plotted against the wave frequency in figure 4.2. The water below $z = -h_d$ plane applies almost no surge force on the cylinder. Note that in figure 4.2 the water cannot be considered as deep for $\omega < 0.25$rad/sec. But since the obtained value of h_d from (4.10) is not far from its value when the water is not assumed deep and also since the sea spectrum energy is not significant for $\omega < 0.25$, the deep water approximation is used for $\omega < 0.25$. When h_d is less than d, the latter may be replaced by the former in (4.5) and (4.6).

For the cylindrical hulls of truss spar platforms for a considerable part of the frequency interval, the cylinder draft is significantly less than the dynamic depth. In such cases the Niedzwecki & Duggal (1992) approximation may be required to be improved. Weggel (1994) used the computer program WaMIT 4.01 (1991), based on the linear diffraction theory, through a parametric study and suggested the following correction factor

$$\left(1 - 0.125\frac{2a}{d}\mathrm{e}^{-1.62ka}\right) \tag{4.11}$$

to be multiplied by the surge force predicted by (4.5). Weggel (1994) found excellent agreement between WaMIT outputs and equations (4.5) and (4.11). He modified (4.5) by mathematical arguments. Our aim here is to modify (4.5) by engineering approximations.

4.3 Approximation considerations

Consider regions $z > -d$ and $z \leq -d$ in figure 4.1a separately to see how the drop of the cylinder CDEF affects the loads acting on the cylinder ABCD. Once the cylinder CDEF is cut and taken out of the water, an amount of water equal to its volume fills its empty place. The water which fills the column CDEF has a kinetic energy. A part of this water comes from $z > -d$ region. Therefore the kinetic energy of waves in region $z > -d$ decreases. This decrease in the kinetic energy of water around the cylinder ABCD causes a reduction in the dynamic pressure and therefore in the surge force acting on the cylinder ABCD. Alternatively, Weggel (1994) argued that the surge force approximated with (4.5) over-predicts the horizontal force since flow is restricted from the passing under the cylinder bottom in the Niedzwecki & Duggal (1992) approximation.

The flow field in figure 4.1b is devided into three regions in the vertical direction, regions I, II and III. As was mentioned, in deep water the velocity potential of waves decay exponentially with depth. Therefore, in region III where the water particles are reasonably far from the rigid body it can be assumed that the scattering potential is damped and the velocity potential is the same as the velocity potential of the incident waves, that is,

$$\phi^{\mathrm{III}} = \phi_I \tag{4.12a}$$

Also in region I which is sufficiently far from the truncation section CD, it is assumed that the velocity potential is the same as the velocity potential of the McCamy & Fuchs (1954) in the similar region of figure 4.1a, i.e.

$$\phi^{\mathrm{I}} = \phi_I + \phi_S, \tag{4.12b}$$

where ϕ_S is the velocity potential of the scattered waves. Region II is the transition section where the velocity potential changes from $\phi_I + \phi_S$ at the interface of regions I and II to ϕ_I at the interface of regions II and III with a possible sharp change or discontinuity at the truncation section CD. In other words, the scattering potential ϕ_S decreases rapidly in region II of figure (4.1b) until it vanishes at the interface of regions II and III while ϕ_S at the same level in figure 4.1a is generally non-zero. It means that ϕ_S corresponding to the region II in figure 4.1b is smaller than ϕ_S corresponding to the same regin in figure 4.1a. Therefore (4.5) over-predicts the surge force. In order to modify the loads given by (4.5) and (4.6) the inline force due to incident and scattering potentials, unlike the McCamy & Fuchs (1954) diffraction theory, must be calculated separately. To this end, the scattering potential for a wave with a node at origin as shown in figure 4.1, can be written as (Rahman 1995)

$$\phi_S = \frac{Hg}{2\omega} \mathrm{e}^{kz} \mathrm{Re} \left\{ \sum_{n=0}^{\infty} \tau_n \mathrm{i}^n \left(\frac{-J_n'(ka)}{H_n'(ka)} H_n(kr) \cos n\theta \right) \mathrm{e}^{-\mathrm{i}\omega t} \right\} \tag{4.13}$$

in which $\tau_0 = 1$ and $\tau_n = 2$, $(n \geq 1)$. For the same wave profile, the force due to incident waves which is the Froude–Krylov force can be given by (Chakrabarti 1987)

$$f_{FK} = C_{FK} \rho \pi a^2 \frac{H}{2} \omega^2 \mathrm{e}^{kz} \cos \omega t \tag{4.14a}$$

in which the Froude–Krylov coefficient C_{FK} is

$$C_{FK} = 2 \frac{J_1(ka)}{ka}. \tag{4.14b}$$

Also the force experssion of McCamy & Fuchs (1954) given in (4.1) can be written in terms of an effective inertia coefficient C_M (Chakrabarti 1987),

$$f_1 = C_M \rho \pi a^2 \frac{gkH}{2} \mathrm{e}^{kz} \cos(\alpha_1 - \omega t) \tag{4.15a}$$

where

$$C_M = \frac{4}{\pi (ka)^2} A(ka). \tag{4.15b}$$

Now we shall derive the force per unit length due to scattering potential. The pressure corresponding to ϕ_S given in (4.13) is,

$$p_S = \frac{-\rho g H}{2} \mathrm{e}^{kz} \sum_{n=0}^{\infty} \tau_n \mathrm{Re} \left\{ \mathrm{i}^{n+1} \left(\frac{J_n'(ka)}{H_n'(ka)} H_n(kr) \right) \mathrm{e}^{-\mathrm{i}\omega t} \right\} \cos n\theta \tag{4.16}$$

and the force per unit length can be obtained from,

$$f_S = \int_0^{2\pi} p_S|_{r=a} n_1 \, a \, \mathrm{d}\theta \,. \tag{4.17}$$

Because $n_1 = -\cos\theta$, substituting for p_S from (4.16) into (4.17) gives us,

$$f_S = \frac{-\rho g H}{2} a\,\mathrm{e}^{kz} \sum_{n=0}^{\infty} \left[\tau_n \mathrm{Re}\left\{ \mathrm{i}^{n+1} \left(\frac{J_n'(ka)}{H_n'(ka)} H_n(kr) \right) \mathrm{e}^{-\mathrm{i}\omega t} \right\} \int_0^{2\pi} \cos n\theta \cos\theta \,\mathrm{d}\theta \right]. \tag{4.18}$$

The integral on the right-hand side of (4.18) is nonzero only for $n = 1$, so that

$$f_S = \frac{-\rho g H}{2} a\,\mathrm{e}^{kz} \left[2\mathrm{Re}\left\{ \mathrm{i}^2 \left(\frac{J_1'(ka)}{H_1'(ka)} H_1(ka) \right) \mathrm{e}^{-\mathrm{i}\omega t} \right\} \pi \right]. \tag{4.19}$$

or

$$f_S = \rho g H \mathrm{e}^{kz} \pi a \frac{J_1'(ka)}{|H_1'(ka)|} \mathrm{Re}\left\{ \mathrm{e}^{\mathrm{i}(\alpha_1 - \omega t)} \mathrm{i} H_1(ka) \right\} \tag{4.20}$$

in which α_1 is defined in (4.2b) and the relation

$$\frac{\mathrm{i}}{H_1'(ka)} = \frac{\mathrm{e}^{\mathrm{i}\alpha_1}}{|H_1'(ka)|} \tag{4.21}$$

is used to derive (4.20) from (4.19). Now from (4.2b) it can simply be deduced that

$$\sin\alpha_1 = \frac{J_1'(ka)}{|H_1'(ka)|}, \tag{4.22a}$$

$$\cos\alpha_1 = \frac{Y_1'(ka)}{|H_1'(ka)|}. \tag{4.22b}$$

Introducing from (4.22a) into (4.20) yields

$$f_S = \frac{2\rho g H}{k} \pi k a \mathrm{e}^{kz} \sin\alpha_1 \,\mathrm{Re}\left\{ H_1(ka) \mathrm{i} \mathrm{e}^{-\mathrm{i}(\omega t - \alpha_1)} \right\}. \tag{4.23}$$

Or since

$$H_1(ka) = |H_1(ka)|\,\mathrm{e}^{\mathrm{i}\beta}, \tag{4.24a}$$

$$\beta = \tan^{-1}\left(\frac{Y_1(ka)}{J_1(ka)} \right), \tag{4.24b}$$

equation (4.23) can be rewritten as follows

$$f_S = \rho\pi a^2 \frac{H}{2} g k e^{kz} \frac{2\,|H_1(ka)|}{ka} \sin\alpha_1 \operatorname{Re}\left\{ \mathrm{i} e^{-\mathrm{i}(\omega t - \alpha_1 - \beta)} \right\} \tag{4.25}$$

or

$$f_S = C_S \rho\pi g\, a^2 \frac{H}{2} k e^{kz} \sin(\omega t - (\alpha_1 + \beta)), \tag{4.26}$$

where scattering coefficient C_S is defined as,

$$C_S = \frac{2\,|H_1(ka)|}{ka} \sin\alpha_1\,. \tag{4.27}$$

Following Mei (1989), by expanding the sine term on the right-hand side of (4.26) the scattering force can be decomposed into inertia and damping components as follows

$$f_S = -C_S \sin(\alpha_1 + \beta) \rho\pi a^2 \frac{H}{2} g\, k e^{kz} \cos\omega t + C_S \cos(\alpha_1 + \beta) \rho\pi a^2 \frac{H}{2} g\, k e^{kz} \sin\omega t \tag{4.28}$$

or

$$f_S = C_{SI}\, \rho\pi a^2 \frac{H}{2} g\, k e^{kz} \cos\omega t + C_{SD}\, \rho\pi a^2 \frac{H}{2} g\, k e^{kz} \sin\omega t \tag{4.29}$$

in which C_{SI} and C_{SD}, the inertia and damping coefficients due to scattering potential, are defined as

$$C_{SI} = -C_S \sin(\alpha_1 + \beta)\,, \tag{4.30a}$$

$$C_{SD} = C_S \cos(\alpha_1 + \beta)\,. \tag{4.30b}$$

The plots of Froude–Krylov and scattering coefficients for three different cylinder radii are given in figure 4.3. The diffraction coefficient is also shown in figure 4.4.

Equation (4.29) can be written as

$$f_S = f_{SI} + f_{SD} = |f_{SI}| \cos\omega t + |f_{SD}| \sin\omega t \tag{4.31}$$

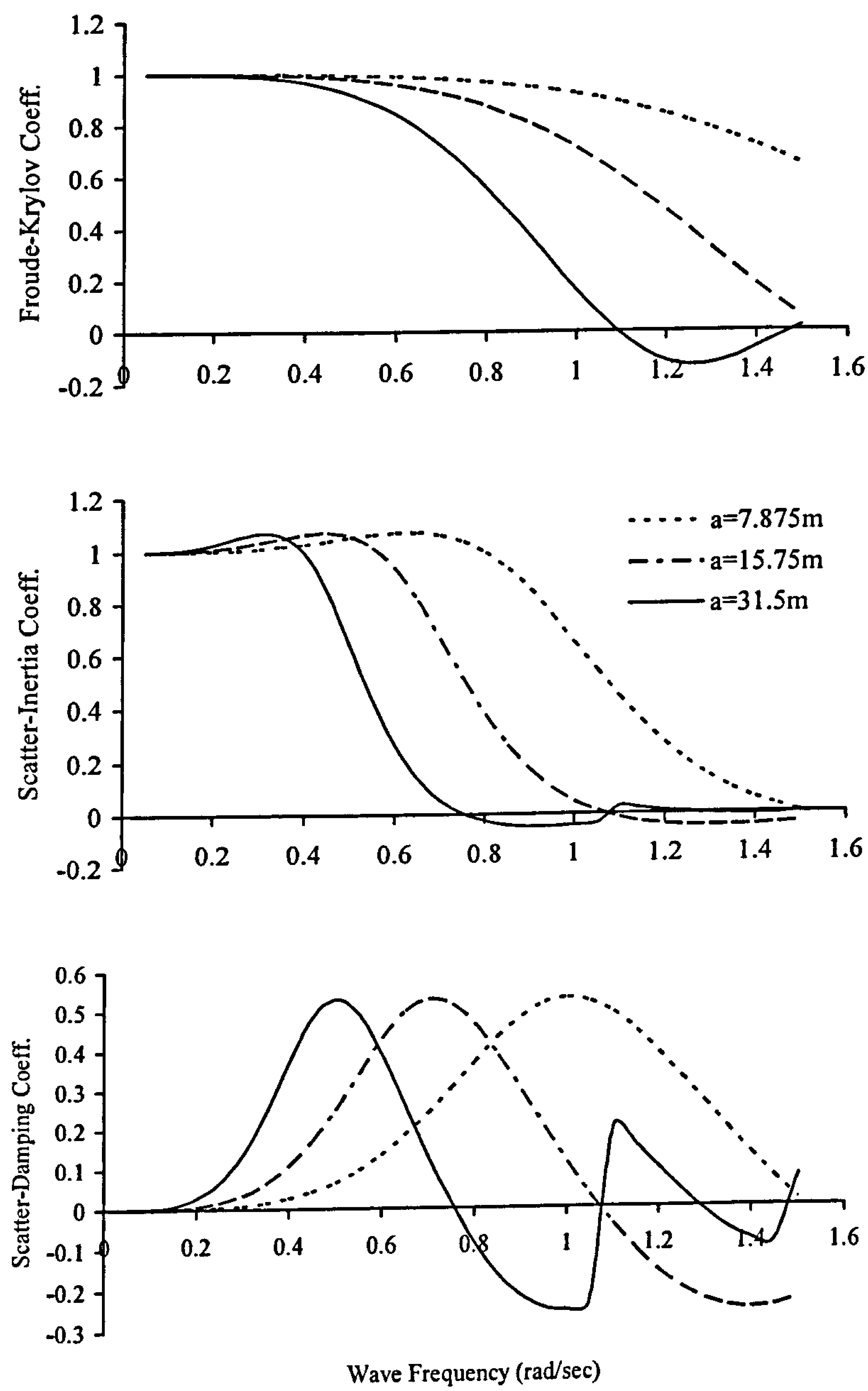

Figure 4.3: C_{FK}, C_{SI} and C_{SD} coefficients

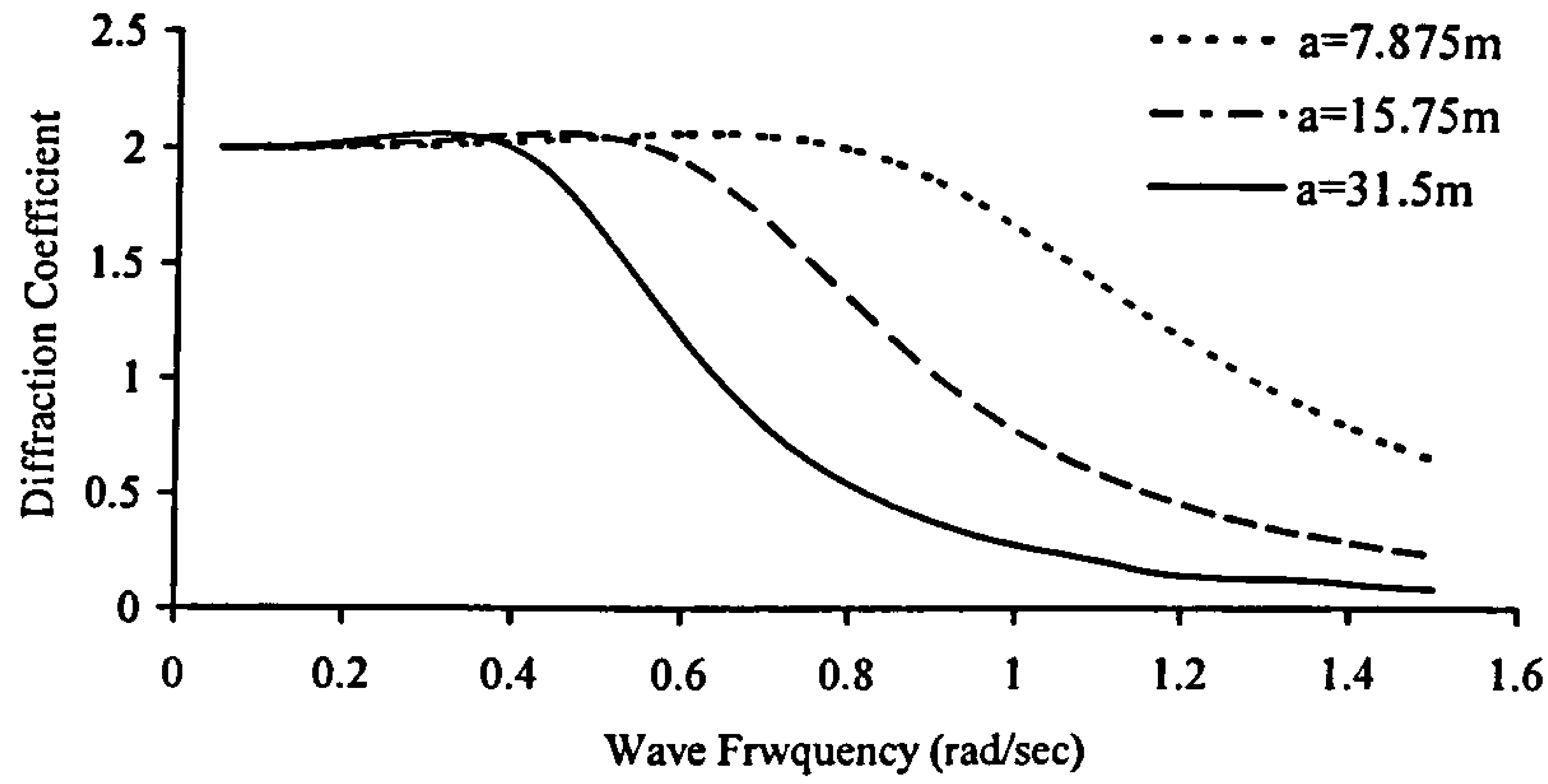

Figure 4.4: C_M coefficient

where

$$f_{SI} = C_{SI}\,\rho\pi a^2 \frac{H}{2} g\, k e^{kz} \cos\omega t\,, \tag{4.32a}$$

$$f_{SD} = C_{SD}\,\rho\pi a^2 \frac{H}{2} g\, k e^{kz} \sin\omega t\,. \tag{4.32b}$$

The total surge force per unit length, that is, the surge force per unit length due to incident and scattering potentials can be obtained by adding the Froude–Krylov force in (4.14a) to the scattering force in (4.29), so that

$$f_S = (C_{FK} + C_{SI})\rho\pi a^2 \frac{H}{2}\, gke^{kz}e^{-i\omega t} + C_{SD}\,\rho\pi a^2 \frac{H}{2}\, gke^{kz}ie^{-i\omega t} \tag{4.33}$$

or

$$f_S = \sqrt{(C_{FK} + C_{SI})^2 + C_{SD}^2}\, e^{i\tan^{-1}\frac{C_{SD}}{C_{FK}+C_{SI}}}\rho\pi a^2 \frac{H}{2}\, gke^{kz}e^{-i\omega t}\,. \tag{4.34}$$

From (4.24) it is easy to show that

$$\sin\beta = \frac{Y_1(ka)}{|H_1(ka)|}\,, \tag{4.35a}$$

$$\cos\beta = \frac{J_1(ka)}{|H_1(ka)|}\,. \tag{4.35b}$$

Then using (4.22), (4.24) and (4.35) it can be shown that

$$\frac{C_{SD}}{C_{FK}+C_{SI}} = \tan\alpha_1 \tag{4.36a}$$

and

$$\sqrt{(C_{FK}+C_{SI})^2 + C_{SD}^2} = C_M \tag{4.36b}$$

Therefore (4.34) is equivalent with the famous equation of linear diffraction theory for a circular cylinder in deep water given by (4.15a). From (4.36b), (4.34) and (4.14a) one can obtain the following relation among the magnitude of force components

$$|f_1| = \sqrt{(|f_{FK}| + |f_{SI}|)^2 + |f_{SD}|^2}. \tag{4.37}$$

By using (4.37), it is possible to find the effect of a change in the scattering forces f_{SI} and f_{SD} on the surge force while in the traditional form of (4.15a), the coefficients C_{FK}, C_{SI} and C_{SD} are combined in one coefficient, C_M, and it is impossible to study the effect of the scattering force alone on the surge force without deriving the scatter potential ϕ_S.

4.4 Improved load approximation

Now turning back to the region II of figure 4.1b, one difficulty in our approximate theory is that the extent of region II in vertical direction is unknown. A meaningful measure of the length of region II could be the cylinder diameter itself. Therefore, as an estimate, it is assumed that the length of region II is equal to the diameter of the truncated cylinder. Consequently, hereafter our approximate theory will be applicable to a truncated cylinder whose draft being greater than its radius. Further, it is assumed that the truncation section CD is located at the middle of region II. In other words, it is assumed that the interface of regions II and III, where the scattering force acting on a circular cylindrical control surface with radius a in figure 4.1b vanishes, is located at $z = -d - a$, that is,

$$f_S^t\Big|_{z=-(d+a)} = 0 \tag{4.38}$$

or

$$f_{SI}^t\Big|_{z=-(d+a)} = 0\,, \tag{4.39a}$$

$$f_{SD}^t\Big|_{z=-(d+a)} = 0\,. \tag{4.39b}$$

where superscript t is used to refer to the truncated cylinder. On the other hand, the magnitude of scattering force for a non-truncated cylinder in figure 4.1a at $z = -d - a$ is,

$$f_S|_{z=-(d+a)} = C_S\frac{H}{2}\,\rho\pi g a^2 k\mathrm{e}^{-k(d+a)}. \tag{4.40}$$

Or equivalently,

$$f_{SI}|_{z=-(d+a)} = C_{SI}\frac{H}{2}\,\rho\pi g a^2 k\mathrm{e}^{-k(d+a)}, \tag{4.41a}$$

$$f_{SD}|_{z=-(d+a)} = C_{SD}\frac{H}{2}\,\rho\pi g a^2 k\mathrm{e}^{-k(d+a)}. \tag{4.41b}$$

Now in region II the correction forces f_{CI} and f_{CD} can be introduced so that when they are added to scattering forces given in (4.41), the scattering forces of the truncated cylinder in (4.39) are obtained, therefore,

$$f_{CI}|_{z=-(d+a)} = -C_{SI}\frac{H}{2}\,\rho\pi g a^2 k\mathrm{e}^{-k(d+a)}, \tag{4.42a}$$

$$f_{CD}|_{z=-(d+a)} = -C_{SD}\frac{H}{2}\,\rho\pi g a^2 k\mathrm{e}^{-k(d+a)}. \tag{4.42b}$$

Now, as a simple approximation, we shall assume that the correction forces f_{CI} and f_{CD} decrease linearly from the values given in (4.42) at the truncation section to a zero value at the interface of regions II and I. Therefore, over

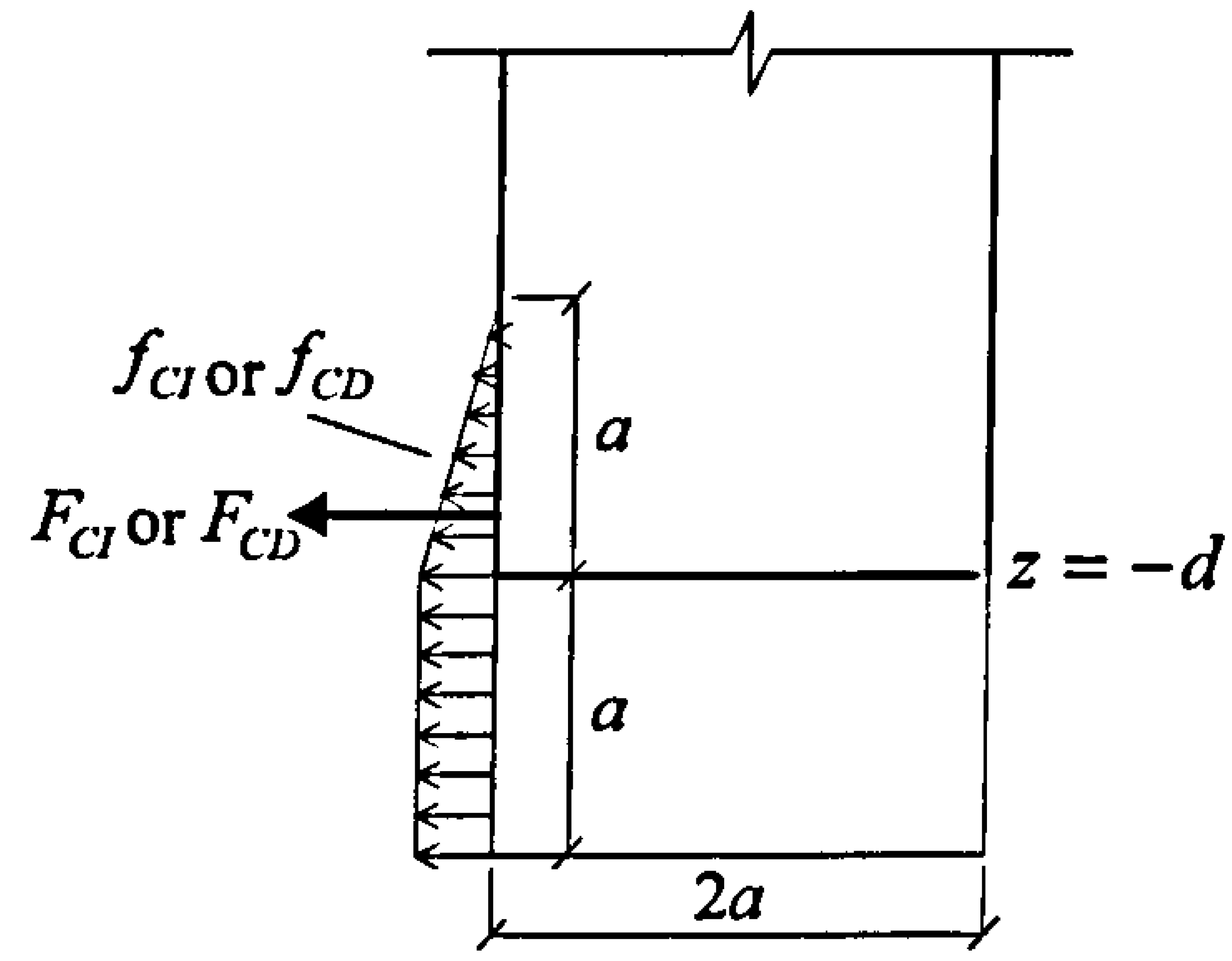

Figure 4.5: Load distribution of correction forces

region II f_{CI} and f_{CD} will have a distribution as shown in figure 4.5. Or mathematically,

$$\begin{Bmatrix} f_{CI} \\ f_{CD} \end{Bmatrix} = \begin{Bmatrix} C_{SI} \\ C_{SD} \end{Bmatrix} \frac{H}{2} \rho \pi g a^2 k e^{-k(d+a)} \frac{z+d-a}{a} \quad : \quad -d \leq z \leq -d+a. \tag{4.43}$$

The integrated form of (4.37) over the interval $z \in [-d,\ -d+a]$ can be written as

$$F_1 = \sqrt{(F_{FK} + F_{SI})^2 + F_{SD}^2}. \tag{4.44}$$

and if F_{CI} and F_{CD} denote the magnitude of total correction forces, then

$$F_1^t = [(F_{FK} + (\underbrace{F_{SI} + F_{CI}}_{F_{SI}^t}))^2 + (\underbrace{F_{SD} + F_{CD}}_{F_{SD}^t})^2]^{1/2} \tag{4.45}$$

where

$$F_{CI} = \int_{-d}^{-d+a} f_{CI}\, \mathrm{d}z = \frac{-H}{4} C_{SI}\, \rho \pi g a^3 k e^{-k(d+a)}, \tag{4.46a}$$

$$F_{CD} = \int_{-d}^{-d+a} f_{CD}\, \mathrm{d}z = \frac{-H}{4} C_{SD}\, \rho \pi g a^3 k e^{-k(d+a)}, \tag{4.46b}$$

$$F_{FK} = \int_{-d}^{-d+a} f_{FK}\, \mathrm{d}z = \frac{H}{2} C_{FK}\, \rho \pi g a^2 k e^{-kd}(e^{ka} - 1), \tag{4.46c}$$

$$F_{SI} = \int_{-d}^{-d+a} f_{SI}\, \mathrm{d}z = \frac{H}{2} C_{SI}\, \rho \pi g a^2 k e^{-kd}(e^{ka} - 1), \tag{4.46d}$$

$$F_{SD} = \int_{-d}^{-d+a} f_{SD}\, \mathrm{d}z = \frac{H}{2} C_{SD}\, \rho \pi g a^2 k e^{-kd}(e^{ka} - 1). \tag{4.46e}$$

Equation (4.44) gives the magnitude of surge force corresponding to the interval $[-d+a, -d]$ of the cylinder in figure 4.1a, or equivalently the surge force corresponding to the same interval of the truncated cylinder in figure 4.1b when the Niedzwecki & Duggal (1992) approximation is used. On the other hand by taking the effects of truncation into account, equation (4.45) improves the Niedzwecki & Duggal (1992) approximation in predicting the loads acting on a vertical truncated cylinder. Since based on our approximate theory F_1 and F_1^t for the region I are the same, the total correction force will be equal to,

$$F_C = F_1 - F_1^t. \tag{4.47}$$

Or,

$$\frac{F_C}{H/2} = \frac{F_1}{H/2} - \frac{F_1^t}{H/2} \tag{4.48}$$

gives the magnitude of surge correction force per unit wave amplitude. Figures 4.6 to 4.10 show the result of (4.48) overlaid with Weggel (1994) correction force and Chan (1990) diffraction program for a cylinder with 59.2m draft in 500m water depth for five different diameter to draft ratios. These results show close agreement between (4.48) and weggel's correction force given by (4.5) and (4.11) for $2a/d$ values of 0.53, 1.06, 1.2, 1.4 and 1.6. The results of (4.48) with respect to the results of Weggel (1994) correction force are even closer to the outputs of Chan (1990) diffraction program. At some high frequencies, the output of Chan's program has a sudden jump which is due to the irregular frequency phenomenon.

