On the homological algebra of clusters, quivers, and triangulations

Thomas Andrew Fisher

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School of Mathematics & Statistics
Newcastle University
Newcastle upon Tyne
United Kingdom

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Contents

Abstract

This thesis is comprised of three parts. Chapter one contains background material detailing some important aspects of category theory and homological algebra. Beginning with abelian categories, we introduce triangulated categories, the homotopy category and give the construction of the derived category. Differential graded algebras, basic Auslander-Reiten theory, and the Cluster Category of Dynkin Type A_n are also introduced, which all play a major role in chapters two and three.

In [21], the cluster category $\mathscr{D}$ of type A_∞, with Auslander-Reiten quiver $\mathbb{Z}A_\infty$, is introduced. Slices in the Auslander-Reiten quiver of $\mathscr{D}$ give rise to direct systems; the homotopy colimit of such direct systems can be computed and these "Prüfer objects" can be adjoined to form a larger category. It is this larger category, $\overline{\mathscr{D}}$, which is the main object of study in chapter two. We show that $\overline{\mathscr{D}}$ inherits a nice geometrical structure from $\mathscr{D}$; "arcs" between non-neighbouring integers on the number line correspond to indecomposable objects, and in the case of $\overline{\mathscr{D}}$ we also have arcs to infinity which correspond to the Prüfer objects. During the course of chapter two, we show that $\overline{\mathscr{D}}$ is triangulated, compute homs, investigate the geometric model, and we conclude by computing the cluster tilting subcategories of $\overline{\mathscr{D}}$.

Frieze patterns of integers were studied by Conway and Coxeter, see [13] and [14]. Let $\mathscr{C}$ be the cluster category of Dynkin type A_n. Indecomposables in $\mathscr{C}$ correspond to diagonals in an $(n+3)$-gon. Work done by Caldero and Chapoton showed that the Caldero-Chapoton map (which is a map dependent on a fixed object R of a category, and which goes from the set of objects of that category to $\mathbb{Z}$), when applied to the objects of $\mathscr{C}$ can recover these friezes, see [10]. This happens precisely when R corresponds to a triangulation of the $(n+3)$-gon, i.e. when R is basic and cluster tilting. Later work (see [6], [22]) generalised this connection with friezes further, now to d-angulations of the $(n+3)$-gon with R basic and rigid. In chapter three, we extend these generalisations further still, to the case where the object R corresponds to a general Ptolemy diagram, i.e. R is basic and add(R) is the most general possible torsion class (where the previous efforts have focused on special cases of torsion classes).

Chapter 1

Introduction

Homological algebra arose out of algebraic topology in the early twentieth century. It was found that the study of chain complexes associated with a topological space could provide useful invariants of that space. For example, topological spaces cannot be homeomorphic to one another if their homologies are different. It was not until the 1940s that homological algebra began to blossom into a subject in its own right, culminating in 1956 with the publication of Cartan and Eilenberg's pioneering book, [12]. At a similar time, category theory was developing, with Mac Lane and Eilenberg both cofounders. Since the 1960s, under Grothendieck's influence, the language of homological algebra has become highly categorical. Abelian categories, kernels and cokernels, projective and injective objects and resolutions all provide powerful tools across much of mathematics. This chapter, the introduction, provides an overview of these notions, and introduces triangulated categories. An important example of a triangulated category is the derived category, which is a very important tool in homological algebra. The introduction ends by describing differential graded algebras, some Auslander-Reiten theory and the Cluster Category of Dynkin Type A_n, first introduced in [11].

There is no claim of originality by the author here in the introduction. The material here is taken from a few standard sources in the literature, but has been interpreted and presented afresh by the author, who also presents his own proofs. The reader is directed to [20], [33] and [2], amongst others, (which where particularly useful are referenced directly), for the background presented in this chapter.

1.1 Categories and some special morphisms

Definition 1.1.1. A *category* $\mathscr{C}$ comprises

(i) a class, $\operatorname{obj}\mathscr{C}$, of *objects* $X, Y, Z, \ldots$ of $\mathscr{C}$,

(ii) for each $X, Y \in \operatorname{obj} \mathscr{C}$, a class $\operatorname{Hom}_{\mathscr{C}}(X, Y)$, which is often written as (X, Y) when it is clear what is meant by context, of *morphisms* from X to Y,

(iii) for each triple $X, Y, Z \in \operatorname{obj} \mathscr{C}$, a *composition* $\circ$ of morphisms,

$$\circ : (X, Y) \times (Y, Z) \to (X, Z), \; (f, g) \mapsto g \circ f.$$

If $f \in (X, Y)$, then we often write this as $f : X \to Y$ or as $X \xrightarrow{f} Y$. We say that f is a morphism, or *arrow*, from (the *domain*) X to (the *codomain*) Y. The composition described above may then be written as follows.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Composition must satisfy the following properties.

(i) **[Associativity].** For every quadruple of objects X_1, X_2, X_3, X_4 and for every triple of morphisms $X_1 \xrightarrow{f} X_2$, $X_2 \xrightarrow{g} X_3$, $X_3 \xrightarrow{h} X_4$, we have that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(ii) **[Identity].** For each $X \in \operatorname{obj} \mathscr{C}$, there exists $1_X = \mathrm{id}_X \in (X, X)$, called the *identity morphism on* X, satisfying

$$f \circ \mathrm{id}_X = f \quad \text{and} \quad \mathrm{id}_Y \circ f = f$$

for all $f \in (X, Y)$.

Additionally, we stipulate that $\mathscr{C}$ satisfies the requirement that (U, V) and (X, Y) are disjoint unless $U = X$ and $V = Y$. ◇

Definition 1.1.2. A category $\mathscr{C}$ is called *small* if both $\operatorname{obj} \mathscr{C}$ and, for all pairs $X, Y \in \operatorname{obj} \mathscr{C}$, $\operatorname{Hom}_{\mathscr{C}}(X, Y)$ are actually sets and not proper classes. In this situation, (X, Y) is called a *Hom-set*. Otherwise, $\mathscr{C}$ is called *large.* A *locally small category* satisfies only the second condition, namely that Hom-classes are actually Hom-sets. ◇

Some examples of categories are listed below.

Example 1.1.3.

(i) $\mathscr{C} = \mathsf{Sets}$. In this category, the objects are sets and morphisms are functions between sets.

(ii) $\mathscr{C} = \mathsf{Grp}$. In this category, the objects are groups and the morphisms are group homomorphisms.

(iii) $\mathscr{C} = \mathsf{Ab}$. In this category, the objects are abelian groups and the morphisms are group homomorphisms.

(iv) $\mathscr{C} = \mathsf{Top}$. In this category, the objects are topological spaces and the morphisms are the continuous functions. $\diamond$

In a category there may be some special morphisms.

Definition 1.1.4. Let $\mathscr{C}$ be a category and let $X, Y, Z \in \operatorname{obj}\mathscr{C}$.

(i) Any morphism $f : X \to X$ is called an *endomorphism* of X, and $\operatorname{End}(X) = (X, X)$.

(ii) A morphism $u : X \to Y$ is a *monomorphism* (or a *monic morphism*, or a *mono*) if for each pair $f, g \in (Z, X)$ such that $u \circ f = u \circ g$, we have that $f = g$ (that is, u is *left cancellative*). An arrow denoting a monomorphism is often written as $\hookrightarrow$.

(iii) A morphism $p : X \to Y$ is an *epimorphism* (or an *epic morphism*, or an *epi*) if for each pair $f, g \in (Y, Z)$ such that $f \circ p = g \circ p$, we have that $f = g$ (that is, p is *right cancellative*). An arrow denoting an epimorphism is often written as $\twoheadrightarrow$.

(iv) A morphism $f : X \to Y$ is called an *isomorphism* if there exists an *inverse* $g : Y \to X$ satisfying
$$g \circ f = 1_X \quad \text{and} \quad f \circ g = 1_Y.$$
If this is the case, then g is also an isomorphism (with inverse f) and we say that the objects X and Y are isomorphic. This is denoted $X \cong Y$. It is a trivial exercise to show that any isomorphism is both a monomorphism and an epimorphism, but it is worth noting that the converse is not always true. A *bimorphism* is a morphism which is both a monomorphism and an epimorphism, and if in $\mathscr{C}$ all bimorphisms are isomorphisms, then $\mathscr{C}$ is said to be *balanced*.

(v) An endomorphism which is an isomorphism is called an *automorphism*.

(vi) A morphism $f : X \to Y$ is called a *constant morphism* (or *left zero morphism*) if for each pair $g, h \in (Z, X)$, we have that $f \circ g = f \circ h$. Dually, f is called a *coconstant morphism* (or *right zero morphism*) if for each pair $g, h \in (Y, Z)$, we have that $g \circ f = h \circ f$. If f is both a constant morphism and a coconstant morphism, then f is called a *zero morphism*. A category is said to *have zero morphisms* if for every pair of objects $X, Y \in \operatorname{obj}\mathscr{C}$ there exists a zero morphism between them, denoted $0_{X,Y} : X \to Y$. $\diamond$

Proposition 1.1.5. *Suppose $\mathscr{C}$ has zero morphisms. Then*

(i) for each pair of objects X, Y, the morphism $0_{X,Y}$ is unique,

(ii) the composition of two zero morphisms is a zero morphism,

(iii) for all triples $X, Y, Z \in \operatorname{obj}\mathscr{C}$ and all $f \in (X,Y)$, $g \in (Y,Z)$, we have

$$0_{Y,Z} \circ f = g \circ 0_{X,Y} = 0_{X,Z}.$$

Proof.

(i) Suppose (X,Y) has two zero morphisms, $0_{X,Y}$ and $0'_{X,Y}$. Then

$$0'_{X,Y} = 0'_{X,Y} \circ 1_X = 0'_{X,Y} \circ 0_{X,X} = 0_{X,Y} \circ 0_{X,X} = 0_{X,Y} \circ 1_X = 0_{X,Y}$$

as required.

(ii) Let $X \xrightarrow{0_{X,Y}} Y \xrightarrow{0_{Y,Z}} Z$ be the composition in question. By part (i) we need only show that $0_{Y,Z} \circ 0_{X,Y}$ is a zero morphism from X to Z. This is trivial: indeed, for any pair $f, g : U \to X$ we have

$$0_{X,Y} \circ f = 0_{X,Y} \circ g \Rightarrow (0_{Y,Z} \circ 0_{X,Y}) \circ f = (0_{Y,Z} \circ 0_{X,Y}) \circ g.$$

Similarly for any pair $f', g' : Z \to V$ we have

$$f' \circ 0_{Y,Z} = g' \circ 0_{Y,Z} \Rightarrow f' \circ (0_{Y,Z} \circ 0_{X,Y}) = g' \circ (0_{Y,Z} \circ 0_{X,Y}).$$

This implies that $0_{Y,Z} \circ 0_{X,Y} = 0_{X,Z}$ as required.

