Particle Detectors in Curved Space Quantum Field Theory

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Abstract

Ambiguities of the particle concept in non-Minkowski spaces are reviewed. To study this and other aspects of quantum field theory y in curved spaces, an operationalist approach is adopted through the use of particle detector models. A precise definition of this general concept is given and shown to include many different types of detector models. Five particular models are studied in detail and their responses in Rindler and Schwarzschild spaced are evaluated. In the Rindler case it is explicitly shown that acceleration radiation is anisotropic and time independent. Direct comparison of detectors' responses is seen to be unsuitable for determining whether two different detectors 'perceive' a given situation identically. A method for comparing different detectors is constructed and applied to the models previously introduced. This leads to the notion of equivalence of different detectors, thereby circumventing the problems of direct comparison of their responses. In addition several general results about quantum fields in non-Minkowski spaces are proven. By studying the details of how particle detectors work, the reasons fordifferent detectors being (in)equivalent are revealed. Model detectors of the charged scalar field and spinor fields are then introduced and several problems of "overly simplistic" models are discussed: in particular problems arising from the fact that these fields contain several species of particles. Particle detector equivalence is then applied to these models and used to construct an elementary symmetry between the charged scalar and spinor field many-particle states in the Minkowski Fock space. Finally, a general discussion of several philosophical and practical aspects of using particle detectors to study quantum fields in curved spaces is presented and some points of general confusion are clarified. The particle detector model is operationalist and as such is seen to be most productive when used with close adherence to the Copenhagen interpretation of quantum mechanics.

The notation and sign conventions used in this thesis follow those adopted in Birrell & Davies (1982).

Contents

1 Introduction

The study of quantum fields in curved space-time has always been classified as a topic of "theoretical physics". There is good reason for this. Almost all (experimentally) observable effects require such extreme physical conditions that there is little prospect, if any, of being able to set up a "laboratory tests" to check the theory either today or in the foreseeable future. The only "laboratories" available at present are the early universe and black holes, both of which are rather distant from us (in space and/or time) making their practical use difficult and inexact. Due to this paucity of direct experimental tests the development of this theory is largely founded on self-consistency, the discovery of unifying links with other fields of physics (notably thermodynamics) and agreement with accepted theories in the "weak field" limit.

A widely used approach for studying the properties of quantum fields in curved spaces is to evaluate and analyse the expectation value of the energy stress tensor $<T_{\mu\nu}>$ for a given quantum state. An alternative technique, originally introduced by Unruh (1976), is to study the responses of model particle detectors of the quantum field in a curved space background. This latter approach avoids some of the calculational problems the stress tensor technique intrinsically suffers. (In particular, the requirement of renormalisation.) Apart from being more convenient, there are good philosophical reasons for utilising particle detector models in this field.

The purpose of studying quantum field theory in curved space-times is to enquire into the nature of <u>quantum</u> phenomena in the presence of gravitational fields. Being quantum in character, we are immediately confronted with interpretational difficulties similar to those faced by the founders of quantum mechanics. Therefore, a "philosophical position" must be taken by us (as researchers) on how to interpret the techniques used and results acquired. In this thesis a fairly strict Copenhagen interpretation shall be adopted, it being the most widely accepted view amongst physicists today. This step elevates the study of particle detectors to an even higher level of importance because, according to the Copenhagen interpretation;

> In a quantum phenomena, …. , measurement does not <u>determine</u> a property of the object so much as essentially <u>define</u> it in the first place, as far as possible. (Scheibe 1973)

From this it is apparent that the Copenhagen view is fundamentally operationalist since a property of quantity has no clear meaning unless it can be measured. Further, such measurements must be using some apparatus setup <u>expressly</u> to measure that <u>particular</u> quantity.

Therefore, in the final analysis, no matter what approach is used to study quantum fields in curved space (including the $<T_{\mu\nu}>$ method), we must study the detector model approach. Our calculations can only ever be related to the physical world through the experiences of detectors. At present such machines as cloud and bubble chambers, scintillation counters, photo-multipliers etc. fulfil this vital role. Unfortunately, as we have already noted, these machines are not quite suited to the regime of physical conditions in which we are interested. So, we must seek other ways of relating the quantum effects we wish to study to measurable quantities which have meaning for us. This is where the Unruh "box" detector and subsequently the DeWitt "monopole" detector (DeWitt 1979) have an important role to play.

In both these models the complications of fine detail have been either stripped away or conveniently pushed aside, leaving only the essential of the interaction between the detector and the field. This is how it should be (at least as an initial step to modelling the "real thing") since it is the interaction Lagrangian relating the quantum field to the detector that makes the apparatus a "measuring device" of the field.

Further, as we shall see, the nature of this coupling between field and device is fundamental to the character of the detector's response to the field.

The Unruh and DeWitt detectors are very similar in that they both couple linearly to the quantum field. It is therefore no surprise that these two devices give identical results. However, there certainly is no reason to expect (a priori) that only linear couplings occur in nature and there are many other interactions available. Considering the fundamental role particle detectors play in developing physical theories (particularly quantum theories) we should also study detectors that couple to quantum field through other Lagrangians. This task is undertaken in the chapters that follow.

Chapter 2 presents a brief survey of basic quantum field theory (of the neutral scalar field) in curved space-time. This survey is used to highlight an initial motivation for introducing particle detector models into the theory. A precise definition of the concept "particle detector" based upon physically realistic notions, is presented. In Chapter 3, five different particle detector models are introduced and their respective responses to a many-particle state in Minkowski space are evaluated. Their responses show that these models do in fact satisfy the definition of a particle detector. Chapters 4 to 7 study each of the five models in detail and their responses to Rindler and Schwarzschild space-times are evaluated. The isotropy of acceleration radiation is also discussed in some of these chapters.

By this stage, it will be quite evident that each detector model responds in a fundamentally different way when placed in the same circumstances. Thus, any attempt to directly compare their responses will be fruitless. In Chapter 8 a method of meaningfully comparing different detector models is introduced and a concept of detector equivalence is defined. These techniques are then used in Chapter 9 to compare the five detector models as well as prove several general theorems about quantum fields in non-Minkowski spaces.

The effect of non-trivial topology on particle detector responses is the subject of Chapter 10. It is shown that linear (DeWitt) detector uniformly accelerating in a $R^1 \times S^1$ flat space-time will respond differently to twisted and untwisted fields. Chapter 11 returns to the question of particle detector equivalence by studying, in close detail, how these devices work and why different detectors may be (in)equivalent.

So far, the thesis has considered only detectors of the neutral scalar field. In Chapter 12, detectors of the charge scalar and spinor fields are defined and discussed. The notion of comparing detectors of different quantum fields is introduced and developed. The possibility of using this detector-equivalence to construct symmetries between scalar and spinor fields is also considered. Finally, in the Conclusion several practical and philosophical aspects of using particle detectors to study quantum field theory in curved space-times are addressed.

During the production of this thesis I published (or jointly published) four research articles (Hinton et al. 1983, Hinton 1983, 1984 and Copeland et al. 1984), all of which form parts of this thesis. The first and last of these four works are contained in Chapters 4 and 10.

I would like to take this opportunity to express my sincere gratitude to my supervisor, Professor P.C.W. Davies for the many and extensive discussions we had, his encouragement and patience. I must acknowledge the initial guidance provided by J. Pfautsch during the year we were both at Newcastle Upon Tyne. Other people who have assisted me during the preparation of this thesis area: S. Bedding, E. Copeland, I. Moss, S. Unwin, W. Walker and A. Wright. I was also fortunate to have participated in helpful discussions with the following people: P. Broadbridge, L. Ford, P. Grove, C. Isham, A. Ottewill, W. Unruh and J. Wheeler. Finally I must acknowledge the endless patience, understanding and encouragement of my wife Rita. This work was funded by the Association of Commonwealth Universities through a British Council Commonwealth Scholarship.

2 Particle Detectors in Quantum Fields in Curved Space-times

2.1 Why Study Particle Detector?

The main aim of quantum field theory, as with most mathematical models of nature, is to represent and describe structures observed in nature. Prime candidates for such tasks are “particles”. Elementary particle theory is a major field of modern theoretical and experimental research.

The study of quantum fields in Minkowski space places significant emphasis on the representation of particles. The importance of the Fock representation of the Hilbert space exemplifies this. Minkowski space, as compared to arbitrary (flat or curved) space-times, possesses special features that greatly facilitate the construction of representations of particles which accord quite well with our intuition. The Poincare symmetries of Minkowski space play a fundamental role in this since the particle states in this space carry the irreducible representation of the Poincare Group (Schweber 1961, Davies 1984). These symmetries and representations lead to the Fock representation in a natural way.

However, in any arbitrary space, such a high degree of symmetry may not always exist, resulting in there being no particular representation which stands out as the above representation does in Minkowski Space.

In this chapter we will show how, in Minkowski Space, the Fock representation leads to a concept of particle which accords well with intuition. In will be shown, however, that in an arbitrary space this intuitive approach is found wanting.

Consider an n-dimensional Minkowski space, the field equation for a scalar field $\phi[x]$ is;

$$\left(\Box+m^2\right)\phi[x]=0 \tag{2.1}$$

where $\Box=\eta^{\alpha\beta}\partial_\alpha\partial_\beta=\eta^{\alpha\beta}\left(\partial/\partial x^\alpha\right)\left(\partial/\partial x^\beta\right)$ is the Minkowski metric and m is the mass of the field quanta. We can express $\phi[x]$ as a mode integral

$$\phi[x]=\int d^{n-1}k\left(u_{\boldsymbol{k}}(x)a_{\boldsymbol{k}}+u_{\boldsymbol{k}}^*(x)a_{\boldsymbol{k}}^*\right) \tag{2.2}$$

In which the field basis functions $u_{\boldsymbol{k}}(x)$ satisfy the scalar equation and can be written in the form

$$u_k(x)=\exp\left(i\boldsymbol{k}.\boldsymbol{x}-i\omega t\right)\Big/\left(2\omega\left(2\pi\right)^{n-1}\right)^{1/2} \tag{2.3}$$

The operators $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^*$ satisfy

$$\left[a_{\boldsymbol{k}},a_{\boldsymbol{k}'}\right]=\left[a_{\boldsymbol{k}}^*,a_{\boldsymbol{k}'}^*\right]=0$$

$$\left[a_{\boldsymbol{k}},a_{\boldsymbol{k}'}^*\right]=\delta^{(n-1)}\left(\boldsymbol{k}-\boldsymbol{k}'\right) \tag{2.4}$$

Where [a,b] = ab – ba and $\delta^{(n-1)}(k\text{-}k')$ is an $(n\text{-}1)$-dimensional Dirac delta function. We define an inner product for the scalar field by

$$\left(\phi_1,\phi_2\right)=-i\int_{S_t}\left\{\phi_1[x]\partial_t\phi_2^*[x]-\left(\partial_t\phi_1[x]\right)\phi_2^*[x]\right\}d^{(n-1)}x \tag{2.5}$$

S_t denoting a space-like hyper-surface of simultaneity at instant t and $\partial_t=\partial/\partial t$.

The functions $u_k(x)$ are orthogonal in that they satisfy

$$\left(u_{\boldsymbol{k}}(x), u_{\boldsymbol{k}'}(x)\right) = \delta^{(n-1)}(\boldsymbol{k} - \boldsymbol{k}')$$

$$\left(u^*_{\boldsymbol{k}}(x), u^*_{\boldsymbol{k}'}(x)\right) = \delta^{(n-1)}(\boldsymbol{k} - \boldsymbol{k}') \tag{2.6}$$

The basis of the Fock representation can be constructed from a unique vector $\left|0_M\right\rangle$ by repeated application of the operators $a^*_{\boldsymbol{k}}$. The state $\left|0_M\right\rangle$ has the unique property

$$a_{\boldsymbol{k}}\left|0_M\right\rangle = 0 \qquad \forall \boldsymbol{k} \tag{2.7}$$

A typical basis vector has the form

$$\left|n_{\boldsymbol{k}_1}, \ldots\ldots, n_{\boldsymbol{k}_j}\right\rangle = \left(n_{\boldsymbol{k}_1}! \ldots n_{\boldsymbol{k}_j}!\right)^{-1/2} \left(a^*_{\boldsymbol{k}_1}\right)^{n_{k_1}} \ldots\ldots \left(a^*_{\boldsymbol{k}_j}\right)^{n_{kj}} \left|0_M\right\rangle \tag{2.8}$$

where the $n_{\boldsymbol{k}_i}$ are positive integers and $\left(n_{\boldsymbol{k}_1}! \ldots\ldots n_{\boldsymbol{k}_j}!\right)^{-1/2}$ is the normalisation factor required to accommodate Bose statistics. The basis vectors are normalised according to

$$\left\langle n_{\boldsymbol{k}_1}, \ldots\ldots, n_{\boldsymbol{k}_j} \mid m_{\boldsymbol{k}_1'}, \ldots\ldots, m_{\boldsymbol{k}_s'}\right\rangle = \delta_{js} \sum_P \delta_{n_{k_1}, m_{\boldsymbol{k}'_{P(1)}}} \ldots\ldots \delta_{n_{kj}, m_{\boldsymbol{k}'_{P(s)}}} \delta^{(n-1)}\left(\boldsymbol{k}_1 - \boldsymbol{k}'_{P(1)}\right) \ldots \delta^{(n-1)}\left(\boldsymbol{k}_j - \boldsymbol{k}'_{P(s)}\right) \tag{2.9}$$

Where the sum is over all perturbations P of integers 1, 2, …..,s. Also we have

$$a^*_{\boldsymbol{k}_i}\left|n_{\boldsymbol{k}_i}\right\rangle = \left(n_{\boldsymbol{k}_i} + 1\right)^{1/2} \left|n_{\boldsymbol{k}_i} + 1\right\rangle$$

$$a_{\boldsymbol{k}_i}\left|n_{\boldsymbol{k}_i}\right\rangle = \left(n_{\boldsymbol{k}_i}\right)^{1/2} \left|n_{\boldsymbol{k}_i} - 1\right\rangle \tag{2.10}$$

The “particle” interpretation of the Fock basis arises from the consideration of energy and momentum of the field. The energy content of the field is found from its Hamiltonian, (Birrell & Davies, 1983)

$$H = \int d^{n-1}\boldsymbol{k} \left(a^*_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{1}{2}\right)\omega$$

Although this expression is formally divergent, it can be made finite by one of several well-known regularisation techniques. (See, for example, Bogolubov & Shirkov, 1980, Bjorken & Drell, 1965). With this done the renormalised Hamlitonian H_{ren}, is given by

$$H = \int d^{n-1}\boldsymbol{k} a^*_{\boldsymbol{k}} a_{\boldsymbol{k}} \omega \tag{2.11}$$

The momentum of the field in the i-th direction is found to be

$$P_i = \int d^{n-1}\boldsymbol{k} a^*_{\boldsymbol{k}} a_{\boldsymbol{k}} k_i \tag{2.12}$$

If we define number operators N_k and N by

$$N_{\boldsymbol{k}} = a^*_{\boldsymbol{k}} a_{\boldsymbol{k}} \tag{2.13}$$

and

$$N = \int d^{n-1}k a^*_{\boldsymbol{k}} a_{\boldsymbol{k}} = \int d^{n-1}k N_{\boldsymbol{k}} \tag{2.14}$$

We have

$$[N,H]=[N,P_i]=0$$

Hence the eigenstates of N are also eigenstates of H and P_i. Furthermore both number operators are Hermittian (being self-adjoint) and so can be used to label (i.e. observe or measure) eigenstates of H and P_i.

Other properties of these number states include:

$$\langle 0_M \mid N_{\boldsymbol{k}} \mid 0_M \rangle = 0 \qquad \forall \boldsymbol{k}$$

$$\left\langle n_{\boldsymbol{k}_1},\ldots,n_{\boldsymbol{k}_j} \mid N_{\boldsymbol{k}_i} \mid n_{\boldsymbol{k}_1},\ldots,n_{\boldsymbol{k}_j} \right\rangle = n_{\boldsymbol{k}_i} \tag{2.15}$$

Therefore the expectation of $N_{\boldsymbol{k}_i}$ for the field in the state $\left| n_{\boldsymbol{k}_1},\ldots\ldots,n_{\boldsymbol{k}_1} \right\rangle$ is the integer $n_{\boldsymbol{k}_i}$, that is, the entry in the ket state vector under the momentum label $\boldsymbol{k}_i$. Similarly

$$\left\langle n_{\boldsymbol{k}_1},\ldots\ldots,n_{\boldsymbol{k}_j} \mid N \mid n_{\boldsymbol{k}_1},\ldots\ldots,n_{\boldsymbol{k}_j} \right\rangle = \int dk^{n-1} n_{\boldsymbol{k}}$$

Now, in Minkowski space quantum field theory it is standard practice to associate energy-momentum with particles, that is, each particle carries with it a momentum $\boldsymbol{k}$ and energy ω. (Bogolubov & Shirkov, 1980, is a good example of this practice.) If we adopt such a standpoint form (2.11), (2.12), (2.13) and (2.14) it can be seen that interpreting $N_{\boldsymbol{k}}$ as the operator which counts the number of particles in state $\boldsymbol{k}$ makes good sense. With such an interpretation of $N_{\boldsymbol{k}}$ (2.11) shows that the total energy of the field is found merely by summing over the contribution to the energy provided by each particle in the field. Similarly the total momentum P_i as just a sum of the contributions k_i of each particle.

From this intuitive approach it follows that:

- $N_{\boldsymbol{k}}$ is the number of particles in state $\boldsymbol{k}$
- $\left| n_{\boldsymbol{k}_1},\ldots,n_{\boldsymbol{k}_j} \right\rangle$ is an n-particle state with $n_{\boldsymbol{k}_1}$ particles in state $\boldsymbol{k}_1$, …. and $n_{\boldsymbol{k}_j}$ particles in state $\boldsymbol{k}_j$.
- $\left| 0_M \right\rangle$ is devoid of particles (which is consistent with $\left\langle 0_M \middle| H \middle| 0_M \right\rangle = \left\langle 0_M \middle| P \middle| 0_M \right\rangle = 0$) and hence called the Minkowski vacuum state.
- $a^*_{\boldsymbol{k}}$ .is the creator of a particle in the state $\boldsymbol{k}$.
- $a_{\boldsymbol{k}}$ is the annihilator of a particle in the state $\boldsymbol{k}$.

This construction provides a representation of the quantum field $\phi[x]$ in Minkowski space that arises naturally due to the high degree of symmetry of the space-time, and further provides a good mathematical representation of "particles".

Now consider a non-Minkowski space (e.g. a space with gravitational curvature or a flat space with non-inertial coordinates). An almost identical procedure can be followed resulting in a Fock representation of the Hilbert space of the quantum field. However, this new Fock representation need not be equivalent to the Fock representation in Minkowski space.

Referring back to (2.1), the operator $\Box$ will have a different mathematical form in the non-Minkowski space since the metric, $g_{\mu\nu}$, is not the Minkowski metric. Therefore the new field basis functions, $v_{\boldsymbol{k}}(x)$, in general will not be the complex exponentials of (2.3). The field can still be expanded in terms of these basis functions (this condition is part of their definition as a basis), so we can write;

$$\phi[x] = \int d^{n-1}k \left(v_{\boldsymbol{k}}(x) b_{\boldsymbol{k}} + v^*_{\boldsymbol{k}}(x) b^*_{\boldsymbol{k}} \right) \tag{2.16}$$

Where

$$\left[b_{\boldsymbol{k}}, b_{\boldsymbol{k}'}\right] = \left[b^*_{\boldsymbol{k}}, b^*_{\boldsymbol{k}'}\right] = 0$$

$$\left[b_{\boldsymbol{k}}, b^*_{\boldsymbol{k}'}\right] = \delta^{(n-1)}\left(\boldsymbol{k} - \boldsymbol{k}'\right) \tag{2.17}$$

As before a vacuum state, $\left|\bar{0}\right\rangle$, can be defined satisfying

$$b_{\boldsymbol{k}}\left|\bar{0}\right\rangle = 0 \qquad\qquad \forall \boldsymbol{k} \tag{2.18}$$

From which a Fock representation can be constructed. We can also define number operators

$$\bar{N}_{\boldsymbol{k}} = b^*_{\boldsymbol{k}} b_{\boldsymbol{k}}$$

$$\bar{N} = \int d^{n-1}k b^*_{\boldsymbol{k}} b_{\boldsymbol{k}} = \int d^{n-1}k \bar{N}_{\boldsymbol{k}} \tag{2.19}$$

To demonstrate how two vacuum states defined over the same space-time (patch) need not be identical, let us consider a particular example. Minkowski coordinates are not the only coordinates with describe flat space. A well-known alternative is the Rindler coordinatisation (Rindler, 1969). The field equation (2.1) can be solved with the Rindler metric and a mode sum of the form (2.16) deduced. From this the Rindler operators $b_{\boldsymbol{k}}$ and $b^*_{\boldsymbol{k}}$ can be defined as well as the Rindler vacuum $\left|\bar{0}\right\rangle$ which satisfies (2.18). Finally the number operators $\bar{N}_{\boldsymbol{k}}$ and $\bar{N}$ are still defined formally by (2.18) and (2.19).

Since the Minkowski and Rindler operators are defined over the same space-time patch, their vacuum states can be compared and, as in general, we find

$$\left\langle 0_M \mid \bar{N} \mid 0_M \right\rangle \neq 0 \qquad\qquad \left\langle \bar{0} \mid N \mid \bar{0} \right\rangle \neq 0$$

Therefore, although there are no particles corresponding to the Minkowski operators $a_{\boldsymbol{k}}$ in the Minkowski vacuum, there are particles corresponding to the Rindler operators $b_{\boldsymbol{k}}$ in that vacuum. Similarly, although the Rindler vacuum is devoid of Rindler particles, it does contain Minkowski particles.

To find the relationship between the corresponding quantities, we use the completeness of the sets of basis field functions. The new modes $v_{\boldsymbol{k}}(x)$ can be expressed in terms of the old (Bogolubov 1958)

$$v_{\boldsymbol{k}}(x) = \int d^{n-1}l\left(\alpha_{\boldsymbol{kl}} u_{\boldsymbol{l}}(x) + \beta_{\boldsymbol{kl}} u^*_{\boldsymbol{l}}(x)\right)$$

and conversely

$$u_{\boldsymbol{k}}(x) = \int d^{n-1}l\left(\alpha^*_{\boldsymbol{lk}}(x) v_{\boldsymbol{l}}(x) + \beta_{\boldsymbol{lk}} v^*_{\boldsymbol{l}}(x)\right) \tag{2.20}$$

These are called Bogolubov transformations and the quantities $\alpha_{\boldsymbol{kl}}$, $\beta_{\boldsymbol{kl}}$, called Bogolubov coefficients. Using the ortho-normality of the modes it follows that

$$\alpha_{kl} = \left(v_k(x), u_l(x)\right) \qquad\qquad \beta_{kl} = -\left(v_k(x), u^*_l(x)\right)$$

The relationship between the operators is

$$a_{\boldsymbol{k}} = \int d^{n-1}l\left(\alpha_{lk} b_l + \beta^*_{lk} b^*_l\right)$$

$$b_{\boldsymbol{k}} = \int d^{n-1}l\left(\alpha_{lk}^{*}a_l - \beta_{lk}^{*}a_l^{*}\right) \tag{2.21}$$

Finally, the Bogolubov coefficients satisfy

$$\int d^{n-1}k\left(\alpha_{lk}\alpha_{jk}^{*} - \beta_{lk}\beta_{jk}^{*}\right) = \delta^{(n-1)}\left(\boldsymbol{l}-\boldsymbol{j}\right)$$
$$\int d^{n-1}k\left(\alpha_{lk}\beta_{jk} - \beta_{lk}\alpha_{jk}\right) = 0$$

Using (2.21), we find

$$\left\langle \overline{0} \mid N_{\boldsymbol{k}} \mid \overline{0} \right\rangle = \int d^{n-1}l\left|\beta_{lk}\right|^2 \tag{2.22}$$

If $\beta_{lk} \neq 0$ identically, then the $\left|\overline{0}\right\rangle$ vacuum will contain $\int d^{n-1}l\left|\beta_{lk}\right|^2$ particles in the *k*th-mode. It is only if $\beta_{lk} = 0$ that the two vacua will be identical.

This result can be understood from the point of view of mixing positive and negative frequencies. The Minkowski modes (2.3) are said to be positive frequency with respect to the time coordinate, t, since they are eigen-functions of the operator $\partial/\partial t$

$$\left(\partial/\partial t\right)u_{\boldsymbol{k}}(x) = -i\omega u_{\boldsymbol{k}}(x) \qquad \omega > 0$$

Similarly, if the modes $v_{\boldsymbol{k}}(x)$ are positive frequency with respect to some time-like Killing vector field, $\partial/\partial\eta$, they satisfy

$$\left(\partial/\partial\eta\right)v_{\boldsymbol{k}}(x) = -i\tilde{\omega}v_{\boldsymbol{k}}(x) \qquad \tilde{\omega} > 0$$

If the $u_{\boldsymbol{k}}(x)$ are a linear combination of the $v_{\boldsymbol{k}}(x)$ alone (no $v_{\boldsymbol{k}}^{*}(x)$), then from (2.20) $\beta_{kl} = 0$. That is, the $u_{\boldsymbol{k}}(x)$ are also positive frequency (mixtures) with respect to $\partial/\partial\eta$. From (2.21), $a_{\boldsymbol{k}}\left|\overline{0}\right\rangle = 0$ and $b_{\boldsymbol{k}}\left|0_M\right\rangle = 0$, thus the two sets of modes have a common vacuum state. However if $\beta_{kl} \neq 0$ identically, then the $u_{\boldsymbol{k}}(x)$ will be a mixture of positive and negative frequency $v_{\boldsymbol{k}}(x)$ modes and, by (2.22), the $\left|\overline{0}\right\rangle$ vacuum is not empty (i.e. devoid of particles) with respect to the $a_{\boldsymbol{k}}$ operators.

In the study of quantum fields in curved or flat space-times, there are available many different coordinatisations of the same space-time (patch). In some cases a high degree of symmetry will lead, in a natural way, to the adoption of a particular coordinatisation. (E.g. Minkowski and De Sitter space). If we desire to adopt the spirit of General Relativity in our investigations; given a space time we should be free to choose any coordinatisation we wish. (Weinberg 1972) However in approaching this topic from such a standpoint, the concept of “particle” immediately becomes ambiguous because the vacuum state of one coordinatisation need not be identical to that of another.

Over the years there has been some debate about the “physicalness” of various vacuum states. (See for example, Parker 1969, Davies 1984, Dray and Renn 1983.) There have been attempts to define, or in some “natural” way determine that, given a space-time a particular vacuum state is “the vacuum” for that space-time. So far, none of these attempts appear to be particularly convincing (Davies 1984).

Referring to the above example, a question which is often asked is: “Do Rindler particles really exist?”, since they seem to fill the Minkowski vacuum. In an attempt to clarify the issues of the physical status of in-equivalent particle definitions, Unruh (1976) and later DeWitt (1979) introduced the use of “particle detectors”. As DeWitt (1979) stated:

> … one has to fall back on operational definitions …, for example, one must ask: How would a given particle detector respond to a given stimulus?

Since most observations and measurement in elementary physics is performed using detectors of some form or another, attempts to assign objective physical significance in the absence of specified detector arrangements, as far as the Copenhagen Interpretation of quantum mechanics is concerned, are mis-guided.

The study of particle detector models in quantum field theory has not been a major area of activity over recent years. That which has occurred has centred on the detectors Unruh and DeWitt invented. (These two detectors are essentially identical since Unruh treats his box detector as a point object with respect to the local space-time length parameter.) There has also been a study of extended detectors by Grove and Ottewill (1983) and some work on spinor detectors in Rindler and Schwarzschild space-times by Iyer and Kunar (1980).

All the above cited works only consider the response of detectors that interact with the quantum field in a particular way. To the author’s knowledge no published works so far have addressed the question of comparing the behaviour of detectors with are coupled to the field through different interaction Lagrangians. Such an analysis will be given here.

2.2 Definition of a particle detector

First, the concept of particle detector must be clarified. This will restrict the selection of mathematical constructs available for us to use as a “particle detector” model. It is important to appreciate that, in the context of this thesis, a particle detector is only a mathematical construct in which we have removed, or put aside, all the complicating details required to model such machines as bubble or cloud chambers. However this in no way detracts from the importance of studying these mathematical constructs since, in the final analysis, it is the interaction Lagrangian between these machines and the fields they detect that dictates how they respond to a given field configuration. Similarly it is through the interaction Lagrangian the particle detectors considered in this thesis are defined.

Here, when using the term “particle detector”, we have in mind a mathematical model involving a point-like entity, M, which can be described by a classical world-line, but which nevertheless possesses internal degrees of freedom having a quantum description provided by energy levels E. Such model detectors can essentially be described by the interaction Lagrangian for the coupling between the internal degrees of freedom and the quantum field. The world-line of the detector is prescribed; it is not part of the dynamics. The detector is initially set in an initial state (e.g. its ground state) E_0, and the probability that, as a result of its interaction, it will eventually be found in a <u>different</u> particular (excited) final state, E, is examined.

To qualify as a “realistic” or “useful” particle detector, such a model must satisfy the following conditions: If the detector’s initial state is its ground state, then:

a) When moving inertially in Minkowski space through the vacuum state $\left|0_M\right\rangle$, the detector must remain in its ground state, thus failing to register the presence of any field quanta.

b) In a many-particle state in Minkowski space in which there are $n_{\boldsymbol{k}}$ particles in mode $\boldsymbol{k}$, the probability that the detector will be excited to energy level E should be a one-to-one function of at least a (weighted) mean energy state occupation number n_k, where k = $|\boldsymbol{k}|$ (i.e. the detector must at least be able to resolve a mean number of particles in each energy state.)

In connection with b) one can divide detectors into two classes: omni-directional detectors in which the response is a function of n_k alone and directional detectors in which the response is also a function of $\tilde{\boldsymbol{k}}$, where $\tilde{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$ (i.e. a function of $n_{\boldsymbol{k}}$).

The DeWitt and Unruh detectors are examples of the former, several examples of the latter type will be introduced below. From their definition it can be seen that an omni-directional detector cannot provide information about any anisotropies of a particle bath into which it has been placed, however a directional detector may.

3 Five Detector Models

With the definition of a "particle detector" given in the previous chapter, quite a wide variety of possible detector models is available. This is so even with restrictions such as PCT-invariance and renormalisability etc.

In this chapter five detector models (i.e. their interaction Lagrangians) will be briefly introduced and shown to satisfy the definition of a particle detector. The models shall subsequently be studied separately and in greater detail by deducing their responses in several different situations.

3.1 The Linear Detector

The first detector to be introduced is the well-known DeWitt detector (DeWitt 1979). We shall, however, call it the "linear detector" since this reflects the structure of its interaction Lagrangian, which is

$$L^1 = cm(\tau)\phi\left[x(\tau)\right] \tag{3.1}$$

In this equation c is a small (dimensionless) coupling constant, $m(\tau)$ is the (point) monopole operator describing the semi-classical entity M introduced above, τ is the detector proper time and $x(\tau)$ is the trajectory its trajectory through space-time. The time evolution of $m(\tau)$ is assumed to be

$$m(\tau) = \exp\left(iH_0\tau\right)m(0)\exp\left(-iH_0\tau\right) \tag{3.2}$$

Where $H_0\left|E\right\rangle = E\left|E\right\rangle$, H_0 being the Hamiltonian for the internal dynamics of the detector with energy states represented by $\left|E\right\rangle$.

Assuming the quantum field is initially in state $\left|\Psi_0\right\rangle$ and the detector in state $\left|E_0\right\rangle$, the transition amplitude, A^1, for a transition of the field to state $\left|\Psi\right\rangle$ and the detector to state $\left|E\right\rangle$, over the extent of its world-line is, to first order perturbation theory (DeWitt 1979)

$$A^1 = ic\left\langle M\right\rangle \int_{-\infty}^{\infty} d\tau e^{i\Delta E\tau}\left\langle \Psi \mid \phi\left[x(\tau)\right] \mid \Psi_0\right\rangle \tag{3.3}$$

Where $\Delta E = E - E_0$, and $\langle M \rangle = \langle E | m(0) | E_0 \rangle$ represents the matrix element for the internal degrees of freedom of the entity M. the transition probability P^1 of the detector to final state $|E\rangle$ for the field in a given state $|\Psi_0\rangle$ is found by summing the modulus squared of (3.3) over a complete set of states $|\Psi\rangle$.

$$P^1 = c^2 \left|\langle M \rangle\right|^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-\Delta E \Delta\tau} \left\langle \Psi_0 | \phi\left[x(\tau)\right] \phi\left[x(\tau')\right] | \Psi_0 \right\rangle \tag{3.4}$$

Where $\Delta\tau = \tau - \tau'$.

Calculating this detector's response to an n-particle state in Minkowski space with the detector stationary enables us to check that it satisfies the definition of a particle detector. Using (2.8) and expanding the field as a mode integral as in (2.2) it is easily shown that

$$\left\langle n_{\boldsymbol{k}_1}, \dots, n_{\boldsymbol{k}_j} | \phi[x] \phi[x'] | n_{\boldsymbol{k}_1}, \dots, n_{\boldsymbol{k}_j} \right\rangle = G^+(x, x') + \int d^{n-1} k n_{\boldsymbol{k}} \left(u_k(x) u_k^*(x') + u_k^*(x) u_k(x') \right) \tag{3.5}$$

Where $x = x(\tau)$ and

$$G^+(x, x') = \left\langle 0_M | \phi[x] \phi[x'] | 0_M \right\rangle \tag{3.6}$$

Is the Wightman Green function of the neutral scalar field in the Minkowski vacuum state.

Since the detector is stationary, plane wave modes (2.3) may be used. Doing this and using (3.5) in (3.4) gives a total transition probability of

$$\begin{aligned} P^1_{n_k} &= c^2 \left|\langle M \rangle\right|^2 2^{(2-n)} \pi^{(3-n)/2} \left\{ \Gamma\left((n-1)/2\right) \right\}^{-1} \\ &\times \left((\Delta E)^2 - m^2 \right)^{(n-3)/2} \bar{n}_{\left((\Delta E)^2 - m^2\right)^{1/2}} \theta(\Delta E - m) \int_{-\infty}^{\infty} d \frac{(\tau + \tau')}{2} \end{aligned} \tag{3.7}$$

In which m is the mass of the field quanta; also we have used

$$\int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\tau = \int_{-\infty}^{\infty} d\Delta\tau \int_{-\infty}^{\infty} d \frac{(\tau + \tau')}{2}$$

And have defined

$$\bar{n}_k = \int d\Omega n_{\boldsymbol{k}} \Big/ \int d\Omega \tag{3.8}$$

In which the $d\Omega$-integral is over all angular directions in $k^{(n-1)}$ space. (NB: $\bar{n}_k$ is an example of the 'mean' energy occupation number referred to in condition (b) of the definition of a particle detector.) It is immediately obvious that $P^1_{n_k}$ is formally divergent due to the $(\tau + \tau')$- integral. The reason for this is well known. Since (3.5) is a function of $\Delta\tau$ only, this divergence represents a constant flux of particles interacting with the detector over all (infinite) time. Factoring out the divergence results in the transition rate: i.e. a transition per unit detector time, which we shall denote as $R^1_{n_k}$.

The quantity $\bar{n}_k$ defined in (3.8) is the average over the n-2 sphere of directions of the total number of particles in energy mode $k = |\boldsymbol{k}|$.

From (3.7) it is obvious that the linear detector satisfied conditions (a) and (b) for an omni-directional detector.

In evaluating (3.7), the $\Delta\tau$ integral was performed before the momentum integral. Swapping the order of integration provides a form for the response that will be of use later.

$$R^1_{n_k} = c^2 \left|\langle M \rangle\right|^2 \int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E \Delta\tau} \left\{ \begin{array}{l} G^+(\Delta\tau) + \\ + \dfrac{2}{(4\pi)^{(n-1)/2}\Gamma((n-1)/2)} \int_m^{\infty} d\omega \bar{n}_{(\omega^2-m^2)^{1/2}} \left(\omega^2 - m^2\right)^{(n-3)/2} \cos(\omega\Delta\tau) \end{array} \right\} \quad (3.9)$$

Where $G^+(\Delta\tau) = G^+(x,x')\big|_{x=x'} = \langle 0_M | \phi[x(\tau)]\phi[x(\tau')] | 0_M \rangle\big|_{x=x'}$ is a function of $\Delta\tau = \tau - \tau'$ only and we have used $k^2 = \omega^2 - m^2$.

3.2 The Quadratic Detector

The quadratic detector is described by the interaction Lagrangian

$$L^2 = cm(\tau)\phi^2\left[x(\tau)\right] \quad (3.10)$$

Where c is a small (dimensionless) coupling constant and $m(\tau)$ has the usual evolution equation (3.2). (Note: the dimensions of $m(0)$ are adjusted in these Lagrangians so that the coupling constants c are always dimensionless.) Repeating the above procedure yields total transition amplitude

$$A^2 = ic\langle M \rangle \int_{-\infty}^{\infty} d\tau e^{i\Delta E\tau} \left\langle \Psi | \phi^2\left[x(\tau)\right] | \Psi_0 \right\rangle \quad (3.11)$$

It is well known that when $|\Psi_0\rangle = |\Psi\rangle$, the integrand in (3.11) is formally divergent. Since (3.11) summed over a complete set $\{|\Psi\rangle\}$, a way must be sought to make that expression meaningful.

This is done by assuming that the particle detectors respond only to renormalised expectation values. Such an assumption is motivated ty the philosophy that in nature we can only ever observe remormalised field quantities. Therefore (3.11) is replaced by

$$A^2 = ic\langle M \rangle \int_{-\infty}^{\infty} d\tau e^{i\Delta E\tau} \left\langle \Psi | \phi^2\left[x(\tau)\right] | \Psi_0 \right\rangle_{ren} \quad (3.12)$$

where the subscript “ren” represents the renormalised expectation value. Summing the modulus squared of (3.12) over a complete set of states yields the total transition probability

$$P^2 = c^2 \left|\langle M \rangle\right|^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-\Delta E\Delta\tau} \left\langle \Psi_0 | \phi^2\left[x(\tau)\right]\phi^2\left[x(\tau')\right] | \Psi_0 \right\rangle_{ren} \quad (3.13)$$

Where the subscript “ren” now means that only the renormalised values in (3.12) have been used.

This mathematical construct can now be tested to see if it satisfies conditions (a) and (b). For this purpose the expectation value in (3.13) can be expressed in the form

$$
\begin{aligned}
&\left\langle n_{k_1},\dots,n_{k_j} \mid \phi^2[x]\phi^2[x'] \mid n_{k_1},\dots,n_{k_j}\right\rangle_{ren} = \\
&= 2\left\{G^+(x,x') + \int d^{n-1}k n_{\mathbf{k}}\left(u_k(x)u_k^*(x') + u_k^*(x)u_k(x')\right)\right\}^2 \\
&\quad + 4\int d^{n-1}k n_{\mathbf{k}}\left|u_{\mathbf{k}}(x)\right|^2 \int d^{n-1}l n_{\mathbf{l}}\left|u_{\mathbf{l}}(x')\right|^2
\end{aligned}
\tag{3.14}
$$

Using modes (2.3), substituting (3.14) into (3.13) and factoring out the $(\tau + \tau')$-integral gives the transition rate

$$
R_{n_k}^2 = \frac{c^2\left|\langle M\rangle\right|^2 (4\pi)^{2-n}}{\left\{\Gamma\left((n-1)/2\right)\right\}^2} \times
\left\{
\begin{array}{l}
\displaystyle\int_m^{\Delta E-m} d\omega \bar{n}_{\left((\Delta E-\omega)^2-m^2\right)^{1/2}} \bar{n}_{\left(\omega^2-m^2\right)^{1/2}} \left((\Delta E-\omega)^2-m^2\right)^{(n-3)/2}\left(\omega^2-m^2\right)^{(n-3)/2}\theta(\Delta E-2m) + \\
\displaystyle + \int_m^{\infty} d\omega\left(\bar{n}_{\left(\omega^2-m^2\right)^{1/2}}+1\right)\bar{n}_{\left((\Delta E+\omega)^2-m^2\right)^{1/2}}\left((\Delta E+\omega)^2-m^2\right)^{(n-3)/2}\left(\omega^2-m^2\right)^{(n-3)/2}\theta(\Delta E-m)
\end{array}
\right\}
\tag{3.15}
$$

Since (3.15) has the form of an autocorrelation of the function $\bar{n}_k k^{(n-3)}$ it is easily seen that the construct (3.10) satisfied the conditions prescribed for an omni-directional particle detector.

Recasting (3.15) into the alternative form gives

$$
R_{n_k}^2 = 2c^2\left|\langle M\rangle\right|^2 \int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E\Delta\tau}
\left\{
\begin{array}{l}
\left[\begin{array}{l} G^+(\Delta\tau) + \\ \displaystyle\frac{2(4\pi)^{(1-n)/2}}{\Gamma\left((n-1)/2\right)}\int_m^{\infty} d\omega \bar{n}_{\left(\omega^2-m^2\right)^{1/2}}\left(\omega^2-m^2\right)^{(n-3)/2}\cos(\omega\Delta\tau)\end{array}\right]^2 \\
\displaystyle + \left[\frac{2(4\pi)^{(1-n)/2}}{\Gamma\left((n-1)/2\right)}\int_m^{\infty} d\omega \bar{n}_{\left(\omega^2-m^2\right)^{1/2}}\left(\omega^2-m^2\right)^{(n-3)/2}\right]^2
\end{array}
\right\}
\tag{3.16}
$$

3.3 The Derivative Detector

The derivative detector is described by

$$
L^2 = cm(\tau)b^\mu \frac{\partial}{\partial x^\mu}\phi\left[x(\tau)\right] \tag{3.17}
$$

Where c is a small coupling constant, $m(\tau)$ has the usual time evolution equation and b^μ is a unit n-vector which gives a fixed, arbitrary orientation in the detector's frame.

For this detector

$$
A^3 = ic\langle M\rangle b^\mu \int_{-\infty}^{\infty} d\tau e^{i\Delta E\tau}\partial_\mu\left\langle \Psi \mid \phi\left[x(\tau)\right] \mid \Psi_0\right\rangle
$$

Where $\partial_\mu = \partial/\partial x^\mu$. The transition probability is

$$P^3 = c^2 \left|\langle M \rangle\right|^2 b^\mu b^\nu \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-\Delta E \Delta \tau} \partial_\mu \partial'_\nu \left\langle \Psi_0 \,|\, \phi\left[x(\tau)\right] \phi\left[x(\tau')\right] |\, \Psi_0 \right\rangle \tag{3.18}$$

In this $\partial'_\nu = \partial / \partial x'^\nu$. Since the detector is point-like, in evaluating (3.18) the derivatives are taken first and then the spatial parts of $x(\tau)$ and $x(\tau')$ are set equal.

Applying the test to see of (3.17) satisfies the conditions for a particle detector, we find

$$\begin{aligned} R^3_{n_k} = & \frac{c^2 \left|\langle M \rangle\right|^2 2^{(2-n)} \pi^{(2-n)/2}}{\Gamma\left((n-1)/2\right)} \left((\Delta E)^2 - m^2\right)^{(n-3)/2} \theta(\Delta E - m) \\ & \times \left\{ \begin{aligned} & \left(b^0\right)^2 \overline{n}_{\left((\Delta E)^2 - m^2\right)^{1/2}} (\Delta E)^2 + \sum_{i,j=1}^{n-1} b^i b^j \overline{n}_{\left((\Delta E)^2 - m^2\right)^{1/2}, ij} \left((\Delta E)^2 - m^2\right) \\ & + 2 b^0 \sum_{i=1}^{n-1} b^i \overline{n}_{\left((\Delta E)^2 - m^2\right)^{1/2}, 0i} \Delta E \left((\Delta E)^2 - m^2\right)^{1/2} \end{aligned} \right\} \end{aligned} \tag{3.19}$$

where the following measure has been used on the (n-2)-sphere

$$d\Omega = \prod_{i=2}^{n-1} \sin^{(n-1-i)} \Omega_i d\Omega_i$$

$$k_i = |\boldsymbol{k}| \cos \Omega_{i+1} \prod_{j=2}^{i} \sin \Omega_j$$

With

$$0 \le \Omega_i < \pi, \quad i = 2, \ldots.., n-2; 0 \le \Omega_{n-1} < 2\pi; \quad \Omega_n \equiv 0$$

Further the quantities $\overline{n}_{k,ij}$, and $\overline{n}_{k,0i}$ are defined by

$$n_{k,ij} \equiv \int d\Omega n_k \cos \Omega_{i+1} \cos \Omega_{j+1} \prod_{p=2}^{i} \sin \Omega_p \prod_{q=2}^{i} \sin \Omega_q \Big/ \int d\Omega \tag{3.20}$$

$$n_{k,0i} \equiv \int d\Omega n_k \cos \Omega_{i+1} \prod_{j=2}^{i} \sin \Omega_j \Big/ \int d\Omega \tag{3.21}$$

Although the response (3.19) of this detector depends on the orientation of the vector b^μ, given this direction the response satisfies the conditions for an omni-directional detector.

As before the response can be rewritten in the form

$$
R^3_{n_k} = c^2 \left|\langle M\rangle\right|^2 \int_{-\infty}^{\infty} d\Delta\tau \left\{ \begin{array}{l} \left(b^0\right)^2 \left[\begin{array}{l} \dfrac{\partial}{\partial\tau\partial\tau'} G^+(\Delta\tau) + \\ + \dfrac{2(4\pi)^{(1-n)/2}}{\Gamma((n-1)/2)} \displaystyle\int_m^\infty d\omega\omega^2 \overline{n}_{(\omega^2-m^2)^{1/2}} \left(\omega^2 - m^2\right)^{(n-3)/2} \cos\omega\Delta\tau \end{array} \right] + \\ + \displaystyle\sum_{i,j=1}^{n-1} b^i b^j \left[\begin{array}{l} \left.\dfrac{\partial}{\partial x^i \partial x'^j} G^+(\Delta\tau)\right| + \\ + \dfrac{2(4\pi)^{(1-n)/2}}{\Gamma((n-1)/2)} \displaystyle\int_m^\infty d\omega \overline{n}_{(\omega^2-m^2)^{1/2},ij} \left(\omega^2 - m^2\right)^{(n-1)/2} \cos\omega\Delta\tau \end{array} \right] + \\ +2b^0 \displaystyle\sum_{i,j=1}^{n-1} b^i \left[\begin{array}{l} \left.\dfrac{\partial}{\partial\tau\partial x'^i} G^+(\Delta\tau)\right| + \\ + \dfrac{2(4\pi)^{(1-n)/2}}{\Gamma((n-1)/2)} \displaystyle\int_m^\infty d\omega\omega \overline{n}_{(\omega^2-m^2)^{1/2},0i} \left(\omega^2 - m^2\right)^{(n-2)/2} \cos\omega\Delta\tau \end{array} \right] \end{array} \right\} \tag{3.22}
$$

In this equation $\left(\partial^2/\partial x^i \partial x'^j\right) G^+(\Delta\tau)\Big|$ denotes the process of taking the derivatives of $G^+\left(x(\tau), x(\tau')\right)$ and then setting the spatial parts of $x(\tau)$ and $x(\tau')$ equal.

3.4 The Spike Detector

The spike detector is constructed so as to respond only to those modes which have a certain fixed direction in momentum space. The interaction Lagrangian can be written as

$$
L^4 = cm(\tau)\phi\left[x(\tau)\right]\Big|_\Omega \tag{3.23}
$$

In which the symbol $\Big|_\Omega$ represents the restriction on the set of modes of the field $\phi\left[(x)\right]$ which can interact with the detector. Mathematically this translates to a restriction on the modes of the field that appear in the calculation of the detector’s response, as shall be seen below.

To check (3.23) qualifies as a particle detector we apply the usual test of evaluating its response to an n-particle state in Minkowski space with the detector at rest.

The field is expressed as a mode integral, however a restriction is placed on the modes that appear in the Lagrangian (3.23). To represent this restriction in a mathematically convenient way a spherical-polar coordinatisation of the Minkowski momentum space is adopted. In this coordinatisation the measure on the (n-2)-sphere of directions in (n-1)-dimensional momentum space is denoted by $d\Omega$. The remaining “magnitude” component of the measure is the usual $dk\, k^{n-2}$. This re-coordinatisation can safely be made because the vacuum state corresponding to the spherical-polar coordinates is identical to the Minkowski vacuum (Pfautsch 1981).

The mode integral (2.2) for the field is now written in the form

$$
\phi\left[(x)\right] = \int dk\, k^{n-2} \int d\Omega \left(u_{k,\Omega}(x) a_{k,\Omega} + u^*_{k,\Omega}(x) a^*_{k,\Omega}\right) \tag{3.24}
$$

The restriction on the direction of the modes which can interact with the detector may now be represented using the Dirac delta function $\delta\left(\Omega - \Omega'\right)$ satisfying

$$\int d\Omega f(\Omega)\delta(\Omega-\Omega')=f(\Omega')$$

The restriction in (3.23) may be written as

$$\begin{aligned}\phi\left[(x)\right]\Big|_{\Omega}&=\int dk\, k^{n-2}\int d\Omega\left(u_{k,\Omega}(x)a_{k,\Omega}+u_{k,\Omega}^{*}(x)a_{k,\Omega}^{*}\right)\delta(\Omega-\Omega')\\&=\int dk\, k^{n-2}\left(u_{k,\Omega'}(x)a_{k,\Omega'}+u_{k,\Omega'}^{*}(x)a_{k,\Omega'}^{*}\right)\end{aligned}\tag{3.25}$$

Using this, the evaluation of A^4 for this detector follows the usual lines;

$$A^4=ic\langle M\rangle\int_{-\infty}^{\infty}d\tau\, e^{i\Delta E\tau}\langle\Psi|\phi\left[x(\tau)\right]\Big|_{\Omega}|\Psi_0\rangle=$$

$$=ic\langle M\rangle\int_{-\infty}^{\infty}d\tau\, e^{i\Delta E\tau}\langle\Psi|\int dk\, k^{n-2}\left(u_{k,\Omega'}(x)a_{k,\Omega'}+u_{k,\Omega'}^{*}(x)a_{k,\Omega'}^{*}\right)|\Psi_0\rangle$$

The transition probability is

$$\begin{aligned}P^4=c^2\left|\langle M\rangle\right|^2\int_{-\infty}^{\infty}d\tau\int_{-\infty}^{\infty}d\tau'\, e^{-i\Delta E\Delta\tau}\langle\Psi_0|\int dk\, k^{n-2}\left(u_{k,\Omega'}(x)a_{k,\Omega'}+u_{k,\Omega'}^{*}(x)a_{k,\Omega'}^{*}\right)\times\\ \times\int dl\, l^{n-2}\left(u_{l,\Omega'}(x)a_{l,\Omega'}+u_{l,\Omega'}^{*}(x)a_{l,\Omega'}^{*}\right)|\Psi_0\rangle\end{aligned}\tag{3.26}$$

Placing the detector in an (anisotropic) n-particle state gives for the expectation value in (3.26)

$$\begin{aligned}&\left\langle n_{\boldsymbol{k}_i},\dots,n_{\boldsymbol{k}_j}\left|\phi[x]\Big|_{\Omega}\phi[x']\Big|_{\Omega}\right|n_{\boldsymbol{k}_i},\dots,n_{\boldsymbol{k}_j}\right\rangle=\\&=\int dk\, k^{n-2}u_{k,\Omega}(x)u_{k,\Omega}^{*}(x')+\int dk\, k^{n-2}n_{k,\Omega}\left(u_{k,\Omega}(x)u_{k,\Omega}^{*}(x')+u_{k,\Omega}^{*}(x)u_{k,\Omega}(x')\right)\end{aligned}\tag{3.27}$$

Using plane wave modes (2.3), only the final term contributes to the response

$$\begin{aligned}R_{n_{\boldsymbol{k}}}^4&=\frac{c^2\left|\langle M\rangle\right|^2}{2(2\pi)^{n-2}}\int\frac{dk}{\omega}\,k^{n-2}n_{k,\Omega}\delta(\omega-\Delta E)\\&=\frac{c^2\left|\langle M\rangle\right|^2}{2(2\pi)^{n-2}}\left((\Delta E)^2-m^2\right)^{(n-3)/2}n_{\left((\Delta E)^2-m^2\right)^{1/2},\Omega}\,\theta(\Delta E-m)\end{aligned}\tag{3.28}$$

The response shows that this construct satisfies the criterion for a particle detector, and furthermore this detector can resolve any directional dependence of the n-particle state. This is, in contrast to the previous detectors, the spike is directional. This detector gives the occupation number, $n_{\boldsymbol{k}}$, for each mode $\boldsymbol{k}$. (Note that $n_{\boldsymbol{k}}$ will be given in spherical coordinates, $n_{k,\Omega}$)

Recasting (3.28),

$$R_{n_{\boldsymbol{k}}}^4=c^2\left|\langle M\rangle\right|^2\int_{-\infty}^{\infty}d\Delta\tau\, e^{i\Delta E\Delta\tau}\left\{G^{+}(\Delta\tau)\Big|_{\Omega}+\frac{(2\pi)^{(2-n)}}{2}\int_{m}^{\infty}d\omega\, n_{\left(\omega^2-m^2\right)^{1/2},\Omega}\left(\omega^2-m^2\right)^{(n-3)/2}\cos\omega\Delta\tau\right\}\tag{3.29}$$

in which

$$G^{+}(\Delta\tau)\Big|_{\Omega}=\int dk\, k^{n-2}u_{k,\Omega}(x)u_{k,\Omega}^{*}(x')\tag{3.30}$$

Is the Minkowski Wightman function with the Ω-integral "removed" and Ω set due to the delta-function introduced into the momentum integral.

3.5 The Cone Detector

A more "realistic" detector (i.e. one that could conceivably be manufactured) is the "cone detector", constructed so as to respond only to those modes within a given range of directions in momentum space. This interaction Lagrangian may be expressed as

$$L^5 = cm(\tau)\phi\left[x(\tau)\right]\Big|_{S_\Omega} \tag{3.31}$$

In this equation the $\Big|_{S_\Omega}$ represents the restriction on the modes that may interact with the detector. As with the spike detector this restriction is on the direction of the modes.