4.5 Approximation of pitch moment

Referring to figure (4.5) the pitch moment due to the surge correction force given by (4.48) about the origin of coordinate system at the free surface can be written as

$$M_c = F_c(d - a/3). \tag{4.49}$$

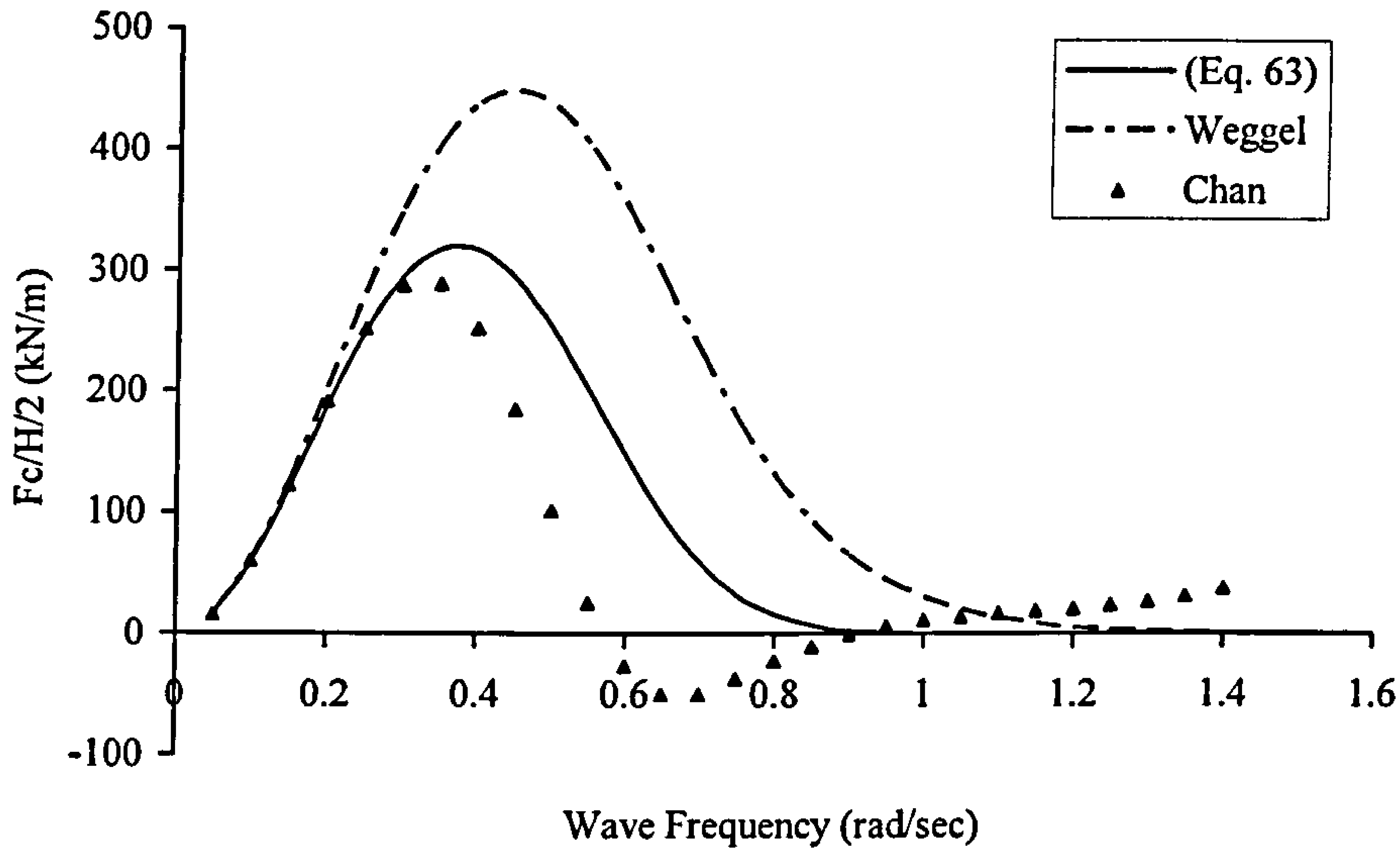

Figure 4.6: Modulus of surge correction force ($2a/d = 0.53$)

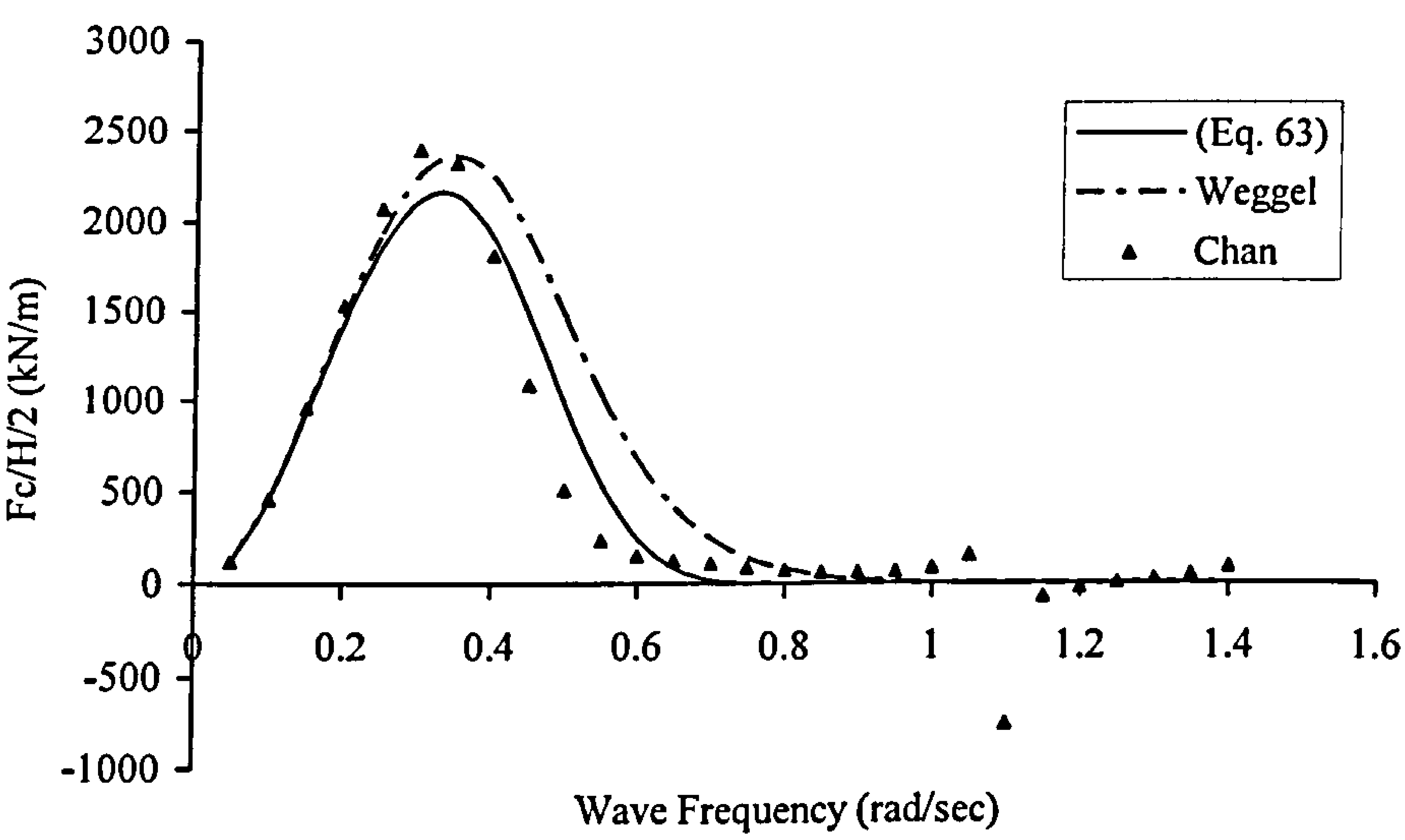

Figure 4.7: Modulus of surge correction force ($2a/d = 1.06$)

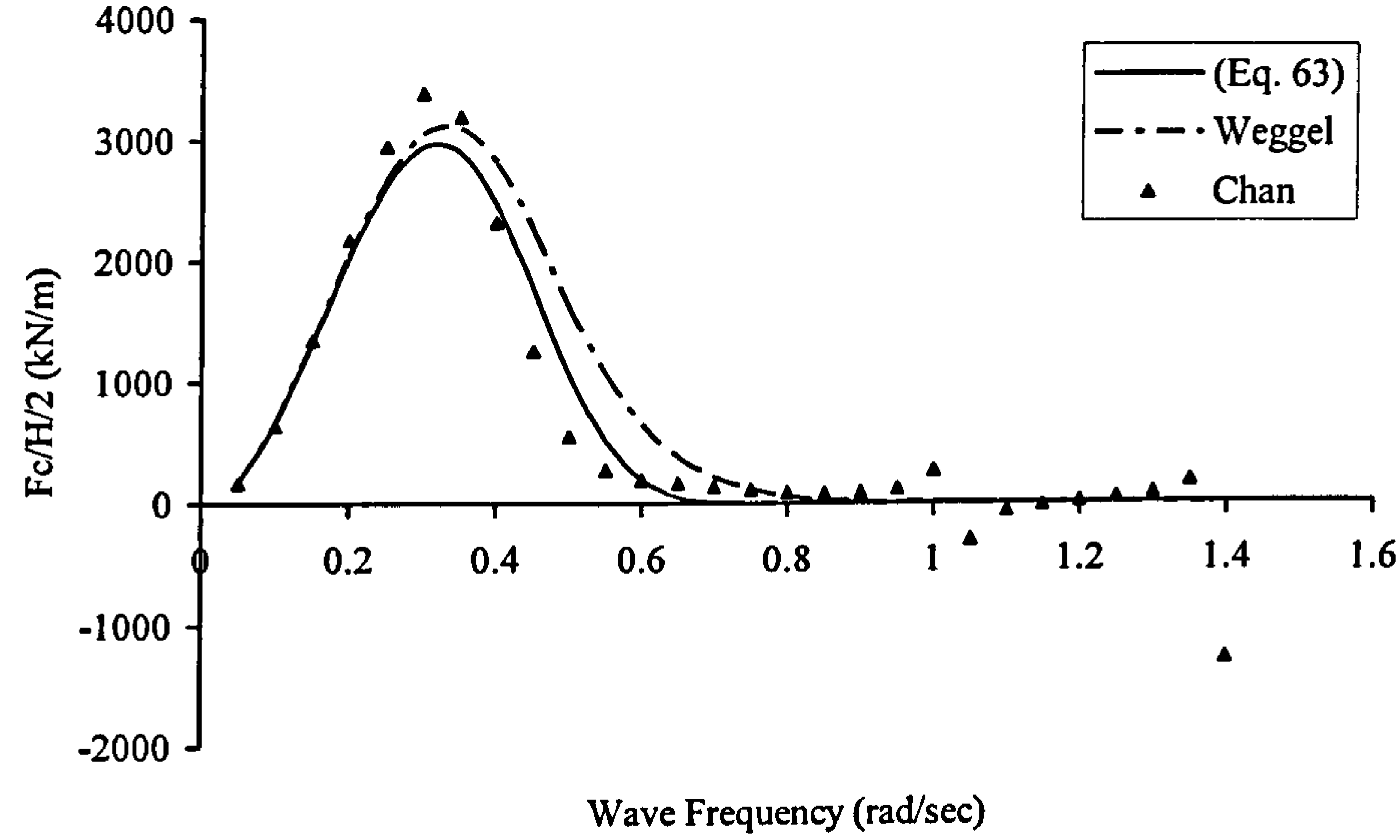

Figure 4.8: Modulus of surge correction force $(2a/d = 1.2)$

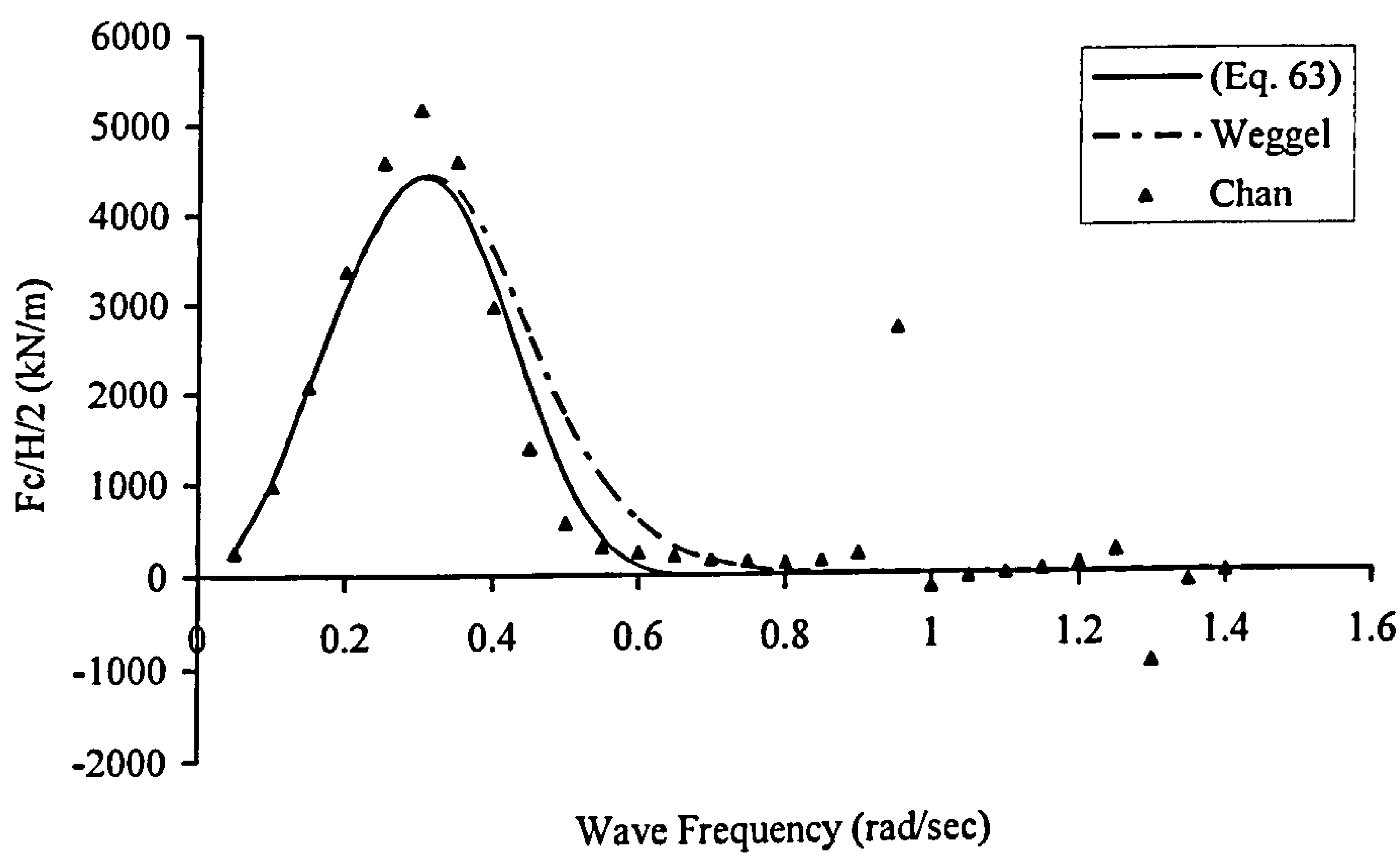

Figure 4.9: Modulus of surge correction force $(2a/d = 1.4)$

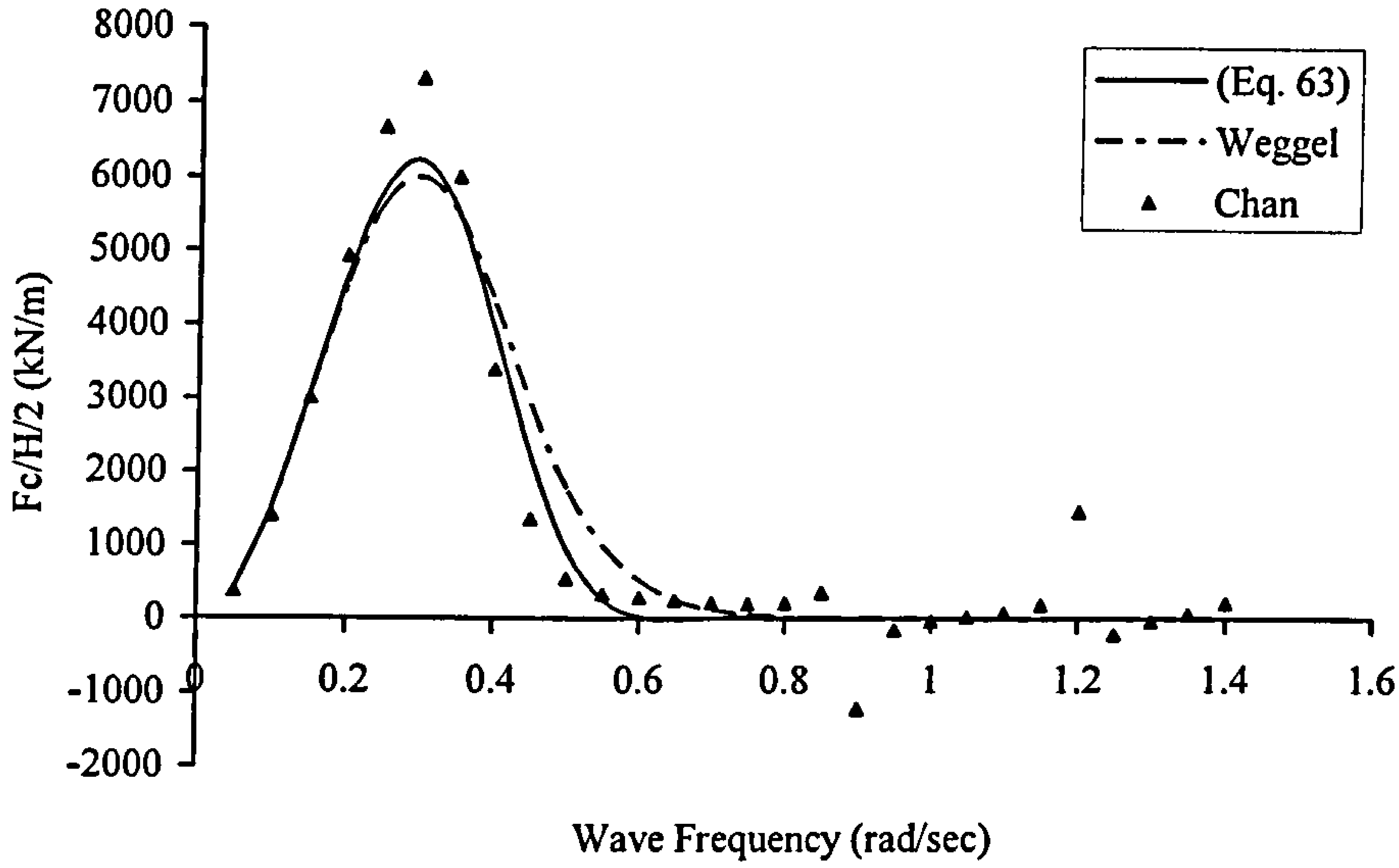

Figure 4.10: Modulus of surge correction force ($2a/d = 1.6$)

Because the correction force obtained in (4.48) is in opposite direction of the surge force given by (4.5), the correction moment in (4.49) must be subtracted from M_2 given in (4.6). Furthermore, on the bottom of a truncated cylinder there is an anti-symmetric pressure distribution, which produces a pitch moment in the direction of M_c. If the pressure at the bottom of truncated cylinder is approximated by the pressure due to incident waves, then for a wave node at origin as shown in figure 4.1 the expression of this moment is,

$$M_{2v} = \rho\pi g\, a^2 \frac{H}{2} e^{-kd} \frac{2J_2(ka)}{ka}\, a \cos\omega t. \tag{4.50}$$

Because the magnitude of correction moments M_c and M_{2v} is small with respect to the magnitude of the pitch moment predicted by (4.6), the effect of phase difference between M_c and M_{2v} can be ignored. Therefore, adding the magnitude of M_{2v} to the magnitude of M_c gives the magnitude of total correction pitch moment which must be subtracted from the moment derived in (4.6),

$$\frac{M_{2c}}{H/2} = \frac{F_c}{H/2}(d - a/3) + \rho\pi g\, a^3 \frac{H}{2} e^{-kd} \frac{2J_2(ka)}{ka}. \tag{4.51}$$

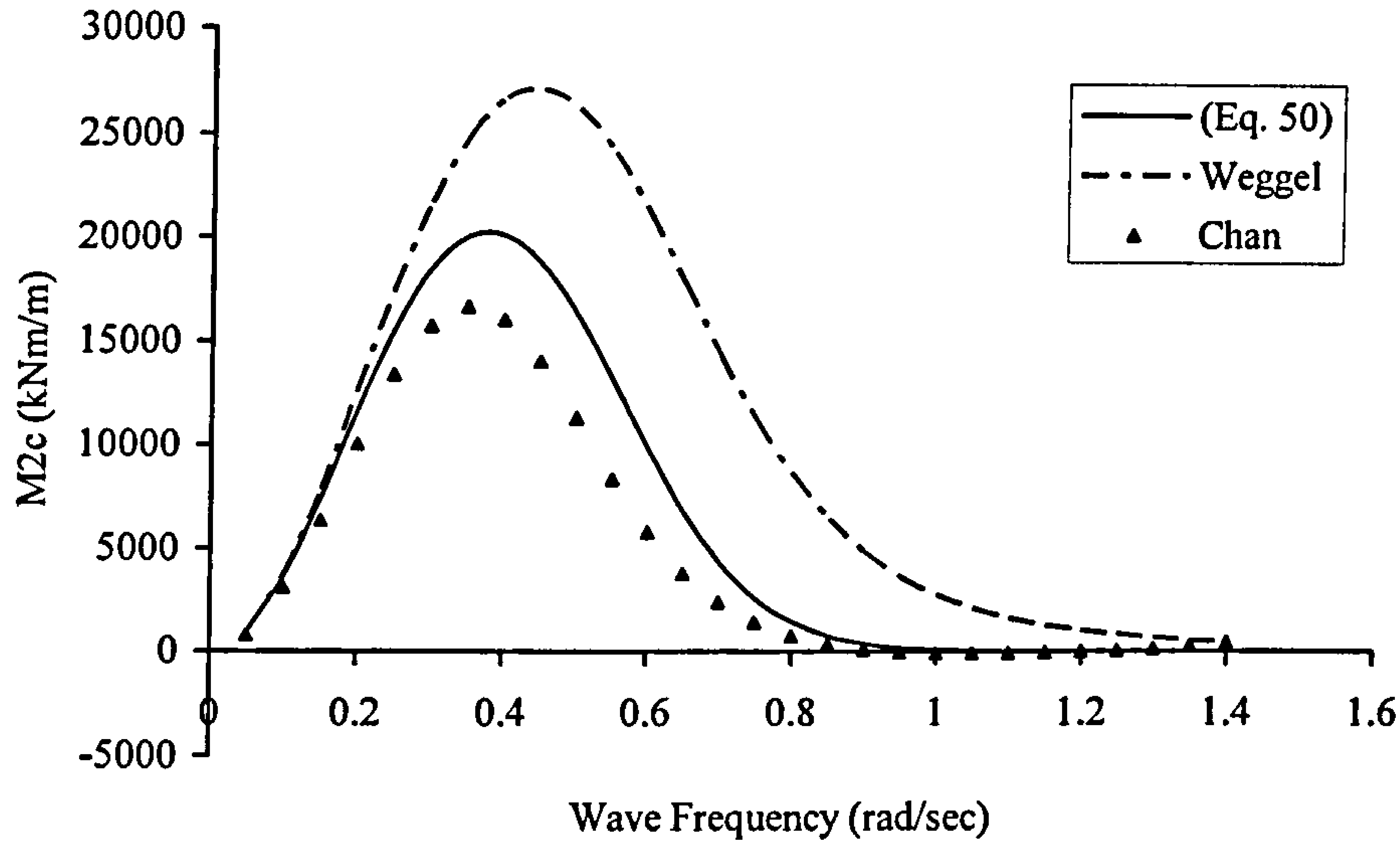

Figure 4.11: Modulus of pitch correction moment ($2a/d = 0.53$)

Alternatively, based on a parametric study of outputs of WAMIT (1991) program Weggel (1994) suggested the reduction factor

$$\left(1 - 0.15\frac{a}{d}\right)\left(1 - 0.4\frac{a}{d}e^{-1.62ka}\right) \tag{4.52}$$

to be multiplied by M_2 given in (4.6) to obtain the pitch moment acting on a truncated cylinder. Figures 4.11 to 4.15 show the results of (4.51) plotted versus Weggel's correction moment obtained by subtracting the product of (4.52) and (4.6) from (4.6) and outputs of Chan (1990) diffraction program. These figures indicate a very good agreement between the results of (4.51) and Chan's diffraction program for most of the diameter to draft ratios considered in here. Equation (4.51) also agrees closely with Weggel's correction moment for $2a/d$ values of 1.06, 1.2, 1.4 and 1.6. As in the case of surge force, results of (4.51) are even closer to Chan (1990) diffraction program.

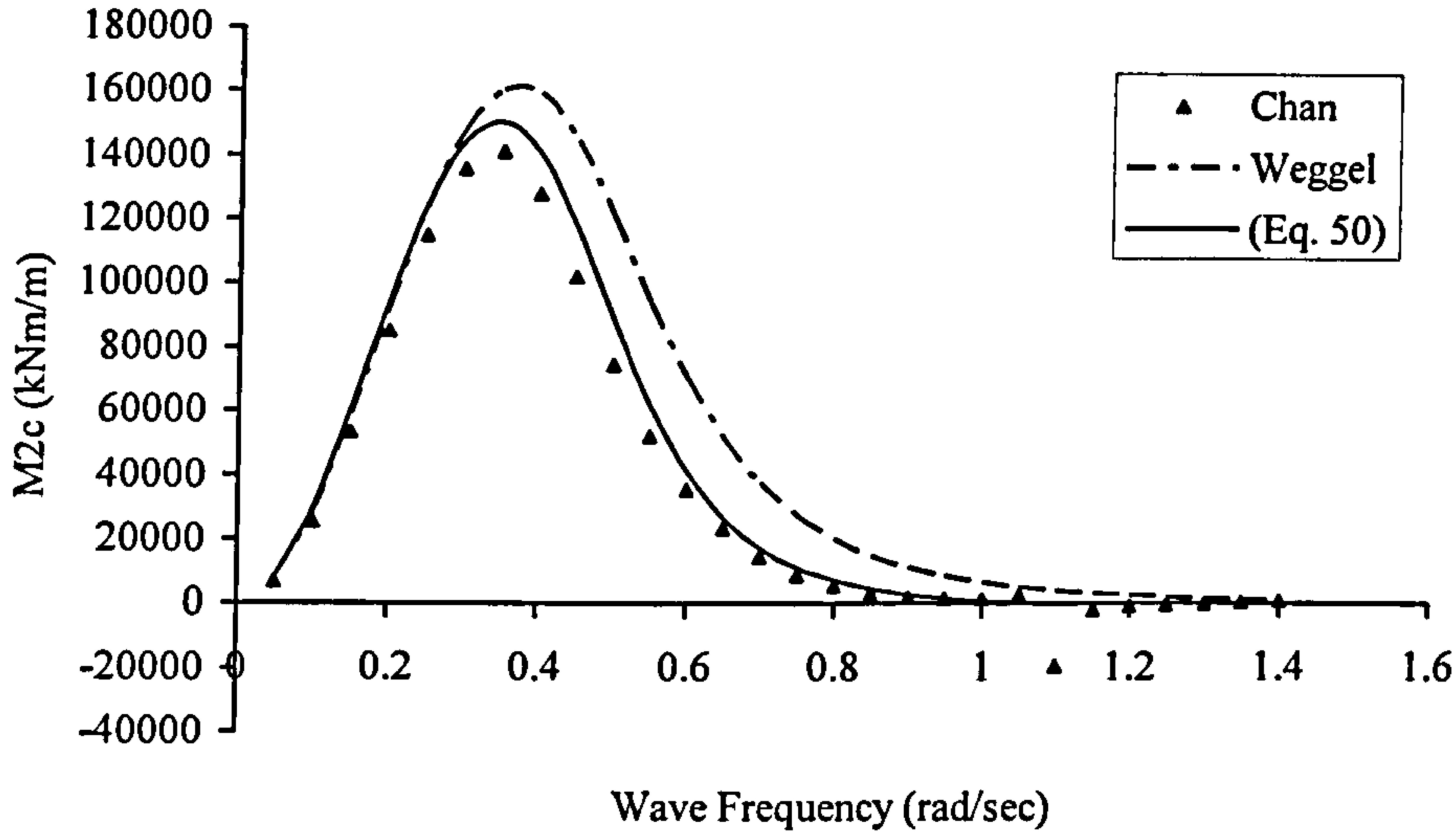

Figure 4.12: Modulus of pitch correction moment ($2a/d = 1.06$)

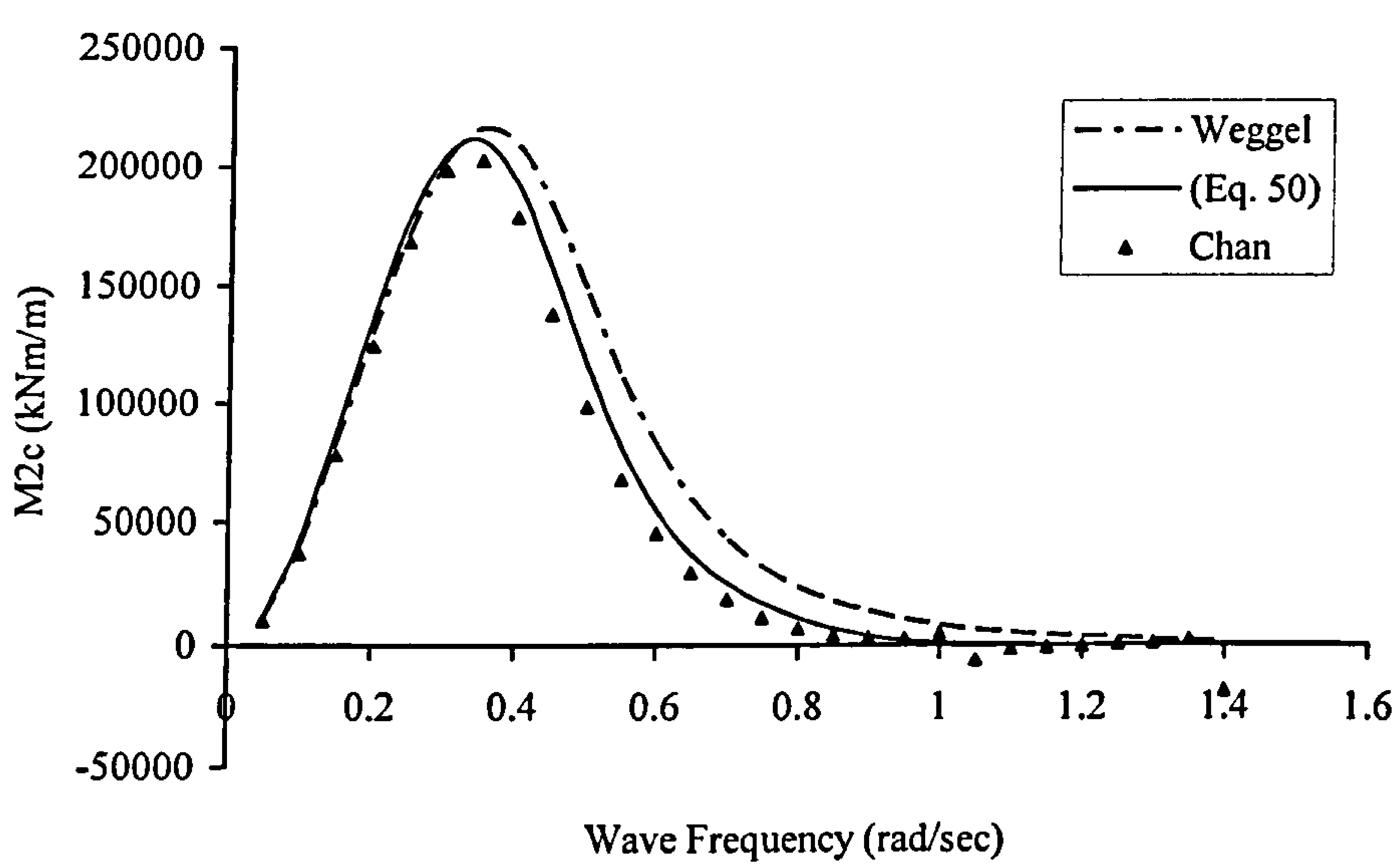

Figure 4.13: Modulus of pitch correction moment ($2a/d = 1.2$)