(iii) Parts (i) and (ii) imply this.

□

Definition 1.1.6. Let $\mathscr{C}$ be a category.

(i) An object I in $\mathscr{C}$ is called an *initial object* if for each $X \in \operatorname{obj}\mathscr{C}$ we have that (I, X) has one element.

(ii) An object T in $\mathscr{C}$ is called a *terminal object* if for each $X \in \operatorname{obj}\mathscr{C}$ we have that (X, T) has one element.

(iii) An object 0 in $\mathscr{C}$ is called a *zero object* if it is both an initial object and a terminal object. ◇

Lemma 1.1.7. *Let $\mathscr{C}$ be a category.*

(i) If $\mathscr{C}$ has an initial object, then that object is unique up to isomorphism.

(ii) If $\mathscr{C}$ has a terminal object, then that object is unique up to isomorphism.

(iii) If $\mathscr{C}$ has a zero object, then that object is unique up to isomorphism.

(iv) If $\mathscr{C}$ has a zero object 0 and $X, Y \in \operatorname{obj}\mathscr{C}$, then $\mathscr{C}$ has zero morphisms and the composition $X \to 0 \to Y$ is the unique zero morphism $0_{X,Y}$ from X to Y.

Proof.

(i) Suppose I_1 and I_2 are two initial objects in $\mathscr{C}$. Then (I_1, I_2) contains precisely one element, i_1. Similarly (I_2, I_1) contains precisely one element, i_2. Then $i_2 \circ i_1 \in (I_1, I_1)$ is the single element in (I_1, I_1), so must be the identity on I_1. Similarly $i_1 \circ i_2$ must be the identity on I_2. Hence i_1 and i_2 are isomorphisms, each other's inverse, and $I_1 \cong I_2$.

(ii) Similar to (i).

(iii) Parts (i) and (ii) imply this.

(iv) We show the composition $X \xrightarrow{f} 0 \xrightarrow{g} Y$ is a zero morphism from X to Y. Suppose we have morphisms $u, u' : U \to X$. We have $f \circ u = f \circ u'$ because 0 is a terminal object. This implies that $(g \circ f) \circ u = (g \circ f) \circ u'$. Therefore $g \circ f$ is a constant morphism. That $g \circ f$ is coconstant is seen similarly and uses that 0 is an initial object. Therefore, $g \circ f$ is the unique zero morphism from X to Y, and hence if $\mathscr{C}$ has a zero object then it has zero morphisms given canonically via factorising through its zero object, illustrated below.

$$\begin{tikzcd}
X \arrow[r, "0_{X,0}"] \arrow[dr, "0_{X,Y}"'] & 0 \arrow[d, "0_{0,Y}"] \\
& Y
\end{tikzcd}$$

∎

We now work towards defining an *abelian category*. First, we give a formal definition of a *diagram* in a category.

Definition 1.1.8. A *diagram* in a category $\mathscr{C}$ is a directed multigraph whose arrows are morphisms and whose vertices are objects (of $\mathscr{C}$). A diagram is said to be commutative if whenever there are two different paths between two fixed vertices, as morphisms in the category $\mathscr{C}$ their composition is equal. For example

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "i"] \\ C \arrow[r, "h"'] \arrow[ur, "j"] & D \end{tikzcd}$$

commutes if $i \circ g = h \circ f$, $j \circ f = g$ and $i \circ j = h$. Unless otherwise stated, diagrams are always assumed to be commutative. $\diamond$

1.2 Abelian categories

Definition 1.2.1. [Ab-category]. A category $\mathscr{C}$ is called an *Ab-category* if every Hom-set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ in $\mathscr{C}$ has the structure of an (additive) abelian group and composition of morphisms distributes over the group addition. That is, given $A, B, C, D \in \mathrm{obj}\,\mathscr{C}$ and diagrams of the form

$$A \xrightarrow{f} B \overset{g}{\underset{g'}{\rightrightarrows}} C \qquad \text{and} \qquad B \overset{g}{\underset{g'}{\rightrightarrows}} C \xrightarrow{h} D$$

we have $(g + g') \circ f = g \circ f + g' \circ f$ and $h \circ (g + g') = h \circ g + h \circ g'$. $\diamond$

Remark 1.2.2. Let $\mathscr{C}$ be an Ab-category. Taking $A = B = C = D$ in the definition above, we see that each $(A, A) = \mathrm{End}(A)$ is an associative ring. $\diamond$

Proposition 1.2.3. *Let $\mathscr{C}$ be an Ab-category. If $X, Y \in \mathrm{obj}\,\mathscr{C}$ then the identity element, $I_{X,Y}$, of the abelian group (X, Y) is a zero morphism (and therefore $I_{X,Y} = 0_{X,Y}$). Therefore Ab-categories have zero morphisms.*

Proof. Let $f : X \to Y$ be a morphism. Let $Z \in \mathrm{obj}\,\mathscr{C}$, and let $g, g' : Y \to Z$ be arbitrary morphisms. Then $f + I_{X,Y} = f$ and composing with $g - g' \in (Y, Z)$ gives

$$(g - g') \circ f = (g - g') \circ (f + I_{X,Y}) = (g - g') \circ f + (g - g') \circ I_{X,Y}.$$

This implies that $(g - g') \circ I_{X,Y} = I_{X,Z}$, that is, $g \circ I_{X,Y} - g' \circ I_{X,Y} = I_{X,Z}$. This gives $g \circ I_{X,Y} = g' \circ I_{X,Y}$, which shows that $I_{X,Y}$ is coconstant. The proof that $I_{X,Y}$ is constant is similar and then, as required, we have that $I_{X,Y} = 0_{X,Y}$ is a zero morphism and hence $\mathscr{C}$ has zero morphisms. $\square$

Proposition 1.2.4. *Let $\mathscr{C}$ be an Ab-category and let I be an initial object of $\mathscr{C}$. Then I is also a terminal object, and hence in Ab-categories, initial, terminal and zero objects are synonymous.*

Proof. Suppose I is an initial object. Then the set (I, I) has one element: $0_{I,I} = \mathrm{id}_I$. Given any object A in $\mathscr{C}$, the set (A, I) has at least one element, the zero morphism $0_{A,I}$ from A to I. Suppose $f \in (A, I)$. Then $f = \mathrm{id}_I \circ f = 0_{I,I} \circ f$, which equals $0_{A,I}$ by part (iii) of Proposition 1.1.5. Hence (A, I) has only one element, $0_{A,I}$, so I is also terminal, as required. By a similar argument, it can be shown that if I is a terminal object, then it is also an initial object. Therefore I is a zero object because it is both an initial object and a terminal object. $\square$

Definition 1.2.5. [Pre-additive category]. A *pre-additive category $\mathscr{C}$* is an Ab-category with a zero object 0. $\diamond$

Definition 1.2.6. Let $\mathscr{C}$ be a pre-additive category. Let $X_1, X_2 \in \mathrm{obj}\,\mathscr{C}$. It is said that an object Y is a *biproduct of* X_1 *and* X_2 if there exist morphisms

$$\begin{tikzcd} X_1 \arrow[r, shift right, "\iota_1"'] & Y \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & X_2 \arrow[l, shift left, "\iota_2"] \end{tikzcd}$$

satisfying $\pi_1 \circ \iota_1 = \mathrm{id}_{X_1}$, $\pi_2 \circ \iota_2 = \mathrm{id}_{X_2}$ and $\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = \mathrm{id}_Y$. In this case, Y is denoted by $X_1 \oplus X_2$. $\diamond$

Remark 1.2.7. The biproduct of a finite set of objects $X_1, X_2, \dots, X_n$ can be defined similarly to the above. It is said that $\oplus_{i=1}^n X_i$ is a biproduct of the X_i if there exist morphisms $\pi_j : \oplus_{i=1}^n X_i \to X_j$ and $\iota_j : X_j \to \oplus_{i=1}^n X_i$ satisfying $\pi_i \circ \iota_i = \mathrm{id}_{X_i}$ for each $i \in \{1, 2, \dots, n\}$ and $\Sigma_{j=1}^n (\iota_j \circ \pi_j) = \mathrm{id}_{\oplus_{i=1}^n X_i}$. $\diamond$

Remark 1.2.8. Biproducts permit the use of "component notation". To see what this means, let $\mathscr{C}$ be a pre-additive category with objects A, B, X_1, X_2 such that X_1 and X_2 have a biproduct $X_1 \oplus X_2$. Consider the following commutative diagram.

$$\begin{tikzcd} & A \arrow[dl, bend right, "\alpha_1"'] \arrow[dr, bend left, "\alpha_2"] \arrow[d, dashed, "\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}"] & \\ X_1 \arrow[r, shift right, "\iota_1"'] \arrow[dr, bend right, "\beta_1"'] & X_1 \oplus X_2 \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] \arrow[d, dashed, "\begin{pmatrix}\beta_1 & \beta_2\end{pmatrix}"] & X_2 \arrow[l, shift left, "\iota_2"] \arrow[dl, bend left, "\beta_2"] \\ & B & \end{tikzcd}$$

Here, the dashed arrows are defined as $\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \iota_1 \circ \alpha_1 + \iota_2 \circ \alpha_2$ and $\begin{pmatrix}\beta_1 & \beta_2\end{pmatrix} = \beta_1 \circ \pi_1 + \beta_2 \circ \pi_2$. Composing in this notation is akin to matrix multiplication. $\diamond$

Definition 1.2.9. Let $\mathscr{C}$ be a category and let $(X_i)_{i \in I}$ be a family of $\mathscr{C}$-objects indexed by a set I.

(i) The *product* of the X_i is an ordered pair $(X, \{p_i : X \to X_i\})$, consisting of an object X and a family of morphisms, called *projections*, $(p_i : X \to X_i)_{i \in I}$, which satisfies the following *universal property*: for every $Y \in \operatorname{obj}\mathscr{C}$ and family of morphisms $(f_i : Y \to X_i)_{i \in I}$, there exists a unique morphism $\theta : Y \to X$ such that for each $i \in I$, we have $p_i \circ \theta = f_i$.