The Lagrangian (3.31), mathematically speaking, is merely the integral of the Lagrangian for the spike, (3.23), over the range of angles described by S_Ω on the (n-2)-sphere. Therefore from (3.23)

$$\phi\left[(x)\right]\Big|_\Omega = \int dk\; k^{n-2}\int_{S_\Omega} d\Omega\left(u_{k,\Omega}(x)a_{k,\Omega} + u^*_{k,\Omega}(x)a^*_{k,\Omega}\right)$$

Giving

$$A^5 = ic\langle M\rangle\int_{-\infty}^{\infty} d\tau\; e^{i\Delta E\tau}\left\langle\Psi\right|\int dk\; k^{n-2}\int_{S_\Omega} d\Omega\left(u_{k,\Omega'}(x)a_{k,\Omega'} + u^*_{k,\Omega'}(x)a^*_{k,\Omega'}\right)\left|\Psi_0\right\rangle \tag{3.32}$$

$$\begin{aligned} P^5 = c^2\left|\langle M\rangle\right|^2 \int_{-\infty}^{\infty} d\tau\int_{-\infty}^{\infty} d\tau'\, e^{-i\Delta E\Delta\tau}\left\langle\Psi_0\right|\int dk\; k^{n-2}\int_{S_\Omega} d\Omega\left(u_{k,\Omega'}(x)a_{k,\Omega'} + u^*_{k,\Omega'}(x)a^*_{k,\Omega'}\right)\times \\ \times\int dl\; l^{n-2}\int_{S_\Omega} d\Theta\left(u_{l,\Theta}(x)a_{l,\Theta} + u^*_{l,\Theta}(x)a^*_{l,\Theta}\right)\left|\Psi_0\right\rangle \end{aligned} \tag{3.33}$$

Placing this detector into an (isotropic) n-particle state gives

$$\begin{aligned} &\left\langle n_{\boldsymbol{k}_i},\ldots,n_{\boldsymbol{k}_j}\right|\phi[x]\big|_{S_\Omega}\,\phi[x']\big|_{S_\Omega}\left|n_{\boldsymbol{k}_i},\ldots,n_{\boldsymbol{k}_j}\right\rangle = \\ &= \int dk\; k^{n-2}\int_{S_\Omega} d\Omega u_{k,\Omega}(x)u^*_{k,\Omega}(x') + \int dk\; k^{n-2}\int_{S_\Omega} d\Omega n_{k,\Omega}\left(u_{k,\Omega}(x)u^*_{k,\Omega}(x') + u^*_{k,\Omega}(x)u_{k,\Omega}(x')\right) \end{aligned}$$

Using plane wave modes the response is

$$R^5_{n_{\boldsymbol{k}}} = \frac{c^2\left|\langle M\rangle\right|^2}{2^{n-2}\pi^{(n-3)/2}\Gamma\left((n-1)/2\right)}\left((\Delta E)^2 - m^2\right)^{(n-3)/2}\,\overline{n}_{\left((\Delta E)^2-m^2\right)^{1/2},S_\Omega}\,\theta(\Delta E - m) \tag{3.34}$$

Where

$$\overline{n}_k = \int_{S_\Omega} d\Omega n_{\boldsymbol{k}}\Big/\int d\Omega \tag{3.35}$$

The cone is seen to satisfy the definition of a particle detector and further can resolve anisotropic over a solid angle larger than S_Ω. If S_Ω were taken to be the entire (n-2)-sphere, such a cone would reduce to the linear detector. On the other hand, if we evaluate the response per unit solid angle and then let S_Ω shrink to a single direction Ω, the response of the spike detector will result. So the cone is a kind of half-way house between the directionality of the spiked detector and the omni-directionality of the linear detector, respectively.

As usual the response is recast

$$R_{n_k}^5 = c^2 \left|\langle M \rangle\right|^2 \int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E\Delta\tau} \left\{ \begin{array}{l} G^+(\Delta\tau)\Big|_{S_\Omega} + \\ \dfrac{2}{(4\pi)^{(n-1)/2}\,\Gamma((n-1)/2)} \displaystyle\int_m^{\infty} d\omega \bar{n}_{(\omega^2-m^2)^{1/2},S_\Omega} \left(\omega^2 - m^2\right)^{(n-3)/2} \cos(\omega\Delta\tau) \end{array} \right\} \quad (3.36)$$

Where

$$G^+(\Delta\tau)\Big|_{S_\Omega} = \int dk\, k^{n-2} \int_{S_\Omega} d\Omega u_{k,\Omega}(x) u_{k,\Omega}^*(x') \quad (3.37)$$

Finally, a variant of the cone detector has been introduced by Israel and Nester (1983), in which an arbitrary angular "screening" function is introduced. This detector is easily recovered from the spike by integrating (3.23) over the (n-2)-sphere with the screening function inserted into the integrand. For the cone this function is merely a characteristic function on S_Ω.

It can be seen from the responses of the various detectors introduced in the chapter that even with identical trajectories in identical particle baths, they will respond differently. This should be expected since their interaction Lagrangians differ. In the following chapters each detector will be studied in greater detail and in situations more interesting than those above. We shall find that the differences in the responses of these detectors cannot be simply factored out nor compensated for.

4 The Linear Detector

4.1 General Remarks

In Section 3.1 the linear detector, described by Lagrangian (3.1), was shown to satisfy the conditions for a particle detector. This detector (as well as others, in subsequent chapters) will now be studied in closer detail. In particular, its response in several well-known situations will be evaluated. These responses will later be compared with those of other detectors and from this the question of detector equivalence will arise.

In all of the cases to be discussed here, the quantum field is in a vacuum state which will be denoted by $|0\rangle$. From (3.4), it follows that the transition probability, P^l is:

$$\begin{aligned} P^l &= c^2 \left|\langle M \rangle\right|^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{i\Delta E\Delta\tau} \langle 0|\phi[x(\tau)]\phi[x(\tau')]|0\rangle \\ &= c^2 \left|\langle M \rangle\right|^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{i\Delta E\Delta\tau} G_\gamma^+(x(\tau), x(\tau')) \end{aligned} \quad (4.1)$$

In which

$$G_\gamma^+(x(\tau), x(\tau')) \equiv \langle 0|\phi[x(\tau)]\phi[x(\tau')]|0\rangle \quad (4.2)$$

Is the positive frequency Wightman function for the state $\left|0\right\rangle$ given the detector trajectory $x^{\mu}(t)$. The subscript indicates the Wightman function must be evaluated along the (detector) trajectory γ. Mathematically, this corresponds to writing G^{+} in a coordinate system that is co-moving with the detector. This follows from the fact that the detector "observes" the state $\left|0\right\rangle$ from this frame.

Let the n-tuple ($\tau,\rho^{1},....,\rho^{n-1}$) represent such a coordinate system where τ is the detector's proper time and $\rho^{1},.....,\rho^{n-1}$ are the spatial coordinates, which shall be abbreviated to ρ. In the (τ,ρ) system the detector's spatial coordinates are fixed for all τ; let them be $\tilde{\rho}^{1},.....,\tilde{\rho}^{n-1}$ (abbreviated to $\tilde{\rho}$). Since the Wightman function in (4.2) is evaluated along the detector world line γ, it follows that we may write

$$G_{\gamma}^{+}\left(x(\tau),x(\tau')\right)\equiv G_{\gamma}^{+}\left(\tau,\tilde{\rho};\tau'\tilde{\rho}'\right) \tag{4.3}$$

However, for a point detector stationary in the (τ,ρ) system, $\tilde{\rho}=\tilde{\rho}'$, giving in (4.2)

$$G_{\gamma}^{+}\left(\tau,\tilde{\rho};\tau',\tilde{\rho}'\right)\equiv G_{\gamma}^{+}\left(\tau,\tau';\tilde{\rho}\right) \tag{4.4}$$

Where

$$G_{\gamma}^{+}\left(\tau,\tilde{\rho};\tau',\tilde{\rho}'\right)\Big|_{\tilde{\rho}=\tilde{\rho}'}\equiv G_{\gamma}^{+}\left(\tau,\tau';\tilde{\rho}\right) \tag{4.5}$$

It is important to note that the (τ,ρ)-coordinate system is not unique. To be suitable as a coordinate patch, the only extra condition it must satisfy is that along the world line, γ, it is co-moving with the detector. This point has caused some confusion in the past, with people claiming that certain coordinatisations "naturally" suite a given observer world line. However, associating an entire coordinate patch with a single trajectory should be done with care. This point is best illustrated by an example. The uniformly accelerating observer may be studied using Rindler coordinates, but equally effective is the so-called K_7 coordinate system introduced by Brown, Ottewill and Silkos (1981). Of course, the field must be in the Minkowski vacuum state when using either coordinatisation.

An extensive discussion of this question appears in Davies (1984) from which one can clearly see the freedom available in the choice of the (τ,ρ)-coordinate system.

Now, given the Wightman function (4.5) corresponding to some situation, and using it to evaluate the response (4.1), it is convenient to split these functions into two classes. The first consists of those functions that are dependent upon τ and τ' only through their difference. That is

$$G_{\gamma}^{+}\left(\tau,\tau';\tilde{\rho}\right)\equiv G_{\gamma}^{+}\left(\Delta\tau;\tilde{\rho}\right) \tag{4.6}$$

From Section 3.1, we have already seen that in such cases the total response of the detector, (4.1), will diverge and must be interpreted as a constant rate of detection.

$$R_{\gamma}^{1}=c^{2}\left|\left\langle M\right\rangle\right|^{2}\int_{-\infty}^{\infty}d\Delta\tau e^{-i\Delta E\Delta\tau}G_{\gamma}^{+}\left(\Delta\tau;\tilde{\rho}\right) \tag{4.7}$$

Wightman functions of the form (4.6) describe time independent situations.

The second class consists of the remaining Wightman functions. For these the concept of transition rate which varies with time by be introduced as follows (Pfautsch 1981).

The transition probability of the detector at proper time τ_0 is

$$P^1(\tau_0) = c^2 \left|\langle M \rangle\right|^2 \int_{-\infty}^{\tau_0} d\tau \int_{-\infty}^{\tau_0} d\tau' e^{-i\Delta E \Delta\tau} G_\gamma^+(\tau,\tau';\tilde{\rho})$$

Therefore the rate per unit proper time is

$$R^1(\tau_0) = \frac{\partial}{\partial\tau} P^1\bigg|_{\tau=\tau_0} = c^2 \left|\langle M \rangle\right|^2 \left\{ e^{i\Delta E\tau_0} \int_{-\infty}^{\tau_0} d\tau e^{-i\Delta E\tau} G_\gamma^+(\tau,\tau_0;\tilde{\rho}) + e^{-i\Delta E\tau_0} \int_{-\infty}^{\tau_0} d\tau e^{i\Delta E\tau} G_\gamma^+(\tau_0,\tau;\tilde{\rho}) \right\}$$

Changing the variable of integration to $\eta = \tau - \tau_0$ in the first integral and $\eta = \tau_0 - \tau$ in the second

$$R^1(\tau_0) = c^2 \left|\langle M \rangle\right|^2 \int_{-\infty}^{\infty} d\eta e^{-i\Delta E\eta} \left\{ G_\gamma^+(\eta+\tau_0,\tau_0;\tilde{\rho})\theta(-\eta) + G_\gamma^+(\tau_0,\tau_0-\eta;\tilde{\rho})\theta(\eta) \right\} \tag{4.8}$$

In considering (4.8), it must be recalled that R^1 is now a function of proper time, τ_0, as well as ΔE. This latter label is supressed for convenience throughout these calculations.

4.2 Response in two and four dimensional Rindler Space

In this and the next section we shall be dealing with a massless scalar field only.

This is the well-known calculation of the response of a detector undergoing uniform acceleration through the Minkowski vacuum. Because the calculation has been thoroughly discussed elsewhere (DeWitt 1979, Birrell & Davies 1982) attention here will be restricted to a number of subtle technical points.

In two dimensions the co-moving frame of a uniformly accelerating detector may be represented by the Rindler coordinates $(\tilde{\tau},\xi)$ related to Minkowski coordinates (t,x) via (Rindler 1969)

$$t = \xi \sinh\tilde{\tau} \qquad\qquad x = \xi\cosh\tilde{\tau} \tag{4.9}$$

The Rindler coordinate $\tilde{\tau}$ is related to proper time τ by

$$\tau = \xi\tilde{\tau} \tag{4.10}$$

The two dimensional Minkowski vacuum Wightman function is (Birrell & Davies 1982)

$$G^+(x,x') = \frac{-1}{4\pi}\ln\left[(\Delta u - i\varepsilon)(\Delta v - i\varepsilon)\right] \tag{4.11}$$

Where $u = t - x$, $v = t + x$ are null coordinates.

Substituting (4.9) into (4.11) and setting $\xi = \xi'$ gives

$$G_\gamma^+(\tilde{\tau},\tilde{\tau}';\xi) = \frac{-1}{4\pi}\ln\left[4\xi^2\left(\begin{array}{l}\sinh^2\frac{\Delta\tilde{\tau}}{2}\cosh^2\left(\frac{\tilde{\tau}+\tilde{\tau}'}{2}\right) \\ -\frac{i\varepsilon}{\xi}\sinh\frac{\Delta\tilde{\tau}}{2}\cosh\left(\frac{\tilde{\tau}+\tilde{\tau}'}{2}\right) - \frac{\varepsilon^2}{4\xi^2} - \sinh^2\frac{\Delta\tilde{\tau}}{2}\sinh^2\left(\frac{\tilde{\tau}+\tilde{\tau}'}{2}\right)\end{array}\right)\right] \tag{4.12}$$

The detector's response, (4.7), is calculated using a contour along the real $\Delta\tilde{\tau}$ - axis. In such a calculation, the positioning of those poles near the real $\Delta\tilde{\tau}$ - axis is crucial, and any manipulation of (4.12) must preserve the pole positions relative to the contour.

In simplifying (4.12) it can be seen that the pole structure is maintained if we absorb the $\cosh\left(\left(\tilde{\tau}+\tilde{\tau}'\right)/2\right)$ term into the ε since the hyperbolic-cosine is positive definite, and the role of the ε terms is to merely (temporarily) marginally shift the poles off the contour. This function is still fulfilled with the hyperbolic-cosine function absorbed. Therefore (4.12) becomes

$$G_{\gamma}^{+}\left(\Delta\tilde{\tau};\xi\right)=\frac{-1}{4\pi}\ln\left[4\xi^{2}\left(\sinh\left(\frac{\Delta\tilde{\tau}}{2}\right)-i\varepsilon\right)^{2}\right]$$

Again the pole structure of this quantity is preserved if we write in tin the more convenient form

$$G_{\gamma}^{+}\left(\Delta\tilde{\tau};\xi\right)=\frac{-1}{4\pi}\ln\left[4\xi^{2}\sinh^{2}\left(\frac{\Delta\tilde{\tau}}{2}-i\varepsilon\right)\right] \tag{4.13}$$

Substituting this into (4.7) gives (See the Appendix for details)

$$R_{\gamma}^{1}=\frac{c^{2}\left|\left\langle M\right\rangle\right|^{2}}{\Delta E\left(\exp\left(\frac{\Delta E}{kT}\right)-1\right)} \tag{4.14}$$

In which

$$kT=\frac{1}{2\pi\xi} \tag{4.15}$$

Referring back to (3.7), setting $n = 2$ and $n_k = 1/(\exp(\omega/kT)-1)$, we see that the response of the linear detector undergoing uniform acceleration through the Minkowski vacuum is identical to that when placed in a bath of radiation, in Minkowski space, having a Planck spectrum with temperature $1/(2\pi\xi k)$ (k being Boltzmann's constant).

The Planckian nature of the response remains true for four dimensions. In this case the Minkowski vacuum Wightman function is

$$G^{+}\left(x,x'\right)=\frac{-1}{4\pi\left[\left(\Delta t-i\varepsilon\right)^{2}-\left|\boldsymbol{x}-\boldsymbol{x}'\right|^{2}\right]}$$

Assuming the detector accelerates in the z-direction, (4.9) can be used with $x = x'$ and $y = y'$. This gives, after a procedure almost identical to the two dimensional case,

$$G_{\gamma}^{+}\left(\Delta\tau;\xi\right)=\frac{-1}{16\pi^{2}\xi^{2}\sinh^{2}\left(\frac{\Delta\tilde{\tau}}{2}-i\varepsilon\right)} \tag{4.16}$$

Placing this into (4.3) results with (see Appendix for details)

$$R_{\gamma}^{1}=\frac{c^{2}\left|\left\langle M\right\rangle\right|^{2}\Delta E}{2\pi\left(\exp\left(\frac{\Delta E}{kT}\right)-1\right)} \tag{4.17}$$

Which can be seen from (3.7) to correspond, once again, to a bath of (isotropic) Planck radiation, in Minkowski space, of temperature given by (4.15).

4.3 Response in two and four dimensional Schwarzschild Space

Several special vacua have been discussed in this space. (For example, see Candelas 1980.) We shall use the Hartle-Hawking vacuum (Hartle & Hawking 1980) and assume the detector's world line is at a fixed radial distance outside the event horizon (with fixed θ and φ in four dimensions). This being a Killing vector trajectory, the resulting Wightman function will have the form $G_{\gamma}^{+}\left(\Delta\tau;\tilde{\rho}\right)$.

For a two dimensional black hole of mass M_S, the appropriate Wightman function is (Birrell & Davies 1982).

$$G_H^+\left(x,x'\right)=\frac{-1}{4\pi}\ln\left[\left(\Delta\bar{u}-i\varepsilon\right)\left(\Delta\bar{v}-i\varepsilon\right)\right] \tag{4.18}$$

Where

$$\bar{u}=\frac{-\exp\left(-\kappa u\right)}{\kappa} \qquad \bar{v}=\frac{\exp\left(\kappa v\right)}{\kappa} \tag{4.19}$$

With $\kappa = 1/4M_S$ (the surface gravity), $u = t - r^*$, $v = t + r^*$ and

$$r^*=r+2M_S\ln\left|\left(\frac{r}{2M_S}\right)-1\right| \tag{4.20}$$

The Hartle-Hawking vacuum is defined with respect to the $\bar{u},\bar{v}$ coordinates. Substituting (4.19) into (4.18) gives

$$G_H^+\left(x,x'\right)=\frac{-1}{4\pi}\ln\left[\frac{4}{\kappa^2}\exp\left(2r^*\kappa\right)\sinh^2\left(\frac{\Delta t\kappa}{2}-i\varepsilon\right)\right]$$

This function has the same structure, apart from a non-contributing additive term, as (4.13), therefore it is easily seen that the transition rate per unit proper time will be

$$R_H^1=\frac{c^2\left|\langle M\rangle\right|^2}{\Delta E\left(\exp\left(\frac{\Delta E}{kT}\right)-1\right)} \tag{4.21}$$

Where

$$kT=\left[64\pi^2{M_S}^2\left(1-\frac{2M_S}{r}\right)\right]^{1/2} \tag{4.22}$$

In this instance the linear detector's response is identical to when it is immersed in a bath of Planckian radiation in Minkowski space, of temperature $T=\left[k^2 64\pi{M_S}^2\left(1-\left(2M_S/r\right)\right)\right]^{-1/2}$. This is often interpreted as the Tolman $(g_{00})^{-1/2}$ redshift factor (Sciama et al. 1981) arising from red-shifting of radiation due to the gravitational field of the black hole.

Again, this identity of responses carries over to four dimensions. In this case the appropriate Wightman function has the form (Christensen & Fulling 1977)

$$G_H^+(x,x') = \sum_{l,m} \int_{-\infty}^{\infty} \frac{d\omega}{4\pi\omega} \left[\begin{array}{l} \exp(-i\omega\Delta t) \dfrac{Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi') \vec{R}_l(\omega|r) \vec{R}_l^*(\omega|r')}{(1-\exp(-2\pi\omega/\kappa))} + \\ + \exp(i\omega\Delta t) \dfrac{Y_{lm}^*(\theta,\varphi) Y_{lm}(\theta',\varphi') \vec{R}_l^*(\omega|r) \vec{R}_l(\omega|r')}{(1-\exp(-2\pi\omega/\kappa))} \end{array} \right] \tag{4.23}$$

In which $Y_{lm}(\theta,\varphi)$ are the usual spherical harmonic functions and the $R_l(\omega|r)$ functions have the asymptotic forms

$$\begin{aligned} \vec{R}_l(\omega|r) &\sim r^{-1}\exp(i\omega r^*) + \vec{A}_l(\omega) r^{-1} \exp(-i\omega r^*) && r \to 2M_S \\ &\sim B_l(\omega) r^{-1} \exp(i\omega r^*) && r \to \infty \end{aligned} \tag{4.24}$$

$$\begin{aligned} \overleftarrow{R}_l(\omega|r) &\sim B_l(\omega) r^{-1} \exp(-i\omega r^*) && r \to 2M_S \\ &\sim r^{-1}\exp(-i\omega r^*) + \overleftarrow{A}_l(\omega) r^{-1} \exp(i\omega r^*) && r \to \infty \end{aligned} \tag{4.25}$$

Setting the spatial coordinates equal in (4.23) gives

$$G_H^+(\Delta t; r,\theta,\varphi) = \int_{-\infty}^{\infty} \frac{d\omega}{16\pi^2\omega} \left[\begin{array}{l} \exp(-i\omega\Delta t) \displaystyle\sum_{l=0}^{\infty} \frac{(2l+1)\left|\vec{R}_l(\omega|r)\right|^2}{(1-\exp(-2\pi\omega/\kappa))} + \\ + \exp(i\omega\Delta t) \displaystyle\sum_{l=0}^{\infty} \frac{(2l+1)\left|\overleftarrow{R}_l(\omega|r)\right|^2}{(\exp(2\pi\omega/\kappa)-1)} \end{array} \right] \tag{4.26}$$

In the asymptotic regions $r \to \infty$ and $r \to 2M$, the following can be shown to hold (Candelas 1980).

$$\begin{aligned} \sum_{l=0}^{\infty} (2l+1)\left|\vec{R}_l(\omega|r)\right|^2 &\sim \frac{4\omega^2}{(1-2M_S/r)} && r \to 2M_S \\ &\sim \frac{1}{r^2} \sum_{l=0}^{\infty} (2l+1)\left|B_l(\omega)\right|^2 && r \to \infty \end{aligned}$$

$$\begin{aligned} \sum_{l=0}^{\infty} (2l+1)\left|\overleftarrow{R}_l(\omega|r)\right|^2 &\sim \frac{1}{4{M_S}^2} \sum_{l=0}^{\infty} (2l+1)\left|B_l(\omega)\right|^2 && r \to 2M_S \\ &\sim 4\omega^2 && r \to \infty \end{aligned}$$

Substituting these expressions into (4.26) which is then used in (4.7) and performing the $\Delta\tau$ integration yields

$$\begin{aligned} R_H^1 &\sim c^2 \left|\langle M \rangle\right|^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\delta\left(\omega + (1-(2M_S/r))^{1/2}\Delta E\right)\omega}{(1-\exp(-2\pi\omega/\kappa))\left(1-\dfrac{2M_S}{r}\right)} && r \to 2M_S \\ &\sim c^2 \left|\langle M \rangle\right|^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\delta(\omega - \Delta E)\omega}{(\exp(2\pi\omega/\kappa)-1)} && r \to \infty \end{aligned}$$

Therefore the detector's response is

$$R_H^1 \sim \frac{c^2 \left|\langle M \rangle\right|^2 \Delta E}{2\pi\left(\exp\left(\Delta E / kT\right) - 1\right)} \tag{4.27}$$

where T is given by (4.22).

From those calculations we can see that immersion in a bath of isotropic Planck radiation in Minkowski space, uniform acceleration through the Minkowski space vacuum and an r, θ, ϕ all constant trajectory in the Hartle-Hawking vacuum all give rise to identical responses (both in two and four dimensions) for the linear detector. We shall now proceed to see if this is also true for the other particle detectors introduced in the previous chapter.

5 The Quadratic Detector

5.1 General Remarks

When the quadratic detector was introduced in Sec. 3.2, the question of the divergence that appears in (3.11) was only briefly addressed. This shall now be considered in greater detail. The discussion in the first part of this chapter will apply to any neutral scalar field, though the explicit examples then given will be restricted to the massless case.

As stated in Sec. 3.2, the motivation for assuming that the (quadratic) detector responds only to renormalised expectation values is the general philosophy that we can only observe renormalised field quantities. This assumption accords with the condition (a) for a particle detector since using the 'bare' expectation values would result in the inertially moving quadratic detector in Minkowski space giving a non-zero (in fact divergent) response. Although in Minkowski space the removal of the vacuum divergence $\langle 0_M | \phi^2[x] | 0_M \rangle$ merely by normal ordering may be sufficient, in non-Minkowski spaces a more general renormalisation method is required.

In summing over a complete set of states $\{|\Psi\rangle\}$ to deduce the total transition probability P^2, (3.13), from the amplitude A^2, (3.11), the 'offending' term that gives rise to the divergences is $\langle 0_M | \phi^2[x] | 0_M \rangle$. For the non-Minkowski space situations considered in this thesis, the state $|\Psi_0\rangle$ will always be a vacuum state $|0\rangle$. Thus we are confronted with the well-known problem of making sense of the formally divergent vacuum expectation value $\langle \phi^2[x] \rangle$.

The standard approach is to split $\langle \phi^2[x] \rangle$ into its divergent, $\langle \phi^2[x] \rangle_{div}$, and finite $\langle \phi^2[x] \rangle_{ren}$, parts. A well-known method (amongst several) is dimensional regularisation. (See Birrell & Davies 1982 for details of this and other methods). Using

$$\langle \theta^2[x] \rangle = -i \lim_{x \to x'} G_F^{DS}(x, x')$$

Where $G_F^{DS}(x, x')$ is the DeWitt Schwinger expansion of the Feynman Green function; we have (Birrell & Davies 1982)

$$\langle \phi^2[x] \rangle_{div} = 2(4\pi)^{-n/2} \left\{ (n-2)^{-1} + \frac{1}{2}\left[\gamma + \ln\left(m^2/\mu^2\right)\right] \right\} \left\{ \frac{-2m^2}{(n-2)} + \left(\frac{1}{6} - \xi\right) R(x) \right\} \tag{5.1}$$

And by definition

$$\left\langle \phi^2[x] \right\rangle_{ren} = \left\langle \phi^2[x] \right\rangle - \left\langle \phi^2[x] \right\rangle_{div} \tag{5.2}$$

In (5.1), m is the mass of the field, n is the dimension of the space-time, $R(x)$ the Ricci scalar at the space-time point x, ξ is the conformal coupling constant and μ an arbitrary scale introduced for dimensional consistency. The renormalised quantity in (5.2) can be expressed as an asymptotic (adiabatic) expansion (Birrell & Davies 1982)

$$\left\langle \phi^2[x] \right\rangle_{ren} = \frac{1}{(4\pi)^2} \sum_{j=2}^{\infty} a_j(x) m^{2-2j} \Gamma(j-1) \tag{5.3}$$

where $a_j(x)$ are curvature dependent quantities which vanish for zero curvature.

From (5.3) we can see that in Minkowski space $\left\langle \phi^2[x] \right\rangle_{ren}$ vanishes as required for the quadratic detector to satisfy condition (a) for a particle detector. Although in any flat space situation there will be no contributions to the quadratic detector's response from the renormalised vacuum expectation, in a non-flat space-time there could be a contribution arising from the curvature.

In the discussion immediately above it can be seen that the philosophical viewpoint adopted in Sec. 2.2 can be mathematically implemented with a procedure well known in the general theory of quantum fields in curved spaces.

Since $\left|0\right\rangle$ is a vacuum state the expectation value in (3.13), recalling that it is a sum over $\left\{ \left|\Psi\right\rangle \right\}$, may be written as

$$\left\langle 0 \left| \phi^2[x] \phi^2[x'] \right| 0 \right\rangle = \sum_{|\Psi\rangle \neq |0\rangle} \left\langle 0 \left| \phi^2[x] \right| \Psi \right\rangle \left\langle \Psi \left| \phi^2[x'] \right| 0 \right\rangle + \left\langle \phi^2[x] \right\rangle_{ren} \left\langle \phi^2[x'] \right\rangle_{ren} \tag{5.4}$$

Expanding $\phi[x]$ as a mode integral, with operators such that $a_{\boldsymbol{k}} \left|0\right\rangle = 0$ for all $\boldsymbol{k}$, gives the first term in (5.4)

$$\sum_{|\Psi\rangle \neq |0\rangle} \left\{ \int d^{n-1}k_r \int d^{n-1}k_s \left\langle 0 \left| a_{\boldsymbol{k}_r} a_{\boldsymbol{k}_s} \right| \Psi \right\rangle u_{\boldsymbol{k}_r}(x) u_{\boldsymbol{k}_s}(x) \int d^{n-1}k_q \int d^{n-1}k_p \left\langle \Psi \left| a^*_{\boldsymbol{k}_q} a^*_{\boldsymbol{k}_p} \right| 0 \right\rangle u^*_{\boldsymbol{k}_q}(x') u^*_{\boldsymbol{k}_p}(x') \right\}$$

All other terms vanish because of mismatching between the number of creators and annihilators. From the various possibilities of creation and annihilation of particle out of the state $\left|0\right\rangle$, and allowing for double counting, we find

$$\begin{aligned}
\left\langle 0 \left| \phi^2[x] \phi^2[x'] \right| 0 \right\rangle_{ren} &= \int d^{n-1}k_r \int d^{n-1}k_s \int d^{n-1}k_p \int d^{n-1}k_q \left[\delta(\boldsymbol{k}_r - \boldsymbol{k}_p) \delta(\boldsymbol{k}_s - \boldsymbol{k}_q) + \delta(\boldsymbol{k}_r - \boldsymbol{k}_q) \delta(\boldsymbol{k}_s - \boldsymbol{k}_p) \right] \\
&\quad u_{\boldsymbol{k}_r}(x) u_{\boldsymbol{k}_s}(x) u^*_{\boldsymbol{k}_p}(x') u^*_{\boldsymbol{k}_q}(x') + \left\langle \phi^2[x] \right\rangle_{ren} \left\langle \phi^2[x'] \right\rangle_{ren} \\
&= 2\left[G^+(x,x') \right]^2 + \left\langle \phi^2[x] \right\rangle_{ren} \left\langle \phi^2[x'] \right\rangle_{ren}
\end{aligned} \tag{5.5}$$

Therefore

$$P^2 = c^2 \left|\langle M\rangle\right|^2 \int_{-\infty}^{\infty} d\tau d\tau' e^{-i\Delta E\Delta\tau} \left\{ 2\left[G_\gamma^+\left(\tau,\tau';\bar{\rho}\right)\right]^2 + \left\langle \phi^2\left[x\right]\right\rangle_{ren} \left\langle \phi^2\left[x'\right]\right\rangle_{ren} \right\} \tag{5.6}$$

As with the linear detector there are time dependent and independent situations. In a time dependent situation the vacuum expectation contribution to (5.6) vanishes because in such cases $\left\langle \phi^2\left[\tau;\bar{\rho}\right]\right\rangle_{ren}$ is not a function of time. That is

$$\left\langle \phi^2\left[\tau;\bar{\rho}\right]\right\rangle_{ren} = \left\langle \phi^2\left[\bar{\rho}\right]\right\rangle_{ren}$$

Hence their appearance in (5.6) has the form $\delta\left(\Delta E\right)\left\langle \phi^2\left[\bar{\rho}\right]\right\rangle_{ren}$ which gives no contribution since, by assumption, $\Delta E \neq 0$. Therefore for time independent situations the transition rate is

$$R^2 = 2c^2 \left|\langle M\rangle\right|^2 \int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E B\tau} \left[G_\gamma^+\left(\Delta\tau;\bar{\rho}\right)\right]^2 \tag{5.7}$$

For a time dependent situation the transition rate is easily seen to be

$$R^2\left(\tau\right) = c^2 \left|\langle M\rangle\right|^2 \int_{-\infty}^{\infty} d\eta e^{-i\Delta E\eta} \left\{ \begin{aligned} &\left(2\left[G_\gamma^+\left(\eta+\tau,\tau;\bar{\rho}\right)\right]^2 + \left\langle \phi^2\left[\eta+\tau;\bar{\rho}\right]\right\rangle_{ren} \left\langle \phi^2\left[\tau;\bar{\rho}\right]\right\rangle_{ren}\right)\theta\left(-\eta\right) \\ &+\left(2\left[G_\gamma^+\left(\tau,\tau-\eta;\bar{\rho}\right)\right]^2 + \left\langle \phi^2\left[\tau;\bar{\rho}\right]\right\rangle_{ren} \left\langle \phi^2\left[\tau-\eta;\bar{\rho}\right]\right\rangle_{ren}\right)\theta\left(\eta\right) \end{aligned} \right\} \tag{5.8}$$

5.2 Response in four and two dimensional Rindler Space

We now consider the response of a quadratic detector, of a massless scalar field, undergoing uniform acceleration through the four dimensional Minkowski vacuum. Since this is a time independent situation, (5.7) is appropriate. From (4.16)

$$R_\gamma^2 = c^2 \left|\langle M\rangle\right|^2 \int_{-\infty}^{\infty} d\Delta\tau \frac{e^{-\Delta E\Delta\tau}}{\left[16\pi\xi^2 \sinh^2\left(\dfrac{\Delta\tilde{\tau}}{2} - i\varepsilon\right)\right]^2} \tag{5.9}$$

Although (5.9) could be calculated directly with the method used to evaluate the corresponding linear detector calculation, we shall take this opportunity to illustrate an alternative procedure which utilises the already known response of the linear detector to the same situation. In doing so, this alternative can avoid some of the calculational problems that may appear when using the direct contour approach. From the linear detector calculation in the Appendix

$$\int d\Delta\tau e^{-i\Delta E\Delta\tau} \sinh^{-2}\left(\frac{\Delta\tau}{2\xi} - i\varepsilon\right) = \frac{-8\pi\Delta E\xi^2}{\exp\left(2\pi\Delta E\xi\right) - 1}$$

Using the convolution theorem for Fourier Transforms the response (5.9) can be evaluated directly

$$\int_{-\infty}^{\infty} d\Delta\tilde{\tau} e^{-i\Delta E\Delta\tilde{\tau}\xi} \sinh^{-4}\left(\frac{\Delta\tilde{\tau}}{2} - i\varepsilon\right) = 64\pi^2 \int_{-\infty}^{\infty} dX \frac{X\left(\xi\Delta E - X\right)}{\left(\exp\left(2\pi X\right) - 1\right)\left(\exp\left(2\pi\left[\xi\Delta E - X\right]\right) - 1\right)}$$

Setting $x = \exp(2\pi X)$ and $b = \exp(2\pi\Delta E\xi)$;

$$\frac{8}{\pi}\int_0^\infty \frac{dx \ln x \ln(x/b)}{(x-1)(x-b)} = \frac{8}{\pi}\left[\frac{4\pi^2 + (\ln b)^2}{6(b-1)}\right]\ln b = \frac{32\pi^2 \Delta E \xi\left(1+(\Delta E\xi)^2\right)}{3\left(\exp(2\pi\xi\Delta E)-1\right)}$$

(See Gradshteyn & Ryzhik (1980) No.4.257.4) Applying this result gives

$$R_\gamma^2 = \frac{16}{3}\frac{c^2 \left|\langle M\rangle\right|^2 (kT)^2 \left(1+(\Delta E/2\pi kT)^2\right)\Delta E}{\left(\exp(\Delta E/kT)-1\right)} \tag{5.10}$$

Using $kT = 1/(2\pi\xi)$

$$R_\gamma^2 = \frac{4}{3\pi}\frac{c^2 \left|\langle M\rangle\right|^2 \left(1+(\Delta E\xi)^2\right)\Delta E}{\xi^2\left(\exp(2\pi\xi\Delta E)-1\right)} \tag{5.11}$$

To compare (5.11) with the response of this detector when immersed in a bath of isotropic Planck radiation, in Minkowski space, we could compute (3.15) with $n_k = 1/(\exp(\omega/kT)-1)$, $m = 0$ and $n = 4$. Unfortunately, the integrals are not easily evaluated. Therefore, an alternative approach is taken. We shall use (3.16), in which the final term makes no contribution since it is independent of $\Delta\tau$. If it can be shown that, with the parameters appropriately set, this first term gives the same result as (5.10), then the response in the two situations will be identical. This reduces to showing that

$$\frac{-1}{4\pi^2(\Delta\tau - i\varepsilon)^2} + \frac{1}{(2\pi)^2}\int_0^\infty d\omega \frac{\omega\cos\omega(\Delta\tau - i\varepsilon)}{\left(\exp(\omega/kT)-1\right)} = \pm\frac{1}{16\pi^2\xi^2\sinh^2\left(\frac{\Delta\tilde{\tau}}{2} - i\varepsilon\right)} \tag{5.12}$$

Which, from (5.9), will give the required result. Evaluating the integral (Gradshteyn & Ryzhik (1980), No. 3.951.5) gives

$$\frac{-1}{4\pi^2(\Delta\tau - i\varepsilon)^2} + \frac{1}{(2\pi)^2}\left[\frac{1}{2(\Delta\tau - i\varepsilon)^2} - \frac{\pi^2(kT)^2}{\sinh^2(\pi kT\Delta\tau - i\varepsilon)}\right] = \frac{-(kT)^2}{4\pi^2\sinh^2(\pi kT\Delta\tau - i\varepsilon)}$$

With $kT = 1/(2\pi\xi)$ and $\Delta\tilde{\tau} = \Delta\tau/\xi$ (5.12) is indeed seen to be satisfied. Therefore the quadratic detector responds identically to immersion in a bath of (isotropic) Planck radiation, in Minkowski space, and uniform accelerated through the Minkowski vacuum.

In this section we have studied the quadratic detector's response only in four dimensional Rindler space. Unfortunately, the calculation of the response of this detector for the two dimensional case is not as straight forward as for the linear detector. Referring to the Appendix, it can be seen that an infinite logarithmic term is discarded in the calculation by exploiting the fact that $\Delta E \neq 0$. Such a manoeuvre is acceptable for the linear detector. However in the case of the quadratic detector that term is multiplied by a non-trivial terms arising from the infinite product representation of the hyperbolic-sine function and there is no obvious justification for discarding all the divergent terms.

This is not related to renormalisation (the vacuum divergence has already been removed). Nor is it related to the infra-red divergences characteristic in two dimensional field theory as may be seen by the compactifying the space-time. This is a common method of removing such divergences, however the Wightman function for an $R^1 \times S^1$ space-time has the form

$$D^{+}(x,x')=\frac{-1}{4\pi}\left\{\ln\left[4\sin\left(\frac{\pi(\Delta u-i\varepsilon)}{L}\right)\sin\left(\frac{\pi(\Delta v-i\varepsilon)}{L}\right)\right]+i\pi\left[1-\frac{(\Delta u+\Delta v)}{L}\right]\right\}$$

which, when placed into (5.7), will still give rise to similar divergent terms in the detector's response. The difficulty in calculating the quadratic detector's response in two dimensional situations is seen to be related to the logarithmic form of the Wightman functions of these situations. To calculate the response of this detector in these cases requires a more indirect method that circumvents the problem of dealing directly with the logarithmic factors.

In point of fact one such method was employed in the above four dimensional calculation. Rather than perform a contour integral to evaluate (5.9) (a method that involves the evaluation of residues of second order poles) we used the convolution theorem. This approach is tantamount to evaluating the auto-correlation of the linear detector's response, thus allowing us to write the quadratic detector's transition rate in terms of that of the linear detector. If the linear detector's response in $R^{l}(\Delta E)$ from the convolution theorem and (5.7) the quadratic detector's response will be,

$$R^{2}(\Delta E)=2\left(R^{1}*R^{1}\right)(\Delta E)=2\int_{-\infty}^{\infty}dX\;R^{1}(X)R^{1}(\Delta E-X) \tag{5.13}$$

which is the auto-correlation of the linear detector transition rate.

When using (5.13), the role played by terms of the form $\delta(\Delta E)$ x (constant) must be considered, especially in view of the possibility of the constant being infinite. This problem is avoided by evaluating the terms in R^{l} for the ranges $0^{-} > \Delta E > -\infty$ and $0^{+} < \Delta E < +\infty$ only. This is in accordance with the statements made in the concept of a "particle detector" was first introduced in Sec. 2.2. In that passage it was stated that interest is attached only to the probability of the detector under-going a **change** from its initial state E_0 to a **different** final state E. As a result terms of the form $\delta(\Delta E)$ x (constant) have no calculational role to play and hence will not appear in (5.13).

Using this auto-correlation technique we can calculate the response of the quadratic detector in a two dimensional Rindler space. Referring to the Appendix, it can be seen that in this case

$$R_{\gamma}^{2}=c^{2}\left|\langle M\rangle\right|^{2}\int_{-\infty}^{\infty}\frac{dX}{\left\{X(\Delta E-X)\left(\exp(X/kT)-1\right)\left(\exp([\Delta E-X]/kT)-1\right)\right\}}$$

This integral is manifestly divergent, but it is now clearly seen to be due to the usual infra-red divergence characteristic of the two dimensional Planck spectrum. Compactification of the space dimension does not remove this divergence from the response.

5.3 Response in four dimensional Schwarzschild space

From (4.23) and the asymptotic values of the Wightman functions we can see that (4.27) is valid at least for $\Delta E \neq 0$. Also, in the evaluation of $\left[G_{H}^{+}(x,x')\right]$ the leading terms in the asymptotic regions area exactly those that produce (4.27). Therefore (5.13) may be used to evaluate R^2 and is easily seen to give (5.11) with

$$kT=\left[64\pi^{2}M_{S}^{\,2}\left(1-\frac{2M_{S}}{r}\right)\right]^{-1/2} \tag{5.14}$$

as with the linear detector.

With Schwarzschild space being curved, it is worth checking that the vacuum expectation does give no contribution to the detector's response. For the Hartle-Hawking vacuum this expectation value is (Candelas 1980)

$$\begin{aligned} \left\langle \phi^2[x] \right\rangle_{ren} &\sim \left(192\pi^2 {M_S}^2\right)^{-1} && r \to 2M_S \\ &\sim \frac{1}{2\pi}\int_0^\infty \frac{d\omega\,\omega}{\exp\left(\left(2\pi\omega/\kappa\right)-1\right)} && r \to \infty \end{aligned}$$

And since the detector's world line is given by r, θ, ϕ all constant no time dependence will appear and so with $\Delta E \neq 0$ there will be no contribution in these asymptotic regions.

Even outside these asymptotic regions, because the detector's trajectory is described by a time-like Killing vector of the space-time, the vacuum expectation will be constant over all time thus not contributing to the response. This relationship between Killing vector fields and the quadratic detector's response is obviously quite general.

From the calculations in this chapter, it is apparent that the quadratic detector responds identically (in four dimensions) in Rindler space and when immersed in a bath of isotropic Planckian radiation in Minkowski space. In these cases this detector exhibits a behaviour in common with the linear detector. On the other hand in four dimensional Schwarzschild space, for the quadratic detector to respond similarly to the linear detector, it must follow a time-like Killing vector trajectory corresponding outside the black hole. Furthermore, in two dimensions the response is undefined due to the form of the two dimensional Planck spectrum.

We shall see below that these similarities and differences are particular cases of rather general statements about the comparative behaviour of different detectors.

6 The Derivative Detector

6.1 General Remarks

The derivative detector differs from the linear and quadratic detectors in that it is "orientable". Inspecting its interaction Lagrangian (3.17) we see that the definition of this detector involves a four-vector b^μ. Although the Lagrangian is covariant (b^μ being contracted with another four-vector), this covariance is "broken" if b^μ is fixed in the co-moving frame. Such a choice defines an "orientation" of the detector in the co-moving frame. When evaluating the response of the detector its orientation (i.e. b^μ) must be specified. With this done the detector is still receptive to the full (n-2)-sphere of momentum mode directions and so an omni-directional detector.

With this detector's orientation specified, the evaluation of its response follows the usual lines. However to make the calculations easier it is convenient to evaluate the response in the frame for which the orientation is fixed (which is usually the co-moving frame). For the previous detectors this manoeuvre is of no major help since those responses involve calculations only with scalar quantities. If b^μ is fixed in, say the co-moving frame, in general it will be time dependent in any other frame.

Fortunately, by using the (τ,ρ)-coordinatisation introduced in Chapter 4, the co-moving frame evaluation of this detector's response is particularly easy for a fixed orientation in that frame. In the co-moving frame a convenient fixed n-ad can be constructed local to the detector's world-line using the approach outlined by Misner, Thorne and Wheeler (1973). For the (τ,ρ)-system the resulting n-ad $\{e^{\mu}\}$ has the form

$$e_0 = \partial_\tau = \frac{\partial}{\partial \tau}, \quad e_i = \partial_{\rho^i}\Big|_{\rho=\tilde{\rho}} = \frac{\partial}{\partial \rho^i}\Bigg|_{\rho=\tilde{\rho}}$$

The (fixed) components of b^{μ} in the n-ad are deduced by contracting the four-vector b with each of the n-ad basis vectors e^{μ}.

As stated by Misner, Thorne and Wheeler, the coordinate systems constructed form the n-ad $\{e^{\mu}\}$ is well behaved only in the region local to the trajectory (which is described by $\rho = \tilde{\rho}$), since in more distant regions the "coordinate lines" sent out form the trajectory may cross. Because the orientation four-vector b^{μ} need be defined only along the detector trajectory the behaviour of these (local) co-moving coordinates far from the trajectory is unimportant. Similarly, the exact nature of the (τ,ρ)-coordinatisation far from the trajectory is of little importance and in general there will be many different possible coordinatisations all of which correspond along the trajectory.

In the following sections of this chapter, comparisons will be made of the derivative detector's response when placed in different situations. To fully describe and compare these situations, the orientation of b^{μ} must be stated in each case. For many cases relating the orientation of the detector in two different situations is most easily done using the vierbein (or n-bein) formalism (Weinberg 1972). This will relate the (τ,ρ)-coordinate system with an appropriate instantaneously co-moving inertial frame. The coordinate derivatives (of the quantum field) in the two frames are related by

$$\partial_\alpha \to \partial_\mu = V_\mu^\alpha \partial_\alpha \tag{6.1}$$

Where the α subscript refers to the inertial frame and the μ subscript to the (τ,ρ)-frame. The use of vierbeins in evaluating the detector's response is best demonstrated by the Rindler space example given below.

From (3.18) with $|\Psi_0\rangle$ vacuum state the total transition amplitude (to first order in c) for this detector moving along a trajectory described by $x^{\mu}(\tau)$ is

$$A^3 = ic\langle M\rangle b^{\mu} \int_{-\infty}^{\infty} d\tau e^{i\Delta E\tau} \partial_\mu \left\langle \Psi \middle| \phi\left[x(\tau)\right] \middle| \Psi_0 \right\rangle \tag{6.2}$$

which gives transition probability

$$P^3 = c^2 \left|\langle M\rangle\right|^2 b^{\mu} b^{\nu} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-i\Delta E\Delta\tau} \partial_\mu \partial'_\nu \left\langle \Psi_0 \middle| \phi\left[x(\tau)\right] \phi\left[(x')\right] \middle| \Psi_0 \right\rangle \tag{6.3}$$

Using the (τ,ρ)-coordinates,

$$P^3 = c^2 \left|\langle M\rangle\right|^2 b^{\mu} b^{\nu} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-i\Delta E\Delta\tau} \partial_\mu \partial'_\nu G_\gamma^+\left(\tau,\rho;\tau',\rho'\right)\Big|_{\rho=\rho'=\tilde{\rho}} \tag{6.4}$$

In (6.4) the derivatives are evaluated at different spatial coordinates ρ and ρ' which are both set equal to $\tilde{\rho}$, the fixed spatial coordinate of the (point) detector. For convenience, henceforth the following abbreviation will be used:

$$\partial_\mu \partial'_\nu G^+_\gamma\left(\tau,\rho;\tau',\rho'\right)\Big|_{\rho=\rho'=\tilde{\rho}} \equiv \partial_\mu \partial'_\nu G^+_\gamma\left(\tau,\tau';\rho'\right)\Big|$$

The vertical bar indicating the method of evaluating the derivatives. The transition rate of this detector, at time τ, is

$$R^3\left(\tau\right)=c^2\left|\left\langle M\right\rangle\right|^2 b^\mu b^\nu \int_{-\infty}^{\infty} d\eta e^{-i\Delta E\eta}\left\{\partial_\mu \partial'_\nu G^+_\gamma\left(\eta+\tau,\tau;\tilde{\rho}\right)\Big|\theta\left(-\eta\right)+\partial_\mu \partial'_\nu G^+_\gamma\left(\tau,\tau-\eta;\tilde{\rho}\right)\Big|\theta\left(\eta\right)\right\} \quad (6.5)$$

For the time independent situations,

$$R^3\left(\tau\right)=c^2\left|\left\langle M\right\rangle\right|^2 b^\mu b^\nu \int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E\Delta\tau}\partial_\mu \partial'_\nu G^+_\gamma\left(\Delta\tau;\tilde{\rho}\right)\Big| \quad (6.6)$$

Before presenting the Rindler and Schwarzschild space examples for this detector, a rather important technical point must be addressed. By definition the derivative detector is non-local. Its response depends upon the derivative of the field which, although defined at a point, is evaluated on a non-local way. This follows from the definition of the derivative. In particular the spatial derivatives involve (the limit of) space-like separations. (For the purposes of this study the detector is assumed to be a point object, this being an idealisation of a detector with infinitesimal spatial extent with respect to the (local) length scaling of the space-time.)

Given (6.6), if we naively evaluate the (spatial) derivatives in the stipulated manner, we find that even in a manifestly stationary situation the detector's response is apparently time dependent. The reason for this can be traced to the Tolman factor (Tolman 1934) in the metric. This factor describes the redshift resulting from the variation of g_{00} in the metric. For stationary situations this variation is purely spatial and will appear when evaluating the spatial derivatives of $G^+_\gamma\left(\Delta\tau;\tilde{\rho}\right)\Big|$.

To make this point more explicit, consider a derivative detector at a fixed radius outside a black hole. Since the detector's trajectory is along a Killing vector of the space-time, it must perceive the situation as time stationary, and hence have a time independent response. The appropriate Wightman function (4.23) is manifestly time independent, having the form

$$G^+_H\left(x,x'\right)=\int_{-\infty}^{\infty}\frac{d\omega}{4\pi\omega}\sum_{l,m}\left[\begin{array}{l}\exp\left(-i\omega\left[1-\dfrac{2M_S}{r}\right]^{-1/2}\Delta\tau\right)\dfrac{Y_{lm}\left(\theta,\varphi\right)Y^*_{lm}\left(\theta',\varphi'\right)\vec{R}_l\left(\omega|r\right)\vec{R}^*_l\left(\omega|r'\right)}{\left(1-\exp\left(-2\pi\omega/\kappa\right)\right)}+\\ +\exp\left(i\omega\left[1-\dfrac{2M_S}{r}\right]^{-1/2}\Delta\tau\right)\dfrac{Y^*_{lm}\left(\theta,\varphi\right)Y_{lm}\left(\theta',\varphi'\right)\vec{R}^*_l\left(\omega|r\right)\vec{R}_l\left(\omega|r'\right)}{\left(1-\exp\left(-2\pi\omega/\kappa\right)\right)}\end{array}\right]$$

In which the Tolman factor is explicit. It is easily seen that taking radial derivatives of this function will introduce factors not solely dependent on $\tau-\tau'$, resulting with this detector's response being time dependent when placed in a manifestly time independent situation.

We shall now demonstrate that the Tolman factor <u>should not</u> be differentiated when evaluating $\partial_\mu\partial'_\nu G^+_\gamma\Big|$. Referring to the well-known Rindler space result, it has been shown that if the quantum field is in the

Minkowski vacuum state the expectation value of observables on the Rindler (or Fulling) Fock space is identical to its ensemble average in thermal equilibrium at temperature T = (proper acceleration)/2πk (Sciama et al. 1981). For example the Rindler number operator $\bar{N}_{\boldsymbol{k}}$ for the momentum mode $\boldsymbol{k}$ has an expectation value given by (Fulling 1973, Davies 1975)

$$\langle 0_M | \bar{N}_{\boldsymbol{k}} | 0_M \rangle = \frac{1}{\exp(2\pi\omega\xi - 1)} \tag{6.7}$$

Where ξ^{-1} is the proper acceleration. Without appreciating the role of the Tolman factor, (6.7) seems to imply that the region close to the event horizon ($\xi \to 0$) in the Rindler wedge cannot be in thermal equilibrium with a region more distant form the horizon ($\xi \to \infty$) due to the apparent temperature gradient with respect to ξ. (This "gradient" arises from the x-derivative of the Tolman factor.) However, as Sciama et al. point out;

> Two bodies can be in thermal equilibrium at different points in a gravitational field if and only if the ratio of their temperatures is equal to the gravitational redshift that light would suffer in travelling from one point to the other. Thus the dependence of temperature on position should cause no great surprise…

This temperature dependence is given by the Tolman factor (Birrell & Davies 1982).

So, referring back to the evaluation of $\partial_\mu \partial'_\nu G^+_\gamma(\Delta\tau;\tilde{\rho})\Big|$, to account for the Tolman "redshift" factor, the following procedure must be followed: Without loss of generality we can assume $G(\tau,\rho;\tau',\rho')$ has the form

$$G^+_\gamma(\tau,\rho;\tau',\rho') = G^+_\gamma(\tau.f(\rho),\rho;\tau'.f(\rho'),\rho')$$

Where $f(\rho)$ is some function of the spatial coordinates ρ. Assume the Tolman factor is given by $g(\rho)$ (i.e. $g(\rho) = (g_{00})^{-1/2}$), then G^+_γ may be written as

$$G^+_\gamma(\tau,\rho;\tau',\rho') = G^+_\gamma(\tilde{\tau}.h(\rho),\rho;\tilde{\tau}'.h(\rho'),\rho') \tag{6.8}$$

Where $\tilde{\tau} = \tau / g(\rho)$ and $h(\rho) = f(\rho)g(\rho)$. The quantity $\tilde{\tau}$ includes the Tolman factor, so when evaluating the spatial derivatives of G^+_γ, $\tilde{\tau}$ is not differentiated with respect to the ρ-coordinates since doing so will fail to take account of the gravitational redshift.