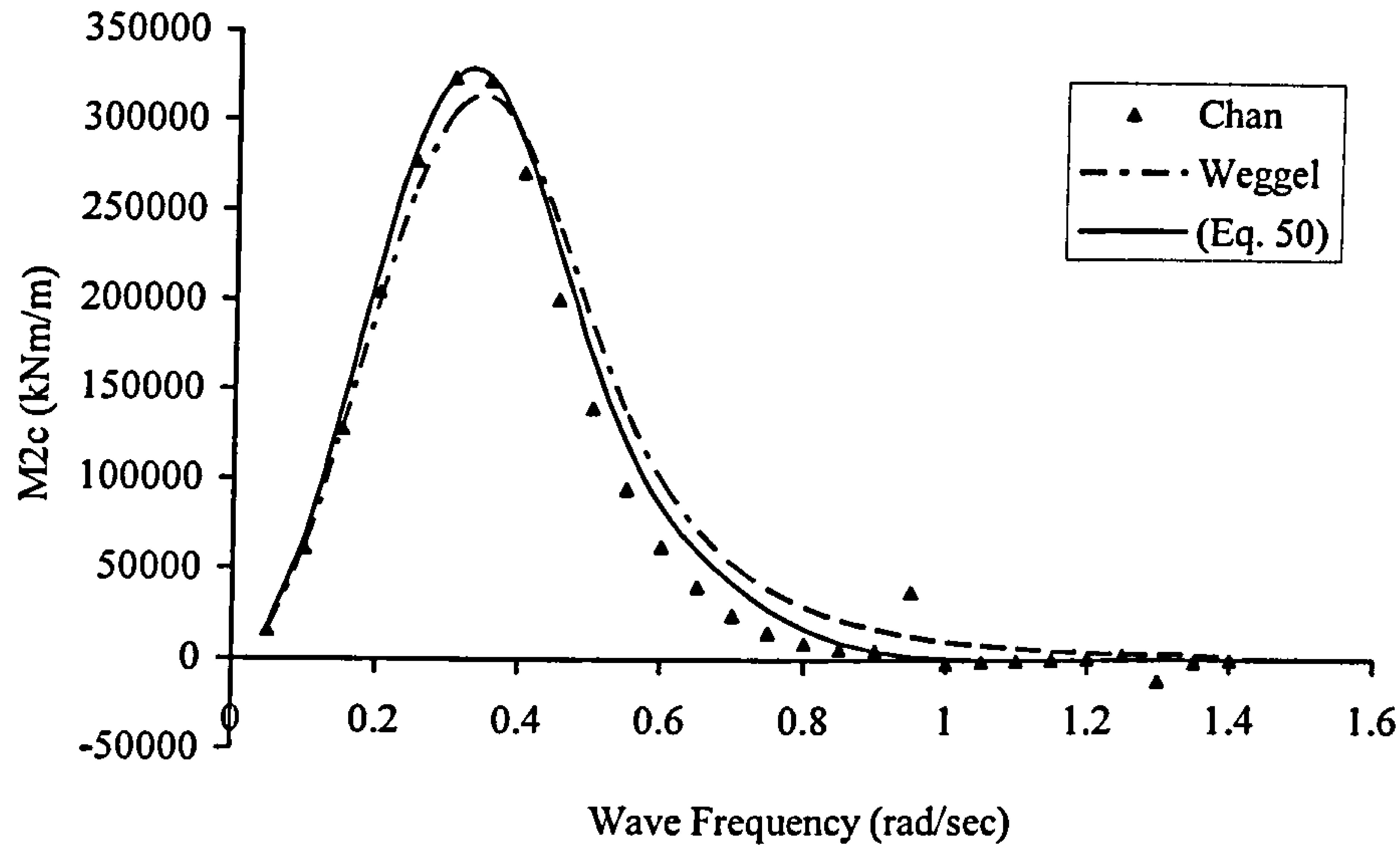

Figure 4.14: Modulus of pitch correction moment ($2a/d = 1.4$)

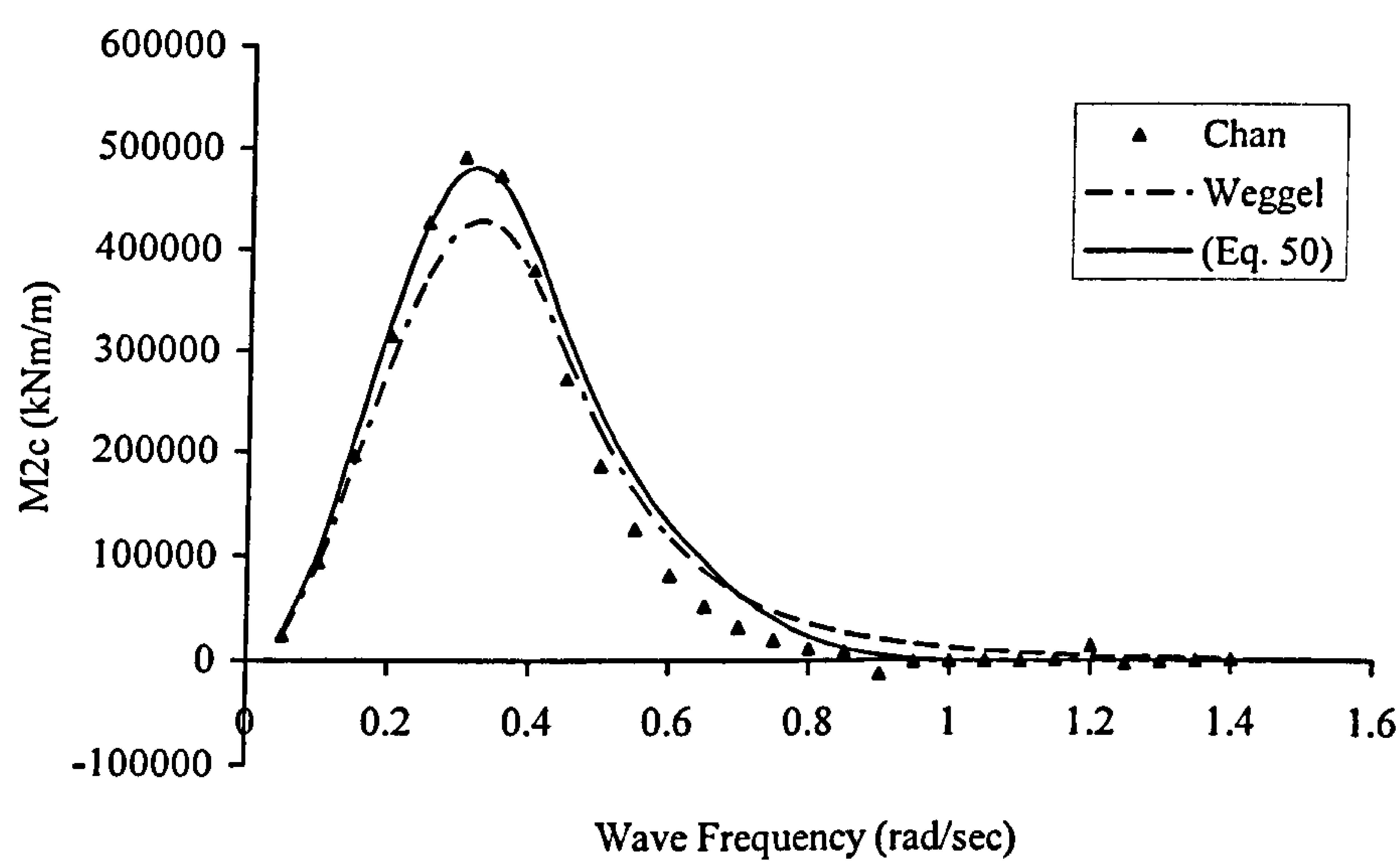

Figure 4.15: Modulus of pitch correction moment ($2a/d = 1.6$)

Chapter 5

Response Analysis of a Truss Spar by Transformation Method

5.1 Introduction

Having established the foundations of the transformation method in chapter 3, a model based on this method is introduced in this chapter which can be used for the response analysis of offshore platforms. This model is presented through the linear response analysis of a truss spar platform. In §§ 5.2 and 5.3 the derivation of linear equations of motion of the platform are presented. In § 5.4 the added mass coefficients of the platform are calculated by transformation approach given by equations (3.18) and (3.47). The linear excitation forces acting on the truss spar are derived in § 5.5. Then, effects of viscosity on the diffraction and radiation problems are studied and the viscous–radiation–diffraction model is introduced in § 5.6. In § 5.7 the non-linear equation of heave is solved by a method different from the conventional one and without any iteration. The coupled equations of surge and pitch is, however, solved by iteration in § 5.8. Finally in § 5.9, numerical results and their comparison with experimental data are presented.

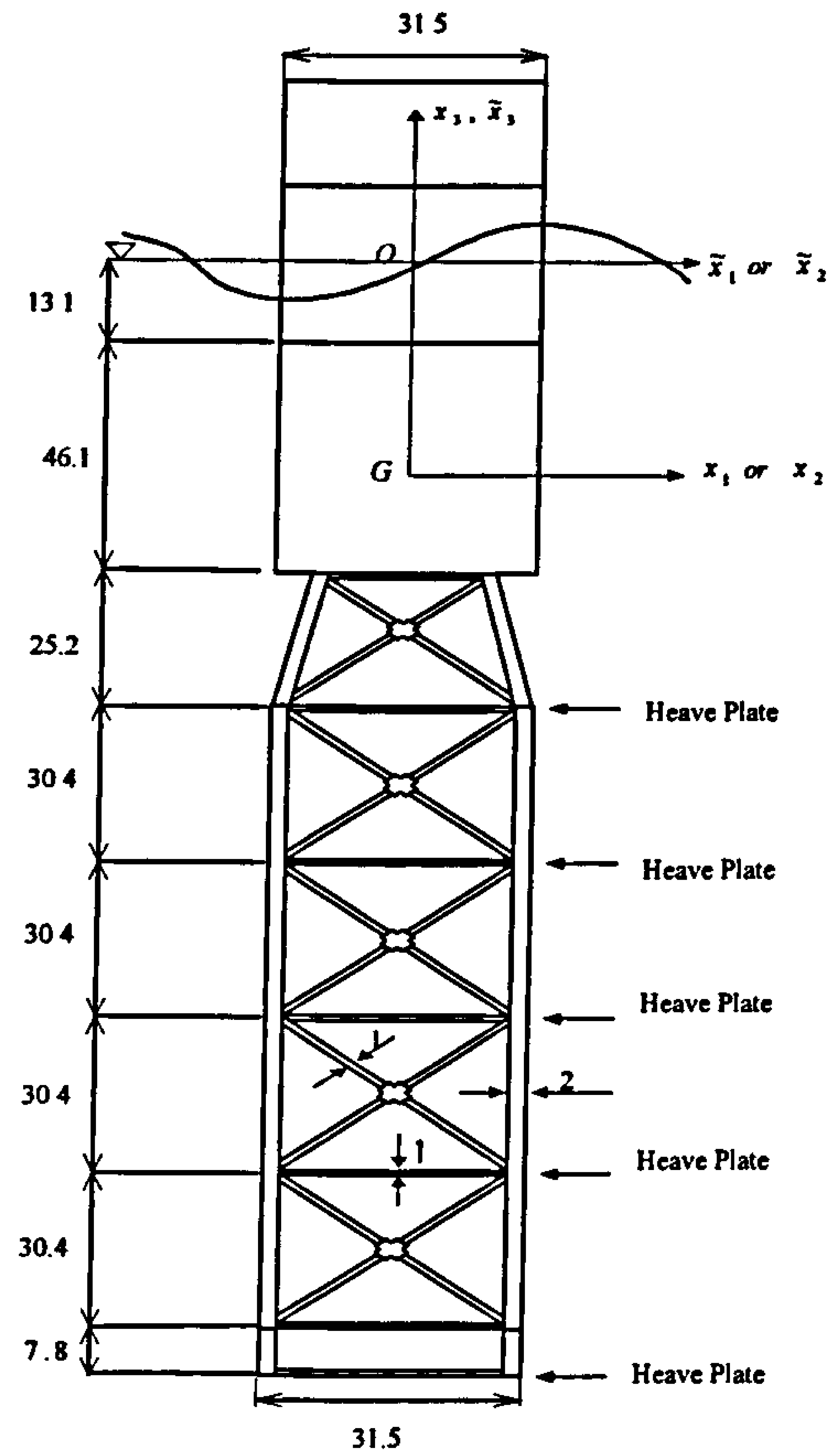

Figure 5.1: Geometry and Dimensions of the Truss Spar

5.2 Equations of motion

The geometry, dimensions and general particulars of the truss spar are given in figure 5.1 and table 5.1. A 1:100 scaled model of this platform has been the subject of few experimental studies (e.g., see Downie et al. 2000, Nygaard et al. 2000 and Stansberg et al. 2001). In the experiments, horizontal heave plates were used to minimize the heave motion. Basically small and large plates were used. Small solid plates were 29.5m×29.5m and large solid plates were 33.5m×33.5m. In figure 5.1 two coordinate systems are used. $Gx_1x_2x_3$ is a right-handed body-fixed coordinate system with its origin at the centre of gravity and its x_3-axis along the centreline of the body. $O\tilde{x}_1\tilde{x}_2\tilde{x}_3$ is a right-handed space-fixed coordinate system with its $\tilde{x}_1\tilde{x}_2$-plane coincident with the plane of the undisturbed free surface, and its $\tilde{x}_3$-axis vertically upward along the x_3-axis. Initially, the equations of motion of the truss spar are presented in

Particular	Unit	Full scale
Cylinder diameter	m	31.5
Cylinder height	m	97.2
Trusswork height	m	154.6
Freeboard	m	38.0
Vertical centre of gravity above base	m	157.0
Metacentric height	m	14.9
ROG[a] roll	m	75.5
ROG[a] pitch	m	75.5
Truss spar dry mass	tonnes	20100
Moment of inertia	tonnes-m^2	0.297×10^9
Mass of ballast water	tonnes	32000
Draft	m	213.8
Length small horizontal plate	m	29.5
Length large horizontal plate	m	33.5

[a]ROG=radius of gyration

Table 5.1: Vessel principal particulars (MARINTEK 2000)

the absence of viscosity, or more precisely in the context of a linear potential theory, and based on the radiation–diffraction problem of the floating body. In other words, the fluid is assumed to be incompressible and irrotational, and the body is modelled as a rigid body with six degrees of freedom.

In the context of a linear theory, it is assumed that the truss spar is composed of two separate bodies, that is, a surface-piercing cylindrical hull and a submerged truss. Also, the hydrodynamic interactions between the cylindrical hull and the truss, and among the various elements of the truss, are assumed to be of higher-order, and therefore are neglected. Because the truss spar is designed to operate in deep water and the truss is far below the free surface, as an approximation in the radiation problem, the truss is modelled as a body in an unbounded fluid. Consequently, the radiation damping due to its oscillations can be neglected, and its added mass coefficients can be assumed to be independent of frequency. Therefore, in the absence of viscosity, the effects of the truss will be confined to its influence on the mass, added mass, and excitation forces. Hence, from (2.36) the six linear coupled differential equations of motion, with the summation convention implied on repeated indices, can be written as

$$\left(\widetilde{M}_{\alpha\beta}+\widehat{M}_{\alpha\beta}+\tilde{a}_{\alpha\beta}+\hat{a}_{\alpha\beta}\right)\ddot{q}_{\beta}+\tilde{b}_{\alpha\beta}\,\dot{q}_{\beta}+\tilde{c}_{\alpha\beta}\,q_{\beta}=\tilde{Q}_{\alpha}+\hat{Q}_{\alpha}, \tag{5.1}$$

where the Greek indices range 1 to 6, and quantities with a tilde refer to the cylindrical hull, while quantities with a hat refer to the submerged truss. Equation (5.1) in matrix form can be written as

$$\left([\widetilde{M}]+[\widehat{M}]+[\tilde{a}]+[\hat{a}]\right)\{\ddot{q}\}+[\tilde{b}]\{\dot{q}\}+[\tilde{c}]\{q\}=\{\tilde{Q}\}+\{\hat{Q}\} \tag{5.2}$$

or more simply

$$([M]+[a])\{\ddot{q}\}+[\tilde{b}]\{\dot{q}\}+[\tilde{c}]\{q\}=\{Q\} \tag{5.3}$$

in which

$$[M]=[\widetilde{M}]+[\widehat{M}], \tag{5.4a}$$

$$[a]=[\tilde{a}]+[\hat{a}], \tag{5.4b}$$

$$\{Q\}=\{\tilde{Q}\}+\{\hat{Q}\}. \tag{5.4c}$$

where quantities without an accent refer to the whole structure. In (5.3) and (5.4), $[\widehat{M}]$, $[\widetilde{M}]$, and $[M]$ are generalized 6×6 mass matrices, $[\hat{a}]$, $[\tilde{a}]$, and $[a]$ are generalized 6×6 added mass matrices, $[\tilde{b}]$ and $[\tilde{c}]$ are radiation damping and hydrostatic restoring matrices, respectively, $\{\hat{Q}\}$, $\{\tilde{Q}\}$, and $\{Q\}$ are excitation vectors, and $\{q\}$ is the generalized displacement vector. Matrices $[\hat{a}]$, $[\tilde{a}]$, and $[\tilde{b}]$ are solutions of the radiation problem, while vectors $\{\hat{Q}\}$ and $\{\tilde{Q}\}$ are solutions of the diffraction problem of the truss spar. In all matrices and vectors, the subscripts 1, 2, 3, 4, 5, and 6 refer to surge, sway, heave, roll, pitch, and yaw, respectively. In (5.3) and (5.4), coefficients and forces relating to rotational degrees of freedom are computed with respect to a unique body coordinate system attached to the centre of gravity, G, of the whole structure.

For the truss spar shown in figure 5.1, two planes of $Gx_1x_2x_3$ coordinate system are planes of symmetry, that is, x_1x_3- and x_2x_3-plane. Therefore, from two symmetry rules derived in section 3.3.3 we can conclude that each matrix on the left–hand side of (5.2) and (5.3) has the following pattern

$$[A] = \begin{bmatrix} A_{11} & 0 & 0 & 0 & A_{15} & 0 \\ 0 & A_{22} & 0 & A_{24} & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 & 0 \\ 0 & A_{24} & 0 & A_{44} & 0 & 0 \\ A_{15} & 0 & 0 & 0 & A_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{66} \end{bmatrix} \tag{5.5}$$

in which $[A]$ could be any of the 6×6 matrices on the left–hand side of (5.2) or (5.3). In addition, because the origin of body–coordinate system is located at G, we have

$$M_{15} = M_{24} = 0$$

or,

$$[M] = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{66} \end{bmatrix}, \tag{5.6}$$

where m is the mass of the whole structure. Besides, due to surge, sway, and yaw, generally no hydrostatic restoring force will be produced, therefore, all elements in first, second, and sixth rows and columns of $[\tilde{c}]$ are equal to zero and from (5.5) we obtain

$$[\tilde{c}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{c}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{c}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{c}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.7}$$

Using (5.5) and substituting from (5.6) and (5.7) into (5.2) gives us

$$\begin{bmatrix} m+a_{11} & 0 & 0 & 0 & a_{15} & 0 \\ 0 & m+a_{22} & 0 & a_{24} & 0 & 0 \\ 0 & 0 & m+a_{33} & 0 & 0 & 0 \\ 0 & a_{24} & 0 & M_{44}+a_{44} & 0 & 0 \\ a_{15} & 0 & 0 & 0 & M_{55}+a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{66}+a_{66} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \\ \ddot{q}_5 \\ \ddot{q}_6 \end{Bmatrix} +$$
$$\begin{bmatrix} \tilde{b}_{11} & 0 & 0 & 0 & \tilde{b}_{15} & 0 \\ 0 & \tilde{b}_{22} & 0 & \tilde{b}_{24} & 0 & 0 \\ 0 & 0 & \tilde{b}_{33} & 0 & 0 & 0 \\ 0 & \tilde{b}_{24} & 0 & \tilde{b}_{44} & 0 & 0 \\ \tilde{b}_{15} & 0 & 0 & 0 & \tilde{b}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{b}_{66} \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{c}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{c}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{c}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} \tag{5.8}$$

Coefficient matrices on the left-hand side of (5.8) are symmetric and in each coefficient matrix only off–diagonal components 15, 51, 24 and 42 are non-zero. As a result, first and fifth, and also second and fourth degrees of freedom are coupled. In other words, surge and pitch are dependent only on each other and so are sway and roll. Also, heave and yaw are independent degrees of freedom.

Consequently, equation (5.8) may be reduced to the following four equations

$$\begin{bmatrix} m+a_{11} & a_{15} \\ a_{15} & M_{55}+a_{55} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_5 \end{Bmatrix} + \begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{15} \\ \tilde{b}_{15} & \tilde{b}_{55} \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_5 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \tilde{c}_{55} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_5 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_5 \end{Bmatrix} \tag{5.9a}$$

$$\begin{bmatrix} m+a_{22} & a_{24} \\ a_{24} & M_{44}+a_{44} \end{bmatrix} \begin{Bmatrix} \ddot{q}_2 \\ \ddot{q}_4 \end{Bmatrix} + \begin{bmatrix} \tilde{b}_{22} & \tilde{b}_{24} \\ \tilde{b}_{24} & \tilde{b}_{44} \end{bmatrix} \begin{Bmatrix} \dot{q}_2 \\ \dot{q}_4 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \tilde{c}_{44} \end{bmatrix} \begin{Bmatrix} q_2 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} Q_2 \\ Q_4 \end{Bmatrix} \tag{5.9b}$$

$$(m+a_{33})\,\ddot{q}_3 + \tilde{b}_{33}\,\dot{q}_3 + \tilde{c}_{33}\,q_3 = Q_3 \tag{5.9c}$$

$$(M_{66}+a_{66})\,\ddot{q}_3 + \tilde{b}_{66}\,\dot{q}_3 = Q_6 \tag{5.9d}$$

The heave, surge and pitch responses of the platform are of our interest, therefore, only (5.9a) and (5.9c) will be considered further.

5.3 Response analysis in heave, surge and pitch

As an approximation, in the radiation problem, it is assumed that the diameter of the hull is not too large to radiate significant waves. In addition, the natural frequencies of the truss spar are relatively low. Consequently, the radiation damping due to the hull can be neglected, and in the context of an irritational flow, the damping term disappears. Also, the added mass coefficients can be assumed to be independent of the frequency (Hooft 1971). Therefore, equations (5.9a) and (5.9c) will be simplified further as equations with constant coefficients, as follows

$$\begin{bmatrix} m+a_{11} & a_{15} \\ a_{15} & M_{55}+a_{55} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_5 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \tilde{c}_{55} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_5 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_5 \end{Bmatrix} \tag{5.10a}$$

and

$$(m+a_{33})\,\ddot{q}_3 + \tilde{c}_{33}\,q_3 = Q_3 \tag{5.10b}$$

For harmonic solutions we have

$$-\begin{bmatrix} \omega^2(m+a_{11}) & \omega^2 a_{15} \\ \omega^2 a_{15} & \omega^2(M_{55}+a_{55})-\tilde{c}_{55} \end{bmatrix} \begin{Bmatrix} q_{10} \\ q_{50} \end{Bmatrix} = \begin{Bmatrix} Q_{10} \\ Q_{50} \end{Bmatrix} \tag{5.11a}$$

and

$$\left[-\omega^2(m+a_{33})+\tilde{c}_{33}\right] q_{30} = Q_{30} \tag{5.11b}$$

The natural frequencies of the system, that is, the frequencies of the free motion, are nontrivial solutions of the homogeneous equations. Therefore,

$$\omega_{1n} = 0 \tag{5.12a}$$

$$\omega_{3n} = \sqrt{\frac{\tilde{c}_{33}}{m+a_{33}}} \tag{5.12b}$$

$$\omega_{5n} = \left[\frac{(m+a_{11})\tilde{c}_{55}}{(m+a_{11})(M_{55}+a_{55})-a_{15}^2}\right]^{0.5} \tag{5.12c}$$

As it is to be expected, in the absence of a mooring system, the surge natural frequency of the structure is equal to zero. The platform is moored by a four–line slack catenary system in 500-m depth (Stansberg et al. 2001). The mooring lines are modelled as linear springs and therefore the mooring forces in surge and pitch can be written as (Mekha et al. 1996)

$$\begin{Bmatrix} Q_1^m \\ Q_5^m \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{15} \\ k_{51} & k_{55} \end{bmatrix} = k_x \begin{bmatrix} 1 & \delta \\ \delta & \delta^2 \end{bmatrix} \tag{5.13}$$

in which $[k]$ is the linear mooring stiffness matrix, k_x is the horizontal stiffness, and δ is the distance from the centre of gravity to fairleads. The value of the horizontal stiffness is 15.5 KN/m (MARINTEK 2000). Adding the linear mooring force on the right-hand side of (5.13) to the radiation side of the unmoored equation of motion (5.10a) yields,

$$\begin{bmatrix} m+a_{11} & a_{15} \\ a_{15} & M_{55}+a_{55} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_5 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{15} \\ k_{51} & \tilde{c}_{55}+k_{55} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_5 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_5 \end{Bmatrix}. \tag{5.14}$$

	Heave			Pitch			
	SSP	LSP	F	SSP	LSP	F	Surge
Estimated	24.4	27.1	17.2	62.4	62.4	–	511
Measured	25.0	30.0	17.5	64.4	61.2	70.5	510
Deviation (%)	-2.4	-9.6	-1.7	-3.1	2.0	–	0.2

SSP, small solid heave plates; LSP, large solid heave plates; F, framework only

Table 5.2: Natural Periods

This is the coupled equation of surge and pitch for the moored truss spar. The natural frequencies of the moored vessel in surge and pitch are as follows:

$$\omega_{1n} = \sqrt{\frac{B - \sqrt{B^2 - 4AC}}{2A}}, \tag{5.15a}$$

$$\omega_{5n} = \sqrt{\frac{B + \sqrt{B^2 - 4AC}}{2A}}, \tag{5.15b}$$

where

$$A = [(m + a_{11})(M_{55} + a_{55}) - a_{15}^2], \tag{5.16a}$$

$$B = [k_{11}(M_{55} + a_{55}) + (k_{55} + \tilde{c}_{55})(m + a_{11}) - 2k_{15}\, a_{15}], \tag{5.16b}$$

$$C = [k_{11}(k_{55} + \tilde{c}_{55})\, k_{15}^2]. \tag{5.16c}$$

In Table 5.2, the natural periods in heave, surge, and pitch calculated from (5.12) and (5.16) are compared with the measured values reported by Stansberg et al. (2001).

5.4 Calculation of added mass coefficients by transformation approach

The added mass coefficients are determined from the solution of the radiation problem. In this regard, the truss is assumed to be an oscillating body in an infinite unbounded and otherwise quiescent fluid. As shown in chapter 3 (see also Sadeghi & Incecik 2005) the added mass coefficients of a rigid body (immersed or floating) forced to oscillate in otherwise calm water are components of three distinct Cartesian second–order tensors. As a result, the transformation laws (3.20) and (3.40) can be applied. To apply these equations, the truss is decomposed into a number of substructures. The main substructure is called the standard truss, which is composed of four identical bays (see Figure 5.1 in p. 74). Excluding the heave plates, which are placed at the bottom of each bay, all substructures of the truss are made of circular cylindrical members. This means that every element of a substructure can be obtained by rotating a single cylinderical element. Therefore, added mass coefficients of each element of a bay can be obtained by transformation of the coefficients of a typical horizontal circular cylinder, whose added mass coefficients are given in Appendix E. Using this technique, the added mass coefficients of one bay of the truss about its local axes is obtained. The results will be valid for the other three identical bays. Then, the parallel–axes theorem is used to derive the coefficients of each bay about the origin of the global coordinate system of the platform at its centre of gravity. The same method is used to add the contributions of the remaining parts of the truss, the heave plates and the hull to obtain the added mass coefficients of the whole structure. Details of this calculation are given by Sadeghi(2001). The added mass coefficients obtained from this method are used to calculate the natural periods of the platform. The good agreement obtained between the calculated and experimental results presented in table 5.2 indicates that the approach is valid and can achieve good accuracy. The larger discrepancy obtained for the large heave plates may be associated with viscous effects due to flow separation from the plate edges projecting beyond the footprint of the truss.

The largest contribution to the added mass is made by the hull, apart from the heave motion where the heave plates do so. In calculating the added mass components of the cylindrical members, the added mass coefficient, C_a, is assumed to be equal to unity which is the value predicted by the potential theory. The effect of viscosity on added mass coefficients is ignored because for the hull the Keulegan–Carpenter number is low, and for the truss the body is assumed to be oscillating in unbounded quiescent fluid. To calculate the added mass coefficient of a solid square plate in heave motion, the formula given by Meyerhoff (1970) is used,

$$a_{33} = 0.579\,\rho/4\,\pi\,a^3, \tag{5.17}$$

where a is the dimension of the side of the plate. The effect of the nonzero added moment of inertia coefficients of the heave plates is not considered in this study. As noted, added mass matrices of a typical horizontal circular cylinder are derived in Appendix E.

5.5 Calculation of excitation forces

The excitation forces are determined from the solution of the diffraction problem. In this regard, the cylindrical hull and the truss section of the platform are considered separately as two fixed structures subjected to waves. Since the truss is composed of slender members whose sizes are small with respect to the wave dimensions, their interaction with the waves may be considered to be negligible. On this basis, the force due to the waves can be assumed to be comprised of the sum of a Froude–Krylov force and an added mass force, or alternatively an inertia force, and the diffraction of the waves by the body is ignored. It would be difficult to justify this engineering approximation for the relatively large cylindrical hull, where the presence of the body affects the shape of the incident wave.