(ii) The *coproduct* of the X_i is an ordered pair $(X, \{q_i : X_i \to X\})$, consisting of an object X and a family of morphisms, called *injections*, $(q_i : X_i \to X)_{i \in I}$, which satisfies the following *universal property*: for every $Y \in \operatorname{obj}\mathscr{C}$ and family of morphisms $(f_i : X_i \to Y)_{i \in I}$, there exists a unique morphism $\phi : X \to Y$ such that for each $i \in I$, we have $\phi \circ q_i = f_i$. $\diamond$

Lemma 1.2.10. *Let $\mathscr{C}$ be a pre-additive category and let $(X_i)_{i=1}^n$ be a finite set of $\mathscr{C}$-objects. Let X be the biproduct of the X_i. Then*

(i) $(X, \{\pi_i : X \to X_i\})$ *is a product of the* X_i*,*

(ii) $(X, \{\iota_i : X_i \to X\})$ *is a coproduct of the* X_i*,*

(iii) $\pi_j \circ \iota_i = 0_{X_i, X_j}$, *if* $i \neq j$.

Proof. This can be found in [33, Chapter VIII, Section 2, Theorem 2]. $\square$

Proposition 1.2.11. *Let $\mathscr{C}$ be a category. Products in $\mathscr{C}$ are unique up to isomorphism, as are coproducts and biproducts.*

Proof. See [20, Theorem II.5.1 and the Definition on page 58]. $\square$

Proposition 1.2.12. *Let $\mathscr{C}$ be a pre-additive category and let $(X_i)_{i=1}^n$ be a finite set of $\mathscr{C}$-objects.*

(i) If $X = \coprod_{i=1}^n X_i$ is a coproduct, then X is a biproduct and hence a product.

(ii) If $X = \prod_{i=1}^n X_i$ is a product, then X is a biproduct and hence a coproduct.

Proof.

(i) Without loss of generality, suppose that $n = 2$. There is a diagram

$$\begin{tikzcd} X_1 \arrow[r, "\iota_1"] \arrow[dr, "\mathrm{id}_{X_1}"'] & X_1 \coprod X_2 & X_2 \arrow[l, "\iota_2"'] \arrow[dl, "0"] \\ & X_1 & \end{tikzcd}$$

which by the universal property of the coproduct, extends via a unique morphism $\pi_1 : X_1 \coprod X_2 \to X_1$ as in the following diagram.

$$\begin{tikzcd} X_1 \arrow[r, "\iota_1"] \arrow[dr, "\mathrm{id}_{X_1}"'] & X_1 \coprod X_2 \arrow[d, dashed, "\pi_1"] & X_2 \arrow[l, "\iota_2"'] \arrow[dl, "0"] \\ & X_1 & \end{tikzcd}$$

Similarly, there is a (unique) morphism $\pi_2 : X_1 \coprod X_2 \to X_2$ which makes the following diagram

$$\begin{tikzcd} X_1 \arrow[r, "\iota_1"] \arrow[dr, "0"'] & X_1 \coprod X_2 \arrow[d, dashed, "\pi_2"] & X_2 \arrow[l, "\iota_2"'] \arrow[dl, "\mathrm{id}_{X_2}"] \\ & X_2 & \end{tikzcd}$$

commute. It is clear that the following equations hold:

(a) $\pi_1 \circ \iota_1 = \mathrm{id}_{X_1}$,

(b) $\pi_1 \circ \iota_2 = 0$,

(c) $\pi_2 \circ \iota_1 = 0$,

(d) $\pi_2 \circ \iota_2 = \mathrm{id}_{X_2}$.

Finally, consider the morphism $\theta = \iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 : X_1 \coprod X_2 \to X_1 \coprod X_2$. We have, via direct computation, that $\theta \circ \iota_1 = \iota_1$ and $\theta \circ \iota_2 = \iota_2$. Hence, θ makes the following diagram

$$\begin{tikzcd} X_1 \arrow[r, "\iota_1"] \arrow[dr, "\iota_1"'] & X_1 \coprod X_2 \arrow[d, dashed, "\theta"] & X_2 \arrow[l, "\iota_2"'] \arrow[dl, "\iota_2"] \\ & X_1 \coprod X_2 & \end{tikzcd}$$

commute. That the dashed morphism is unique implies that $\theta = \mathrm{id}_{X_1 \coprod X_2}$, because the morphism $\mathrm{id}_{X_1 \coprod X_2}$ also makes the above diagram commute. Hence, $\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = \mathrm{id}_{X_1 \coprod X_2}$, and this, together with the four equations (a) - (d) above, shows that $X_1 \coprod X_2$ is a biproduct as in Definition 1.2.6, and hence a product by Lemma 1.2.10 (with projections given by the collection $(\pi_i)_{i=1}^n$).

(ii) Proved using a similar (dual) argument to (i).

□

Definition 1.2.13. [Additive category]. An *additive category* $\mathscr{C}$ is a pre-additive category where each pair of objects X, Y in $\mathscr{C}$ has a biproduct $X \oplus Y$. ◇

Definition 1.2.14. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *functor* $F : \mathscr{C} \to \mathscr{D}$ is a mapping which

(i) for each object $X \in \mathscr{C}$, associates an object $F(X) \in \mathscr{D}$,

(ii) for each $f : X \to Y$ in $\mathscr{C}$, associates a morphism $F(f) : F(X) \to F(Y)$ in $\mathscr{D}$ such that

  (a) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every object $X \in \operatorname{obj} \mathscr{C}$,

  (b) either $F(g \circ f) = F(g) \circ F(f)$ for all pairs $f : X \to Y$ and $g : Y \to Z$, in which case the functor is said to be *covariant*, or $F(g \circ f) = F(f) \circ F(g)$ for all pairs $f : X \to Y$ and $g : Y \to Z$, in which case the functor is said to be *contravariant*. ◇

Definition 1.2.15. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and let F and G be functors from $\mathscr{C}$ to $\mathscr{D}$ with the same variance. A *natural transformation* η from the functor F to the functor G is a family of morphisms which satisfy the following.

(i) The natural transformation $\eta : F \to G$ associates to every $\mathscr{C}$-object X a $\mathscr{D}$-morphism $\eta_X : F(X) \to G(X)$. This morphism is called the *component* of η at X.

(ii) For each $\mathscr{C}$-morphism $f : X \to Y$, components satisfy

$$G(f) \circ \eta_X = \eta_Y \circ F(f),$$

if F and G are covariant, and

$$\eta_X \circ F(f) = G(f) \circ \eta_Y,$$

if F and G are contravariant.

If, for every $\mathscr{C}$-object X, it is the case that η_X is an isomorphism in the category $\mathscr{D}$, then η is called a *natural isomorphism* or a *natural equivalence*. $\diamond$

Definition 1.2.16. Let $\mathscr{C}$ and $\mathscr{D}$ be pre-additive categories. Then an *additive functor* $F : \mathscr{C} \to \mathscr{D}$ is a functor which preserves the abelian group structure on Hom-sets. That is, given objects X, Y in $\mathscr{C}$, the set map $(X, Y) \to (F(X), F(Y))$ which F induces is a group homomorphism. $\diamond$

Definition 1.2.17. Let $\mathscr{C}$ be a category and $X, Y \in \operatorname{obj} \mathscr{C}$. Let $f, g \in (X, Y)$.

(i) The *equaliser of f and g* consists of an object $E(f, g)$ and a morphism $eq(f, g) : E \to X$ satisfying $f \circ eq(f, g) = g \circ eq(f, g)$. Note, when it is clear by the context which morphisms are under consideration, $E(f, g)$ and $eq(f, g)$ will be shortened simply to E and eq respectively. The pair (E, eq) also satisfies the following universal property: given any object O and a morphism $\sigma : O \to X$, if $f \circ \sigma = g \circ \sigma$, then there exists a unique morphism $u : O \to E$ such that $eq \circ u = \sigma$. A morphism $\sigma : O \to X$ is said to *equalise f and g* if $f \circ \sigma = g \circ \sigma$.

(ii) The *coequaliser of f and g* consists of an object $Q(f, g)$ and a morphism $coeq(f, g) : Y \to Q$ satisfying $coeq(f, g) \circ f = coeq(f, g) \circ g$. Note, when it is clear by the context which morphisms are under consideration, $Q(f, g)$ and $coeq(f, g)$ will be shortened simply to Q and $coeq$ respectively. The pair $(Q, coeq)$ also satisfies the following universal property: given any object R and a morphism $r : Y \to R$, if $r \circ f = r \circ g$, then there exists a unique morphism $v : Q \to R$ such that $v \circ coeq = r$. A morphism $r : Y \to Q$ is said to *coequalise f and g* if $r \circ f = r \circ g$. $\diamond$

Proposition 1.2.18. *In any category, the morphisms eq and coeq are a monomorphism and epimorphism, respectively.*

Proof. Let (E, eq) be the equaliser of f and g, and suppose that for a pair of morphisms $u_1, u_2 : O \longrightarrow E$ it is the case that $eq \circ u_1 = eq \circ u_2 = \sigma$. Consider the following commutative diagram.

$$\begin{tikzcd}
O \arrow[d, shift right, "u_1"'] \arrow[d, shift left, "u_2"] \arrow[dr, "\sigma"] & & \\
E \arrow[r, "eq"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

We know that $f \circ \sigma = g \circ \sigma$ and therefore, by the universal property of the equaliser, there is a unique $u : O \to E$ such that $\sigma = eq \circ u$. Hence, $u_1 = u = u_2$, which shows that eq is a monomorphism. That $coeq$ is an epimorphism is proved using a dual argument. $\square$

Definition 1.2.19. [Pre-abelian category]. Let $\mathscr{C}$ be an additive category. Then $\mathscr{C}$ is called a *pre-abelian category* if for each pair $X, Y \in \operatorname{obj}\mathscr{C}$ and each morphism $f \in (X, Y)$,

(i) the equaliser of f and $0_{X,Y}$ exists, in which case this is called the *kernel of* f, and is denoted $\operatorname{Ker} f$,

(ii) the coequaliser of f and $0_{X,Y}$ exists, in which case this is called the *cokernel of* f, and is denoted $\operatorname{Coker} f$. $\diamond$

Remark 1.2.20. Given $f : X \to Y$ in a pre-abelian category $\mathscr{C}$, the universal properties of the kernel $(\operatorname{Ker} f, u)$ and the cokernel $(\operatorname{Coker} f, p)$ are illustrated below. From now on, we will always write the zero morphism from A to B as $0 : A \to B$ (instead of $0_{A,B} : A \to B$) if the domain and codomain of the zero morphism are clear by the context.