6.2 Response in two and four dimensional Rindler space

The two dimensional Wightman function for Rindler space is given by (4.9) - (4.11). The derivatives are to be evaluated in the manner stipulated above. The acceleration occurs in the (only) spatial direction and in the co-moving frame this corresponds to the ξ-direction. (This follows from (6.1) with the veirbein V = diag(ξ,1)). Also using (6.8), $g(\xi) = \xi$ and $\tilde{\tau} = \tau / \xi$. Taking the τ-derivative, we can set $\xi = \xi'$ giving

$$\partial_\tau \left\{ (-1/4\pi) \ln\left[4\xi^2 \sinh^2\left(\frac{\Delta\tilde{\tau}}{2} - i\varepsilon \right) \right] \right\} = \frac{-1}{4\pi\xi} \coth\left(\frac{\Delta\tilde{\tau}}{2} - i\varepsilon \right)$$

Thus

$$\partial_{\tau}\partial_{\tilde{\tau}}G_{\gamma}^{+}\left(\Delta\tau;\xi\right)\Big| = \frac{1}{8\pi^2\xi^2}\sinh^{-2}\left(\frac{\Delta\tilde{\tau}}{2} - i\varepsilon\right) \tag{6.9}$$

From the Appendix the transition rate is seen to be

$$R_{\gamma,00}^{3} = \frac{c^2\left|\langle M\rangle\right|^2\left(b^0\right)^2\Delta E}{\left(\exp\left(\Delta E/kT\right)-1\right)} \tag{6.10}$$

Where $kT = 1/2\pi\xi$. The evaluation of $R_{\gamma,11}^{3}$ is much more drawn out, so only the major steps will be presented. With the points split, the Wightman function is

$$G_{\gamma}^{+}\left(\tau,\xi;\tau',\xi'\right) = \frac{-1}{4\pi}\ln\left[\left(\xi\sinh\tilde{\tau} - \xi'\sinh\tilde{\tau}' - i\varepsilon\right)^2 - \left(\xi\cosh\tilde{\tau} - \xi'\cosh\tilde{\tau}'\right)^2\right] \tag{6.11}$$

The ξ – and ξ– derivatives are evaluated with the points split, resulting in

$$\partial_{\xi}\partial_{\xi'}G_{\gamma}^{+}\left(\tau,\xi;\tau',\xi'\right) =$$

$$\frac{1}{4\pi}\left\{\begin{array}{l} \dfrac{-2\sinh\tilde{\tau}'\sinh\tilde{\tau} + 2\cosh\tilde{\tau}'\cosh\tilde{\tau}}{\left(\xi\sinh\tilde{\tau} - \xi'\sinh\tilde{\tau}' - i\varepsilon\right)^2 - \left(\xi\cosh\tilde{\tau} - \xi'\cosh\tilde{\tau}'\right)^2} \\ -\left[2\sinh\tilde{\tau}\left(\xi\sinh\tilde{\tau} - \xi'\sinh\tilde{\tau}' - i\varepsilon\right) - 2\left(\xi\cosh\tilde{\tau} - \xi'\cosh\tilde{\tau}'\right)\cosh\tilde{\tau}\right]\times \\ \times\dfrac{\left[-2\sinh\tilde{\tau}'\left(\xi\sinh\tilde{\tau} - \xi'\sinh\tilde{\tau}' - i\varepsilon\right)' + 2\left(\xi\cosh\tilde{\tau} - \xi'\cosh\tilde{\tau}'\right)\cosh\tilde{\tau}'\right]}{\left(\left(\xi\sinh\tilde{\tau} - \xi'\sinh\tilde{\tau}' - i\varepsilon\right)^2 - \left(\xi\cosh\tilde{\tau} - \xi'\cosh\tilde{\tau}'\right)^2\right)^2} \end{array}\right\}$$

After a rather large amount of manipulation and pairing of terms this expression reduces to

$$\partial_{\xi}\partial_{\xi'}G_{\gamma}^{+}\left(\Delta\tau,\xi\right)\Big| = \frac{-1}{8\pi\xi^2}\sinh^{-2}\left(\frac{\Delta\tau}{2} - i\varepsilon\right)$$

Which is identical to (6.10) and so

$$R_{\gamma,11}^{3} = \frac{c^2\left|\langle M\rangle\right|^2\left(b^1\right)^2\Delta E}{\left(\exp\left(\Delta E/kT\right)-1\right)} \tag{6.12}$$

The 0,1 - and 1,0 – cross terms in the response are evaluated by differentiating (6.11) with respect to τ, then with respect to ξ and then setting $\xi = \xi$. After a calculation similar to the $\partial_{\xi}\partial_{\xi'}$ evaluation, we find

$$\partial_{\tau}\partial_{\xi'}G_{\gamma}^{+}\left(\Delta\tilde{\tau},\xi\right)\Big| = 0$$

Therefore

$$R_{\gamma,01}^{3} + R_{\gamma,10}^{3} = 0$$

Referring back to (3.19) we see that (6.10) and (6.12) are thermal responses for this detector. Therefore the derivative detector responds thermally (for all orientations) in two-dimensional Rindler space.

To calculate this detector's responses in four dimensional Rindler space the approach adopted by Unruh (1976) will be used. This involves evaluating the transition amplitude (6.2) instead of (6.6) and makes the calculation much neater.

The massless scalar field equation in Rindler space is written in the form (Pfautsch 1981)

$$\left[\xi^{-2}\frac{\partial^2}{\partial\tilde{\tau}^2}-\xi^{-1}\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi}-\left(\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}\right)\right]\phi[x]=0$$

which gives mode solutions, in the right hand wedge,

$$u_R\left(x;\tilde{\omega},k_1,k_2\right)=\frac{\left(\sinh\pi\tilde{\omega}\right)^{1/2}}{2\pi^2}e^{-i\tilde{\omega}\tilde{\tau}}e^{i\left(k_1x+k_2y\right)}K_{i\tilde{\omega}}\left(Q\xi\right) \tag{6.13}$$

Where $K_{i\omega}(Q\xi)$ is a Macdonald function (Titchmarsh 1958), $Q^2=k_1^2+k_2^2$ and $\tilde{\omega}>0$. The field can be expanded in terms of Rindler modes and operators $a_R\left(\tilde{\omega},k_1,k_2\right)$, $a_R^*\left(\tilde{\omega},k_1,k_2\right)$.

$$\phi[x]=\int d\tilde{\omega}dk_1dk_2\left[a_R\left(\tilde{\omega},k_1,k_2\right)u_R\left(x;\tilde{\omega},k_1,k_2\right)+a_R^*\left(\tilde{\omega},k_1,k_2\right)u_R^*\left(x;\tilde{\omega},k_1,k_2\right)\right] \tag{6.14}$$

Substituting this into (3.3) (for the linear detector) and following the calculation through as in the Appendix gives the already known response for the linear detector in four-dimensional Rindler space. Note that in the calculation, a logarithmic divergence characteristic of such equilibrium situations has been removed (Birrell &Davies 1982). We should expect to remove identical divergences from the response of the derivative detector.

Substituting (6.13) and (6.14) into the transition amplitude expression (6.2) gives

$$\begin{aligned}A_\gamma^3=&\frac{ic\langle M\rangle}{2\pi^2}\xi b^\mu\int_{-\infty}^{\infty}d\tilde{\tau}e^{i\Delta E\xi\tilde{\tau}}\int d\tilde{\omega}dk_1dk_2\times\\&\langle\Psi|\Big\{a_R\left(\tilde{\omega},k_1,k_2\right)\partial_\mu\left[\left(\sinh\pi\tilde{\omega}\right)^{1/2}e^{-i\tilde{\omega}\tau}e^{i\left(k_1x+k_2y\right)}K_{i\omega}\left(Q\xi\right)\right]+\\&+a_R^*\left(\tilde{\omega},k_1,k_2\right)\partial_\mu\left[\left(\sinh\pi\tilde{\omega}\right)^{1/2}e^{i\tilde{\omega}\tau}e^{-i\left(k_1x+k_2y\right)}K_{-i\omega}\left(Q\xi\right)\right]\Big\}|0_M\rangle\end{aligned} \tag{6.15}$$

Choosing the orientation of the detector in the co-moving frame to be, in turn, $\partial_\tau=\left(1/\xi\right)\partial_{\tilde{\tau}}=\partial_0$, $\partial_x=\partial_1$ $\partial_y=\partial_2$, $\partial_z=\partial_3$ and taking them in sequence;

$$A_{\gamma,0}^3=\frac{c\langle M\rangle b^0}{\pi}\int d\tilde{\omega}dk_1dk_2\tilde{\omega}\left(\sinh\pi\tilde{\omega}\right)^{1/2}e^{i\left(k_1x+k_2y\right)}K_{i\omega}\left(Q\xi\right)\delta\left(\Delta E\xi-\tilde{\omega}\right)\langle\Psi|a_R\left(\tilde{\omega},k_1,k_2\right)|0_M\rangle$$

Where the $\Delta\tilde{\tau}$ - integration has been performed. The second term in (6.15) makes no contribution because of the resulting delta function $\delta\left(\xi\Delta E+\tilde{\omega}\right)$ cannot attain its argument since $\xi\Delta E>0$. Performing the $\tilde{\omega}$ - integration immediately gives

$$A_{\gamma,0}^3=\frac{c\langle M\rangle b^0}{\pi}\Delta E\xi\left(\sinh\pi\Delta E\xi\right)^{1/2}\int dk_1dk_2e^{i\left(k_1x+k_2y\right)}K_{i\Delta E\xi}\left(Q\xi\right)\langle\Psi|a_R\left(\Delta E\xi,k_1,k_2\right)|0_M\rangle \tag{6.16}$$

Using a Bogolubov transformation the Rindler operators a_R, a_R^* can be related to the Minkowski operators a_M, a_M^* by (Pfaustch 1981)

$$a_R(\tilde{\omega},k_1,k_2)=\int d^3k'\left[2\pi\omega'\left(1-e^{-2\pi\tilde{\omega}}\right)\right]^{-1/2}\left[\frac{\omega'+k_3'}{Q}\right]^{i\tilde{\omega}}\delta(k_1-k_1')\delta(k_2-k_2') \\ \times\left\{a_M(k_1',k_2',k_3')+e^{-\pi\tilde{\omega}}a_M^*(k_1',k_2',k_3')\right\} \tag{6.17}$$

Which in (6.16) yields

$$A_{\gamma,0}^3=\frac{c\langle M\rangle b^0}{\pi}\Delta E\xi\left(\sinh\pi\Delta E\xi\right)^{1/2}e^{-\pi\Delta E\xi}\left[2\pi\left(1-e^{-2\pi\Delta E\xi}\right)\right]^{-1/2} \\ \times\int\frac{d^3k'}{\omega'^{1/2}}\left[\frac{\omega'+k_3'}{Q'}\right]^{i\Delta E\xi}e^{i(k_1'x+k_2'y)}K_{i\Delta E\xi}(Q\xi)\langle\Psi|1_{\underset{\sim}{k}'}\rangle$$

To arrive at the transition probability $P_{\gamma,00}^3$ we have

$$P_{\gamma,00}^3=\sum_{|\Psi\rangle}A_{\gamma,0}^3\left[A_{\gamma,0}^3\right]^*$$

Where $|\Psi\rangle$ is a complete set of states. Only the one particle states can contribute to the transition amplitude and probability. That is, $|\Psi\rangle=|1_{\underset{\sim}{k}}\rangle$ hence $\langle\Psi|1_{\underset{\sim}{k}'}\rangle=\delta^3(\underset{\sim}{k}-\underset{\sim}{k}')$. Therefore it follows that

$$P_{\gamma,00}^3=\frac{c^2|\langle M\rangle|^2\left(b^0\right)^2(\Delta E\xi)^2\sinh(\pi\Delta E\xi)\,e^{-2\pi\Delta E\xi}}{2\pi^3\left(1-\exp(-2\pi\Delta E\xi)\right)}\int\frac{d^3k}{\omega}\left[K_{i\Delta E\xi}(Q\xi)\right]^2$$

Where we have used $K_{-n}(x)=K_n(x)$. Simplifying and using $Q=\left(k_1^2+k_2^2\right)^{1/2}=k\sin\theta$, where $k=|\underset{\sim}{k}|$, $\omega=k$ since the field is massless. Writing the integral in spherical coordinates,

$$P_{\gamma,00}^3=\frac{c^2|\langle M\rangle|^2\left(b^0\right)^2(\Delta E\xi)^2\,e^{-2\pi\Delta E\xi}}{4\pi^3}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\int_0^{\infty}dk\,k\sin\theta\left[K_{i\Delta E\xi}(k\xi\sin\theta)\right]^2$$

The φ – integral is straight forward and the k – integral is evaluated in the Appendix. The result is

$$P_{\gamma,00}^3=\frac{c^2|\langle M\rangle|^2\left(b^0\right)^2(\Delta E\xi)^2\,e^{-2\pi\Delta E\xi}}{4\pi^3}\Gamma(1+i\Delta E\xi)\Gamma(1-i\Delta E\xi)\int_0^{\pi}\frac{d\theta}{\sin\theta}$$

Manipulating the Gamma function as with the linear detector calculation gives

$$P_{\gamma,00}^3=\frac{c^2|\langle M\rangle|^2\left(b^0\right)^2(\Delta E)^3}{2\pi\left(e^{2\pi\Delta E\xi}-1\right)}\int_0^{\pi}\frac{d\theta\xi}{\sin\theta}$$

The logarithmic divergence is identical to that for the linear detector as calculated in the Appendix, therefore the transition rate per unit detector time is

$$P^3_{\gamma,00} = \frac{c^2 \left|\langle M\rangle\right|^2 \left(b^0\right)^2 \left(\Delta E\right)^3}{2\pi\left(e^{\Delta E/kT}-1\right)} \tag{6.18}$$

With $T = 1/2\pi\xi$.

The same procedure is followed with the x - and y – derivative responses and using the symmetry of the situation, these will be identical. From (6.15)

$$A^3_{\gamma,1} = -c\langle M\rangle b^1 \xi\left(\sinh \pi\Delta E\xi\right)^{1/2}\int dk_1 dk_2 e^{i(k_1 x+k_2 y)} K_{i\Delta E\xi}\left(Q\xi\right) k_1 \langle \Psi\left|a_R\left(\tilde{\omega},k_1,k_2\right)\right|0_M\rangle$$

Using the Bogolubov transformation

$$\begin{aligned} A^3_{\gamma,1} &= -c\langle M\rangle b^1 \xi\left(\sinh \pi\Delta E\xi\right)^{1/2} e^{-\pi\Delta E\xi}\left[2\pi\left(1-e^{-2\pi\Delta E\xi}\right)\right]^{1/2}\times \\ &\times\int \frac{dk'}{\omega'^{1/2}} k_1'\left[\frac{\omega'+k_3'}{Q'}\right]^{i\Delta E\xi} e^{i(k_1' x+k_2' y)} K_{i\Delta E\xi}\left(Q'\xi\right) k_1 \langle \Psi\left|1_{\underline{k}'}\right.\rangle \end{aligned}$$

Giving

$$P^3_{\gamma,11} = \frac{c^2\left|\langle M\rangle\right|^2\left(b^1\right)^2 \xi^2 e^{-\pi\Delta E\xi}}{4\pi^3}\int_0^{2\pi} d\phi \cos^2\phi \int_0^{\pi} d\theta \int_0^{\infty} dk k^3 \sin^3\theta\left[K_{i\Delta E\xi}\left(k\xi\sin\theta\right)\right]^2$$

Calculating the ϕ – and k – integrals

$$P^3_{\gamma,11} = \frac{c^2\left|\langle M\rangle\right|^2\left(b^1\right)^2 e^{-\pi\Delta E\xi}\Gamma\left(2+i\Delta E\xi\right)\Gamma\left(2-i\Delta E\xi\right)}{12\pi^2\xi^2}\int_0^{\pi}\frac{d\theta}{\sin\theta}\xi$$

Using $\Gamma(1+x)=x\Gamma(x)$, the Gamma function terms can be evaluated as before

$$P^3_{\gamma,11} = \frac{c^2\left|\langle M\rangle\right|^2\left(b^1\right)^2 \Delta E\left(1+\left(\Delta E\xi\right)^2\right)}{6\pi\xi^2\left(e^{2\pi\Delta E\xi}-1\right)}\int_0^{\pi}\frac{d\theta}{\sin\theta}\xi$$

Therefore

$$\begin{aligned} R^3_{\gamma,11} &= \frac{c^2\left|\langle M\rangle\right|^2\left(b^1\right)^2 \Delta E\left(1+\left(\Delta E\xi\right)^2\right)}{6\pi\xi^2\left(e^{2\pi\Delta E\xi}-1\right)} \\ &= \frac{2\pi}{3}\frac{c^2\left(b^1\right)^2\left|\langle M\rangle\right|^2 \Delta E\left(1+\left(\Delta E/2\pi kT\right)^2\right)\left(kT\right)^2}{\left(e^{\Delta E/kT}-1\right)} \end{aligned} \tag{6.19}$$

$$R^3_{\gamma,22} = \left(b^2\right)^2 R^3_{\gamma,11}\Big/\left(b^1\right)^2 \tag{6.20}$$

The cross term $P^3_{\gamma,01}$ is given by

$$P^3_{\gamma,10} = \Re e\left\{\sum_{|\Psi\rangle} A^3_{\gamma,0}\left[A^3_{\gamma,1}\right]^*\right\}$$

$$= -\xi^2 c^2 \left|\langle M\rangle\right|^2 b^0 b^1 e^{-\pi\Delta E\xi}\int_0^{2\pi} d\phi \cos\phi \int_0^{\pi} d\theta \int_0^{\infty} dk k^2 \sin^2\theta \left[K_{i\Delta E\xi}\left(Q\xi\right)\right]^2 = 0$$

Where $\Re e\{.\}$ is the real part of the argument. Thus there are no 0,1 – or 0,2 – cross terms in the detector's response. Similarly there are no 1,2 – cross terms, as can be seen from the form of $A^3_{\gamma,i}$ for *i* = 1,2.

To evaluate the response in the direction of motion, the z-direction in the commoving frame, the vierbein approach discussed above is used to deduce which derivatives must be calculated in the Rindler coordinate system to correspond to the z-direction in an instantaneous commoving frame. For the Rindler metric

$$ds = \xi^2 \left(d\tau\right)^2 - \left(dx\right)^2 - \left(dy\right)^2 - \left(d\xi\right)^2$$

The vierbein is $V^\alpha_\mu = diag\left(\xi,1,1,1\right)$ giving

$$\partial_z \to \partial_\xi$$

Therefore, from (6.15), the only contributing term is

$$A^3_{\gamma,3} = \frac{ic\langle M\rangle}{2\pi^2} b^3 \int_{-\infty}^{\infty} \xi d\tilde{\tau} e^{i\Delta E\xi\tilde{\tau}} \int d\tilde{\omega} dk_1 dk_2 \left(\sinh\pi\tilde{\omega}\right)^{1/2} e^{-i\tilde{\omega}\tau} e^{i\left(k_1 x + k_2 y\right)} \langle\Psi\left|a_R\left(\tilde{\omega},k_1,k_2\right)\right|0_M\rangle \partial_\xi K_{i\omega}\left(Q\xi\right)$$

Where $\tilde{\tau}$ has not been differentiated with respect to ξ in line with the prescription above. From Gradshteyn and Ryhzik (1980)

$$\left(\frac{\partial}{\partial x}\right) K_\nu\left(x\right) = \frac{-\left\{K_{\nu-1}\left(x\right) + K_{\nu+1}\left(x\right)\right\}}{2}$$

Using this, performing the $\tilde{\tau}$ - and $\tilde{\omega}$ - integrals and using the Bogolubov transformations (6.17) gives

$$A^3_{\gamma,3} = \frac{-c\langle M\rangle b^3\xi}{2\pi}\left(\sinh\pi\Delta E\xi\right)^{1/2} e^{-\pi\Delta E\xi}\left[2\pi\left(1-e^{-2\pi\Delta E\xi}\right)\right]^{-1/2} \times$$

$$\times \int \frac{d^3k'}{\omega'^{1/2}} Q'\left[\frac{\omega' + k_3'}{Q'}\right]^{i\Delta E\xi} e^{i\left(k_1 x + k_2 y\right)}\left\{K_{i\Delta E\xi+1}\left(Q'\xi\right) + K_{i\Delta E\xi-1}\left(Q'\xi\right)\right\}$$

For x real, $K_{\nu^*}\left(x\right) = \left(K_\nu\left(x\right)\right)^*$ and $K_\nu\left(x\right) = K_{-\nu}\left(x\right)$, therefore

$$P^3_{\gamma,33} = c^2\left|\langle M\rangle\right|^2\left(b^3\right)^2 \xi^2 e^{-\pi\Delta E\xi}\int_0^{\pi} d\theta\int_0^{\infty} dk k^3 \sin^3\theta\left\{\left[K_{i\Delta E\xi+1}\left(\xi k\sin\theta\right)\right]^2\right.$$

$$\left. + 2K_{i\Delta E\xi+1}\left(\xi k\sin\theta\right)K_{i\Delta E\xi-1}\left(\xi k\sin\theta\right) + \left[K_{i\Delta E\xi-1}\left(\xi k\sin\theta\right)\right]^2\right\}$$

Again using Gradshteyn and Ryhzik as above

$$P^3_{\gamma,33} = \frac{c^2\left|\langle M\rangle\right|^2}{24\pi^2}\left(b^3\right)^2 e^{-\pi\Delta E\xi}\left\{4\Gamma(2-i\Delta E\xi)\Gamma(2+i\Delta E\xi)+\Gamma(3+i\Delta E\xi)\Gamma(1-i\Delta E\xi)+\right.$$
$$\left.+\Gamma(1+i\Delta E\xi)\Gamma(3-i\Delta E\xi)\right\}\int_0^\pi \frac{d\theta}{\sin\theta}\xi$$

Evaluating the Gamma function terms and removing the logarithmic divergence

$$\begin{aligned} R^3_{\gamma,33} &= c^2\left|\langle M\rangle\right|^2\left(b^3\right)^2 \frac{\Delta E\left(4+(\Delta E\xi)^2\right)}{12\pi\xi^2\left(e^{2\pi\Delta E\xi}-1\right)} \\ &= \frac{\pi}{3}c^2\left|\langle M\rangle\right|^2\left(b^3\right)^2\frac{\Delta E\left(4+(\Delta E/kT)^2\right)(kT)^2}{\left(e^{\Delta E/kT}-1\right)} \end{aligned} \tag{6.21}$$

The 1,3 – and 2,3 – cross terms are zero due to the angular integrals. However the 0,3 – cross term is nontrivial;

$$\begin{aligned} P^3_{\gamma,03} &= \sum_{|\Psi\rangle} A^3_{\gamma,3}\left[A^3_{\gamma,0}\right]^* \\ &= \frac{-ic^2\left|\langle M\rangle\right|^2}{4\pi^2} b^0 b^3\xi^2\Delta E e^{-\pi\Delta E\xi}\int_0^\pi d\theta\int_0^\infty dk k^2\sin^2\theta K_{i\Delta E\xi}(k\xi\sin\theta)\times \\ &\times\left\{K_{i\Delta E\xi+1}(k\xi\sin\theta)+K_{i\Delta E\xi-1}(k\xi\sin\theta)\right\} \\ &= -ic^2\left|\langle M\rangle\right|^2 b^0 b^3\xi^2\Delta E e^{-\pi\Delta E\xi}\left\{\Gamma(2-i\Delta E\xi)\Gamma(1+i\Delta E\xi)+\Gamma(2+i\Delta E\xi)\Gamma(1-i\Delta E\xi)\right\}\times \\ &\times\int_0^\pi\frac{d\theta}{\sin\theta}\xi \\ &= \frac{-ic^2\left|\langle M\rangle\right|^2 b^0 b^3(\Delta E)^2}{2\pi\xi\left(e^{-2\pi\Delta E\xi}-1\right)}\int_0^\pi\frac{d\theta}{\sin\theta}\xi \end{aligned}$$

Thus the total cross term contribution to the response is

$$R^3_{\gamma,03}+R^3_{\gamma,30} = R^3_{\gamma,03}+\left[R^3_{\gamma,03}\right]^* = 0$$

Comparing the various responses with their corresponding components of a derivative detector's response in an isotropic Planck bath in Minkowski space (see (3.19), (3.20) and (3.21)), we see that although the 0,0 – component and the total cross terms of the response are identical, the others are not.

Therefore, it seems that immersion in an isotropic Planck radiation bath and uniform acceleration through the Minkowski vacuum give rise to the same response in this detector only if it is coupled solely to the time derivative of the field.

6.3 Response in Two and Four Dimensional Schwarzschild Space

The full Hartle-Hawking vacuum Wightman function in two-dimensions is

$$G_H^+(x,x') = \frac{-1}{4\pi}\ln\left\{\kappa^{-2}e^{(r^*+r^{*\prime})\kappa}\left[2\cosh\Delta\tilde{\tau}\kappa - \cosh(r^*-r^{*\prime})\kappa - 2i\varepsilon\left(e^{-r^*\kappa}\sinh\tilde{\tau}\kappa - e^{-r^*\kappa}\sinh\tilde{\tau}'\kappa\right) - \varepsilon^2\right]\right\} \tag{6.22}$$

Where κ=1/4M_S and $\tilde{\tau} = \tau\left(1-2M_S/r\right)^{1/2}$ is the time coordinate with the Tolman factor included. To evaluate the 0,0-component response, for a detector stationary outside the black hole, the logarithmic function is used to split away and discard any $\tilde{\tau}$-independent terms, since they do not contribute, leaving

$$\tilde{G}_H^+(x,x') = \frac{-1}{4\pi}\ln\left\{\cosh\Delta\tilde{\tau}\kappa - \cosh(r^*-r^{*\prime})\kappa - i\varepsilon\left(e^{-r^*\kappa}\sinh\tilde{\tau}\kappa - e^{-r^*\kappa}\sinh\tilde{\tau}'\kappa\right) - \varepsilon^2\right\}$$

Since the detector is at a fixed distance from the black hole,

$$\partial_\tau = \left(1-\frac{2M}{r}\right)^{-1/2}\partial_{\tilde{\tau}} = (\text{constant})\times\partial_{\tilde{\tau}} \tag{6.23}$$

Proceeding as before

$$\partial_{\tilde{\tau}}\tilde{G}_H^+(x,x') = \left(\frac{-1}{4\kappa}\right)\frac{\kappa\sinh\Delta\tilde{\tau}\kappa - i\varepsilon\kappa e^{-r^{*\prime}\kappa}\cosh\tilde{\tau}\kappa}{\cosh\Delta\tilde{\tau}\kappa - \cosh(r^*-r^{*\prime})\kappa - i\varepsilon\left(e^{-r^{*\prime}\kappa}\sinh\tilde{\tau}\kappa - e^{-r^*\kappa}\sinh\tilde{\tau}'\kappa\right) - \varepsilon^2}$$

$$\partial_{\tilde{\tau}'}\partial_{\tilde{\tau}}\tilde{G}_H^+(x,x') = \frac{-1}{4\pi}\left\{\begin{array}{c}\dfrac{\left(-\kappa^2\cosh\Delta\tilde{\tau}\kappa\right)}{\cosh\Delta\tilde{\tau}\kappa - \cosh(r^*-r^{*\prime})\kappa - i\varepsilon\left(e^{-r^{*\prime}\kappa}\sinh\tilde{\tau}\kappa - e^{-r^*\kappa}\sinh\tilde{\tau}'\kappa\right) - \varepsilon^2} \\ -\dfrac{\left(\kappa\sinh\Delta\tilde{\tau}\kappa - i\varepsilon\kappa e^{-r^{*\prime}\kappa}\cosh\tilde{\tau}\kappa\right)\left(-\kappa\sinh\Delta\tilde{\tau}\kappa + i\varepsilon\kappa e^{-r^*\kappa}\cosh\tilde{\tau}'\kappa\right)}{\left(\cosh\Delta\tilde{\tau}\kappa - \cosh(r^*-r^{*\prime})\kappa - i\varepsilon\left(e^{-r^{*\prime}\kappa}\sinh\hat{\tau}\kappa - e^{-r^*\kappa}\sinh\tilde{\tau}\kappa\right) - \varepsilon^2\right)^2}\end{array}\right\}$$

Setting $r^* = r^{*\prime}$, absorbing some positive functions into the ε term and using some hyperbolic identities results in

$$\partial_\tau\partial_{\tau'}\tilde{G}_H^+(x,x')\Big| = \frac{-\kappa^2}{8\pi\sinh^2\left(\dfrac{\Delta\tilde{\tau}\kappa}{2} - i\varepsilon\right)} \tag{6.24}$$

Using (6.23) and (6.24) the 0,0-component of the response is

$$R_{H,00}^3 = \frac{c^2\left|\langle M\rangle\right|^2\left(b^0\right)^2\Delta E}{e^{\Delta E/kT}-1} \tag{6.25}$$

Where $kT = [64\pi^2M^2(1\text{-}2M/r)]^{-1/2}$. Looking back at (3.19), this component of the response is identical to that due to immersion in a Planck bath of radiation in two-dimensional Minkowski space.

For the spatial component of the response, the r-derivative is used. We have

$$\partial_r = \left(\frac{dr^*}{dr}\right)\partial_{r^*} = \left(1-\frac{2M_S}{r}\right)^{-1}\partial_{r^*}$$

Using

$$\partial_{r*}\tilde{G}_H^+(x,x') = \left(\frac{-1}{4\pi}\right)\frac{\left(-\kappa\sinh(r*-r*')\kappa - i\varepsilon e^{-r*'\kappa}\sinh\tilde{\tau}\kappa\right)}{\cosh\Delta\tilde{\tau}\kappa - \cosh(r*-r*')\kappa - i\varepsilon\left(e^{-r*'\kappa}\sinh\tilde{\tau}\kappa - e^{-r*\kappa}\sinh\tilde{\tau}'\kappa\right) - \varepsilon^2} \quad (6.26)$$

$$\partial_{r*'}\partial_{r*}\tilde{G}_H^+(x,x') = \left(\frac{-\kappa^2}{4\pi}\right)\left\{\begin{array}{c}\dfrac{\cosh(r*-r*')\kappa}{\cosh\Delta\tilde{\tau}\kappa - \cosh(r*-r*')\kappa - i\varepsilon\left(e^{-r*'\kappa}\sinh\tilde{\tau}\kappa - e^{-r*\kappa}\sinh\tilde{\tau}'\kappa\right)} \\ -\dfrac{\left(-\sinh(r*-r*')\kappa - i\varepsilon e^{-r*\kappa}\sinh\tilde{\tau}'\kappa\right)\left(\sinh(r*-r*')\kappa + i\varepsilon e^{-r*'\kappa}\sinh\tilde{\tau}\kappa\right)}{\left(\cosh\Delta\tilde{\tau}\kappa - \cosh(r*-r*')\kappa - i\varepsilon\left(e^{-r*'\kappa}\sinh\tilde{\tau}\kappa - e^{-r*\kappa}\sinh\tilde{\tau}'\kappa\right) - \varepsilon^2\right)}\end{array}\right\}$$

setting $r^* = r^{*\prime}$ and discarding the ε and ε^2 terms in the numerator, since they make no contribution once ε is set to zero, gives

$$\partial_{r'}\partial_r\tilde{G}_H^+(x,x')\Big| = \frac{-\kappa^2}{8\pi^2\left(1-\dfrac{2M}{r}\right)^2\sinh\left(\dfrac{\Delta\tilde{\tau}\kappa}{2} - i\varepsilon\right)}$$

which in turn gives

$$R^3_{H,11} = \frac{c^2\left|\langle M\rangle\right|^2\left(b^1\right)^2\Delta E}{\left(1-\dfrac{2M}{r}\right)\left(e^{\Delta E/kT}-1\right)} \quad (6.27)$$

This is not identical to the 1,1-component of this detector's response in a Planck bath of radiation in two dimensions. Note that agreement does occur in the region $r \to \infty$. Following an identical procedure, we see directly from (6.26) that

$$R^3_{H,10} + R^3_{H,01} = 0$$

This is also identical to the corresponding responses to a Planck bath of radiation in two dimensions.

For the four-dimensional case, the Wightman function is given in (4.23). The evaluation of $R^3_{H,00}$ is straight forward

$$\partial_{\tilde{\tau}}\partial_{\tilde{\tau}'}G_H^+(x,x')\Big| = \int_{-\infty}^{\infty}\frac{d\omega}{16\pi^2}\omega\left\{\mathrm{e}^{-i\omega\Delta\tilde{\tau}}\sum_{l=0}^{\infty}\frac{(2l-1)\left|\vec{R}_l(\omega\,|\,r)\right|^2}{1-e^{-2\pi\omega/\kappa}} + \mathrm{e}^{i\omega\Delta\tilde{\tau}}\sum_{l=0}^{\infty}\frac{(2l-1)\left|\bar{R}_l(\omega\,|\,r)\right|^2}{e^{2\pi\omega/\kappa}-1}\right\}$$

Using the asymptotic values of the summation yields

$$\begin{aligned}R^3_{H,00} &\sim c^2\left|\langle M\rangle\right|^2\left(b^0\right)^2\left(1-\frac{2M}{r}\right)^{-3/2}\int_{-\infty}^{\infty}d\omega\frac{\omega^3\delta\left(\omega+\left(1-\dfrac{2M_S}{r}\right)^{1/2}\Delta E\right)}{2\pi\left(1-e^{-2\pi\omega/\kappa}\right)} && r \to 2M_S \\ &\sim c^2\left|\langle M\rangle\right|^2\left(b^0\right)^2\int_{-\infty}^{\infty}d\omega\frac{\omega^3\delta(\omega-\Delta E)}{2\pi\left(e^{2\pi\omega/\kappa}-1\right)} && r \to \infty\end{aligned}$$

giving

$$R^3_{H,00} \sim \frac{c^2 \left|\langle M \rangle\right|^2 \left(b^0\right)^2 \left(\Delta E\right)^3}{2\pi\left(e^{\Delta E/kT} - 1\right)} \tag{6.28}$$

Where $kT=[64\pi^2 M_S^2(1\text{-}2M_S/r)]^{-1/2}$. Comparing this with (3.19) we see that the Hartle-Hawking vacuum and isotropic Planck radiation, in Minkowski space, produce the same response for the time-derivative detector. For $R^3_{H,11}$ we require $\partial_r \partial_{r'} G_H^+\left(x,x'\right)\Big|$

To evaluate this quantity in the asymptotic regions, we follow the method used by Candelas (1980). The quantities of interest are

$$\frac{\partial^2}{\partial r \partial r'} \sum_{l=0}^{\infty} \left(2l+1\right) \overleftarrow{R}_l^*\left(\omega \,|\, r\right) \overleftarrow{R}_l\left(\omega \,|\, r'\right)\Bigg|_{r=r'}$$

$$\frac{\partial^2}{\partial r \partial r'} \sum_{l=0}^{\infty} \left(2l+1\right) \overrightarrow{R}_l\left(\omega \,|\, r\right) \overrightarrow{R}_l^*\left(\omega \,|\, r'\right)\Bigg|_{r=r'}$$

From (4.24) for $r \to 2M_S$

$$\frac{\partial}{\partial r} \overleftarrow{R}_l\left(\omega \,|\, r\right) \sim \frac{-B_l\left(\omega\right)}{r} e^{-i\omega r^*} \left\{ \frac{1}{r} - \frac{i\omega}{\left(1 - \frac{2M_S}{r}\right)} \right\}$$

$$\sim \frac{-iB_l\left(\omega\right) e^{-i\omega r^*} \omega}{2M_S\left(1 - \frac{2M_S}{r}\right)} \qquad r \to 2M_S$$

where

$$\frac{dr^*}{dr} = \left(1 - \frac{2M_S}{r}\right)^{-1}$$

Therefore

$$\frac{\partial^2}{\partial r \partial r'} \overleftarrow{R}_l^*\left(\omega \,|\, r\right) \overleftarrow{R}_l\left(\omega \,|\, r'\right)\Bigg|_{r=r'} \sim \frac{\left|B_l\left(\omega\right)\right|^2 \omega^2}{4M^2\left(1 - \frac{2M_S}{r}\right)^2} \qquad r \to \infty \tag{6.29}$$

The asymptotic form for this quantity for r $\to \infty$ requires some work to calculate, so the details are presented in the Appendix. The result is

$$\frac{\partial^2}{\partial r \partial r'} \sum_{l=0}^{\infty} \left(2l+1\right) \overleftarrow{R}_l^*\left(\omega \,|\, r\right) \overleftarrow{R}_l\left(\omega \,|\, r'\right)\Bigg|_{r=r'} \sim \frac{4}{3}\omega^4 \qquad r \to \infty \tag{6.30}$$

Using (4.24)

$$\frac{\partial}{\partial r}\vec{R}_l(\omega|r) \sim \frac{B_l(\omega)e^{i\omega r^*}}{r}\left\{\frac{1}{r}+\frac{i\omega}{\left(1-\frac{2M_S}{r}\right)}\right\}$$

$$\sim \frac{iB_l(\omega)\omega e^{i\omega r^*}}{r} \qquad\qquad r\to\infty$$

Giving

$$\left.\frac{\partial^2}{\partial r\partial r'}\vec{R}_l^*(\omega|r)\vec{R}_l(\omega|r')\right|_{r=r'} \sim \frac{\left|B_l(\omega)\right|^2\omega^2}{r^2} \qquad r\to\infty \tag{6.31}$$

From the Appendix the other asymptotic form is

$$\left.\frac{\partial^2}{\partial r\partial r'}\sum_{l=0}^{\infty}(2l+1)\vec{R}_l^*(\omega|r)\vec{R}_l(\omega|r')\right|_{r=r'} \sim \frac{\left(1-4M_S{}^2\omega^2\right)\omega^2}{3M^2\left(\frac{r}{2M_S}-1\right)^3} \qquad r\to 2M_S \tag{6.32}$$

Using these forms in (6.6) yields

$$\begin{aligned}
R_{H,11}^3 &\sim c^2\left|\langle M\rangle\right|^2\left(b^1\right)^2\int_{-\infty}^{\infty}d\omega\frac{\omega\delta\left(\omega+\left(1-\frac{2M_S}{r}\right)^{1/2}\Delta E\right)\left(1+4M_S{}^2\omega^2\right)}{24M_S{}^2\left(1-e^{-2\pi\omega/\kappa}\right)\left(\frac{r}{2M_S}-1\right)^{5/2}}\\
&\sim c^2\left|\langle M\rangle\right|^2\left(b^1\right)^2\frac{\Delta E\left(1+4M_S{}^2\left(1-\frac{2M_S}{r}\right)(\Delta E)^2\right)}{24\pi M_S{}^2\left(e^{\Delta E/kT}-1\right)}\\
&\sim c^2\left|\langle M\rangle\right|^2\left(b^1\right)^2\frac{\Delta E}{24\pi M_S{}^2\left(\frac{r}{2M_S}-1\right)^2\left(e^{\Delta E/kT}-1\right)}
\end{aligned} \tag{6.33}$$

Where $kT=[64\pi^2M_S^2(1\text{-}2M_S/r)]^{-1/2}$. For the region $r\to\infty$

$$\begin{aligned}
R_{H,11}^3 &\sim c^2\left|\langle M\rangle\right|^2\left(b^1\right)^2\int_{-\infty}^{\infty}d\omega\frac{\omega^3\delta(\Delta E-\omega)}{6\pi\left(e^{2\pi\omega/\kappa}-1\right)}\\
&\sim c^2\left|\langle M\rangle\right|^2\left(b^1\right)^2\frac{(\Delta E)^3}{6\pi\left(e^{\Delta E/kT}-1\right)}
\end{aligned} \tag{6.34}$$

Next we consider the angular derivatives. The quantities of interest are

$$\left.\frac{\partial^2}{\partial\theta\partial\theta'}\sum_{m=-l}^{l}Y_{lm}^*(\theta,\varphi)Y_{lm}(\theta',\varphi)\right|_{\theta=\theta'} \tag{6.35}$$

$$\frac{\partial^2}{\partial\varphi\partial\varphi'}\sum_{m=-l}^{l}Y_{lm}^{*}(\theta,\varphi)Y_{lm}(\theta,\varphi')\Big|_{\varphi=\varphi'} \tag{6.36}$$

(This can be confirmed using the appropriate vierbein.) Using the identity

$$\sum_{m=-l}^{l}Y_{ml}^{*}(\theta,\varphi)Y_{ml}(\theta',\varphi')=\frac{(2l+1)}{4\pi}P_l(\cos\gamma)$$

Where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$ and $P_l(\cos\gamma)$ is a Legendre Function (Gradshteyn & Rhyzik, 1980, No.8.820). (6.35) can be written in the form

$$\frac{\partial^2}{\partial\theta\partial\theta'}\frac{(2l+1)}{4\pi}P_l(\cos\theta\cos\theta'+\sin\theta\sin\theta')\Big|_{\theta=\theta'}=\frac{(2l+1)}{4\pi}\frac{\partial}{\partial x}P_l(x)\Big|_{x=1}$$

To evaluate this we note that

$$P_l(x)=F\left(-l,l+1;1;\frac{1-x}{2}\right)$$

Where $F(.)$ is the hyper-geometric function (Gradshteyn & Rhizik,1980, No.9.100)

$$\frac{\partial}{\partial x}P_l(x)\Big|_{x=1}=\frac{-1}{2}\frac{\partial}{\partial z}F(-l,l+1;1;z)\Big|_{z=0}=\frac{l(l+1)}{2}$$

Hence

$$\frac{\partial^2}{\partial\theta\partial\theta'}\sum_{m=-l}^{l}Y_{ml}^{*}(\theta,\varphi)Y_{ml}(\theta',\varphi)\Big|_{\theta=\theta'}=\frac{l(l+1)(2l+1)}{8\pi} \tag{6.37}$$

For (6.36)

$$\frac{\partial^2}{\partial\varphi\partial\varphi'}\frac{(2l+1)}{4\pi}P_l\left(\cos^2\theta+\sin^2\theta\cos(\varphi-\varphi')\right)\Big|_{\varphi=\varphi'}=\frac{(2l+1)}{2\pi}\sin^2\theta\frac{\partial P_l(x)}{\partial x}\Big|_{x=1}$$
$$=\frac{l(l+1)(2l+1)\sin^2\theta}{8\pi}$$

Therefore

$$\frac{\partial^2}{\partial\varphi\partial\varphi'}\sum_{m=-l}^{l}Y_{ml}^{*}(\theta,\varphi)Y_{ml}(\theta,\varphi')\Big|_{\varphi=\varphi'}=\frac{l(l+1)(2l+1)\sin^2\theta}{8\pi} \tag{6.38}$$

The equations (6.37) and (6.38) are actually equivalent since the $\sin\theta$ term represents the fact that coordinates of the two-sphere has a coordinate singularity at $\theta = 0$, π. These two expressions are related in a manner we would expect due to the spherical symmetry of the situation. Only the θ-derivative terms will be evaluated, those for the φ-derivative follow automatically. These components of the detector's response are deduced using

$$\frac{\partial^2}{\partial\theta\partial\theta'}G_H^+\left(x,x'\right)\bigg| = \frac{1}{r^2}\int_{-\infty}^{\infty}\frac{d\omega}{32\pi^2\omega}\left\{\begin{array}{l} e^{-i\omega\Delta\tilde{\tau}}\sum_{l=0}^{\infty}\frac{l\left(l+1\right)\left(2l+1\right)\left|\vec{R}_l\left(\omega\,|\,r\right)\right|^2}{1-e^{-2\pi\omega/\kappa}} \\ +e^{i\omega\Delta\tilde{\tau}}\sum_{l=0}^{\infty}\frac{l\left(l+1\right)\left(2l+1\right)\left|\vec{R}_l\left(\omega\,|\,r\right)\right|^2}{e^{2\pi\omega/\kappa}-1} \end{array}\right\}$$

The $1/r^2$ term has been introduced into the expression because (from the vierbein) it is to this that the detector is coupled. Just as with the above asymptotic forms

$$\frac{1}{r^2}\sum_{l=0}^{\infty}l\left(l+1\right)\left(2l+1\right)\left|\vec{R}_l\left(\omega\,|\,r\right)\right|^2 \sim \frac{1}{r^4}\sum_{l=0}^{\infty}l\left(l+1\right)\left(2l+1\right)\left|B_l\left(\omega\right)\right|^2 \qquad r\to\infty$$

$$\frac{1}{r^2}\sum_{l=0}^{\infty}l\left(l+1\right)\left(2l+1\right)\left|\vec{R}_l\left(\omega\,|\,r\right)\right|^2 \sim \frac{1}{16M^4}\sum_{l=0}^{\infty}l\left(l+1\right)\left(2l+1\right)\left|B_l\left(\omega\right)\right|^2 \qquad r\to 2M_S$$

The other two asymptotic forms are more difficult to evaluate and the calculation is done in the Appendix. The result is

$$\frac{1}{r^2}\sum_{l=0}^{\infty}l\left(l+1\right)\left(2l+1\right)\left|\vec{R}_l\left(\omega\,|\,r\right)\right|^2 \sim \frac{8\omega^4}{3} \qquad r\to\infty$$

$$\frac{1}{r^2}\sum_{l=0}^{\infty}l\left(l+1\right)\left(2l+1\right)\left|\vec{R}_l\left(\omega\,|\,r\right)\right|^2 \sim \frac{\left(1+16M^2\omega^2\right)\omega^2}{6M_S^{\;2}\left(\dfrac{r}{2M_S}-1\right)^2} \qquad r\to 2M_S$$

Using these quantities in (6.6) yields for the response in the asymptotic regions

$$R_{H,22}^3 \sim \frac{c^2\left|\langle M\rangle\right|^2\left(b^2\right)^2\Delta E}{96M_S^{\;2}\pi\left(1-\dfrac{2M_S}{r}\right)\left(e^{\Delta E/kT}-1\right)} \qquad r\to 2M_S \tag{6.39}$$

$$\sim \frac{c^2\left|\langle M\rangle\right|^2\left(b^2\right)^2\left(\Delta E\right)^3}{6\pi\left(e^{\Delta E/kT}-1\right)} \qquad r\to\infty \tag{6.40}$$

with kT as before with the Hartle-Hawking vacuum. For the φ-derivative terms the $\sin^2\theta$ dependence is only due to the coordinate system adopted. Thus

$$R_{H,22}^3 = R_{H,33}^3 \tag{6.41}$$

Finally, for the cross terms, since

$$\frac{\partial}{\partial\theta}\sum_{m=-l}^{l}Y_{ml}^*\left(\theta,\varphi\right)Y_{ml}\left(\theta',\varphi\right)\bigg|_{\theta=\theta'} = 0$$

$$\frac{\partial}{\partial\varphi}\sum_{m=-l}^{l}Y_{ml}^*\left(\theta,\varphi\right)Y_{ml}\left(\theta,\varphi'\right)\bigg|_{\varphi=\varphi'} = 0$$

There will be no such terms involving the angular derivatives. This leaves the 0,1-cross term. For the limit $r\to\infty$, from the Appendix

$$\frac{\partial}{\partial r} G_B^+(x,x') \sim \int_0^\infty \frac{d\omega}{16\pi^2\omega} e^{-i\omega\Delta\tilde{\tau}} \left\{ \begin{array}{l} \left.\sum_{l=0}^{\infty}(2l+1)\vec{R}_l(\omega|r)\vec{R}_l^*(\omega|r')\right|_{r=r'} \\ \left.+\sum_{l=0}^{\infty}(2l+1)\vec{R}_l^*(\omega|r)\vec{R}_l(\omega|r')\right|_{r=r'} \end{array} \right\}$$

$$\sim \left.\frac{(r-r')}{2\pi\left[(\Delta\tilde{\tau}-i\varepsilon)^2-(r-r')^2\right]^2}\right|_{r=r'} = 0 \qquad r \to \infty$$

Therefore

$$\frac{\partial}{\partial r}\left.\sum_{l=0}^{\infty}(2l+1)\vec{R}_l^*(\omega|r)\vec{R}_l(\omega|r')\right|_{r=r'} = 0 \qquad r \to \infty$$

Also, from the Appendix, for $r \to 2M_S$,

$$\frac{\partial}{\partial r}\left.\sum_{l=0}^{\infty}(2l+1)\vec{R}_l^*(\omega|r)\vec{R}_l(\omega|r')\right|_{r=r'} \sim \sum_{l=0}^{\infty}\frac{i(2l+1)|B_l(\omega)|^2\omega}{4M_S{}^2\left(1-\frac{2M^S}{r}\right)} \qquad r \to 2M_S$$

$$\frac{\partial}{\partial r}\left.\sum_{l=0}^{\infty}(2l+1)\vec{R}_l(\omega|r)\vec{R}_l^*(\omega|r')\right|_{r=r'} \sim \frac{\omega^2}{M_S\left(\frac{r}{2M_S}-1\right)} \qquad r \to 2M_S$$

Giving

$$R_{H,10}^3 + R_{H,01}^3 \approx c^2|\langle M\rangle|^2 b^0 b^1 \sum_{l=0}^{\infty}\frac{(2l+1)\left|B_l\left(\Delta E\left(1-\frac{2M_S}{r}\right)^{1/2}\right)\right|^2\Delta E}{16\pi M_S{}^2\left(e^{\Delta E/kT}-1\right)\left(1-\frac{2M_S}{r}\right)^{1/2}} \qquad r \to 2M_S \qquad (6.42)$$

Comparing the respective responses with those in chapter three, we see that in the limit $r \to \infty$ this detector's response to the Hartle-Hawking vacuum is identical to that resulting from immersion in an isotropic bath of Planck radiation. However, as $r \to 2M_S$ the spatial components become non-Planckian, as shown by (6.33) and (6.39), also the cross term (6.42) appears. So, although for the derivative detector in the four-dimensional Hartle-Hawking vacuum and isotropic Planck radiation appear identical in the region $r \to \infty$, only for the 0,0-component does this identity also hold for $r \to 2M_S$.

7 The Cone and Spike Detectors

7.1 General Remarks

Due to the close similarity of these two detectors, they shall be discussed together. The key difference between the cone and spike detectors and the other three detectors introduced above is the restricted access the former have to the $(n\text{-}2)$-sphere of momentum mode directions. The introduction of this "screening" gives rise to directionality in these two detectors and (as with the derivative detector) the resulting problems of "orientation" of the detector. Also, with the screening and directionality, the question of their time variance in a general reference frame must be addressed. Unfortunately, the answer to this question is far from straight forward and (to the author's knowledge) no complete resolution has been published.

The transition amplitude, to first order, for the cone detector moving along trajectory $x^{\mu}(\tau)$, with the field in vacuum state $|0\rangle$ is (formally)

$$A^5 = ic\langle M\rangle \int_{-\infty}^{\infty} d\tau e^{i\Delta E\tau} \langle\Psi|\phi\left[x(\tau)\right]\Big|_{S_{\Omega(\tau)}}|0\rangle \tag{7.1}$$

where $\Big|_{S_{\Omega(\tau)}}$ represents the restriction on the modes that can interact with the detector. For the spike detector the set of directions $S_{\Omega(\tau)}$ in the (n-2)-sphere of momentum space is restricted to a single direction, $\Omega(\tau)$. As stated in Sec. 3.4, the screening is best represented by a restriction of the modes over which the field $\phi[x]$ in (7.1) is expanded. In general representing this restriction mathematically can be very difficult and has been a major stumbling block for the common use of this detector model. However, proceeding at a purely formal level, the transition probability for this detector is

$$P^5 = c^2\left|\langle M\rangle\right|^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-i\Delta E\Delta\tau}\, G_{\gamma}^{+}\left(\tau,\tau';\tilde{\rho}\right)\Big|_{S_{\Omega}(\tau,\tau')}$$

Where $\Big|_{S_{\Omega}(\tau,\tau')}$ represents the fact that G_{γ}^{+} has been evaluated with a restriction of the momentum modes to $S_{\Omega(\tau)}$ at time τ and $S_{\Omega(\tau')}$ at time τ'. That is

$$\begin{aligned} G_{\gamma}^{+}\left(\tau,\tau';\tilde{\rho}\right)\Big|_{S_{\Omega}(\tau,\tau')} &= \langle 0|\phi\left[x(\tau)\right]\Big|_{S_{\Omega}(\tau)}\phi\left[x(\tau')\right]\Big|_{S_{\Omega}(\tau')}|0\rangle \\ &= \int_{S_{\Omega}(\tau)\cap S_{\Omega}(\tau')} d^{n-1}k\, u_{\boldsymbol{k}}(x)\, u_{\boldsymbol{k}}^{*}(x') \end{aligned} \tag{7.2}$$

Where $u_{\boldsymbol{k}}(x)$ are the field mode equations. The general time dependence of the screening is manifest in (7.2). Of course, this dependence vanishes in the frame for which the screening S_{Ω} is fixed. Unfortunately, usually the integral in (7.2) is not manageable in that frame. It is because of this difficulty that this detector will only be discussed in Rindler space. In Schwarzschild space the restriction over the momentum space integral in (7.2) becomes a very complicated restriction over the l, m-summation of the spherical harmonics.

The relationship between the responses of the cone (R^5) and spike (R^4) detector may be represented by the following equation

$$R^4(\Omega) = \lim_{S_{\Omega}\to 0} \frac{R^5(S_{\Omega})}{S_{\Omega}}$$

Where $R^4(\Omega)$ is the response of the spike detector with its orientation given by Ω and $R^5(S_{\Omega})$ is the response of the cone which has access to the hyper-solid-angle S_{Ω} of momentum modes around the hyper-angle Ω. Thus the spike's response equals the response per unit solid-angle of the cone (as its aperture shrinks to zero). Given the response of the cone (as a function of S_{Ω}) the response of the spike detector can be deduced and vice versa.

7.2 The Spike and Cone Detectors in four dimensional Rindler space

Using the directional discrimination of these detectors we can now discover whether or not acceleration radiation is truly isotropic, as some have implied. (See for example, Gerlach 1983). In fact the study of the isotropy of this radiation is of greater interest than the actual detector response and so we shall initially aim our enquiry more so at revealing the directional nature of the radiation than at a particle detector model. The mathematics is greatly simplified by using the spike detector model which has a fixed orientation, Ω, in an inertial frame and so varying orientation in the co-moving frame. A discussion of the detectors with a fixed orientation in the co-moving Rindler frame appears later in this chapter.

If the acceleration radiation bath is isotropic in the co-moving frame, then the response per unit solid angle will be constant and independent of orientation. Hence the total transition probability will be infinite, reflecting the time independence. However, if the radiation is anisotropic the spike detector's response will be time dependent due to the changing orientation of the detector in the co-moving frame.