The total surge force and pitch moment acting on the hull have been approximated using the linear diffraction theory of McCamy & Fuchs (1954), based on the Niedzwecki & Duggal (1992) approximation for a truncated cylinder (see Chapter 4). On this basis, the force and moment can be written as,

$$\tilde{Q}_1 = \frac{2\rho g H_w}{k^2} A(kr)[1 - \mathrm{e}^{-kd}]\cos(\alpha_1 - \omega t) \tag{5.18a}$$

$$\tilde{Q}_5 = \frac{2\rho g H_w}{k^2} A(kr)[(\tilde{d} - \frac{1}{k}) + \mathrm{e}^{-kd}(\hat{d} + \frac{1}{k})]\cos(\alpha_5 - \omega t) \tag{5.18b}$$

where $\tilde{d}$ and $\hat{d}$ are the distance from G to the free surface and the bottom of the hull, respectively, and r is the radius of the hull. In deriving this equation, the water is assumed to be deep, and

$$A(kr) = [J_1'^2(kr) + Y_1'^2(kr)]^{-1/2} \tag{5.19a}$$

$$\alpha_1 = \tan^{-1}\left[\frac{J_1'(kr)}{Y_1'(kr)}\right] \tag{5.19b}$$

in which $J_1(kr)$ and $Y_1(kr)$ are first-order Bessel functions of the first and second kind, respectively. Following Weggel & Roesset (1994), the total heave force acting on the hull is assumed to be the product of a diffraction coefficient and the Froude–Krylov force, where the Froude–Krylov force can be obtained by integrating the dynamic pressure over the bottom of the hull. Therefore,

$$\tilde{Q}_3 = \rho g H_w \pi r^2 [1 - \frac{1}{2}\sin(kr)]\frac{J_1(kr)}{kr}\mathrm{e}^{-kd}\cos(\alpha_2 - \omega t) \tag{5.20}$$

where

$$\alpha_2 = 31.0\frac{\pi}{180}(kr)^{1.3} \tag{5.21}$$

As mentioned in § 3.4, the common approach in calculating the excitation forces acting on a structure composed of slender members, such as a truss, is the use of Morrison's equation (1950) with the normal acceleration components approach (Sarpkaya & Isaacson 1981, Chakrabarti 1987, Patel & Witz 1991). Because the added mass coefficients of the sub-structures of the truss have already been calculated with the use of transformation approach, it would be advantageous if the wave excitation and viscous forces acting on the truss can also be obtained by a method different from the Morison's equation (1950) and consistent with the transformation approach.

Since the truss section of the spar is far below the free surface, the added mass coefficients of its parts are independent of frequency. For the same reason, the added mass coefficient of a differential element at the top of a member can be assumed to be equal to that of a similar element at the bottom of that member. In addition, for the substructures considered, the $\tilde{x}_1\tilde{x}_3-$ and $\tilde{x}_2\tilde{x}_3-$ planes are planes of symmetry. Therefore, the inertia surge and heave forces can be approximated by the following formulae

$$\widehat{Q}_1 = \hat{a}_{11}\bar{\ddot{u}}_1 \tag{5.22a}$$

$$\widehat{Q}_3 = \hat{a}_{33}\bar{\ddot{u}}_3 \ . \tag{5.22b}$$

The terms $\hat{a}_{11}$ and $\hat{a}_{33}$ are the added mass coefficients of the sub-structure in question in surge and heave, respectively, obtained by the transformation method, and $\bar{\ddot{u}}_1$ and $\bar{\ddot{u}}_3$ are the mean values of the water particles' acceleration components acting on the substructure.

Therefore, in order to find surge and heave forces acting on the substructures of the truss, it is only necessary to find the mean values of the acceleration components $\ddot{u}_1$ and $\ddot{u}_3$ on them. The calculation simplifies still further because, for a circular cylinder, the Froude–Krylov force is equal to the acceleration force, so that only one of them need to be calculated explicitly to obtain the inertia forces acting on the truss. In other words, the inertia forces acting on each element or sub-structure of the truss are approximated by the product of the relevant inertia coefficient of that element and the mean value of the water particle's acceleration at the center of that element. The overall pitch moment about the centre of gravity of the platform is calculated as the sum of the pitch moments due to surge and heave forces acting on the sub-structures of the truss.

It should be mentioned that, the Froude–Krylov force acting on the heave plates is ignored because the displaced volume of the plates are negligible.

5.6 Viscous effects

In the solution presented so far, the simplifying assumption of incompressible irrotational flow allows the effects of viscosity to be omitted, and the motions to be predicted using a linear potential analysis. The effect of viscosity will now be considered in the context of a viscous-radiation-diffraction model. In other words, the influence of viscous effects on the radiation and diffraction problems of the truss spar will be dealt with separately.

5.6.1 Effect of viscosity in the diffraction problem

Two types of force normally acts on members of offshore structures. One is the potential force and the other is viscous force. The potential force can be evaluated by two general approaches, i.e., the radiation–diffraction theory or the inertia term of Morison's equation. The viscous force, however, has been evaluated primarily by the drag term of Morison's equation. Morison equation is a semi-empirical equation and is not valid for all structures. For the inertia force, the range of validity of Morison's equation can be stated by a non-dimensional parameter known as diffraction parameter $\pi D/L$ where D is the cylinder diameter and L is the wave length. For a $\pi D/L$ value greater than 0.5, diffraction effects are important and Morison's equation should not be used. Therefore for large diameter structures usually a diffraction analysis should be employed.

On the other hand, for blunt bodies like circular cylinders, the drag force is primarily due to the separation of flow and formation of substantial vortex flow. The drag force can be generated in the unidirectional flow more easily than the oscillating flow. This is because in oscillations the flow may not remain unidirectional long enough for the separation and the formation of large vorticies to be initiated (Patel 1989). A non-dimensional parameter for assessing this and the importance of viscous forces in oscillating flow is the relative amplitude or the viscosity parameter H/D where H is the wave height and D is the cylinder diameter. In deep water H/D represents the ratio of the water particle orbit width to cylinder diameter. Therefore, the larger H/D is, the more time is available for the separation and development of large vorticies and the greater the viscous drag force will be. The viscosity parameter is valid close to the water surface. A more rational non-dimensional parameter which is valid for any depth is Keulegan–Carpenter number, defined as KC= $U_m T/D$ where U_m is the maximum normal velocity in the oscillatory flow of period T. Experiments showed that for KC< 5 (Chakrabarti 1987) or for H/D< 1.5 (Patel 1989) viscous forces are very small and can be neglected. In addition for bodies with large πD/L values, H/D cannot be large owing to the height limitation for stable waves. Therefore as the diffraction becomes more important viscous forces becomes less important. With this introduction we shall now focus on the diffraction problem of the truss spar in the presence of viscosity.

In the viscous-diffraction problem, the platform is considered to be fixed and subjected to waves. With the configuration of slender members and heave plates deeply submerged and a large hull at the free surface, the truss spar produces almost no viscous forces when subjected to waves. This is firstly because the drag force is proportional to the square of the water particle velocity, which decays exponentially with depth, and secondly because the relative importance of the drag force decreases as the size of the body increases. The latter can be investigated in more detail. Assuming the wave spectrum as a JONSWAP wave spectrum with a significant wave height of 15m and a peak spectral period of 15sec, the maximum magnitude of the fluid particle velocity normal to the cylinder in the absence of the body will be 3.15 m/sec. Therefore for the cylindrical hull with 31.5m diameter the Keulegan–Carpenter number corresponding to the spectral peak period will be 1.5 which is well below KC= 5. This indicates that the viscous excitation force due to the hull can be neglected. For the deeply submerged truss, as noted, the square of the water particle velocity will be negligible and the viscous excitation force can be ignored. Consequently, the total drag forces due to incident and scattered waves acting on the stationary platform can be assumed to be negligible in comparison to the dominant terms. As a result, there will be no contribution from the drag forces on the right-hand side of motion equations.

5.6.2 Effect of viscosity in the radiation problem

In the viscous-radiation problem, because the viscous damping force is proportional to the square of the platform velocity relative to the fluid, this force is expected to be significant in particular for the truss section which is composed of slender members.

As yet there is no simple theoretical analysis for calculation of the viscous damping force. In the radiation problem, the definition of the viscosity parameter has to be changed from that given in the diffraction problem (H/D) to $2\delta/D$ where δ is the motion amplitude of the structure. It should be noted that in the radiation problem unlike the diffraction problem the relative amplitude has almost the same value in any depth. That is because the heave, surge and pitch velocities for different members of the structure are either the same or not much different. Therefore, the viscosity parameter in the radiation problem is almost as accurate as Keulegan–Carpenter number in the diffraction problem.

Now recalling that compliant offshore platforms are designed to have small first-order wave-frequency motions, it follows that the surge motion amplitude of a truss spar can be assumed to be an order of magnitude less than its diameter. Therefore, $2\delta/D$ for the hull will be much less than 1.5. Consequently, the viscous damping force due to the hull in the coupled motion of surge and pitch will be negligible. However, the hull viscous damping in heave motion can not be ignored because of flow separation from the sharp edges of the underside of the hull.

Now we shall consider the damping effects of the truss. As mentioned in the potential radiation problem, the radiated waves generated by the hull can be assumed to be not important. In addition, the truss section is immersed far below the free surface. In deep water, therefore, the radiated waves die away before reaching to the truss. This is the assumption implicitly made in the potential problem that allowed us to consider the truss as a structure oscillating in unbounded fluid. Thus, assuming that, viscous damping forces acting on an immersed body oscillating in an otherwise calm water can be approximated by the viscous forces acting on the fixed body subjected to an oscillating flow where the flow velocity is the same as the velocity of the platform, the viscous damping force can be approximated by the drag term of the original Morison's equation as follows

$$F_3^d = B_{33}|\dot{q}_3|\dot{q}_3 \tag{5.23a}$$

and

$$\begin{Bmatrix} F_1^d \\ F_5^d \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{15} \\ B_{51} & B_{55} \end{bmatrix} \begin{Bmatrix} |\dot{q}_1|\dot{q}_1 \\ |\dot{q}_5|\dot{q}_5 \end{Bmatrix}. \tag{5.23b}$$

To be consistent with the transformation approach, the viscous damping coefficient of the hull and the truss structure are calculated by the simple "projected area" method rather than the Morison's equation and the cross-flow approach. For instance, based on the projected-area method, the damping coefficient of the platform in heave in the case of large heave plates may be written as

$$B_{33} = \frac{1}{2}\rho[n_p C_d^p A_p + n_{XB} C_d^{XB} A_{XB}] + B_{33}^h \tag{5.24}$$

where indices p and XB are for the heave plates and X-bracing, respectively, A_p and A_{XB} are projected areas in the heave direction, n_p and n_{XB} are the number of heave plates and X-bracings in the structure, and B_{33}^h is the damping coefficient of the hull.

The viscous damping forces given by (5.23a) and (5.23b) can simply be added to the radiation side of the inviscid equation of motions (5.10b) and (5.14) to take viscous effects into account. Therefore,

$$(m + a_{33})\ddot{q}_3 + B_{33}|\dot{q}_3|\dot{q}_3 + \tilde{c}_{33}q_3 = Q_3 \tag{5.25a}$$

and

$$\begin{bmatrix} m + a_{11} & a_{15} \\ a_{15} & M_{55} + a_{55} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_5 \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{15} \\ B_{51} & B_{55} \end{bmatrix} \begin{Bmatrix} |\dot{q}_1|\dot{q}_1 \\ |\dot{q}_5|\dot{q}_5 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{15} \\ k_{51} & \tilde{c}_{55} + k_{55} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_5 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_5 \end{Bmatrix}. \tag{5.25b}$$

In contrast to the common drag terms for moving bodies, based on the relative velocity, the drag terms in these equations are separated into two terms, one containing the structure velocity only, and the other containing the water particle velocity only. As noted in § 5.6.1, the latter is neglected here owing to the particular configuration of the truss spar.

In estimating the damping coefficients of the truss, the drag coefficients of cylindrical members are chosen to be equal to 0.7, and for a square plate aligned normal to the plane of motion, C_D is assumed to be equal to 2.0, which is the minimum value suggested by Prislin et al. (1998). The drag coefficient for the hull in heave motion has been approximated as being half the drag coefficient of a circular disk reported by Hoerner (1965), and which is equal to 0.57.

5.7 Solution of the heave equation of motion

Equation (5.25a) is nonlinear owing to the viscous damping term. The usual frequency domain technique for solving this equation is to linearize the viscous damping term and then solve the approximate equation of motion iteratively (Patel & Witz 1991, Berthelsen 2000). Assuming q_3 and Q_3 to be

$$q_3 = q_{30}\cos(\omega t - \beta_3) \tag{5.26a}$$

$$Q_3 = Q_{30}\cos(\omega t - \theta_3), \tag{5.26b}$$

equation (5.25a) can be written as

$$\begin{aligned}&[\tilde{c}_{33} - \omega^2(m + a_{33})]q_{30}\cos(\omega t - \beta_3) - \\ &\omega^2 B_{33} q_{30}^2 |\sin(\omega t - \beta_3)| \sin(\omega t - \beta_3) = Q_{30}\cos(\omega t - \theta_3)\end{aligned} \tag{5.27}$$

Using the first term of the Fourier series expansion of $|\sin(\omega t-\beta_3)|\sin(\omega t-\beta_3)$, this nonlinear term can be approximated as

$$|\sin(\omega t - \beta_3)|\sin(\omega t - \beta_3) \approx \frac{8}{3\pi}\sin(\omega t - \beta_3) \tag{5.28}$$

Therefore, an approximate form of (5.27) can be written as

$$\begin{aligned}&[\tilde{c}_{33} - \omega^2(m + a_{33})]q_{30}\cos(\omega t - \beta_3) - \\ &\frac{8}{3\pi}\omega^2 B_{33} q_{30}^2 \sin(\omega t - \beta_3) = Q_{30}\cos(\omega t - \theta_3)\end{aligned} \tag{5.29}$$

This equation is still nonlinear since the damping term is proportional to the square of the motion amplitude. If an equivalent damping term B_{33}^{eq} is defined as

$$B_{33}^{\text{eq}} = \frac{8}{3\pi}\omega B_{33} q_{30}, \tag{5.30}$$

then (5.29) can be written as

$$(m + a_{33})\ddot{q}_3 + B_{33}^{\text{eq}}\dot{q}_3 + \tilde{c}_{33} q_3 = Q_3 \tag{5.31}$$

which looks like a linear differential equation, and therefore the approximation of $B_{33}|\dot{q}_3|\dot{q}_3$ by $B_{33}^{\text{eq}}\dot{q}_3$ is somtimes referred to as linearization in the literature, although, as has been mentioned, $B_{33}^{\text{eq}}\dot{q}_3$ is still a nonlinear term with respect to the motion amplitude.

The mathematical approximation used in (5.28) is physically equivalent to the approximation of the energy dissipated by the drag force $B_{33}|\dot{q}_3|\dot{q}_3$ at every frequency ω, with the energy dissipated by the equivalent damping force $B_{33}^{eq}\dot{q}_3$ at that frequency. A particular case of this approximation is when the energy dissipated by the nonlinear damping force is approximated by that of the equivalent damping force only at the resonance frequency. In this case, B_{33}^{eq} is

$$B_{33}^{eq} = \frac{8}{3\pi}\omega_{3n} B_{33} q_{30} \tag{5.32}$$

where ω_{3n} is the natural frequency in heave. The approximation given by (5.30) rather than the one given by (5.32) is used here since it is more general.

As pointed out, the conventional method of solving (5.29) or its equivalent, (5.31), is by iteration, however, it can be solved without any iteration. In complex notation (5.29) can be written as

$$\begin{aligned} &\left[\tilde{c}_{33} - \omega^2(m + a_{33})\right] q_{30}\,\mathrm{Re}\left\{e^{-i(\omega t-\beta_3)}\right\} - \frac{8}{3\pi}\omega^2 B_{33} q_{30}^2\,\mathrm{Re}\left\{i\,e^{-i(\omega t-\beta_3)}\right\} \\ &= Q_{30}\,\mathrm{Re}\left\{e^{-i(\omega t-\theta_3)}\right\} \end{aligned} \tag{5.33a}$$

or

$$\begin{aligned} \mathrm{Re}\Big\{\left[\tilde{c}_{33} - \omega^2(m + a_{33})\right] q_{30}e^{-i\omega t}e^{i\beta_3} - \frac{8}{3\pi}i\omega^2 B_{33} q_{30}^2 e^{-i\omega t}e^{i\beta_3} - \\ Q_{30}e^{-i\omega t}e^{i\theta_3}\Big\} = 0, \end{aligned} \tag{5.33b}$$

or

$$\mathrm{Re}\left\{\left(\left[\tilde{c}_{33} - \omega^2(m + a_{33})\right] q_{30}e^{i\beta_3} - i\omega^2\bar{B}_{33}q_{30}^2e^{i\beta_3} - Q_{30}e^{i\theta_3}\right)e^{-i\omega t}\right\} = 0 \tag{5.33c}$$

where $\bar{B}_{33} = 8B_{33}/3\pi$. Assuming that

$$\begin{aligned} Z_1 &= \left[\tilde{c}_{33} - \omega^2(m + a_{33})\right] q_{30}e^{i\beta_3} - i\omega^2\bar{B}_{33}q_{30}^2e^{i\beta_3} - Q_{30}e^{i\theta_3} \\ Z_2 &= e^{-i\omega t}, \end{aligned} \tag{5.34}$$

equation (5.33c) can be written as

$$\operatorname{Re}(Z_1 Z_2) = 0 \tag{5.35}$$

which yields

$$\operatorname{Re}(Z_1)\cos\omega t + \operatorname{Im}(Z_1)\sin\omega t = 0. \tag{5.36}$$

In order for the above equation to be true for an arbitrary ω, it is necessary that $\operatorname{Re}(Z_1) = 0$ and $\operatorname{Im}(Z_1) = 0$. Or equivalently,

$$Z_1 = 0. \tag{5.37}$$

Now substituting for Z_1 in (5.34), yields

$$\left[\tilde{c}_{33} - \omega^2(m + a_{33})\right] q_{30}e^{i\beta_3} - i\omega^2 \bar{B}_{33} q_{30}^2 e^{i\beta_3} = Q_{30}e^{i\theta_3}$$

or

$$\left[\tilde{c}_{33} - \omega^2(m + a_{33})\right] q_{30} - i\omega^2 \bar{B}_{33} q_{30}^2 = Q_{30}e^{i(\theta_3 - \beta_3)}. \tag{5.38}$$

In this equation Q_{30} is the magnitude of the heave force due to a wave of height H_w, and therefore q_{30} is the heave response and not the response amplitude operator (RAO) in heave. Taking the modulus of both sides of (5.38) yields

$$\sqrt{\left[\tilde{c}_{33} - \omega^2(m + a_{33})\right]^2 q_{30}^2 + \left[\omega^2 \bar{B}_{33}\right]^2 q_{30}^4} = Q_{30}$$

or

$$\left[\omega^2 \bar{B}_{33}\right]^2 q_{30}^4 + \left[\tilde{c}_{33} - \omega^2(m + a_{33})\right]^2 q_{30}^2 - Q_{30}^2 = 0. \tag{5.39}$$

Introducing the substitution

$$x = q_{30}^2 \tag{5.40}$$

allows (5.39) to be written as a quadratic equation in x, which can be solved without iteration.

$$ax^2 + bx - c = 0 \tag{5.41}$$

in which

$$\begin{aligned} a &= \left[\omega^2 \bar{B}_{33}\right]^2 \\ b &= \left[\tilde{c}_{33} - \omega^2(m + a_{33})\right]^2 \\ c &= Q_{30}^2. \end{aligned} \tag{5.42}$$

Therefore,

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}.$$

Finally, from (5.40) we obtain

$$q_{30} = \sqrt{\frac{-b + \sqrt{b^2 + 4ac}}{2a}}. \tag{5.43}$$

The RAO in heave now becomes

$$\mathrm{RAO} = \sqrt{\frac{-b + \sqrt{b^2 + 4ac}}{2a}} \Big/ (H_w/2). \tag{5.44}$$

The RAO defined in (5.44) is obtained by direct solution of the approximate nonlinear equation of motion for a particular wave height, as opposed to the more traditional RAO defined as the solution of the linear equation of motion for unit wave amplitude. Since the force term on the right-hand side of (5.38) varies with wave height, the RAO defined in this way, unlike the traditional RAO, also varies with wave height. For the present purposes, it will therefore be termed a pseudo–response amplitude operator or in shorthand a PRAO.

If a linear damping as well as a nonlinear viscous damping exists, then in terms of motion amplitude, the equation of motion will be a quartic equation rather than a quadratic equation, which can be solved by a technique which is different from the one presented here.

5.8 Solution of the coupled equation of surge and pitch

Using the same approximation for surge and pitch as heave, that is,

$$|\sin(\omega t-\beta_1)|\sin(\omega t-\beta_1)\approx\frac{8}{3\pi}\sin(\omega t-\beta_1) \tag{5.45a}$$
$$|\sin(\omega t-\beta_5)|\sin(\omega t-\beta_5)\approx\frac{8}{3\pi}\sin(\omega t-\beta_5), \tag{5.45b}$$

equation (5.25b) in complex form can be written as

$$\begin{bmatrix} k_{11}-\omega^2(m+a_{11}) & k_{11}\delta-\omega^2 a_{15} \\ k_{11}\delta-\omega^2 a_{15} & \tilde{c}_{55}+k_{11}\delta^2-\omega^2(M_{55}+a_{55}) \end{bmatrix}\begin{Bmatrix} q_{10}\mathrm{e}^{\mathrm{i}\beta_1} \\ q_{50}\mathrm{e}^{\mathrm{i}\beta_5} \end{Bmatrix}$$
$$-\mathrm{i}\omega^2\frac{8}{3\pi}\begin{bmatrix} B_{11} & B_{15} \\ B_{51} & B_{55} \end{bmatrix}\begin{Bmatrix} q_{10}^2\mathrm{e}^{\mathrm{i}\beta_1} \\ q_{50}^2\mathrm{e}^{\mathrm{i}\beta_5} \end{Bmatrix}=\begin{Bmatrix} Q_{10}\mathrm{e}^{\mathrm{i}\theta_1} \\ Q_{50}\mathrm{e}^{\mathrm{i}\theta_5} \end{Bmatrix} \tag{5.46}$$

or more simply as

$$\begin{bmatrix} C_{11} & C_{15} \\ C_{15} & C_{55} \end{bmatrix}\begin{Bmatrix} q_{10}\,\mathrm{e}^{\mathrm{i}\beta_1} \\ q_{50}\,\mathrm{e}^{\mathrm{i}\beta_5} \end{Bmatrix}-\mathrm{i}\begin{bmatrix} D_{11} & D_{15} \\ D_{51} & D_{55} \end{bmatrix}\begin{Bmatrix} q_{10}^2\,\mathrm{e}^{\mathrm{i}\beta_1} \\ q_{50}^2\,\mathrm{e}^{\mathrm{i}\beta_5} \end{Bmatrix}=\begin{Bmatrix} Q_{10}\,\mathrm{e}^{\mathrm{i}\theta_1} \\ Q_{50}\,\mathrm{e}^{\mathrm{i}\theta_5} \end{Bmatrix} \tag{5.47}$$

where matrices $[C]$ and $[D]$ are defined in (5.46). Equation (5.47) can be written as

$$\begin{bmatrix} C_{11} & C_{15} \\ C_{15} & C_{55} \end{bmatrix}\begin{Bmatrix} q_{10}\,\mathrm{e}^{\mathrm{i}\beta_1} \\ q_{50}\,\mathrm{e}^{\mathrm{i}\beta_5} \end{Bmatrix}-\mathrm{i}\begin{bmatrix} D_{11}q_{10} & D_{15}q_{50} \\ D_{51}q_{10} & D_{55}q_{50} \end{bmatrix}\begin{Bmatrix} q_{10}\,\mathrm{e}^{\mathrm{i}\beta_1} \\ q_{50}\,\mathrm{e}^{\mathrm{i}\beta_5} \end{Bmatrix}=\begin{Bmatrix} Q_{10}\,\mathrm{e}^{\mathrm{i}\theta_1} \\ Q_{50}\,\mathrm{e}^{\mathrm{i}\theta_5} \end{Bmatrix} \tag{5.48a}$$

or

$$\left(\begin{bmatrix} C_{11} & C_{15} \\ C_{15} & C_{55} \end{bmatrix}-\mathrm{i}\begin{bmatrix} D_{11}q_{10}^k & D_{15}q_{50}^k \\ D_{51}q_{10}^k & D_{55}q_{50}^k \end{bmatrix}\right)\begin{Bmatrix} q_{10}^{k+1}\,\mathrm{e}^{\mathrm{i}\beta_1} \\ q_{50}^{k+1}\,\mathrm{e}^{\mathrm{i}\beta_5} \end{Bmatrix}=\begin{Bmatrix} Q_{10}\,\mathrm{e}^{\mathrm{i}\theta_1} \\ Q_{50}\,\mathrm{e}^{\mathrm{i}\theta_5} \end{Bmatrix} \tag{5.48b}$$

or

$$\begin{Bmatrix} q_{10}^{k+1}\,\mathrm{e}^{\mathrm{i}\beta_1} \\ q_{50}^{k+1}\,\mathrm{e}^{\mathrm{i}\beta_5} \end{Bmatrix} = \left(\begin{bmatrix} C_{11} & C_{15} \\ C_{15} & C_{55} \end{bmatrix} - \mathrm{i} \begin{bmatrix} D_{11}q_{10}^k & D_{15}q_{50}^k \\ D_{51}q_{10}^k & D_{55}q_{50}^k \end{bmatrix} \right)^{-1} \begin{Bmatrix} Q_{10}\,\mathrm{e}^{\mathrm{i}\theta_1} \\ Q_{50}\,\mathrm{e}^{\mathrm{i}\theta_5} \end{Bmatrix} \tag{5.48c}$$

or

$$\begin{Bmatrix} q_{10}^{k+1}\mathrm{e}^{\mathrm{i}\beta_1} \\ q_{50}^{k+1}\mathrm{e}^{\mathrm{i}\beta_5} \end{Bmatrix} = \begin{bmatrix} G_{11}^k & G_{12}^k \\ G_{21}^k & G_{22}^k \end{bmatrix} \begin{Bmatrix} Q_{10}\,\mathrm{e}^{\mathrm{i}\theta_1} \\ Q_{50}\,\mathrm{e}^{\mathrm{i}\theta_5} \end{Bmatrix} \tag{5.48d}$$

where the complex matrix $[G^k]$ is defined on the right-hand side of (5.48c). From the last equation, it follows that

$$\begin{aligned} q_{10}^{k+1}\,\mathrm{e}^{\mathrm{i}\beta_1} &= G_{11}^k\,Q_{10}\,\mathrm{e}^{\mathrm{i}\theta_1} + G_{12}^k\,Q_{50}\,\mathrm{e}^{\mathrm{i}\theta_5}, \\ q_{50}^{k+1}\,\mathrm{e}^{\mathrm{i}\beta_5} &= G_{21}^k\,Q_{10}\,\mathrm{e}^{\mathrm{i}\theta_1} + G_{22}^k\,Q_{50}\,\mathrm{e}^{\mathrm{i}\theta_5}. \end{aligned}$$

Taking the modulus of the above equations yields

$$q_{10}^{k+1} = \left| G_{11}^k\,Q_{10}\,\mathrm{e}^{\mathrm{i}\theta_1} + G_{12}^k\,Q_{50}\,\mathrm{e}^{\mathrm{i}\theta_5} \right|, \tag{5.49a}$$

$$q_{50}^{k+1} = \left| G_{21}^k\,Q_{10}\,\mathrm{e}^{\mathrm{i}\theta_1} + G_{22}^k\,Q_{50}\,\mathrm{e}^{\mathrm{i}\theta_5} \right|. \tag{5.49b}$$

Equation (5.49) can be solved by iteration to find the absolute values of the surge and pitch responses due to a wave height H_w. If q_{10}^n and q_{50}^n are convergent solutions of (5.49) after n iterations, then the PRAOs in surge and pitch can be calculated from

$$\mathrm{PRAO}_1 = \frac{q_{10}^n}{H_w/2}, \tag{5.50a}$$

$$\mathrm{PRAO}_5 = \frac{q_{50}^n}{H_w/2}. \tag{5.50b}$$

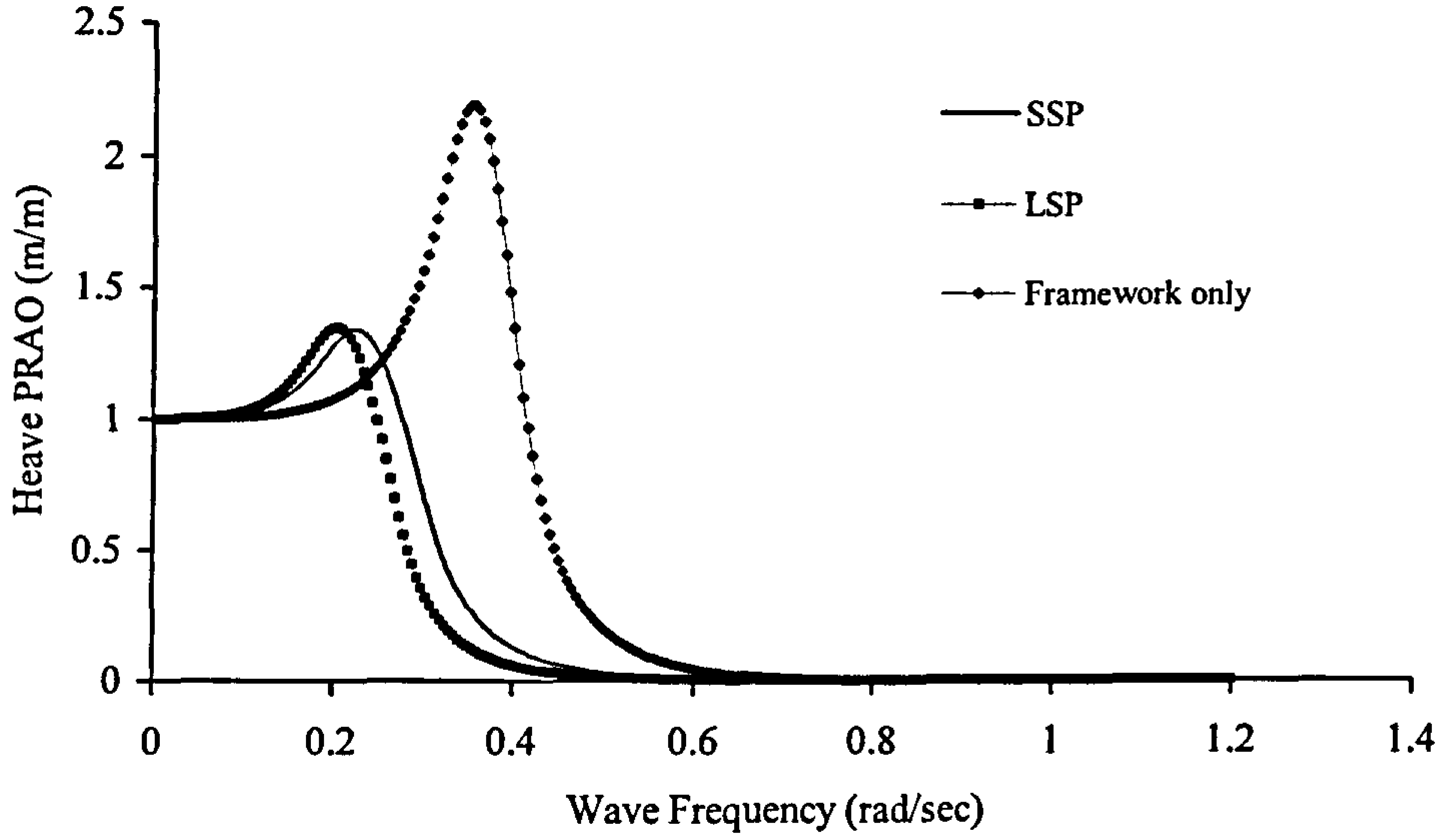

Figure 5.2: Heave PRAOs. SSP, small solid heave plates; LSP, large solid heave plates