$$\begin{tikzcd}
K' \arrow[dr, dashed, "!h'"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "h"'] & & \\
& \operatorname{Ker} f \arrow[r, "0"] \arrow[d, "u"] & Y \\
& X \arrow[ur, "f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& Y \arrow[d, "p"] \arrow[dr, bend left, "g"] & \\
X \arrow[ur, "f"] \arrow[r, "0"'] \arrow[drr, bend right, "0"'] & \operatorname{Coker} f \arrow[dr, dashed, "!g'"] & \\
& & Q'
\end{tikzcd}$$

The morphisms u and p, by virtue of respectively being an equaliser and a coequaliser, are respectively monic and epic. $\diamond$

Definition 1.2.21. Let $\mathscr{C}$ be a pre-abelian category and let $f \in (A, B)$. Then the two diagrams

(i) $\operatorname{Ker} f \xrightarrow{u} A \xrightarrow{p'} \operatorname{Coker} u$,

(ii) $\operatorname{Ker} p \xrightarrow{u'} B \xrightarrow{p} \operatorname{Coker} f$,

exist and we define the *coimage of* f, $\operatorname{Coim} f = \operatorname{Coker} u = \operatorname{Coker} \operatorname{Ker} f$, and the *image of* f, $\operatorname{Im} f = \operatorname{Ker} p = \operatorname{Ker} \operatorname{Coker} f$. $\diamond$

Proposition 1.2.22. *Let $\mathscr{C}$ be a pre-abelian category. Then every morphism $f : A \to B$ has a canonical decomposition, shown below.*

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r, "u"] & A \arrow[r, "f"] \arrow[d, "p'"'] & B \arrow[r, "p"] & \operatorname{Coker} f \\
& \operatorname{Coim} f \arrow[r, "\bar{f}"'] & \operatorname{Im} f \arrow[u, "u'"'] &
\end{tikzcd}$$

In particular, there is a unique connecting morphism $\bar{f} : \operatorname{Coim} f \to \operatorname{Im} f$ *making the above diagram commutative (the reason it is called a 'connecting' morphism is that it connects the two diagrams seen above in Definition 1.2.21).*

Proof. Construct the following diagram

$$\begin{tikzcd}
A \arrow[dr, dashed, "!f'"'] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "f"'] & & \\
& \operatorname{Ker} p \arrow[r, "0"] \arrow[d, "u'"'] & \operatorname{Coker} f \\
& B \arrow[ur, "p"'] &
\end{tikzcd}$$

and note that, because $p \circ f = 0$, there exists a unique morphism $f' : A \to \operatorname{Ker} p$ such that $f = u' \circ f'$. Now, $u' \circ f' \circ u = f \circ u = 0 = u' \circ 0$ and because u' is a monomorphism, we have $f' \circ u = 0$. Now consider the following commutative diagram.

$$\begin{tikzcd}
& A \arrow[d, "p'"] \arrow[ddr, bend left, "f'"] & \\
\operatorname{Ker} f \arrow[ur, "u"] \arrow[r, "0"'] \arrow[drr, bend right, "0"'] & \operatorname{Coker} u \arrow[dr, dashed, "!\bar{f}"] & \\
& & \operatorname{Ker} p
\end{tikzcd}$$

Now, f' induces a unique morphism $\bar{f} : \operatorname{Coker} u \to \operatorname{Ker} p$ such that $\bar{f} \circ p' = f'$, because $f' \circ u = 0$. Hence, $f = u' \circ f' = u' \circ \bar{f} \circ p'$, as required. $\square$

Proposition 1.2.23. *Let* $\mathscr{C}$ *be a pre-abelian category,* $A, B \in \operatorname{obj} \mathscr{C}$ *and let* $f \in (A, B)$*. Let* $u : \operatorname{Ker} f \to A$ *be the canonical morphism from the kernel to* A*, and* $p : B \to \operatorname{Coker} f$ *be the canonical morphism from* B *to the cokernel. Suppose* $\bar{f} : \operatorname{Coim} f \to \operatorname{Im} f$ *is an isomorphism. Then*

(i) if f *is a monomorphism, then it is the kernel of some map,*

(ii) if f *is an epimorphism, then it is the cokernel of some map.*

Proof.

(i) Because $f \circ u = 0 = f \circ 0$ and f is a monomorphism, we have that $u = 0$ and hence by the universal property of the kernel, $\operatorname{Ker} f$ is the zero object. Hence $A \cong \operatorname{Ker} p$

because $A = \operatorname{Coker} 0_{0,A}$ (seen by the fact $(A, 1_A)$ is a cokernel for the zero morphism). Hence the canonical decomposition of f reduces to

$$\begin{tikzcd} 0 \arrow[r, "0"] & A \arrow[r, "f"] \arrow[d, "1_A"'] & B \arrow[r, "p"] & \operatorname{Coker} f \\ & A \arrow[r, "\cong"'] & A \arrow[u, "u'"'] & \end{tikzcd}$$

whence $u' = f$ and $A \xrightarrow{f} B$ is the kernel of $B \xrightarrow{p} \operatorname{Coker} f$.

(ii) This is proved using a similar method to part (i) and shows that if $f \in (A, B)$ is epic, then $A \xrightarrow{f} B$ is the cokernel of $\operatorname{Ker} f \xrightarrow{u} A$.

□

Proposition 1.2.24. *In a pre-abelian category $\mathscr{C}$, the following are equivalent:*

(i) For every pair of objects X, Y in $\mathscr{C}$ and every morphism $f : X \to Y$, the canonical morphism $\bar{f} : \operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism,

(ii) Every monomorphism in $\mathscr{C}$ is a kernel and every epimorphism in $\mathscr{C}$ is a cokernel.

Proof. That (i) implies (ii) was proved above in Proposition 1.2.23. That (i) is implied by (ii) can be found in, for example, [33, Chapter VIII]. □

Definition 1.2.25. [Abelian category]. Let $\mathscr{C}$ be a pre-abelian category. Then $\mathscr{C}$ is said to be an *abelian category* if it satisfies the pair of equivalent conditions

(i) For every pair of objects X, Y in $\mathscr{C}$ and every morphism $f : X \to Y$, the canonical morphism $\bar{f} : \operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism,

(ii) Every monomorphism in $\mathscr{C}$ is a kernel and every epimorphism in $\mathscr{C}$ is a cokernel. ◇

Remark 1.2.26. In an abelian category, every morphism factors into the composition of an epimorphism followed by a monomorphism, and in particular we can always decompose a morphism $f : X \to Y$ as

$$\begin{tikzcd} \operatorname{Ker} f \arrow[r, hook, "u"] & X \arrow[r, two heads, "\bar{f} \circ p'"] \arrow[rr, bend right, "f"'] & \operatorname{Im} f \arrow[r, hook, "u'"] & Y \arrow[r, two heads, "p"] & \operatorname{Coker} f \end{tikzcd}$$

with $\bar{f} \circ p'$ epic and u' monic. ◇

Proposition 1.2.27. *Let $\mathscr{C}$ be an abelian category. Let $f : X \to Y$ be a morphism in $\mathscr{C}$ and suppose f is both epic and monic. Then f is an isomorphism.*

Proof. On the one hand, $A \xrightarrow{f} B$ epic implies that (B, f) is the cokernel of $\operatorname{Ker} f \to A$. On the other, $A \xrightarrow{f} B$ monic implies that (A, f) is the kernel of $B \to \operatorname{Coker} f$. Hence the canonical decomposition of $f : A \to B$ can be written as

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r] & A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[r] & \operatorname{Coker} f \\
 & B \arrow[r, "\bar{f}"'] & A \arrow[u, "f"'] &
\end{tikzcd}$$

showing that $f \circ \bar{f} \circ f = f = f \circ 1_A = 1_B \circ f$. Now using the fact that f is both epic and monic gives $\bar{f}$ as the inverse of f, showing that f is an isomorphism, as required. □

Definition 1.2.28. Let $\mathscr{C}$ be a category. Then a category $\mathscr{D}$ is a *subcategory of* $\mathscr{C}$ if

(i) $X \in \operatorname{obj} \mathscr{D} \Rightarrow X \in \operatorname{obj} \mathscr{C}$,

(ii) for all $A, B \in \operatorname{obj} \mathscr{D}$, we have $\operatorname{Hom}_{\mathscr{D}}(A, B) \subseteq \operatorname{Hom}_{\mathscr{C}}(A, B)$,

(iii) if $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are morphisms in $\mathscr{D}$, then the composite $g \circ f$ is in $\operatorname{Hom}_{\mathscr{D}}(A, C)$ and is equal to the composite $g \circ f$ in $\operatorname{Hom}_{\mathscr{C}}(A, C)$,

(iv) for all objects A in $\mathscr{D}$, the identity 1_A is in $\operatorname{Hom}_{\mathscr{D}}(A, A)$ and is equal to the identity 1_A in $\operatorname{Hom}_{\mathscr{C}}(A, A)$.

A subcategory is said to be *full* if for each pair of objects $A, B \in \operatorname{obj} \mathscr{D}$, we have that $\operatorname{Hom}_{\mathscr{D}}(A, B) = \operatorname{Hom}_{\mathscr{C}}(A, B)$. ◇

Definition 1.2.29. Let $\mathscr{C}$ be a category. Then the *opposite category* $\mathscr{C}^{\mathsf{op}}$ is defined by setting

(i) $\operatorname{obj} \mathscr{C}^{\mathsf{op}} = \operatorname{obj} \mathscr{C}$,

(ii) $\operatorname{Hom}_{\mathscr{C}^{\mathsf{op}}}(X, Y) = \operatorname{Hom}_{\mathscr{C}}(Y, X)$.

Composition in the opposite category works as follows. Let $X, Y, Z \in \operatorname{obj} \mathscr{C}^{\mathsf{op}}$, $f \in \operatorname{Hom}_{\mathscr{C}^{\mathsf{op}}}(X, Y)$ and $g \in \operatorname{Hom}_{\mathscr{C}^{\mathsf{op}}}(Y, Z)$. Then $f \in \operatorname{Hom}_{\mathscr{C}}(Y, X)$ and $g \in \operatorname{Hom}_{\mathscr{C}}(Z, Y)$, and hence $f \circ_{\mathscr{C}} g \in \operatorname{Hom}_{\mathscr{C}}(Z, X)$ which is, by definition, $\operatorname{Hom}_{\mathscr{C}^{\mathsf{op}}}(X, Z)$. Thus composition in $\mathscr{C}^{\mathsf{op}}$ is defined such that $g \circ_{\mathscr{C}^{\mathsf{op}}} f = f \circ_{\mathscr{C}} g$. ◇

Remark 1.2.30. Let $\mathscr{C}$ be a category and consider its opposite category, $\mathscr{C}^{\mathsf{op}}$. Given a morphism $f \in (A, B)$ in $\mathscr{C}$, its domain in $\mathscr{C}$ is its codomain in $\mathscr{C}^{\mathsf{op}}$; this is A, and its codomain in $\mathscr{C}$ is its domain in $\mathscr{C}^{\mathsf{op}}$; this is B. If $g : B \to C$ is a $\mathscr{C}$-morphism, then in $\mathscr{C}$ it is the composition $g \circ f : A \to C$ which is defined. However, in $\mathscr{C}^{\mathsf{op}}$, it is the composition $f \circ g : C \to A$ which is defined. ◇

Remark 1.2.31. It is clear from Definition 1.2.29 that $(\mathscr{C}^{\mathsf{op}})^{\mathsf{op}} = \mathscr{C}$. ◇

We now introduce the concept of subobjects and quotient objects.