For the spike detector, in an inertial frame $G^+(x,x')\big|_{\Omega'}$ with orientation Ω' is

$$G^+(x,x')\big|_{\Omega'} = \int d^3k u_{\boldsymbol{k}}(x) u_{\boldsymbol{k}}^*(x') \delta(\Omega - \Omega')$$

Using plane wave modes (2.3), and orienting the spike to accept only modes in the θ, φ direction gives

$$G^+(x,x')\big|_{\Omega'} = \frac{1}{16\pi^3}\int_0^{2\pi} d\varphi\delta(\varphi-\varphi')\int_0^{\pi} d\theta \sin\theta\delta(\cos\theta-\cos\theta')\int dk\, k e^{ik\left[|\boldsymbol{x}-\boldsymbol{x}'|\cos\theta - t - t' + i\varepsilon\right]}$$

where θ is the angle between the propagation vector $\boldsymbol{k}$ and $\boldsymbol{x}$-$\boldsymbol{x}'$. The result of the integration is

$$\frac{1}{16\pi^3}\left\{|\boldsymbol{x}-\boldsymbol{x}'|\cos\theta - (t-t'-i\varepsilon)\right\}$$

Substituting in the Rindler trajectory

$$t = \xi\sinh\tilde{\tau} \qquad x = x' \qquad y = y' \qquad z = \xi\cosh\tilde{\tau} \tag{7.3}$$

Where $\tilde{\tau} = (\text{propertime})/\xi$ and fixing θ and φ in the inertial frame at $\tilde{\tau} = 0$ yields,

$$G_\gamma^+(\tilde{\tau},\tilde{\tau}';\xi)\big|_{\Omega} = \frac{-1}{16\pi^3\xi^2\left\{\left|\sinh\frac{\Delta\tilde{\tau}}{2}\sinh\overline{\tau}\right|\cos\theta - \left(\sinh\frac{\Delta\tilde{\tau}}{2}\cosh\overline{\tau} - i\varepsilon\right)\right\}^2} \tag{7.4}$$

where $\overline{\tau} = (\tilde{\tau}+\tilde{\tau}')/2$. Due to the presence of the moduli when (7.4) is substituted into (3.26), to give the transition probability, the ranges of the integration over $\Delta\tilde{\tau}$ and $\overline{\tau}$ must be appropriately split. For $\Delta\tilde{\tau} > 0, \overline{\tau} > 0$ and $\Delta\tilde{\tau} < 0, \overline{\tau} < 0$

$$G_\gamma^+(\tilde{\tau},\tilde{\tau}';\xi)\big|_{\Omega} = \frac{-1}{16\pi^3\xi^2\left\{\sinh\frac{\Delta\tilde{\tau}}{2}\sinh\overline{\tau}\cos\theta - \left(\sinh\frac{\Delta\tilde{\tau}}{2}\cosh\overline{\tau} - i\varepsilon\right)\right\}^2} \tag{7.5}$$

and for $\Delta\tilde{\tau} > 0, \overline{\tau} < 0$ and $\Delta\tilde{\tau} < 0, \overline{\tau} > 0$

$$G_\gamma^+\left(\tilde{\tau},\tilde{\tau}';\xi\right)\Big|_\Omega = \frac{-1}{16\pi^3\xi^2\left\{-\sinh\frac{\Delta\tilde{\tau}}{2}\sinh\overline{\tau}\cos\theta-\left(\sinh\frac{\Delta\tilde{\tau}}{2}\cosh\overline{\tau}-i\varepsilon\right)\right\}^2} \tag{7.6}$$

Considering (7.5), the pole in the complex $\Delta\tilde{\tau}$ -plane has been shifted up by the $-i\varepsilon$ term. The same effect is achieved by writing that expression in the form

$$\frac{-1}{64\pi^3\xi^2\left(\cosh\overline{\tau}-\cos\theta\sinh\overline{\tau}\right)^2\sinh^2\left(\frac{\Delta\tilde{\tau}}{2}-i\varepsilon\right)} \tag{7.7}$$

Repeating this manipulation with (7.6) gives the form

$$\frac{-1}{64\pi^3\xi^2\left(\cosh\overline{\tau}+\cos\theta\sinh\overline{\tau}\right)^2\sinh^2\left(\frac{\Delta\tilde{\tau}}{2}-i\varepsilon\right)} \tag{7.8}$$

From (7.7) and (7.8) the detector's response will be

$$P_\gamma^4 = \frac{-c^2\left|\langle M\rangle\right|^2}{64\pi^3}\left\{\begin{array}{l}\displaystyle\int_{-\infty}^{0}\frac{d\Delta\tilde{\tau}e^{-i\Delta E\Delta\tilde{\tau}\xi}}{\sinh^2\left(\frac{\Delta\tilde{\tau}}{2}-i\varepsilon\right)}\int_{-\infty}^{0}\frac{d\overline{\tau}}{\left(\cosh\overline{\tau}-\cos\theta\sinh\overline{\tau}\right)^2}\\ \displaystyle+\int_{0}^{\infty}\frac{d\Delta\tilde{\tau}e^{-i\Delta E\Delta\tilde{\tau}\xi}}{\sinh^2\left(\frac{\Delta\tilde{\tau}}{2}-i\varepsilon\right)}\int_{0}^{\infty}\frac{d\overline{\tau}}{\left(\cosh\overline{\tau}-\cos\theta\sinh\overline{\tau}\right)^2}\\ \displaystyle+\int_{-\infty}^{0}\frac{d\Delta\tilde{\tau}e^{-i\Delta E\Delta\tilde{\tau}\xi}}{\sinh^2\left(\frac{\Delta\tilde{\tau}}{2}-i\varepsilon\right)}\int_{0}^{\infty}\frac{d\overline{\tau}}{\left(\cosh\overline{\tau}+\cos\theta\sinh\overline{\tau}\right)^2}\\ \displaystyle+\int_{0}^{\infty}\frac{d\Delta\tilde{\tau}e^{-i\Delta E\Delta\tilde{\tau}\xi}}{\sinh^2\left(\frac{\Delta\tilde{\tau}}{2}-i\varepsilon\right)}\int_{-\infty}^{0}\frac{d\overline{\tau}}{\left(\cosh\overline{\tau}+\cos\theta\sinh\overline{\tau}\right)^2}\end{array}\right\}$$

Substituting $\overline{\tau}\to-\overline{\tau}$ in the first and fourth integrals gives

$$P_\gamma^4 = \frac{-c^2\left|\langle M\rangle\right|^2}{32\pi^3}\left\{\begin{array}{l}\displaystyle\int_{-\infty}^{0}\frac{d\Delta\tilde{\tau}e^{-i\Delta E\Delta\tilde{\tau}\xi}}{\sinh^2\left(\frac{\Delta\tilde{\tau}}{2}-i\varepsilon\right)}\int_{0}^{\infty}\frac{d\overline{\tau}}{\left(\cosh\overline{\tau}+\cos\theta\sinh\overline{\tau}\right)^2}\\ \displaystyle+\int_{0}^{\infty}\frac{d\Delta\tilde{\tau}e^{-i\Delta E\Delta\tilde{\tau}\xi}}{\sinh^2\left(\frac{\Delta\tilde{\tau}}{2}-i\varepsilon\right)}\int_{0}^{\infty}\frac{d\overline{\tau}}{\left(\cosh\overline{\tau}-\cos\theta\sinh\overline{\tau}\right)^2}\end{array}\right\}$$

Evaluating the $\overline{\tau}$ - integrals

$$\int_0^\infty\frac{d\overline{\tau}}{\left(\cosh\overline{\tau}\pm\cos\theta\sinh\overline{\tau}\right)^2}=\int_0^\infty\frac{d\overline{\tau}}{\left(\frac{1}{2}\left(1+\cos^2\theta\right)\cosh 2\overline{\tau}\pm\cos\theta\sinh 2\overline{\tau}+\frac{1}{2}\sin^2\theta\right)}$$

Using Gradshteyn and Ryhzik (1980) No.3.513.4

$$=\frac{1}{(1\pm\cos\theta)}$$

Thus

$$P_{\gamma}^{4}=\frac{-c^{2}\left|\langle M\rangle\right|^{2}}{32\pi^{3}}\left\{\begin{array}{l}\frac{1}{1+\cos\theta}\int\limits_{0}^{\infty}\frac{d\Delta\tilde{\tau}e^{-i\Delta E\Delta\tilde{\tau}\xi}}{\sinh^{2}\left(\frac{\Delta\tilde{\tau}}{2}-i\varepsilon\right)}\\+\frac{1}{1-\cos\theta}\int\limits_{-\infty}^{0}\frac{d\Delta\tilde{\tau}e^{-i\Delta E\Delta\tilde{\tau}\xi}}{\sinh^{2}\left(\frac{\Delta\tilde{\tau}}{2}-i\varepsilon\right)}\end{array}\right\}$$

Although this expression is difficult to evaluate at it stands, the nature of the acceleration radiation can be adequately demonstrated by using a "double spike" consisting of two spikes, one having orientation θ and the other $\pi-\theta$. The total response of this bi-spike is

$$\begin{aligned}P_{\gamma}^{4\_Dble}&=\frac{-c^{2}\left|\langle M\rangle\right|^{2}}{32\pi^{3}}\left\{\int\limits_{-\infty}^{\infty}\frac{d\Delta\tau e^{-i\Delta E\Delta\tilde{\tau}\xi}}{\sinh^{2}\left(\frac{\Delta\tilde{\tau}}{2}-i\varepsilon\right)}\left[\frac{1}{1+\cos\theta}+\frac{1}{1-\cos\theta}\right]\right\}\\&=\frac{-c^{2}\left|\langle M\rangle\right|^{2}}{16\pi^{2}\sin^{2}\theta}\int\limits_{-\infty}^{\infty}\frac{d\Delta\tau e^{-i\Delta E\Delta\tilde{\tau}\xi}}{\sinh^{2}\left(\frac{\Delta\tilde{\tau}}{2}-i\varepsilon\right)}\\&=\frac{-c^{2}\left|\langle M\rangle\right|^{2}\xi\Delta E}{2\pi^{2}\sin^{2}\theta\left(e^{2\pi\Delta E\xi}-1\right)}\end{aligned}\tag{7.9}$$

So, provided $\theta\neq 0$, π, this detector's response is finite and hence the acceleration radiation is anisotropic. For a bi-cone accepting modes with $0<\varphi<2\pi$, $\alpha_1<\theta<\alpha_2$ fixed in the $\tilde{\tau}=0$ inertial frame

$$P_{\gamma}^{5}=\int\limits_{\alpha_1}^{\alpha_2}d\theta\int\limits_{0}^{2\pi}d\varphi\sin\theta P_{\gamma}^{4}=\frac{c^{2}\left|\langle M\rangle\right|^{2}\Delta E\xi}{2\pi\left(e^{2\pi\Delta E\xi}-1\right)}\left[\ln\tan\left(\frac{\alpha_2}{2}\right)-\ln\tan\left(\frac{\alpha_1}{2}\right)\right]\tag{7.10}$$

To recover the linear detectors we require $\alpha_1\to 0$ and $\alpha_2\to\pi/2$. The latter limit gives zero ($\ln(\tan(\pi/4))$), but the α_1 limit gives the characteristic logarithmic divergence intrinsic in the linear detector (equilibrium) response to uniform acceleration.

Although the spike detector used above demonstrates in a simple and direct way, the anisotropy of the acceleration radiation, it does not address the question of its time dependence. The detector itself introduces a time dependence (in a co-moving frame) due to its changing orientation in that frame. Intuitively, we would expect the radiation to be time independent (since the trajectory is stationary) and anisotropic. Israel and Nester (1983) have published a result that supports this expectation; however their calculation assumes time independence of the detector's response. It is not mathematically deduced. We shall now correct this oversight, and study the question of time dependence of the cone detector or any "screened" detector on a Rindler trajectory.

Consider a monopole detector linearly coupled to a quantum field but through a directionally dependent screen which has a transmission coefficient S (such that $0 < S < 1$) held constant **in the detectors rest frame**. Assume that $S = S(\cos\theta_0)$ where θ_0 is the angle of the incident modes in the co-moving frame with respect to the direction of acceleration (in the z-direction). This gives for the total transition probability of the screened detector

$$P = c^2 \left|\langle M \rangle\right|^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-i\Delta E \Delta\tau} G_S^+\left(x(\tau), x(\tau')\right) \tag{7.11}$$

Where G_S^+ is the screened Wightman function given by

$$G_S^+\left(x, x'\right) = \int d^3k u_{\boldsymbol{k}}\left(x\right) u_{\boldsymbol{k}}^*\left(x'\right) S\left(\cos\theta_0\left(\tau\right)\right) S\left(\cos\theta_0\left(\tau'\right)\right)$$

For the detector's response to be time independent G_S^+ must be a function of $\Delta\tau$ only. Using the plane wave modes and evaluating the momentum integral in spherical coordinates

$$G_s^+\left(x(\tau), x(\tau')\right) = \frac{-1}{8\pi} \int_0^{\pi} d\theta \frac{\sin\theta S\left(\cos\theta_0\left(\tau\right)\right) S\left(\cos\theta_0\left(\tau'\right)\right)}{\left(t - t' - i\varepsilon - \left|x(\tau) - x(\tau')\right| \cos\theta\right)^2}$$

It is at this point that Israel and Nester assume time independence and fail to include the $|x(t)\text{-}x(t')|\cos\theta$ term. This equation is written in an inertial frame (due to the use of plane wave modes) and the $S(\cos\theta_0(t))$ functions represent the co-moving fixed screening function as observed in the co-moving frame.

Substituting in the Rindler trajectory (7.3) gives

$$\frac{-1}{32\pi^2\xi^2} \int_0^{\pi} d\theta \frac{\sin\theta S\left(\cos\theta_0\left(\tau\right)\right) S\left(\cos\theta_0\left(\tau'\right)\right)}{\left(\sinh\frac{\Delta\tilde{\tau}}{2}\cosh\overline{\tau} - i\varepsilon - \left|\sinh\frac{\Delta\tilde{\tau}}{2}\sinh\overline{\tau}\right|\cos\theta\right)^2} \tag{7.12}$$

Again, the $-i\varepsilon$ term in the denominator merely relates the direction in which the poles of G_S^+ are shifted (i.e. upward in the complex $\Delta\tilde{\tau}$ - plane) when performing the contour integral to evaluate (7.11). Alternatively, the contour may be deformed around (below) the poles achieving the same result. (See Fig.13.) Taking this approach G_S^+ may be written as follows: for $\Delta\tilde{\tau} > 0, \overline{\tau} > 0$ and $\Delta\tilde{\tau} < 0, \overline{\tau} < 0$

$$G_S^+\left(\Delta\tilde{\tau}, \overline{\tau}; \xi =\right) \frac{-1}{32\pi^2\xi^2 \sinh^2\left(\frac{\Delta\tilde{\tau}}{2} - i\varepsilon\right)} \int_0^{\pi} d\theta \frac{\sin\theta S\left(\cos\theta_0\left(\tau\right)\right) S\left(\cos\theta_0\left(\tau'\right)\right)}{\left(\cosh\overline{\tau} - \cos\theta\sinh\overline{\tau}\right)^2} \tag{7.13}$$

For $\Delta\tilde{\tau} < 0, \overline{\tau} > 0$ and $\Delta\tilde{\tau} > 0, \overline{\tau} < 0$

$$G_S^+\left(\Delta\tilde{\tau}, \overline{\tau}; \xi =\right) \frac{-1}{32\pi^2\xi^2 \sinh^2\left(\frac{\Delta\tilde{\tau}}{2} - i\varepsilon\right)} \int_0^{\pi} d\theta \frac{\sin\theta S\left(\cos\theta_0\left(\tau\right)\right) S\left(\cos\theta_0\left(\tau'\right)\right)}{\left(\cosh\overline{\tau} + \cos\theta\sinh\overline{\tau}\right)^2} \tag{7.14}$$

For G_S^+ to be a function of $\Delta\tilde{\tau}$ only, we require its value with $\Delta\tilde{\tau} > 0$ to be the same for both $\bar{\tau} > 0$ and $\bar{\tau} < 0$. That is for these values of $\Delta\tilde{\tau}$ and $\bar{\tau}$, (7.13) and (7.14) may be equated. Re-arranging this equality gives

$$\int_0^\pi d\theta \frac{\sin\theta S\left(\cos\theta_0\left(\bar{\tau}+\frac{\Delta\tau}{2}\right)\right)S\left(\cos\theta_0\left(\bar{\tau}-\frac{\Delta\tau}{2}\right)\right)}{\left(\cosh\bar{\tau}-\cos\theta\sinh\bar{\tau}\right)^2} = \int_0^\pi d\theta \frac{\sin\theta S\left(\cos\theta_0\left(\bar{\tau}+\frac{\Delta\tau}{2}\right)\right)S\left(\cos\theta_0\left(\bar{\tau}-\frac{\Delta\tau}{2}\right)\right)}{\left(\cosh\bar{\tau}+\cos\theta\sinh\bar{\tau}\right)^2}$$

Since $\bar{\tau} > 0$ in the left hand side and $\bar{\tau} < 0$ on the right, we replace $\bar{\tau}$ in the right with $-\bar{\tau}$. This yields

$$\int_0^\pi d\theta \frac{\sin\theta}{\left(\cosh\bar{\tau}-\cos\theta\sinh\bar{\tau}\right)^2}\begin{bmatrix} S\left(\cos\theta_0\left(\bar{\tau}+\frac{\Delta\tau}{2}\right)\right)S\left(\cos\theta_0\left(\bar{\tau}-\frac{\Delta\tau}{2}\right)\right) \\ -S\left(\cos\theta_0\left(-\bar{\tau}+\frac{\Delta\tau}{2}\right)\right)S\left(\cos\theta_0\left(-\bar{\tau}-\frac{\Delta\tau}{2}\right)\right)\end{bmatrix} = 0 \qquad (7.15)$$

This equation applies when $\Delta\tilde{\tau} = 0$, giving

$$\int_0^\pi d\theta \frac{\sin\theta\left[S^2\left(\cos\theta_0\left(\bar{\tau}\right)\right)-S\left(\cos\theta_0\left(-\bar{\tau}\right)\right)\right]}{\left(\cosh\bar{\tau}-\cos\theta\sinh\bar{\tau}\right)^2} = 0$$

This condition is satisfied

$$S\left(\cos\theta_0\left(\tau\right)\right) = S\left(\cos\theta_0\left(-\tau\right)\right) \qquad (7.16)$$

We now show that (7.16) is satisfied by all screening functions, S, held constant in the co-moving frame. This function S is defined in the three spatial dimensions and so, when referred to an inertial frame, it must represent the effect of Lorentz contractions on the physical structure of the screen. This should not be confused with the aberration of the momentum modes in (four dimensions) momentum space. However, either can be used to represent the effects of the detector's acceleration on its accessibility to the field modes.

Writing $\cos\theta_0\left(\tilde{\tau}\right)$ as a function of $\tilde{\tau}$, with respect to the inertial frame defined by $\tilde{\tau} = 0$, gives

$$\cos\theta_0\left(\tilde{\tau}\right) = \left(1-v\left(\tau\right)^2\right)^{1/2}\cos\theta_0 = \left(1-\tanh^2\tilde{\tau}\right)^{1/2}\cos\theta_0 = \operatorname{sech}\tilde{\tau}\cos\theta_0$$

which is even in $\tilde{\tau}$. Therefore (7.15) is satisfied for any screening function fixed in the co-moving frame. From (7.13) and (7.14) we may write G_S^+ in the form

$$G_S^+\left(\Delta\tilde{\tau};\xi\right) = \frac{-F_S\left(\Delta\tilde{\tau}\right)}{32\pi^2\xi^2\sinh^2\left(\frac{\Delta\tilde{\tau}}{2}-i\varepsilon\right)} \qquad (7.17)$$

where $F_S\left(\Delta\tilde{\tau}\right)$ represents the screening function terms in (7.13) and (7.14). Comparing (7.17) with the Wightman function for the unscreened linear detector, given in (4.16), we see that the screened Wightman function is (7.17) which consists of a thermal part and the screening term F_S. Using the convolution theorem and noting the form of the unscreened linear detector's response (4.17) we see the response of the screened detector has the form

$$R = \frac{c^2 \left|\langle M \rangle\right|^2}{2\pi} \int_{-\infty}^{\infty} dx \frac{x\overline{F}_S\left(\Delta E - x\right)}{e^{2\pi\xi x} - 1}$$

where

$$\overline{F}_S\left(x\right) = \int_{-\infty}^{\theta} d\Delta\tilde{\tau} e^{-ix\Delta\tilde{\tau}\xi} F_S\left(\Delta\tilde{\tau}\right)$$

The screened detector's response is a linear superposition of Planck responses of varying (positive and negative) energies x. The Planck nature of the response is still evident, although in the total response it may be hidden due to the F_S terms. This is consistent with the general thermal nature of Rindler observables operating on the Minkowski vacuum state. (Sciama et al. 1981) The Israel, Nester conclusion that the detector's response is non-thermal, although correct in that G_S^+ generally does not have a thermal form (and need not even be periodic in imaginary time), fails to expose the deeper characteristics of the response.

We now consider a cone detector for which

$$\begin{aligned} S\left(\cos\theta_0\right) &= 1 \qquad && 0 \le \theta_0 \le \alpha \\ &= 0 \qquad && \alpha \le \theta_0 \le \pi \end{aligned}$$

is held constant in the co-moving frame. This gives

$$F_S = \frac{1 - M\left(\tilde{\tau}'\right)}{\cosh^2\overline{\tau} \pm \left(1 + M\left(\tilde{\tau}'\right)\right)\cosh\overline{\tau}\sinh\overline{\tau} + M\left(\tilde{\tau}'\right)\sinh^2\overline{\tau}} \tag{7.18}$$

Where the addition in the denominator corresponds to the range $\Delta\tilde{\tau} < 0,\ \overline{\tau} < 0$ and the subtraction to the range $\Delta\tilde{\tau} < 0,\ \overline{\tau} > 0$ and

$$M\left(x\right) = \frac{\cos\alpha - \tanh x}{1 - \cos\alpha \tanh x}$$

For the other regions

$$F_S = \frac{1 - M\left(\tilde{\tau}\right)}{\cosh^2\overline{\tau} \pm \left(1 + M\left(\tilde{\tau}\right)\right)\cosh\overline{\tau}\sinh\overline{\tau} + M\left(\tilde{\tau}\right)\sinh^2\overline{\tau}} \tag{7.19}$$

Where the addition corresponds to $\Delta\tilde{\tau} > 0,\ \overline{\tau} > 0$ and the subtraction to $\Delta\tilde{\tau} > 0,\ \overline{\tau} < 0$. After an extensive algebraic manipulation, both (7.18) and (7.19) reduce to

$$F_S\left(\Delta\tilde{\tau}\right) = \frac{2}{1 + e^{-\left|\Delta\tilde{\tau}\right|}\left(\dfrac{1+\cos\alpha}{1-\cos\alpha}\right)} \tag{7.20}$$

From this we can see that the transition rate per unit solid-angle of the cone is proportional to

$$\int_{-\infty}^{\infty} d\Delta\tilde{\tau} \frac{e^{-i\Delta E\Delta\tilde{\tau}\xi}}{\sinh^2\left(\frac{\Delta\tilde{\tau}}{2} - i\varepsilon\right)\left[1 + e^{-|\Delta\tilde{\tau}|} + \cos\alpha\left(e^{-|\Delta\tilde{\tau}|} - 1\right)\right]} \tag{7.21}$$

If the acceleration radiation was isotropic (7.21) would be α-independent. Thus, again the anisotropy of acceleration radiation is demonstrated.

8 The Definition of Particle Detector Equivalence

The five mathematical constructs introduced above all qualify as "particle detectors" since they satisfy the required conditions. To ascertain whether or not they concur on what they "see" when placed in the same situation we must somehow compare their respective responses. It is clear from the foregoing that in a given situation their actual responses will differ due to their different interaction Lagrangains. To compare the "effective particle content" that these detectors perceive, we must introduce some reference standpoint that avoids direct comparison of their responses.

In a more general context, since the use of particle detectors in this field is based on an "operationalisation" standpoint (c.f. DeWitt (1979) as quoted in chapter 2) any method of comparing particle detectors should also be operationalist in the same sense. Therefore the method of comparison should, as much as possible, be based on detector responses and not on the (mathematical) details of studying them. This approach has the added advantage that it frees the method of comparison from being tied to the "state of the art" of studying quantum field theory in curved spaces. It is a general hope that ultimately the present perturbative techniques used in this field will be replaced by exact methods. An operationalist approach to comparing detector models will be equally suited to either of these mathematical techniques.

The response of a detector will depend upon a variety of factors:

1. The state of the quantum field
2. The motion of the detector
3. The space-time geometry
4. The space-time topology
5. The orientation of the detector (if it has some sort of directional dependence).

The approach adopted here, to avoid direct comparison of detector responses, will be to compare the various responses of a given detector when placed in a variety of situations.

Let $\mathcal{D}$ be the detector under consideration. Define the set $\mathcal{S}$ by

$\mathcal{S}$ = {all situations into which $\mathcal{D}$ can be placed} = $\{s_i: i \in I\}$

where I is an index set. Let $\mathcal{D}(s_i)$ be the response of detector $\mathcal{D}$ when placed in situation $s_i \in \mathcal{S}$. Define "detector equivalence of this situation" as follows;

Definition: Let s_i, $s_j \in \mathcal{S}$. Then s_i and s_j are said to be $\mathcal{D}$-equivalent if and only if

$$\mathcal{D}(s_i) = \mathcal{D}(s_j) \tag{8.1}$$

$\mathcal{D}$-equivalence is an equivalence relation and hence partitions $\mathcal{S}$ into disjoint equivalence classes (Birchoff & MacLane 1963). Denote these equivalence classes by S_a, $a \in A$ an index set. It follows that

$$\forall s_i, s_j \in S_a, \mathcal{D}(s_i) = \mathcal{D}(s_j)$$

and

$$\mathcal{S} = \bigcup_{a \in A} S_a \qquad S_a \bigcap S_b = 0 \qquad \forall a \neq b$$

Now define the set $\mathcal{S}^{\mathcal{D}}$ by

$$\mathcal{S}^{\mathcal{D}} = \{S_a : a \in A\} = \{\mathcal{D}\text{-equivalent classes in } \mathcal{S}\}$$

$\mathcal{S}^{\mathcal{D}}$ is the quotient set of $\mathcal{S}$ produced by the $\mathcal{D}$-equivalence relation.

Now suppose we have a second detector, $\mathcal{D}'$. Repeating the above will yield another quotient set $\mathcal{S}^{\mathcal{D}'}$

Definition: Two detectors $\mathcal{D}$ and $\mathcal{D}'$ are $\mathcal{S}$-equivalent if and only if

$$\mathcal{S}^{\mathcal{D}} = \mathcal{S}^{\mathcal{D}'} \tag{8.2}$$

This will be our definition of "particle detector equivalence on $\mathcal{S}$". Although the definition is stated for the set $\mathcal{S}$ of all situations into which the detectors can be placed, it can obviously be restated for a subset $\hat{\mathcal{S}}$ of $\mathcal{S}$.

From this definition it follows that if s_i and s_j give the same response for $\mathcal{D}$ (i.e. $\mathcal{D}(s_i) = \mathcal{D}(s_j)$), then they belong to the same equivalence class $S_a \in \mathcal{S}^{\mathcal{D}}$. Further, if $\mathcal{D}$ and $\mathcal{D}'$ are $\mathcal{S}$-equivalent, then $S_a \in \mathcal{S}^{\mathcal{D}'}$ and $\mathcal{D}'(s_i) = \mathcal{D}'(s_j)$. Therefore s_i and s_j also give identical responses with $\mathcal{D}'$. Clearly the converse is also true. If two detectors are $\mathcal{S}$-equivalent then the sets of situations which give the same response for each detector is identical. If the two detectors are not $\mathcal{S}$-equivalent, then they will be referred to as "$\mathcal{S}$-in-equivalent".

Note that there has been no direct comparison of detector responses. Instead, the comparison is in the way each detector splits $\mathcal{S}$ into quotient subsets, S_a. An important consequence of this is that it is meaningless to compare two detectors for a single situation (e.g. S_a having only one element for some $a \in A$.) This may be understood in the following way.

Provided a model satisfies conditions a) and b) in Sec. 2.2, it qualifies as a particle detector. There is no guarantee that the $\langle M \rangle$-dependent term (i.e. $\langle E | m(0) | E_0 \rangle = \langle M \rangle$ in the above detectors) can be separated out from the terms dependent only on the coupling to the quantum field. The detector could merely be a "black box". Therefore, given only single situation, one cannot tell whether or not the different responses are due to different $\langle M \rangle$-dependent terms or differing coupling between the detectors and the field. Only when there is more than one situation in the equivalence classes S_a can the $\langle M \rangle$-dependent terms be identified, allowing meaningful comparison between detectors. Unless such a standpoint is adopted, only detectors with exactly identical construction can be equivalent. This is undesirable because in the use of particle detectors to study quantum fields in curved spaces, importance is attached primarily to the $\langle M \rangle$-independent terms in the detector's response (Birrell & Davies 1982).

To illustrate this point, consider the linear and derivative detectors in Rindler space. (In using this as an illustrative example, we must recognise and put aside the unusually high degree of symmetry special to the Rindler situation and thermal particle states.) Assume the orientation of the derivative detector dipole is fixed in the co-moving frame. If we compare the responses of these two detectors in the situation **alone**, we could not know the linear detector is responding thermally because we are only studying their responses in the Rindler situation. Therefore, since we already expect the two detectors to give different responses, there is no way we can determine whether or not the derivative detector is also responding thermally. As a result it cannot be ascertained if the two detectors perceive this situation as the same thing (i.e. both responding thermally).

However, our predicament changes significantly once a second situation which gives rise to the same response for one of the detectors is included. For the purposes of this illustration let this second situation be immersion of the detector in an isotropic bath of Planckian radiation in Minkowski space. Now we will be aware of the thermal nature of the linear detector's response in the Rindler situation. Further we can evaluate response of the derivative detector to the particle bath and decide whether or not it is responding identically as it did to the Rindler situation. Only now are we able to determine if the two detectors perceive the same particle content in the Rindler situation.

DeWitt (1965) introduced the distinction between so-called "elementary" and "complete" measurements. Elementary measurements deal merely with the correlations between the system on which the measurement is being performed and the response of the apparatus. Complete measurements are concerned with analysing the response of the apparatus through comparison with other measurements. It is from complete measurements that conclusions about the system (such as its particle content) are drawn. From the discussion immediately above, it can be seen that the definition of particle detector equivalence involves these comparisons and hence defines detector equivalence at a "complete" level. Only when the equivalence classes contain more than one member can comparisons be made at this level. If the equivalence classes contain only a single element, then the measurements are elementary and no conclusions can be drawn about the system.

Another aspect of the concept of detector equivalence introduced above is its applicability to fields of arbitrary spin. Since no specialisation to a particular quantum field occurs in the formalism used to define "particle detector equivalence", it follows that this method of comparing detectors can be applied to detectors of any quantum field. Further it is the quotient sets $\mathcal{S}^{\mathcal{D}}$ produced by the detector responses that are compared. If it is possible to avoid mentioning the field statistics when applying the concept, can we take the extra step and discuss the equivalence of detectors of fields of different spin? To do this we must be able to describe all the situations in $\mathcal{S}$ without referring to the statistics of the field involved. Further discussion of this question will be deferred to Chapter 12 where detectors of higher spin fields will be considered.

9 Application of the equivalence criterion

Five different "particle detectors' were introduced in chapter 3 and in the subsequent chapters it was shown that they each respond differently even in a single given situation. An application of the criterion of detector equivalence introduced in the previous chapter will now be demonstrated by using it to study the equivalence of these detectors in the following sets of situations:

$\mathcal{S}_1$ = {time independent particle eigen-states in Minkowski space with the detector stationary}

$\mathcal{S}_2$ = {time-like trajectories through Minkowski space in the Minkowski vacuum}

These two sets have the following subsets;

$\mathcal{S}_1'$ = {isotropic particle states in $\mathcal{S}_1$}

$\mathcal{S}_2'$ = {stationary trajectories in $\mathcal{S}_2$}

Finally $\mathcal{S}_3$ = {time-like trajectories through non-flat (empty) space-time vacuum states}

Through this chapter we shall frequently encounter the following integral transform:

$$F\{f(x),y\} = \int_m^\infty f(x)\cos xy\, dx \tag{9.1}$$

where $0 \le m < \infty$ and $f(x)$ is a function defined only for $m \le x < \infty$. This This is a Cosine-Fourier transform truncated at the lower limit m. One can easily show that the inverse of this this transform is

$$F^{-1}\{h(y),x\} = \frac{2}{\pi}\int_0^\infty h(y)\cos xy\, dy \tag{9.2}$$

due to the special nature of $f(x)$. This inverse transform shall be used several times.

9.1 Comparing Omni-directional detectors

Due to the intrinsic structural differences between omni-directional and directional detectors, we shall first apply the equivalence formalism to compare only the omni-directional detectors. The directional detectors will be discussed below.

In the following sub-sections the linear, quadratic and derivative detectors will be compared using the sets of situations introduced at the beginning of this chapter.

9.1.1 Equivalence in $\mathcal{S}_1'$

The response of the three detectors for the situations in $\mathcal{S}_1'$ are given by (3.7), (3.15) and (3.19). For $\mathcal{S}_1'$, (3.8) gives

$$\overline{n}_k = n_k \tag{9.3}$$

Similarly from (3.20) and (3.21)

$$n_{k,ij} = n_k \Gamma\left(\frac{(n-1)}{2}\right)\delta_{ij} \bigg/ 2\Gamma\left(\frac{(n+1)}{2}\right) = n_k \delta_{ij}\left(n-1\right) \tag{9.4}$$

$$n_{k,0i} = 0 \tag{9.5}$$

From (9.3), (9.4) and (9.5), we see that for the set $\mathcal{S}_1'$, the detector equivalence classes S_a for all three detectors contain only one element each since the responses are uniquely determined by the occupation numbers n_k. Therefore, for the reasons given in the previous chapter, the M-dependence of the detectors cannot be factored out and hence one cannot apply the concept of detector equivalence on the set of isotropic particle states $\mathcal{S}_1'$.

9.1.2 Equivalence in $\mathcal{S}_1$

There will now be more than one element in each equivalence class since one can envisage a variety of differing anisotropic particle states, n_k, which when placed in the integrands of (3.8), (3.20) and (3.21) give identical results, respectively, upon performing the integrals. In fact, the following result holds for **any** omni-directional detector.

Theorem 1: Let $\mathcal{D}$ be an omni-directional detector. Then the set of isotropic particle baths, $n_k \in \mathcal{S}_1'$ form a transversal of $\mathcal{S}_1^{\mathcal{D}}$. Furthermore, every $S_a \in \mathcal{S}_1^{\mathcal{D}}$ contains at least two elements.

Proof: For $\{n_k \in \mathcal{S}_1'\}$ to form a transversal of $\mathcal{S}_1^{\mathcal{D}}$, there must be one and only one n_k in each $S_a \in \mathcal{S}_1^{\mathcal{D}}$. Consider an equivalence class S_a for some $a \in A$. Let $n_{\boldsymbol{k}} \in S_a$ be an (anisotropic) particle state. By definition, the response of the detector $\mathcal{D}$ is a function of the average $n_{\boldsymbol{k}}$ over the $(n$-2)-sphere of directions. Let this averaging of $n_{\boldsymbol{k}}$ be denoted by $A(n_{\boldsymbol{k}})$ where A is an average over the (n-2)-sphere of directions.

This means $A(n_{\boldsymbol{k}})$ can be written in the form

$$A(n_{\boldsymbol{k}}) = \int W(\Omega) n_{k,\Omega} d\Omega$$

Where the integral $\int d\Omega$ is over the (n-2)-sphere of directions and k,Ω is the magnitude, angle decomposition of momentum vector $\boldsymbol{k}$. Finally, $W(\Omega)$ is the weighting function for the averaging over the (n-2)-sphere of directions in n-dimensional momentum space.

Therefore we may write for the response to $n_{\boldsymbol{k}}$

$$\mathcal{D}(n_{\boldsymbol{k}}) = \mathcal{D}(A(n_{\boldsymbol{k}})) \tag{9.6}$$

Define the particle state

$$\widehat{n}_k = A(n_{\boldsymbol{k}}) / A(1) = \int n_{k,\Omega} W(\Omega) d\Omega \Big/ \int W(\Omega) d\Omega \tag{9.7}$$

Where $A(1) = A(n_{\boldsymbol{k}} = 1)\ \forall\ \boldsymbol{k}$. Now $A(1) \neq 0$, otherwise $\mathcal{D}$ does not satisfy conditions (a) and (b) of a particle detector, so $\widehat{n}_k$ is well defined. Furthermore, $\widehat{n}_k$ is isotropic since, by its definition, it has no directional dependence. The detector's response to $\widehat{n}_k$ is

$$\mathcal{D}(\widehat{n}_{\boldsymbol{k}}) = \mathcal{D}\left(A\left(\tfrac{A(n_k)}{A(1)}\right)\right) = \mathcal{D}\left(\int \left(\frac{\int n_{k,\Omega} W(\Omega') d\Omega'}{\int W(\Omega'') d\Omega''}\right) W(\Omega) d\Omega\right)$$

$$= \mathcal{D}\left(\int n_{k,\Omega} W(\Omega') d\Omega' \frac{\int W(\Omega) d\Omega}{\int W(\Omega'') d\Omega''}\right) = \mathcal{D}(A(n_{\boldsymbol{k}}))$$

Therefore $\widehat{n}_k \in S_a$; each equivalence class S_a has at least one isotropic particle state as an element. Now, suppose there are two isotropic states $\widehat{n}_k, \tilde{n}_k \in S_a$. Then from (9.6) and the assumption that they are $\mathcal{D}$-equivalent

$$\mathcal{D}(\widehat{n}_k) = \mathcal{D}(\tilde{n}_k)$$

From requirement (b) in the definition of a particle detector, $\mathcal{D}(.)$ is a one-to-one function of the weighted mean of particle states $n_{\boldsymbol{k}} \in \mathcal{S}_1$. Given the definition of $\widehat{n}_k$ above, it is a weighted mean of particle states in

$\mathcal{S}_1$. Therefore D(.) is one-to-one for states $\hat{n}_k, \tilde{n}_k \in S_a$ and so has an inverse. Therefore we have $\hat{n}_k = \tilde{n}_k$. Hence the $\hat{n}_k$ are unique and $\mathcal{S}_1'$ forms a transversal of $\mathcal{S}_1^{\mathcal{D}}$.

To prove that every $S_a \in \mathcal{S}_1^{\mathcal{D}}$ has more than one element, assume there is some S_a for which this claim is not true. (i.e. $\exists\, a \in A$ such that S_a has only one element). If this element, $n_{\boldsymbol{k}}$, is anisotropic, then using (9.7) we can define an isotropic particle state $\hat{n}_k \in S_a$. If the lone element, $\tilde{n}_k$ of S_a is isotropic we can construct an isotropic state $m_{\boldsymbol{k}}$ defined by

$$m_{\boldsymbol{k}} = m_{k,\Omega} = \tilde{n}_k W'(\Omega)$$

Where $W'(\Omega)$ is a directionally dependent weighting function such that $A(W'(\Omega)) = A(1)$.

Q.E.D.

Returning to the three detectors under study, since $\mathcal{S}_1'$ forms a transversal of $\mathcal{S}_1^{\mathcal{D}}$, (9.7) can be used to deduce $\mathcal{D}$-equivalence of situations. That is, the $\mathcal{D}$-equivalent groupings $n_{\boldsymbol{k}} \in \mathcal{S}_1$ are made on the basis of which isotropic states $n_k \in \mathcal{S}_1'$ give the same response in $\mathcal{D}$. For the linear and quadratic detectors, the definition corresponding to (9.7) is statement (3.8)

$$\bar{n}_k = \int d\Omega n_{\boldsymbol{k}} \Big/ \int d\Omega \tag{9.8}$$

For the derivative detector, from (3.19), (3.20) and (3.21), it is

$$\hat{n}_k = \frac{\int d\Omega n_{\boldsymbol{k}} \left\{ \omega b^0 + \sum_{i=1}^{n-1} b^i \left(\omega^2 - m^2\right)^{1/2} \cos\Omega_{i+1} \prod_{p=2}^{i} \sin\Omega_p \right\}^2}{\left\{ \left(\omega b^0\right)^2 + \left(\omega^2 - m^2\right)\left(n-1\right)^{-1} \sum_{j=1}^{n-1} \left(b^j\right)^2 \right\} \int d\Omega} \tag{9.9}$$

Since the conditions for linear and quadratic detector equivalence in $\mathcal{S}_1$ are identical, these two detectors are $\mathcal{S}_1$-equivalent. Neither of these two are $\mathcal{S}_1$-equivalent to the general derivative detector. However, if the derivative detector has $b^0 = 1$, $b^i = 0$, $i = 1, .., n-1$, then (9.9) reduces to (9.8). So a pure time-derivative detector is $\mathcal{S}_1$-equivalent to the linear and quadratic detectors. A space-derivative detector is obviously not.

9.1.3 Equivalence in $\mathcal{S}_2$

The elements of $\mathcal{S}_2$ include situations that give a time-dependent response. The comparison of detector responses for such situations must be made along the entire world-line since it may occur that , although their total transition probabilities, P_γ, agree in the asymptotic regions $\tau \to \infty$, there transition rates, R_γ, are not identical for all points along the world-line γ. Therefore (as with the equilibrium situations) it is the transition rate per unit detector proper-time that is utilised when comparing detectors in time-dependent situations.

Using the notation and results from Sec's 4.1, 5.1 and 6.1 we can compare the three detectors for situations in $\mathcal{S}_2$ as follows. From (4.8) the response of the linear detector to the element γ of $\mathcal{S}_2$ is

$$R_\gamma^1(\tau) = c^2 \left|\langle M \rangle\right|^2 \int_{-\infty}^{\infty} d\eta e^{-i\Delta E \eta} \left\{ G_\gamma^+\left(\eta+\tau,\tau;\tilde{\rho}\right)\theta(-\eta) + G_\gamma^+\left(\tau,\tau-\eta;\tilde{\rho}\right)\theta(\eta) \right\} \tag{9.10}$$

For two trajectories γ_1, $\gamma_2 \in \mathcal{S}_2$ to be linear detector equivalent, we require $R_{\gamma_1}(\tau) = R_{\gamma_2}(\tau+\beta)$ for all τ and some fixed, finite β which represents a time translation relating the origins of the proper-time coordinates of the two trajectories. We are free to set $\beta = 0$ my merely shifting the origin of the proper-time scales on either of the trajectories. From (9.10), taking an inverse Fourier transform, this requirement reduces to

$$G^+_{\gamma_1}(\eta+\tau,\tau;\tilde{\rho}) = G^+_{\gamma_2}(\eta+\tau+\beta,\tau+\beta;\tilde{\xi}) \qquad \forall \eta > 0, \forall \tau$$

Where $\tilde{\rho}$ denotes the spatial coordinates of the detector in the coordinate frame corresponding to the trajectory γ_1 and $\tilde{\xi}$ is likewise for the trajectory γ_2. The arguments of the Wightman functions in this equality span the entire world-line, therefore this condition reduces further to

$$G^+_{\gamma_1}(\tau,\tau';\tilde{\rho}) = G^+_{\gamma_2}(\tau+\beta,\tau'+\beta;\tilde{\xi}) \tag{9.11}$$

At this point a check must be made that (9.11) does not imply $\gamma_1 = \gamma_2$ since this means the equivalence classes S_a in $\mathcal{S}_2^{\mathcal{D}}$ contain only one element each. As demonstrated above, this means the concept of detector equivalence (applied to the linear detector at least) is void on $\mathcal{S}_2$.

The Wightman function in Minkowski space is a function of the point separation σ where

$$\sigma = (t-t')^2 - |\boldsymbol{x}-\boldsymbol{x}'|^2 \tag{9.12}$$

In which t and x are Minkowski coordinates. For any $\gamma \in \mathcal{S}_2$, the relationship between the Minkowski coordinates and co-moving coordinates can be expressed as

$$t = f(\tau,\rho) \qquad x^i = x^i(\tau,\rho) \tag{9.13}$$

As functions of the point separation, for time-like trajectories, the Wightman functions are one-to-one (Bogolubov & Shirkov 1980). Hence for the Wightman functions corresponding to the two trajectories to satisfy (9.11) the point separations must be equal, that is

$$\sigma_1(\tau,\tau';\tilde{\rho}) = \sigma_2(\tau+\beta,\tau'+\beta;\tilde{\xi}) \tag{9.14}$$

where σ_i is the point separation for the trajectory γ_i. Let (9.13) correspond to γ_1 and for γ_2 write

$$t = h(\tau;\tilde{\xi}) \qquad y^i = y^i(\tau;\tilde{\xi}) \tag{9.15}$$

Further, because β represents a time translation, it follows that

$$h(\tau+\beta;\xi) = h(\tau;\xi) + \text{constant}$$

Substituting (9.13) and (9.15) into (9.14) gives

$$\left\{f(\tau;\tilde{\rho}) - f(\tau';\tilde{\rho})\right\}^2 - \sum_{i=1}^{n-1}\left\{x^i(\tau;\tilde{\rho}) - x^i(\tau';\tilde{\rho})\right\}^2 = \left\{h(\tau;\tilde{\xi}) - h(\tau';\tilde{\xi})\right\}^2 - \sum_{i=1}^{n-1}\left\{y^i(\tau;\tilde{\xi}) - y^i(\tau';\tilde{\xi})\right\}^2$$

Therefore, γ_1 and γ_2 are related by a Poincare transformation. Treating trajectories related by such transforms as distinct, the concept of detector equivalence can be applied to $\mathcal{S}_2$. Note, however, that the concept cannot be applied to $\mathcal{S}_2$(modulo Poincare transformations of the trajectories). This symmetry in the detector responses is quite expected because the Minkowski vacuum is invariant under Poincare transformations. A more general transformation mixes the positive and negative frequency modes with resulting particle creation. (See Chapter 2.) With this technicality settled, the form (9.11) can now be applied as the condition for detector equivalence of trajectories in $\mathcal{S}_2$.

For the quadratic detector, following the same approach, we use (5.8) with the renormalised vacuum contributions set to zero (since $\mathcal{S}_2$ consists exclusively of flat space situations). Thus for this detector

$$R_\gamma^2(\tau) = 2c^2 \left|\langle M\rangle\right|^2 \int_{-\infty}^{\infty} d\eta e^{-i\Delta E\eta} \left\{\left[G_\gamma^+(\eta+\tau,\tau;\tilde{\rho})\right]^2 \theta(-\eta) + \left[G_\gamma^+(\tau,\tau-\eta;\tilde{\rho})\right]^2 \theta(\eta)\right\} \qquad (9.16)$$

From this, for $\gamma_1, \gamma_2 \in \mathcal{S}_2$ to be quadratic detector equivalent the condition is

$$\left[G_{\gamma_1}^+(\tau,\tau';\tilde{\rho})\right]^2 = \left[G_{\gamma_2}^+(\tau,\tau';\tilde{\xi})\right]^2$$

which obviously reduces to (9.11). Thus the linear and quadratic detectors are $\mathcal{S}_2$-equivalent.

For the derivative detector, (6.5) gives the transition rate for $\gamma \in \mathcal{S}_2$.

$$R_\gamma^3(\tau) = c^2 \left|\langle M\rangle\right|^2 b^\mu b^\nu \int_{-\infty}^{\infty} d\eta e^{-i\Delta E\eta} \left\{\partial_\mu \partial'_\nu G_\gamma^+(\eta+\tau,\tau;\tilde{\rho})\Big|\theta(-\eta) + \partial_\mu \partial'_\nu G_\gamma^+(\tau,\tau-\eta;\tilde{\rho})\Big|\theta(\eta)\right\}$$

From this response, the condition for derivative detector equivalence of $\gamma_1, \gamma_2 \in \mathcal{S}_2$ is

$$b^\mu b^\nu \partial_\mu \partial'_\nu G_{\gamma_1}^+(\tau,\tau';\tilde{\rho})\Big| = d^\mu d^\nu \partial_\mu \partial'_\nu G_{\gamma_2}^+\left(\tau+\beta,\tau'+\beta;\tilde{\xi}\right)\Big| \qquad (9.17)$$

In (9.17), b^μ need not equal d^ν since orientation of the detector is part of the description of the situation in which the detector is placed.

From (9.17) it is not obvious that the concept of derivative detector equivalence can be applied to the elements of $\mathcal{S}_2$. Even so, by comparison with (9.11) it is straightforward to see that the linear and derivative detectors are $\mathcal{S}_2$-inequivalent. Considering the time-derivative detector, from (9.17) the condition for detector equivalence of $\gamma_1, \gamma_2 \in \mathcal{S}_2$ is given by setting $b^0 = d^0 = 1$ and $b^i = d^i = 0$ for $i = 1, \ldots, n\text{-}1$. This yields

$$G_{\gamma_1}^+(\tau,\tau';\tilde{\rho}) = G_{\gamma_2}^+(\tau+\beta,\tau'+\beta;\tilde{\rho}) + H(\tau) + K(\tau')$$

However, the time-derivative detector responds only to the $G_{\gamma_1}^+$ and $G_{\gamma_2}^+$ terms (the $H(\tau)$ and $K(\tau')$ give $\delta(\Delta E)$ terms) and so the condition for detector equivalence reduces to (9.11) as for the linear detector. Therefore these two detectors are $\mathcal{S}_2$-equivalent.

9.1.4 Equivalence in $\mathcal{S}_2'$

Since the trajectories in $\mathcal{S}_2'$ are stationary, it follows that the corresponding Wightman function will be a function of $\Delta\tau$.

$$G_\gamma^+(\tau,\rho;\tau',\rho') = G_\gamma^+(\Delta\tau;\rho,\rho') \qquad (9.18)$$

More precisely they will have the form

$$\int d^{n-1}ke^{\pm i\omega\Delta\tau}H_\gamma^+\left(\rho,\rho';k\right) \tag{9.19}$$

Where ω is the frequency parameter in the co-moving frame and H($\rho,\rho';k$) is a function only of space-like coordinates and momentum components (i.e. no τ or ω dependence). Since $\mathcal{S}_2' \subset \mathcal{S}_2$, $\mathcal{S}_2'$-equivalence of the linear and quadratic detectors automatically follows. What of the linear and derivative detectors? From (9.11), using (9.19), the condition for linear detector equivalence on $\mathcal{S}_2'$ can be written as

$$\int d^{n-1}ke^{\pm i\omega\Delta\tau}H_{\gamma_1}^+\left(\tilde{\rho},\tilde{\rho}';k\right)=\int d^{n-1}ke^{\pm i\tilde{\omega}\Delta\tau}H_\gamma^+\left(\tilde{\xi},\tilde{\xi}';k\right) \tag{9.20}$$

and from (9.17) the condition for derivative detector equivalence is

$$b^\mu b^\nu\partial_\mu\partial'_\nu\left[\int d^{n-1}ke^{\pm i\omega\Delta\tau}H_{\gamma_1}^+\left(\tilde{\rho},\tilde{\rho}';k\right)\right]=d^\mu d^\nu\partial_\mu\partial'_\nu\left[\int d^{n-1}ke^{\pm i\tilde{\omega}\Delta\tau}H_{\gamma_2}^+\left(\tilde{\xi},\tilde{\xi}';k\right)\right] \tag{9.21}$$

As with $\mathcal{S}_2$, these two equations in general do not reduce to the same statement. However, for the time-derivative detector we have $b^0 = d^0 = 1$ and $b^i = d^i = 0$ for $i = 1, \ldots, n\text{-}1$ and (9.21) becomes

$$\int d^{n-1}k\omega^2e^{\pm i\omega\Delta\tau}H_{\gamma_1}^+\left(\tilde{\rho},\tilde{\rho}';k\right)=\int d^{n-1}k\tilde{\omega}^2e^{\pm i\tilde{\omega}\Delta\tau}H_\gamma^+\left(\tilde{\xi},\tilde{\xi}';k\right) \tag{9.22}$$

One can see that (9.22) and (9.20) are, in fact, equivalent. Therefore, the linear and time-derivative detectors are $\mathcal{S}_2'$-equivalent. This result concurs with that of the previous section and those in Chapters 4 and 6.

9.1.5 Equivalence in $\mathcal{S}_1 \cup \mathcal{S}_2'$

So far the application of the definition of equivalence has only been within simple sets. We shall now use it to consider how the detectors under study associate the elements of $\mathcal{S}_1$ with those of $\mathcal{S}_2'$.

The response of the linear detector in $n_{\boldsymbol{k}} \in \mathcal{S}_1$ is given in (3.9) and to $\gamma \in \mathcal{S}_2$ is given in (4.7). From these two equations the condition for $n_{\boldsymbol{k}}$ to be linear detector equivalent to γ, which is given by $R^1_{n_{\boldsymbol{k}}} = R^1_\gamma$, reduces to

$$G_\gamma^+\left(\Delta\tau;\tilde{\rho}\right)=G^+\left(\Delta\tau\right)+\frac{2\left(4\pi\right)^{(1-n)/2}}{\Gamma\left(\left(n-1\right)/2\right)}\int_m^\infty d\omega\,\overline{n}_{\left(\omega^2-m^2\right)^{1/2}}\left(\omega^2-m^2\right)^{(n-3)/2}\cos\omega\Delta\tau \tag{9.23}$$

Since the linear detector equivalence classes in $\mathcal{S}_1$ and $\mathcal{S}_2'$ contain more than one element, those in $\mathcal{S}_1 \cup \mathcal{S}_2'$ must likewise contain more than one element. Consequentially the concept of linear detector equivalence is applicable to $\mathcal{S}_1 \cup \mathcal{S}_2'$.

For the quadratic detector, the response to $n_{\boldsymbol{k}} \in \mathcal{S}_1$ is given in (3.16) and to $\gamma \in \mathcal{S}_2'$ in (5.7), with the vacuum term set to zero. Therefore the condition for quadratic detector equivalence on $\mathcal{S}_1 \cup \mathcal{S}_2'$ is

$$\left[G_\gamma^+\left(\Delta\tau;\tilde{\rho}\right)\right]^2=\left[G^+\left(\Delta\tau\right)+\frac{2\left(4\pi\right)^{(1-n)/2}}{\Gamma\left(\left(n-1\right)/2\right)}\int_m^\infty d\omega\,\overline{n}_{\left(\omega^2-m^2\right)^{1/2}}\left(\omega^2-m^2\right)^{(n-3)/2}\cos\omega\Delta\tau\right]^2 \tag{9.24}$$

Where the final term in (3.16) has been discarded because it is independent of $\Delta\tau$ and therefore will not contribute to the detector's response since $\Delta E \neq 0$.

It is immediately seen from (9.23) and (9.24) that the linear and quadratic detectors are $\mathcal{S}_1 \cup \mathcal{S}_2'$-equivalent. Further, from (9.23) the following theorem can be proved;

Theorem 2: In four- and higher-dimensional space-times, linear detector equivalence on $\mathcal{S}_1'\cup\mathcal{S}_2'$ produces an isomorphism between

$$\left\{ n_k \in \mathcal{S}_1' : \int_m^\infty d\omega\, k^{n-3} n_k < \infty \right\}$$

and

$$\left\{ \gamma \in \mathcal{S}_2' \left(\text{modulo Poincare transformations on the trajectories}\right)\right\}$$

Proof: Inspecting (9.23) we note that the right-hand side of this is a Cosine Fourier transform of the type discussed at the beginning of this chapter, hence taking the inverse transform gives

$$n_{\left(\omega^2-m^2\right)^{1/2}}\left(\omega^2-m^2\right)^{(n-3)/2} = (4\pi)^{(n-1)/2}\,\Gamma\left((n-1)/2\right)\int_0^\infty d\Delta\tau\,\left\{G_\gamma^+\left(\Delta\tau;\tilde{\rho}\right)-G^+\left(\Delta\tau\right)\right\}\cos\omega\Delta\tau\big/\pi \quad (9.25)$$

For the transform to provide an isomorphism between n_k and γ, the only requirement that must be checked is the absolute integrability of the arguments (Spiegel 1968, Poularikas 2000). That is, we require

$$\int_m^\infty d\omega\, k^{n-3} n_k < \infty \quad (9.26)$$

Where $k^2 = \omega^2 - m^2$, and

$$\int_0^\infty d\Delta\tau\left|G_\gamma^+\left(\Delta\tau;\tilde{\rho}\right)-G^+\left(\Delta\tau\right)\right| < \infty \quad (9.27)$$

The condition (9.26) is assumed and (9.27) will now be shown to hold for the set designated in the theorem.

Firstly, it must be noted that in evaluating the response of the linear detector to a trajectory $\gamma\in\mathcal{S}_2'$, the integral in (4.7) is improper since the Wightman function has a pole at $\Delta\tau = 0$ which is shifted, by amount ε, into the upper-half complex $\Delta\tau$-plane. This quantity ($-i\varepsilon$) is set to zero only at the end of the calculation (Birrell & Davies 1982). Therefore (9.27) must be evaluated with the pole shifted. This will give convergence of the integral at the lower limit. For $0 < \Delta\tau < \infty$, since the trajectory is time-like both Wightman functions are finite for this range. Convergence at the upper limit is guaranteed for $n \geq 4$ on dimensional grounds since the Wightman functions have dimensions (length)$^{2-n}$.

Q.E.D.

Turning our attention to the derivative detector, the response of this detector to $n_k\in\mathcal{S}_1$ is given in (3.22) and to $\gamma\in\mathcal{S}_2'$ by (6.6). Equating these two responses produces a condition for derivative detector equivalence which is manifestly not identical to that for linear detector equivalence on $\mathcal{S}_1\cup\mathcal{S}_2'$. Therefore these two detectors are $\mathcal{S}_1\cup\mathcal{S}_2'$-inequivalent.

This need not mean that there exists no subset of $\mathcal{S}_1\cup\mathcal{S}_2'$ for which these two detectors are equivalent. In fact, we can use the definition of equivalence to enquire as the existence of such sets. This shall now be done.

Assume that such a subset $\hat{\mathcal{S}}$ of $\mathcal{S}_1'\cup\mathcal{S}_2'$ exists, $\hat{\mathcal{S}}$ -equivalence of the linear and derivative detectors can be used to study the properties of $\gamma\in\hat{\mathcal{S}}$. Note that $\hat{\mathcal{S}}$ is a subset of $\mathcal{S}_1'\cup\mathcal{S}_2'$ **not** $\mathcal{S}_1\cup\mathcal{S}_2'$. Since the general derivative detector and linear detectors are $\mathcal{S}_1$-inequivalent, it is not possible for them to be $\mathcal{S}_1\cup\mathcal{S}_2'$-equivalent. However, we can use $\mathcal{S}_1'$ because, by itself, the concept of detector equivalence is void in that set, hence the two detectors are not $\mathcal{S}_1'$-inequivalent. The conditions for linear detector equivalence in $\hat{\mathcal{S}}$ will be (9.23) since $\mathcal{S}_1\supset\mathcal{S}_1'$. For the purposes of studying the implication of $\hat{\mathcal{S}}$ -equivalence of the two detectors, we use (9.25) for the linear detector.