5.9 Numerical results

The heave, pitch, and surge responses of the platform are derived for a sea-state defined by the JONSWAP wave spectrum, with a significant wave height of 15 m and a peak spectral period of 15 sec. Figure 5.2 shows the PRAOs of the truss spar with small and large solid heave plates, as well as without any heave plate. The effect of heave plates in reducing the amplitude and natural frequency of the heave motion is evident from this figure. The heave response is moderately sensitive to the assumed value for the drag coefficient for heave plates. The change in the heave standard deviation obtained for the case where $C_D = 7.0$ (Magee et al. 2000) is shown in Table 5.3. In figures 5.3 to 5.5, the PRAOs estimated by (5.44) and (5.48d) to (5.50) for the structure with small solid heave plates are compared with the experimental results reported by Downie et al. (2000). From figures 5.3, 5.4 and 5.5 it can be seen that there is a slight difference between the predicted and measured responses of the truss spar. Because the prediction of the first-order respones, in particular in the heave mode, should be fairly accurate, the error could partly be attributed to our simplifying assumption that the hull added mass coeffi-

	Present model	Stansberg et al.	
	Estimated	Measured	Estimated
	SD(m or deg)	SD(m or deg)	SD(m or deg)
Heave			
SSP	0.5450 (0.4682)*	0.531	0.3659
LSP	0.2714 (0.2463)*	0.224	0.1959
F	3.7344	—	—
Pitch			
SSP	0.8823	1.287	0.8810
LSP	0.8830	1.168	0.8797
Surge			
SSP	2.033	—	—
LSP	2.033	—	—

SSP, small solid heave plates; LSP, large solid heave plates; F, framework only

* Standard deviation (SD) in heave when $C_D = 7.0$ is chosen for the heave plates

Table 5.3: Heave, pitch, and surge standard deviations ($H_s = 15$m, $T_p = 15$s)

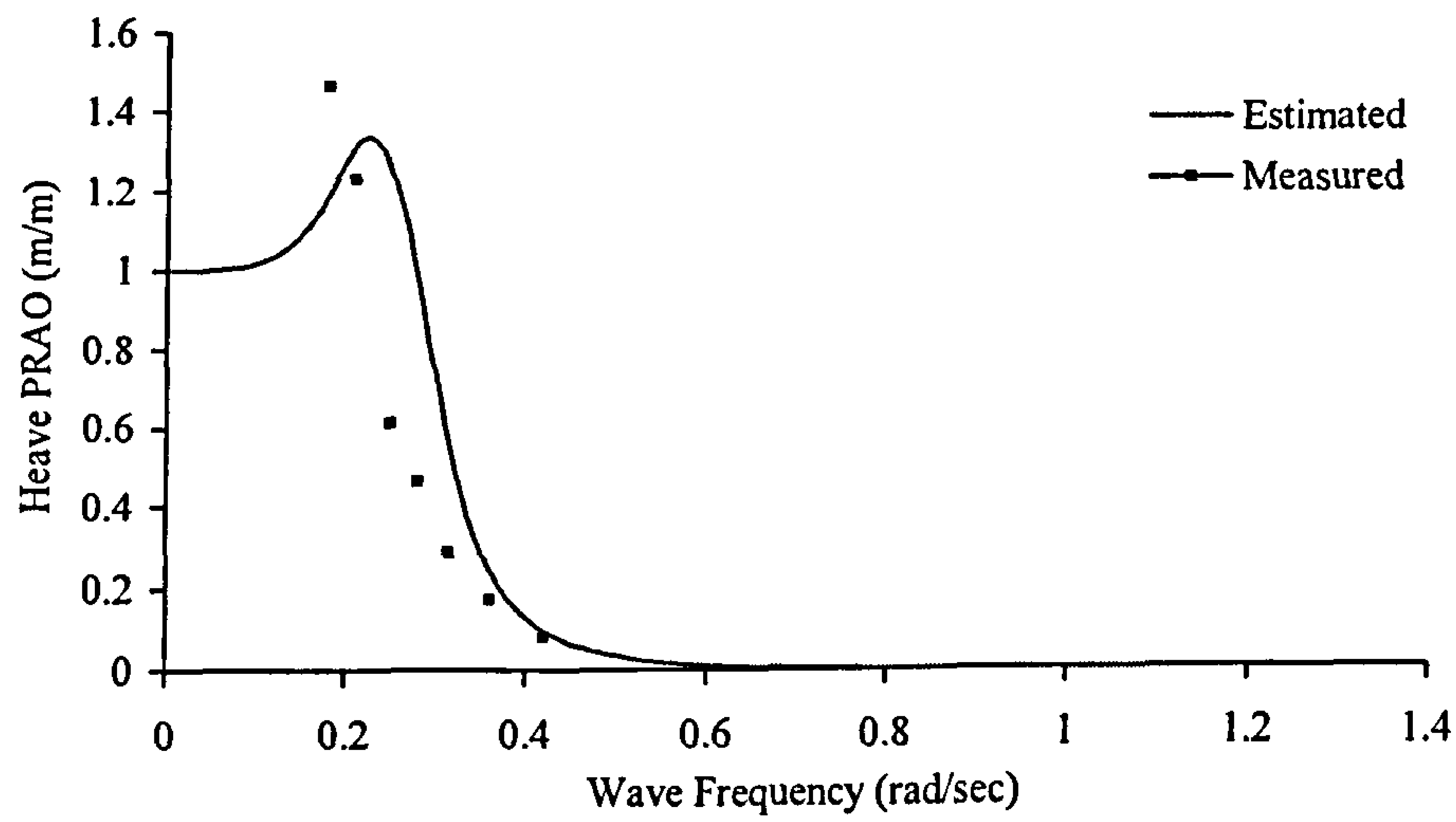

Figure 5.3: Estimated and measured heave PRAO for truss spar with small solid heave plates

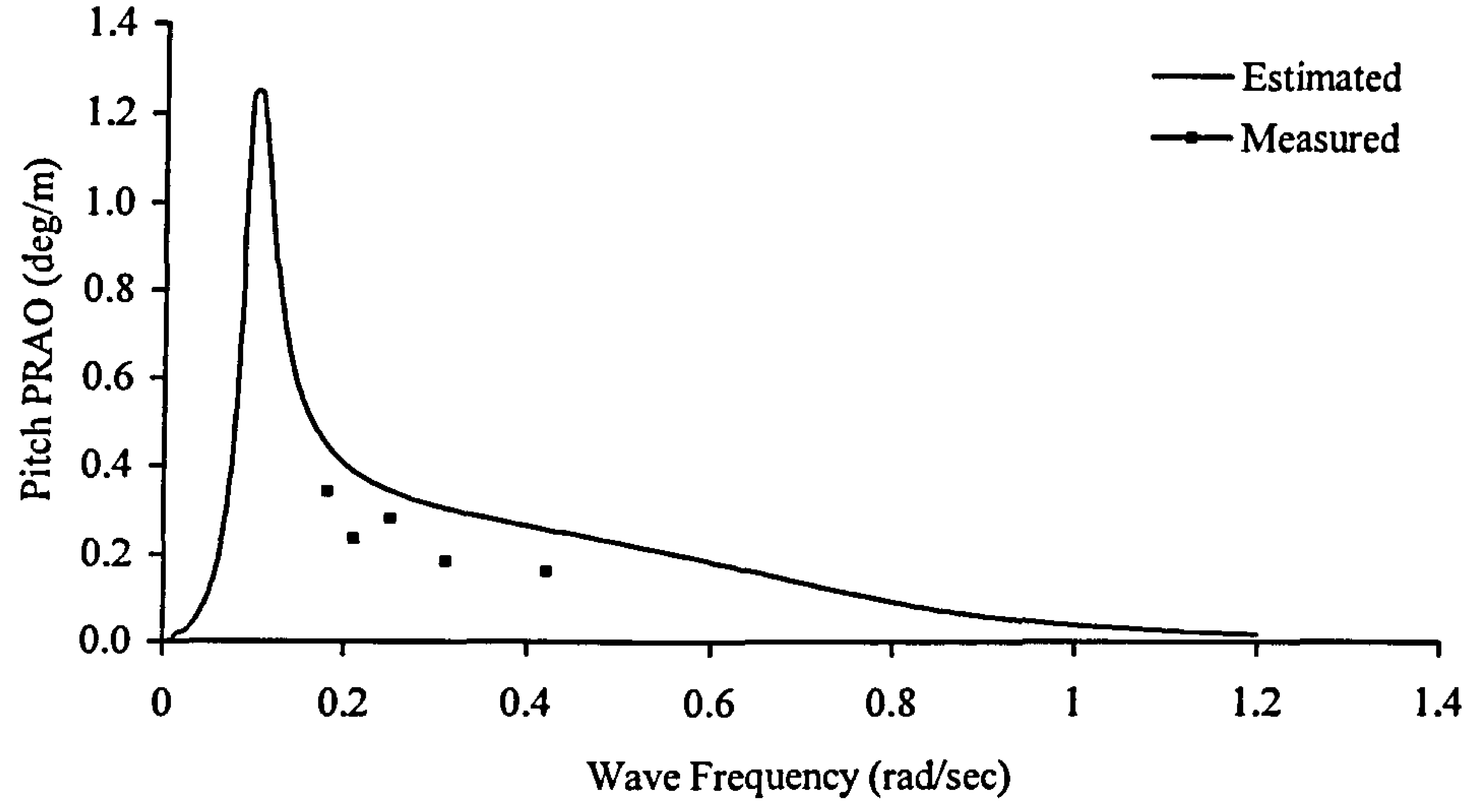

Figure 5.4: Estimated and measured pitch PRAO for truss spar with small solid heave plates

cients are independent of frequency. On the other hand, the difference between the measured and estimated results could be associated with the accuracy of the measurements, particularly in regard to the horizontal stiffness of mooring lines.

From the plot of the PRAOs for the coupled motions of surge and pitch, it can be seen that at the natural period of pitch there is a local maximum in surge response. The measured responses of the platform in wave frequencies around its pitch and surge natural frequencies were not reported by Stansberg et al. (2001), but they did present the standard deviation of the measured heave and pitch of the truss spar. They also estimated these values by a numerical model based on time–domain analysis. In that model, the forces acting on the spar hull and heave plates were calculated by the WAMIT (1995) program based on a potential theory, and a slender–body model with $C_D = 0.7$ was used to estimate the hydrodynamic forces acting on the truss. The standard deviations were derived for a sea–state defined by the JONSWAP wave spectrum mentioned above. In Table 5.3, the estimated standard deviations obtained from the present model based on the frequency domain approach are compared with the measured and simulated values reported by Stansberg et

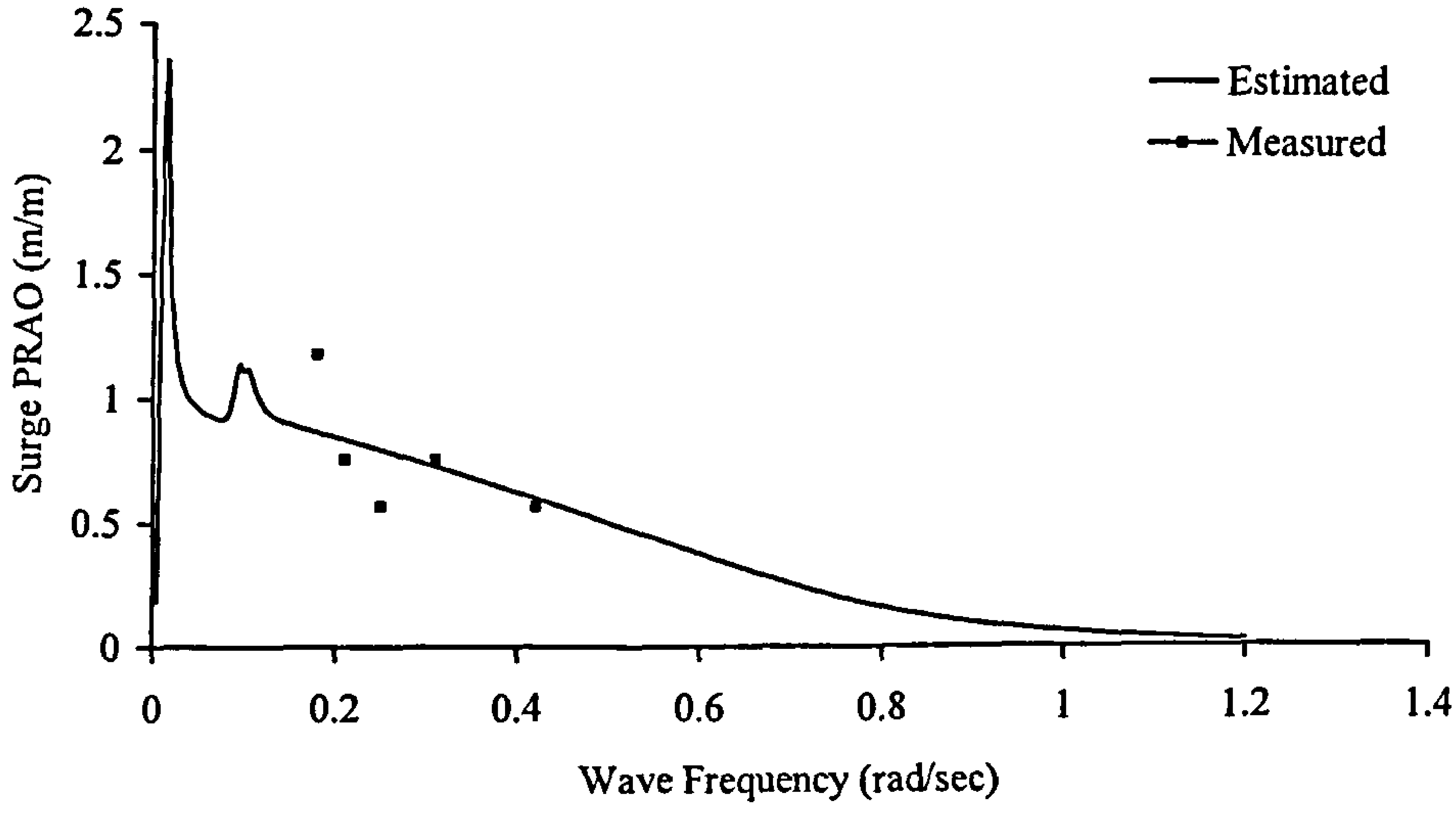

Figure 5.5: Estimated and measured surge PRAO for truss spar with small solid heave plates

	Present model[1]	Stansberg et al.[1]	
	Estimated	Measured	Estimated
Pitch	0.913*	1.287	0.8810
	0.882		
Surge	1.959*	—	—
	2.033		

[1] Results for the truss spar with small solid heave plates

* Obtained after modification of hull loads

Table 5.4: Pitch, and surge standard deviations before and after hull load modification

al. (2001). In heave, this method gives results that agree closely with the experimental results of Stansberg et al. (2001), and are significantly higher than their numerical results. In coupled motions of surge and pitch, the results of response analysis presented in Table 5.3 are obtained based on a model used by Sadeghi et al. (2004) where, as mentioned in § 5.5, the in-line force intensity of McCamy & Fuchs (1954) is used to predict the surge and pitch loads acting on the hull. This gives a pitch standard deviation very close to the simulated value of Stansberg et al. (2001) but lower than their experimental result. No experimental or estimated data have been reported for the surge motions, however the calculations show that the pitch and surge motions are inversely proportional. The results presented in Table 5.3 show that the simplifying assumptions made do not detract the accuracy of the response analysis when compared with more complex approaches. In the present model, the McCamy & Fuchs (1954) diffraction loads are modified to take the effect of cylinder truncation into account (see Chapter 4 or Sadeghi et al. 2003). As results in Table 5.4 show, these modifications improved the standard deviation of the surge and pitch responses. However, for the truss spar considered in this thesis, the improvements are not significant.

Chapter 6

Second-order Surge Response of a Truss Spar

6.1 Introduction

One of the most important hydrodynamic problems in the designing of offshore platforms is the prediction of magnitude of the low-frequency horizontal excursions of the platform due to second-order excitation forces. This problem has been extensively studied in the literature. A survey on the early literature on this subject has been given by Ogilvie (1983) where the fundamental hydrodynamic and dynamic problems in connection with the second-order responses of floating platforms has been presented. Since then, numerical procedures have been developed. Reviews of these developments may be found in Faltinsen (1990) and Molin (1993). Another recent developement is the rigorous treatment of the problem by a multi-scale perturbation technique. In § 6.2 the drift phenomena is reviewed briefly. High frequency motions such as springing and ringing are not considered and the attention is focused on the low frequency motions. This is because of the low natural frequencies of the spar type platforms. The literature on fluid, dynamic and statistical aspects of the drift motion is too extensive to be cited comprehensively in § 6.2. Therefore, only those parts of the theory which are more relevant to fundamental and approximate methods of analysis are studied in order to provide a foundation for the approximate analysis performed in §§ 6.3 and 6.4. The results of the analysis and comparison with experimental results are also presented in these sections.

6.2 An overview of the drift phenomenon

A few decades ago, as the world's energy need was increasing, offshore industry started to explore and exploite oil from the sea bottom by using floating structures. These structures despite ships should have been moored in the open sea even under storm conditions. Therefore, to reduce wave induced motions on floating offshore platforms, they were designed so that their natural frequencies to be far away from the peak frequency of the wave spectrum. This ensured low motions in wave frequencies, however, in addition to wave frequency horizontal motions, large motions at frequencies much lower than the frequency range of the sea occured (Remery & Hermans 1971). This phenomenon was important because large excursions that occured could cause large forces in anchor lines and limitations in drilling operations (Faltinsen 1980). At the same time, the invention of big tankers required their loading and discharging to be carried out in open sea where a similar large amplitude low frequency horizontal motion interfered with the operation (Hermans 1999). In the words of Hsu & Blenkarn (1970) this caused major difficulties in mooring under storm condition followed by extensive financial loss for the offshore oil industry. Consequently, the problems concerning the mooring of vessels in the open sea gained much attention and attempts have been made to explore the cause of the drift phenomenon.

Linear theory was not able to predict nor to fully describe slow-drift oscillations of floating bodies. Because according to the linear theory, the frequency of the exciting force and therefore the frequency of the predicted motion of a moored vessel is the same as the frequency of the incident wave. But at frequencies of slow-drift motion the incident waves involve insignificant energy. Therefore, model tank experiments were conducted to study the slow-drift phenomenon.

Because vessels usually are moored with a soft mooring system, the stiffness of the mooring system is very low with respect to the mass and added mass of a large vessel. As a result, the natural frequency of the vessel in horizontal motions is very low. During experiments it was observed that the model undergoes an oscillating slow motion with a frequency in the close range of the natural frequency of the moored vessel. Considering that the damping at low speeds is very small, it was suggested that the slow-drift motion is a resonance motion at the surge natural frequency of the platform and therefore it can be excited by a small force at that frequency (see Verhagen & Sluijs 1970).

On the other hand, earlier research (Havelock 1940, Maruo 1960, Newman 1967) showed that for a small amplitude regular wave, second-order nonlinear steady forces which are too small to influence the first-order motions are responsible for a steady drift force that causes a static shift of the average position of the moored vessel. Therefore, Hsu & Blenkarn (1970) and Remery & Hermans (1971) used the constant drift forces in regular waves to predict the slow-drift motion of moored vessels and compared the results of their calculations with the results of the model tests. Based on their observations, Hsu & Blenkarn (1970) were able to express a simplified explanation for the phenomenon of the slow-drift oscillations of a moored vessel.

They considered a sea wave as an irregular wave and discretized that wave to a number of half cycles and then approximated each irregular half cycle with a half cycle of a regular wave with the same peak amplitude. Therefore, the peak amplitude of the wave changes from one half cycle to the adjacent half cycle. In addition, considering the sea as a continuous medium this change cannot be large. In other words, in a sea represented by a narrow-banded spectrum the change in peak amplitude is gradual and small. Therefore, assigning a constant drift force to each half cycle, a slowly varying drift force can be assigned to an irregular wave (Faltinsen & Loken 1980). This force being proportional to the square of the small wave amplitudes has a very small magnitude but due to its low frequency can exite the resonance motions of a moored platform in the horizontal plane. Results of calculations based on this model were verified by experimental results. As a result of this finding, most investigators have approached the problem of slow-drift motion of a moored vessel by a straight forward perturbation theory and considered the problem up to the second-order. One of the most thorough analyses by this approach has been presented in a series of papers by Pinkster (see Pinkster 1980).

In a perturbation analysis, the basic assumptions of the linear theory are retained, i.e., the fluid flow is assumed to be incompressible and irrotational, the wave amplitudes are assumed to be small compared to the wave length and characteristic vessel dimensions and a long-crested sea is assumed as the superposition of regular waves. Newman (1974) used the latter assumption to show that in a discrete wave spectrum the square of the wave amplitude has a component which oscillates with low frequencies equal to the difference frequency of each pair of frequencies present in the discrete wave system. Therefore, low frequency excitation always exsits. Newman's (1974) approach will be considered in detail in the following.

According to the linear theory, an irregular small amplitude wave elevation in a point can be assumed as the superposition of regular small amplitude wave elevations at that point, i.e.,

$$\eta(t) = \sum_{i=1}^{N} \eta_i \cos(\omega_i t + \epsilon_i), \tag{6.1}$$

where $\eta_i = \eta_i(\omega_i)$ is the amplitude of the i–th harmonic. Now from Trigonometry it is well known that the product of two cosine functions with arguments A and B can be written as the sum of two cosine functions with arguments

$A-B$ and $A+B$. Therefore, the square of wave elevation $\eta(t)$ in (6.1) can be written as

$$\begin{aligned}\eta^2(t) &= \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2}\eta_i\eta_j\cos[(\omega_i-\omega_j)t+(\epsilon_i-\epsilon_j)]+\\ &\quad \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2}\eta_i\eta_j\cos[(\omega_i+\omega_j)t+(\epsilon_i+\epsilon_j)]. \end{aligned} \tag{6.2}$$

In other words, the square of the wave elevation has two parts. One a difference-frequency part and the other a sum-frequency part. Denoting them by $\eta_d^2(t)$ and $\eta_s^2(t)$, respectively, the difference frequency part of the square wave elevation can be written as

$$\eta_d^2(t) = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2}\eta_i\eta_j\cos[(\omega_i-\omega_j)t+(\epsilon_i-\epsilon_j)]. \tag{6.3}$$

If the resolution of the discrete set of frequencies representing the sea spectrum is chosen to be very high, the set of difference frequencies $\omega_i-\omega_j$, $(i,j=1,\ldots,N)$, will always include very small frequencis such as the natural frequencies of a moored floating platform in horizontal motions. Therefore, a second-order force proportional to the square wave amplitude always has a component at natural frequencies of the platform and can excite the resonance horizontal motions of the platform. In addition, it should be noted that for $i=j$, $\eta_d^2(t)$ in (6.3) contains a steady term which can be viewed as the time-average of $\eta_d^2(t)$ and corresponds to a steady mean drift force.

Now using a perturbation expansion the wave elevation ζ can be written as

$$\zeta = \zeta^{(0)} + \epsilon\zeta^{(1)} + \epsilon^2\zeta^{(2)} + O(\epsilon^3) \tag{6.4}$$

where ϵ is the wave steepness and

$$\begin{aligned}\zeta^{(0)} &\equiv 0,\\ \zeta^{(1)} &\equiv \eta,\end{aligned} \tag{6.5}$$

$$\zeta^{(2)} \equiv \eta^2.$$

Similarly the force acting on the platform due to wave ζ can be written as

$$F = F^{(0)} + \epsilon F^{(1)} + \epsilon^2 F^{(2)} + O(\epsilon^3) \tag{6.6}$$

where $F^{(0)}$ is a constant buoyancy force in the absence of waves, $F^{(1)}$ is proportional with the wave elevation η and $F^{(2)}$ is proportional with the squared wave elevation η^2. Therefore using (6.1) and (6.3), $F^{(1)}$ and $F_d^{(2)}$ may be written as follows

$$F^{(1)} = \sum_{i=1}^{N} \eta_i H_i \cos(\omega_i t + \epsilon_i + \delta_i), \tag{6.7a}$$

$$F_d^{(2)} = \sum_{i=1}^{N}\sum_{j=1}^{N} \eta_i \eta_j Q_{ij}^d \cos[(\omega_i - \omega_j)t + (\epsilon_i - \epsilon_j) + \delta_{ij}^d], \tag{6.7b}$$

$F_d^{(2)}$ is the difference frequency wave excitation due to second-order interaction of N harmonic waves. In (6.7) $H_i = H(\omega_i)$ and $Q_{ij}^d = Q^d(\omega_i, \omega_j)$ are the magnitude of linear and quadratic transfer functions, and $H_i e^{i\delta_i}$ and $Q_{ij}^d e^{i\delta_{ij}^d}$ are the linear and quadratic transfer functions, respectively. For $N = 2$ and $\eta_i = \eta_j = 1$, $F_d^{(2)}$ becomes

$$\begin{aligned} F_d^{(2)} &= (Q_{ii}^d + Q_{jj}^d) + Q_{ij}^d \cos[(\omega_i - \omega_j)t + (\epsilon_i - \epsilon_j) + \delta_{ij}^d] + \\ &\quad Q_{ji}^d \cos[(\omega_j - \omega_i)t + (\epsilon_j - \epsilon_i) + \delta_{ji}^d], \end{aligned} \tag{6.8}$$

Because the designation of two indices i and j to represent two waves is arbitrary, the symmetry of the quadratic transfer function needs to be prescribed (Newman 1974), i.e.,

$$Q_{ij}^d e^{i\delta_{ij}^d} = (Q_{ij}^d e^{i\delta_{ij}^d})^* \tag{6.9}$$

where an asterisk is used to denote complex conjugate. It follows from (6.9) that

$$Q_{ji}^d = Q_{ij}^d, \tag{6.10a}$$

$$\delta_{ji}^d = -\delta_{ij}^d \tag{6.10b}$$

Substituting from (6.10) into (6.8) yields

$$F_d^{(2)} = (Q_{ii}^d + Q_{jj}^d) + 2Q_{ij}^d \cos[(\omega_i - \omega_j)t + (\epsilon_i - \epsilon_j) + \delta_{ij}^d], \tag{6.11}$$

The first two terms on the right-hand side of (6.11) are steady forces and the last term represents an oscillating force. In (6.11) Q_{ij}^d is the difference frequency force due to second-order interactions of two harmonic waves with unit wave amplitude and frequencies ω_i and ω_j, respectively. From (6.7b) the time-average of the drift force will be obtained as

$$\bar{F}^{(2)} = \sum_{i=1}^{N} \eta_i^2 Q_{ii}^d, \tag{6.12}$$

where $\bar{F}^{(2)}$ is used to denote $\bar{F}_d^{(2)}$. In (6.12) Q_{ii}^d is the steady mean drift force acting on the platform in regular waves of unit amplitude and frequency ω_i. Because in $Q_{ii}^d = Q^d(\omega_i, \omega_i)$ both frequencies are the same it can be denoted by Q_i^d,

$$\bar{F}^{(2)} = \sum_{i=1}^{N} \eta_i^2 Q_i^d. \tag{6.13}$$

If one excludes the steady mean drift force, the remaining oscillating part of the drift force,

$$\tilde{F}^{(2)} = F^{(2)} - \bar{F}^{(2)}, \tag{6.14}$$

is known as slow-drift force. For a pair of waves with amplitudes η_i and η_j it follows from (6.11) that

$$\tilde{F}_d^{(2)} = 2\eta_i\eta_j Q_{ij}^d \cos[(\omega_i - \omega_j)t + (\epsilon_i - \epsilon_j) + \delta_{ij}^d]. \tag{6.15}$$

The quadratic interactions of the linear effects are not the only source of the second-order excitations. The pressure due to the second-order velocity potential also exerts second-order forces on a platform's wetted surface. Therefore, we shall distinguish the force components due to the second-order potential and quadratic interactions, that is,

$$Q_{ij}^d = Q_{ij}^p + Q_{ij}^q. \tag{6.16}$$

The force components Q_{ij}^q are due to the quadratic interactions of the first-order effects whereas the force components Q_{ij}^p can be obtained by solving the boundary value problem of the second-order velocity potential. Forces due to second-order velocity potential are small at low difference frequencies and vanish more rapidly as the water depth increases. This means that if the natural frequencies of the moored platform are low and the water is deep then Q_{ij}^p can generally be ignored (Newman 1974, Aranha & Pesce 1986, Barltrop 1998), i.e.,

$$Q_{ij}^d \approx Q_{ij}^q. \tag{6.17}$$

In addition, from the form of Bernoulli's integral it follows directly that the mean second-order pressure is not affected by the second-order velocity potential. Therefore, equation (6.13) can be written more precisely as

$$\bar{F}^{(2)} = \sum_{i=1}^{N} \eta_i^2 Q_i^q. \tag{6.18}$$

The full computation of the quadratic transfer function Q_{ij}^d is difficult due to the amount of computations and technical numerical problems. For instance, Aranha & Pesce (1986) mentioned that for a sea approximated by N harmonic waves, in the consistent method of Faltinsen & Loken (1980) $N^2 + N$ second-order combinations of possible low frequency must be calculated. To overcome this difficulty, different approximations are proposed. One of the most popular approximations is that of Newman (1974). If one considers Taylor expansion of the quadratic transfer function about zero difference frequency,

$$Q^d(\omega_i, \omega_j) = Q^d(\omega_i, \omega_i) + (\omega_j - \omega_i) \left. \frac{\partial Q^d(\omega_i, \omega_j)}{\partial \omega_j} \right|_{\omega_j = \omega_i} + \dots . \tag{6.19}$$

Newman's approximation can be thought of as the first term of the expansion. When $(\omega_j - \omega_i)$ is very small one may neglect the second and higher terms in the Taylor series and approximates $Q^d(\omega_i, \omega_j)$ with $Q^d(\omega_i, \omega_i) = Q_i^q$ (Faltinsen 1993). Therefore, the smaller the difference frequency is, the better Q_i^q predicts Q_{ij}^d. This means that Newman's approximation is expected to give good results when the natural frequency of the moored vessel is very low and the water is deep. If ω and μ denote ω_i and $\omega_j - \omega_i$, respectively,

$$Q_{ij}^d = Q^d(\omega, \omega + \mu), \tag{6.20}$$

then Newman's approximation can be written in one of the following forms

$$Q^d(\omega, \omega + \mu) \approx Q^q(\omega) \approx Q^q(\omega + \mu) \approx Q^q(\omega + \frac{\mu}{2}). \tag{6.21}$$

Aranha & Fernandes (1996) showed that within the family of Newman's approximation the accuracy of $Q^q(\omega + \frac{\mu}{2})$ is better than the others in calculating the spectrum of the difference-frequency force.