Definition 1.2.32. Let $\mathscr{C}$ be a category and let $A_1 \to B$ and $A_2 \to B$ be monomorphisms in $\mathscr{C}$. Then $A_1 \to B$ and $A_2 \to B$ are said to be *equivalent* if there are morphisms $A_1 \to A_2$ and $A_2 \to A_1$ such that the following diagrams

$$\begin{tikzcd} A_1 \arrow[dd] \arrow[dr] & \\ & B \\ A_2 \arrow[ur] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A_1 \arrow[dr] & \\ & B \\ A_2 \arrow[uu] \arrow[ur] & \end{tikzcd}$$

commute. Then a *subobject* of B is an equivalence class of monomorphisms with codomain B. It is clear that if two monomorphisms represent the same subobject of B, then they have isomorphic domains. *Quotient objects* are defined dually, by reversing the arrows above and requiring that $B \to A_1$ and $B \to A_2$ are epimorphisms. A quotient object of B is then an equivalence class of epimorphisms with domain B. ◇

It is clear that if $\mathscr{C}$ is a pre-abelian category, and $f : A \to B$ is a morphism in $\mathscr{C}$, then the equivalence class of $\operatorname{Ker} f$ is a subobject of A and the equivalence class of $\operatorname{Coker} f$ is a quotient object of B. Next, we introduce the notion of an exact sequence.

Proposition 1.2.33. *Let $\mathscr{C}$ be an abelian category and let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of morphisms in $\mathscr{C}$ such that the composite $g \circ f$ is zero. Then there is a natural map from* $\operatorname{Im} f$ *to* $\operatorname{Ker} g$ *obtained as in the following diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi_f"'] & & Y \arrow[r, "g"] & Z \\ & \operatorname{Im} f \arrow[ur, "\iota_f"'] \arrow[r, dashed, "!w"'] & \operatorname{Ker} g \arrow[u, "k"'] & \end{tikzcd}$$

with π_f and ι_f the canonical morphisms seen in Remark 1.2.26.

Proof. Because π_f is an epimorphism, $g \circ \iota_f \circ \pi_f = g \circ f = 0 = 0 \circ \pi_f$ implies that $g \circ \iota_f = 0$. Hence, by the definition of the kernel, there is a unique morphism $w : \operatorname{Im} f \to \operatorname{Ker} g$ such that the diagram commutes. □

Definition 1.2.34. [Exact sequence]. Suppose $\mathscr{C}$ is an abelian category. A sequence (either finite or infinite) of morphisms

$$\cdots \longrightarrow X_{n+1} \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n-1} \longrightarrow \cdots$$

in $\mathscr{C}$ is called *exact* if, for all $i \in \mathbb{Z}$, we have that

(i) $f_i \circ f_{i+1} = 0$,

(ii) the natural map $\operatorname{Im} f_{i+1} \to \operatorname{Ker} f_i$ is an isomorphism.

An exact sequence of the form

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is called a *short exact sequence*. ◇

Definition 1.2.35. Let $\mathscr{C}$ be a category and let $f : A \to B$ be a morphism in $\mathscr{C}$.

(i) Suppose that f is a monomorphism. If there exists a morphism $g : B \to A$ such that $g \circ f = 1_A$, then f is called a *split monomorphism*, or a *section*.

(ii) Suppose that f is an epimorphism. If there exists a morphism $g : B \to A$ such that $f \circ g = 1_B$, then f is called a *split epimorphism*, or a *retraction*.

It can be seen that in (i), g is a retraction (of f), and in (ii), f is a section (of g). ◇

Definition 1.2.36. A short exact sequence $0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 0$ in an abelian category $\mathscr{C}$ is called *split* if the following equivalent conditions are satisfied.

(i) There exists a map $j : C \to B$ such that $\pi \circ j = 1_C$.

(ii) There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r] \arrow[d, "\cong"] & C \arrow[r] \arrow[d, "\mathrm{id}_C"] & 0 \\
0 \arrow[r] & A \arrow[r, "\begin{pmatrix} \mathrm{id}_A \\ 0 \end{pmatrix}"'] & A \oplus C \arrow[r, "\begin{pmatrix} 0 & \mathrm{id}_C \end{pmatrix}"'] & C \arrow[r] & 0
\end{tikzcd}$$

where the morphism $B \to A \oplus C$ is an isomorphism.

(iii) There exists a map $q : B \to A$ such that $q \circ \iota = 1_A$. $\diamond$

That these conditions are equivalent can be found in [38, Exercise 7.17].

Definition 1.2.37. Let $\mathscr{C}$ be an additive category and k a commutative ring. Then $\mathscr{C}$ is said to be a *k-linear category* if each Hom-set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ carries a k-module structure and composition is a k-bilinear map. $\diamond$

1.3 The module category

Now we introduce an important example of an abelian category.

Definition 1.3.1. Let k be a field and let A be a k-algebra. Then the *category of left A-modules* is the category with objects given by the left modules over A and morphisms given by left-A-module homomorphisms. This category can be denoted by either ${}_A\mathsf{Mod}$ or $\mathsf{Mod}(A)$ and is a standard example of an abelian category. The category of right A-modules is defined analogously and is denoted by either Mod_A or $\mathsf{Mod}(A^{\mathsf{op}})$. $\diamond$

Remark 1.3.2. The kernels and cokernels in ${}_A\mathsf{Mod}$ are given as follows. Given a homomorphism $f : X \to Y$, we have $\mathrm{Ker}\, f = \{x \in X \mid f(x) = 0\}$ and $\mathrm{Coker}\, f = Y/\,\mathrm{Im}(f)$, where $\mathrm{Im}(f) = \{f(x) \mid x \in X\}$. $\diamond$

Definition 1.3.3. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then F is said to be

(i) *faithful* if for all $A, B \in \mathrm{obj}\,\mathscr{C}$, the set map

$$F_{A,B} : (A, B) \to (F(A), F(B)),$$

defined by $f \mapsto F(f)$, is injective,

(ii) *full* if each $F_{A,B}$ is surjective,

(iii) *essentially surjective*, or *dense*, if each $Y \in \mathrm{obj}\,\mathscr{D}$ is isomorphic to an object of the form $F(X)$ with $X \in \mathrm{obj}\,\mathscr{C}$,

(iv) *exact* if F is additive and given a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathscr{C}$, the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is exact in $\mathscr{D}$ if F is covariant, or the sequence $0 \to F(Z) \to F(Y) \to F(X) \to 0$ is exact in $\mathscr{D}$ if F is contravariant. $\diamond$

Theorem 1.3.4. ***[Freyd-Mitchell Embedding Theorem (1964)].*** *Let $\mathscr{C}$ be a small abelian category. Then there exists a ring R (with identity, but not necessarily commutative) and a faithful, full, and exact functor $F : \mathscr{C} \to {}_R\mathsf{Mod}$ embedding $\mathscr{C}$ into ${}_R\mathsf{Mod}$.*

Proof. See [34, Theorem VI.7.2]. □

Remark 1.3.5. Diagrams in abelian categories which involve kernels, cokernels and exact sequences can be embedded via the Freyd-Mitchell embedding into a module category. This allows the use of arguments which involve *diagram chasing* (that is, arguments on the level of elements inside objects). ◇

Lemma 1.3.6. ***[The five lemma].*** *Let $\mathscr{C}$ be an abelian category and let*

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r, "\beta_1"'] & B_2 \arrow[r, "\beta_2"'] & B_3 \arrow[r, "\beta_3"'] & B_4 \arrow[r, "\beta_4"'] & B_5 \end{tikzcd}$$

be a commutative diagram in $\mathscr{C}$. Suppose that the two rows are exact, f_1 is epic, f_5 is monic, and f_2, and f_4 are isomorphisms. Then f_3 is also an isomorphism.

Proof. By the Freyd-Mitchell embedding theorem (1.3.4), the diagram can be treated as though it is in the category ${}_R\mathsf{Mod}$. Hence the lemma may be proved by diagram chasing. The diagram chase will consist of two parts. Here is the first.

- Let $b_3 \in B_3$.
- There exists an element $a_4 \in A_4$ with $f_4(a_4) = \beta_3(b_3)$, because f_4 is surjective.
- The diagram commutes, hence $\beta_4(f_4(a_4)) = f_5(\alpha_4(a_4))$.
- By exactness, $f_5(\alpha_4(a_4)) = \beta_4(f_4(a_4)) = \beta_4(\beta_3(b_3)) = 0$.
- Because f_5 is injective, $\alpha_4(a_4) = 0$, hence $\alpha_4 \in \operatorname{Ker} \alpha_4 = \operatorname{Im} \alpha_3$.
- Hence there exists $a_3 \in A_3$ such that $\alpha_3(a_3) = a_4$.
- Then $\beta_3(f_3(a_3)) = f_4(\alpha_3(a_3)) = f_4(a_4) = \beta_3(b_3)$. Therefore $\beta_3(b_3 - f_3(a_3)) = 0$.
- Because $b_3 - f_3(a_3) \in \operatorname{Ker} \beta_3$, by exactness it must be the case that $b_3 - f_3(a_3) \in \operatorname{Im} \beta_2$. Therefore there exists $b_2 \in B_2$ such that $\beta_2(b_2) = b_3 - f_3(a_3)$.
- There exists $a_2 \in A_2$ such that $b_2 = f_2(a_2)$, because f_2 is surjective.
- The diagram commutes, hence $f_3(\alpha_2(a_2)) = \beta_2(f_2(a_2)) = \beta_2(b_2) = b_3 - f_3(a_3)$.
- Because f_3 is a module homomorphism, $f_3(\alpha_2(a_2) + a_3) = f_3(\alpha_2(a_2)) + f_3(a_3) = b_3 - f_3(a_3) + f_3(a_3) = b_3$. Hence, f_3 is surjective.