The response of the derivative detector to $n_k\in\mathcal{S}_1'$ is given by (3.22) and to $\gamma\in\mathcal{S}_2'$ by (6.6). Hence the condition for derivative detector equivalence on $\hat{\mathcal{S}}$ will be $R^3_{n_k}=R^3_\gamma$ which gives

$$
\begin{aligned}
b^\mu b^\nu\,\partial_\mu\partial'_\nu\,G^+_\gamma\left(\Delta\tau;\tilde{\rho}\right)\Big| &= d^\alpha d^\beta\,\partial_\alpha\partial'_\beta\,G^+\left(\Delta\tau\right)\Big| + \\
&+\frac{2\left(4\pi\right)^{(1-n)/2}}{\Gamma\left(\left(n-1\right)/2\right)}\int_m^\infty d\omega\left(\omega^2-m^2\right)^{(n-3)/2}\left\{\begin{array}{l}\left(d^0\omega\right)^2\overline{n}_{\left(\omega^2-m^2\right)^{1/2}}+\left(\omega^2-m^2\right)\sum\limits_{i,j=1}^{n-1}d^id^j\overline{n}_{\left(\omega^2-m^2\right)^{1/2},ij}\\ +2\omega\left(\omega^2-m^2\right)^{1/2}d^0\sum\limits_{i=1}^{n-1}d^i\overline{n}_{\left(\omega^2-m^2\right)^{1/2},0i}\end{array}\right\}
\end{aligned}
\tag{9.28}
$$

Where the $\overline{n}_{\boldsymbol{k},\alpha\beta}$ are given by (3.20) and (3.21) with $\overline{n}_{\boldsymbol{k}}$ is defined in (3.8). Also, in (9.28), the unit b^μ vector represents the orientation in the co-moving frame for $\gamma\in\mathcal{S}_2'$, where-as d^α represents the orientation in the (non-moving) Minkowski space frame. (Recall that the specification of $n_{\boldsymbol{k}}$ is frame dependent due to Doppler effects.)

Re-writing (9.28) for $n_{\boldsymbol{k}}$ isotropic will give, from (9.3) to (9.5)

$$
\begin{aligned}
&\overline{n}_k=n_k\equiv n_kN_{00} && \\
&\overline{n}_{k,ij}=\delta_{ij}n_k/\left(n-1\right)\equiv n_kN_{ij} && i,j=1,...,n-1\\
&\overline{n}_{k,0i}=0\equiv n_kN_{0i} && i=1,...,n-1
\end{aligned}
\tag{9.29}
$$

Due to the isotropy of these states, the spatial orientation of d^a is irrelevant. Therefore, there is no loss of generality if we introduce a pre-defined co-moving coordinate system into both $\mathcal{S}_1'$ and $\mathcal{S}_2'$ situations such that

$$
b^\mu=\delta^\mu_\alpha d^\alpha \tag{9.30}
$$

Just such a coordinate system was introduced in Sec. 6.1 and corresponds to the transform

$$
\partial_\alpha\to\partial_\mu=V^\alpha_\mu\partial_\alpha
$$

In which V^α_μ is the n-bein discussed in that section. Thus (9.28) may be re-expressed in the form

$$
\begin{aligned}
&\int_m^\infty d\omega\left(\omega^2-m^2\right)^{(n-3)/2}\overline{n}_{\left(\omega^2-m^2\right)^{1/2}}\left\{\left(b^0\omega\right)^2+N_{ij}b^ib^j\left(\omega^2-m^2\right)\right\}\cos\omega\Delta\tau=\\
&\frac{\left(4\pi\right)^{(n-1)/2}}{2}\Gamma\left(\left(n-1\right)/2\right)b^\mu b^\nu\left\{\partial_\mu\partial'_\nu\,G^+_\gamma\left(\Delta\tau;\tilde{\rho}\right)\Big|-\delta^\alpha_\mu\delta^\beta_\nu\,\partial_\alpha\partial'_\beta\,G^+\left(\Delta\tau\right)\Big|\right\}
\end{aligned}
$$

Taking the inverse transform gives

$$\overline{n}_{\left(\omega^2-m^2\right)^{1/2}}\left(\omega^2-m^2\right)^{(n-3)/2}=$$

$$\frac{(4\pi)^{(n-1)/2}}{\pi}\Gamma\left((n-1)/2\right)b^{\mu}b^{\nu}\frac{\int\limits_0^{\infty}d\Delta\tau\left\{\partial_{\mu}\partial'_{\nu}\,G_{\gamma}^{+}\left(\Delta\tau;\tilde{\rho}\right)\right|-\delta_{\mu}^{\alpha}\delta_{\nu}^{\beta}\,\partial_{\alpha}\partial'_{\beta}\,G^{+}\left(\Delta\tau\right)\left|\right\}\cos\omega\Delta\tau}{\left\{\left(b^0\omega\right)^2+N_{ij}b^ib^j\left(\omega^2-m^2\right)\right\}} \tag{9.31}$$

By the assumption of $\hat{\mathcal{S}}$ -equivalence of the two detectors, the n_k in (9.25) for the linear detector must be identical to the n_k given by (9.31), so equating these two gives

$$\int\limits_0^{\infty}d\Delta\tau\left\{G_{\gamma}^{+}\left(\Delta\tau;\tilde{\rho}\right)-G^{+}\left(\Delta\tau\right)\right\}\cos\omega\Delta\tau=$$

$$b^{\mu}b^{\nu}\frac{\int\limits_0^{\infty}d\Delta\tau\left\{\partial_{\mu}\partial'_{\nu}\,G_{\gamma}^{+}\left(\Delta\tau;\tilde{\rho}\right)\right|-\delta_{\mu}^{\alpha}\delta_{\nu}^{\beta}\,\partial_{\alpha}\partial'_{\beta}\,G^{+}\left(\Delta\tau\right)\left|\right\}\cos\omega\Delta\tau}{\left\{\left(b^0\omega\right)^2+N_{ij}b^ib^j\left(\omega^2-m^2\right)\right\}} \tag{9.32}$$

Although the functions $n_{\left(\omega^2-m^2\right)^{1/2}}$ are not defined for $\omega< m$, since they have been eliminated from the equation immediately above, we may allow ω to have values less than m provided both sides of (9.32) remain well defined. Taking the inverse Cosine Fourier transform of both sides, with the integral from zero to infinity, gives

$$G_{\gamma}^{+}\left(\Delta\tau';\tilde{\rho}\right)-G^{+}\left(\Delta\tau'\right)=$$

$$\frac{2b^{\mu}b^{\nu}}{\pi}\int\limits_0^{\infty}d\Delta\tau\left\{\partial_{\mu}\partial'_{\nu}\,G_{\gamma}^{+}\left(\Delta\tau;\tilde{\rho}\right)\right|-\delta_{\mu}^{\alpha}\delta_{\nu}^{\beta}\,\partial_{\alpha}\partial'_{\beta}\,G^{+}\left(\Delta\tau\right)\left|\right\}\int\limits_0^{\infty}d\omega\frac{\cos\omega\Delta\tau\cos\omega\Delta\tau'}{\left\{\left(b^0\omega\right)^2+N_{ij}b^ib^j\left(\omega^2-m^2\right)\right\}} \tag{9.33}$$

Fortunately the ω-integral is well defined and can be evaluated as follows. Define $\chi=\Delta\tau-\Delta\tau'$ and $\overline{\chi}=\Delta\tau+\Delta\tau'$, this gives for that integral

$$\frac{1}{2A^2}\int\limits_0^{\infty}d\omega\frac{\cos\chi\omega+\cos\overline{\chi}\omega}{\left(\omega^2-B^2\right)}$$

where A and B are defined as

$$A^2=\left(b^0\right)^2+N_{ij}b^ib^j=N_{\alpha\beta}b^{\alpha}b^{\beta}$$
$$B\ =m^2N_{ij}b^ib^j\big/N_{\alpha\beta}b^{\alpha}b^{\beta}$$

From Gradshteyn and Ryzhik (1980) No.3.723.9

$$\int\limits_0^{\infty}d\omega\frac{\cos b\omega}{\left(\omega^2-B^2\right)}=\frac{-\pi}{2B}\sin bB\qquad\qquad b>0,\,B>0 \tag{9.34}$$

Therefore, for $\chi>0,\,\overline{\chi}>0$

$$\int_0^\infty d\omega \frac{\cos\chi\omega + \cos\bar{\chi}\omega}{\left(\omega^2 - B^2\right)} = \frac{-\pi}{2B}\left(\sin\chi B + \sin\bar{\chi} B\right)$$

and for $\chi < 0, \bar{\chi} > 0$

$$\int_0^\infty d\omega \frac{\cos\chi\omega + \cos\bar{\chi}\omega}{\left(\omega^2 - B^2\right)} = \frac{-\pi}{2B}\left(-\sin\chi B + \sin\bar{\chi} B\right)$$

Placing these terms into (9.33) and using some trigonometry results with

$$G_\gamma^+\left(\Delta\tau';\tilde{\rho}\right) - G^+\left(\Delta\tau'\right) = \frac{-b^\mu b^\nu}{A^2 B}\begin{Bmatrix} \sin\left(B\Delta\tau'\right)\int_0^{\Delta\tau'} d\Delta\tau\left\{\partial_\mu\partial'_\nu G_\gamma^+\left(\Delta\tau;\tilde{\rho}\right)\right| - \delta_\mu^\alpha\delta_\nu^\beta\,\partial_\alpha\partial'_\beta\, G^+\left(\Delta\tau\right)\left|\right\}\cos\left(B\Delta\tau\right) + \\ +\cos\left(B\Delta\tau'\right)\int_{\Delta\tau'}^{\infty} d\Delta\tau\left\{\partial_\mu\partial'_\nu G_\gamma^+\left(\Delta\tau;\tilde{\rho}\right)\right| - \delta_\mu^\alpha\delta_\nu^\beta\,\partial_\alpha\partial'_\beta\, G^+\left(\Delta\tau\right)\left|\right\}\sin\left(B\Delta\tau\right) \end{Bmatrix} \qquad (9.35)$$

(N.B. This equation is not a tautology reflecting the $\mathcal{S}_1 \cup \mathcal{S}_2'$-inequivalence of linear and derivative detectors.) This equality (9.35) is a mathematical representation of the condition required to be satisfied by all trajectories $\gamma \in \hat{\mathcal{S}}$, for the linear and derivative detectors to be $\hat{\mathcal{S}}$-equivalent.

In its present form, it is not obvious from (9.35) what the structure of $G_\gamma^+\left(\Delta\tau;\rho\right)$ must be for the equality to hold. To simplify matters, for the general derivative detector, we shall take the massless limit of (9.35), hence giving the condition on the massless field Wightman function $D_\gamma^+\left(\Delta\tau;\rho\right)$. Before this can be done, a check must be made that all the steps deducing (9.35) from (9.32) are valid in the limit m = 0. In particular, the integral (9.34) as written requires m > 0. However, taking the limit $m \to 0$ both sides of that equation yields

$$\int_0^\infty d\omega \frac{\cos b\omega}{\omega^2} = \frac{-\pi b}{2} \qquad b > 0 \qquad (9.36)$$

This result is easily verified since the left-hand side is the Fourier transform of ω^{-2}.

$$\int_0^\infty d\omega \frac{\cos b\omega}{\omega^2} = \frac{1}{2}\int_{-\infty}^\infty d\omega \frac{e^{ib\omega}}{\omega^2}$$

Evaluating the Fourier transforms by a contour integral gives

$$\int_0^\infty d\omega \frac{\cos b\omega}{\omega^2} = \frac{-\pi}{2}|b| \qquad (9.37)$$

Which agrees with (9.36). Since all the other manipulations used to deduce (9.35) are manifestly valid for the massless field and since 0 ≤$\Delta\tau$<∞, for the massless field (9.35) yields

$$D_\gamma^+\left(\Delta\tau';\tilde{\rho}\right)-D^+\left(\Delta\tau'\right)=$$
$$\frac{-b^\mu b^\nu}{b^\alpha b^\alpha N_{\alpha\beta}}\left\{\begin{array}{l}\Delta\tau'\int_0^{\Delta\tau'}d\Delta\tau\left\{\partial_\mu\partial'_\nu D_\gamma^+\left(\Delta\tau;\tilde{\rho}\right)\Big|-\delta_\mu^\sigma\delta_\nu^\kappa\,\partial_\sigma\partial'_\kappa D^+\left(\Delta\tau\right)\Big|\right\}+\\+\int_{\Delta\tau'}^{\infty}d\Delta\tau\,\Delta\tau\left\{\partial_\mu\partial'_\nu D_\gamma^+\left(\Delta\tau;\tilde{\rho}\right)\Big|-\delta_\mu^\sigma\delta_\nu^\kappa\,\partial_\sigma\partial'_\kappa D^+\left(\Delta\tau\right)\Big|\right\}\end{array}\right\} \quad (9.38)$$

Differentiating with respect to $\Delta\tau'$ and dropping the primes;

$$\frac{\partial^2}{\partial\left(\Delta\tau\right)^2}\left(D_\gamma^+\left(\Delta\tau';\tilde{\rho}\right)-D^+\left(\Delta\tau'\right)\right)=\frac{-b^\mu b^\nu}{b^\alpha b^\alpha N_{\alpha\beta}}\left\{\partial_\mu\partial'_\nu D_\gamma^+\left(\Delta\tau;\tilde{\rho}\right)\Big|-\delta_\mu^\sigma\delta_\nu^\kappa\,\partial_\sigma\partial'_\kappa D^+\left(\Delta\tau\right)\Big|\right\}$$

Since $\Delta\tau = \tau - \tau'$, this can be written as

$$\frac{\partial^2}{\partial\tau\partial\tau'}\left(D_\gamma^+\left(\Delta\tau';\tilde{\rho}\right)-D^+\left(\Delta\tau'\right)\right)=\frac{-b^\mu b^\nu}{b^\alpha b^\alpha N_{\alpha\beta}}\left\{\partial_\mu\partial'_\nu D_\gamma^+\left(\Delta\tau;\tilde{\rho}\right)\Big|-\delta_\mu^\sigma\delta_\nu^\kappa\,\partial_\sigma\partial'_\kappa D^+\left(\Delta\tau\right)\Big|\right\} \quad (9.39)$$

Although (9.39) was deduced for $0 \le \Delta\tau < \infty$, the steps used to produce this equality can easily be repeated for $-\infty \le \Delta\tau < \infty$ by using the fact that cos is even in $\Delta\tau$. Therefore (9.39) holds for all $\Delta\tau$.

In an n-dimensional Minkowski space, the Wightman function $D^+(x,x')$ has the form

$$\begin{aligned}D^+\left(x,x'\right)&=-F(n)\left[\left(\Delta\tau-i\varepsilon\right)^2-\left|\boldsymbol{x}-\boldsymbol{x}'\right|^2\right]^{(2-n)/2} && n>2\\&=-F(2)\ln\left[\left(\Delta\tau-i\varepsilon\right)^2-\left|\boldsymbol{x}-\boldsymbol{x}'\right|^2\right]^{(2-n)/2} && n=2\end{aligned}$$

Where $F(n)$ is a constant dependent upon the space-time dimensionality.

Therefore

$$\begin{aligned}\frac{\partial^2}{\partial\tau\partial\tau'}D^+\left(\Delta\tau\right)=\frac{\partial^2}{\partial\tau\partial\tau'}D^+\left(\Delta\tau\right)\Big|&=\\&=F(n)(n-1)(n-2)\left(\Delta\tau-i\varepsilon\right)^{-n} && n>2\\&=-2F(2)\left(\Delta\tau-i\varepsilon\right)^{-2} && n=2\end{aligned}$$

And for $i,j = 1,\ldots, n$-1

$$\begin{aligned}\frac{\partial^2}{\partial x^i\partial x'^j}D^+\left(\Delta\tau\right)\Big|&=F(n)(n-2)\delta_{ij}\left(\Delta\tau-i\varepsilon\right)^{-n} && n>2\\&=-2F(2)\left(\Delta\tau-i\varepsilon\right)^{-2} && n=2\end{aligned}$$

$$\frac{\partial^2}{\partial x^i\partial\tau'}D^+\left(\Delta\tau\right)\Big|=\frac{\partial^2}{\partial\tau\partial x'^i}D^+\left(\Delta\tau\right)\Big|=0$$

Substituting these into (9.39) gives for n>2

$$b^{\sigma}b^{\kappa}N_{\sigma\kappa}\frac{\partial^2}{\partial\tau\partial\tau'}D_{\gamma}^{+}\left(\Delta\tau;\tilde{\rho}\right)-b^{\sigma}b^{\kappa}\,\partial_{\sigma}\partial'_{\kappa}\,D^{+}\left(\Delta\tau;\tilde{\rho}\right)\Big| =$$
$$=\frac{F(n)(n-2)(n-1)}{\left(\Delta\tau-i\varepsilon\right)^{n}}b^{\sigma}b^{\kappa}\left\{N_{\sigma\kappa}-\delta_{\sigma}^{0}\delta_{\kappa}^{0}-\frac{1}{(n-1)}\sum_{i=1}^{n-1}\delta_{\sigma}^{i}\delta_{\kappa}^{i}\right\}$$

Using (9.29), the right-hand side of this equality vanishes. This leaves

$$\sum_{i=1}^{n-1}\left(b^{i}\right)^{2}\frac{\partial^2}{\partial\tau\partial\tau'}D_{\gamma}^{+}\left(\Delta\tau;\tilde{\rho}\right)=(n-1)\left\{b^{0}\sum_{i=1}^{n-1}b^{i}\left(\frac{\partial^2}{\partial\tau\partial\rho'^{i}}+\frac{\partial^2}{\partial\rho^{i}\partial\tau'}\right)+\sum_{i,j=1}^{n-1}b^{i}b^{j}\left(\frac{\partial^2}{\partial\rho^{i}\partial\rho'^{j}}\right)\right\}D_{\gamma}^{+}\left(\Delta\tau;\tilde{\rho}\right)\Big| \quad (9.40)$$

In general this condition will not be automatically satisfied, hence if there exists a set $\hat{\mathcal{S}} \subset \mathcal{S}_1' \cup \mathcal{S}_2'$ such that the linear and derivative detectors are $\hat{\mathcal{S}}$ -equivalent, then $\hat{\mathcal{S}}$ is a proper subset of $\mathcal{S}_1' \cup \mathcal{S}_2'$.

For the two-dimensional case, the condition of equivalence is

$$\left(b^{1}\right)^{2}\frac{\partial^2}{\partial\tau\partial\tau'}D_{\gamma}^{+}\left(\Delta\tau;\tilde{\rho}\right)=\left\{b^{0}b^{1}\left(\frac{\partial^2}{\partial\tau\partial\rho'}+\frac{\partial^2}{\partial\rho\partial\tau'}\right)+\left(b^{1}\right)^{2}\frac{\partial^2}{\partial\rho\partial\rho'}\right\}D_{\gamma}^{+}\left(\Delta\tau;\tilde{\rho}\right)\Big| \qquad (9.41)$$

Referring back to Sec. 6.2, we see that the Wightman function for two-dimensional Rindler space satisfies (9.41). This is consistent with the results of that section where the derivative was found to respond thermally in two-dimensional Rindler space. Therefore, although the linear and general derivative detectors of the massless field may not be $\mathcal{S}_1' \cup \mathcal{S}_2'$-equivalent in two dimensions, the subset $\hat{\mathcal{S}}$ of $\mathcal{S}_1' \cup \mathcal{S}_2'$ for which they are equivalent contains at least two members (i.e. Planck radiation and Rindler space).

This example beautifully illustrates the fundamental role these Killing vectors play in determining the response of a detector. It responds only to those modes that are not positive frequency with respect to its own trajectory. The importance of these vector fields goes beyond the above example because it is only in such stationary (or time independent) situations that the concept of “particle content” can be associated with the detector’s response.

Consider a space-time in which there is no time-like Killing vector field, except in the asymptotic future and past regions. Then an “in-vacuum” may be defined with respect to the Killing vector in the asymptotic past and an “out-vacuum” with respect to the Killing vector in the asymptotic future. A detector with registers no particles in the in-vacuum will register particles in the out region (Birrell & Davies 1982). Furthermore, the response (rate) of the detector in the intermediate region, between the two vacua, <u>will not</u> be a measurement of the particle content or rate of creation of particles in that region. In fact, in the intermediate region the concepts of “particle content” and “particle creation rate” become vague if not meaningless.

The reason for this comes directly from the role of the time-like Killing vectors. To define “particles” (time independently) a Fock space is required which, in turn, requires a well-defined vacuum state. Without a time-like Killing vector, no such state can be constructed. Even if, for reasons of symmetry, a particular definition of particles can be achieved, the particle number will not be constant and hence its measurement is intrinsically uncertain. Parker (1969) has shown that if the average particle creation rate, of particles mass m, over an interval Δt is A, then the total uncertainty in the particle number, ΔN, over Δt is

$$\Delta N \geq \left(m\Delta t\right)^{-1}+\left|A\right|\Delta t$$

Which has a minimum value $2(|A|/m)^{1/2}$ when $\Delta t = (m|A|)^{-1/2}$. Therefore, provided $A \neq 0$ and $m \neq \infty$, the inherent uncertainty in N is non-zero. So, it can be seen from the above that, except in time independent situations, the response of a particle detector at some time t cannot be considered an accurate measure of the (rate of change of) particle content at that instant.

Having noted this, it must be remembered that here we are primarily interested in comparing the responses of different detectors, rather than trying to associate direct physical meaning to these responses. So we can apply the criterion of detector equivalence to these time dependent situations.

The actual application of the criterion is quite simple. In Chapter 5 the quadratic detector was shown to couple to the (renormalised) vacuum expectation $\left\langle \phi^2\left[x(\tau)\right]\right\rangle_{ren}$, whereas the linear detector and derivative detectors do not. Further, in Sec. 5.1, we saw that only in time independent did this vacuum expectation <u>not</u> contribute to the quadratic detector's response. In the set of such situations, from comparing (4.7) and (5.7), the linear and quadratic detectors are equivalent. Although these two detectors are not $\mathcal{S}_3$-equivalent, defining $\mathcal{S}_3' \subset \mathcal{S}_3$ by

$\mathcal{S}_3'$ = {all time independent situations in $\mathcal{S}_3$}

the linear and quadratic detectors are $\mathcal{S}_3'$-equivalent. The derivative detector's $\mathcal{S}_3$-inequivalence to the linear detector follows immediately from the respective responses (4.7) and (6.5) of these two detectors in $\mathcal{S}_2$. These responses correspond to conditions for detector equivalence given by (9.11) and (9.17) respectively. Again, for the same reason as with $\mathcal{S}_2$, the time-derivative and linear detectors are $\mathcal{S}_3$-equivalent.

9.2 Comparing orientable and directional detectors

Orientable detectors, such as the derivative detector, and directional detectors, such as the cone and spike, have in common the fact that they can be "pointed" in a certain direction (with respect to a suitable chosen reference). Therefore, when describing a situation into which these types of detectors are placed, the orientation of the detector must be specified.

9.2.1 Equivalence in $\mathcal{S}_1'$

Since $\mathcal{S}_1'$ consists of isotropic particle states the idea of orientation is, strictly speaking, meaningless since all orientations are equivalent due to the symmetry of the state. However, by introducing a (spherical) coordinate system this symmetry can be (artificially) broken. For each of the detectors under consideration there are now an infinite number of orientations which, for a given n_k, may be viewed as different situations.

This is basically the same manoeuvre as that used with $\mathcal{S}_2$ above. In that case situations that were identical modulo Poincare transformations were viewed as distinct. In this case, situations that are identical modulo rotations of the detector orientation are being viewed as distinct. This step was of little use when comparing omni-directional detectors in Sec. 9.1.1 since two of the detectors being studied were non-orientable.

For the derivative detector, from (3.19), (9.4) and (9.5) with a given (isotropic) particle state, n_k, R_{n_k} will be uniquely determined. The same is true for the spike detector (from (3.28)) and the cone (from (3.34)). Therefore it follows that for all three detectors each detector-equivalent class of situations will contain an infinite number of elements. The classes will be indexed by the particle state n_k and their elements by the orientation of the detectors. Furthermore, it also follows that the detector-equivalent classes for each detector will be identical to those for the other two detectors and so these three detectors are $\mathcal{S}_1'$-equivalent.

Although detector equivalence on $\mathcal{S}_1'$ can be made meaningful for orientable detectors and directional detectors, it is rather artificial.

9.2.2 Equivalence in $\mathcal{S}_1$

For the set of anisotropic particle states, these three detectors are in-equivalent. This follows from the dis-similar forms of (3.20), (3.21) for the derivative detector, (3.28) for the spike and (3.35) for the cone. These differences are basically due to the fact that each detector has access to different subsets of the mode directions in momentum space.

Due to the calculational difficulties of explicitly describing the spike and cone detectors for situations in $\mathcal{S}_2$ and $\mathcal{S}_3$, equivalence on these sets will not be discussed at this point. However, in Chapter 11 we shall see that by studying the details of how detectors work, general statements about the cone and spike detectors on these sets can be made.

10 Topological effects

In Chapter 8 a list of five factors which describe any situation into which a particle detector may be places was presented. Item four of that list was the topology of the space-time. In this chapter, the effect of a non-trivial topology will be investigated by considering the response of a uniformly accelerating detector in a flat two-dimensional space-time with $\mathcal{R}^1 \times S^1$ topology.

This situation is interesting in its own right because a uniformly accelerating observer in this space-time has no event horizon. So, we can gain some insight into the relationship between the thermal character of the acceleration radiation and the presence of event horizons in the frame of the observer. Gibbons and Hawking (1977) as well as Sciama et al. (1981) state that there is an intimate association between the two, whereas Sanchez (1981) apparently disagrees.

In a normal $\mathcal{R}^2$ flat space-time, a uniformly accelerating observer follows the hyperbolic trajectory shown in Figure 1. For such an observer, the lines $u = 0$ and $v = 0$ represent the future and past event horizons, respectively. The region to the right of these two lines is often called the Rindler Wedge and it has been generally assumed that the thermal acceleration radiation is best regarded as a global phenomenon owing its origin to the causal structure of this region. On the other hand, in an $\mathcal{R}^1 \times S^1$ space-time shown in Figure 2, due to the compact spatial section, the global causal structure of the space is drastically altered. Locally, however, it is still identical to $\mathcal{R}^2$ Rindler space.

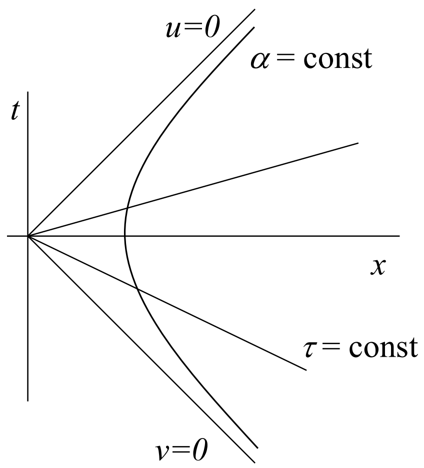


Figure 1 The Rindler wedge is the right hand region between the null lines u = 0 (future event horizon) and v = 0 (past event horizon). The detector trajectory is represented by the hyperbola which is the world-line of the particle moving with constant acceleration (α = const). The line τ = const is a line of constant proper time.

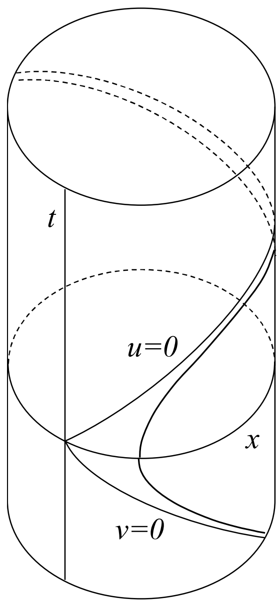


Figure 2 For the $\mathcal{R}^1$xS^1 space-time, the uniformly accelerating trajectory winds around the "cylinder". The winding results in regions of the space-time being causally connected to the trajectory of the detector. Hence there is no event horizon.

With the compactified dimension, the space no longer possesses event horizons in the frame of the accelerated observer. This fact is easily appreciated from Figure 2 since the asymptotic null ray, to which the detector's world-line tends at infinite proper-time, winds around the space-time cylinder. Thus there are no events causally disconnected from the detector's world-line lying above u = 0 or below v = 0. By contrast, in the Rindler Wedge some of the information about the quantum state in the region to the left of the diagram will forever be inaccessible to the accelerating detector. This forfeiture of information is neatly identified with the thermal nature of the detector's response. The question then arises as to what extent the thermal acceleration radiation can be attributed to the existence of the event horizon. In this example it will be shown that in the absence of the event horizon, a uniformly accelerating linear detector still responds as though immersed in a time independent bath of radiation with a continuous Planck spectrum if the field is untwisted (i.e. periodic boundary conditions $\phi[x+nL]$ = $\phi[x]$, where n is an integer and L the length associated with S^1). However, for a twisted field (in which case the boundary conditions are $\phi[x+nL]$

= $(-1)^n\phi[x]$), there appears to be an extra time dependent term in the response in addition to the steady Planck factor.

The continuous nature of the detector's response was emphasised above because the particle states in $\mathcal{R}^1 \text{x} S^1$ space have a discrete energy spectrum due to the compactification. Therefore, strictly speaking, the response is not an $\mathcal{R}^1 \text{x} S^1$ Planck response since this has occupation numbers, n_k, with k discrete ($k = 2\pi n/L$ for untwisted, $2\pi(n+1/2)/L$ for twisted). This is another example of the effects of non-trivial topologies on detectors. In this particular case it arises from the fact that although the momentum modes for the inertial observer may be discrete, the mixing of the positive and negative frequency modes, due to the non-inertial motion, can result in a continuous spectrum.

This effect is not unique to these topologies; it occurs even in flat space in the Rindler situation. Calculation of the massive scalar field linear detector response, when undergoing uniform acceleration, shows that there is no mass cut-off in the resulting spectrum (i.e. the $\theta(\Delta E\text{-}m)$ term) in (3.7) does not appear). Although the response of the stationary detector to an isotropic Planck bath has this cut-off, that of the uniformly accelerating detector does not. (Midorikowa 1981, Walker 1983) This effect is also due to the mixing of positive and negative frequency modes not respecting restrictions on the modes in the inertial frame.

To evaluate the linear detector's response when undergoing uniform acceleration in flat R^1 x S^1 space, we can use the equations in Chapter 4, however in this case the vacuum state $|0\rangle$ must be the Casimir vacuum (Birrell & Davies 1982). Using (3.3)

$$A^1 = ic\langle M\rangle \int_{-\infty}^{\infty} d\tau e^{i\Delta E\tau} \langle \Psi \,|\, \phi\left[x(\tau)\right] |\, \Psi_0 \rangle \tag{10.1}$$

In this equation the field $\phi[x]$ can be expanded in terms of discrete modes

$$u_k(x) = \exp(ikx - i\omega t)/(2L\omega)^{1/2} \tag{10.2}$$

with

$$k = 2\pi n/L \qquad\qquad n = 0, \pm 1, \pm 2, ... \tag{10.3}$$

Introducing null coordinates $u = t - x$ and $v = t + x$ we have

$$\begin{aligned}
\vec{u}_k(t,x) &= \exp\left[2\pi ni(x-t)/L\right]/(2L\omega)^{1/2} \qquad\qquad \omega = k, n > 0 \\
&= \exp\left[-2\pi niu/L\right]/(2L\omega)^{1/2} \\
\overleftarrow{u}_k(t,x) &= \exp\left[-2\pi ni(x+t)/L\right]/(2L\omega)^{1/2} \qquad\qquad \omega = -k, n > 0 \\
&= \exp\left[-2\pi niv/L\right]/(2L\omega)^{1/2}
\end{aligned}$$

In this equation, $\vec{u}_k$ represents modes propagating to the right and $\overleftarrow{u}_k$ represents the modes propagating to the left. Splitting the ϕ-field in a corresponding way, one may write for the expectation value (10.1)

$$\begin{aligned}
\langle \vec{1}_k \left| \phi[x] \right| 0 \rangle &= \exp\left[2\pi inu/L\right]/(4\pi n)^{1/2} \\
\langle \overleftarrow{1}_k \left| \phi[x] \right| 0 \rangle &= \exp\left[2\pi inv/L\right]/(4\pi n)^{1/2}
\end{aligned} \tag{10.4}$$

In which $\left|\vec{1}_k\right\rangle$ represents a one particle right moving momentum state and $\left|\overleftarrow{1}_k\right\rangle$ represents the corresponding left moving state. The world-line of a uniformly accelerating detector in R^1 x S^1 space is described piecewise by

$$\begin{aligned} x &= \xi \cosh \tilde{\tau} - mL \\ t &= \xi \sinh \tilde{\tau} \end{aligned} \tag{10.5}$$

where the proper-time $\tau = \xi\tilde{\tau}$, ξ^1 is the proper acceleration and m in integer with the interpretation of "winding number" (i.e. the number of times the detector has orbited the cylinder). The compactification of the space breaks the Lorentz symmetry and introduces a privileged frame. We choose the winding number m = 0 to include the portion of the trajectory (10.5) at which the detector is at rest with respect to the privileged frame ($\tilde{\tau} = 0$). So m > 0 corresponds to $\tilde{\tau} > 0$. The integral in (10.1) decomposes as follows

$$\int_{-\infty}^{\infty} d\tau \Rightarrow \xi \sum_{m=-\infty}^{\infty} \int_{\tilde{\tau}_m}^{\tilde{\tau}_{m+1}} d\tilde{\tau} = \xi \sum_{m=0}^{\infty} \left\{ \int_{\tilde{\tau}_m}^{\tilde{\tau}_{m+1}} d\tilde{\tau} + \int_{-\tilde{\tau}_{m+1}}^{-\tilde{\tau}_m} d\tilde{\tau} \right\} \tag{10.6}$$

Where $\tilde{\tau}_m = \operatorname{arccosh}\left(mL/\xi\right)$ for $m > 0$; $\tilde{\tau}_m = -\operatorname{arccosh}\left(mL/\xi\right)$ for $m < 0$. Substituting (10.5) into (10.4) which is then placed into (10.1) and using (10.6) we obtain

$$\vec{A}^1 = ic\langle M\rangle \xi \sum_{m=-\infty}^{\infty} \int_{\tilde{\tau}_m}^{\tilde{\tau}_{m+1}} \frac{d\tilde{\tau}}{\left(4\pi n\right)^{1/2}} \left\{ \exp\left[\frac{2\pi i n \xi\left(\sinh\tilde{\tau} - \cosh\tilde{\tau}\right)}{L} + 2\pi i n m \right] \exp\left(i\Delta E \xi \tilde{\tau}\right) \right\} \tag{10.7}$$

$$\overleftarrow{A}^1 = ic\langle M\rangle \xi \sum_{m=-\infty}^{\infty} \int_{\tilde{\tau}_m}^{\tilde{\tau}_{m+1}} \frac{d\tilde{\tau}}{\left(4\pi n\right)^{1/2}} \left\{ \exp\left[\frac{2\pi i n \xi\left(\sinh\tilde{\tau} + \cosh\tilde{\tau}\right)}{L} + 2\pi i n m \right] \exp\left(i\Delta E \xi \tilde{\tau}\right) \right\} \tag{10.8}$$

where $\vec{A}^1$ and $\overleftarrow{A}^1$ are the contributions to the amplitude from the modes $\left|\vec{1}_k\right\rangle$ and $\left|\overleftarrow{1}_k\right\rangle$ respectively.

To evaluate (10.7) and (10.8) the sums are split into two parts; the m = 0 strip and $|m|$ > 0. The m = 0 part for $\vec{A}^1$ gives

$$\int_{-\operatorname{arccosh}(L/\xi)}^{\operatorname{arccosh}(L/\xi)} d\tilde{\tau} \exp\left[-2\pi i n \xi e^{-\tilde{\tau}}/L\right] \exp\left(i\xi\Delta E\tilde{\tau}\right) \tag{10.9}$$

Let $z = \exp\left(-\tilde{\tau}\right)$ and $b = \operatorname{arccosh}\left(L/\xi\right)$, then (10.9) becomes

$$\int_{-b}^{b} dz\, z^{-i\xi\Delta E - 1} \exp\left(-2\pi i n \xi z/L\right)$$

Using

$$\int_{y}^{\infty} dx\, x^{\mu-1} e^{\pm iax} = a^{-\mu} e^{\pm i\mu\pi/2} \Gamma\left(\mu, \mp iay\right) \qquad \operatorname{Re}\mu < 1$$

(10.9) yields

$$\exp\left(-\pi\xi\Delta E/2\right) a^{i\Delta E\xi} \left[\Gamma\left(-i\xi\Delta E, iae^{-b}\right) - \Gamma\left(-i\xi\Delta E, iae^{b}\right) \right] \tag{10.10}$$

Where $a = 2\pi n\xi/L$. For $\vec{A}^1$ component, the m = 0 term is

$$\int_{-\operatorname{arccosh}(L/\xi)}^{\operatorname{arccosh}(L/\xi)} d\tilde{\tau}\exp\left[2\pi in\xi e^{-\tilde{\tau}}/L\right]\exp\left(i\xi\Delta E\tilde{\tau}\right)$$

Using $z = \exp(-\tilde{\tau})$ this becomes

$$\int_{-b}^{b} dz\, z^{i\xi\Delta E-1}\exp\left(2\pi in\xi z/L\right) = \exp\left(-\pi\xi\Delta E/2\right)a^{-i\Delta E\xi}\left[\Gamma\left(i\xi\Delta E,-iae^{-b}\right)-\Gamma\left(i\xi\Delta E,-iae^{b}\right)\right] \quad (10.11)$$

We note that (10.10) and (10.11) are complex conjugates. The m > 0 contributions to $\vec{A}^1$ is, from (10.6) and (10.7),

$$\sum_{m=1}^{\infty}\int_{\tilde{\tau}_m}^{\tilde{\tau}_{m+1}} d\tilde{\tau}\exp\left(-2\pi in\xi e^{-\tilde{\tau}}/L\right)\exp\left(i\xi\Delta E\tilde{\tau}\right)+\int_{-\tilde{\tau}_{m+1}}^{\tilde{\tau}_m} d\tilde{\tau}\exp\left(-2\pi in\xi e^{-\tilde{\tau}}/L\right)\exp\left(i\xi\Delta E\tilde{\tau}\right)$$

The integrals in this sum can be evaluated in the same manner as those for the m = 0 strip.

$$\exp\left(-\pi\xi\Delta E/2\right)a^{i\Delta E\xi}\sum_{m=1}^{\infty}\begin{bmatrix}\Gamma\left(-i\xi\Delta E, iaf(m)\right)-\Gamma\left(-i\xi\Delta E, ia/f(m)\right)\\ -\Gamma\left(-i\xi\Delta E, iaf(m+1)\right)+\Gamma\left(-i\xi\Delta E, ia/f(m+1)\right)\end{bmatrix} \quad (10.12)$$

Were $f(m)$=exp[arccosh(mL/ξ)]. Similarly, the m > 0 contribution to (10.8) is

$$\exp\left(-\pi\xi\Delta E/2\right)a^{-i\Delta E\xi}\sum_{m=1}^{\infty}\begin{bmatrix}\Gamma\left(i\xi\Delta E, -iaf(m)\right)-\Gamma\left(i\xi\Delta E, -ia/f(m)\right)\\ -\Gamma\left(i\xi\Delta E, -iaf(m+1)\right)+\Gamma\left(i\xi\Delta E, -ia/f(m+1)\right)\end{bmatrix} \quad (10.13)$$

These sums are readily evaluated by noting that the first two terms in the summands, for each m, cancel the second two terms for the previous value of m. Evaluating the infinite sums using $\sum_{m=1}^{\infty}X_m = \lim_{N\to\infty}\sum_{m=1}^{N}X_m$ and noting that the remaining m = 1 terms are exactly cancelled by the m = 0 terms, we find (10.7) and (10.8) give

$$\begin{aligned}\vec{A}^1 &= ic\xi\langle M\rangle e^{-\pi\xi\Delta E/2}\lim_{m\to\infty}\left[\Gamma\left(-i\Delta E\xi, ia/f(m+1)\right)-\Gamma\left(-i\Delta E\xi, iaf(m+1)\right)\right]a^{-i\Delta E\xi}\Big/\left(4\pi n\right)^{1/2}\\ &= -\left[\overleftarrow{A}^1\right]^*\end{aligned}$$

Using the integral representation of the Gamma function, we analytically continue into the complex plane as shown in Figure 3. Taking the limit gives

$$\begin{aligned}\vec{A}^1 &= ic\xi\langle M\rangle e^{-\pi\xi\Delta E/2}\,\Gamma\left(-i\Delta E\xi\right)\big/\left(4\pi n\right)^{1/2}\\ &= -\left[\overleftarrow{A}^1\right]^*\end{aligned} \quad (10.14)$$

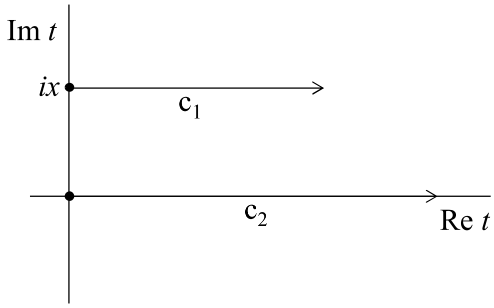


Figure 3 Using the integral representation of the incomplete Gamma function given in the text, the contour c_2 represents the complete Gamma function and the contour c_1 represents the analytic continuation of the incomplete Gamma function into the complex t-plane, used to evaluate $\Gamma(a,ix)$ where $x \in R$.

The transition probability is given by

$$P^1 = \sum_n \left|\vec{A}^1\right|^2 + \left|\overleftarrow{A}^1\right|^2 = \frac{c^2 \xi \left|\langle M \rangle\right|^2}{2\pi} e^{-\pi\xi\Delta E} \left|\Gamma\left(i\Delta E\xi\right)\right|^2 \sum_{n=1}^{\infty}\left(\xi/n\right) \tag{10.15}$$

Using the identity

$$\left|\Gamma\left(ix\right)\right|^2 = \frac{\pi}{x \sinh \pi x}$$

(10.15) reduces to

$$P^1 = \frac{c^2 \left|\langle M \rangle\right|^2}{\Delta E\left(e^{2\pi\Delta E\xi} - 1\right)} \sum_{n=1}^{\infty}\left(\xi/2n\right) \tag{10.16}$$

The total transition probability is divergent, as expected since the detector is accelerating (and responding) for all time. Dividing out the $\sum_{n=1}^{\infty}\left(\xi/2n\right)$ yields a constant, finite transition rate identical to the usual Planckian spectrum and is continuous. In summary, in spite of the absence of an event horizon, the (thermal) Rindler character of the detector's response emerges.

The existence of a non-trivial topology in this model allows the possibility of anti-periodic boundary conditions, or a twisted field configuration (Isham 1978). Interest then attaches to the response of the accelerated detector to the twisted vacuum. In particular, will the detector differentiate between the twisted and un-twisted vacua.

The anti-periodic boundary conditions may be achieved by replacing (10.3) by

$$k = 2\pi\left(n + \tfrac{1}{2}\right)/L \tag{10.17}$$

The calculation of the detector's response then proceeds along the lines of the un-twisted case above. However, in this case the transition amplitude summations analogous to (10.7) and(10.8) cannot be evaluated by simply noting the cancellation of successive terms, due to the alternating sign arising from the anti-periodicity of the boundary conditions. Never the less, an approximate calculation of the transition rate for L >> ξ can be performed.

In the twisted case we have, in place of (10.4)

$$\left\langle \vec{1}_k \left| \phi[x] \right| 0 \right\rangle = \exp\left[\pi i (2n+1) u / L \right] \Big/ (4\pi n)^{1/2}$$
$$\left\langle \overleftarrow{1}_k \left| \phi[x] \right| 0 \right\rangle = \exp\left[\pi i (2n+1) v / L \right] \Big/ (4\pi n)^{1/2} \tag{10.18}$$

Using (10.5) and (10.6) we obtain

$$\vec{A}^1 = \frac{ic\langle M \rangle \xi}{\left(2\pi(2n+1)\right)^{1/2}} \sum_{m=-\infty}^{\infty} \left\{ \int_{\tilde{\tau}_m}^{\tilde{\tau}_{m+1}} d\tilde{\tau} (-1)^m \exp\left[\frac{-\pi i (2n+1) \xi e^{-\tilde{\tau}}}{L} \right] \exp\left(i\Delta E \xi \tilde{\tau} \right) \right\} \tag{10.19}$$

$$\overleftarrow{A}^1 = \frac{ic\langle M \rangle \xi}{\left(2\pi(2n+1)\right)^{1/2}} \sum_{m=-\infty}^{\infty} \left\{ \int_{\tilde{\tau}_m}^{\tilde{\tau}_{m+1}} d\tilde{\tau} (-1)^m \exp\left[\frac{-\pi i (2n+1) \xi e^{-\tilde{\tau}}}{L} \right] \exp\left(i\Delta E \xi \tilde{\tau} \right) \right\} \tag{10.20}$$

where $\tilde{\tau}_m$ is defined as before. These expressions are evaluated as before by splitting the sums into the $m = 0$ term and the $|m| > 0$ terms. The former is

$$\exp(-\pi\xi\Delta E/2) d^{i\Delta E\xi} \left[\Gamma\left(-i\xi\Delta E, ide^{-b}\right) - \Gamma\left(-i\xi\Delta E, ide^{b}\right) \right]$$

and its complex conjugate respectively, with $d = \pi(2n+1)\xi/L$. The $m > 0$ sums are identical to (10.12) and (10.13) respectively, with "a" replaced by "d" and a factor of $(-1)^m$ now appears in the summands arising from the anti-periodicity of the boundary conditions. This presence precludes the immediate evaluation of the sums. We proceed by splitting the sums into even and odd parts; re-arranging we find the two sums become

$$\exp(-\pi\xi\Delta E/2) d^{i\Delta E\xi} \left\{ \begin{array}{l} \displaystyle\sum_{m=1}^{\infty} \left[\begin{array}{l} \Gamma\left(-i\xi\Delta E, idf(m)\right) - \Gamma\left(-i\xi\Delta E, id/f(m)\right) \\ -\Gamma\left(-i\xi\Delta E, idf(m+1)\right) + \Gamma\left(-i\xi\Delta E, id/f(m+1)\right) \end{array} \right] \\ \displaystyle +2\sum_{m=1}^{\infty} \left[\begin{array}{l} \Gamma\left(-i\xi\Delta E, idf(2m)\right) - \Gamma\left(-i\xi\Delta E, id/f(2m)\right) \\ -\Gamma\left(-i\xi\Delta E, idf(2m-1)\right) + \Gamma\left(-i\xi\Delta E, id/f(2m-1)\right) \end{array} \right] \end{array} \right\} \tag{10.21}$$

and its complex conjugate respectively.

The first sum in (10.21) is identical to the corresponding sum in the un-twisted case (10.13) and leads to a term similar to (10.14). The total transition amplitudes are therefore

$$\begin{aligned} \vec{A}^1 &= \\ &\frac{ic\xi\langle M \rangle e^{-\pi\xi\Delta E/2}}{\left(2\pi(2n+1)\right)^{1/2}} \left(\pi(2n+1)\xi/L\right)^{i\xi\Delta E} \left\{ \begin{array}{l} \Gamma(-i\xi\Delta E) \\ \displaystyle +2\sum_{m=1}^{\infty} \left[\begin{array}{l} \Gamma\left(-i\xi\Delta E, idf(2m)\right) - \Gamma\left(-i\xi\Delta E, id/f(2m)\right) \\ -\Gamma\left(-i\xi\Delta E, idf(2m-1)\right) + \Gamma\left(-i\xi\Delta E, id/f(2m-1)\right) \end{array} \right] \end{array} \right\} \\ &= -\left[\overleftarrow{A}^1 \right]^* \end{aligned} \tag{10.22}$$

Unfortunately the remaining sum in (10.22) cannot be evaluated exactly, however for $L >> \xi$ an approximate expression can be obtained. In this limit

$$f(m) = \exp\left[\operatorname{arccosh}(mL/\xi) \right] \approx (mL/\xi) + \left((mL/\xi)^2 - 1 \right)^{1/2} \approx 2mL/\xi$$

From the definition of d this implies

$$id\,f(m) \approx 2\pi im(2n+1)$$
$$id/f(m) \approx -2\pi im(2n+1)$$

Using the analytically continued integral representation of the Gamma function gives an approximation for the summand (10.22)

$$\int_0^\infty dt\, e^{-t} \left\{ \begin{array}{l} \left[t+4\pi im(2n+1)\right]^{-i\xi\Delta E-1} - \left[t-4\pi im(2n+1)\right]^{-i\xi\Delta E-1} \\ -\left[t+4\pi i(2m-1)(2n+1)\right]^{-i\xi\Delta E-1} + \left[t-4\pi i(2m-1)(2n+1)\right]^{-i\xi\Delta E-1} \end{array} \right\} b^{i\Delta E\xi} \tag{10.23}$$

The sum in (10.22) may now be evaluated explicitly using

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{(m+a)^s} = 2^{-s}\left[\zeta(s,a/2) - \zeta(s,(a+1)/2)\right] \tag{10.24}$$

Where ζ is the Riemann Zeta function (Gradshteyn & Ryzhik 1980, 9.521)

With (10.23), the sum in (10.22) can now be recast in the form

$$2\left(2\pi i(2n+1)\right)^{-i\Delta E\xi-1} \int_0^\infty dt \left\{ \begin{array}{l} \sum_{m=1}^{\infty}(-1)^m \left(m+\dfrac{t}{2\pi i(2n+1)}\right)^{-i\Delta E\xi-1} + \\ +(-1)^{-i\Delta E\xi-1} \sum_{m=1}^{\infty}(-1)^m \left(m-\dfrac{t}{2\pi i(2n+1)}\right)^{-i\Delta E\xi-1} \end{array} \right\}$$

Then (10.23) gives

$$\left[2\pi i(2n+1)\right]^{-i\Delta E\xi-1} \left[\begin{array}{l} \int_0^\infty dt \left\{ e^{-t}2^{-i\Delta E\xi} \left[\begin{array}{l} \varsigma\left(i\Delta E\xi+1, t/4\pi i(2n+1)\right) \\ -\varsigma\left(i\Delta E\xi+1, \left[t+2\pi i(2n+1)\right]/4\pi i(2n+1)\right) \\ -\left(2\pi i(2n+1)/t\right)^{i\xi\Delta E+1} \end{array} \right] \right\} \\ +\int_0^\infty dt \left\{ e^{-t}2^{-i\Delta E\xi}(-1)^{-i\Delta E\xi-1} \left[\begin{array}{l} \varsigma\left(i\Delta E\xi+1, -t/4\pi i(2n+1)\right) \\ -\varsigma\left(i\Delta E\xi+1, \left[-t+2\pi i(2n+1)\right]/4\pi i(2n+1)\right) \\ -\left(2\pi i(2n+1)/t\right)^{i\xi\Delta E+1} \end{array} \right] \right\} \end{array} \right]$$

$$\begin{aligned} &= \left[2\pi i(2n+1)\right]^{-i\Delta E\xi-1} 2^{-i\Delta E\xi} \int_0^\infty dt e^{-t} \left[\begin{array}{l} \varsigma\left(i\Delta E\xi+1, t/4\pi i(2n+1)\right) \\ -\varsigma\left(i\Delta E\xi+1, \left[t+2\pi i(2n+1)\right]/4\pi i(2n+1)\right) \end{array} \right] \\ &+ \left[-2\pi i(2n+1)\right]^{-i\Delta E\xi-1} 2^{-i\Delta E\xi} \int_0^\infty dt\, e^{-t} \left[\begin{array}{l} \varsigma\left(i\Delta E\xi+1, -t/4\pi i(2n+1)\right) \\ -\varsigma\left(i\Delta E\xi+1, \left[-t+2\pi i(2n+1)\right]/4\pi i(2n+1)\right) \end{array} \right] \\ &-2\Gamma(i\xi\Delta E) \end{aligned} \tag{10.25}$$

The integrals in this expression are manifestly convergent and from (10.25) $\overleftarrow{A}^1$ is easily found using its relationship to $\vec{A}^1$. Let the first integral be denoted by $B(\xi\Delta E,(2n+1))$ and the second by $B(\xi\Delta E,-(2n+1))$, then (10.22) may be written as

$$\vec{A}^1 = ic\xi\langle M\rangle e^{-\pi\xi\Delta E/2}\frac{\left(\pi(2n+1)\xi/L\right)^{i\xi\Delta E}}{\left(2\pi(2n+1)\right)^{1/2}}\left\{\begin{array}{l}-\Gamma(i\xi\Delta E)+\left[2\pi i(2n+1)\right]^{-i\xi\Delta E-1}2^{-i\xi\Delta E}B\left(\xi\Delta E,(2n+1)\right)+\\ +\left[-2\pi i(2n+1)\right]^{-i\xi\Delta E-1}2^{-i\xi\Delta E}B\left(\xi\Delta E,-(2n+1)\right)\end{array}\right\} \quad (10.26)$$

From the relationship between $\vec{A}^1$ and $\overleftarrow{A}^1$, we have

$$P_1 = \sum_n \left|\vec{A}^1\right|^2 + \left|\overleftarrow{A}^1\right|^2 = 2\sum_n \left|\vec{A}^1\right|^2$$

Using (10.26) this gives

$$P^1 = \frac{\xi c^2\left|\langle M\rangle\right|^2}{\pi}\times$$
$$e^{-\pi\xi\Delta E}\sum_{n=1}^{\infty}\left\{\begin{array}{l}\left|\Gamma(i\xi\Delta E)\right|^2\dfrac{\xi}{(2n+1)}+\dfrac{\xi\left[e^{\pi\xi\Delta E}\left|B\left(\xi\Delta E,(2n+1)\right)\right|^2+e^{-\pi\xi\Delta E}\left|B\left(\xi\Delta E,-(2n+1)\right)\right|^2\right]}{4\pi^2(2n+1)^3}\\ -\xi e^{\pi\xi\Delta E/2}\dfrac{\mathrm{Im}\left\{\Gamma(-i\xi\Delta E)\left[4\pi(2n+1)\right]^{-i\xi\Delta E}\left(B\left(\xi\Delta E,(2n+1)\right)-e^{-\pi\xi\Delta E}B\left(\xi\Delta E,-(2n+1)\right)\right)\right\}}{\pi(2n+1)^2}\\ -\xi\dfrac{\mathrm{Re}\left\{B\left(\xi\Delta E,(2n+1)\right)B\left(\xi\Delta E,-(2n+1)\right)\right\}}{2\pi(2n+1)^3}\end{array}\right\} \quad (10.27)$$

All of the sums in this response are finite except for the first which, as with the un-twisted field, represents a constant flux of particles. Comparing this term with (10.15) we see that the total response has the form

$$P^1 = c^2\left|\langle M\rangle\right|^2\left\{\frac{\sum_{n=1}^{\infty}\left(\xi/(2n+1)\right)}{\Delta E\left(\exp(2\pi\xi\Delta E)-1\right)}+\text{finite terms}\right\}$$

As a result the transition rate is

$$R^1 = c_2\left|\langle M\rangle\right|^2\left\{\frac{1}{\Delta E\left(\exp(2\pi\xi\Delta E)-1\right)}+\text{transient terms}\right\} \quad (10.28)$$

The transient terms must be located around $\tilde{\tau}=0$, since its integral over all proper-time is finite. Therefore the detector's response consists of the usual steady Planckian flux plus a time dependent term which goes to zero as $\tilde{\tau}\to\pm\infty$.