Basically, there are two ways of calculation of quadratic transfer functions. The first one known as the far-field method starts from the consideration on the change of momentum of the fluid within a control surface surrounding the body and remote from it so that the components of mean drift force can be obtained (Maruo 1960, Newman 1967). The other method known as the near-field method, uses the direct integration of the second-order pressure over the body instantaneous wetted surface to calculate second-order forces (Pinkster & Oortmersen 1977). Each method has its own advantages. The far field method yields a simple formula for the mean forces and is known to be computationally more efficient and robust than the near field method. The far field method, however, is only suitable for evaluation of steady second-order forces principally in surge, sway and yaw mode. The near-field method works well for calculation of unsteady forces, although fluid velocities along the hull have to be calculated. This should be performed with care otherwise numerically inaccurate results may be obtained. The near field method when applied through lower order panel programs is sensitive to the discretization of the body and the convergence is slow for bodies with sharp corners (Ferreira & Lee 1994, Le Boulluec et al. 1994). On the other hand, the near-field method can be used to evaluate local variables such as pressure or free surface elevation. The latter is important for calculation of air-gap between the waves and the underside of the platform. The near-field method also provides a better physical insight into the problem than the far-field method (Hermans & Sierevogel 1996, Grue & Palm 1993, Lee & Newman 1994). Quite recently, Chen (1988) developed a middle-field method, where merits of near-field method together with the computational efficiency of the far-field methd are claimed to be retained.

It has been known that second-order drift forces on a hull are influenced by the velocity of the slow drift motion. In other words if U denotes the platform slow-drift velocity, then (6.7b) can be written more precisely as

$$F_d^{(2)}(U) = \sum_{i=1}^{N}\sum_{j=1}^{N} \eta_i \eta_j Q_{ij}^d(U) \cos[(\omega_i - \omega_j)t + (\epsilon_i - \epsilon_j) + \delta_{ij}^d], \tag{6.22}$$

Now using the Taylor expansion for the left-hand side of the equation, one can write

$$F_d^{(2)}(U) = F_d^{(2)}(0) + \left.\frac{\partial F_d^{(2)}}{\partial U}\right|_{U=0} U + \left.\frac{\partial^2 F_d^{(2)}}{\partial U^2}\right|_{U=0} U^2 + \ldots. \tag{6.23}$$

On the other hand, it is well known that the velocity of drift motion is small. Therefore for a small U, the first two terms of the Taylar series in (6.23) will be sufficient to approximate the difference-frequency drift force, i.e,

$$F_d^{(2)}(U) \approx F_d^{(2)}(0) + \left.\frac{\partial F_d^{(2)}(U)}{\partial U}\right|_{U=0} U. \tag{6.24}$$

$F_d^{(2)}(U)$ tends to $F_d^{(2)}(0)$ as $U \to 0$, with the leading order correction proportional to the velocity U. The $\partial F_d^{(2)}(U)/\partial U\big|_{U=0}$ term in (6.24) is in phase with the velocity. Provided that the drift force increases by a small forward speed against the incoming wave direction, which is usually the case, the linear term in U can be interpreted as a damping term. Therefore, considering the derivative of the force with respect to U, evaluated at $U = 0$, as a damping coefficient,

$$\left.\frac{\partial F_d^{(2)}(U)}{\partial U}\right|_{U=0} = B_d^{(2)}, \tag{6.25}$$

equation (6.24) takes the following form

$$F_d^{(2)}(U) \approx F_d^{(2)}(0) + B_d^{(2)} U. \tag{6.26}$$

Now using a similar Taylor expansion on the right-hand side of (6.22) it follows that

$$\left.\frac{\partial F_d^{(2)}(U)}{\partial U}\right|_{U=0} = \sum_{i=1}^{N}\sum_{j=1}^{N} \eta_i \eta_j \left.\frac{\partial Q_{ij}^d(U)}{\partial U}\right|_{U=0} \cos[(\omega_i - \omega_j)t + (\epsilon_i - \epsilon_j) + \delta_{ij}^d]. \tag{6.27}$$

or

$$B_d^{(2)} = \sum_{i=1}^{N}\sum_{j=1}^{N} \eta_i \eta_j D_{ij}^d \cos[(\omega_i - \omega_j)t + (\epsilon_i - \epsilon_j) + \delta_{ij}^d], \tag{6.28}$$

where

$$D_{ij}^d = \left.\frac{\partial Q_{ij}^d(U)}{\partial U}\right|_{U=0} \tag{6.29}$$

is defined as the quadratic transfer function of damping. The damping coefficient $B_d^{(2)}$ derived from the drift force $F_d^{(2)}$ in (6.26) is called wave drift damping. The existence of the wave drift damping was first discovered exprementally in a study by Wichers & Sluijs (1979), where it is observed that the damping increases in the presence of waves. From a series of drift decay tests Wichers & Sluijs (1979) indicated that the wave drift damping is linear with respect to the velocity of the low frequency motion and is proportional to the square of wave amplitude.

From (6.28) the steady part of wave drift damping can be written as

$$\bar{B}_d^{(2)} = \sum_{i=1}^{N} \eta_i^2 D_i^d, \tag{6.30}$$

where D_i^d is used to denote $D_{ii}^d = D^d(\omega_i, \omega_i)$. The mean drift damping coefficient $\bar{B}_d^{(2)}$ can be written more precisely as

$$\bar{B}_d^{(2)} = \sum_{i=1}^{N} \eta_i^2 D_i^q, \tag{6.31}$$

because, as was mentioned, from Bernoulli's integral the mean value of Q_{ij}^p vanishes. In other words, we have

$$D_i^d = D_i^q = \left. \frac{\partial Q_i^q(U)}{\partial U} \right|_{U=0}. \tag{6.32}$$

$F_d^{(2)}(0)$ in (6.26) can be obtained by solving the zero-speed second-order wave-body interaction problem. This has been achieved by many authors. Among them one can refer to Kim & Yue (1990) and Lee et al. (1991) who solved the problem for a vertical circular cylinder and a body with arbitrary shape, respectively. To evaluate wave drift damping $B_d^{(2)}$ on the right-hand side of (6.26) the forward-speed second-order wave-body interaction problem should be solved. For this purpous, rigorous methods were derived by a number of authors. One can mention the works of Huijsmans & Hermans (1985), Zhao & Faltinsen (1989), Wu & Eatock-Taylor (1990), Nossen et al. (1991), Newman (1993), Grue & Palm (1993, 1996) and Finne & Grue (1998) for bodies of arbitrary geometry and Emmerhoff & Sclavounos (1992, 1996) and Malenica et al. (1995) for vertical circular cylinders. In order to find $B_d^{(2)}$ one should compute the wave drift damping quadratic transfer functions D_{ij}^d which by using (6.16) can be written as follows

$$D_{ij}^{d} = \left.\frac{\partial Q_{ij}^{q}(U)}{\partial U}\right|_{U=0} + \left.\frac{\partial Q_{ij}^{p}(U)}{\partial U}\right|_{U=0} . \tag{6.33}$$

Using the above methods, D_{ij}^{d} may be obtained either by analytical differentiation of drift force quadratic transfer functions if a closed form solution exists or by solving the forward-speed problem for a small speed increment and performing a numerical differentiation. This is a formidable and tedious task to do. If Newman's approximation is used, one can neglect Q_{ij}^{p} in (6.33) and replace Q_{ij}^{q} with Q_{i}^{q}, therefore, it follows that

$$D_{ij}^{d} \approx \left.\frac{\partial Q_{i}^{q}(U)}{\partial U}\right|_{U=0} , \tag{6.34}$$

or using (6.32),

$$D_{ij}^{d} \approx D_{i}^{q}. \tag{6.35}$$

Applying (6.35) in (6.28) yields,

$$B_{d}^{(2)} \approx \sum_{i=1}^{N}\sum_{j=1}^{N} \eta_i \eta_j D_i^q \cos[(\omega_i - \omega_j)t + (\epsilon_i - \epsilon_j) + \delta_{ij}^{d}] \tag{6.36}$$

which represents Newman's approximation for the wave drift damping coefficient. Wichers (1982) and Zhao & Faltinsen (1988) reported that the calculation of $B_d^{(2)}$ with respect to $\bar{B}_d^{(2)}$ has little effect on the standard deviation of the drift motion (Barltrop 1998). Therefore, in the equation of slow drift motion the wave drift damping coefficient $B_d^{(2)}$ may be approximated by the mean wave drift damping coefficient $\bar{B}_d^{(2)}$ (Barltrop 1998). Even the calculation of $\bar{B}_d^{(2)}$ by rigorous methods is still computationally demanding. One may evaluate the mean wave drift damping coefficient by the so called 'added resistance gradient method' (Hearn & Tong 1986, Hearn et al. 1987). This method uses a two- or three-dimensional diffraction theory with a simplified treatment of the forward speed effects (Barltrop 1998). A more simple approximation, known as the 'wave drift gradient method' was proposed by Standing et al. (1987). In this method, the effect of a forward speed is confined to the frequency of encounter ω_e and by using the Chain rule of differentiation the mean wave drift force derivative with respect to the forward velocity is represented in terms of the encounter frequency, i.e.,

$$\bar{B}_d^{(2)} = \left.\frac{\partial \bar{F}_d^{(2)}(U)}{\partial U}\right|_{U=0} \approx \left.\left(\frac{\partial \bar{F}_d^{(2)}}{\partial \omega_e}\frac{\partial \omega_e}{\partial U}\right)\right|_{U=0} \tag{6.37}$$

Because in deep water the encounter frequency ω_e is related to the wave frequency ω by

$$\omega_e = \omega - U\frac{\omega^2}{g}\cos\beta, \tag{6.38}$$

it follows from (6.37) and (6.38) that

$$\bar{B}_d^{(2)} \approx \left.\frac{\omega^2}{g}\frac{\partial \bar{F}_d^{(2)}(U)}{\partial \omega_e}\right|_{U=0}, \tag{6.39}$$

where β, the angle between the the forward speed and incident wave direction, is assumed to be zero. In addition, because the forward speed is small, one may replace ω_e in (6.39) with ω,

$$\bar{B}_d^{(2)} \approx \frac{\omega^2}{g}\frac{\partial \bar{F}_d^{(2)}(0)}{\partial \omega}. \tag{6.40}$$

This formula approximates the mean wave drift damping coefficient with the derivative of the mean drift force with respect to the wave radial frequency evaluated at zero forward speed. The accuracy of the wave drift gradient approach is found to be not satisfactory (Hearn & Tong 1988, Clark et al. 1993). On the other hand, Aranha (1991, 1994) used the principle of conservation of wave action and obtained the mean wave drift damping coefficient on a two-dimensional body in infinitely deep water. His formula was used for a three-dimensional body by Clark et al. (1993) who restated it as follows

$$\bar{B}_d^{(2)} = \frac{\omega^2}{g}\frac{\partial \bar{F}_d^{(2)}}{\partial \omega} + \frac{4\omega}{g}\bar{F}_d^{(2)}. \tag{6.41}$$

Surprisingly, this simple formula generates exactly the same results as the complicated theory developed by Emmerhoff & Sclavounos (1992) for a vertical circular cylinder in deep water where the radiation is not considered (Clark et al. 1993). In (6.41) the mean drift damping coefficient is the sum of two terms. The first term is proportional to the drift force derivative with respect to the radial frequency and the second term is proportional to the drift force itself. By substituting from (6.18) into (6.41) $\bar{B}_d^{(2)}$ can be written in terms of mean drift quadratic transfer function Q_i^q,

$$\bar{B}_d^{(2)} = \frac{\omega^2}{g}\sum_{i=1}^{N}\eta_i^2\left(\frac{\partial}{\partial \omega} + \frac{4}{\omega}\right)Q_i^q(0). \tag{6.42}$$

Therefore it is not required to calculate D_i^q from (6.34). Comparing (6.42) with (6.31), the Aranha (1991) and Clark et al. (1993) formula can be rewritten in terms of the quadractic transfer functions as follows

$$D_i^q = \frac{\omega^2}{g}\left(\frac{\partial}{\partial \omega} + \frac{4}{\omega}\right)Q_i^q(0). \tag{6.43}$$

The beauty of (6.41) is that it represents $\bar{B}_d^{(2)}$ in terms of $\bar{F}_d^{(2)}$ for the zero-speed second-order problem and therefore it is not required to solve the forward-speed problem. Some discrepancies between the computed results with the results of Aranha and Clark et al. formula are reported in the literature (Finne & Grue 1998, Hermans 1999). However, Trassoudaine & Naciri (1999) used model test results for a tanker and two barges and from the comparison of results with (6.41) they concluded that this formula yields very useful results for engineering applications. They argued that the relatively small discrepancies obtained for two barges can be viewed to be due to the sharp deviation of the geometry of the vessel with respect to that of an infinite vertical cylinder, where (6.41) is very accurate. Because the light barges used in the experiments had shallow draft and in addition presence of sharp edges made it difficult to experimentally determine the contribution of the wave drift damping in the measured total damping. Therefore, as a simple rule, the degree of similarity between a vessel shape and an infinite vertical circular cylinder determines the extent of use and the accuracy of Aranha and Clark et al. formula.

As was mentioned the slow drift motion is a resonance motion, consequently, low frequency motion amplitudes are very dependent on the system damping. Therefore, it is important to consider all damping components present in the problem. So far the wave drift damping as a non-viscous potential mechanism for dissipation of energy has been considered. In addition to the wave drift damping there are two other major sources for the damping of the slow drift motion. One is the hull damping and the other is the mooring-line damping. Both of these damping mechanisms are associated with viscous effects. The hull damping is the drag force on the hull due to motion of the platform. The mooring-line damping is the damping due to the drag forces on the mooring lines, risers and tethers, arisen from the motions of these elements and also due to the friction between the mooring lines and the sea floor. Until recently it has been customary to neglect the mooring-line damping. It was based on the argument that the drag area of mooring lines is negligible compared to that of the platform. However, Huse (1986) and Huse et al. (1988, 1989) present convincing evidence that the mooring-line damping can be an appreciable part of the total damping. The physical explanation is that the transverse line motion can be much larger than the slow-drift surge motion. In addition, the energy dissipation due to drag force is proportional to the third power of the amplitude. Thus, in spite of their small drag area, the mooring lines may still represent a dominent contribution to the total surge damping (Huse 1988).

Viscous dissipation is proportional to the drift velocity squared and wave drift damping is proportional to the drift velocity and the square of the wave amplitude. In that sense, all three major damping components of the slow drift motion are small quantities. Viscous damping is small because the square of the velocity of slow-drift motion is a second-order quantity. Wave drift damping is small because the square of wave amplitude is a second-order quantity and the mooring-line damping is small because the drag area of mooring lines is a second-order quantity. However, in different sea states one of these three damping mechanisms can become dominant. For instance in high sea states the square of wave amplitude is greater than the square of slow drift velocity, therefore, the wave drift damping is the dominant damping mechanism. On the other hand, in moderate sea states the damping is dominated by viscous effects. In lower sea states, the mooring-line damping could become the dominant damping mechanism. Also, for vessels with very little inherent damping and for vessels moored in ultra deep waters, mooring-induced damping can play a significant role in limiting the surge response. (Wichers 1988, Aranha 1994, 1996, Hermans 1999, Webster 1995).

Viscosity effects do not generate only damping forces. Huse (1977) showed that for structures with components that are small compared with wave height, viscous drift forces may not be negligible. As Ogilvie (1983) mentioned, for some kinds of ocean platforms the effect of viscosity and of wave diffraction are of the same order. Chakrabarti (1984) showed that for many semi-submersible and some TLP platforms the viscous drift force is significant. Therefore, the second-order viscous drift forces must be added to the potential drift forces. This force is usually modeled by the drag term of Morison equation. Where diffraction effects are insignificant, it is more efficient to use the inertia term of Morison equation than the diffraction theory. Consequently, in addition to a radiation-diffraction theory, the Morison's equation presents a second approach for the prediction of drift forces on offshore platforms.

Standing et al. (1981) and Chakrabarti (1984) discussed the regions of importance of viscous and potential drift force for fixed vertical cylinders.

Two parameters used for this analysis were diffraction parameter ka (wave number×cylinder radius) and viscosity parameter H/D (wave height/cylinder diameter). As was mentioned in Chapter 5 a parameter more rational than viscosity parameter (relative amplitude) is the Keulegan–Carpenter number. At low values of diffraction parameter, the inertia term of Morison equation can be used in place of a diffraction theory. In either approach, when Keulegan–Carpenter number is less than 5 viscous drift forces are often negligible (Chakrabarti 1987, Patel 1989).

The nonlinear wave forces acting on an offshore platform can be expressed by a Volterra series expansion (Volterra 1931, Schetzen 1980). The wave elevation is assumed as the input. The first term of the Volterra series represents the linear or first-order force and the second term of the series represents the second-order force. The series is usually truncated after the second term. In the frequency domain, the Fourier transform of kernels of integrals in the first and second term of the Volterra series are the linear and quadratic transfer functions. Expressions for the Kernels can be obtained if the nonlinearity has a polynomial form. If it is not the case, as the nonlinear drag term of Morison equation, a polynomial expansion of the water velocity can be used to approximate the nonlinearity. The unknown polynomial coefficients can be found by minimizing the error, stochastically, in a mean square sense. Depending on the order of approximating polynomial, equivalent stochastic linearization, quadratization or cubicization techniques may be used. Comparisons with simulations showed that the wave frequency responses and the mean second-order responses may be predicted by stochastic linearization (Donley & Spanos 1990, Chitrapu et al. 1993). However, responses at frequencies outside the wave spectrum frequencies are not predicted by linearization. Therefore, in order to predict low and high frequency responses due to quadratic effects, at least stochastic quadratization technique must be used (Donley & Spanos 1990). In addition, because for large diameter structures, second-order diffraction effects cannot be predicted by Morison's equation, complementary terms can be used to generate these nonlinear effects.. Temporal and convective acceleration forces, Rainey's (1989) axial divergence term and forces due to wetted surface calculation can generate difference frequency forces. Kim (1994) showed that for vertical circular cylinders the effect of these modifications is equivalent with forces obtained from a second-order diffraction analysis. In addition Li & Kareem (1992) showed that calculation of forces in the displaced position of structure is a source of drift force in addition to potential and viscous drift forces. More or less, this procedure represents the state of the art on predicting the drift force by using Morison's equation.

In using Morison's equation for second-order response analysis of vertical truncated cylinders like the hull of spar platforms, Mekha et al. (1996) claimed that the free surface calculations and the convective acceleration terms tend to cancel each other and the effect of axial divergence term is negligible but calculation of forces in the displaced position is important to capture the trend of second-order response in time domain.

In the frequency domain, to compute the mean and slow-drift force in irregular waves we employ a method proposed by Pinkster (1975, 1979). If the irregular sea is represented by a spectrum, the wave elevation η_i and sea spectrum $S(\omega)$ are related by the following well known equation

$$\frac{1}{2}\eta_i^2 = S(\omega_i)\Delta\omega, \tag{6.44}$$

where $\Delta\omega$ is a constant difference frequency (Faltinsen 1990). Using (6.44) every quantity that is a function of wave elevation can be represented in terms of the sea spectrum. Therefore, the mean drift force and wave drift damping coefficient given by (6.18) and (6.31), respectively, in terms of sea spectrum $S(\omega)$

become,

$$\bar{F}^{(2)} = 2\int_0^\infty S(\omega)Q^q(\omega)\mathrm{d}\omega, \tag{6.45}$$

$$\bar{B}_d^{(2)} = 2\int_0^\infty S(\omega)D^q(\omega)\mathrm{d}\omega, \tag{6.46}$$

where the summation in (6.18) and (6.31) is replaced with an integral. Similarly by substituting from (6.44) into (6.15) and taking into account the contribution from all pairs of frequencies with non-zero difference frequency, the spectrum of the slow-drift force in integral form becomes

$$S_F(\mu) = 8\int_0^\infty S(\omega+\mu)S(\omega)Q^{d^2}(\omega,\omega+\mu)\mathrm{d}\omega. \tag{6.47}$$

If Newman's approximation is used the spectrum of the slow-drift force can be obtained from the mean drift quadratic transfer function,

$$S_F(\mu) = 8\int_0^\infty S(\omega+\mu)S(\omega)Q^{q2}(\omega+\frac{\mu}{2})\mathrm{d}\omega. \tag{6.48}$$

It must be noted that (6.45) to (6.48) are derived for potential drift forces where the drift force in regular waves is proportional to the wave amplitude squared. Because viscous drift forces are approximately proportional to the wave amplitude squared, these equations are approximate for the drag terms in the Morison equation (see Chitrapu et al. 1993)

The literature on the prediction of slow-drift response of offshore platforms is mainly concerned with the computation of drift excitation forces. Once this task is completed, the traditional way to derive the slow-drift response is to model the platform as a single-degree-of-freedom mass-spring-damper system as follows

$$a\ddot{x} + b\dot{x} + cx = F^{(2)}, \tag{6.49}$$

where a is the summation of mass and added mass, b is the damping coefficient, c is the summation of hydrostatic and mooring restoring stiffness and

x is the displacement of the slow response in the particular mode of motion. From (6.49) the magnitude of mechanical admittance will be

$$H(\mu) = \frac{1}{\sqrt{(c - \mu^2 a)^2 + b^2}}. \tag{6.50}$$

Using the spectrum of the slow-drift force, $S_F(\mu)$, and the mechanical admittance of the system, $H(\mu)$, the spectrum of the slow-drift response can be obtained from

$$S_x(\mu) = H^2(\mu) S_F(\mu), \tag{6.51}$$

which yields the significant value of the slow-drift response as follows

$$x_{sig} = 2\sqrt{\int_0^\infty S_x(\mu) \mathrm{d}\mu}. \tag{6.52}$$

Alternatively, associating S_F at natural frequency of the surge motion to a white noise process, the significant value of the slow-drift response can be obtained from the following formula

$$x_{sig} = 2\sqrt{S_F(\omega_n)}\sqrt{\int_0^\infty H^2(\mu) \mathrm{d}\mu}. \tag{6.53}$$

6.3 Drift response of a truss spar platform

In this section we shall consider the problem of mean and slow drift responses of a truss spar platform. We shall evaluate these responses by an approximate method in the frequency domain. Our approximate method uses a simplified model. This model is based on a series of assumptions as follows.

The problem will be studied in the context of a perturbation analysis. This means that strongly nonlinear phenomena such as steep or breaking waves will not be considered. The water will be assumed to be deep. The directional spreading of the sea will be neglected and the sea will be considered to be longcrested. The effect of current will be ignored in the analysis. The possible vortex induced vibration (VIV) will not be considered too. An uncoupled hull–mooring analysis will be applied. Moreover, following the common assumption in the literature, the geometric nonlinearities in the mooring restoring forces will be neglected since they are less important than the wave force nonlinearities (Donley & Spanos 1990).

The truss spar will be modeled as a rigid body with six degrees of freedom. For a flow parallel to the surge axis of the platform, the hull responds in three degrees of freedom of surge, pitch and heave as shown in Chapter 5. As mentioned in Chapter 1, a truss spar has a much larger heave damping than a comparable classic spar and is therefore less sensitive to heave excitations. Consequently, the second-order low-frequency response in heave mode will not be studied herein. For the truss spar surge and pitch motions are coupled. However, the natural frequencies of surge and pitch of the platform are not very different from those obtained when the off-diagonal terms in the mass and stiffness matrices are neglected. This indicates that the coupling between surge and pitch is weak. In addition, the second-order pitch response will be balanced by hydrostatic as well as restoring stiffness. Such small pitch motion, weakly coupled to the surge motion, represents a pitch-induced surge motion which is an order of magnitude less than the slow-drift surge motion. So the slow-drift surge motion will be analysed as an uncoupled motion and the effect of the second-order pitch response will be ignored.

Following the above argument, the second-order response analysis of the truss spar will be considered in the surge mode only. As mentioned earlier, there are two methods for the prediction of mean and slow drift forces on offshore platforms. One uses a diffraction analysis where the viscous effects will be added to the analysis by the nonlinear drag term of Morison's equation and the other uses the Morison equation where the inertia term can represent the diffraction effects. To be consistent with the method used in the response analysis of first-order motions in chapter 5, we shall use a second-order radiation-diffraction model. In other words, the mean and slow drift surge response of a truss spar platform will be analysed in the context of a potential theory. Furthermore, the effect of viscosity will be ignored first and the problem will be solved in the absence of drag forces. Then viscous effects will be added to the problem. To exemplify this model, a truss spar the same as the one used in chapter 5 will be used.

The second-order radiation–diffraction problem is modeled as follows. The slow-drift diffraction problem is considered as a second-order problem for a body either fixed or undergoing first-order motions. This model was used by Kim & Yue (1990). In this way, the forcing due to the diffraction potential represents the combined effect of the second-order velocity potential and all quadratic interactions of the first-order effects. This is consistent with the method presented in the previous section. On the other hand, the slow-drift radiation problem is considered as a second-order problem for a body undergoing large-amplitude low-frequency motions in otherwise calm water. So the second-order radiation problem resembles the first-order radiation problem. Based on this model, the equation of slow-drift surge motion can be written as

$$(m + a_{11})\ddot{x} + b_{11}\dot{x} + k_{11}x = F_d^{(2)}(0). \tag{6.54}$$

In (6.54) x is the displacement of the slow-drift surge motion. Radiation damping b_{11} due to slow-drift motions of a platform is usually negligible. On the other hand, as shown in § 6.2, the influence of forward speed $\dot{x}$ on drift force can be represented by the wave-drift damping force $B_d^{(2)}\dot{x}$. Therefore, equation (6.54) can be modified as follows

$$(m + a_{11})\ddot{x} + B_d^{(2)}\dot{x} + k_{11}x = F_d^{(2)}(0). \tag{6.55}$$

In (6.55) $\dot{x}$, the velocity of the slow-drift motion, is equal to $-U$ in (6.24). Difference frequency force $F_d^{(2)}(0)$ on the right-hand side of (6.55) is the second-order force due the second-order velocity potential and all quadratic interactions of the first-order effects. Using Newman's approximation the effect of the second-order velocity potential can be neglected and for the calculation of the forces due to quadratic interactions, precise values of the first-order velocity potentials must be used. For this purpose an exact analytical formula for a truncated vertical cylinder or a numerical diffraction analysis can be used. Fortunately, the results of the latter are readily available in the literature. Mekha et al. (1996) used a formula derived from the parametric study of the outputs of WAMIT (1995) program (Weggel 1997). This formula in terms of mean drift quadratic transfer functions can be written as follows,

$$Q^q = \begin{cases} \rho g R[-0.1kR + 0.51(kR)^2 - 0.15(kR)^3] & : \quad 0 < kR \leq 2 \\ \rho g R[0.645] & : \quad 2 < kR \leq 4 \end{cases} \tag{6.56}$$

where k is the wave number and R is the cylinder radius. Equation (6.56) predicts the mean drift quadratic transfer function for $kR \leq 4$. For the same cylinder a larger kR means a larger wave number and the latter implies smaller wavelengths. Following Faltinsen (1990) the asymptotic value of the mean drift quadratic transfer function for small wavelengths for a body with vertical sides at the waterplane can be written as

$$Q^q = \frac{\rho g}{2} \int_l \sin^2(\theta + \beta) \sin\theta \mathrm{d}l \tag{6.57}$$

where l is the non-shadow part of the waterline curve, β is the wave propagation direction with respect to the x-direction and θ is the angle between the direction of dl and the x-direction. For a wave in the direction of surge, $\beta = 0$, it follows that

$$Q^q = \frac{\rho g}{2} \int_l \sin^3(\theta) \mathrm{d}l \tag{6.58}$$

and for a circular water plane area

$$Q^q = \frac{\rho g}{2} \int_0^\pi \sin^3(\theta) R \mathrm{d}\theta = \frac{\rho g R}{2} \int_0^\pi \sin^3(\theta) \mathrm{d}\theta \tag{6.59}$$

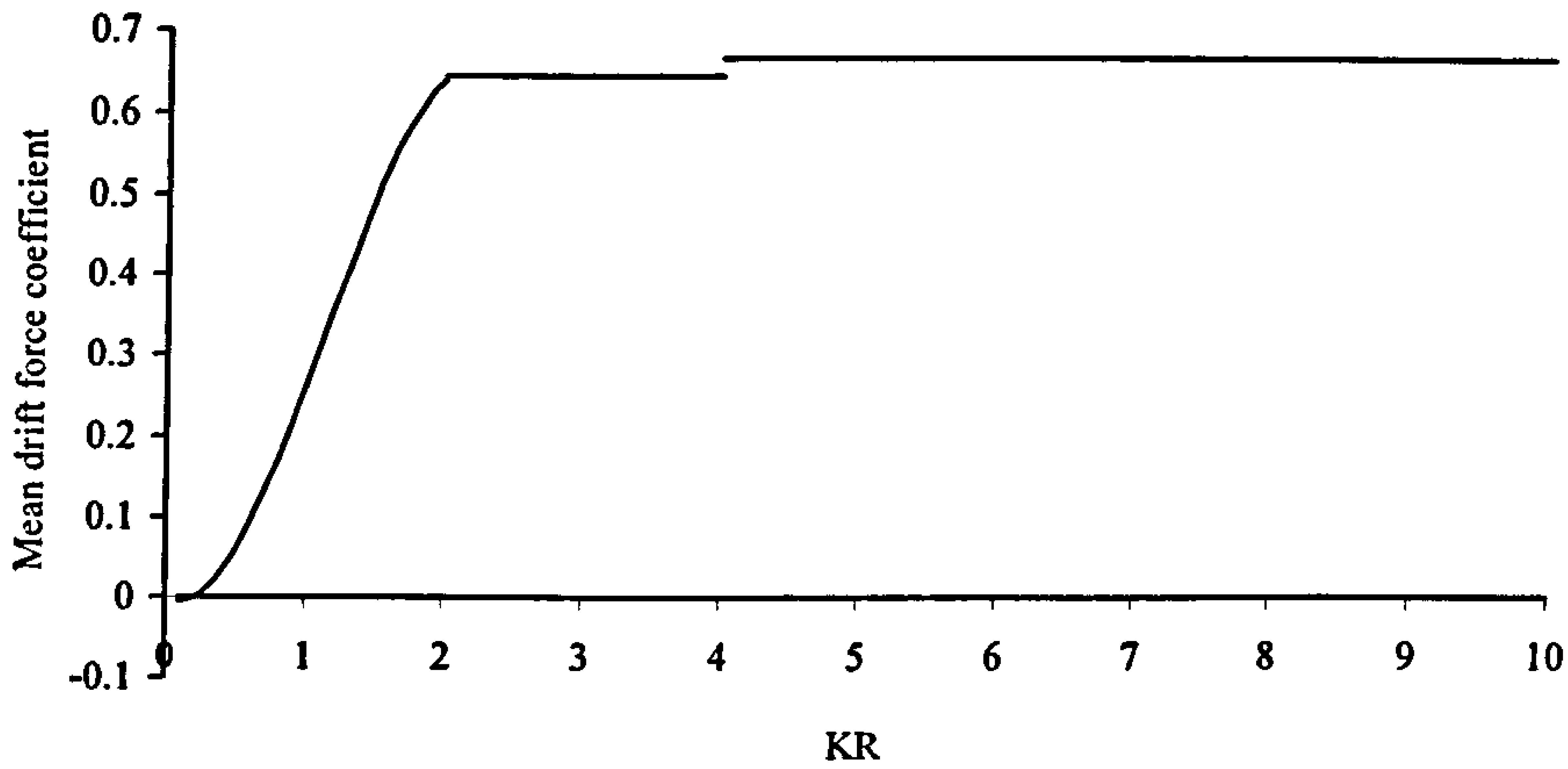

Figure 6.1: Mean drift quadratic transfer function

or,

$$Q^q = \frac{2}{3}\rho g R \tag{6.60}$$

Using (6.60) for short waves corresponding to $kR > 4$, the mean drift force quadratic transfer function given by (6.56) can be extended as follows

$$Q^q = \begin{cases} \rho g R[-0.1kR + 0.51(kR)^2 - 0.15(kR)^3] & : \ 0 < kR \leq 2 \\ \rho g R[0.645] & : \ 2 < kR \leq 4 \\ \rho g R[0.666] & : \ kR > 4 \end{cases} \tag{6.61}$$