The second part is similar and instead shows that f_3 is injective, and putting the two parts together shows that f_3 is an isomorphism. $\square$

There are two more diagram lemmas we will mention here. The nine lemma can be proved by utilising an appropriate diagram chase. The same is true of the snake lemma, but there are parts of the proof which are slightly more involved; the reader is directed to Lemma 5 on Page 206 of [33] for the details.

Lemma 1.3.7. ***[The nine lemma].*** *Let $\mathscr{C}$ be an abelian category and let*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_2 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_3 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

be a commutative diagram in $\mathscr{C}$. Suppose that the three columns are exact. Then if any two of the three rows are exact, then the other one is exact too. $\diamond$

Lemma 1.3.8. ***[The snake lemma].*** *Let $\mathscr{C}$ be an abelian category and let*

$$\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[r, "g"] \arrow[d, "q"] & Z \arrow[r] \arrow[d, "r"] & 0 \\
0 \arrow[r] & X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' &
\end{tikzcd}$$

be a commutative diagram with exact rows. Then there is a long exact sequence

$$\operatorname{Ker} p \longrightarrow \operatorname{Ker} q \longrightarrow \operatorname{Ker} r \xrightarrow{\varphi} \operatorname{Coker} p \longrightarrow \operatorname{Coker} q \longrightarrow \operatorname{Coker} r.$$

$\diamond$

Definition 1.3.9. The morphism φ in the snake lemma above is called the *connecting homomorphism*. The way this is defined is given by $\varphi(z) = (f')^{-1}(q(g^{-1}(z))) + \operatorname{Im} p$. $\diamond$

1.4 Triangulated categories

Now we define triangulated categories.

Definition 1.4.1. [Triangulated category]. A *triangulated category* is a triple $(\mathscr{C}, \Sigma, \Delta)$, where $\mathscr{C}$ is an additive category, $\Sigma : \mathscr{C} \to \mathscr{C}$ is an automorphism called the *suspension functor*, and Δ is a collection of diagrams of the form $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$, known as *distinguished triangles*, which satisfy the following axioms.

- **[TR1].** [TR1] has three parts.
  - Each morphism $\alpha : A \to B$ can be embedded into a distinguished triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$.
  - For each object A, we have that $A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0 \longrightarrow \Sigma A$ is a distinguished triangle.
  - The collection Δ is closed under isomorphisms.
- **[TR2].** The diagram $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$ is a distinguished triangle if and only if the diagram $B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A \xrightarrow{-\Sigma\alpha} \Sigma B$ is a distinguished triangle.
- **[TR3].** If

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] \arrow[d, "g"'] & C \arrow[r, "\gamma"] & \Sigma A \arrow[d, "\Sigma f"] \\ A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' \arrow[r, "\gamma'"] & \Sigma A' \end{tikzcd}$$

  is a commutative diagram where the two rows are distinguished triangles, then there exists a morphism $h : C \to C'$ with the property that $h \circ \beta = \beta' \circ g$ and $(\Sigma f) \circ \gamma = \gamma' \circ h$.
- **[TR4].** [TR4] is called The Octahedral Axiom. If $A \to B \to C \to \Sigma A$, $B \to X \to Y \to \Sigma B$, $A \to X \to Z \to \Sigma A$ are distinguished triangles which make the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & B \arrow[r] \arrow[d] & C \arrow[r] & \Sigma A \\ & X \arrow[d] \arrow[dr] & & \\ & Y \arrow[d] & Z \arrow[dr] & \\ & \Sigma B & & \Sigma A \end{tikzcd}$$

commute, then there is a distinguished triangle $C \to Z \to Y \to \Sigma C$ compatible with the commutativity of the diagram. $\diamond$

Remark 1.4.2. The suspension functor is often called the *translation functor* or the *shift functor*. $\diamond$

Remark 1.4.3. Strictly speaking, a triangulated category is a triple $(\mathscr{C}, \Sigma, \Delta)$. However, when Σ and Δ are known by context, often, in an abuse of notation, the category $\mathscr{C}$ (by itself) is referred to as being a triangulated category. $\diamond$

Definition 1.4.4. Let $\mathscr{T}$ be a triangulated category with suspension functor Σ. Then $\mathscr{G}$, a full, additive subcategory of $\mathscr{T}$, is a *triangulated subcategory* if it is closed under the suspension functor Σ, closed under isomorphisms, and if $x \to y \to z \to \Sigma x$ is a distinguished triangle and $x, y \in \mathscr{G}$, then $z \in \mathscr{G}$. $\diamond$

Definition 1.4.5. Let $\mathscr{T}$ and $\mathscr{T}'$ be triangulated categories with respective suspension functors Σ and Σ'. Let $F : \mathscr{T} \to \mathscr{T}'$ be an additive functor. Suppose that there is an associated natural transformation $\sigma : F\Sigma \to \Sigma' F$ such that, given a distinguished triangle $x \to y \to z \to \Sigma x$ in $\mathscr{T}$, it is the case that $Fx \to Fy \to Fz \to \Sigma' Fx$ is a distinguished triangle in $\mathscr{T}'$ (where $Fz \to \Sigma' Fx$ decomposes as $Fz \to F\Sigma x \stackrel{\sigma_x}{\to} \Sigma' Fx$). Then the pair (F, σ) is called a *triangulated functor*. If F is a natural equivalence, then F is said to be a *triangle equivalence*. $\diamond$

Definition 1.4.6. Let $\mathscr{C}$ be an abelian category and let $\mathscr{T}$ be a triangulated category with suspension functor Σ. Let $F : \mathscr{T} \to \mathscr{C}$ be an additive, covariant functor. Then F is called a *homological functor* if $F(A) \to F(B) \to F(C)$ is exact at $F(B)$ whenever $A \to B \to C \to \Sigma A$ is a distinguished triangle in $\mathscr{T}$. If F is instead contravariant, then if $F(C) \to F(B) \to F(A)$ is exact at $F(B)$ whenever $A \to B \to C \to \Sigma A$ is a distinguished triangle in $\mathscr{T}$, then F is called a *cohomological functor*. $\diamond$

Remark 1.4.7. Let $\mathscr{T}$ be a triangulated category and $A \to B \to C \to \Sigma A$ a distinguished triangle. Let $\mathscr{C}$ be an abelian category. A homological functor $F : \mathscr{T} \to \mathscr{C}$ induces a natural long exact sequence

$$\cdots \to F(\Sigma^{n-1}C) \to F(\Sigma^n A) \to F(\Sigma^n B) \to F(\Sigma^n C) \to F(\Sigma^{n+1}A) \to \cdots .$$

Analogously, a cohomological functor $G : \mathscr{T} \to \mathscr{C}$ induces a natural long exact sequence

$$\cdots \to G(\Sigma^{n+1}A) \to G(\Sigma^n C) \to G(\Sigma^n B) \to G(\Sigma^n A) \to G(\Sigma^{n-1}C) \to \cdots .$$

If F is homological, then $F_n(X)$ is defined to be $F(\Sigma^{-n}X)$, and if G is cohomological, then $G^n(X)$ is defined to be $G(\Sigma^{-n}X)$. $\diamond$

Proposition 1.4.8. *Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ be a distinguished triangle in a triangulated category $\mathscr{T}$. Then $h \circ g = 0$ and $g \circ f = 0$. That is, two consecutive morphisms in a distinguished triangle compose to zero.*

Proof. By [TR2], it suffices to prove that $g \circ f = 0$, since any distinguished triangle can be "rolled". By [TR2], there is a distinguished triangle $B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B$. Consider the following diagram

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "g"'] & C \arrow[r, "h"] \arrow[d, "1_C"] & \Sigma A \arrow[r, "-\Sigma f"] & \Sigma B \arrow[d, "\Sigma g"] \\
C \arrow[r, "1_C"'] & C \arrow[r] & 0 \arrow[r] & \Sigma C
\end{tikzcd}$$

whose bottom row is a distinguished triangle by [TR1]. By [TR3], This diagram can be completed to a morphism of triangles, yielding the right-hand-most square

$$\begin{tikzcd}
\Sigma A \arrow[r, "-\Sigma f"] \arrow[d, "0"'] & \Sigma B \arrow[d, "\Sigma g"] \\
0 \arrow[r, "0"'] & \Sigma C
\end{tikzcd}$$

which says that $\Sigma g \circ (-\Sigma f) = -\Sigma(g \circ f) = 0$, which implies that $g \circ f = 0$, as required, since Σ is an automorphism. $\square$

Proposition 1.4.9. *Let $\mathscr{T}$ be a triangulated category and let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ be a distinguished triangle in $\mathscr{T}$. Let $K \in \operatorname{obj} \mathscr{T}$ and suppose there is a morphism $u : K \to Y$ such that $\beta \circ u = 0$. Then there is a (not necessarily unique) morphism $u' : K \to X$ such that $\alpha \circ u' = u$. That is, u' can be found such that the following diagram*

$$\begin{tikzcd}
 & K \arrow[dl, dashed, "u'"'] \arrow[d, "u"] \arrow[dr, "0"] & & \\
X \arrow[r, "\alpha"'] & Y \arrow[r, "\beta"'] & Z \arrow[r, "\gamma"] & \Sigma X
\end{tikzcd}$$

commutes.

Proof. By [TR1], there is a distinguished triangle $K \xrightarrow{\mathrm{id}_K} K \longrightarrow 0 \longrightarrow \Sigma A$ and by [TR2], this can be rolled to yield the distinguished triangle $0 \longrightarrow K \xrightarrow{\mathrm{id}_K} K \longrightarrow 0$. Further to this, [TR2] also gives a distinguished triangle $\Sigma^{-1} Z \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$. Form the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & K \arrow[r, "1_K"] & K \arrow[r] \arrow[d, "u"'] & 0 \arrow[d, "0"] \\
\Sigma^{-1} Z \arrow[r] & X \arrow[r, "\alpha"'] & Y \arrow[r, "\beta"'] & Z
\end{tikzcd}$$

and by [TR2] and [TR3] complete this to a morphism of triangles

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & K \arrow[r, "1_K"] \arrow[d, dashed, "u'"'] & K \arrow[r] \arrow[d, "u"'] & 0 \arrow[d, "0"] \\
\Sigma^{-1} Z \arrow[r] & X \arrow[r, "\alpha"'] & Y \arrow[r, "\beta"'] & Z
\end{tikzcd}$$

which gives u', as required. □

The following is a well-known theorem (see, for example, [42, Example 10.2.8 and Exercise 10.2.3]) which follows from Propositions 1.4.8 and 1.4.9.