The presence of this extra term in the twisted field case might be taken to suggest that the detector could determine the R^1 x S^1 topology of the space (by detecting the presence of the twisted terms) from a local experiment. It must be remembered though that the treatment discussed here refers to the detector's response over the entire world-line ($-\infty < \tilde{\tau} < \infty$) during which the detector will encircle the "cylinder" repeatedly. In that sense it will be aware of the non-trivial topology, even though the "twist" term is independent of L.

It may be argued, on grounds of physical consistency, that for the twisted field one should use a detector coupling that is "gauge invariant" in the sense that it remains unchanged as the field twist flips the sign of ϕ with successive circuits around the cylinder. The quadratic detector satisfies this condition of "gauge invariance", however it is easily shown that formally this detector also responds to the field twists.

The transition rate per unit proper-time, $\tilde{\tau}$, for the linear detector was given in (4.8). From (10.27) we know that this detector's response at time $\tilde{\tau}$ consists of $\tilde{\tau}$-dependent and $\tilde{\tau}$-independent components. Furthermore we can split away the $\tilde{\tau}$-dependent twist terms of the response. Therefore, for the linear detector in this situation (4.8) has the form

$$c^2 \left|\langle M \rangle\right|^2 \int_{-\infty}^{\infty} d\eta \, e^{-i\Delta E \eta} \left\{ \left[A(\eta) + C(\eta,\tau) \right] \theta(-\eta) + \left[A(\eta) + C(-\eta,\tau) \right] \theta(\eta) \right\}$$

Where $A(\eta)$ contains the time-independent (thermal) part of the response and $C(\eta,\tau)$ is the twisted τ-dependent part. For the quadratic detector, from (5.8), the transition rate per unit detector time at $\tilde{\tau}$ is

$$2c^2 \left|\langle M \rangle\right|^2 \int_{-\infty}^{\infty} d\eta \, e^{-i\Delta E \eta} \left\{ \left[A(\eta) + C(\eta,\tau) \right]^2 \theta(-\eta) + \left[A(\eta) + C(-\eta,\tau) \right]^2 \theta(\eta) \right\}$$

The vacuum expectation makes no contribution since it is translation invariant (c.f. Chapter 5). Now for the quadratic detector to remain unaware of the field twists, it must respond to the twisted and un-twisted fields in the same way. This requires taking the inverse Fourier transform and assuming $C \neq 0$,

$$2A(\eta) + C(\eta,\tau) = 0 \qquad \forall \eta$$

which is obviously not the case. Therefore, in general, if the linear detector responds to the field twists, the "gauge invariant" quadratic detector also recognises whether or not the field is twisted.

We can understand why an apparently "gauge invariant" detector will respond to the twits by noting that a detector's non-zero response occurs when there is mixing of positive and negative frequency field modes with respect to the detector's "zero-response" vacuum state. Although the quadratic detector responds to ϕ^2, this does not mean the modes mix in even pairs so as to cancel out the effect of the twist. In fact there is mixing of modes with even m (giving $(-1)^m = 1$) with modes with odd m (giving $(-1)^m = -1$). This mixing of even and modes will cause the quadratic detector respond to the twisted field differently to the untwisted field. Although the quadratic detector may be gauge invariant with respect to the field for inertial motion, it does not remain so for non-inertial motion due to the mixing of positive and negative frequency modes.

11 How particle detectors work

From the above chapters, it is obvious that not all particle detectors are equivalent. To appreciate, in detail, why this is so we shall now look at how particle detectors work. From this study we shall see that detectors do not merely count particles. The fact that detectors are not merely "particle counters" will lead to an understanding of why different detectors are in-equivalent.

In carrying out this study we will initially confine out considerations to situations in S_1, S_2' and S_3'. This restriction merely simplifies the discussion and the conclusions drawn will easily be seen to be quite general.

11.1 Omni-directional detectors

Consider the set $S_4 = S_2' \cup S_3'$. The set S_4 consists of time independent situations which means the wave equation can be separated giving mode solutions of the from

$$u_k(x) = e^{-i\omega\tau}\chi_k(\rho) \tag{11.1}$$

Where the (τ,ρ)-coordinates have been used and $\chi_k(\rho)$ is the spatial part of the mode solution of the wave equation expressed in that coordinate system. The quantity ω is naturally defined as the energy of the k-mode (Pfautsch 1981). Also, in (11.1) the subscript "k" is merely a mode label. In general the requirement of positive norm, (2.5), for the modes is not identical to the condition $\omega > 0$. These conditions correspond if the space-time is static and the surfaces τ = constant are Cauchy surfaces (Pfautsch 1982, Grove & Ottewill 1983).

Expanding the field as a mode integral with operators b_k and, b_k^*,

$$\phi[x] = \int dk\left(u_k(x)b_k + u_k^*(x)b_k^*\right) \tag{11.2}$$

For the set S_4, the transition amplitude of the linear detector for a transition from its initial state, E_0, to a final state, E, has the form given in (3.3). Using (11.1) and (11.2) gives

$$A^1 = ic\langle M\rangle\int_{-\infty}^{\infty}d\tau\int d^{n-1}k\left\{e^{i(\Delta E-\omega)\tau}\chi_k(\rho)\langle\Psi|b_k|\Psi_0\rangle + e^{i(\Delta E+\omega)\tau}\chi_k^*(\rho)\langle\Psi|b_k^*|\Psi_0\rangle\right\}$$

Thus the transition probability is

$$P^1 = c^2\left|\langle M\rangle\right|^2\sum_{|\Psi\rangle}\int_{-\infty}^{\infty}d\frac{(\tau+\tau')}{2}\int_{-\infty}^{\infty}d\Delta\tau\int d^{n-1}k\int d^{n-1}l$$
$$\left(\begin{aligned}&\left\{e^{-i(\Delta E-\omega)\tau}\chi_k^*(\rho)\langle\Psi_0|b_k^*|\Psi\rangle + e^{-i(\Delta E+\omega)\tau}\chi_k(\rho)\langle\Psi_0|b_k|\Psi\rangle\right\}\times\\&\left\{e^{i(\Delta E-\varpi)\tau'}\chi_l(\rho)\langle\Psi|b_l|\Psi_0\rangle + e^{i(\Delta E+\varpi)\tau'}\chi_k^*(\rho)\langle\Psi|b_k^*|\Psi_0\rangle\right\}\end{aligned}\right)$$

where the $|\Psi\rangle$ sum is over a complete set of states and

$$\int_{-\infty}^{\infty}d\tau\int_{-\infty}^{\infty}d\tau' = \int_{-\infty}^{\infty}d\frac{(\tau+\tau')}{2}\int_{-\infty}^{\infty}d\Delta\tau$$

has been used. Performing the $|\Psi\rangle$ sum, only the terms with one annihilation and one creation operator survive, so

$$P^1 = c^2\left|\langle M\rangle\right|^2\int_{-\infty}^{\infty}d\frac{(\tau+\tau')}{2}\int_{-\infty}^{\infty}d\Delta\tau\int d^{n-1}k\int d^{n-1}l\left\{\begin{aligned}&e^{-i(\Delta E-\omega)\tau+i(\Delta E-\varpi)\tau'}\chi_k^*(\rho)\chi_l(\rho)\langle\Psi_0|b_k^*b_l|\Psi_0\rangle +\\&+e^{-i(\Delta E+\omega)\tau+i(\Delta E+\varpi)\tau'}\chi_k(\rho)\chi_l^*(\rho)\langle\Psi_0|b_kb_l^*|\Psi_0\rangle\end{aligned}\right\} \tag{11.3}$$

For calculational convenience we assume the state $\left|\Psi_0\right\rangle$ is an eigen-state of the number operator $N_k = b_k^* b_k$. (This assumption does not restrict the generality of the conclusions drawn.) With this assumption the expectation values on the right-hand side of (11.3) above can be written as

$$\begin{aligned} \left\langle \Psi_0 \middle| b_k^* b_l \middle| \Psi_0 \right\rangle &= \delta^{n-1}(k-l)\, n_k(\Psi_0) \\ \left\langle \Psi_0 \middle| b_k b_l^* \middle| \Psi_0 \right\rangle &= \delta^{n-1}(k-l)\left(1+n_k(\Psi_0)\right) \end{aligned} \tag{11.4}$$

Where $n_k(\Psi_0) = \left\langle \Psi_0 \middle| N_k \middle| \Psi_0 \right\rangle = \left\langle \Psi_0 \middle| b_k^* b_k \middle| \Psi_0 \right\rangle$ is the number of "b-particles" of the k-th mode in the state $\left|\Psi_0\right\rangle$. Using (11.4) in (11.3) and performing the l-integration;

$$\begin{aligned} P^1 = c^2 \left|\langle M \rangle\right|^2 \int_{-\infty}^{\infty} d\frac{(\tau+\tau')}{2} \int_{-\infty}^{\infty} d\Delta\tau \int d^{n-1}k \\ \left\{ e^{i(\omega-\Delta E)\Delta\tau} \left|\chi_k(\rho)\right|^2 n_k(\Psi_0) + e^{-i(\omega+\Delta E)\Delta\tau} \left|\chi_k(\rho)\right|^2 \left(1+n_k(\Psi_0)\right) \right\} \end{aligned} \tag{11.5}$$

As expected with a time independent situation, (11.5) is divergent due to the $(\tau+\tau')/2$-integral. Factoring this out in the usual way gives the transition rate. Also the $\Delta\tau$-integration may be performed;

$$R^1 = 2\pi c^2 \left|\langle M \rangle\right|^2 \int d^{n-1}k \left\{ \delta(\omega-\Delta E) \left|\chi_k(\rho)\right|^2 n_k(\Psi_0) + \delta(\omega+\Delta E) \left|\chi_k(\rho)\right|^2 \left(1+n_k(\Psi_0)\right) \right\} \tag{11.6}$$

If the detector merely toted up the number of b-particles in the modes with energy $\omega = \Delta E$ in the state $\left|\Psi_0\right\rangle$, its response would have the form

$$R^1 = (\text{constant}) \times c^2 \left|\langle M \rangle\right|^2 \int d^{n-1}k\, \delta(\omega-\Delta E)\, n_k(\Psi_0)$$

Which manifestly does not correspond to the actual response given in (11.6). Therefore we see that the linear detector does not merely "count particles". To see exactly what it does do (to first order perturbation in the coupling constant), refer back to (11.6). To first order there are two terms which contribute to R^1 depending upon the relative ranges of ω and ΔE. The first term in (11.6), i.e. the $\delta(\omega\text{-}\Delta E)$ term, shall be called the "absorption" term and the second the "emission" term. These names arise from the fact that, in the detector's frame, they may be represented by Feynman diagrams which correspond to the process of b-particle absorption and emission respectively. (See Figure 4.)

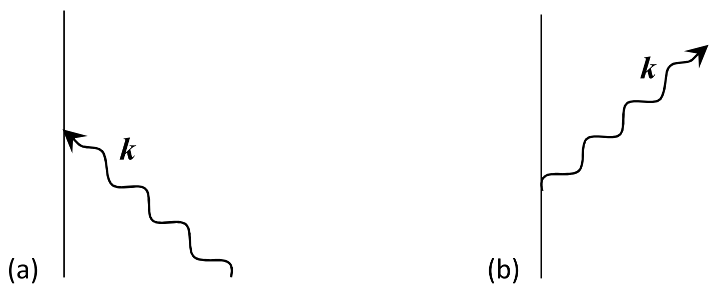


Figure 4 (a) The Feynman diagram corresponding to the absorption term in (11.6). In this diagram the world-line of the detector is the solid straight line representing its classical trajectory. The quantum mode is represented by the wavy line. (b) The Feynmann diagram corresponding to the emission term in (11.6)

The range of values ΔE may take is determined by the choice of the initial state $|E_0\rangle$ of the detector. If $|E_0\rangle$ is not the ground state, the detector will be able to de-excite by emitting a b-particle. This emission will be stimulated by quantum fluctuations of the field (Sciama et al. 1981). It is usual practice to avoid this process by preparing the detector initially in its ground state. This then requires $\Delta E > 0$. However, this precaution does not necessarily prevent emission term contributions to the response. Situations for which ω may go negative will give emission term contributions. (An example of such a situation is a detector undergoing constant rotation about an inertial axis in the Minkowski vacuum. In this case the detector's response arises solely from the emission terms. See Pfautsch 1981, Grove & Ottewill 1983).

Even if $\omega > 0$ and $\Delta E > 0$ so that only absorption terms contribute (11.6) shows the detector's response is, in fact, a weighted sum over the modes with energy ΔE. For the linear detector this weighting is provided by the functions $\chi_k(\rho)$. Thus, in general the linear detector's response will not be proportional to the b-particle content (or b-particle number) of a quantum state. In the several well-known situations where its response is found to be proportional to this quantity (e.g. Rindler, Robertson-Walker space-times) close inspection of the calculation reveals symmetries in the quantities involved (i.e. $\chi_k(\rho)$, β_{ij}).These symmetries are special to those situations and conspire to yield direct proportionality. (For examples of this see the Appendix for the Rindler space-time calculation and see Birrell & Davies (1982) for the Robert-Walker space-time calculation.)

If $|\Psi_0\rangle$ is not an eigen-state of the number operator, the weighted averaging process still occurs, in addition to other contributions to the detector's response. Thus it is seen that in general the linear detector does not merely count the number of particles in a given momentum mode(s).

For the quadratic detector we have

$$\begin{aligned} A^2 = ic\langle M\rangle \int_{-\infty}^{\infty} d\tau \int d^{n-1}k \int d^{n-1}l e^{i\Delta E\tau} \\ \langle\Psi|\left(e^{-i\omega\tau}\chi_k(\rho)b_k + e^{i\omega\tau}\chi_k^*(\rho)b_k^*\right)\left(e^{-i\varpi\tau}\chi_l(\rho)b_l + e^{i\varpi\tau}\chi_l^*(\rho)b_l^*\right)|\Psi_0\rangle_{ren} \end{aligned} \tag{11.7}$$

Due to the assumption of time independence, there will be no contribution from the $\langle\phi^2\rangle_{ren}$ term. Removal of this contribution from (11.7) merely requires excluding the $|\Psi\rangle = |\Psi_0\rangle$ term from the sum over the set of field states. (See Sec. 5.1 for details.) Therefore, from (11.7)

$$\begin{aligned} P^2 = c^2\left|\langle M\rangle\right|^2 \int_{-\infty}^{\infty} d\frac{(\tau+\tau')}{2} \int_{-\infty}^{\infty} d\Delta\tau \int d^{n-1}k_s \int d^{n-1}k_r \int d^{n-1}k_p \int d^{n-1}k_q e^{-i\Delta E(\tau-\tau')} \\ \sum_{|\Psi\rangle\neq|\Psi_0\rangle} \begin{Bmatrix} B(r,s;-p,-q)\langle\Psi_0|b_{k_r}^* b_{k_s}^* b_{k_p} b_{k_q}|\Psi_0\rangle + B(r,-s;-p,q)\langle\Psi_0|b_{k_r}^* b_{k_s} b_{k_p} b_{k_q}^*|\Psi_0\rangle + \\ +B(r,-s;p,-q)\langle\Psi_0|b_{k_r}^* b_{k_s} b_{k_p}^* b_{k_q}|\Psi_0\rangle + B(-r,s;p,-q)\langle\Psi_0|b_{k_r} b_{k_s}^* b_{k_p}^* b_{k_q}|\Psi_0\rangle + \\ +B(-r,s;-p,q)\langle\Psi_0|b_{k_r} b_{k_s}^* b_{k_p} b_{k_q}^*|\Psi_0\rangle + B(-r,-s;p,q)\langle\Psi_0|b_{k_r} b_{k_s} b_{k_p}^* b_{k_q}^*|\Psi_0\rangle \end{Bmatrix} \end{aligned} \tag{11.8}$$

where

$$B(r,s;-p,-q) = e^{i(\omega_r+\omega_s)\tau - i(\omega_r+\omega_s)\tau'} \chi_{k_r}^*(\rho)\chi_{k_s}^*(\rho)\chi_{k_p}(\rho)\chi_{k_q}(\rho)$$

The expectation values in this equation are evaluated as follows:

$$\left\langle \Psi_0 \left| b_{k_r}^* b_{k_s}^* b_{k_p} b_{k_q} \right| \Psi_0 \right\rangle = n_{k_r}\left(\Psi_0\right) n_{k_s}\left(\Psi_0\right) \begin{pmatrix} \delta^{n-1}\left(k_p - k_r\right)\delta^{n-1}\left(k_q - k_s\right) + \\ +\delta^{n-1}\left(k_p - k_s\right)\delta^{n-1}\left(k_q - k_r\right) \end{pmatrix}$$

$$\left\langle \Psi_0 \left| b_{k_r}^* b_{k_s} b_{k_p} b_{k_q}^* \right| \Psi_0 \right\rangle = n_{k_r}\left(\Psi_0\right)\left(n_{k_s}\left(\Psi_0\right)+1\right)\delta^{n-1}\left(k_p - k_r\right)\delta^{n-1}\left(k_q - k_s\right)$$

$$\left\langle \Psi_0 \left| b_{k_r}^* b_{k_s} b_{k_q}^* b_{k_q} \right| \Psi_0 \right\rangle = n_{k_r}\left(\Psi_0\right)\left(n_{k_s}\left(\Psi_0\right)+1\right)\delta^{n-1}\left(k_p - k_s\right)\delta^{n-1}\left(k_q - k_r\right)$$

$$\left\langle \Psi_0 \left| b_{k_r} b_{k_s}^* b_{k_p}^* b_{k_q} \right| \Psi_0 \right\rangle = \left(n_{k_r}\left(\Psi_0\right)+1\right) n_{k_s}\left(\Psi_0\right)\delta^{n-1}\left(k_p - k_r\right)\delta^{n-1}\left(k_q - k_s\right)$$

$$\left\langle \Psi_0 \left| b_{k_r} b_{k_s}^* b_{k_p} b_{k_q}^* \right| \Psi_0 \right\rangle = \left(n_{k_r}\left(\Psi_0\right)+1\right) n_{k_s}\left(\Psi_0\right)\delta^{n-1}\left(k_p - k_s\right)\delta^{n-1}\left(k_q - k_r\right)$$

$$\left\langle \Psi_0 \left| b_{k_r} b_{k_s} b_{k_p}^* b_{k_q}^* \right| \Psi_0 \right\rangle = \left(n_{k_r}\left(\Psi_0\right)+1\right)\left(n_{k_s}\left(\Psi_0\right)+1\right) \begin{pmatrix} \delta^{n-1}\left(k_p - k_r\right)\delta^{n-1}\left(k_q - k_s\right) + \\ +\delta^{n-1}\left(k_p - k_s\right)\delta^{n-1}\left(k_q - k_r\right) \end{pmatrix}$$

Substituting these into (11.8), performing the k_p- and k_q-integrals as well as the $\left|\Psi\right\rangle$ sum

$$P^2 = 2c^2 \left|\left\langle M \right\rangle\right|^2 \int_{-\infty}^{\infty} d\frac{\left(\tau+\tau'\right)}{2} \int_{-\infty}^{\infty} d\Delta\tau \int d^{n-1}k_s \int d^{n-1}k_r \left|\chi_k\left(\rho\right)\right|^2 \left|\chi_k\left(\rho\right)\right|^2$$
$$\begin{Bmatrix} e^{i\left(\omega+\bar{\omega}-\Delta E\right)\Delta\tau} n_k\left(\Psi_0\right) n_l\left(\Psi_0\right) + e^{i\left(\omega-\bar{\omega}-\Delta E\right)\Delta\tau} n_k\left(\Psi_0\right)\left(n_l\left(\Psi_0\right)+1\right) + \\ +e^{i\left(\bar{\omega}-\omega-\Delta E\right)\Delta\tau}\left(n_k\left(\Psi_0\right)+1\right) n_l\left(\Psi_0\right) + e^{-i\left(\omega+\bar{\omega}+\Delta E\right)\Delta\tau}\left(n_k\left(\Psi_0\right)+1\right)\left(n_l\left(\Psi_0\right)+1\right) \end{Bmatrix}$$

This gives for the response of the quadratic detector

$$R^2 = 4\pi c^2 \left|\left\langle M \right\rangle\right|^2 \int d^{n-1}k_s \int d^{n-1}k_r \left|\chi_k\left(\rho\right)\right|^2 \left|\chi_k\left(\rho\right)\right|^2$$
$$\begin{Bmatrix} \delta\left(\omega+\bar{\omega}-\Delta E\right) n_k\left(\Psi_0\right) n_l\left(\Psi_0\right) + \delta\left(\omega-\bar{\omega}-\Delta E\right) n_k\left(\Psi_0\right)\left(n_l\left(\Psi_0\right)+1\right) + \\ +\delta\left(\bar{\omega}-\omega-\Delta E\right)\left(n_k\left(\Psi_0\right)+1\right) n_l\left(\Psi_0\right) + \delta\left(\omega+\bar{\omega}+\Delta E\right)\left(n_k\left(\Psi_0\right)+1\right)\left(n_l\left(\Psi_0\right)+1\right) \end{Bmatrix} \qquad (11.9)$$

From (11.9) it is seen that in general the quadratic detector's response is not proportional to the square of the b-particle number $[n_k(\Psi_0)]^2$. More importantly, to any given order of perturbation, the quadratic detector's interaction with the field involves processes which have no analogue in the linear detector. Figure 5 depicts the various interactions that may occur between this detector and the field (to first order). Although there is an obvious correspondence between Figure 4(a) and Figure 5(a), in that they both depict only absorption of b-particles, and similarly between Figure 4(b) and Figure 5(b), the interaction in Figure 5(c) has no analogue in the linear detector. Even at higher orders of perturbation, this process has no analogue in the linear detector.

This extra process will occur with $\omega > 0, \bar{\omega} > 0$ and $\Delta E > 0$ since it only requires $\omega > \bar{\omega} + \Delta E$ or $\bar{\omega} > \omega + \Delta E$. That is, the quadratic detector can absorb a mode of energy ω (or $\bar{\omega}$) and decay into a lower internal energy state (of ΔE), emitting a mode of energy $\bar{\omega}$ (or ω respectively). Although this does not occur in the linear detector, the equivalence of these two detectors should not be surprising. Viewing (11.3) and (11.8), it can be seen that, in general, both detectors' responses involve the same basic "unit" which has the form $\chi_k\left(\rho\right)\chi_l^*\left(\rho\right) b_k b_l^*$. Since these units are determined by the situation into which the detectors are placed, we would expect the detector equivalence sets (of equilibrium situations) to be identical.

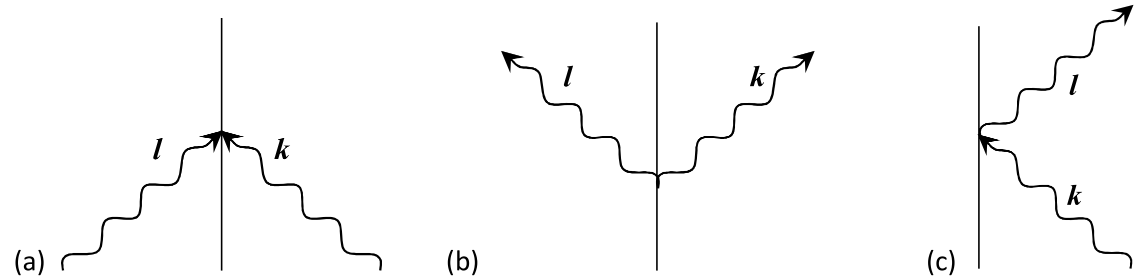


Figure 5 (a) The Feynman diagram corresponding to the pure absorption term in (11.9) which contains the $\delta\left(\omega+\bar{\omega}-\Delta E\right)$ term. (b) The Feynman diagram corresponding to the pure emission term in (11.9) which contains the $\delta\left(\omega+\bar{\omega}+\Delta E\right)$ term. (c) The Feynman diagram corresponding to the processes involving both emission and absorption. This process corresponds to the $\delta\left(\omega-\bar{\omega}-\Delta E\right)$ and $\delta\left(\bar{\omega}-\omega-\Delta E\right)$ terms in (11.9).

This conclusion can be intuitively guessed from the fact that the quadratic detector is very similar in form to the second order perturbation term of the linear detector's transition amplitude. We have (DeWitt 1979)

$$A^{1}=\left\langle E,\Psi\right|T\left[\exp\left(i\int_{-\infty}^{\infty}d\tau\; cm(\tau)\phi\left[x(\tau)\right]\right)\right]\left|E_0,\Psi_0\right\rangle$$

Where T[..] is the chronological ordering operator. Expanding to second order in the coupling constant gives

$$\begin{aligned}A^{1}&=ic\int_{-\infty}^{\infty}d\tau\,\left\langle E\right|m(\tau)\left|E_0\right\rangle\left\langle\Psi\right|\phi\left[x(\tau)\right]\left|\Psi_0\right\rangle-\\&-c^{2}T\left[\int_{-\infty}^{\infty}d\tau\int_{-\infty}^{\infty}d\tau'\left\langle E\right|m(\tau)m(\tau')\left|E_0\right\rangle\left\langle\Psi\right|\phi\left[x(\tau)\right]\phi\left[x(\tau')\right]\left|\Psi_0\right\rangle\right]\end{aligned}\tag{11.10}$$

The first term in (11.10) is the usual linear detector transition amplitude. Comparing the second term with (3.11) for the quadratic detector, the similarity is obvious.

Referring now to the derivative detector, the transition amplitude for this detector is given by

$$A^{3}=ic\left\langle M\right\rangle\int_{-\infty}^{\infty}d\tau\begin{Bmatrix}e^{i(\Delta E-\omega)\tau}\left(-ib^{0}\omega+b^{i}\partial_{\rho^{i}}\right)\chi_k(\rho)\left\langle\Psi\right|b_k\left|\Psi_0\right\rangle+\\+e^{i(\Delta E+\omega)\tau}\left(ib^{0}\omega+b^{i}\partial_{\rho^{i}}\right)\chi_k^{*}(\rho)\left\langle\Psi\right|b_k^{*}\left|\Psi_0\right\rangle\end{Bmatrix}$$

From this, using the identical approach as with the linear detector, we find

$$\begin{aligned}R^{3}&=2\pi c^{2}\left|\left\langle M\right\rangle\right|^{2}\int d^{n-1}k\left\{\delta(\omega-\Delta E)n_k(\Psi_0)+\delta(\omega+\Delta E)\left(n_k(\Psi_0)+1\right)\right\}\times\\&\times\left\{\left(-ib^{0}\omega+b^{i}\partial_{\rho^{i}}\right)\chi_k(\rho)\left(ib^{0}\omega+b^{i}\partial_{\rho'^{i}}\right)\chi_k^{*}(\rho')\right\}\Big|_{\rho=\rho'}\end{aligned}\tag{11.11}$$

The reason for derivative and linear detector in-equivalence is now obvious. The weighting functions involved in the summing process is in (11.6) and (11.11) are quite different.

For two situations to be linear detector equivalent, the weighted sums (given in (11.6)) for those two situations must be identical. However, that does not necessarily mean that the weighted sums given by

(11.11) for those situations will also be identical. Note, however, that for the time-derivative detector, (11.11) gives

$$R^3 = 2\pi c^2 \left|\langle M\rangle\right|^2 \left(b^0\right)^2 \int d^{n-1}k\omega^2 \left\{\delta(\omega-\Delta E)n_k(\Psi_0)+\delta(\omega+\Delta E)\left(n_k(\Psi_0)+1\right)\right\}\left|\chi_k(\rho)\right|^2 \quad (11.12)$$

Due to the delta functions (and using $x\delta(x-a) = a\delta(x-a)$) the averaging occurs only over the spatial components k of the momentum modes in both (11.6) and (11.12). Therefore the weighting function for the time-derivative detector is identical to that for the linear detector. Thus these two detectors are equivalent.

It is now clear why different detectors may be in-equivalent. Since detector responses are derived from weighted sums (such as (11.6), (11.9) and (11.11)) different detectors using different weightings will resolve sets of situations into different equivalence classes. Furthermore, interaction processes may occur in some detectors that do not occur in others, as with the quadratic detector. These extra processes may involve couplings between the detector and quantities such as, say $\langle\phi^2\rangle_{ren}$. These detectors will obviously be in-equivalent to detectors that do not couple to such quantities.

A slightly different way of viewing the workings of a detector when placed in situations in $\mathcal{S}_2'$ is Sciama's "fluctuometer" approach (Sciama 1979). In this approach the linear detector coupled to the quantum field vacuum fluctuations and its response is a measure of the power spectrum $\mathcal{P}(E)$ of these fluctuations which is given by

$$\mathcal{P}(E) \propto \int d\tau\, e^{-iE\tau} \langle 0_M|\phi(\tau)\phi(0)|0_M\rangle$$

Referring to (4.7) the similarity is obvious. Using this notation, the quadratic detector's response is related to the auto-correlation of $\mathcal{P}(E)$. That is, the quadratic detectors measures

$$\int d\tau e^{-iE\tau} \left[\langle 0_M|\phi(\tau)\phi(0)|0_M\rangle\right]^2$$

From this we would expect $\mathcal{S}_2'$- equivalence of these two detectors. However, the derivative detector's response is related to

$$b^\mu b^\nu \int d\tau\, e^{-iE\tau} \partial_\mu \partial'_\nu \langle 0_M|\phi(\tau)\phi(0)|0_M\rangle|$$

which would lead us to expect this detector to be $\mathcal{S}_2'$-in-equivalent to the other two. This fluctuometer interpretation appears to be applicable only in time independent situations.

11.2 Directional and orientable detectors

So far in this thesis the terms "omni-directional" and "orientable" and "directional" have loosely used to classify the detectors discussed. The screening function concept introduced in Chapter 7 will now be used to make these classifications precise. Firstly, the idea of a "screening function" S(Ω) will be made definite;

<u>Definition</u>: Let $\mathcal{D}_M$ represent a hypothetical monopole detector such that when placed in an n-particle state $n_{\boldsymbol{k}}$ in Minkowski space (stationary with respect to $n_{\boldsymbol{k}}$), the response of $\mathcal{D}_M$ has the form

$$\mathcal{D}_M(n_{\boldsymbol{k}}) = \mathcal{D}_M\left(\int d\Omega\, n_{\boldsymbol{k}}\right)$$

where the Ω-integral is over the entire (n-2)-sphere of momentum space. Now consider a second detector $\mathcal{D}$ that differs from $\mathcal{D}_M$ in that

$$\mathcal{D}(n_{\boldsymbol{k}}) = \mathcal{D}_M\left(\int d\Omega W_{\mathcal{D}}(\Omega) n_{\boldsymbol{k}}\right)$$

Then $W_{\mathcal{D}}(\Omega)$ is defined to be the "weighting" or "screening" function of the detector $\mathcal{D}$ and $\mathcal{D}_M$ is the "hypothetical monopole" detector corresponding to $\mathcal{D}$.

This definition is a generalisation of the concept of receptivity used in microwave engineering (Ramo, Whinnery & Van Duzer 1965). Some examples may help clarify this definition;

Example (1): It automatically follows from the definition that, for the hypothetical monopole detector, $W_{\mathcal{D}_M}(\Omega) = 1$ (See Figure 6(a).)

(2): For the cone detector, the hypothetical detector $\mathcal{D}_M$ is the linear detector and $W_{\mathcal{D}}(\Omega)$ is given by

$$\begin{aligned} W_{\mathcal{D}}(\Omega) &= 1 \qquad \Omega \in S_\Omega \\ &= 0 \qquad \Omega \notin S_\Omega \end{aligned}$$

Where S_Ω is as shown in Figure 6(b) for 2 spatial dimensions.

(3): For the spike detector, $\mathcal{D}_M$ is the linear detector and

$$W_{\mathcal{D}}(\Omega) = \delta(\Omega - \Omega')$$

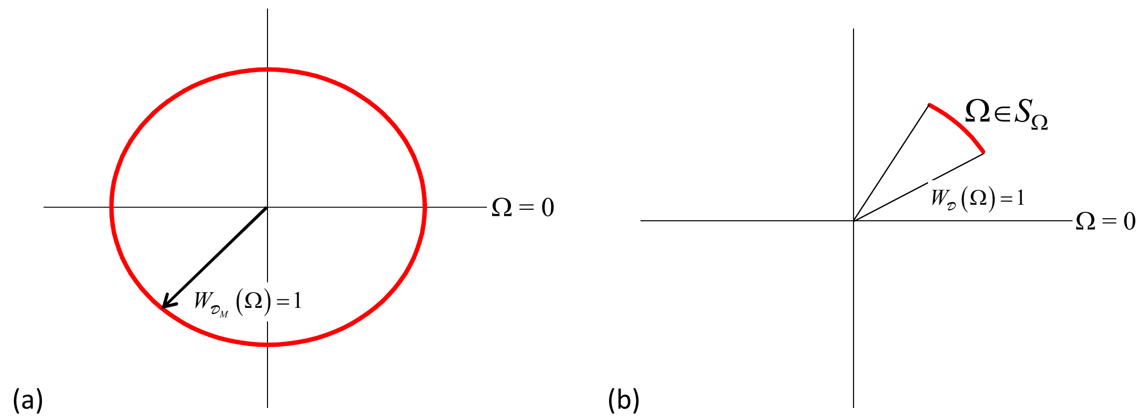
Where Ω' is the single mode direction accessible to the detector. (See Figure 6(c).)

(4): The linear detector is not the $\mathcal{D}_M$ for the derivative detector as can be seen on dimensional grounds. The time-derivative detector is the $\mathcal{D}_M$ for this detector. For the spatial-derivative detector the screening function is

$$W_{\mathcal{D}}(\Omega) = \cos^2 \theta_\Omega$$

Where θ_Ω is the hyper-angle between b^μ and the direction Ω in the (n-2)-sphere. (See Figure 6(d).)

(5): The quadratic detector is its own hypothetical monopole detector because it has $W(\Omega) = 1$.



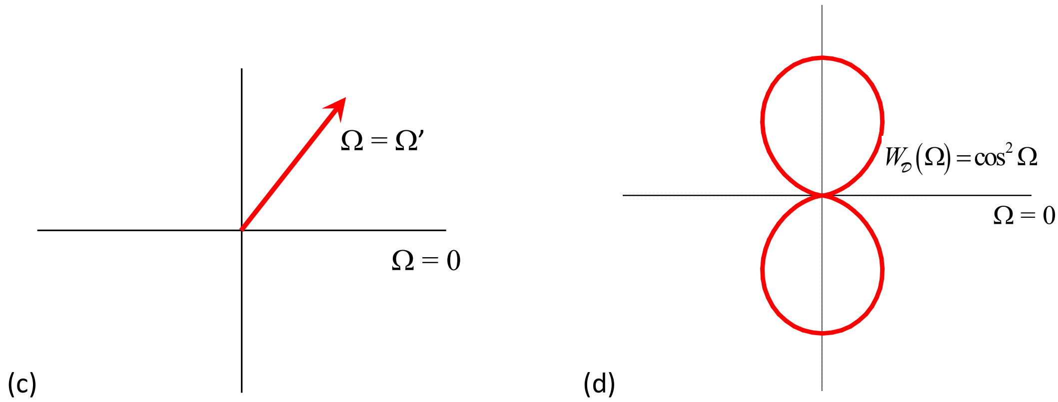


Figure 6 (a) A graphical representation of the screening function $W(\Omega)$ of a monopole detector (in two spatial dimensions). The function $W(\Omega)$ is given by the red line indicating equal receptivity of momentum modes from all directions. (b) The screening function of the cone detector in two spatial dimensions. The set S_Ω represents the subset of the (n-2)-sphere of momentum mode directions that may interact with the detector. (c) The screening function for the spike detector which may interact only with modes of direction Ω'. (d) The screening function of the spatial derivative detector in two spatial dimensions.

With the screening function defined three types of detectors can also be defined.

Definition: A detector $\mathcal{D}$ is omni-directional if its screening function $W_{\mathcal{D}}(\Omega)$ is non-zero almost everywhere. (That is, the Lebesgue measure of the set of directions such that $W_{\mathcal{D}}(\Omega) = 0$ is zero.)

Definition: A detector that is not omni-directional is said to be directional. That is, a detector is directional if the subset of $(n\text{-}2)$-sphere for which $W_{\mathcal{D}}(\Omega) = 0$ has non-zero measure and is not pathological. (This last condition is inserted to avoid screening functions which are unphysical and effectively omni-directional. An example of such a function is $W_{\mathcal{D}}(\Omega) = 0$, $\Omega \in \{(x_1, x_2, ..., x_{n-1}) \in \mathcal{R}^{n-2}: x_i$ is irrational$\}$ and $W_{\mathcal{D}}(\Omega) \neq 0$ otherwise. We will not side track into discussing the technicalities of what is "pathological" since it adds little to the main argument.)

Definition: A detector is orientable if it is omni-directional and $W_{\mathcal{D}}(\Omega)$ is not constant for all $\Omega \in (n\text{-}2)$-sphere.

Examples of omni-directional detectors included the linear, quadratic and derivative detectors. Examples of directional detectors include the cone and spike detectors. The derivative detector is orientable and, finally, the linear and quadratic detectors are non-orientable.

The distinction between directional and omni-directional detectors is fundamental when comparing or discussing the use of particle detectors. Omni-directional, non-orientable detectors cannot be used to make statements about the isotropy or otherwise of a particle state. (It was most likely a failure to recognise this fact that led to much confusion regarding the isotropy of acceleration radiation.) On the other hand, it is possible to use directional and orientable detectors to probe anisotropies of a quantum state.

The equivalence criterion introduced above, when applicable, reflects these fundamental differences since the detector equivalence classes of an omni-directional detector will always be different from those of a directional detector resulting in their in-equivalence. This fact can be made manifest by using Theorem 1.

- For an omni-directional, non-orientable detector, $\mathcal{D}_1$, the screening function $W(\Omega)$ = constant, so

$$\begin{aligned}\mathcal{D}_1(n_{\boldsymbol{k}}) &= \mathcal{D}_1\left(\int d\Omega\, n_{\boldsymbol{k}}\right)\\ &= \mathcal{D}_1(\hat{n}_{\boldsymbol{k}})\end{aligned}$$

where $\hat{n}_k = \int d\Omega\, n_{\boldsymbol{k}} \Big/ \int d\Omega$ is the corresponding isotropic state. No matter how the detector is "held" with respect to some chosen direction in the particle bath $n_{\boldsymbol{k}}$, the response will be identical to that for $\hat{n}_k$. So, as stated above, no information on the anisotropy of $n_{\boldsymbol{k}}$ can be deduced.

- For an orientable detector, $\mathcal{D}_2$, $W(\Omega)$ is not constant and an "orientation" of the detector may be defined by, say, the solid angle Ω' for which $W(\Omega') = \operatorname*{Sup}_{\Omega} W(\Omega)$. For a given (anisotropic) particle state $n_{\boldsymbol{k}}$, the corresponding isotropic particle state is defined by

$$\mathcal{D}_2(\hat{n}_k) = \mathcal{D}_2\left(\int d\Omega\, W(\Omega) n_{\boldsymbol{k}}\right) \tag{11.13}$$

will now depend upon the orientation Ω' of the detector, since for different orientations the weight of the "screening" given to each mode will differ. So, (11.13) should be re-written defining $\hat{n}_k(\Omega')$ by;

$$\hat{n}_k(\Omega') \equiv \int d\Omega\, W(\Omega,\Omega') n_{\boldsymbol{k}} \Big/ \int d\Omega\, W(\Omega,\Omega')$$

where $W(\Omega,\Omega')$ represents the screening function $W(\Omega)$ with orientation Ω'. It can be seen that for a given $n_{\boldsymbol{k}}$, different orientations Ω' will (by Theorem 1) give different $\mathcal{D}_2$-equivalent isotropic states $\hat{n}_k$. Only when $n_{\boldsymbol{k}}$ itself is isotropic does the Ω' dependence of the response disappear. Thus orientable detectors can discern between isotropic and anisotropic states. (In fact this is how the derivative detector was used in Sec. 6.2 to show that acceleration radiation is anisotropic.) Because orientable detectors are receptive to modes from all directions in momentum space, although they can discern anisotropy, they cannot probe the modes of a quantum state only in a particular direction or set directions.

- A directional detector, $\mathcal{D}_3$, can be used to probe n_k in only a particular direction or set of directions. Since for $\mathcal{D}_3$, the screening function has the form

$$\begin{aligned} W(\Omega') &\neq 0 \quad \Omega' \in S_\Omega \\ &= 0 \quad \Omega' \notin S_\Omega \end{aligned}$$

where S_Ω is a proper subset of the $(n\text{-}2)$-sphere of directions in momentum space, this detector can be used to make statements about the modes incident to the detector from directions S_Ω only. This is how the cone and spike detectors were used to demonstrate the anisotropy of acceleration radiation.

From the above it follows that directional detectors can provide information about the quantum state that other detector types cannot. Similarly, orientable detectors can provide information non-orientable detectors cannot.

We shall now relate the general screening function $W(\Omega,\Omega')$ which the averaging procedure introduced in Theorem 1, denoted by $A(.)$ and the weighting functions introduced in Sec. 11.1. Firstly, from Theorem 1, it is seen that for a screening function $W(\Omega,\Omega')$, the averaging $A(.)$ is defined by

$$A(n_k) = \int d\Omega\, W(\Omega,\Omega') n_{\boldsymbol{k}} \Big/ \int d\Omega \tag{11.14}$$

This does not mean that all averaging functions, $A(.)$, can be written in terms of a screening function even though (11.14) shows the converse is true.

Turning now to the weighted sums introduced in the previous section of this chapter, for the purposes of comparison let us assume that the detector's response for time independent situations can be written in the form (c.f. (11.6), (11.11) and (11.12))

$$R \propto c^2 \left|\langle M \rangle\right|^2 \int d^{n-1}k \left\{ \delta(\xi - \Delta E) n_{\boldsymbol{k}}(\Psi_0) + \delta(\xi + \Delta E)\left(n_{\boldsymbol{k}}(\Psi_0) + 1\right) \right\} W_{\boldsymbol{k}} \tag{11.15}$$

where $W_{\boldsymbol{k}}$ is the weighting function. Due to the delta function in the mode energy the momentum integral is effectively over the spatial modes $\boldsymbol{k}$. Assuming this detector has screening function $W(\Omega)$ (ignoring orientation), for a particle state $n_{\boldsymbol{k}}$ in Minkowski space with the detector, we may write

$$R \propto c^2 \left|\langle M \rangle\right|^2 \int d^{n-1}k\, \delta(\omega - \Delta E) n_{\boldsymbol{k}} W(\Omega) F(\omega) \tag{11.16}$$

$$= c^2 \left|\langle M \rangle\right|^2 F(\Delta E) k^{n-2} \int d\Omega\, W(\Omega) n_{\boldsymbol{k}} \tag{11.17}$$

where $n_{\boldsymbol{k}}$ is such that $k^2 + m^2 = (\Delta E)^2$ and $F(\omega)$ is a function of the mode energy ω only. Note that in Minkowski space $\omega > 0$ thus only one of the delta functions in (11.15) contributes, giving (11.16). However, in general ω may be negative (Pfautsch 1981) and both delta functions may contribute. We shall ignore this complication since our main interest is with the directional screening of the modes, which is represented by $W_{\boldsymbol{k}}$ in (11.15) and $W(\Omega)F(\omega)$ in (11.16).

The step from (11.16) to (11.17) uses the high symmetry of Minkowski space (represented by the quality $k^2 + m^2 = \omega^2$). However, in the more general response (11.15) such symmetries need not exist in the mode functions that contribute to $W_{\boldsymbol{k}}$ and so the screening component of $W_{\boldsymbol{k}}$ cannot always be expressed as explicitly as in (11.17). In fact, even though the momentum space integral in (11.15) can be formally represented as $\int d^{n-1}k$, in general the actual form will be such that it cannot be split into "magnitude" and "angular" integrals as in Minkowski space. This fact is nicely illustrated by the four-dimensional Rindler space calculations in which the momentum space splits into the form (c.f. (6.14),

$$\int d^3k \rightarrow \int_{-\infty}^{\infty} d\tilde{\omega} \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2$$

where there is no relation between the mode energy $\tilde{\omega}$ and the momentum components k_1 and k_2. Also, the modes in this space do not have a well-defined direction in the 2-sphere of momentum space. As a result the angular functions and integrals are not well defined in Rindler momentum space. In particular a screening function cannot be represented in this momentum space.

From this it can be seen that although the screening function may be explicitly expressible in a simple form for situations in $\mathcal{S}_1$, in a more general situation (represented by (11.15)), this will not be possible. In other words, in general we cannot expect to be able to write an equation analogous to (11.14) explicitly and directly relating $W_{\boldsymbol{k}}$ and $W(\Omega)$. This problem is referred to in Chapter 7 where it was noted that, in general, one cannot find a simple mathematical for representing the restriction on the modes accessible to the cone and spike detectors.

Given the significance of the screening functions of a detector in determining its response, we shall now see how strong the nexus is between screening functions and detector equivalence. Although the relationship between these two is close, it is not direct. To demonstrate this point, consider the following detectors;

1. The linear detector (which has $W(\Omega) = 1$)
2. The time derivative detector ($W(\Omega) =1$)
3. The quadratic detector ($W(\Omega) = 1$)
4. The spatial-derivative detector (for which, from the above, $W(\Omega) = \cos^2\Omega$)
5. A screened linearly coupled detector with $W(\Omega) = \cos^2\Omega$.

We already know that the linear and time-derivative detectors are equivalent and the both have $W(\Omega) = 1$. However, the quadratic detector also has $W(\Omega) = 1$ but is not equivalent to the other two because it couples to $\langle \phi^2 \rangle_{ren}$. The reason for this in-equivalence is that the quadratic detector's response includes processes the other two do not, i.e. coupling to $\langle \phi^2 \rangle_{ren}$. This is further supported by the equivalence of all three of these detectors for situations in which $\langle \phi^2 \rangle_{ren}$ does not contribute.

It has already been demonstrated that these three detectors are in-equivalent to the spatial-derivative detector due to the different screening and weighting functions which determine their responses. This leaves comparing the spatial-derivative detector with the screened linear detector introduced in 5 above. Can the comparison of screening functions be used to prove equivalence or otherwise of these two detectors?

Theorem 4: The spatial-derivative detector and screened linear detector (with $W(\Omega) = \cos^2\Omega$) are $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$- equivalent.

Proof: Firstly, from the forms of the interaction Lagrangians of these two detectors, the same quantum field annihilation and creation processes contribute to their responses. (i.e. both their Lagrangians are linear in the quantum field.) Due to their simple design this only leaves consideration of the screening functions of these two detectors to determine equivalence. Let $\mathcal{D}_d$ and $\mathcal{D}_s$ represent the spatial-derivative and screened linear detectors respectively. The theorem is proven if it can be shown that for all s_i, $s_j \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$,

$$\mathcal{D}_d(s_i) = \mathcal{D}_d(s_j) \tag{11.18}$$

if and only if

$$\mathcal{D}_s(s_i) = \mathcal{D}_s(s_j) \tag{11.19}$$

Let W represent the screening function of these two detectors. Also note that the linear detector (denoted $\mathcal{D}_l$) and the time-derivative detector (denoted $\mathcal{D}_t$) are the "hypothetical monopole" detectors corresponding to $\mathcal{D}_s$ and $\mathcal{D}_d$ respectively. Therefore for all $s_k \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ we can write

$$\mathcal{D}_d(s_k) = \mathcal{D}_t(W(s_k)) \qquad \mathcal{D}_s(s_k) = \mathcal{D}_l(W(s_k))$$

Where $W(s_k)$ represents the effect of the screening of modes in situation s_k. Using the equalities in (11.18) and (11.19) gives

$$\mathcal{D}_d\left(s_i\right)=\mathcal{D}_t\left(W\left(s_i\right)\right) \qquad \mathcal{D}_d\left(s_j\right)=\mathcal{D}_t\left(W\left(s_j\right)\right)$$
$$\mathcal{D}_s\left(s_i\right)=\mathcal{D}_l\left(W\left(s_i\right)\right) \qquad \mathcal{D}_s\left(s_j\right)=\mathcal{D}_l\left(W\left(s_j\right)\right)$$

Now, from Chapter 9, we know that the linear detector and time-derivative detectors are $\mathcal{S}_1\cup\mathcal{S}_2\cup\mathcal{S}_3$-equivalent. Thus we have

$$\mathcal{D}_t\left(W\left(s_i\right)\right)=\mathcal{D}_t\left(W\left(s_j\right)\right) \text{ iff } \mathcal{D}_l\left(W\left(s_i\right)\right)=\mathcal{D}_l\left(W\left(s_j\right)\right)$$

So it follows that

$$\mathcal{D}_d\left(s_i\right)=\mathcal{D}_d\left(s_j\right) \text{ iff } \mathcal{D}_s\left(s_i\right)=\mathcal{D}_s\left(s_j\right)$$

Q.E.D.

Although for these simple detector models the screening function is a powerful tool, this need not be true in general. As was seen with the quadratic detector different detector's responses may involve fundamentally different processes and so even with the same screening functions, such detectors will be in-equivalent.

With the simple detector models considered so far in this thesis, questions of how these detectors work and their equivalence have been effectively reduced to questions about their corresponding hypothetical monopole and screening function. The hypothetical monopole describes the quantum annihilation and creation processes that contribute to the detector's response and the screening function gives the weighting of the (spatial) mode contributions. For more complicated detector models there may be other factors involved in determining their responses.

As an example, we note at this juncture that in this thesis only minimally coupled fields are considered. More generally the scalar field wave equation could have the form

$$\left(\Box+\xi R\left(x\right)+m^2\right)\phi\left[x\right]=0$$

which results with a direct coupling between the field modes and the background curvature $R(x)$. Although in a flat space-time the responses of a linear detector of this field may be identical to those of the minimally coupled field detector, this will not be true in a curved background. Of course, we are comparing detectors of different fields (see the following chapter for a discussion of this), however this example clearly illustrates that there may be more to describing a detector than just its screening function and hypothetical monopole. (The two detectors described here have identical screening functions and hypothetical monopole detectors.) Even so, the fundamental concepts of particle detector equivalence are still applicable.

12 Detectors and quantum fields

Although the definitions of a particle detector and detector equivalence adopted in this thesis are quite general, so far only detectors of the neutral scalar field have been discussed. This quantum field is special in that only one "species" of particles is present, that is, there is only on set of creators $a_{\boldsymbol{k}}^*$ for constructing particle states in the Fock space. However, for charged and higher spin fields there are at least "species" present and particle states can be mixtures of particles of different charge and/or spin. The presence of these extra species has a subtle but significant impact on the use of (overly simple) particle detector models.

For any "particle detector" (be it a mathematical model or actual device), if it is coupled to some field it will respond to all species of particles present of that field. For example, a machine that "clicks" when it detects an electron will also click with a positron. This follows from the PCT-invariance of the interaction Lagrangian of the detector which, in turn, arises from the requirements of invariance under Lorentz transformations and Hermiticity (Pauli 1968, Bogolubov & Shirkov 1980). Of course, this is not to say that no particle detectors can distinguish between particles and anti-particles. Modern detectors used in high energy physics obviously do. However, such detectors are quite complicated in design; far too complicated to model with the simple mathematical constructs discussed in this thesis.

Fairly simple detector models which can distinguish between, say, electrons and positrons may be constructed. For example, a (three dimensional) lattice of (regularly spaced) monopole detectors with a uniform background magnetic field would do the job. Although this model may be simple, its mathematical analysis is not. To study this particular model requires the analysis of correlations between the transition probabilities of the various monopoles in order to deduce the nature of the particle detected.

On the other hand, if we proceed using the simple monopole type detectors it will quickly be realised that such models will generally fail to satisfy condition (b) for a particle detector. (See Sec. 2.2) This condition requires that the response of the detector be (at least) a one-to-one function of the mean energy state occupation number n_k (where $k = |\boldsymbol{k}|$). To demonstrate this problem and a way of partially circumventing it, we shall consider simple monopole detectors of the charged scalar and spinor fields.

12.1 The charged scalar field

The charged scalar field is represented by a complex operator $\Phi[x]$ which satisfies the Klein-Gordon equation

$$\left(\Box + m^2\right)\Phi\left[x\right] = 0$$

The conjugate operator $\Phi^*[x]$ also satisfied the field equation and is linearly independent of $\Phi[x]$. Therefore this field consists of two oppositely charged species of particles (Bogolubov & Shirkov 1980). As with the neutral field, Φ and Φ^* may be expressed as mode integrals

$$\begin{aligned}\Phi\left[x\right] &= \int d^{n-1}k\left(b_{\boldsymbol{k}}u_{\boldsymbol{k}}\left(x\right) + a_{\boldsymbol{k}}^{*}u_{\boldsymbol{k}}^{*}\left(x\right)\right)\\ \Phi^{*}\left[x\right] &= \int d^{n-1}k\left(b_{\boldsymbol{k}}^{*}u_{\boldsymbol{k}}^{*}\left(x\right) + a_{\boldsymbol{k}}u_{\boldsymbol{k}}\left(x\right)\right)\end{aligned} \tag{12.1}$$

where $u_{\boldsymbol{k}}(x)$ are mode solutions of the field equation and

$$\left[a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}^{*}\right] = \left[b_{\boldsymbol{k}}, b_{\boldsymbol{k}'}^{*}\right] = \delta^{(n-1)}\left(\boldsymbol{k} - \boldsymbol{k}'\right)$$

with all other commutators equal to zero. The construction of a Fock space follows the same lines as for the neutral field, however there are now two sets of operators; $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^{*}$ which may be respectively interpreted as annihilation and creation operators of the anti-particles of the field, while $b_{\boldsymbol{k}}$ and $b_{\boldsymbol{k}}^{*}$ play the same respective roles for the particles. The vacuum state $\left|0\right\rangle$ is now defined by

$$a_{\boldsymbol{k}}\left|0\right\rangle = b_{\boldsymbol{k}}\left|0\right\rangle = 0$$

(For details about the Fock space representation of this field see Itzykson & Zuber 1982.)

When constructing a particle detector of the charged field, the interaction Lagrangian must satisfy charge conservation which requires invariance under simultaneous phase transformations (Bogolubov & Shirkov 1980).

$$\Phi[x] \to e^{i\alpha}\Phi[x] \qquad\qquad \Phi^*[x] \to e^{-i\alpha}\Phi^*[x]$$

This requires the interaction Lagrangian to be bilinear in Φ and Φ*, hence the simplest form is

$$L^6 = cm(\tau)\Phi^*\left[x(\tau)\right]\Phi\left[x(\tau)\right] \tag{12.2}$$

The Lagrangian (12.2) must be checked against conditions (a) and (b) required of a particle detector.