The graph of the mean drift quadratic transfer function obtained from (6.61) is plotted in figure 6.1. Now following the discussion on p. 114, we shall use the mean wave drift damping coefficient, $\bar{B}_d^{(2)}$, instead of the slowly varying drift damping coefficient, $B_d^{(2)}$, in (6.55), i.e.,

$$(m + a_{11})\ddot{x} + \bar{B}_d^{(2)}\dot{x} + k_{11}x = F_d^{(2)}(0). \tag{6.62}$$

In addition, as discussed in § 6.2, the approximate formula of Aranha (1991) and Clark et al. (1993) for the prediction of $\bar{B}_d^{(2)}$ is very accurate for vertical circular cylinders and thus for the truss spar hull. Therefore, the mean drift

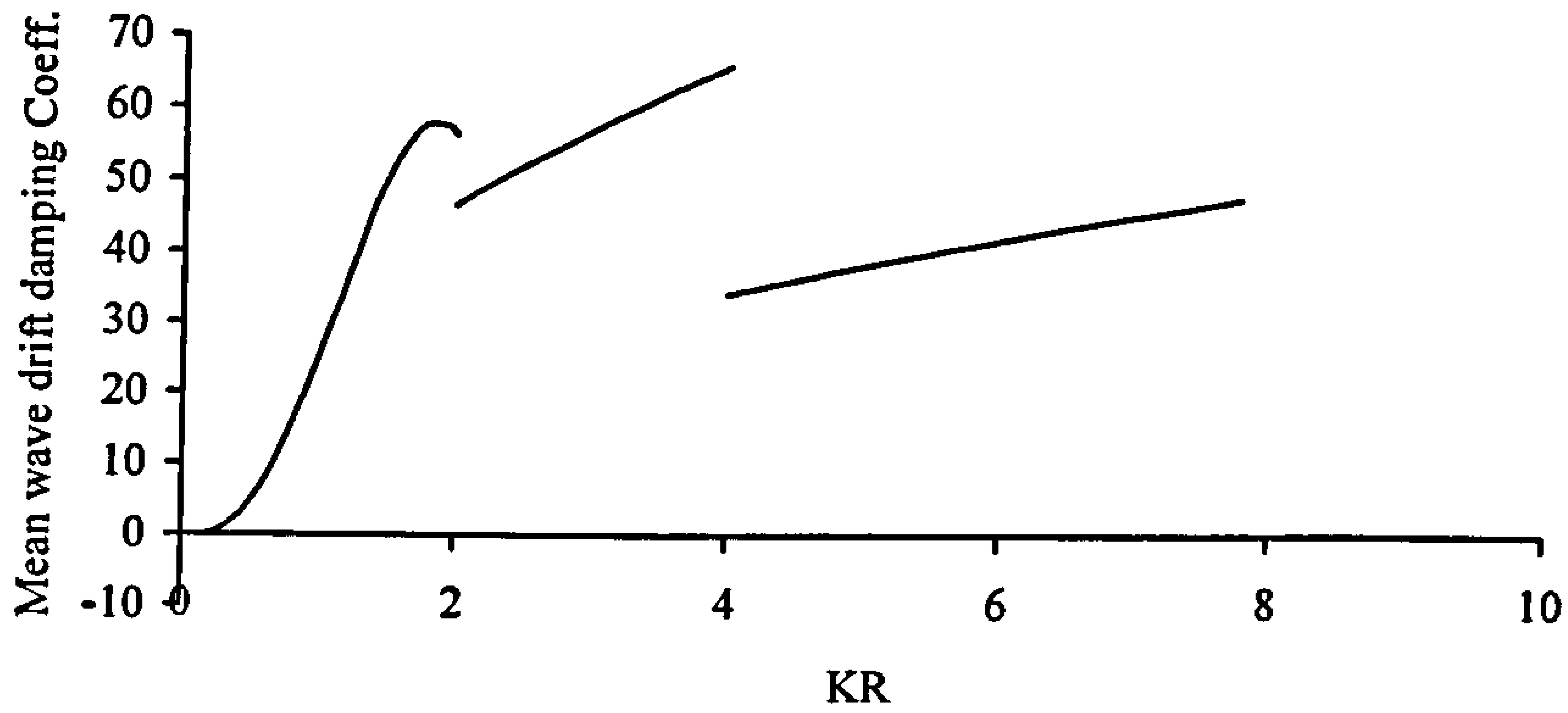

Figure 6.2: Mean wave drift damping quadratic transfer function

damping coefficient of the truss spar can be obtained from (6.41)

$$\bar{B}_d^{(2)} = \frac{\omega^2}{g}\frac{\partial \bar{F}_d^{(2)}}{\partial \omega} + \frac{4\omega}{g}\bar{F}_d^{(2)}. \tag{6.63}$$

or in terms of mean quadratic transfer functions

$$D^q = \frac{\omega^2}{g}\frac{\partial Q^q}{\partial \omega} + \frac{4\omega}{g}Q^q. \tag{6.64}$$

Substituting (6.61) into (6.64), the following simple expression will be derived for the wave drift damping quadratic transfer function

$$D^q = \begin{cases} \rho\omega R[-0.6kR + 4.08(kR)^2 - 1.5(kR)^3] & : \quad 0 < kR \leq 2 \\ \rho\omega R[2.58] & : \quad 2 < kR \leq 4 \\ \rho\omega R[2.66] & : \quad kR > 4 \end{cases} \tag{6.65}$$

The graph of (6.65) is plotted in figure 6.2. Now substituting (6.65) and (6.61) into (6.46) and (6.48), respectively, and evaluating the integrals numerically, the mean drift damping coefficient $\bar{B}_d^{(2)}$ and the spectrum of the slow drift force can be obtained. Using these values in (6.62) and applying the values of added mass and surge restoring coefficients of the first-order response analysis to the second-order problem all coefficients of (6.62) will be known. Finally, using (6.50) to (6.52), the significant value of the second-order slow-drift surge response can be estimated. For the truss spar in figure 4.1 the slow drift surge response calculated based on this model is equal to 19.31 meters. In addition, substituting (6.61) into (6.45) gives a mean surge force of 262.3 kN which for a mooring stiffness coefficient of 15.5 kN/m yields a mean drift surge response of 16.92 m. The measured values of mean-drift and slow-drift surge responses was reported as 15 m and 9.22 m (MARINTEK 2000, Berthelsen 2000), respectively. Comparison of results shows that the simplified model used predicted the mean-drift surge response well and even without taking the viscous effects into account the predicted slow-drift response is comparable with the measured one. In the next section the effect of viscosity will be studied.

6.4 Viscous effects on slow-drift surge motion

In order to take viscous effects on the slow-drift surge motion of the truss spar into account, we shall follow a model very similar to that used for the first-order wave frequency motions of the platform. That is, we shall consider the effect of viscosity on the slow-drift surge motion by a viscous-radiation–diffraction model. In other words, the effects of viscosity will be dealt with in the second-order radiation and diffraction problems separately.

6.4.1 Viscous-diffraction problem

As was mentioned earlier, the second-order diffraction problem was considered as a second-order problem for a body either fixed or undergoing first-order motions. Therefore, viscous-diffraction problem can be assumed to be consisted of two sub-problems. In the first problem we shall consider the viscous excitation forces acting on the fixed platform due to second-order incident and diffracted waves. In the second problem we shall consider the effect of first-order motions in correcting the viscous excitation forces of the first problem.

In the first problem, for the truss spar with the configuration of a large body at free surface and slender members deeply submerged, no viscous forces will be produced. As was discussed in § 5.6.1 this is because the Keulegan–Carpenter number for hull is small and the square of the water particle's velocity in deep water for the truss members is negligible.

In the second problem, because first-order motions are an order of magnitude smaller than the second-order slow-drift motion the Keulegan–Carpenter number or the viscosity parameter for the hull will be again very small and therefore viscous excitation forces due to the hull can be neglected. Now let us consider the second problem for the truss. The truss is a body immersed far below free surface in deep water, therefore, unlike the hull, first-order heave motion of the structure cannot generate a surface which is part of the time in the water and part of the time out of the water. So a significant source of the second-order force, due to first-order motions, is absent in the second problem for the truss. No other important source of second-order excitation is present for the truss. On the other hand, first-order oscillations of the truss do produce a damping force. However, that force is the damping force employed in the first-order problem and is not a second-order force and in that sense should not be considered in the second-order problem. As a result, viscous excitation forces due to truss in the second problem will be negligible too. Consequently, the total second-order viscous excitation forces predicted by the viscous-diffraction problem will be negligible.

6.4.2 Viscous-radiation problem

The second-order radiation problem was considered as a radiation problem for a body undergoing large-amplitude low-frequency motions in otherwise calm water. Therefore, in the second-order viscous-radiation problem the drag force will be the viscous damping force produced by the slow-surge motion of the platform. Following the first-order viscous-radiation model in § 5.6.2, we can use the relative amplitude, $2\delta/D$, to evaluate the importance of viscous damping force for the hull and the truss. Here, in the relative amplitude δ is the amplitude of the slow-drift motion and D is the diameter of the cylinder.

Neglecting the small pitch motion with respect to the large amplitude surge motion, δ can be assumed to be constant for the whole structure. Because the slow-drift surge motion is a resonance motion with a large amplitude at the natural frequency of the surge motion, one can assume that the amplitude of the slow-surge motion and the diameter of the hull are of the same order of magnitude. So if d denotes the diameter of the largest cylindrical member of the truss, then the relative amplitude for the cylindrical members of the truss can be approximated by D/d which will be much greater than 1.5. So, as expected, viscous effects are important for the truss. On the other hand, for the hull $2\delta/D$ will be close to the limit value of 1.5 where the viscous damping could become negligible. However, because the slow-drift motion is a resonance motion and in resonance every source of damping is important, we shall consider the damping due to the hull. In chapter 5, ω_{1n} was found to be equal 0.0123 rad/sec. A motion with a natural frequency that small has a very large period. Therefore, its characteristics will be close to a unidirectional flow. As a result, the drag coefficient of the hull in the unidirectional flow can be used as a measure of that coefficient in the slow-drift motion. In other words, the hull damping coefficient in the slow-drift motion should be close to that of unidirectional motion. In low speed unidirectional flow past circular cylinders the drag coefficient is a function of Reynold's number. In the radiation problem the latter can be defined as

$$Re = \frac{\dot{x}D}{\nu}$$

where $\dot{x}$ is the speed of the slow-drift motion, D is the cylinder diameter and ν is the kinematic viscosity of water. For the truss spar hull with 31.5 m diameter, Re number will be large enough for the flow to be assumed as turbulent. The minimum value of the drag coefficient in turbulant flow is that of a smooth cylinder which is 0.3. Using this value as a crude estimate, the drag coefficient of the hull in slow-drift motion is assumed to be $C_D = 0.3$. For the cylindrical members of the truss we shall use the same C_D value as used in the first-order motion, i.e., 0.7.

We shall also use an approximation the same as the one used in the first-order viscous-radiation problem, that is, we shall assume that the drag forces acting on the body oscillating in otherwise calm water can be approximated by the drag forces acting on a fixed body subjected to an oscillatory flow, where the flow velocity is the same as the slow-velocity of the platform. The fact that slow-surge motions of the truss spar radiates negligible waves and the slender truss members are deeply submerged are in favour of this approximation. As a result of this assumption, the viscous damping forces due to the second-order viscous-radiation problem can be written in terms of the slow-surge velocity of the platform as $B_{11}|\dot{x}|\dot{x}$, where B_{11} is the viscous damping coefficient of the platform in surge mode. Similar to the first-order viscous-radiation problem of § 5.6.2, the viscous damping coefficient can be calculated by the simple projected area method rather than the Morison's equation and the cross flow approach. As a result, we shall use the viscous damping coefficient of the truss in the first-order surge problem for the second-order problem.

The viscous radiation damping force obtained from this model can be added to the radiation side of the incompressible irrotational slow-drift equation of surge motion, equation (6.62), i.e.,

$$(m + a_{11})\ddot{x} + \bar{B}_d^{(2)}\dot{x} + B_{11}|\dot{x}|\dot{x} + k_{11}x = F_d^{(2)}(0). \qquad (6.66)$$

The advantage of this equation is that it is written solely in terms of the slow-drift surge motion. This is the result of the proposed second-order viscous-radiation–diffraction model. The simple equation given in (6.66) allows the slow-surge motion to be predicted in the frequency domain easily.

As (6.66) shows the damping due to the wave drift and viscous effects limits the amplitude of the second-order resonance motion in surge. In order to compare the effect of wave drift damping estimated in § 6.3 with the viscous damping obtained in this section on the slow-drift surge motion of the truss spar, we shall first consider the effect of viscous damping alone. In other words we shall consider the following equation,

$$(m + a_{11})\ddot{x} + B_{11}|\dot{x}|\dot{x} + k_{11}x = F_d^{(2)}(0). \qquad (6.67)$$

This equation is similar to the first-order heave equation of motion solved without iteration in § 5.7. The same technique can be used to find the slow-drift surge response from (6.67). However, this equation will be solved by iteration herein. Again, the spectrum of the slow-drift force can be obtained by substituting (6.61) into (6.48). Then using the value obtained and applying the values of the coefficients on the left-hand side of (6.67) from the first-order surge equation of motion, the significant slow-drift surge response can be evaluated from (6.50) to (6.52). This procedure yields a significant slow-drift surge respone equal 15.39 m. Comparison of this significant value with the one obtained in § 6.3 due to the wave drift damping alone, indicates that the viscous damping and the wave drift damping are almost equally important in limiting the slow-drift surge resonance motion of the truss spar.

When both wave drift damping and viscous damping are considered, equation (6.66) can be solved by linearizing the drag term. The solution can then be found by iteration. For the problem in hand, the result of this procedure is a significant slow-drift surge response of 12.97 m which is close to the measured value of 9.22 m. Results of the above calculations for different damping components are summarized in table 6.1. Comparison of the predicted results with experimental data indicates that the simplified viscous-radiation–diffraction model can be used to predict the significant slow-drift surge response of truss spar platforms in the frequency domain.

From Table 6.1 it can be seen that the viscous-radiation–diffraction model slightly overpredicts both the mean and slow drift responses. In this regard, the model is consistent. As the prediction of the mean drift force should be fairly accurate the difference between the estimated and measured mean

	Mean Drift (m)	Slow Drift (m)
Estimated	16.92	19.31[a]
		15.39[b]
		12.97[c]
Measured	15.00	9.22

[a]Wave-drift damping only

[b]Viscous damping only

[c]Wave-drift plus viscous damping

Table 6.1: Significant values of mean and slowly varying surge responses

drift response could be attributed to the error in measuring the horizontal stiffness of the mooring lines. On the other hand, it is important to notice that the experimental low frequency response spectra were obtained by Fast Fourier Transform of the time series (MARINTEK 2000, Berthelsen 2000) where numerical errors in transformations are possible.

Chapter 7

Conclusions

7.1 General conclusions

Methods of analytical mechanics were introduced into the marine hydrodynamics at the late nineteenth century. However, their use in marine hdrodynamics did not become widespread during the last century due to the complexity of their application to the fluid-body interaction problems in comparison with the application of the methods of Newtonian mechanics. Some of the difficulties were related to the treatment of the unbounded domain of fluid, the deformable free surface and the moving surface of the rigid body in the framework of the variational methods. In this work, the energy and analytical methods were applied to the problem of fluid-body interaction by a new approach which overcame some of those complexities and made their application to marine hydrodynamics problems comparable with the application of the dominant methods of Newtonian mechanics in terms of simplicity and ease of use. The key idea that made this simplification possible was the notion that the fluid kinetic and potential energy associated with the wetted surface of a body should be sufficient to formulate the dynamic problem of the fluid-body interaction. To verify this hypothesis, the linear radiation problem of a floating body was studied by combined application of Newtonian and Lagrangian approches and by employing real velocity potentials in place of their common complex counterparts. By this approach it became possible to show that the radiation damping force is derivable from a part of the kinetic energy which has a bilinear form. This was then used to prove that, for small amplitude oscillations, the fluid total mechanical energy corresponding to the body wetted surface is sufficient to find all fluid radiation effects on the motion equations of a body—immersed or floating. This result has not been reached by previous workers as they all used a surface integral in the farfield to express the radiation damping.

The study of the motion problem of a floating body by this method provided a unique example in analytical mechanics in which in place of a dissipation function a part of the kinetic energy was used to produce the damping force. As a consequence, Rayliegh's dissipation function were replaced with the bilinear kinetic energy of damping in Lagrange's equations of motion. This was done in the context of a mechanical system with a form of energy constitutive relations the same as that of the linear radiation problem. This change resulted in a variant of Lagrange's equations of motion. The derived equation was based on the total mechanical energy rather than the Lagrangian. These equations, though confined to a limited class of mechanical systems with a certain energy constitutive relation, are able to formulate the equations of motion of the system with one less scalar function in comparison with the Lagrange's equations with dissipation.

To complete the discussion, a conjugate variational operator was defined from the operator of first variation to introduce a variant of the Hamilton's principle as the variational generator of the variant Lagrange's equations of motion. The power of these equations and the simplicity of their application on the fluid-body interaction problem was shown in finding the linear differential equations of motion of a floating body. The derivation of the variants of the Hamilton principle and Lagrange's equations of motion is a new developement in the analytical mechanics.

In the next step, the derived expressions of the kinetic and potential energy associated with the wetted surface of a body were used to show that each of the added mass and radiation damping coefficients associated with a body contains three second-order tensors. As a result, each of the 6×6 radiation matrices associated with an immersed or a floating body can be partitioned into four 3×3 sub–matrices. The diagonal sub–matrices correspond to symmetric second–order tensors. The off–diagonal sub–matrices are transpose of each other and correspond to a second-order pseudo–tensor. In case of an immersed body results similar to those stated here were obtained by Happel & Brenner (1965). However, their work has remained unknown to marine hydrodynamicists. In this thesis, those results were derived by an energy method. In addition, for the first time, it was shown that the same results are also valid for a floating body oscillating with small amplitude motions. As a result of this finding, the transformation method was applied to radiation coefficients. The application of the transformation method, based on the transformation laws of second-order tensors, for calculation of added mass coefficients of offshore platforms was shown for a truss spar platform. It was shown that the transformation method can be more efficient than the more commonly used methods for the calculation of added mass and radiation damping coefficients.

Consistent with the transformation method, a simplified technique for the calculation of dynamic responses of an offshore platform to wave loads was introduced. This technique started first by assuming the fluid flow to be a potential flow and neglecting the viscosity. Then equations of motion were written by using a radiation–diffraction decomposition. To include viscosity effects in the otherwise non-viscous equations of motion, a viscous-radiation–diffraction model was proposed where the effects of viscosity were considered in the radiation and diffraction problems separately. As an engineering approximation and to simplify the model, the viscous damping forces acting on a platform oscillating in otherwise calm water were approximated by the viscous damping forces acting on a fixed platform subjected to an oscillatory flow, where the flow velocity was assumed to be the same as the oscillatory velocity of the platform in the radiation problem. As a result, the use of the relative velocity was eliminated from the calculation of viscous excitation and damping forces. In comparison with the traditional methods, this simplified the solution of the equations of motion greatly. In addition, the simple projected area method was employed for the calculation of viscous damping coefficients of a structure. By using the projected area method, the viscous coefficients associated with the radiation and diffraction problems were calculated for the whole structure at once. Then the viscous forces were added to the radiation and diffraction sides of the non-viscous equations of motion.

In the first-order diffraction problem, to be consistent with the transformation method and to obviate the use of the Morison's equation, the excitation forces acting on the slender members of the structure were approximated as the product of the mean value of the water particles' acceleration of those members and their inertia coefficient. It was also shown that, contrary to the conventional approach, in case of uncoupled modes of motion the non-linear equation of motion derived from the viscous-radiation–diffraction model can be put into the form of a quadratic equation which can be solved without iteration. This approach is more efficient than the iterative method of solution commonly seen in the literature and is believed to be new.

The viscous-radiation–diffraction model were developed for both first- and second-order dynamic response analysis of an offshore platform. Assumptions and approximations similar to those applied in the first-order model were employed to develop the second-order model. This leads to an equation of motion for the slow-drift surge motion where damping terms are written solely in terms of the slow velocity of the platform. Results of this approximate model were compared with the experimental results where the validity of the model for a truss spar platform were verified.

In predicting of first-order loads acting on truncated vertical cylinders, an engineering approximation were employed. The method modifies the traditional surge and pitch loads estimated by McCamy & Fuchs (1954) inline force per unit length in the vicinity of the truncation section to take the effect of truncation into account. To obtain the magnitude of the correction loads, the diffraction force was decomposed into a scattering and a Froude-Krylov force. Then expressions for the inertia and damping forces due to the scattering force were derived. The relation between these forces with the conventional diffraction force was also presented. It was shown that the accuracy of the proposed approximate method for a range of slenderness ratios is fairly good.

7.2 Concluding remarks on truss spar platforms

One of the objectives of the research undertaken in this thesis was the developement of a simplified dynamic response analysis technique which can be applied on truss spar platforms. This was achieved for both the first- and second-order dynamic response analysis of truss spar platforms.

Decomposing the truss spar to a surface piercing floating hull and a submerged truss and considering the effect of each structure separately simplified the dynamic analysis. Assuming the deeply submerged truss as a body oscillating in otherwise unbounded fluid was adequate for modelling the truss structure. This simplified the dynamic analysis greatly. The proposed transformation method together with the viscous-radiation–diffraction model worked very well as a simplified approximate method for the truss spar platforms in both first- and second-order dynamic response analysis. In the first- and second-order problems, good predictions were obtainred despite the fact that mooring lines were considered as linear restoring elements and an uncoupled hull/mooring analysis was performed. For the truss structure with a lot of slender members located in different directions, the use of projected area method increased the efficiency in predicting viscous damping coefficients of the structure and therefore increased the efficiency of the dynamic response analysis. Observing the results of the proposed approximate method for the wave and low frequency motions shows that in an over-simplified model in the surge and pitch motion the truss can be replaced by a damper neglecting all of its inertia and excitation effects. However, in heave motion the effect of the added mass has to be considered.

For the truss spar platform considered the standard deviation of the first-order heave and pitch motions were small. In heave, the first-order viscous-radiation–diffraction model predicted the standard deviation of motion much better than the more sophisticated methods in the time domain. The results of the model agreed very closely with the experimental results. In the coupled motions of surge and pitch the accuracy of the model was the same as that obtained from the method of time domain but results of both methods were rather poor with respect to the experimental results. In calculating the first-order surge and pitch loads of the hull, the proposed approximate method of including the truncation effects was found to have little effect in correcting loads acting on the truss spar considered in this research. Therefore, the more common approximation in which the effect of cylinder truncation is neglected can be used for this particular platform.

Classic spar platforms are resonance-dominant structures (Weggel 1997). In comparison with classic spars, truss spar platforms have larger damping coefficients in all modes of motion, therefore, they are less susceptible to resonance oscillations. Natural frequencies of these structures are usually lower than the frequency range of sea waves. Therefore, the effect of the sum-frequency part of the quadratic transfer functions are negligible in all modes of motion and only the difference-frequency terms have to be considered for these structures. Among the resonant motions the slow-drift surge motion generates the largest excursions to the platform and therefore exerts the design load on the mooring system. In heave motion, the resonance due to swell can be more important than the resonance due to difference frequency excitation.

For the truss spar and the wave environment considered in this work, the nonlinear slow-drift surge response was affected predominantly by the nonlinearities in the forcing functions and in the damping. Nonlinearities in the restoring and inertia terms were neglected without generating a significant error.

The quadratic transfer functions used for the mean drift and the mean wave-drift damping were simple algebraic expressions in terms of the diffraction parameter kR. Those expressions were derived by patching the available formulae in the literature for the mean drift force in the low frequency and asymptotic high frequency ranges. The derived formula performed well in predicting the second-order mean drift force and simplified the calculations enormously. As expected, the approximate formula of Aranha and Clark et al. worked well in predicting the mean wave-drift damping coefficient. In the second-order viscous-radiation–diffraction model, each of the drag and wave-drift damping alone decreased the amplitude of the resonant motion to a value comparable with the experimental results. As expected, the amount of damping was important at the resonance.

7.3 Recommendations for the future work

The methods of analytical mechanics and tensor analysis were applied to the hydrodynamics of floating and immersed bodies in this work. The effort was confined to the linear problem in the frequency domain. Therefore, it is worthwhile to continue this effort to more general cases. For instance, one can extend the analytical methods so that the problem of fluid-body interaction can be formulated where the nonlinear effects are included in the energy formulation of the problem. In this regard one may investigate if the total mechanical energy associated with the wetted surface of a body is sufficient to express all fluid effects on a body even in the case of the large amplitude nonlinear oscillations. Then one can study the tensor properties of the radiation coefficients in general case of nonlinear oscillations.

In terms of the application, the analytical methods led to the application of the transformation methods in the dynamic response analysis of the offshore structures. In this regard, a viscous-radiation–diffraction model was proposed as an alternative approach for the dynamic response analysis of offshore structures. For truss spar platforms these methods predict first- and second-order responses with fairly good accuracy and very good efficiency. This makes the proposed model attractive for the repetitive analysis and for the priliminary design. Therefore, from a practical point of view, one may investigate if these techniques can deliver the same level of accuracy and efficiency for other types of offshore structures.

Appendix A

Work-energy relation for a floating body

In Chapter 2 it is shown that for a mechanical system governed by (2.49),

$$T = T^Q + T^B, \tag{A.1a}$$

$$T^Q = T^Q(\dot{q}_\alpha) = \frac{1}{2}\dot{q}_\alpha\, a_{\alpha\beta}\, \dot{q}_\beta, \tag{A.1b}$$

$$T^B = T^B(\dot{q}_\alpha,\, q_\alpha) = \frac{1}{2}q_\alpha b_{\alpha\beta}\, \dot{q}_\beta, \tag{A.1c}$$

$$V = V(q_\alpha) = \frac{1}{2}q_\alpha c_{\alpha\beta}\, q_\beta, \tag{A.1d}$$

the equations of motion can be stated by (2.57) where these equations are derived from (2.53),

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^Q}{\partial \dot{q}_\gamma} + \frac{\partial V}{\partial q_\gamma} + 2\frac{\partial T^B}{\partial q_\gamma} = Q_\gamma. \tag{A.2}$$

Multiplying both sides of (A.2) by $\dot{q}_\gamma$ yields

$$\dot{q}_\gamma\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^Q}{\partial \dot{q}_\gamma} + \dot{q}_\gamma\frac{\partial V}{\partial q_\gamma} + 2\dot{q}_\gamma\frac{\partial T^B}{\partial q_\gamma} = \dot{q}_\gamma\, Q_\gamma. \tag{A.3}$$

Because $V = V(q_\alpha) = V(q_\alpha(t))$, we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial q_\gamma}\frac{\mathrm{d}q_\gamma}{\mathrm{d}t} = \dot{q}_\gamma \frac{\partial V}{\partial q_\gamma}. \tag{A.4}$$

In addition the first term on the left-hand side of (A.3) can be written as

$$\dot{q}_\gamma \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^Q}{\partial \dot{q}_\gamma} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}_\gamma \frac{\partial T^Q}{\partial \dot{q}_\gamma}\right) - \ddot{q}_\gamma \frac{\partial T^Q}{\partial \dot{q}_\gamma}. \tag{A.5}$$

But from (A.1b) we have

$$\frac{\partial T^Q}{\partial \dot{q}_\gamma} = a_{\alpha\gamma}\dot{q}_\alpha,$$

therefore,

$$\dot{q}_\gamma \frac{\partial T^Q}{\partial \dot{q}_\gamma} = 2T^Q. \tag{A.6}$$

Furthermore, because $T^Q = T^Q(q_\alpha) = T^Q(q_\alpha(t))$, it follows that

$$\frac{\mathrm{d}T^Q}{\mathrm{d}t} = \frac{\partial T^Q}{\partial \dot{q}_\gamma}\frac{\mathrm{d}\dot{q}_\gamma}{\mathrm{d}t} = \ddot{q}_\gamma \frac{\partial T^Q}{\partial q_\gamma}. \tag{A.7}$$

Now substituting from (A.6) and (A.7) into the right-hand side of (A.5) gives

$$\dot{q}_\gamma \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^Q}{\partial \dot{q}_\gamma} = \frac{\mathrm{d}}{\mathrm{d}t}\left(2T^Q\right) - \frac{\mathrm{d}T^Q}{\mathrm{d}t} = \frac{\mathrm{d}T^Q}{\mathrm{d}t}. \tag{A.8}$$

Introducing from the left-hand side of (A.4) and the most right-hand side term of (A.8) into (A.3) yields

$$\frac{\mathrm{d}T^Q}{\mathrm{d}t} + \frac{\mathrm{d}V}{\mathrm{d}t} + 2\dot{q}_\gamma \frac{\partial T^B}{\partial q_\gamma} = \dot{q}_\gamma Q_\gamma. \tag{A.9}$$

Now from (A.1c) we have

$$\frac{\partial T^B}{\partial q_\gamma} = \frac{1}{2}b_{\alpha\gamma}\dot{q}_\alpha,$$

therefore

$$2\dot{q}_\gamma \frac{\partial T^B}{\partial \dot{q}_\gamma} = \dot{q}_\gamma b_{\alpha\gamma}\dot{q}_\alpha = 2R \tag{A.10}$$

where R is Rayleigh's dissipation function defined in (2.44). In addition, $\dot{q}_\gamma Q_\gamma$ is the power associated with the non-conservative force Q_γ, so

$$\dot{q}_\gamma Q_\gamma = \frac{\mathrm{d}W^{nc}}{\mathrm{d}t}. \tag{A.11}$$

Substituting from (A.10) and (A.11) into (A.9) gives us

$$\frac{\mathrm{d}\left(T^Q + V\right)}{\mathrm{d}t} + 2R = \frac{\mathrm{d}W^{nc}}{\mathrm{d}t}. \tag{A.12}$$

Finally, using $T^Q + V = E^Q$ yields the work-energy relation for a mechanical system governed by (A.1) as follows

$$\frac{\mathrm{d}E^Q}{\mathrm{d}t} + 2R = \frac{\mathrm{d}W^{nc}}{\mathrm{d}t}. \tag{A.13}$$

For a rigid floating body the constitutive relations of the kinetic and potential energy of the water associated with the wetted surface of the body are the same as (A.1). In addition, for a floating body, apart from the fluid radiation actions, $T^B = 0$ and constitutive relations (A.1) reduce to the damping free consititutive relations (2.59). Therefore for a floating body R_B will vanish and $E_B^Q = E_B$ and we obtain the common work-energy relation

$$\frac{\mathrm{d}E_B}{\mathrm{d}t} = \frac{\mathrm{d}W^{nc}}{\mathrm{d}t}. \tag{A.14}$$

Now using the subscript $_{S_B}$ to denote quantities related to water and adding the contribution due to fluid radiation action to (A.14) yields

$$\frac{\mathrm{d}(E_B + E_{S_B}^Q)}{\mathrm{d}t} + 2R_{S_B} = \frac{\mathrm{d}W^{nc}}{\mathrm{d}t}. \tag{A.15}$$

This is the work-energy relation for a floating body which can also be derived directly from (2.65).