Theorem 1.4.10. *Let $\mathscr{T}$ be a triangulated category. The functors $\mathscr{T}(A, -)$ and $\mathscr{T}(-, A)$ are (respectively) homological and cohomological for all $A \in \operatorname{obj} \mathscr{T}$.* ◇

Now we shall explore some of the properties that triangulated categories have.

Proposition 1.4.11. *Let $\mathscr{T}$ be a triangulated category.*

(i) In [TR3], if f and g are isomorphisms, then the morphism h which completes the diagram is an isomorphism as well (this is known as the triangulated five lemma).

(ii) If $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$ is a distinguished triangle, then f is an isomorphism if and only if $Z \cong 0$.

Proof. The proof of (i) can be found in [19, Proposition I.1.1], and the proof of (ii) is presented as follows. Suppose $X \xrightarrow{f} Y \to Z \to \Sigma X$ is a distinguished triangle and suppose that f is an isomorphism. By [TR3], the following diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "1_Y"] & Z \arrow[r] & \Sigma X \arrow[d, "\Sigma f"] \\
Y \arrow[r, "1_Y"'] & Y \arrow[r] & 0 \arrow[r] & \Sigma Y
\end{tikzcd}$$

can be completed to a morphism of triangles, and by part (i), the morphism $Z \to 0$ must be an isomorphism. □

Definition 1.4.12. A distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ in a triangulated category $\mathscr{T}$ is called a *split triangle,* or it is said that the triangle *splits*, if f is a split monomorphism and g is a split epimorphism. ◇

Proposition 1.4.13. *In a triangulated category $\mathscr{T}$, the distinguished triangles*

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{h} \Sigma X$$

and

$$X \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} X \oplus Z \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} Z \xrightarrow{0} \Sigma X$$

are isomorphic if and only if h is the zero morphism.

Proof. This can be found in [36, Corollary 1.2.7]. □

Corollary 1.4.14. *Let $\mathscr{T}$ be a triangulated category. Any distinguished triangle in $\mathscr{T}$ of the form*

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{0} \Sigma X$$

is a split triangle. ◇

Definition 1.4.15. Let $\mathscr{A}$ be an abelian category. If every short exact sequence in $\mathscr{A}$ is a split short exact sequence, then $\mathscr{A}$ is called *semisimple*. ◇

Theorem 1.4.16. *Let $\mathscr{T}$ be a category which is both abelian and triangulated. Then every short exact sequence in $\mathscr{T}$ splits; that is, $\mathscr{T}$ is semisimple.*

Proof. Suppose $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a short exact sequence in $\mathscr{T}$. Embed f into a distinguished triangle $\Sigma^{-1}U \xrightarrow{u} X \xrightarrow{f} Y \xrightarrow{u'} U$. By Proposition 1.4.8, $f \circ u = 0$. Hence

$u = 0$, because f is a monomorphism by virtue of it being the first map in a short exact sequence. The following distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{u'} U \xrightarrow{\Sigma u} \Sigma X$$

is obtained by rolling, with $\Sigma u = \Sigma 0 = 0$. By Corollary 1.4.14, this distinguished triangle splits. Hence, f is a section, which (by definition) means that $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a split short exact sequence. □

Definition 1.4.17. Let $\mathscr{T}$ be a triangulated category. An object $X \in \mathscr{T}$ is called *compact* if, for any coproduct of a family $\{Y_\lambda : \lambda \in \Lambda\}$, the natural map

$$\coprod_{\lambda \in \Lambda} \operatorname{Hom}(X, Y_\lambda) \longrightarrow \operatorname{Hom}\Big(X, \coprod_{\lambda \in \Lambda} Y_\lambda\Big)$$

is an isomorphism. See [36, Example 4.1.2] (where instead of saying that X is $\aleph_0$-small, we say that X is compact). ◇

1.5 The category of chain complexes

Definition 1.5.1. Let R be a ring with identity. A *chain complex* C over R is a sequence of R-modules $\{C_n\}_{n\in\mathbb{Z}}$ which are connected by R-module homomorphisms, called *differentials*, $d_n : C_n \to C_{n-1}$ such that each composite $d_{n-1} \circ d_n$ is zero. Note that all long exact sequences in ${}_R\mathsf{Mod}$ can be realised as chain complexes, but not all chain complexes are exact. This is because all that is required is $\operatorname{Im} d_n \subset \operatorname{Ker} d_{n-1}$, for all $n \in \mathbb{Z}$ (whereas a long exact sequence would have $\operatorname{Im} d_n = \operatorname{Ker} d_{n-1}$, for all $n \in \mathbb{Z}$). ◇

Definition 1.5.2. Let R be a ring with identity. A *cochain complex* C over R is a sequence of R-modules $\{C^n\}_{n\in\mathbb{Z}}$ which are connected by R-module homomorphisms $d^n : C^n \to C^{n+1}$ such that each composite $d^{n+1} \circ d^n$ is zero. ◇

Remark 1.5.3. It is easy to switch between chain complex (homological) notation and cochain complex (cohomological) notation for a given complex C. We apply the convention that $C_n = C^{-n}$ and $d_n = d^{-n}$. ◇

Definition 1.5.4. Let C and D be chain complexes over a ring R with identity. A *chain map* (or *morphism of chain complexes*) $f : C \to D$ is a collection of morphisms $\{f_n : C_n \to D_n\}_{n\in\mathbb{Z}}$ satisfying $f_{n-1} \circ d_n^C = d_n^D \circ f_n$ for all $n \in \mathbb{Z}$; that is, the following diagram

$$\begin{tikzcd}
\cdots \arrow[r, "d^C_{n+2}"] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "d^D_{n+2}"] & D_{n+1} \arrow[r, "d^D_{n+1}"] & D_n \arrow[r, "d^D_n"] & D_{n-1} \arrow[r, "d^D_{n-1}"] & \cdots
\end{tikzcd}$$

commutes. ◇

Chain complexes together with chain maps form an abelian category.

Definition 1.5.5. Let $\mathscr{A}$ be an abelian category. Then define $C(\mathscr{A})$ to be the category of chain complexes over $\mathscr{A}$ with morphisms given by chain maps. If $\mathscr{A} = {}_R\mathsf{Mod}$, where R is a ring with identity, then the category $C({}_R\mathsf{Mod})$ of chain complexes of R-modules is instead denoted as $C(R)$. ◇

Remark 1.5.6. Let $\mathscr{A}$ be an abelian category. Then $C(\mathscr{A})$ is an abelian category (see, for example, [42, Chapter 1, Theorem 1.2.3]). Kernels of chain maps are defined degreewise,

$$\begin{tikzcd}
\operatorname{Ker} f_{i+1} \arrow[r, "h'_{i+1}"] \arrow[d, "u_{i+1}"'] & \operatorname{Ker} f_i \arrow[d, "u_i"] \\
A_{i+1} \arrow[r, "d^A_{i+1}"] \arrow[d, "f_{i+1}"'] & A_i \arrow[d, "f_i"] \\
B_{i+1} \arrow[r, "d^B_{i+1}"'] & B_i
\end{tikzcd}$$

with differentials arising from the universal property of the kernel, seen below (that $f_i \circ d^A_{i+1} \circ u_{i+1} = 0$ can be seen by the fact that $f_i \circ d^A_{i+1} \circ u_{i+1} = d^B_{i+1} \circ f_{i+1} \circ u_{i+1} = d^B_{i+1} \circ 0$).

$$\begin{tikzcd}
\operatorname{Ker} f_{i+1} \arrow[dr, dashed, "h'_{i+1}"'] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "d^A_{i+1} \circ u_{i+1}"'] & & \\
& \operatorname{Ker} f_i \arrow[r, "0"] \arrow[d, "u_i"'] & B_i \\
& A_i \arrow[ur, "f_i"'] &
\end{tikzcd}$$

Cokernels are similarly defined. Short exact sequences of complexes are exact in each degree. ◇

Definition 1.5.7. A *graded ring* is a ring that decomposes as a direct sum of abelian groups

$$R = \bigoplus_{n \in \mathbb{Z}} R_n = \cdots \oplus R_{-2} \oplus R_{-1} \oplus R_0 \oplus R_1 \oplus R_2 \oplus \cdots$$

such that if $r_i \in R_i$ and $r_j \in R_j$, then $r_i r_j \in R_{i+j}$. A *graded module* is a left module M over a graded ring R that decomposes as a direct sum of abelian groups

$$M = \bigoplus_{i \in \mathbb{Z}} M_i = \cdots \oplus M_{-2} \oplus M_{-1} \oplus M_0 \oplus M_1 \oplus M_2 \oplus \cdots$$

such that if $r_i \in R_i$ and $m_j \in M_j$, then $r_i m_j \in M_{i+j}$. $\diamond$

Definition 1.5.8. Let

$$C = \cdots \xrightarrow{d_{n+3}} C_{n+2} \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \xrightarrow{d_{n-2}} \cdots$$

be a chain complex over a ring with identity. Define the graded module

$$H(C) = \{H_n(C)\}_{n \in \mathbb{Z}} = \left\{ \frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}} \right\}_{n \in \mathbb{Z}},$$

called the *homology* of C. The n^{th} graded piece, $H_n(C)$, is called the n^{th} *homology module* of C. Further, define

(i) *n-chains*, which are the elements of C_n,

(ii) *n-cycles*, which are the elements of $\operatorname{Ker} d_n$,

(iii) *n-boundaries*, which are the elements of $\operatorname{Im} d_{n+1}$.