The particle states of a charged scalar quantum field are particle and anti-particle mixtures which may be represented by

$$\begin{aligned}\left|n_{\boldsymbol{k}_1}, n_{\boldsymbol{k}_2}, ..., n_{\boldsymbol{k}_j}, m_{\boldsymbol{k}_1}, m_{\boldsymbol{k}_2}, ..., m_{\boldsymbol{k}_s}\right\rangle &= \left(n_{\boldsymbol{k}_1}!n_{\boldsymbol{k}_2}!...,n_{\boldsymbol{k}_j}!m_{\boldsymbol{k}_1}!m_{\boldsymbol{k}_2}!...,m_{\boldsymbol{k}_s}!\right)^{-1/2} \times \\ &\times \left(b^*_{\boldsymbol{k}_1}\right)^{n_{k1}}\left(b^*_{\boldsymbol{k}_2}\right)^{n_{k2}}...\left(b^*_{\boldsymbol{k}_j}\right)^{n_{kj}}\left(a^*_{\boldsymbol{k}_1}\right)^{m_{k1}}\left(a^*_{\boldsymbol{k}_2}\right)^{m_{k2}}...\left(a^*_{\boldsymbol{k}_s}\right)^{m_{ks}}|0\rangle\end{aligned}$$

where $n_{\boldsymbol{k}_i}$ are the number of particles in the momentum mode $\boldsymbol{k}_i$ and $m_{\boldsymbol{k}_i}$ the number of anti-particles in that mode. Since the interaction Lagrangian is quadratic in the field operator we assume (as with the quadratic detector) that it responds only to the renormalised expectation values of the quantum state. Thus

$$A^6 = ic\langle M\rangle\int_{-\infty}^{\infty}d\tau\, e^{i\Delta E\tau}\left\langle\Psi\right|\Phi^*\left[x(\tau)\right]\Phi\left[x(\tau)\right]\left|\Psi_0\right\rangle_{ren}$$

where the subscript "*ren*" signifies the renormalised expectation value. The transition probability is

$$P^6 = c^2\left|\langle M\rangle\right|^2\int_{-\infty}^{\infty}d\tau\int_{-\infty}^{\infty}d\tau'\, e^{-i\Delta E\Delta\tau}\left\langle\Psi_0\right|\Phi\left[x(\tau)\right]\Phi^*\left[x(\tau)\right]\Phi^*\left[x(\tau')\right]\Phi\left[x(\tau')\right]\left|\Psi_0\right\rangle_{ren} \tag{12.3}$$

For a particle state in Minkowski space, the expectation value in (12.3) is

$$\left\langle n_{\boldsymbol{k}_1},....,n_{\boldsymbol{k}_j},m_{\boldsymbol{k}_1},....,m_{\boldsymbol{k}_s}\right|\Phi[x]\Phi^*[x]\Phi^*[x']\Phi[x']\left|n_{\boldsymbol{k}_1},....,n_{\boldsymbol{k}_j},m_{\boldsymbol{k}_1},....,m_{\boldsymbol{k}_s}\right\rangle_{ren}$$

Following the same approach used with the quadratic detector and noting that the particle states now contain two species, this expectation value reduces to

$$\begin{aligned}&\int d^{n-1}k\; n_k u_k^*(x)\,u_k(x')\int d^{n-1}l\; m_l u_l^*(x)\,u_l(x')+\\ &+\int d^{n-1}k\; n_k u_k^*(x)\,u_k(x')\int d^{n-1}l\,(m_l+1)\,u_l(x)u_l^*(x')+\\ &+\int d^{n-1}k\,(n_k+1)\,u_k(x)u_k^*(x')\int d^{n-1}l\; m_l u_l^*(x)\,u_l(x')+\\ &+\int d^{n-1}k\,(n_k+1)\,u_k(x)u_k^*(x')\int d^{n-1}l\,(m_l+1)\,u_l(x)u_l^*(x')\end{aligned} \tag{12.4}$$

For a detector initially in its ground state, only the first three terms will contribute to the detector's response which, using plane wave modes (2.3), is given by

$$R^{6}_{n_{\boldsymbol{k}}m_{l}}=\frac{c^{2}\left|\left\langle M\right\rangle\right|^{2}}{2\left(4\pi\right)^{n-2}\left[\Gamma\left(\left(n-1\right)/2\right)\right]^{2}}\left\{\begin{array}{l}\int\limits_{m}^{\infty}d\omega\left(\left(\Delta E+\omega\right)^{2}-m^{2}\right)^{(n-3)/2}\left(\omega^{2}-m^{2}\right)^{(n-3)/2}\times\\ \times\left[\begin{array}{l}\left(\bar{n}_{\left(\left(\Delta E+\omega\right)^{2}-m^{2}\right)^{1/2}}+1\right)\bar{n}_{\left(\omega^{2}-m^{2}\right)^{1/2}}+\\ +\left(\bar{m}_{\left(\left(\Delta E+\omega\right)^{2}-m^{2}\right)^{1/2}}+1\right)\bar{m}_{\left(\omega^{2}-m^{2}\right)^{1/2}}\end{array}\right]\theta\left(\Delta E\right)+\\ +\int\limits_{m}^{\Delta E-m}d\omega\left(\left(\Delta E-\omega\right)^{2}-m^{2}\right)^{(n-3)/2}\left(\omega^{2}-m^{2}\right)^{(n-3)/2}\times\\ \times\left[\bar{n}_{\left(\left(\Delta E-\omega\right)^{2}-m^{2}\right)^{1/2}}\bar{m}_{\left(\omega^{2}-m^{2}\right)^{1/2}}\right]\theta\left(\Delta E-2m\right)\end{array}\right\}\qquad(12.5)$$

Where

$$\bar{n}_{k}=\int d\Omega\, n_{\boldsymbol{k}}\Big/\int d\Omega\qquad \bar{m}_{k}=\int d\Omega\, m_{\boldsymbol{k}}\Big/\int d\Omega\qquad(12.6)$$

From (12.5) the Lagrangian satisfies condition (a) (i.e. no response for $n_{\boldsymbol{k}}=m_{\boldsymbol{k}}=0\ \forall\boldsymbol{k}$, the Minkowski vacuum), but it is also obvious that condition (b) is not satisfied. The detector's response is not a one-to-one function of the mean occupation numbers $\bar{n}_{k}$ and $\bar{m}_{k}$. (We could view (12.5) as an equation in two unknowns admitting an infinite number of $\bar{n}_{k}$ and $\bar{m}_{k}$ giving the same response.)

However if the set of situations into which this construct (N.B. it is not a particle detector) is placed, is restricted by the requirement $\bar{n}_{k}=\bar{m}_{k}$ then condition (b) is satisfied and (12.2) may be used as particle detector. Only situations which satisfy this condition will be considered.

Although this restriction is rather strong, we adopt it for the following reason: Firstly, all the elements of $\mathcal{S}_2'$ satisfy this condition and a major part of this chapter discusses situations in that set. Secondly, the spinor field detector introduced below has a form analogous to (12.2) and so fails to satisfy condition (b) in the same way. This detector has already been studied by Iyer and Kunar (1980).

Using $\bar{n}_{k}=\bar{m}_{k}$ in (12.4) the response of the detector may be expressed as

$$R^{6}_{n_{\boldsymbol{k}}}=c^{2}\left|\left\langle M\right\rangle\right|^{2}\int\limits_{-\infty}^{\infty}d\Delta\tau e^{-i\Delta E\tau}\left\{G^{+}\left(\Delta\tau\right)+\frac{2\left(4\pi\right)^{(1-n)/2}}{\Gamma\left(\left(n-1\right)/2\right)}\int\limits_{m}^{\infty}d\omega\,\bar{n}_{\left(\omega^{2}-m^{2}\right)^{1/2}}\left(\omega^{2}-m^{2}\right)^{(n-3)/2}\cos\omega\Delta\tau\right\}^{2}\qquad(12.7)$$

(N.B. without assuming $\bar{n}_{k}=\bar{m}_{k}$, the response of this construct is much more complicated.) The response of this detector when on a trajectory $\gamma\in\mathcal{S}_2'$ is

$$R^{6}_{\gamma}=c^{2}\left|\left\langle M\right\rangle\right|^{2}\int\limits_{-\infty}^{\infty}d\eta e^{-i\Delta E\eta}\left\{\left[G^{+}_{\gamma}\left(\eta+\tau,\tau;\tilde{\rho}\right)\right]^{2}\theta\left(-\eta\right)+\left[G^{+}_{\gamma}\left(\tau,\tau-\eta;\tilde{\rho}\right)\right]^{2}\theta\left(\eta\right)\right\}\qquad(12.8)$$

For a stationary trajectory

$$R^{6}_{\gamma}=c^{2}\left|\left\langle M\right\rangle\right|^{2}\int\limits_{-\infty}^{\infty}d\Delta\tau e^{-i\Delta E\Delta\tau}\left[G^{+}_{\gamma}\left(\Delta\tau;\tilde{\rho}\right)\right]^{2}\qquad(12.9)$$

For an element of S_3, the following vacuum expectation values must be added to the response (12.8)

$$c^2\left|\langle M\rangle\right|^2\int_{-\infty}^{\infty}d\eta\, e^{-i\Delta E\eta}\begin{Bmatrix}\left\langle\Phi\left[x(\eta+\tau)\right]\Phi^*\left[x(\eta+\tau)\right]\right\rangle_{ren}\left\langle\Phi^*\left[x(\tau)\right]\Phi\left[x(\tau)\right]\right\rangle_{ren}\theta(-\eta)+\\+\left\langle\Phi\left[x(\tau)\right]\Phi^*\left[x(\tau)\right]\right\rangle_{ren}\left\langle\Phi^*\left[x(\tau-\eta)\right]\Phi\left[x(\tau-\eta)\right]\right\rangle_{ren}\theta(\eta)\end{Bmatrix}\quad(12.10)$$

The G_γ^+ functions appearing in (12.7) to (12.9) are identical to those appearing in the corresponding equations for the quadratic detector of the neutral field. This follows from the fact that the mode functions in both cases satisfy the same (scalar field) wave equation. It is easily that this particle detector will associate any given trajectory $\gamma\in S_2$ with the same particle occupation number n_k (= m_k) that the quadratic (and linear) detector associates. In particular, a uniformly accelerating charged scalar field detector described in (12.2) will respond as though immersed in an isotropic bath of Planck radiation of both species (particles and anti-particles), in Minkowski space, with temperature T = (proper acceleration)/($2\pi k$). (N.B. since the field under consideration is a free field, there will be no particle anti-particle annihilation or scattering which would cause modification of the spectra.)

Applying the definition of particle detector equivalence to the quadratic detector (of a neutral scalar field) and the charged scalar field detector, we can also see that these two are S_2-equivalent even though they couple to different fields. (See Sec. 12.3 below.) It immediately follows that the charged scalar field detector is also S_2-equivalent to the linear and time-derivative detectors. Obviously, the charged field scalar detector is not S_1-equivalent to those detectors.

Equivalence in S_3 of the quadratic and charged scalar field detectors also follows provided

$$\left\langle\phi^2[x]\right\rangle_{ren}=\left\langle\Phi[x]\Phi^*[x]\right\rangle_{ren}$$

This equality does hold because the same Green function $G^{(1)}(x,x')$ can be used to construct both quantities. Apart from this, both these expectation values have the same functional dependence on the gravitational curvature (Birrell & Davies 1982).

12.2 The spinor field

The spinor field may be represented by the operator Υ[x] which satisfies the equation

$$\left(i\gamma^\mu\nabla_\mu-m\right)\Upsilon[x]=0\qquad(12.11)$$

This field consists of spin ½ particles with mass m, the operator Υ carrying a spinor label Υ^a (a = 1, 2, 3, 4) which has been suppressed. Similarly the Dirac matrices gm have suppressed labels γ^μ_{ab}. These matrices satisfy

$$\left\{\gamma^\mu,\gamma^\nu\right\}=2g^{\eta\nu}I$$

Where {a,b} = $ab+ba$ and I is the unit matrix. The adjoint operator $\bar{\Upsilon}[x]$ can be defined by $\bar{\Upsilon}=\Upsilon^+\gamma^0$ where Υ^+ is the transposed conjugate of Υ and satisfies

$$\bar{\Upsilon}[x]\left(i\overleftarrow{\nabla}_\mu\gamma^\mu+m\right)=0$$

where $\overleftarrow{\nabla}_\mu$ means the covariant derivative operates to the left. For an extensive introduction to spinor fields see Bogolubov & Shirkov (1980) or Itzykson & Zuber (1982).

The quantisation of this field in Minkowski and non-Minkowski spaces is well covered elsewhere (a good starting point is Birrell & Davies 1982) and so will not be presented in detail. The solution $\Upsilon[x]$ to (12.11) can be expressed as the mode integral

$$\Upsilon^a[x] = \int d^{n-1}k \sum_{\pm s}\left(b_{\boldsymbol{k},s}u^a_{\boldsymbol{k},s}(x) + d^*_{\boldsymbol{k},s}v^a_{\boldsymbol{k},s}(x)\right)$$

where $u^a_{\boldsymbol{k},s}(x), v^a_{\boldsymbol{k},s}(x)$ are spinor mode functions and $b_{\boldsymbol{k},s}, d^*_{\boldsymbol{k},s}$ are field operators. The only non-vanishing anti-commutators between the operators are

$$\left\{b_{\boldsymbol{k},s}, b^*_{\boldsymbol{k}',s'}\right\} = \left\{d_{\boldsymbol{k},s}, d^*_{\boldsymbol{k}',s'}\right\} = \delta^{(n-1)}(\boldsymbol{k}-\boldsymbol{k}')\delta_{s,s'}$$

It is immediately seen that, like the charged field, the spinor field contains more than one species of particles.

As before, the Fock space can be built up from a vacuum state $|0\rangle$, which satisfies

$$b_{\boldsymbol{k},s}|0\rangle = d_{\boldsymbol{k},s}|0\rangle = 0 \qquad \forall\boldsymbol{k}, \forall s,$$

by using creation operators $b^*_{\boldsymbol{k},s}$ and $d^*_{\boldsymbol{k},s}$. These field operators may be interpreted as follows: $b_{\boldsymbol{k},s}$ as the annihilation operator of a particle with momentum $\boldsymbol{k}$ and spin s; $b^*_{\boldsymbol{k},s}$ as its the creation operator. Similarly $d_{\boldsymbol{k},s}$ can be interpreted as the annihilation operator of an anti-particle with momentum $-\boldsymbol{k}$ and spin s with $d^*_{\boldsymbol{k},s}$ as its creation operator. (Thus there are in fact four species of particles in this field.)

The spinors $u^a_{\boldsymbol{k},s}(x)$ and $v^a_{\boldsymbol{k},s}(x)$ respectively represent positive and negative energy mode solutions and are normalised according to

$$\overline{u}^a_{\boldsymbol{k},s}(x)u^a_{\boldsymbol{k},s'}(x) = -\overline{v}^a_{\boldsymbol{k},s}(x)v^a_{\boldsymbol{k},s'}(x) = \delta_{s,s'} \tag{12.12}$$

Where $\overline{u}^a_{\boldsymbol{k},s}(x)$ represents the adjoint spinor of $u^a_{\boldsymbol{k},s}(x)$. Also

$$\begin{aligned}
\sum_{\pm s} u^a_{\boldsymbol{k},s}(x)\overline{u}^b_{\boldsymbol{k},s}(x) &= \left[\frac{\gamma^\mu k_\mu + m}{2m}\right]^{ab} = \left[\frac{\not{k}+m}{2m}\right]^{ab} = \Lambda^+(k)^{ab} \qquad m \neq 0 \\
&= \left[\frac{\gamma^\mu k_\mu}{2|k|}\right]^{ab} = \left[\frac{\not{k}}{2|k|}\right]^{ab} = \Lambda^+(k)^{ab} \qquad m \neq 0 \\
\sum_{\pm s} v^a_{\boldsymbol{k},s}(x)\overline{v}^b_{\boldsymbol{k},s}(x) &= \left[\frac{\gamma^\mu k_\mu - m}{2m}\right]^{ab} = \left[\frac{\not{k}-m}{2m}\right]^{ab} = \Lambda^-(k)^{ab} \qquad m \neq 0 \\
&= \left[\frac{\gamma^\mu k_\mu}{2|k|}\right]^{ab} = \left[\frac{\not{k}}{2|k|}\right]^{ab} = \Lambda^-(k)^{ab} \qquad m \neq 0
\end{aligned} \tag{12.13}$$

The particle states are constructed as usual except that, due to the anti-commuting operators, the states must satisfy Fermi statistics. A particle state may be represented by

$$\begin{aligned}&\left|n_{\boldsymbol{k}_1,s_1},n_{\boldsymbol{k}_2,s_2},\ldots,n_{\boldsymbol{k}_j,s_j},m_{\boldsymbol{k}_1,s_1},m_{\boldsymbol{k}_2,s_2},\ldots,m_{\boldsymbol{k}_i,s_i}\right\rangle=\\&\qquad\left(b^*_{\boldsymbol{k}_1,s_1}\right)^{n_{\boldsymbol{k}_1,s_1}}\left(b^*_{\boldsymbol{k}_2,s_2}\right)^{n_{\boldsymbol{k}_2,s_2}}\ldots\left(b^*_{\boldsymbol{k}_j,s_j}\right)^{n_{\boldsymbol{k}_j,s_j}}\left(d^*_{\boldsymbol{k}_1,s_1}\right)^{m_{\boldsymbol{k}_1,s_1}}\left(d^*_{\boldsymbol{k}_2,s_2}\right)^{m_{\boldsymbol{k}_2,s_2}}\ldots\left(d^*_{\boldsymbol{k}_i,s_i}\right)^{m_{\boldsymbol{k}_i,s_i}}\left|0\right\rangle\end{aligned}\tag{12.14}$$

Where the particle occupation numbers (for the state $\boldsymbol{k},s$) $n_{\boldsymbol{k},s}$, $m_{\boldsymbol{k},s}$ are either one or zero. The state is anti-symmetric in the operators hence if $\boldsymbol{k}_i=\boldsymbol{k}_j$ and $s_i=s_j$ for any $i\neq j$, then the action of b's and d's on the state is zero. The states (12.14) form the basis of the Fock space and are normalised according to

$$\begin{aligned}&\left\langle n_{\boldsymbol{k}_1,s_1},n_{\boldsymbol{k}_2,s_2},\ldots,n_{\boldsymbol{k}_j,s_j},m_{\boldsymbol{k}_1,s_1},m_{\boldsymbol{k}_2,s_2},\ldots,m_{\boldsymbol{k}_i,s_i}\right|p_{\boldsymbol{k}'_1,s'_1},p_{\boldsymbol{k}'_2,s'_2},\ldots,p_{\boldsymbol{k}'_r,s'_r},q_{\boldsymbol{k}'_1,s'_1},q_{\boldsymbol{k}'_2,s'_2},\ldots,q_{\boldsymbol{k}'_t,s'_t}\Big\rangle=\\&=\delta_{ir}\delta_{jt}\left\{\sum_{\wp_1}\delta_{n_{\boldsymbol{k}_1},p_{\boldsymbol{k}'_{\wp_1(1)}}}\ldots\delta_{n_{\boldsymbol{k}i},p_{\boldsymbol{k}'_{\wp_1(r)}}}\delta\left(\boldsymbol{k}_1-\boldsymbol{k}'_{\wp_1(1)}\right)\delta_{s_1s'_{\wp_1(1)}}\ldots\delta\left(\boldsymbol{k}_i-\boldsymbol{k}'_{\wp_1(r)}\right)\delta_{s_is'_{\wp_1(r)}}\right\}\times\\&\times\left\{\sum_{\wp_2}\delta_{m_{\boldsymbol{k}_1},q_{\boldsymbol{k}'_{\wp_2(1)}}}\ldots\delta_{m_{\boldsymbol{k}_j},p_{\boldsymbol{k}'_{\wp_2(j)}}}\delta\left(\boldsymbol{k}_1-\boldsymbol{k}'_{\wp_2(1)}\right)\delta_{s_1s'_{\wp_2(1)}}\ldots\delta\left(\boldsymbol{k}_j-\boldsymbol{k}'_{\wp_2(t)}\right)\delta_{s_js'_{\wp_2(t\theta)}}\right\}\end{aligned}$$

where the sums are over signed permutations $\wp_1$ and $\wp_2$ of the integers 1,….,r and 1,….,t respectively. The number operators are given by

$$N_{\boldsymbol{k},s}=b^*_{\boldsymbol{k},s}b_{\boldsymbol{k},s}\ \text{ for particles}$$

$$M_{\boldsymbol{k},s}=d^*_{\boldsymbol{k},s}d_{\boldsymbol{k},s}\ \text{ for anti-particles}$$

which give the number of particles (anti-particles) in momentum state $\boldsymbol{k}$ with spin s. For the total number operators

$$N=\int d^{n-1}k\sum_{\pm s}N_{\boldsymbol{k},s}\ \text{ for particles}$$

$$M=\int d^{n-1}k\sum_{\pm s}M_{\boldsymbol{k},s}\ \text{ for anti-particles}$$

The interaction Lagrangian must satisfy the usual requirements of invariance. The simplest interaction Lagrangain is

$$L^7=cm(\tau)\overline{\Upsilon}\left[x(\tau)\right]\Upsilon\left[x(\tau)\right]=cm(\tau)\overline{\Upsilon}^a\left[x(\tau)\right]\Upsilon^a\left[x(\tau)\right]\tag{12.15}$$

where the spinor index is summed over. Since this detector is quadratically coupled to the field we assume, as usual, that it responds only to the renormalised field expectation values. To first order in the coupling constant the transition amplitude is

$$A^7=ic\langle M\rangle\int_{-\infty}^{\infty}d\tau\,e^{i\Delta E\tau}\left\langle\Psi\right|\overline{\Upsilon}^a\left[x(\tau)\right]\Upsilon^a\left[x(\tau)\right]\left|\Psi_0\right\rangle_{ren}$$

Where $\left|\Psi\right\rangle$ and $\left|\Psi_0\right\rangle$ are now spinor field states. The transition probability is

$$P^7 = c^2 \left|\left\langle M \right\rangle\right|^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{-i\Delta E \Delta\tau} \left\langle \Psi_0 \right| \Upsilon^b\left[x(\tau)\right] \overline{\Upsilon}^b\left[x(\tau)\right] \overline{\Upsilon}^a\left[x(\tau')\right] \Upsilon^a\left[x(\tau')\right] \left|\Psi_0\right\rangle_{ren} \quad (12.16)$$

To calculate the response of this detector to a particle state in Minkowski space we write

$$\left\langle n_{\boldsymbol{k}_1,s_1},\ldots,n_{\boldsymbol{k}_j,s_j},m_{\boldsymbol{k}_1,s_1},\ldots,m_{\boldsymbol{k}_i,s_i} \right| \Upsilon^b[x] \overline{\Upsilon}^b[x] \overline{\Upsilon}^a[x'] \Upsilon^a[x'] \left| n_{\boldsymbol{k}_1,s_1},\ldots,n_{\boldsymbol{k}_j,s_j},m_{\boldsymbol{k}_1,s_1}\ldots,m_{\boldsymbol{k}_i,s_i} \right\rangle$$

Using the anti-commutation relations and removing the vacuum divergence yields

$$\begin{aligned}
&-\int d^{n-1}k \sum_{\pm r} n_{\boldsymbol{k},r} \overline{u}^a_{\boldsymbol{k},r}(x) u^a_{\boldsymbol{k},r}(x) \int d^{n-1}l \sum_{\pm s} n_{\boldsymbol{l},s} \overline{u}^a_{\boldsymbol{l},s}(x') u^a_{\boldsymbol{l},s}(x') \\
&+\int d^{n-1}k \sum_{\pm r} n_{\boldsymbol{k},r} \overline{u}^b_{\boldsymbol{k},r}(x) u^a_{\boldsymbol{k},r}(x') \int d^{n-1}l \sum_{\pm s} \left(n_{\boldsymbol{l},s}-1\right) u^b_{\boldsymbol{l},s}(x) \overline{u}^a_{\boldsymbol{l},s}(x') \\
&+\int d^{n-1}k \sum_{\pm r} n_{\boldsymbol{k},r} u^b_{\boldsymbol{k},r}(x) \overline{u}^b_{\boldsymbol{k},r}(x) \int d^{n-1}l \sum_{\pm s} m_{\boldsymbol{l},s} \overline{v}^a_{\boldsymbol{l},s}(x') v^a_{\boldsymbol{l},s}(x') \\
&+\int d^{n-1}k \sum_{\pm r} \left(n_{\boldsymbol{k},r}-1\right) u^b_{\boldsymbol{k},r}(x) \overline{u}^a_{\boldsymbol{k},r}(x') \int d^{n-1}l \sum_{\pm s} \left(m_{\boldsymbol{l},s}-1\right) \overline{v}^b_{\boldsymbol{l},s}(x) v^a_{\boldsymbol{l},s}(x') \\
&+\int d^{n-1}k \sum_{\pm r} n_{\boldsymbol{k},r} \overline{u}^b_{\boldsymbol{k},r}(x) u^a_{\boldsymbol{k},r}(x') \int d^{n-1}l \sum_{\pm s} m_{\boldsymbol{l},s} v^b_{\boldsymbol{l},s}(x) \overline{v}^a_{\boldsymbol{l},s}(x') \\
&+\int d^{n-1}k \sum_{\pm r} m_{\boldsymbol{k},r} v^a_{\boldsymbol{k},r}(x) \overline{v}^a_{\boldsymbol{k},r}(x) \int d^{n-1}l \sum_{\pm s} n_{\boldsymbol{l},s} \overline{u}^b_{\boldsymbol{l},s}(x') u^b_{\boldsymbol{l},s}(x') \\
&-\int d^{n-1}k \sum_{\pm r} m_{\boldsymbol{k},r} v^a_{\boldsymbol{k},r}(x) \overline{v}^a_{\boldsymbol{k},r}(x) \int d^{n-1}l \sum_{\pm s} m_{\boldsymbol{l},s} \overline{v}^b_{\boldsymbol{l},s}(x') v^b_{\boldsymbol{l},s}(x') \\
&+\int d^{n-1}k \sum_{\pm r} m_{\boldsymbol{k},r} v^b_{\boldsymbol{k},r}(x) \overline{v}^a_{\boldsymbol{k},r}(x') \int d^{n-1}l \sum_{\pm s} \left(m_{\boldsymbol{l},s}-1\right) \overline{v}^b_{\boldsymbol{l},s}(x) v^a_{\boldsymbol{l},s}(x')
\end{aligned} \quad (12.17)$$

From (12.13), all the terms involving quantities of the form $\overline{u}^a_{\boldsymbol{k},s}(x) u^a_{\boldsymbol{k},s}(x)$ and $v^a_{\boldsymbol{k},s}(x) \overline{v}^a_{\boldsymbol{k},s}(x)$ are independent of τ and τ' and so make no contribution to the response. These terms shall be discarded. Introducing plane wave modes

$$u^a_{\boldsymbol{k},s}(\boldsymbol{x}) = \frac{u^a_{\boldsymbol{k},r} \exp\left(i\boldsymbol{k}.\boldsymbol{x} - i\omega t\right)}{\left[(2\pi)^{n-1}(\omega/m)\right]^{1/2}}$$

$$v^a_{\boldsymbol{k},s}(\boldsymbol{x}) = \frac{v^a_{\boldsymbol{k},r} \exp\left(-i\boldsymbol{k}.\boldsymbol{x} + i\omega t\right)}{\left[(2\pi)^{n-1}(\omega/m)\right]^{1/2}}$$

with the detector the detector stationary, the contributing terms (12.17) reduce to

$$\int d^{n-1}k\int d^{n-1}l\,\frac{m^2e^{i(\omega-\rho)\Delta\tau}}{\omega\rho(2\pi)^{2n-2}}\sum_{\pm r}n_{\boldsymbol{k},r}\overline{u}^b_{\boldsymbol{k},r}u^a_{\boldsymbol{k},r}\sum_{\pm s}\left(n_{\boldsymbol{l},s}-1\right)u^b_{\boldsymbol{l},s}\overline{u}^a_{\boldsymbol{l},s}$$
$$+\int d^{n-1}k\int d^{n-1}l\,\frac{m^2e^{-i(\omega+\rho)\Delta\tau}}{\omega\rho(2\pi)^{2n-2}}\sum_{\pm r}\left(n_{\boldsymbol{k},r}-1\right)u^b_{\boldsymbol{k},r}\overline{u}^a_{\boldsymbol{k},r}\sum_{\pm s}\left(m_{\boldsymbol{l},s}-1\right)\overline{v}^b_{\boldsymbol{l},s}v^a_{\boldsymbol{l},s}$$
$$+\int d^{n-1}k\int d^{n-1}l\,\frac{m^2e^{i(\omega+\rho)\Delta\tau}}{\omega\rho(2\pi)^{2n-2}}\sum_{\pm r}n_{\boldsymbol{k},r}\overline{u}^b_{\boldsymbol{k},r}u^a_{\boldsymbol{k},r}\sum_{\pm s}m_{\boldsymbol{l},s}v^b_{\boldsymbol{l},s}\overline{v}^a_{\boldsymbol{l},s}$$
$$+\int d^{n-1}k\int d^{n-1}l\,\frac{m^2e^{i(\omega-\rho)\Delta\tau}}{\omega\rho(2\pi)^{2n-2}}\sum_{\pm r}m_{\boldsymbol{k},r}v^b_{\boldsymbol{k},r}\overline{v}^a_{\boldsymbol{k},r}\sum_{\pm s}\left(m_{\boldsymbol{l},s}-1\right)\overline{v}^b_{\boldsymbol{l},s}v^a_{\boldsymbol{l},s}$$

Assuming the probability of a particle being in the +s spin state equals that of being in the –s spin state, we have $n_{\boldsymbol{k},s}$ $(=n_{\boldsymbol{k}}/2)$ and $m_{\boldsymbol{k},s}=(m_{\boldsymbol{k}}/2)$. Also, using (12.13) the above expression can be written as

$$\begin{aligned}&\int d^{n-1}k\int d^{n-1}l\,\frac{m^2e^{i(\omega-\rho)\Delta\tau}}{\omega\rho(2\pi)^{2n-2}}n_{\boldsymbol{k}}\left(n_{\boldsymbol{l}}-2\right)\mathrm{Tr}\left[\Lambda^+(k)\Lambda^+(l)\right]\\&-\int d^{n-1}k\int d^{n-1}l\,\frac{m^2e^{-i(\omega+\rho)\Delta\tau}}{\omega\rho(2\pi)^{2n-2}}\left(n_{\boldsymbol{k}}-2\right)\left(m_{\boldsymbol{l}}-2\right)\mathrm{Tr}\left[\Lambda^+(k)\Lambda^-(l)\right]\\&+\int d^{n-1}k\int d^{n-1}l\,\frac{m^2e^{i(\omega+\rho)\Delta\tau}}{\omega\rho(2\pi)^{2n-2}}n_{\boldsymbol{k}}m_{\boldsymbol{l}}\mathrm{Tr}\left[\Lambda^+(k)\Lambda^-(l)\right]\\&+\int d^{n-1}k\int d^{n-1}l\,\frac{m^2e^{i(\omega-\rho)\Delta\tau}}{\omega\rho(2\pi)^{2n-2}}m_{\boldsymbol{k}}\left(m_{\boldsymbol{l}}-2\right)\mathrm{Tr}\left[\Lambda^-(k)\Lambda^-(l)\right]\end{aligned}\tag{12.18}$$

Again using (12.13) and noting that (Itzykson & Zuber 1982)

$$\mathrm{Tr}\left[\not{a}\not{b}\right]=4g^{\mu\nu}a_\mu b_\nu=4a^\mu b_\mu$$

The traces can be evaluated

$$\begin{aligned}&\mathrm{Tr}\left[\Lambda^+(k)\Lambda^+(l)\right]=\mathrm{Tr}\left[\Lambda^-(k)\Lambda^-(l)\right]\\&\quad=\frac{k^\mu l_\mu}{m^2}+1\qquad m\neq0\\&\quad=\frac{k^\mu l_\mu}{\omega\rho}\qquad m=0\\&-\mathrm{Tr}\left[\Lambda^+(k)\Lambda^-(l)\right]\\&\quad=\frac{k^\mu l_\mu}{m^2}-1\qquad m\neq0\\&\quad=\frac{k^\mu l_\mu}{\omega\rho}\qquad m=0\end{aligned}\tag{12.19}$$

To proceed further it is convenient to use the spherical coordinatisation of Minkowski space

$$\int d^{n-1}k=\int dk\,k^{n-1}\int d\Omega\tag{12.20}$$

Also, define the angular quantity θ_{kl} by

$$\boldsymbol{k}.\boldsymbol{l} = kl\cos\theta_{kl} \tag{12.21}$$

Where $k = |\boldsymbol{k}|$. Using (12.19) to (12.21) in (12.18) yields

$$\frac{(4\pi)^{(1-n)}}{\left[\Gamma\left((n-1)/2\right)\right]^2}\int dk\int dl\,\frac{k^{n-2}l^{n-2}}{\omega\rho}\left\{\begin{aligned}&\left[\bar{n}_k\left(\bar{n}_l-2\right)\left(\omega\rho+m^2\right)-klA_{kl}\left(n_{\boldsymbol{k}},n_{\boldsymbol{l}}\right)\right]e^{i(\omega-\rho)\Delta\tau}\\&+\left[\left(\bar{n}_k-2\right)\left(\bar{m}_l-2\right)\left(\omega\rho-m^2\right)-klA_{kl}\left(n_{\boldsymbol{k}},m_{\boldsymbol{l}}\right)\right]e^{-i(\omega+\rho)\Delta\tau}\\&+\left[\bar{n}_k\bar{m}_l\left(\omega\rho-m^2\right)-klA_{kl}\left(n_{\boldsymbol{k}},m_{\boldsymbol{l}}\right)\right]e^{i(\omega+\rho)\Delta\tau}\\&+\left[\bar{m}_k\left(\bar{m}_l-2\right)\left(\omega\rho-m^2\right)-klA_{kl}\left(m_{\boldsymbol{k}},m_{\boldsymbol{l}}\right)\right]e^{i(\omega-\rho)\Delta\tau}\end{aligned}\right\} \tag{12.22}$$

where

$$\bar{n}_k = \int d\Omega\, n_{\boldsymbol{k}}\Big/\int d\Omega$$

and

$$A_{kl}\left(n_k,m_l\right) = \int d\Omega_k\int d\Omega_l n_{\boldsymbol{k}}m_{\boldsymbol{l}}\cos\theta_{kl}\Big/\left[\int d\Omega\right]^2 \tag{12.23}$$

From (12.22), the response of this detector when initially in its ground state is

$$R^7_{n_{\boldsymbol{k}},m_l} = \frac{c^2\left|\langle M\rangle\right|^2}{2(4\pi)^{n-2}\left[\Gamma\left((n-1)/2\right)\right]^2}\times \left\{\begin{aligned}&\int_m^\infty d\omega\, k^{n-3}l^{n-3}\left\{\begin{aligned}&\left[\bar{n}_k\left(\bar{n}_l-2\right)+\bar{m}_k\left(\bar{m}_k-2\right)\right]\left(\omega\left(\Delta E+\omega\right)+m^2\right)\\&-kl\left[A_{kl}\left(n_{\boldsymbol{k}},n_{\boldsymbol{l}}\right)+A_{kl}\left(m_{\boldsymbol{k}},m_{\boldsymbol{l}}\right)\right]\end{aligned}\right\}\theta\left(\Delta E\right)\\&+\int_m^{\Delta E-m} d\omega\, k^{n-3}\hat{l}^{n-3}\left\{\bar{n}_k\bar{m}_{\hat{l}}\left(\omega\left(\Delta E-\omega\right)-m^2\right)-k\hat{l}A_{k\hat{l}}\left(n_{\boldsymbol{k}},m_{\hat{\boldsymbol{l}}}\right)\right\}\theta\left(\Delta E-2m\right)\end{aligned}\right\} \tag{12.24}$$

in which $l = \left(\left(\Delta E+\omega\right)^2-m^2\right)^{1/2}$ and $\hat{l} = \left(\left(\Delta E-\omega\right)^2-m^2\right)^{1/2}$. For the massless case

$$R^7_{n_{\boldsymbol{k}},m_l} = \frac{c^2\left|\langle M\rangle\right|^2\theta\left(\Delta E\right)}{8(4\pi)^{n-2}\left[\Gamma\left((n-1)/2\right)\right]^2}\times \left\{\begin{aligned}&\int_0^\infty d\omega\,\omega^{n-2}\left(\Delta E+\omega\right)^{n-2}\left\{\begin{aligned}&\left[\bar{n}_k\left(\bar{n}_k-2\right)+\bar{m}_k\left(\bar{m}_k-2\right)\right]\\&-A_{\omega\rho}\left(n_{\boldsymbol{k}},n_{\boldsymbol{l}}\right)-A_{\omega\rho}\left(m_{\boldsymbol{k}},m_{\boldsymbol{l}}\right)\end{aligned}\right\}\\&+\int_0^{\Delta E} d\omega\,\omega^{n-2}\left(\Delta E-\omega\right)^{n-2}\left\{\bar{n}_k\bar{n}_{\hat{l}}-A_{\omega\rho}\left(n_{\boldsymbol{k}},m_{\hat{\boldsymbol{l}}}\right)\right\}\theta\left(\Delta E-2m\right)\end{aligned}\right\} \tag{12.25}$$

From (12.24) and (12.25), the Lagrangian (12.15) fails to qualify as a particle detector for the same reason as the charged field detector. Therefore, the situations are restricted by requiring $n_{\boldsymbol{k}} = m_{\boldsymbol{k}}$. Further, since this detector is omni-directional, by Theorem 1 its responses in this set of particle states can be labelled by its response to the isotropic particle states in that set. Requiring $n_{\boldsymbol{k}}$ (= $m_{\boldsymbol{k}}$) to be isotropic greatly simplifies the form of the response since, from (12.23) $A_{kl}(n_k,m_l)$ vanishes for such states. For these states the response can be written in the form

$$R^7_{n_k} = 4c^2\left|\langle M\rangle\right|^2 \int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E\Delta\tau}\left\{\begin{array}{l}\left[\int\frac{d^{n-1}k}{(2\pi)^{n-1}}n_k\cos\omega\Delta\tau - 2i\frac{d}{d\tau}G^+(\Delta\tau)\right]^2 \\ +m^2\left[\int\frac{d^{n-1}k}{(2\pi)^{n-1}\omega}n_k\sin\omega\Delta\tau - 2iG^+(\Delta\tau)\right]^2\end{array}\right\} \tag{12.26}$$

where we have used

$$S^\pm(x,x') = \left(i\gamma^\mu\partial_\mu + m\right)G^\pm(x,x')$$

The momentum integrals can be converted to spherical form (c.f. (12.7) for the charged scalar field detector), but there is little advantage in doing this for our purposes.

With these simplifying assumptions it is still not immediately obvious that condition (b) for a particle detector is satisfied (although (12.25) does have the form of an auto-correlation function of n_k). In particular, (12.26) does not have the simple form characteristic of the corresponding scalar field detector response (12.7) which manifestly satisfied condition (b). However, it will now be demonstrated that (12.26) does satisfy that condition. This requires proving that if $R^7_{n_k} = R^7_{\hat{n}_k}$ then $n_k = \hat{n}_k$. From (12.26), if we assume $R^7_{n_k} = R^7_{\hat{n}_k}$, then

$$\begin{aligned}&\left[\int\frac{d^{n-1}k}{(2\pi)^{n-1}}n_k\cos\omega\Delta\tau - 2i\frac{d}{d\tau}G^+(\Delta\tau)\right]^2 + m^2\left[\int\frac{d^{n-1}k}{(2\pi)^{n-1}\omega}n_k\sin\omega\Delta\tau - 2iG^+(\Delta\tau)\right]^2 \\ &= \left[\int\frac{d^{n-1}k}{(2\pi)^{n-1}}\hat{n}_k\cos\omega\Delta\tau - 2i\frac{d}{d\tau}G^+(\Delta\tau)\right]^2 + m^2\left[\int\frac{d^{n-1}k}{(2\pi)^{n-1}\omega}\hat{n}_k\sin\omega\Delta\tau - 2iG^+(\Delta\tau)\right]^2\end{aligned} \tag{12.27}$$

This equation is meaningful only if the integrals converge, which in turn requires

$$\int_m^\infty d\omega k^{n-3}n_k < \infty \qquad \int_m^\infty d\omega k^{n-3}\hat{n}_k < \infty$$

We shall assume these inequalities are satisfied. Expanding (12.27) and cancelling like terms yields an identity which contains real and imaginary parts. The imaginary part of the identity can be written as

$$\int d^{n-1}k\left(n_k - \hat{n}_k\right)\cos\omega\Delta\tau\,\Re e\left\{\frac{d}{dt}G^+(\Delta\tau)\right\} = m^2\int\frac{d^{n-1}k}{\omega}\left(n_k - \hat{n}_k\right)\sin\omega\Delta\tau\,\Re e\left\{G^+(\Delta\tau)\right\} \tag{12.28}$$

In Minkowski space the Wightman function is (Birrell & Davies 1982)

$$G^+(\Delta\tau) = \frac{\pi}{(4\pi i)^{n/2}}\left(\frac{2im}{\Delta\tau}\right)^{(n-2)/2} H^{(2)}_{\frac{n}{2}-1}(m\Delta\tau) \tag{12.29}$$

The function $\Re e\left\{G^+(\Delta\tau)\right\}$ depends upon the dimension of the space-time, so the case $n = 4$ will be considered. Even so, the argument will be seen to hold for al l $n > 2$. In four dimensions, (12.29) becomes

$$G^{+}(\Delta\tau)=\frac{im}{8\pi\Delta\tau}\left[J_1(m\Delta\tau)-iN(m\Delta\tau)\right]$$

And

$$\mathrm{Re}\left\{G^{+}(\Delta\tau)\right\}=\frac{m}{8\pi\Delta\tau}N_1(m\Delta\tau)$$
$$\mathrm{Re}\left\{\frac{d}{d\tau}G^{+}(\Delta\tau)\right\}=\frac{d}{d\tau}\mathrm{Re}\left\{G^{+}(\Delta\tau)\right\}=\frac{-m}{4\pi(\Delta\tau)^2}N_1(m\Delta\tau)+\frac{m^2}{8\pi\Delta\tau}N_0(m\Delta\tau)$$

Using these identities in (12.28)

$$\begin{aligned}&\left[\frac{-N_1(m\Delta\tau)}{\Delta\tau}+\frac{m}{2}N_0(m\Delta\tau)\right]\int d^{n-1}k\,(n_k-\hat{n}_k)\cos\omega\Delta\tau=\\&=-m^2N_1(m\Delta\tau)\int\frac{d^{n-1}k}{\omega}(n_k-\hat{n}_k)\sin\omega\Delta\tau\end{aligned}\qquad(12.30)$$

Assuming $n_k\neq\hat{n}_k$, (12.30) should hold for all $\Delta\tau$. However, in the region $\Delta\tau\sim 0$ the Bessel functions behave like;

$$N_1(x)\sim-2/\pi x\qquad N_0(x)\sim\frac{2}{\pi}\ln x\qquad x\sim0^{+}$$

Which when placed into (12.30) yields

$$\begin{aligned}&\left[\frac{2}{m(\Delta\tau)^2}+\frac{m}{2}\ln(m\Delta\tau)\right]\int d^{n-1}k\,(n_k-\hat{n}_k)\cos\omega\Delta\tau=\\&=\frac{2}{\Delta\tau}\int\frac{d^{n-1}k}{\omega}(n_k-\hat{n}_k)\sin\omega\Delta\tau\end{aligned}$$

The only way for this equality to be satisfied if is for both integrals to be zero, which in turn requires $n_k=\hat{n}_k$.Thus condition (b) is satisfied.

Turning now to the set $\mathcal{S}_2$, this detector's response will be given by

$$P^7=c^2\left|\langle M\rangle\right|^2\int_{-\infty}^{\infty}d\tau\int_{-\infty}^{\infty}d\tau'\,e^{-i\Delta E\Delta\tau}\langle 0_M|\Upsilon^a[x(\tau)]\overline{\Upsilon}^a[x(\tau)]\overline{\Upsilon}^b[x(\tau')]\Upsilon^b[x(\tau')]|0_M\rangle$$

which gives

$$R_\gamma^7=c^2\left|\langle M\rangle\right|^2\int_{-\infty}^{\infty}d\eta\,e^{-i\Delta E\eta}\left\{\begin{array}{l}\mathrm{Tr}\left[\left(S_\gamma^{+}(\eta+\tau,\tau;\tilde{\rho})\right)^2\right]\theta(-\eta)+\\+\mathrm{Tr}\left[\left(S_\gamma^{+}(\tau,\tau-\eta;\tilde{\rho})\right)^2\right]\theta(\eta)\end{array}\right\}$$

For stationary trajectories

$$R_\gamma^7=c^2\left|\langle M\rangle\right|^2\int_{-\infty}^{\infty}d\eta\,e^{-i\Delta E\Delta\tau}\mathrm{Tr}\left[\left(S_\gamma^{+}(\Delta\tau;\tilde{\rho})\right)^2\right]\qquad(12.31)$$

where $S_{\gamma}^{+}(x, x')$ is the spinor field Green function expressed in the detector's frame.

Using (12.26) and (12.31) a correspondence between trajectories in $\mathcal{S}_2'$ and isotropic particle states n_k (with $n_k = m_k$) which satisfy

$$\int_{m}^{\infty} d\omega k^{n-3} n_k < \infty$$

can be constructed for the spinor field. Using the results from Sections 9.1.5 and 12.1, a very similar correspondence can also be constructed for the charged scalar field. We shall now study these correspondences in more detail, particularly the possibility of constructing symmetries between fields of different statistics.

12.3 Particle detectors and field statistics

When defining particle detector equivalence in Chapter 8, the concept of "detector equivalent situations" was introduced and utilised in that definition. This concept is applicable to any particle detector, irrespective of its complexity and the fields to which it is coupled. One could day that, at this level, we are merely comparing detector read-outs when placed in different situations. In fact the actual nature of the quantum field being detected appears only in the specification of the "situation" into which the detector is placed. Again, referring back to Chapter 8, a list of five factors required to specify a detector's situation was presented. In reviewing that list, we see that only the first factor (the state of the quantum field) actually refers to the field to which the detector is coupled. If a quantum state could be found that may, in some way, be considered identical for all quantum fields, then the specification of that situation will become truly field independent. This may then lead to the construction of correspondences or symmetries between different fields and possibly fields of different statistics.

There is (at least) one very good candidate for this quantum state; the Minkowski vacuum. The mathematically rigorous construction of the Minkowski vacuum is now well understood (see Segal 1963, Bogolubov, Logunov & Todorov 1975), as is the construction of the Fock space in Minkowski quantum field theory. The Fock space for a quantum field can be represented by the Hilbert space $\mathcal{H}$tha1975)t is constructed as a direct sum of (n-particle) Hilbert spaces $\hbar_n$ (Bogolubov, Logunov & Todorov)

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \hbar_n$$

For a bosonic field, the $\hbar_n$ are symmetrised direct products of one particle states

$$\hbar_n = \mathrm{sym}\left(\underbrace{\hbar_1 \otimes \dots\dots \otimes \hbar_1}_{n \; spaces} \right)$$

For a fermionic field the re anti-symmetrised direct products

$$\hbar_n = \text{anti-sym}\left(\underbrace{\hbar_1 \otimes \dots\dots \otimes \hbar_1}_{n \; spaces} \right)$$

For both types of fields, the (Minkowski) vacuum is represented by the space $\hbar_0$ which is isomorphic to the complex numbers C. Therefore, although the Fock spaces of bosonic and fermionic fields are different, the vacuum states (i.e. their Minkowski vacua) are isomorphic.

From the nature of their Fock spaces, the field statistics are reflected in the construction of all particle state except the vacuum state. Hence, in this context, the Minkowski vacuum can be considered field independent. With this standpoint, the situations in $\mathcal{S}_2$ (and more particularly $\mathcal{S}_2'$) are now field independent. Thus the equivalence (or in-equivalence) of detectors of different quantum fields in $\mathcal{S}_2$ can be discussed. To illustrate this point, consider the charged scalar field and spinor field detectors introduced above. In Sec. 9.1.3 it was shown that a quadratic detector equivalence class in $\mathcal{S}_2$ consists of all trajectories related by Poincare transformations. Now the condition for $\gamma_1, \gamma_2 \in \mathcal{S}_2$ to be spinor detector equivalent is

$$\mathrm{Tr}\left[\left(S_{\gamma_1}^{+}\left(\tau,\tau';\tilde{\rho}\right)\right)^2\right]=\mathrm{Tr}\left[\left(S_{\gamma_2}^{+}\left(\tau+\beta,\tau'+\beta;\tilde{\xi}\right)\right)^2\right] \tag{12.32}$$

For the charged scalar field detector to be $\mathcal{S}_2$-equivalent to this detector, the condition (12.32) must be satisfied if and only if γ_1 and γ_2 are related by a Poincare transformation. A proof that this requirement is satisfied cannot be constructed in the same way as for the scalar field detector because of the more complicated structure of the spinor Green function S_{γ}^{+} and the presence of the trace.

For the scalar Green function, the transformation resulting from a change to new coordinates $\overline{x}$ from old x is a bi-scalar transformation;

$$G^{+}\left(x,x'\right)\to G^{+}\left(x\left(\overline{x}\right),x\left(\overline{x}'\right)\right)$$

However, the quantity $S^{+}(x,x')$ is a coordinate bi-scalar and a Lorentz bi-spinor (Weinberg 1972). Therefore it transforms as

$$S^{+}\left(x,x'\right)\to D\left(\Lambda\left(\overline{x}\right)\right)D\left(\Lambda\left(\overline{x}'\right)\right)S^{+}\left(x\left(\overline{x}\right),x\left(\overline{x}'\right)\right) \tag{12.33}$$

where $D(\Lambda(x))$ is the spin ½ irreducible representation of the Lorentz group

$$D\left(\Lambda\right)=1+\frac{1}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}$$

and

$$\Sigma_{\mu\nu}=\frac{1}{4}\left[\overline{\gamma}_{\mu},\overline{\gamma}_{\nu}\right]$$

The matrices γ_μ are the appropriate Dirac matrices for the new coordinatisation and are given by

$$\overline{\gamma}^{\mu}=V_{\alpha}^{\mu}\gamma^{\alpha}$$

In this V_{α}^{μ} is the n-bein and γ^{α} the Minkowski space Dirac matrices (Weinberg 1972).

Now if

$$S_{\gamma_1}^{+}\left(\tau,\tau';\tilde{\rho}\right)=S_{\gamma_2}^{+}\left(\tau+\beta,\tau'+\beta;\tilde{\xi}\right) \tag{12.34}$$

then (12.32) is satisfied and further, using (12.33), it follows that γ_1 and γ_2 are related by a Poincare transformation. This result corresponds to that for the scalar field. However, the equality (12.32) does not exclusively imply (12.34) since for matrices A and B, with A ≠ B, we can still have $A^2 = B^2$ and/or Tr[A] = Tr[B]. Thus the spinor detector equivalent classes could quite conceivably be different from the charged scalar detector equivalent classes.

This $\mathcal{S}_2$-inequivalence of these two detector seems counter-intuitive for the following reasons;

1. Assume that two trajectories γ_1 and γ_2 are not related by a Poincare transformation but satisfy (12.32). With this we have, from Sec. 9.1.3 $G^+_{\gamma_1} \neq G^+_{\gamma_2}$. Both G^+_{γ} and S^+_{γ} transform as coordinate scalars (where G^+_{γ} also transforms as a Lorentz scalar where-as S^+_{γ} transforms as a Lorentz spinor). It is difficult to see how the Lorentz spinor transform relating γ_1 and γ_2 can somehow "cancel out" the effect of the coordinate scalar transformation relating the two trajectories so as to give (12.32). Especially in view of the fact that the spinor transformation is merely a (faithful) representation of the transformation relating the two trajectories (Weinberg 1972).
2. In Sec. 9.1.3 we saw how for γ_1 and γ_2 with $G^+_{\gamma_1} = G^+_{\gamma_2}$, the relationship between the two trajectories can at most be a Poincare transform since only with these transformations is the Minkowski vacuum invariant. (All other transformations result in mixing of positive and negative frequencies.) It is difficult to see how this would not also be the cased for the spinor field.
3. Considering the stationary trajectories in interpreting $S^+_{\gamma}(x,x')$ as a measure of the vacuum fluctuations it seems counter-intuitive that two trajectories that are not related by a Poincare transformation should have identical vacuum fluctuations power spectra. However, this point is not as strong as the above two because the detector does not respond directly to $S^+_{\gamma}(x,x')$ but to the trace of its square.

Apart from the above reasons for believing the charged scalar and spinor field detectors are $\mathcal{S}_2$-equivalent, their equivalence (if it is true) has some interesting consequences. Let us assume $\mathcal{S}_2$-equivalence and consider some of these consequences. Since $\mathcal{S}_2' \subset \mathcal{S}_2$, these two detectors will then also be $\mathcal{S}_2'$-equivalent. From Theorem 2, the charged scalar field detector's response can be used to construct an isomorphism between $\mathcal{S}_2'$ and the set $\mathcal{S}_1^b$ defined by:

$$\mathcal{S}_1^b = \left\{ \begin{array}{l} \text{isotropic charged scalar field particle eigen-states with } n_k = m_k \\ \text{and} \int\limits_m^\infty d\omega k^{n-3} n_k < \infty \end{array} \right\}$$

Similarly, defining a corresponding set, $\mathcal{S}_1^f$, for the spinor field detector

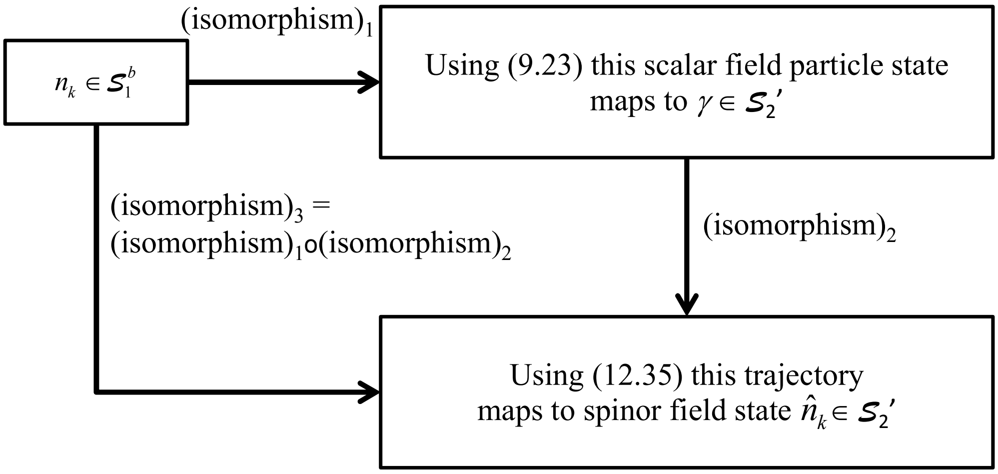
$$\mathcal{S}_1^f = \left\{ \begin{array}{l} \text{isotropic spinor field particle eigen-states with } \hat{n}_k = \hat{m}_k \\ \text{and} \int\limits_m^\infty d\omega k^{n-3} \hat{n}_k < \infty \end{array} \right\}$$

Equating the responses (12.26) and (12.31) sets up a transformation between $\mathcal{S}_1^f$ and $\mathcal{S}_2'$ given by $R^7_{\gamma} = R^7_{n_k}$, which gives;

$$\mathrm{Tr}\left[\left(S_\gamma^+\left(\Delta\tau;\tilde{\rho}\right)\right)^2\right]=\left[\int\frac{d^{n-1}k}{\left(2\pi\right)^{n-1}}n_k\cos\omega\Delta\tau-2i\frac{d}{d\tau}G^+\left(\Delta\tau\right)\right]^2+ \\ +m^2\left[\int\frac{d^{n-1}k}{\left(2\pi\right)^{n-1}\omega}n_k\sin\omega\Delta\tau-2iG^+\left(\Delta\tau\right)\right]^2 \tag{12.35}$$

This relation can be seen to be an isomorphism given the following; both sides of this equation and their integrals with respect to Dt are finite; for $\gamma_1 \neq \gamma_2$ (12.31) gives different responses and similarly for (12.26) with $n_k \neq n'_k$.