Appendix B

Transformation law for an arbitrary three-dimensional body

Consider an arbitrary three dimentional body and assume that the radiation tensor components R_{ij} are known in $x_1x_2x_3$ coordinate system. An arbitrary coordinate system can be obtained by rotating $x_1x_2x_3$ about its origin. In general, this new coordinate system can be obtained by three rotations about coordinte axes. One may choose the three rotations as roll, pitch and yaw. The order of rotations is arbitrary but for a systematic description of kinematics, a unique order should be used. We shall use yaw, pitch, roll in order. Therefore by rotating $x_1x_2x_3$ coordinate system about x_3-axis through an angle α, $x'_1x'_2x'_3$ coordinate system will be obtained. A second rotation through an angle θ about x'_2-axis produces $x''_1x''_2x''_3$ coordinate system and finally the third rotation, through an angle β about x''_1-axis produces $x'''_1x'''_2x'''_3$ coordinate system. The tranformation matrices for these three rotations are as follows

$$[a_1] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{B.1}$$

$$[a_2] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \tag{B.2}$$

and

$$[a_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix}. \tag{B.3}$$

The composite rotation matrix is obtained by applying these three rotations in order, i.e.,

$$[a] = [a_3]\,[a_2]\,[a_1] \tag{B.4}$$

Now by using the transformation law (5.19), the components of the radiation tensor in the triple-primed coordinate system can be obtained from the following equation

$$[R'''] = [a]\,[R]\,[a]^T \tag{B.5}$$

in which all six independent components of $[R]$ are non-zero in general and $[a]$ is given by (B.4) . For the sake of brevity, we shall derive the component form of the transformation law for two rotations of yaw and pitch. The equations for the general case of three rotations can be obtained by repeating matrix multiplications for the roll. Therefore, the combined rotation matrix will be

$$[a] = [a_2]\,[a_1] = \begin{bmatrix} \cos\theta\cos\alpha & \cos\theta\sin\alpha & -\sin\theta \\ -\sin\alpha & \cos\alpha & 0 \\ \sin\theta\cos\alpha & \sin\theta\sin\alpha & \cos\theta \end{bmatrix}, \tag{B.6}$$

and the transformation law is

$$[R''] = [a]\,[R]\,[a]^T, \tag{B.7}$$

where $[a]$ is given by (B.6). Substituting from (B.6) into (B.7) and performing matrix multiplications yields the following transformation equations

$$
\begin{aligned}
R''_{11} &= R_{11}\cos^2\theta\cos^2\alpha + R_{22}\cos^2\theta\sin^2\alpha + R_{33}\sin^2\theta + \\
&\quad R_{12}\cos^2\theta\sin 2\alpha - (R_{13}+R_{23})\sin 2\theta\cos\alpha && \text{(B.8a)}\\
R''_{22} &= R_{11}\sin^2\alpha + R_{22}\cos^2\alpha - R_{12}\sin 2\alpha && \text{(B.8b)}\\
R''_{33} &= R_{11}\sin^2\theta\cos^2\alpha + R_{22}\sin^2\theta\sin^2\alpha + R_{33}\cos^2\theta + \\
&\quad R_{12}\sin^2\theta\sin 2\alpha + (R_{13}+R_{23})\sin 2\theta\cos\alpha && \text{(B.8c)}\\
R''_{12} = R''_{21} &= \frac{1}{2}(R_{22}-R_{11})\cos\theta\sin 2\alpha + R_{12}\cos\theta\cos 2\alpha + \\
&\quad (R_{13}\sin\alpha - R_{23}\cos\alpha)\sin\theta && \text{(B.8d)}\\
R''_{23} = R''_{32} &= \frac{1}{2}(R_{22}-R_{11})\sin\theta\sin 2\alpha + R_{12}\sin\theta\cos 2\alpha + \\
&\quad (R_{23}\cos\alpha - R_{13}\sin\alpha)\cos\theta && \text{(B.8e)}\\
R''_{13} = R''_{31} &= \frac{1}{2}(R_{11}\cos^2\alpha - R_{22}\sin^2\alpha)\sin 2\theta + \frac{1}{2}R_{12}\sin 2\theta\sin 2\alpha + \\
&\quad (R_{13}\cos\alpha + R_{23}\sin\alpha)\cos 2\theta - \frac{1}{2}R_{33}\sin 2\theta && \text{(B.8f)}
\end{aligned}
$$

Now consider the application of (B.8) for added mass coefficients of an immersed cylinder with arbitrary cross section as shown in figure B.1 in p. 148. If the origin of the coordinate system is assumed in the mid cross-section of the cylinder, the x_1x_2-plane will be the plane of symmetry and from the first symmetry rule in § 3.3.2 the added mass matrix takes the following form

$$
[m] = \begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{12} & m_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \text{(B.9)}
$$

In figure B.1 the primed coordinate system is obtained from the unprimed coordinate system by a yaw. Therefore, by setting $\theta = 0$ in (B.8) and replacing R''_{ij} and R_{ij} with m'_{ij} and m_{ij}, respectively, we obtain

$$
m'_{11} = m_{11}\cos^2\alpha + m_{22}\sin^2\alpha + m_{12}\sin 2\alpha, \qquad \text{(B.10a)}
$$

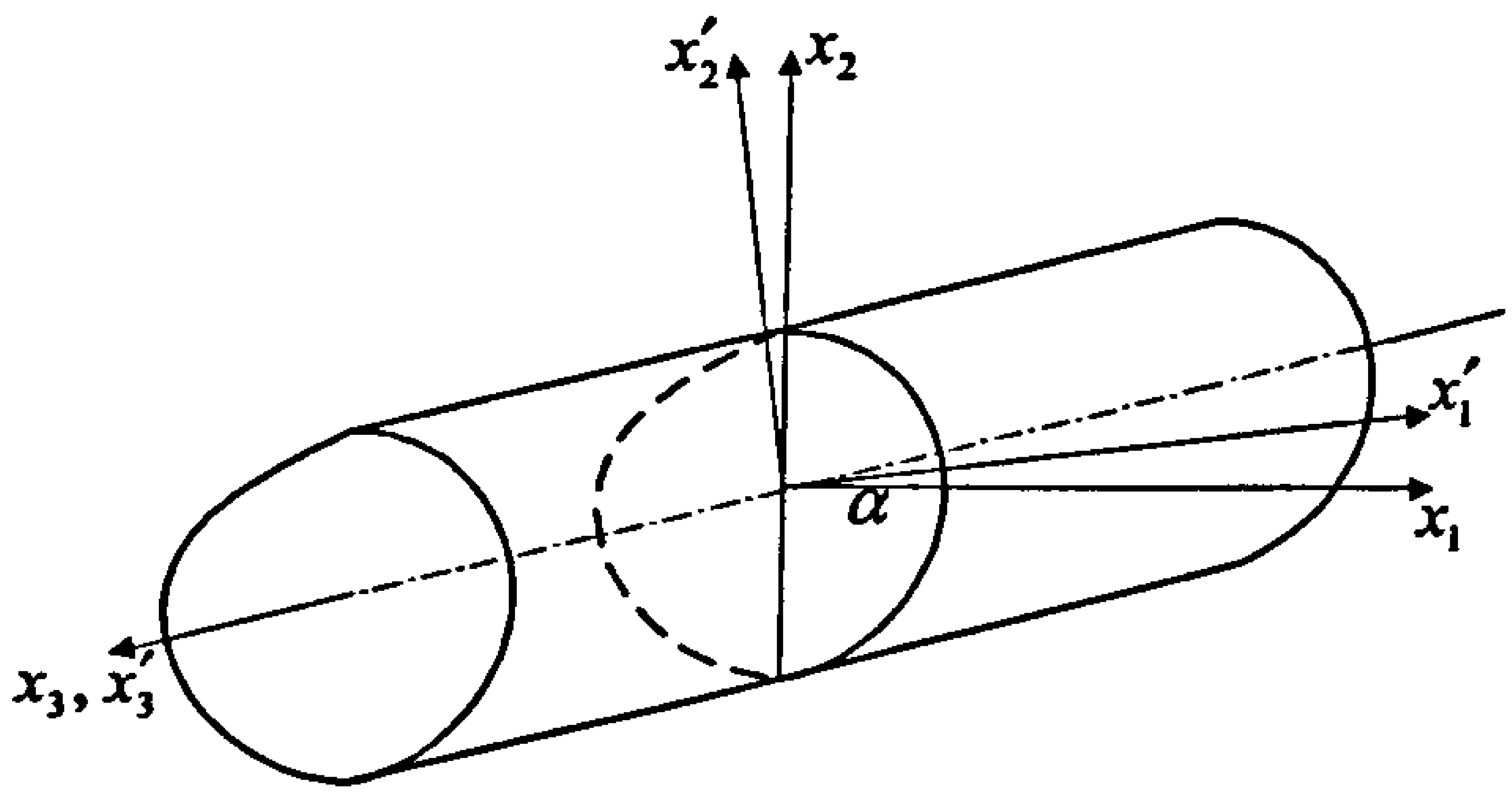

Figure B.1: A cylinder with arbitrary cross section

$$m'_{22} = m_{11}\sin^2\alpha + m_{22}\cos^2\alpha - m_{12}\sin 2\alpha, \tag{B.10b}$$

$$m'_{12} = m'_{21} = \frac{1}{2}(m_{22} - m_{11})\sin 2\alpha + m_{12}\cos 2\alpha, \tag{B.10c}$$

and

$$m'_{33} = m_{33}, \tag{B.11a}$$

$$m'_{13} = m'_{31} = m'_{23} = m'_{32} = 0. \tag{B.11b}$$

Equations (B.10) are the same as equations that Sedov (1965) derived for an arbitrary two–dimensional body in the complex plane and Sarpkaya & Isaacson (1981) restated them. Method of Sedov (1965) is not applicable to a three–dimensional body. The transformation method can, however, be used for an arbitrary three dimensional body in an arbitrary rotation of coordinate system.

Appendix C

Derivation of translation law of radiation tensors of a floating body

In order to derive the translation law of radiation tensors of a floating body, we shall consider (3.15)

$$T^Q = \frac{1}{2}U_i A_{ij} U_j + \frac{1}{2}\Omega_i S_{ji} U_j + \frac{1}{2}U_i S_{ij}\Omega_j + \frac{1}{2}\Omega_i X_{ij}\Omega_j, \quad \text{(C.1a)}$$

$$T^B = \frac{1}{2}q_i B_{ij} U_j + \frac{1}{2}\theta_i D_{ji} U_j + \frac{1}{2}q_i D_{ij}\Omega_j + \frac{1}{2}\theta_i E_{ij}\Omega_j,. \quad \text{(C.1b)}$$

Following (3.35) we have

$$U_i = U_i' + H_{ik}\Omega_k \quad \text{(C.2a)}$$

$$U_j = U_j' + H_{jl}\Omega_l. \quad \text{(C.2b)}$$

In (C.1b) θ_i is the component of the infinitesimal rotation vector, therefore, an equation similar to (C.2) can be written for the components of the displacement vector, i.e.,

$$q_i = q_i' + H_{ik}\theta_k \quad \text{(C.3a)}$$

$$q_j = q'_j + H_{jl}\theta_l \,. \tag{C.3b}$$

Second order tensors U_iU_j, Ω_iU_j and $U_i\Omega_j$ on the right-hand side of (C.1a) can be expressed in the primed coordinate system by using (C.2). The result is given in (3.36). Therefore, equations (3.40) will also be valid for added mass tensors of a floating body, i.e.,

$$A'_{ij} = A_{ij}, \tag{C.4a}$$

$$S'^T_{ij} = S^T_{ij} + H^T_{ik}A_{kj}, \tag{C.4b}$$

$$S'_{ij} = S_{ij} + A_{ik}H_{kj}, \tag{C.4c}$$

$$X'_{ij} = H^T_{ik}A_{kl}H_{lj} + H^T_{ik}S_{kj} + S^T_{ik}H_{kj} + X_{ij}. \tag{C.4d}$$

Equation (C.4) represents the parallel-axes theorem for the added mass tensors of a floating body. Now consider the second-order tensor components q_iU_j, θ_iU_j and $q_i\Omega_j$ on the right-hand side of (C.1b). Substituting from (C.2b) and (C.3a) for U_j and q_i, respectively, yields

$$q_iU_j = q'_iU'_j + q'_iH_{jl}\Omega_l + H_{ik}\theta_kU'_j + H_{ik}H_{jl}\theta_k\Omega_l \tag{C.5a}$$

$$\theta_iU_j = \theta_iU'_j + \theta_iH_{jl}\Omega_l \tag{C.5b}$$

$$q_i\Omega_j = q'_i\Omega_j + H_{ik}\theta_k\Omega_j \,. \tag{C.5c}$$

In addition, because θ_i and Ω_j are free vectors, the second-order tensor $\theta_i\Omega_j$ remains invariant in the primed coordinate system. Now introducing from (C.5) into the right-hand side of (C.1b) and changing the dummy indices i and j, respectively, with dummy indices k and l yields

$$\begin{aligned} T^B = \frac{1}{2}q'_iB_{ij}U'_j + \frac{1}{2}\theta_i\left\{D^T_{ij} + H^T_{ik}B_{kj}\right\}U'_j + \frac{1}{2}q'_i\left\{D_{ij} + B_{il}H_{lj}\right\}\Omega_j \\ +\frac{1}{2}\theta_i\left\{H^T_{ik}B_{kl}H_{lj} + H^T_{ik}D_{kj} + D^T_{il}H_{lj} + E_{ij}\right\}\Omega_j. \end{aligned} \tag{C.6}$$

On the other hand, T^B in the primed coordinate system takes the following form

$$T^B = \frac{1}{2}q'_i B'_{ij} U'_j + \frac{1}{2}\theta_i D'^T_{ij} U'_j + \frac{1}{2}q'_i D'_{ij}\Omega_j + \frac{1}{2}\theta_i E'_{ij}\Omega_j. \tag{C.7}$$

Now equating the right-hand sides of (C.6) and (C.7) and taking into account that θ_i, Ω_i, U'_i and q'_i are independent, arbitrary and generally non-zero, it follows that

$$B'_{ij} = B_{ij}, \tag{C.8a}$$

$$D'^T_{ij} = D^T_{ij} + H^T_{ik} B_{kj}, \tag{C.8b}$$

$$D'_{ij} = D_{ij} + B_{ik} H_{kj}, \tag{C.8c}$$

$$E'_{ij} = H^T_{ik} B_{kl} H_{lj} + H^T_{ik} D_{kj} + D^T_{ik} H_{kj} + E_{ij}. \tag{C.8d}$$

Equation (C.8) represents the parallel-axes theorem for the radiation damping tensors of a floating body.

Appendix D

Translation Law in Component Form

In order to obtain the parallel–axes–theorem (3.47) in component form, assume that the components d_i in (3.34) are given by

$$\mathbf{d} = d_i \, \mathbf{e}_i \equiv (a,\, b,\, c). \tag{D.1}$$

Substituting (D.1) into $H_{ij} = -\epsilon_{ijk} d_k$ gives the H_{ij} components which in matrix form can be written as

$$[H] = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}. \tag{D.2}$$

Now for an immersed body by inserting tensors $\mathbf{m}$, $\mathbf{J}$ and $\mathbf{I}$ for $\mathbf{R}^0$, $\mathbf{R}^1$ and $\mathbf{R}^2$ repectively, in (3.45) and then using (D.2) and performing the matrix multiplications, the parallel–axes–theorem in component form for added mass coefficients can be stated as follows

$$J'_{11} = c\,m_{12} - b\,m_{13} + J_{11},$$

$$J'_{12} = -c\,m_{11} + a\,m_{13} + J_{12},$$

$$
\begin{aligned}
J'_{13} &= b\,m_{11} - a\,m_{12} + J_{13},\\
J'_{21} &= c\,m_{22} - b\,m_{23} + J_{21},\\
J'_{22} &= -c\,m_{12} + a\,m_{23} + J_{22},\\
J'_{23} &= b\,m_{12} - a\,m_{22} + J_{23},\\
J'_{31} &= c\,m_{23} - b\,m_{33} + J_{31},\\
J'_{32} &= -c\,m_{13} + a\,m_{33} + J_{32},\\
J'_{33} &= b\,m_{13} - a\,m_{23} + J_{33},
\end{aligned}
\tag{D.3}
$$

and

$$
\begin{aligned}
I'_{11} &= c^2\,m_{22} + b^2\,m_{33} - 2bc\,m_{23} + 2(c\,J_{21} - b\,J_{31}) + I_{11},\\
I'_{22} &= c^2\,m_{11} + a^2\,m_{33} - 2ac\,m_{13} + 2(a\,J_{32} - c\,J_{12}) + I_{22},\\
I'_{33} &= b^2\,m_{11} + a^2\,m_{22} - 2ab\,m_{12} + 2(b\,J_{13} - a\,J_{23}) + I_{33},\\
I'_{12} &= -c^2 m_{12} + bc m_{13} + ac m_{23} - ab m_{33} + c(J_{22} - J_{11}) - bJ_{32} + aJ_{31} + I_{12},\\
I'_{13} &= bc m_{12} - b^2 m_{13} - ac m_{22} + ab m_{23} + b(J_{11} - J_{33}) - aJ_{21} + cJ_{23} + I_{13},\\
I'_{23} &= -bc m_{11} + ac m_{12} + ab m_{13} - a^2 m_{23} + a(J_{33} - J_{22}) - cJ_{13} + bJ_{12} + I_{23}.
\end{aligned}
\tag{D.4}
$$

In order to obtain the parallel-axes-theorem or the translation law from the primed- to unprimed-coordinate system in the component form, one should substitute (D.2) into (3.48). The results are

$$
\begin{aligned}
J_{11} &= -c\,m'_{12} + b\,m'_{13} + J'_{11},\\
J_{12} &= c\,m'_{11} - a\,m'_{13} + J'_{12},\\
J_{13} &= -b\,m'_{11} + a\,m'_{12} + J'_{13},\\
J_{21} &= -c\,m'_{22} + b\,m'_{23} + J'_{21},\\
J_{22} &= c\,m'_{12} - a\,m'_{23} + J'_{22},\\
J_{23} &= -b\,m'_{12} + a\,m'_{22} + J'_{23},
\end{aligned}
\tag{D.5}
$$

$$J_{31} = -c\,m'_{23} + b\,m'_{33} + J'_{31},$$

$$J_{32} = c\,m'_{13} - a\,m'_{33} + J'_{32},$$

$$J_{33} = -b\,m'_{13} + a\,m'_{23} + J'_{33},$$

and

$$\begin{aligned}
I_{11} &= c^2\,m'_{22} + b^2\,m'_{33} - 2bc\,m'_{23} + 2(b\,J'_{31} - c\,J'_{21}) + I'_{11},\\
I_{22} &= c^2\,m'_{11} + a^2\,m'_{33} - 2ac\,m'_{13} + 2(c\,J'_{12} - a\,J'_{32}) + I'_{22},\\
I_{33} &= b^2\,m'_{11} + a^2\,m'_{22} - 2ab\,m'_{12} + 2(a\,J'_{23} - b\,J'_{13}) + I'_{33},\\
I_{12} &= -c^2m'_{12} + bcm'_{13} + acm'_{23} - abm'_{33} + c(J'_{11} - J'_{22}) + bJ'_{32} - aJ'_{31} + I'_{12},\\
I_{13} &= bcm'_{12} - b^2m'_{13} - acm'_{22} + abm'_{23} + b(J'_{33} - J'_{11}) + aJ'_{21} - cJ'_{23} + I'_{13},\\
I_{23} &= -bcm'_{11} + acm'_{12} + abm'_{13} - a^2m'_{23} + a(J'_{22} - J'_{33}) + cJ'_{13} - bJ'_{12} + I'_{23}.
\end{aligned} \tag{D.6}$$

Analogous equations for added mass and damping coefficients of a floating body can be obtained from (3.45) and (3.46) if the relevant tensors are substituted for $\mathbf{R}^0$, $\mathbf{R}^1$, $\mathbf{R}'^1$, $\mathbf{R}^2$ and $\mathbf{R}'^2$. For a symmetric body by using two symmetry rules of § 3.3.2 zero components of tensors can be found and therefore (D.3) to (D.6) will be simplified.

To verify the parallel-axes-theorem (3.47), we shall derive the well known parallel-axes-theorem for the moment of inertia tensor of a rigid body mass from (3.40d)

$$I'_{jm} = H^T_{ji} m_{ik} H_{km} + H^T_{ji} J_{im} + J^T_{ji} H_{im} + I_{jm}. \tag{D.7}$$

but for a rigid body mass, m_{ik} corresponds to an isotropic tensor such that

$$m_{ik} = m\,\delta_{ik} \tag{D.8}$$

where m is the mass of the body. Therefore,

$$I'_{jm} = H^T_{ji} m\, \delta_{ik} H_{km} + H^T_{ji} J_{im} + J^T_{ji} H_{im} + I_{jm}. \tag{D.9}$$

To make the derivations simple, we assume that the origin of the unprimed coordinate system coincides the centre of mass of the body. This implies that $J_{im} = 0$ but does not affect the generality of the final result. Therefore (D.9) simplifies as follows

$$I'_{jm} = H^T_{ji} m\, \delta_{ik} H_{km} + I_{jm} = m\, H^T_{ji} H_{im} + I_{jm}. \tag{D.10}$$

However, we have (see page 45)

$$H_{ij} = -\epsilon_{ijk} d_k. \tag{D.11}$$

Therefore,

$$\begin{aligned} H^T_{ji} H_{im} &= H_{ij} H_{im} = (-\epsilon_{ijk} d_k)(-\epsilon_{iml} d_l) = \epsilon_{ijk}\epsilon_{iml} d_k d_l \\ &= (\delta_{jm}\delta_{kl} - \delta_{jl}\delta_{km}) d_k d_l = d_l d_l \delta_{jm} - d_j d_m. \end{aligned} \tag{D.12}$$

Introducing from (D.12) into (D.10) and denoting $d_l d_l$ by d^2, it follows that

$$I'_{jm} = I_{jm} + m(d^2 \delta_{jm} - d_j d_m). \tag{D.13}$$

Equation (D.13) is the general form of Steiner's (1796–1863) parallel-axes-theorem (e.g., see Marion & Thornton 1995 p.420, Norwood 1979 p.185). As shown herein, equation (D.13) is a particular form of a more general parallel-axes theorem (3.40d) when $m_{ij} = m\delta_{ij}$.

Appendix E

Added-mass Matrices of a Circular Cylinder

Consider a circular cylinder as shown in figure E.1 in p. 160. Because the three planes of the coordinate system $x_1x_2x_3$ are symmetry planes of the cylinder, the application of symmetry rules of § 3.3.2 results in the following relations

$$[m] = \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{bmatrix}, \tag{E.1}$$

$$[I] = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}, \tag{E.2}$$

and

$$[J] = 0 \tag{E.3}$$

On the other hand, a circular cylinder is an axisymmetric body and here the x_1-axis is the axis of circular symmetry. Thus (E.1) and (E.2) will be more simplified as follows

$$[m] = \begin{bmatrix} \hat{m}' & 0 & 0 \\ 0 & \hat{m} & 0 \\ 0 & 0 & \hat{m} \end{bmatrix}, \tag{E.4}$$

and

$$[m] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \hat{I} & 0 \\ 0 & 0 & \hat{I} \end{bmatrix}, \tag{E.5}$$

in which

$$\hat{m} = \rho \pi r^2 l, \tag{E.6}$$

$$\hat{m}' = \frac{8}{3}\rho r^3, \tag{E.7}$$

and

$$\hat{I} = I_{22} = I_{33}. \tag{E.8}$$

Now we shall use the parallel-axes theorem to calculate $\hat{I}$. To this end, we first consider a slice of thickness dx_1 of the cylinder as shown in figure E.1. From parallel-axes theorem, equation (D.6), we have

$$I_{22} = c^2 m'_{11} + a^2 m'_{33} - 2acm'_{13} + 2(cJ'_{12} - aJ'_{32}) + I'_{22} \tag{E.9}$$

where primed quantities on the right-hand side of (E.9) are added mass coefficients of the circular disk with respect to its local coordinate system $x'_1x'_2x'_3$ and (a, b, c) is the position vector of o' with respect to o. Because the three coordinate planes of $x'_1x'_2x'_3$ are also symmetry planes of the circular disk, it follows that

$$m'_{13} = J'_{12} = J'_{32} = 0. \tag{E.10}$$

Therefore,

$$I_{22} = c^2 m'_{11} + a^2 m'_{33} + I'_{22}. \tag{E.11}$$

Also from figure E.1 we have

$$b = c = 0, \tag{E.12a}$$

$$a = x_1. \tag{E.12b}$$

Using (E.12) and assuming the circular disk as a differential element of the cylinder, equation (E.11) can be written as follows

$$\mathrm{d}I_{22} = x_1^2 m'_{33} + \mathrm{d}I'_{22}. \tag{E.13}$$

Using (E.6) it follows that

$$\mathrm{d}I_{22} = x_1^2 \rho\pi r^2 \mathrm{d}x_1 + \mathrm{d}I'_{22}. \tag{E.14}$$

Integrating over the length of the cylinder, gives us

$$I_{22} = \int_{-l/2}^{l/2} \mathrm{d}I_{22} = \int_{-l/2}^{l/2} x_1^2 \rho\pi r^2 \mathrm{d}x_1 + \int_{-l/2}^{l/2} \mathrm{d}I'_{22} \tag{E.15}$$

or

$$I_{22} = \frac{1}{12}\rho\pi r^2 l^3 + I'_{22} \tag{E.16}$$

or using (E.6) and (E.8) we obtain

$$\hat{I} = \frac{1}{12}\hat{m}l^2 + \hat{I}'. \tag{E.17}$$

On the other hand mass moment of inertia of a circular cylinder is equal to

$$I = \frac{1}{12}ml^2 + \frac{1}{4}mr^2 \tag{E.18}$$

where m is the mass of the cylinder. Comparing (E.17) and (E.18), by analogy one may assume

$$\hat{I}' = \frac{1}{4}\hat{m}r^2. \tag{E.19}$$

Thus,

$$\hat{I} = \frac{1}{12}\hat{m}l^2 + \frac{1}{4}\hat{m}r^2. \tag{E.20}$$

The ratio of the second term to the first term on the right-hand side of (E.20) is $(r/l)^2$. Therefore, for slender cylinders, where $r/l \ll 1$, the second term is negligible with respect to the first one, and we have

$$\hat{I} = \frac{1}{12}\hat{m}l^2 = \frac{1}{12}\rho\pi r^2 l^3. \tag{E.21}$$

In a structure made of slender circular cylinders like a truss, bases of the cylindrical members are usually covered by the adjacent members and have no contact with water. Excluding the effect of bases is equivalent to neglecting $\hat{m}'$ given by (E.7) in the added mass matrices $[m]$ and $[I]$. Hence, the added mass matrices can be considered as follows

$$[m] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \hat{m} & 0 \\ 0 & 0 & \hat{m} \end{bmatrix}, \tag{E.22}$$

$$[I] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \hat{I} & 0 \\ 0 & 0 & \hat{I} \end{bmatrix}. \tag{E.23}$$

If instead of x_1-axis the x_2-axis is the axis of circular symmetry, we obtain

$$[m] = \begin{bmatrix} \hat{m} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hat{m} \end{bmatrix}, \tag{E.24}$$

$$[I] = \begin{bmatrix} \hat{I} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hat{I} \end{bmatrix}, \tag{E.25}$$

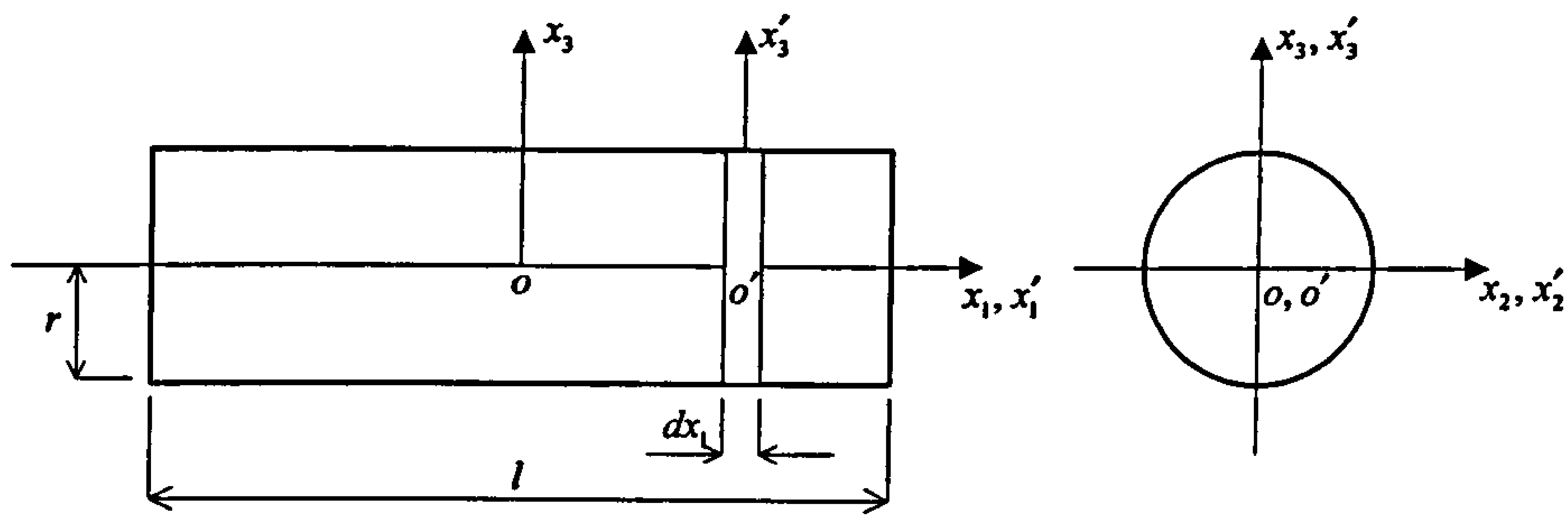

Figure E.1: A typical circular cylinder

and for x_3-axis as the axis of circular symmetry we have

$$[m] = \begin{bmatrix} \hat{m} & 0 & 0 \\ 0 & \hat{m} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{E.26}$$

$$[I] = \begin{bmatrix} \hat{I} & 0 & 0 \\ 0 & \hat{I} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{E.27}$$

In (E.22) to (E.27), $\hat{m}$ and $\hat{I}$ are given by (E.6) and (E.21).

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