Using $\mathcal{S}'$ as an intermediary, an isomorphism between $\mathcal{S}_1^b$ and $\mathcal{S}_1^f$ can be constructed. This isomorphism can be represented diagrammatically;



This isomorphism between $\mathcal{S}_1^b$ and $\mathcal{S}_1^f$ works for the Rindler trajectory. From Sec. 5.2 and Sec. 12.1 it is easily seen that n_k will be Planckian for the charged scalar field and Iyer & Kunar (1980) have shown that the spinor detector under-going uniform acceleration also perceives Planck radiation. Boyer (1980) has shown that for the photon field, a uniformly accelerating detector does not perceive radiation with a Planck spectrum.

It must be emphasised that this isomorphism is (in its present form) rather speculative because it has been intuitively assumed that (12.35) produces an isomorphism between $\mathcal{S}_2'$ and $\mathcal{S}_1^f$. Further, even if this isomorphism does hold it must be noted that;

- Had different (in-equivalent) detector models been used, the resulting isomorphism would be quite different. Hence this correspondence between scalar and spinor fields is possibly one of many which may be constructed.
- The correspondence is only between isotropic states. Although in principle it may be extended by using different detectors, in practice this would not be easy, primarily due to the complicated form of the spinor field detector's response to anisotropic states. Further, for omni-directional detectors, this correspondence would no longer be an isomorphism if extended to anisotropic states due to Theorem 1.
- The correspondence is presently restricted to sets of particle eigen-states $\mathcal{S}_1^b$ and $\mathcal{S}_1^f$ which do not include all elements of their respective Fock spaces, nor mixed states.

Although the isomorphism between scalar and spinor field states constructed here is not very general, it does raise the possibility of using non-Minkowski quantum field theory methods to probe possible symmetries between fields of different spin. In this particular case, the quantum fluctuations of the fields for non-inertial trajectories through the Minkowski vacuum have been used to construct such an isomorphism (or symmetry). A more sophisticated use of this (rather natural) correspondence could provide a more sound basis to present super-symmetric theories which so far have been completely founded on an "incredible postulate" (Salam 1984).

13 Conclusions

An initial motivation for the use of particle detectors in curved space-time quantum field theory was an attempt to use operational methods to overcome the ambiguities the "particle" concept suffers in non-Minkowskian spaces. Although not successful in this aim, the use of detector models has provided researchers with important insight into the subtleties and complexities of quantum phenomena occurring in background gravitational fields. Due to these achievements, and for various philosophical reasons, particle detector models have seen wide use for a variety of purposes. In particular, they have often been used to deduce what an observer will "see" (i.e. measure) when placed in some situation. (See, for example, Isham 1977, Sciama et al. 1981, Birrell & Davies 1982, Hinton, Davies & Pfautsch 1983, Israel & Nester 1984.) Although most researchers who use detectors do so to give some idea of what a situation "looks like" from the detector's rest frame, none have considered in detail what this application entails. Statements such as:

> ".. the equilibrium between the accelerated detector and the field in the state $|0_M\rangle$ is the same as that which would have been achieved had the detector remain un-accelerated, but immersed in a bath of thermal radiation.."

(Birrell & Davies 1982, also see Unruh 1976, Sciama et al. 1981) tacitly construct a correspondence between some non-Minkowskian situation and (pure or mixed) particle states in Minkowski Fock space. In other words, using a detector to discuss what a situation "looks like" is tantamount to constructing a detector-equivalence mapping between the set of situations $\mathcal{S}$ into which the detector may be placed and the set of situations $\hat{\mathcal{S}}$ of placing the detector inertially in some quantum state of the field in Minkowski space.

This is a more general case of the isomorphism constructed in Theorems 2 and 3. There the mapping was between $\mathcal{S}_2'$ of stationary trajectories through Minkowski space and $\mathcal{S}_1'$ of isotropic particle states in Minkowski space. The set $\mathcal{S}_1'$ consists only of pure particle states in the Fock space, more generally all pure and mixed quantum states should be included. Either way, this use of particle models represents an attempt to utilise (particle) concepts that are well defined and suited to our every-day (Minkowski space-time) experience in (non-Minkowskian) situations where the concept becomes ambiguous. This is very similar, in essence, to the ambiguities that result from using (classical) concepts well defined and suited to our (macroscopic) experience in (sub-microscopic) situations where these concepts breakdown. (Bohr 1928, Schiebe 1973)

The problems with the particle concept in non-Minkowskian spaces have been widely discussed (Unruh 1976, Isham 1977, Sciama et al. 1981, Grove & Ottewill 1983, Davies 1984). In line with the Copenhagen interpretation most researchers believe that these problems and their resulting paradoxes arise;

only if one attempts to use classical language to describe what is a strictly operational purely quantum mechanical process. Thus questions such as "are the particles there?" must be understood using quantum not classical notions of reality" (Isham 1977)

Adhering even more strictly to the Copenhagen doctrine Davies (1984) asserts that;

> Any discussion about what is a "real physical vacuum" must … be related to the behaviour of real, physical measuring devices, in this case particle-number detectors. Armed with such heuristic devices, we assert the following. There are quantum states and there are particle detectors. Quantum field theory enables us to predict probabilistically how a particle detector will respond to that state. That is all there can ever be in physics, …. We can't meaningfully talk about whether such-and-such a state contains particles except in the context of a specific particle detector measurement."

In the uniformly accelerating observer case the Minkowski vacuum, which is a pure state for an inertial observer, is a mixed thermal state in the Fulling-Rindler Fock space (Sciama et al. 1981). The use of a particle detector in this case associates with that state, via its detector-equivalence classes, a set of (mixed and pure) thermal states in Minkowski space. It is through this association that we decide what a uniformly accelerating observer would "see". Such an interpretation of the detector's response raises an issue that has been widely discussed in the literature. Whether or not the "particles" observed by a non-inertially moving observer (in Minkowski space) are "real". For example Isham (1977) states;

> "The particles measured by the accelerating observer are not "fictitious"; indeed, if at the same time he measured <u>his</u> normally ordered energy momentum tensor he finds genuine energy."

On the other hand according to Sciama et al. (1981)

> "… while it is sometimes convenient to describe the readings of the detector by saying that the detector perceives itself to be immersed in black-body radiation, this is in part just a form of words that need not be interpreted as meaning that the detected "particles" are "real"."

Sciama et al. support their claim by noting that the acceleration radiation does not exhibit any Doppler shifting when comparing measurements of two neighbouring observers with the same acceleration but different velocities.

This result is true for all non-inertial stationary motions through the Minkowski vacuum, a fact that follows from the discussion in Sec. 9.1.3. There it was shown that two trajectories $\gamma_1, \gamma_2 \in \mathcal{S}_2'$ give the same response (in the linear detector case) provided they are at most related by a Poincare transform. This result, in turn, follows from the Poincare invariance of the Minkowski and its Wightman function. Observers moving along different trajectories through $|0_M\rangle$ are merely observing the same (coordinate free) state from different reference frames. Therefore observers whose reference frames are related by a Poincare transform must 'see" the same thing without any Doppler effects.

To this extent the "particles" observed by an observer travelling along $\gamma \in \mathcal{S}_2'$ are different from the usual concept of particle (which break the Poincare invariance of the Minkowski vacuum and hence suffer Doppler shifts). Note, however, a unique Poincare invariant many-particle state can be constructed in Minkowski space (Boyer 1980). Still, the point made by Isham (and later expanded upon by Davies (1984)) is valid, in that by adopting an operationalist Copenhagen viewpoint, objects that register on apparatus built to detect "particles" are, for all intents and purposes, particles.

We must also realise that our definition of a "particle detector" is based upon Minkowski concepts and intuition. As Davies (1984) states;

> If somebody's model detector failed to give a zero response in this situation [i.e. inertial motion through the Minkowski vacuum], we should reject the model as "an unreliable instrument". There is an element of circularity here. If the state $\left|0_M\right\rangle$ is defined physically to be that which produced zero response in a "reliable" detector, what does it mean to say that a "reliable detector" is one which doesn't respond to $\left|0_M\right\rangle$? This situation is common to all branches of physics and is resolved by external criteria of a professional and philosophical nature."

Furthermore, this "professional and philosophical nature" has been shaped (perhaps even dictated) by the fact that we are classical beings living in a weak gravitational field. When we try to apply (operational) concepts defined in this (special) situation to phenomena which involve quantum and (general) relativistic processes we should expect ambiguities.

So we can see that, as with all quantum phenomena, we must acknowledge the ambiguities resulting from applying classical notions at the quantum level. This is true even in Minkowski quantum field theory. In addition to this, in non-Minkowskian spaces further ambiguities arise because we are attempting to apply concepts constructed in and based upon essentially Minkowski space-time experience to non-inertial and general relativistic situations. Even if there is a high degree of symmetry (e.g. DeSitter or Robertson-Walker spaces) these notions can become misleading. A good example is the use of a linear detector in DeSitter space. This detector responds thermally, but the energy stress tensor is far from thermal in form. (See Sciama et al. 1982 for details, also see Birrell & Davies 1982).

Another widely debated result arising from the use of particle detectors, in flat space-time in particular, is the apparent disagreement between a detector defined vacuum (i.e. states for which the detector does not respond) and the canonically defined vacuum (based upon the Bogolubov coefficients with respect to the Minkowski vacuum). After studying and comparing five different stationary coordinatisations of flat space, Letaw and Pfautsch (1981) concluded that;

> The correspondence between vacuum states defined via canonical quantum field theory and via a detector is thus broken for more general stationary motions, and we must conclude that the two definitions are in-equivalent."

Contrary to this Myhrvold (1984) states;

> It is not possible that the response of a particle detector and the particle content under canonical quantisation differ because the formalism allows us, even compels us, to directly relate the two.

Statements like the above two arise from the misunderstanding of what particle detectors do and how they operate. As shown in Chapter 11, particle detectors do not merely count canonically defined "particles". When considering situations in $\mathcal{S}_2'$, this means that detectors do not respond solely to the β_{ij} Bogolubov coefficients. This fact alone means that the experience of a specific detector is in general no guide to the canonically defined "particle content". Davies (1984) also warns that;

"… one must resist the temptation to divide up detector motions into "reasonable" and "screwy" on the basis of their compatibility with Bogolubov transformations. Let me repeat; there are quantum states and detector measurements. What we mean by a "particle" cannot sensibly be expressed without reference to a detector. All we can predict and discuss (as far as the physical world is concerned) are experiences of detectors."

Grove and Ottewill (1983) made some progress in sorting out the confusion displayed by Letaw, Pfautsch and Myhrvold, but unfortunately succumb to the "temptation" Davies warned against by splitting detector responses into "spurious radiation effects" (which correspond to the emission terms introduce in Chapter 11) and "particle detections" (which correspond to the β_{ij} terms).

The apparent disagreement between detector responses and the canonical formalism is resolved by noting two facts. Firstly, the calculations of detector responses and of Bogolubov coefficients are quite distinct. One concerns the experience of a (point) object along a given world-line, the other relates "particle states" that are not localised but defined over a whole space-time patch (Davies 1984). Secondly, due to the non-Minkowskian nature of these non-inertial situations, there need not be any systematic relationship between the state for which $|\beta_{ij}|^2 = 0$ and that with zero "particle" content as perceived by a particle detector (Pfautsch 1981). This latter point was addressed in Sec. 11.1 where it was shown that if ω can attain negative values, then a detector will give a non-zero response even if $n_k(\Psi_0)$=0. Inspecting (11.6) we see that the occurrence of this effect depends solely upon the range of values ω may acquire. This, in turn, is dictated by the Killing trajectory of the detector (i.e. its four-velocity $\partial/\partial t$) and (at least for the linear detector) is quite independent of the spatial coordinates chosen (i.e. $\chi_k(\rho)$ functions). Thus attempts to remove this effect by choosing specially adapted spatial coordinate systems are misguided (Myhrvold 1984). This result also follows from the discussion in Sec. 4.1 where the (τ,ρ)-coordinates were introduced. This point has also been made by Davies (1984) who remarks further, in a discussion about the use of Rindler coordinates as the "natural" coordinatisation to use for uniformly accelerating observers, that;

"In any case, in curved space-time, no such "naturally adapted" coordinatisation will generally exist and so we must be careful not to base too much on it in the special case of the Rindler system."

This is another danger that must be guarded against when using particle detectors to study quantum fields in curved spaces.

One of the largest obstacles inhibiting the more rapid development of this field is the computational difficulties that so often arise. As a result most research efforts are concentrated on a few highly symmetric space-time models (e.g. DeSitter, Robertson-Walker, conformally flat spaces, etc.). Once a problem has been solved in one of these special cases, a mixture of intuition and approximation is applied to deduce results for less specialised situations. A danger of this approach, particularly when using particle detectors, is to rely too much on intuition founded on these special (highly symmetric) cases. In the final analysis much of the confusion that has occurred with the use of particle detectors could be said to have stemmed from this temptation.

Grove and Ottewill (1983) proposed a scheme involving several detectors, which they contend will enable one to deduce the "particle content" as given by the canonical approach. They claim that;

"… after a particle detection the number of particles in the external field decreases so that on average the probability of a subsequent detection falls. After a radiation recoil the number of particles in the external field increases so that on average the probability of a subsequent detection rises. Thus, by using several detectors (as one realistically would) and calculating the correlations between them, one can determine which excitations are spurious radiation and which correspond to particle detections."

In this passage a "particle" is defined by the canonical quantisation of the field.

Unruh and Wald (1984) have subsequently shown this procedure will not work, primarily because of the evolution of the quantum state $|\Psi_0\rangle$ resulting from the interaction between the detector and the field. Also because the quantum field state is most likely not an eigenstate of the number operator. Regarding the effect of the detector on the field state, from the analysis of Unruh and Wald, it can be seen that if the frequency of the field modes in the detector's frame can go negative, the final state of the field, $|\Psi_0(\tau=\infty)\rangle$, has the form

$$|\Psi_0(\tau=\infty)\rangle = |n\rangle|E_0\rangle - i\left\{\Gamma_1\sqrt{n}\,|n-1\rangle|E\rangle + \Gamma_2\sqrt{(n+1)}\,|n+1\rangle|E\rangle\right\}$$

Where Γ_1 and Γ_2 are functions dependent on the quantum field and the modes of the state to which the detector has responded. The kets $|n\rangle$ represent a state of the field which contains n particles in the detected mode. From this equation we see that, assuming a detection has occurred, the number of particles in the external field has neither decreased nor increased in the sense that Grove and Ottewill imply. The fact is, the linear detector (to first order) interacts with the field not solely via an absorption interaction nor solely via an emission interaction (see Chapter 11 for an explanation of these terms), but through a linear superposition of the two. This can be seen directly from the interaction Lagrangian and is explicitly demonstrated by the analysis of Unruh and Wald. Therefore, the absorption and emission interactions cannot be split apart because they occur concurrently and (unless one is suppressed by restrictions on the range of the frequency) both will contribute to the detector's response and the final state of the field.

With respect to the field not being in an eigen-state of the number operator, Unruh and Wald state;

> "The act of detection not only removes a particle of energy Ω from the field but also performs a (partial) measurement of the state of the field, since the detector is most likely to be excited if a large number of particles were initially present …. Thus the fact that the detector became excited weights the high particle number states more heavily in the distribution and indicates that a larger number of particles than originally expected were present initially. As a dramatic example of this effect, suppose that the initial state of the quantum field were chosen to be $|0\rangle + \left(1/\sqrt{n}\right)|n\rangle$ with n large. Then the initial energy is ≈E, where E is the energy of a single quantum. However, if a detection occurs, then the expected field energy becomes $(n\text{-}1)E >> E$."

Their energy measurement argument can obviously be repeated for a particle number measurement. Thus the scheme put forward by Grove and Ottewill will not work even if the quantum state of the field is an eigen-state of the number operator.

From the discussion above, to date, all attempts to use particle detectors to (directly) measure the particle content of a situation, as given by canonical quantisation, have failed. In fact, for the simple detector models so far discussed in the literature, there is good reason to believe that such models are intrinsically unable to fulfil this task.

It was demonstrated in Chapter 11 that detector models of the type we have been considering can be described by their corresponding hypothetical monopole and screening function. However, as previously stated, it is the detector's trajectory that determines whether or not emission terms also contribute to its response, not details of the internal construction. This can be seen directly from (11.15) and the subsequent discussion.

These simple detector models (and most likely particle detector models in general) must be understood in terms of their initial purpose; to give an operational means of observing quantum effects in non-Minkowski situations. Even when being used in this manner, we must appreciate their short-comings. At the beginning of this chapter we saw that a common use of particle detectors is to deduce what a co-moving observer, ***O***, would perceive in some situation. It was noted that this use is tantamount to employing detectors to provide a correspondence between some (time independent) situation, s, and the set of all (pure or mixed) quantum states in the Minkowski space, which has been denoted by $\hat{\mathcal{S}}$. Although this may be fairly straight forward in a conceptual sense, the actual application of the ideas is not so simple. Ideally, we would like the mapping relating the situation s to the state in $\hat{\mathcal{S}}$ to be a one-to-one function thus giving a unique $|\Psi_s\rangle \in \hat{\mathcal{S}}$ representing what ***O*** perceives when in s. However, there are several hurdles to be overcome before this can be attempted. In particular the following;

1. There is no guarantee that $|\Psi_s\rangle$ is detector independent. In fact a strict Copenhagen view would lead one to expect it to be intrinsically detector dependent. If this in fact is the case, then depending upon the detector used, the state $|\Psi_s\rangle$ perceived by ***O*** when in s will differ. A one-to-one function between s and the elements of $\hat{\mathcal{S}}$ is obviously not possible in this case.
2. Even if we assume $|\Psi_s\rangle$ is detector independent, it is still not guaranteed to be unique. Since $\hat{\mathcal{S}}$ includes states that are not eigen-states of the number operator as well as mixed states, the detector's action on $|\Psi_s\rangle$ may merely be as a projection operator and so many states in $\hat{\mathcal{S}}$ may give the same response as $|\Psi_s\rangle$. (This follows from the fact that to fully specify a quantum state, in a Hilbert space, generally requires a complete set of mutually commuting observables.)
3. Assuming that the problems posed in the above two points do not materialise, or can be circumvented, there are still calculational difficulties to be overcome. In particular, to attain most information about the state $|\Psi_s\rangle$, it is sensible to use a detector (or detectors) with the angular discrimination of the spike detector (or a narrow cone). However, in Chapter 7 we saw how these detectors will generally be difficult to mathematically represent in non-Minkowski situations.
4. Given the above disadvantage of directional detectors, we will generally be forced to use omni-directional detectors to deduce $|\Psi_s\rangle$. Here again problems arise since, from Theorem 1, these detectors will associate the situation s with an entire class of detector-equivalent situations $S_s \subset \hat{\mathcal{S}}$, such that $|\Psi_s\rangle \in S_s$. So, how do we discern the state $|\Psi_s\rangle$ from all the other elements of S_s?

A possible solution to the last two points is the following: Consider a set of n+1 detectors, denoted $\mathcal{D}_1$, $\mathcal{D}_1'$, $\mathcal{D}_2$, …., $\mathcal{D}_n$. This set, if need be, is constructed from a (complete) set of observables. This step is an important and significant generalisation of the concept of "detector". In Chapter 2 the notion of "particle detector" was made precise by relating it to the particle number operator in Minkowski space. To define "detectors" of other observables we would have to follow a similar procedure of making that concept precise. Now assume that $\mathcal{D}_1$ and $\mathcal{D}_1'$ are equivalent where-as $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$ are mutually in equivalent. Each detector will associate with a situation s a detector equivalent class of states in $\hat{\mathcal{S}}$. Thus we have for each

$$\mathcal{D}_i(s) = \mathcal{D}_i(|\Psi\rangle) \qquad |\Psi\rangle \in S_{s,i} \tag{13.1}$$

where $|\Psi\rangle \in \hat{\mathcal{S}}$ and $S_{s,i}$ is the class of situations in $\hat{\mathcal{S}}$ that are $\mathcal{D}_i$-equivalent to the situation s. Since we have assumed $|\Psi_s\rangle$ is detector independent we must have

$$|\Psi_s\rangle \in S_{s,i} \qquad \forall \mathcal{D}_i \tag{13.2}$$

and hence

$$|\Psi_s\rangle \in S_s = \bigcap_{i=1}^{n} S_{s,i} \tag{13.3}$$

Therefore, we can "close in" on $|\Psi_s\rangle$ by studying all the $S_{s,i} \subset \hat{\mathcal{S}}$. The in-equivalence of the various detectors used is of fundamental importance for the success of this method. If we use equivalent detectors (i.e. $\mathcal{D}_1$ and $\mathcal{D}_1'$) then

$$S_{s,1} = S_{s,1'}$$

And so even though (13.1) to (13.3) are all true for $\mathcal{D}_1$ and $\mathcal{D}_1'$, we also have

$$S_{s,1} \cap S_{s,1'} = S_{s,1}$$

Thus due to its equivalence to $\mathcal{D}_1$ the use of $\mathcal{D}_1'$ has added no new information about $|\Psi_s\rangle$.

If this procedure is to be used to find $|\Psi_s\rangle$ (assuming the problems (1) and (2) above do not eventuate) a family of in-equivalent detectors are required. Furthermore, this family of detectors must be large enough so that the set S_s has only one element, namely $|\Psi_s\rangle$.

This leads to the question of how many in-equivalent detectors (of a given quantum field) can be constructed. In other words, how many different interaction Lagrangians between the quantum field and the entity M (representing the detector) can we produce? There are several restrictions on the choice of Lagrangians available. Obviously they must satisfy conditions of locality, general covariance, PCT-invariance and the like. At present there are also the further restrictions arising from the "state of the art" of quantum field theory. In particular restrictions of renormalisability and the fact that all calculations involving interactions presently use perturbation theory will constrain the available Lagrangian forms. These latter factors offer the most severe limitations on the selection of models available since they stem from difficulties that are calculational rather than conceptual.

In particular, the renormalisability condition excludes interactions of the form $m(t)\phi^n[x(t)]$ where $n > 4$. (Although Ford (1983) has suggested a technique for renormalising field theories for this range of n, the problems encountered are still far from solved.) This restriction excludes high order polynomial interactions, thus even interactions that may be representable by a well behaved Taylor series are excluded, let alone anything more exotic.

Above we saw how (families of in-equivalent) particle detectors can be used to construct a mapping between situations (in flat or curved space-times) and the set of quantum states in Minkowski space. The motivation for such a construction, as explained at the beginning of this chapter, being based essentially on a desire to find out what a situation would "look like" when related to concepts well-defined in our (almost) Minkowskian environment. We now consider a more general interpretation of the response of a particle detector in some general situation, s, in a space-time $\mathcal{M}$.

A detector's response is determined by the five factors listed in Chapter 8 on page 55. Consider an observer ***O*** with a detector moving on a Killing trajectory γ through the space-time $\mathcal{M}$. (We assume the situation is stationary.) The choice of γ and $\mathcal{M}$ determine the factors 2, 3 and 4 (assume 5 has also been determined). In her frame, ***O*** can construct a representation of the quantum field with operators a_k, a_k^*, vacuum state $|0\rangle$ and with field modes $u_{\boldsymbol{k}}(x)$ in the following manner: She chooses a Killing vector field which must be time-like in a space-time region containing all the trajectory γ and have an integral curve identical to γ. (I.e. g must be an integral curve of the Killing field.) This allows ***O*** to then construct a representation of the quantum field in the usual manner (see Chapter 2). For example, a uniformly accelerating observer may use either a Rindler coordinate system (in which the Killing trajectories are time-like only in the left and right hand wedges), or a K_7 conformal Killing vector field (Brown, Ottewill & Silkos 1981) which is time-like everywhere. The only condition is that the one chosen must contain an integral curve identical to the observer's trajectory.

Now the states of the quantum field (the choice of which determines factor 1) are coordinate free. Denote the set of these states by $\mathcal{A}$ = Fock space $\cup$ (mixed states) On the other hand; the operators a_k, a_k^* and vacuum state $|0\rangle$ are determined by the trajectory γ and ***O***'s quantisation procedure. The observer can use these operators and the vacuum state, together with appropriate scalar coefficient (distributional) functions, to construct a representation $\mathcal{A}_\gamma$, of all these states of the quantum field in that space-time. (N.B. In this construction we are allowing more than merely a representation of the Fock space, as mixed states are also being included.) Assuming the representation is faithful, it is isomorphic to $\mathcal{A}$. Thus we may write

$$\begin{aligned} &\mathcal{R}_\gamma : \mathcal{A} \rightarrow \mathcal{A}_\gamma \\ &\mathcal{R}_\gamma\left(|\Psi\rangle\right) = F\left(a_k^*\right)|0\rangle \end{aligned} \tag{13.4}$$

where the isomorphism $\mathcal{R}_\gamma$ is ***O***'s representation of the elements $|\Psi\rangle$ of the set $\mathcal{A}$. The $F\left(a_{\boldsymbol{k}}^*\right)$ is a function of the field operators $a_{\boldsymbol{k}}^*$ which acts on the vacuum state $|0\rangle$ to give an element of $\mathcal{A}_\gamma$ which represents $|\Psi\rangle$. Also the $a_{\boldsymbol{k}}^*$ may represent a set of creation operators.

As an example, consider the Minkowski vacuum $|0_M\rangle$. Being a quantum state it is coordinate free and an element of $\mathcal{A}$ (with $\mathcal{M}$ flat four-dimensional space-time). For a Rindler observer, the representation of $|0_M\rangle$ in terms of Rindler operators $a_{\nu,\boldsymbol{k}}^*$ (for the right hand wedge in Figure 1) and $b_{\nu,\boldsymbol{k}}^*$ (for the left hand wedge) and the Fulling vacuum $|0_F\rangle$ is (Sciama et al. 1981)

$$\left|0_M\right\rangle = (constant) \times \prod_{\nu,\boldsymbol{k}} \sum_{n_{\nu,\boldsymbol{k}}=0}^{\infty} \exp\left(-\pi n_{\nu,\boldsymbol{k}}\right) \frac{\left(a_{\nu,\boldsymbol{k}}^*\right)^{n_{\nu,\boldsymbol{k}}}}{\left(n_{\nu,\boldsymbol{k}}!\right)^{1/2}} \frac{\left(b_{\nu,\boldsymbol{k}}^*\right)^{n_{\nu,\boldsymbol{k}}}}{\left(n_{\nu,\boldsymbol{k}}!\right)^{1/2}} \left|0_F\right\rangle \tag{13.5}$$

Where, strictly speaking, we have

$$\left|0_F\right\rangle = \left|0_F^R\right\rangle \left|0_F^L\right\rangle$$

With $\left|0_F^i\right\rangle$ the vacuum state in the appropriate wedge,

$$a_{\nu,\boldsymbol{k}}\left|0_F^R\right\rangle = 0 \qquad b_{\nu,\boldsymbol{k}}\left|0_F^L\right\rangle = 0$$

In this example, referring to (13.4)

$$\left|\Psi\right\rangle = \left|0_M\right\rangle$$

$$F\left(a_{\boldsymbol{k}}^*\right)\left|0\right\rangle = (constant) \times \prod_{\nu,\boldsymbol{k}} \sum_{n_{\nu,\boldsymbol{k}}=0}^{\infty} \exp\left(-\pi n_{\nu,\boldsymbol{k}}\right) \frac{\left(a_{\nu,\boldsymbol{k}}^*\right)^{n_{\nu,\boldsymbol{k}}}}{\left(n_{\nu,\boldsymbol{k}}!\right)^{1/2}} \frac{\left(b_{\nu,\boldsymbol{k}}^*\right)^{n_{\nu,\boldsymbol{k}}}}{\left(n_{\nu,\boldsymbol{k}}!\right)^{1/2}} \left|0_F\right\rangle$$

Also see Davies (1978) for the two-dimensional maximally extended Schwarzschild black hole case.

To understand the role of particle detectors in the representation of $\mathcal{A}$ by $\mathcal{A}_\gamma$ we refer back to detector-equivalence of situations.

Detector responses, being trajectory dependent, can be considered as forming and equivalence relation on the set $\mathcal{A}_\gamma$, resolving this set into a (quotient) set of detector-equivalence classes. Denoting this quotient by $\mathcal{A}_\gamma^{\mathcal{D}}$ and letting $\mathcal{E}_{\mathcal{D}}$ represent the projection operator that acts on $\mathcal{A}_\gamma$ to give the equivalence classes in $\mathcal{A}_\gamma^{\mathcal{D}}$, we can write

$$\mathcal{E}_{\mathcal{D}} : \mathcal{A}_\gamma \to \mathcal{A}_\gamma^{\mathcal{D}} \tag{13.6}$$

From this a "quasi-representation", $\mathcal{R}_\gamma^{\mathcal{D}}$, of $\mathcal{A}$ can be defined by

$$\mathcal{R}_\gamma^{\mathcal{D}} = \mathcal{E}_{\mathcal{D}} \circ \mathcal{R}_\gamma \tag{13.7}$$

such that

$$\mathcal{R}_\gamma^{\mathcal{D}} : \mathcal{A} \to \mathcal{A}_\gamma^{\mathcal{D}} \tag{13.8}$$

The function $\mathcal{R}_\gamma^{\mathcal{D}}$ is called a "quasi-representation" because it is not an isomorphism (being a many-to-one function) and so the set $\mathcal{A}_\gamma^{\mathcal{D}}$ is not a faithful representation of $\mathcal{A}$.

Thus a detector may be viewed as a piece of apparatus that implements (a restriction of) the representation of the states of the quantum field with respect to the quantum operators and vacuum state defined by an observer. An example of this is the linear detector giving a thermal response when uniformly accelerating through the Minkowski vacuum. This should be expected because, in the Rindler frame representation, the Minkowski vacuum is a thermal state.

Of course, as seen from (13.7) and (13.8), the detector mapping $\mathcal{R}_{\gamma}^{\mathcal{D}}$ is not as "fine graded" as the actual representation since detectors group states together in their equivalence classes (i.e. the $\mathcal{A}_{\gamma}^{\mathcal{D}}$). To attain a fine graded representation using detectors would require a family of in-equivalent detectors.

Using the above approach to study particle detectors we could construct a general formal theory of particle detectors and particle detector equivalence, which complements the theory and applications studied in this thesis.

A factor common to all the detectors discussed in detail in this thesis is that their quantum field interactions all involve the operator $N_{\boldsymbol{k}} = a_{\boldsymbol{k}}^{*} a_{\boldsymbol{k}}$ (and/or $1+N_{\boldsymbol{k}}$). Due to the simplicity of the interaction Lagrangians the operators $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^{*}$ always appear only linearly or quadratically. The calculational restrictions on the Lagrangians are the main reason for such simplicity. If this limitation could be removed detectors whose responses are related to quantum processes other than those of the number operator $N_{\boldsymbol{k}}$ could be constructed. In fact, this will most likely be necessary if we intend to use detectors for the purposes discussed on pages 113 to 115.

For example, it may be possible to construct a detector that couples directly to the energy-momentum stress tensor ($T_{\mu\nu}(x)$) of the field. There is an interesting side note here. Given that all our experiences of nature are related to us by "particle detectors", is it ever possible to measure exactly a quantity such as $\langle T_{\mu\nu}(x) \rangle$ which is defined at a space-time point "x"? Regarding $\langle T_{\mu\nu}(x) \rangle$ in particular, such a measurement seems contrary to the Heisenberg Uncertainty relations. The components of $\langle T_{\mu\nu}(x) \rangle$ are (measures of) energy and momentum at the space-time point "x". Thus exact measurement of $\langle T_{\mu\nu}(x) \rangle$ and its positional dependence violates the Uncertainty principle.

Quite apart from the particular constraint on the energy-momentum tensor, the use of a detector introduces other restrictions on general measurements. Taking the linear detector as an example, from (4.8) its response at time τ is dependent upon its entire history (i.e. on all $\tau' < \tau$), rather than at a single point. This problem cannot be solved merely by rapidly turning the detector "on" and "off" arbitrarily close to the point "x" at which we wish to conduct a measurement. If the switching is done too rapidly the process of turning the detector on and off itself will excite the detector, even if the quantity to be measured is zero. (Unruh & Wald 1984) Therefore it seems that, at best, a detector can only give some average measure of a (local) quantity over some non-zero period T which is much larger than the "relaxation time" of the detector, (see Unruh & Wald for details). So, truly local measurements, of $\langle T_{\mu\nu}(x) \rangle$ say, will be meaningless in an operationalist approach since such experiments cannot be conducted.

Although the discussion so far in this chapter has concentrated on the scalar field, the general points made also apply to higher spin fields. We may therefore ask how it is that all the detectors so far used in high energy physics agree on what they "perceive" or "measure" in sub-atomic interactions. The reason is that all of these machines operate in situations that are almost Minkowskian. The gravitational curvature on the Earth's surface is very small and their motion almost inertial when compared with the extreme conditions required for the quantum effects discussed above to become significant. For example, an acceleration of 10^{20}m/s^2 is required to detect acceleration radiation of the order 1^{o}K. Thus for the regime of physical conditions in which modern particle detectors are used today, the particle concept is quite useful. However, Bell & Leinnaas (1983) have shown that non-inertial quantum field effects may not be entirely negligible when analysing the results of high energy particle experiments. This will be even more-so the as higher energy machines come into use.

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Appendix A

A.1 Calculation of (4.14)

The quantity to be evaluated is given by (4.13)

$$\frac{-1}{4\pi}\int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E\Delta\tau}\ln\left[4\pi\xi^2\sinh^2\left(\frac{\Delta\tau}{2\xi}-i\varepsilon\right)\right]= \\ =\frac{-1}{4\pi}\int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E\Delta\tau}\left\{\ln\left[4\pi\xi^2\right]+2\ln\left[\sinh\left(\frac{\Delta\tilde{\tau}}{2}-i\varepsilon\right)\right]\right\} \tag{A.1}$$

The initial term is discarded because, by assumption, $\Delta E \neq 0$. To evaluate the remaining integral, the method used by Birrell & Davies (1982) is adopted. With the identity

$$\sinh x = x\prod_{m=1}^{\infty}\frac{(m\pi - ix)(m\pi + ix)}{(m\pi)^2}$$

(A.1) becomes

$$\frac{-1}{4\pi}\int_{-\infty}^{\infty} d\Delta\tau e^{-i\Delta E\Delta\tau}\left\{\begin{array}{l}\ln\left(\frac{\Delta\tau}{2\xi}-i\varepsilon\right)+\ln\prod_{m=1}^{\infty}\left(m\pi-\frac{i\Delta\tau}{2\xi}-\varepsilon\right)+ \\ +\ln\prod_{m=1}^{\infty}\left(m\pi+\frac{i\Delta\tau}{2\xi}+\varepsilon\right)-2\ln\prod_{m=1}^{\infty}(m\pi)\end{array}\right\}$$

The first and final terms make no contribution. The first may be evaluated using a contour closed in the lower half complex $\Delta\tau$-plane and since the (shifted) pole and branch-cut are in the upper half plane, the integral is zero. The final term makes no contribution because $\Delta E \neq 0$. Using the following identity on the remaining terms

$$\lim_{\sigma\to 0}\int_{\sigma}^{\infty}\frac{d\omega\, e^{-i\omega x}}{\omega\left(e^{\beta\omega}-1\right)}\to -\ln\left\{\prod_{m=1}^{\infty}\sigma e^{\gamma}\left(\beta m + ix\right)\right\}$$

where γ is Euler's constant, yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\Delta\tau\, e^{-i\Delta E\Delta\tau}\int_{0}^{\infty}\frac{d\omega\cos\left(\Delta\tau/2\xi\right)}{\omega\left(e^{\pi\omega}-1\right)}$$

Interchanging the order of integration gives a sum of two delta functions, only one of which contributes to the w-integral. Thus the result is

$$\frac{1}{\Delta E\left(\exp\left(2\pi\Delta E\xi\right)-1\right)}$$

A.2 Calculation of (4.17)

The quantity to be evaluated is given by (4.16)

$$-\int_{-\infty}^{\infty} \frac{e^{-i\Delta E\Delta\tau}\, d\Delta\tau}{16\pi^2\xi^2 \sinh^2\left(\frac{\Delta\tau}{2\xi} - i\varepsilon\right)}$$

We use the contour in Figure 7 in the complex $\Delta\tau$-plane. The contour consists of C_1, which corresponds to the quantity to be evaluated, and C_2 which is a distance $2\pi\xi$ from C_1 as shown.

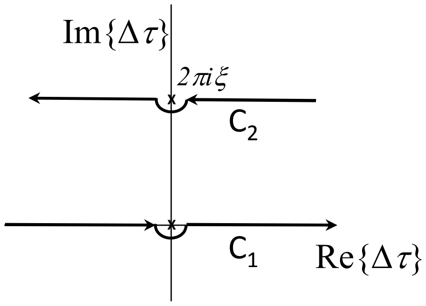

Figure 7 The contour C, represented by a heavy line, consists of C_1 which lies along $\mathrm{Re}\Delta\tau$ axis and C_2 which is a distance $2\pi\xi$ above C_1 as shown.

The hyperbolic-sine function is anti-periodic in the following sense

$$\sinh\left\{\left(\frac{\Delta\tau}{2\xi} + 2\pi i\right)\right\} = -\sinh\left\{\left(\frac{\Delta\tau}{2\xi}\right)\right\} \tag{A.2}$$

Using the Cauchy contour integral theorem, only the pole at the origin contributes to the contour integral around C. Thus

$$\int_C d\Delta\tau\, e^{-i\Delta E\Delta\tau} \sinh^{-2}\left(\frac{\Delta\tau}{2\xi}\right) = \int_{C_1} d\Delta\tau\, e^{-i\Delta E\Delta\tau} \sinh^{-2}\left(\frac{\Delta\tau}{2\xi}\right) + \int_{C_2} d\Delta\tau\, e^{-i\Delta E\Delta\tau} \sinh^{-2}\left(\frac{\Delta\tau}{2\xi}\right)$$

Using (A.2)

$$\begin{aligned} &= \left(1 - e^{2\pi\Delta E\xi}\right) \int_{-\infty}^{\infty} d\Delta\tau\, e^{-\Delta E\Delta\tau} \sinh^{-2}\left(\frac{\Delta\tau}{2\xi}\right) = \\ &= 2\pi i\, \mathrm{Res}\left\{ e^{-\Delta E\Delta\tau} \sinh^{-2}\left(\frac{\Delta\tau}{2\xi}\right)\right\}\Bigg|_{\Delta\tau=0} \end{aligned} \tag{A.3}$$

A power series expansion of the terms in the Residue gives

$$\left[1 - i\Delta E\Delta\tau - \left(\frac{\Delta E\Delta\tau}{2}\right) + \ldots\right]\left[\left(\frac{2\xi}{\Delta\tau}\right)^{-1} - \frac{1}{6} + \ldots.\right]$$

which gives as the coefficient of $(\Delta\tau)^{-1}$;

$$\mathrm{Res}\left\{ e^{-\Delta E\Delta\tau}\sinh^{-2}\left(\frac{\Delta\tau}{2\xi}\right)\right\}\Bigg|_{\Delta\tau=0}=-4i\Delta E\xi^{2}$$

Substituting this into (A.3) gives

$$\int_{-\infty}^{\infty} d\Delta\tau\, e^{-i\Delta E\Delta\tau}\sinh^{-2}\left(\frac{\Delta\tau}{2\xi}-i\varepsilon\right)=\frac{-8\pi\Delta E\xi^{2}}{\left(\exp\left(2\pi\Delta E\xi\right)-1\right)}$$

A.3 An alternative calculation of the Rindler space result

An alternative method of calculating the response of a linear detector uniformly accelerating through Minkowski vacuum is to use the transition amplitude (3.3) and Rindler quantum field operators. This calculation will now be presented for use in derivative detector calculations. It demonstrates the appearance and form of the logarithmic divergence characteristic to this method of calculating equilibrium situation responses. The massless scalar field equation in Rindler space can be case into the form

$$\left[\xi^{-2}\frac{\partial^{2}}{\partial\tilde{\tau}^{2}}-\xi^{-1}\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi}-\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}\right)\right]\phi[x]=0$$

where $\tilde{\tau}=\left(\text{proper-time}\right)/\xi$. This gives for the mode functions, in the appropriate wedge

$$u_{R}\left(x;\tilde{\omega},k_{1},k_{2}\right)=\frac{\left(\sinh\pi\tilde{\omega}\right)^{1/2}}{2\pi^{2}}e^{-i\tilde{\omega}\tilde{\tau}}e^{i\left(k_{1}x+k_{2}y\right)}K_{i\tilde{\omega}}\left(Q\xi\right) \tag{A.4}$$

Where $Q^2 = k_1^2 + k_2^2$ and $\tilde{\omega}>0$. (Note: For a Rindler observer the energy of a mode $u_{R}\left(x;\tilde{\omega},k_{1},k_{2}\right)$ is $\tilde{\omega}$, which is independent of the "momenta" k_1 and k_2. In fact, even for a massive field, the only restriction on $\tilde{\omega}$ is $\tilde{\omega}>0$. Also there is no well-defined momentum component in the direction of acceleration; the z-direction.

The Bogolubov transformations relating the Rindler operators a_R, a_R^* to the Minkowski operators a_M, a_M^* are,

$$\begin{aligned} a_{R}\left(\tilde{\omega},k_{1},k_{2}\right)=\int\frac{d^{3}k'}{\left[2\pi\omega'\left(1-e^{-2\pi\tilde{\omega}}\right)\right]^{1/2}}\left[\frac{\omega'+k_{3}'}{Q}\right]^{i\tilde{\omega}}\delta\left(k_{1}-k_{1}'\right)\delta\left(k_{2}-k_{2}'\right)\times \\ \times\left[a_{M}\left(k_{1}',k_{2}',k_{3}'\right)+e^{-\pi\tilde{\omega}}a_{M}^{*}\left(k_{1}',k_{2}',k_{3}'\right)\right] \end{aligned} \tag{A.5}$$

Where $\omega'^2 = k_1'^2+k_2'^2+k_3'^2$. Using (A.3) in (3.3) gives

$$A_{\gamma}^{1}=ic\left\langle M\right\rangle\xi\int_{-\infty}^{\infty}d\tilde{\tau}\,e^{i\Delta E\xi\tilde{\tau}}\int d\tilde{\omega}dk_{1}dk_{2}\frac{\left(\sinh\pi\tilde{\omega}\right)^{1/2}}{2\pi^{2}}\left\{\begin{matrix} e^{-i\tilde{\omega}\tilde{\tau}}e^{i\left(k_{1}x+k_{2}y\right)}K_{i\tilde{\omega}}\left(Q\xi\right)\left\langle\Psi\right|a_{R}\left(\tilde{\omega},k_{1},k_{2}\right)\left|0_{M}\right\rangle+ \\ +e^{i\tilde{\omega}\tilde{\tau}}e^{-i\left(k_{1}x+k_{2}y\right)}K_{-i\tilde{\omega}}\left(Q\xi\right)\left\langle\Psi\right|a_{R}^{*}\left(\tilde{\omega},k_{1},k_{2}\right)\left|0_{M}\right\rangle\end{matrix}\right\}$$

Performing the $\tilde{\tau}$-integral, only the first term contributes due to the restriction on $\tilde{\omega}$. Substituting (A.5) into this expression, allowing the field operators to act on the Minkowski vacuum and performing the $\tilde{\omega}$, k_1 and k_2 integrals

$$A_\gamma^1 = \frac{ic\langle M\rangle}{\pi}\xi\left(\frac{\sinh \pi\Delta E\xi}{2\pi\left(1-e^{-2\pi\Delta E\xi}\right)}\right)^{1/2}\times$$
$$\times e^{-\pi\Delta E\xi}\int\frac{d^3k'}{(\omega')^{1/2}}\left[\frac{\omega'+k_3'}{Q'}\right]^{i\Delta E\xi} e^{i(k_1'x+k_2'y)}K_{i\Delta E\xi}(Q'\xi)\langle\Psi|1_{\boldsymbol{k}'}\rangle$$

The transition probability P_γ^1 is given by

$$P_\gamma^1 = \sum_{|\Psi\rangle}\left|A_\gamma^1\right|^2 \tag{A.6}$$

where $|\Psi\rangle$ sum is over a complete set of states. Only the one particle states contribute to this sum and with $|\Psi\rangle = |1_{\boldsymbol{k}}\rangle$, we have $\langle\Psi|1_{\boldsymbol{k}'}\rangle = \delta(\boldsymbol{k}-\boldsymbol{k}')$. Thus (A.6) gives, using spherical coordinates,

$$P_\gamma^1 = \frac{c^2\left|\langle M\rangle\right|^2}{4\pi^2}\xi^2 e^{\pi\Delta E\xi}\int_0^\pi d\theta\sin\theta\int_0^\infty dk\, k\left[K_{i\Delta E\xi}(k\xi\sin\theta)\right]^2\int_0^{2\pi}d\varphi$$

In this we have used $K_{-\nu}(x) = K_\nu(x)$. To evaluate the k-integral we use (Gradshteyn & Ryhzik 1980, No.6.576.4)

$$\int_0^\infty dx\, x^{-\lambda}K_\mu(ax)K_\nu(bx) =$$
$$= \frac{2^{-2-\lambda}a^{-\nu+\lambda-1}b^\nu}{\Gamma(1-\lambda)}\Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right)\times$$
$$\times F\left(\frac{1-\lambda+\mu+\nu}{2},\frac{1-\lambda-\mu+\nu}{2};1-\lambda;1-\frac{b^2}{a^2}\right)$$

$\mathrm{Re}\{a+b\}>0$, $\mathrm{Re}\{\lambda\} < 1 - |\mathrm{Re}\{\mu\}| - |\mathrm{Re}\{\nu\}|$.

This results with

$$\frac{c^2\left|\langle M\rangle\right|^2}{4\pi^2}e^{-\pi\Delta E\xi}\xi\Gamma(1+i\Delta E\xi)\Gamma(1-i\Delta E\xi)\int_0^\pi\frac{d\theta\xi}{\sin\theta}$$

The Gamma functions satisfy the identity (Gradshteyn & Ryhzik 1980,No.8.332.3

$$\Gamma(1+ix)\Gamma(1-ix) = \pi x/\sinh\pi x$$

giving

$$P_\gamma^1 = \frac{c^2\left|\langle M\rangle\right|^2\Delta E}{2\pi\left(\exp(2\pi\Delta E\xi)-1\right)}\int_0^\pi\frac{d\theta\,\xi}{\sin\theta}$$

The remaining integral is logarithmically divergent, and once removed this results agrees with the previous calculation of the detector's response in this situation. Therefore, in other calculations of detector response in this (time independent) situation we should expect identical logarithmic divergences to appear.

Finally, by using this method to calculate a detector's response in the Rindler situation, the high degree of symmetry of this particular case is easily seen. The delta functions in the Bogolubov transformations and the fortuitous cancelling of various exponential factors result with the detector's response being directly proportional to the quantity

$$\langle 0_M | a_R^* a_R | 0_M \rangle = 1 \big/ \left[\exp(2\pi\tilde{\omega}) - 1 \right]$$

Which is the number of Rindler particles (with "Rindler energy" $\tilde{\omega}$) in the Minkowski vacuum. Only in such highly symmetric situations does this direct proportionality occur.

A.4 Calculation of asymptotic terms in the Hartle-Hawking vacuum

The quantities required are

$$\frac{\partial^2}{\partial r \partial r'} \sum_{l=0}^{\infty} (2l+1) \overleftarrow{R}_l^* (\omega | r) \overleftarrow{R}_l (\omega | r') \Big|_{r=r'} \qquad r \to \infty \tag{A.7}$$

$$\frac{\partial^2}{\partial r \partial r'} \sum_{l=0}^{\infty} (2l+1) \vec{R}_l (\omega | r) \vec{R}_l^* (\omega | r') \Big|_{r=r'} \qquad r \to 2M_S \tag{A.8}$$

Following the approach of Candelas (1980), we note that we can write

$$\frac{\partial^2}{\partial r \partial r'} G_B^+ (t, r, \theta, \varphi; 0, r', \theta, \varphi) \Big|_{r=r'} =$$

$$= \int_0^{\infty} \frac{d\omega e^{-i\omega t}}{16\pi^2 \omega} \frac{\partial^2}{\partial r \partial r'} \sum_{l=0}^{\infty} (2l+1) \left[\begin{array}{l} \vec{R}_l (\omega | r) \vec{R}_l^* (\omega | r') + \\ + \overleftarrow{R}_l^* (\omega | r) \overleftarrow{R}_l (\omega | r') \end{array} \right] \Bigg|_{r=r'} \sim \frac{1}{2\pi^2 t^4} \qquad r \to \infty$$

Using

$$\int_0^{\infty} d\omega \, \omega^3 e^{-i\omega t} = 6t^{-4}$$

We find

$$\frac{\partial^2}{\partial r \partial r'} \sum_{l=0}^{\infty} (2l+1) \overleftarrow{R}_l^* (\omega | r) \overleftarrow{R}_l (\omega | r') \Big|_{r=r'} \sim \frac{4\omega^4}{3} \qquad r \to \infty \tag{A.9}$$

For (A.8) we use the result that to leading order

$$\sum_{l=0}^{\infty} (2l+1) \vec{R}_l (\omega | r) \vec{R}_l^* (\omega | r') \xrightarrow[r \to 2M_S]{} \frac{2}{M_S{}^2 \Gamma(iq) \Gamma(-iq)} \int_0^{\infty} dl \, l K_{iq} \left(2l\alpha^{1/2} \right) K_{iq} \left(2l\alpha'^{1/2} \right) \tag{A.10}$$

Where $q = 4M_S\omega$ and $\alpha = (r/2M_S)-1$. By use of the chain rule, the quantity we desire is

$$\int_0^\infty dl\, l \frac{\partial}{\partial\alpha} K_{iq}\left(2l\alpha^{1/2}\right) \frac{\partial}{\partial\alpha'} K_{iq}\left(2l\alpha'^{1/2}\right)\bigg|_{\alpha=\alpha'}$$

$$= \frac{1}{4\alpha}\int_0^\infty dl\, l^3 \left[K_{iq+1}\left(2l\alpha^{1/2}\right) + K_{iq-1}\left(2l\alpha^{1/2}\right)\right]^2$$

$$= \frac{\left(4+q^2\right)}{96\alpha^3}\Gamma\left(1+iq\right)\Gamma\left(1-iq\right)$$

which gives

$$\sum_{l=0}^\infty (2l+1)\vec{R}_l\left(\omega \mid r\right)\vec{R}_l^*\left(\omega \mid r'\right) \sim \frac{\omega^2\left(1+4\omega^2 M_S{}^2\right)}{3M_S{}^2\left(\frac{r}{2M_S}-1\right)^3} \qquad r \to 2M_S$$

Next, we evaluate the angular derivatives of the Hartle-Hawking Wightman function. The quantities we seek are

$$\frac{\partial^2}{r^2\partial\theta\partial\theta'} G_H^+\left(t,r,\theta,\varphi;t,r,\theta,\varphi\right)\bigg|_{\theta=\theta'}$$

and

$$\frac{\partial^2}{r^2\sin^2\theta\partial\varphi\partial\varphi'} G_H^+\left(t,r,\theta,\varphi;t,r,\theta,\varphi\right)\bigg|_{\varphi=\varphi'}$$

Using (4.23) and

$$\frac{\partial^2}{\partial\theta\partial\theta'}\sum_{m=-l}^{l} Y_{lm}^*\left(\theta,\varphi\right)Y_{lm}\left(\theta',\varphi\right)\bigg|_{\theta=\theta'} = \frac{l\left(l+1\right)\left(2l+1\right)}{8\pi}$$

We require

$$\sum_{l=0}^\infty l\left(l+1\right)\left(2l+1\right)\left|\vec{R}_l\left(\omega \mid r\right)\right|^2 \qquad r \to \infty$$

$$\sum_{l=0}^\infty l\left(l+1\right)\left(2l+1\right)\left|\vec{R}_l\left(\omega \mid r\right)\right|^2 \qquad r \to 2M_S$$

Using the spherical symmetry, and the fact that Schwarzschild space is asymptotically flat, we can write

$$\frac{\partial^2}{r^2\partial\theta\partial\theta'} G_H^+\left(t,r,\theta,\varphi;t,r,\theta,\varphi\right)\bigg|_{\theta=\theta'} \sim \frac{1}{2\pi^2 t^4} \qquad r \to \infty$$

hence, following exactly the same approach as above, we get

$$\frac{\partial^2}{r^2\partial\theta\partial\theta'}\sum_{l=0}^\infty l\left(l+1\right)\left(2l+1\right)\left|\vec{R}_l\left(\omega \mid r\right)\right|^2 \sim \frac{8\omega^4}{3} \qquad r \to \infty$$

For the other limit we have, to leading order

$$r^{-2}\sum_{l=0}^{\infty} l(l+1)(2l+1)\left|\vec{R}_l(\omega\,|\,r)\right|^2 \xrightarrow[r\to 2M_S]{} \frac{2}{M_S{}^4\Gamma(iq)\Gamma(-iq)}\int_0^{\infty} dl\, l\left[{}^3K_{iq}\left(2l\alpha^{1/2}\right)\right]^2$$
$$\sim \frac{\left(1+16M_S{}^2\omega^2\right)\omega^2}{6M_S{}^2\left(\dfrac{r}{2M_S}-1\right)^2}$$

Finally, for the 0,1 – cross terms, the quantities required are

$$\frac{\partial}{\partial r}\sum_{l=0}^{\infty}(2l+1)\overleftarrow{R}_l^*(\omega\,|\,r)\,\overleftarrow{R}_l(\omega\,|\,r')\Big|_{r=r'} \qquad r\to 2M_S$$
$$\frac{\partial}{\partial r}\sum_{l=0}^{\infty}(2l+1)\vec{R}_l(\omega\,|\,r)\,\vec{R}_l^*(\omega\,|\,r')\Big|_{r=r'} \qquad r\to 2M_S$$

The former follows from (4.25) and the chain rule (see the derivation of (6.29)). The latter is evaluated (to both orders in l) as before. It is easily seen that the quantity required is

$$\frac{1}{M^3\Gamma(iq)\Gamma(-iq)}\int_0^{\infty} dl\left(l+\tfrac{1}{2}\right)\frac{\partial}{\partial\alpha}K_{iq}\left(2l\alpha^{1/2}\right)K_{iq}\left(2l\alpha'^{1/2}\right)\Big|_{\alpha=\alpha'}$$

which gives

$$\frac{\partial}{\partial r}\sum_{l=0}^{\infty}(2l+1)\vec{R}_l(\omega\,|\,r)\,\vec{R}_l^*(\omega\,|\,r')\Big|_{r=r'} \sim$$
$$\sim \frac{\omega^2}{M_S\left(\dfrac{r}{2M_S}-1\right)^2} - \frac{\omega\tanh\left(4\pi\omega M_S\right)}{16\pi M_S{}^2\left(\dfrac{r}{2M_S}-1\right)^{3/2}} \qquad r\to 2M_S$$