The graded module $H(C)$ is sometimes written as $H_*(C)$ to avoid ambiguity with the completely analogous construction $H^*(C)$ which involves cochain complexes. $\diamond$

Remark 1.5.9. The graded module $H(C)$ can be given the structure of a chain complex by setting each of its differentials to be zero. $\diamond$

Remark 1.5.10. Let $\mathscr{C}$ be an abelian category. Given a short exact sequence $0 \to A \to B \to C \to 0$ of complexes A, B, C of objects of $\mathscr{C}$, the snake lemma 1.3.8 can be used to obtain a long exact sequence of homology

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\varphi_n} H_{n-1}(A) \to \cdots$$

with φ_n the connecting homomorphism, see [20, Theorem IV.2.1]. There is also an obvious, analogous construction involving cohomology. $\diamond$

Definition 1.5.11. Let $\mathscr{A}$ be an abelian category and consider the category $C(\mathscr{A})$ of chain complexes over $\mathscr{A}$. Let $A, B \in \operatorname{obj} C(\mathscr{A})$ and let $f : A \to B$ be a chain map. Define the map $H_n(f) : H_n(A) \to H_n(B)$, called the *$n^{th}$ homology map of f,* in the following way. Set $H_n(f)(a_n + \operatorname{Im} d^A_{n+1}) = f_n(a_n) + \operatorname{Im} d^B_{n+1}$ for $a_n \in \operatorname{Ker} d^A_n$. ◇

Lemma 1.5.12. *The map $H_n(f)$ in Definition 1.5.11 is well-defined.*

Proof. Suppose that $a_n - a'_n \in \operatorname{Im} d^A_{n+1}$ and $a_n, a'_n \in \operatorname{Ker} d^A_n$. Then there exists $a_{n+1} \in A_{n+1}$ such that $d^A_{n+1}(a_{n+1}) = a_n - a'_n$. Hence we have that

$$f_n(a_n) - f_n(a'_n) = f_n(a_n - a'_n) = f_n(d^A_{n+1}(a_{n+1})) = d^B_{n+1}(f_{n+1}(a_{n+1})),$$

which gives that $f_n(a_n) - f_n(a'_n) \in \operatorname{Im} d^B_{n+1}$ as required. □

Proposition 1.5.13. *Let R be a ring with identity. The n^{th} homology map is a covariant functor from $C(R)$ to ${}_R\mathsf{Mod}$.*

Proof. This is [42, Exercise 1.1.2]. □

Definition 1.5.14. Let $\mathscr{A}$ be an abelian category and let $f : C \to D$ be a chain map in $C(\mathscr{A})$. Then f is called a *quasi-isomorphism* if, for each $n \in \mathbb{Z}$, the homomorphisms $H_n(f) : H_n(C) \to H_n(D)$ are isomorphisms. ◇

Lemma 1.5.15. *Let $\mathscr{A}$ be an abelian category and let $A \in \operatorname{obj} C(\mathscr{A})$. Then the following are equivalent.*

(i) A is exact,

(ii) $H_n(A) = 0$ for all $n \in \mathbb{Z}$,

(iii) the zero map $0 \to A$ is a quasi-isomorphism.

Proof. This is trivial. □

Definition 1.5.16. Let $\mathscr{C}$ be an abelian category and let C and D be chain complexes over $\mathscr{C}$ and f a chain map between them, illustrated below.

$$\begin{tikzcd}
\cdots \arrow[r, "d^C_{n+2}"] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "d^D_{n+2}"] & D_{n+1} \arrow[r, "d^D_{n+1}"] & D_n \arrow[r, "d^D_n"] & D_{n-1} \arrow[r, "d^D_{n-1}"] & \cdots
\end{tikzcd}$$

Define the *mapping cone of f* to be the chain complex $E = E(f)$ with $E_n = C_{n-1} \oplus D_n$ and $d^E_n((x, y)) = (-d^C_{n-1}(x), f_{n-1}(x) + d^D_n(y))$. ◇

Lemma 1.5.17. *Let $\mathscr{C}, C, D, f, E$ be as in Definition 1.5.16 above. Then the diagrams*

$$
\begin{tikzcd}
D_n \arrow[r, "d_n^D"] \arrow[d, "\iota_n"'] & D_{n-1} \arrow[d, "\iota_{n-1}"] \\
C_{n-1} \oplus D_n \arrow[r, "d_n^E"'] & C_{n-2} \oplus D_{n-1}
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
C_{n-1} \oplus D_n \arrow[r, "d_n^E"] \arrow[d, "\pi_{n-1}"'] & C_{n-2} \oplus D_{n-1} \arrow[d, "\pi_{n-2}"] \\
C_{n-1} \arrow[r, "-d_{n-1}^C"'] & C_{n-2},
\end{tikzcd}
$$

with ι the canonical inclusion and π the canonical projection, commute.

Proof. A straightforward diagram chase gives the required result. □

Definition 1.5.18. Let $\mathscr{A}$ be an abelian category and let C be a chain complex in $C(\mathscr{A})$. Define ΣC to be the chain complex with degree n part equal to C_{n-1} and the n^{th} differential given by $-d_{n-1}^C$. ◇

Lemma 1.5.19. *Let $f : C \to D$ be a chain map in $C(\mathscr{A})$, where $\mathscr{A}$ is an abelian category. Recall the $\iota : D \to E(f)$ and $\pi : E(f) \to \Sigma C$ maps of Lemma 1.5.17. Then there is a short exact sequence $0 \longrightarrow D \xrightarrow{\iota} E(f) \xrightarrow{\pi} \Sigma C \longrightarrow 0$ in $C(\mathscr{A})$.* ◇

Lemma 1.5.20. *Let $\mathscr{A}$ be an abelian category and let $f : C \to D$ be a chain map in $C(\mathscr{A})$. Then f is a quasi-isomorphism if and only if the mapping cone $E(f)$ is exact.*

Proof. By Lemma 1.5.19, there is a short exact sequence $0 \to D \to E(f) \to \Sigma C \to 0$, and by Lemma 1.5.10, this gives a long exact sequence of homology

$$\cdots \to H_{n+1}(\Sigma C) \to H_n(D) \to H_n(E(f)) \to H_n(\Sigma C) \to H_{n-1}(D) \to \cdots$$

with connecting homomorphism given by f_*, see [42, Lemma 1.5.3]. Now, by virtue of how ΣC is defined, it is the case that $H_{n+1}(\Sigma C) \cong H_n(C)$ (see, for example, [42, Exercise 1.2.8]). Hence this long exact sequence can be rewritten as

$$\cdots \to H_n(C) \to H_n(D) \to H_n(E(f)) \to H_{n-1}(C) \to H_{n-1}(D) \to \cdots .$$

It now follows from this long exact sequence that f is a quasi-isomorphism if and only if each $H_n(E(f))$ is 0, that is, if and only if $E(f)$ is exact. □

1.6 The homotopy category

Now we begin the introduction of the homotopy category.

Definition 1.6.1. Let $\mathscr{C} = C(\mathscr{A})$ be the category of chain complexes over an abelian category $\mathscr{A}$. Let $f : C \to D$ be a chain map in $C(\mathscr{A})$, illustrated below.

$$\begin{tikzcd}
\cdots \arrow[r, "d^C_{n+2}"] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "d^D_{n+2}"] & D_{n+1} \arrow[r, "d^D_{n+1}"] & D_n \arrow[r, "d^D_n"] & D_{n-1} \arrow[r, "d^D_{n-1}"] & \cdots
\end{tikzcd}$$

Then if there are $\mathscr{A}$-morphisms $s_n : C_n \to D_{n+1}$ for all $n \in \mathbb{Z}$ such that

$$f_n = s_{n-1} \circ d^C_n + d^D_{n+1} \circ s_n,$$

then f is called *null-homotopic*. Two chain maps f and g are said to be *chain homotopic* (or, simply, *homotopic*) if $f - g$ is null-homotopic. This is denoted $f \sim g$. $\diamond$

Proposition 1.6.2. *Let $\mathscr{C} = C(\mathscr{A})$ be the category of chain complexes over an abelian category $\mathscr{A}$. Then chain homotopy is an equivalence relation on $\mathrm{Hom}_{C(\mathscr{A})}(C, D)$ for any pair of chain complexes C and D.*

Proof. See [20, Lemma IV.3.2]. $\square$

Proposition 1.6.3. *Let $\mathscr{C} = C(\mathscr{A})$ be the category of chain complexes over an abelian category $\mathscr{A}$. Let $f \sim g$ be two chain-homotopic chain maps in $C(\mathscr{A})$ between chain complexes C and D. Then*

(i) if $\mu : M \to C$ is a chain map, then $f \circ \mu \sim g \circ \mu$,

(ii) if $\nu : D \to N$ is a chain map, then $\nu \circ f \sim \nu \circ g$.

Equivalence classes of chain maps can therefore be composed, and this is well-defined.

Proof. We prove part (i) only, because part (ii) is very similar. Because $f \sim g$, there are maps $\{s_n : C_n \to D_{n+1}\}_{n\in\mathbb{Z}}$ such that $(f - g)_n = s_{n-1} \circ d^C_n + d^D_{n+1} \circ s_n$. We are required to show that $(f - g) \circ \mu$ is null-homotopic, that is, that there are maps $\{t_n : M_n \to D_{n+1}\}_{n\in\mathbb{Z}}$ such that $(f - g)_n \circ \mu_n = t_{n-1} \circ d^M_n + d^D_{n+1} \circ t_n$. We claim that $t_n = s_n \circ \mu_n$ works. Indeed, $(f - g)_n \circ \mu_n = (s_{n-1} \circ d^C_n + d^D_{n+1} \circ s_n) \circ \mu_n = s_{n-1} \circ d^C_n \circ \mu_n + d^D_{n+1} \circ s_n \circ \mu_n = s_{n-1} \circ \mu_{n-1} \circ d^M_n + d^D_{n+1} \circ s_n \circ \mu_n = t_{n-1} \circ d^M_n + d^D_{n+1} \circ t_n$, as required. $\square$

Definition 1.6.4. Let $\mathscr{A}$ be an abelian category. Then $K(\mathscr{A})$ is the *homotopy category of $\mathscr{A}$*, which is defined as follows. The objects of $K(\mathscr{A})$ are the chain complexes over $\mathscr{A}$ of $C(\mathscr{A})$ and the morphisms of $K(\mathscr{A})$ are the homotopy equivalence classes of ordinary chain maps in $C(\mathscr{A})$. $\diamond$

Proposition 1.6.5. *Let A be an abelian category. If $f, g : C \to D$ are homotopic chain maps in $C(\mathscr{A})$, then $H_n(f) = H_n(g)$ for each $n \in \mathbb{Z}$.*

Proof. Consider the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r, "d^C_{n+2}"] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "(f-g)_{n+1}"'] & C_n \arrow[r, "d^C_n"] \arrow[dl, "s_n"'] \arrow[d, "(f-g)_n"] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[dl, "s_{n-1}"] \arrow[d, "(f-g)_{n-1}"] & \cdots \\
\cdots \arrow[r, "d^D_{n+2}"'] & D_{n+1} \arrow[r, "d^D_{n+1}"'] & D_n \arrow[r, "d^D_n"'] & D_{n-1} \arrow[r, "d^D_{n-1}"'] & \cdots
\end{tikzcd}$$

We have that $f_n - g_n = s_{n-1} \circ d^C_n + d^D_{n+1} \circ s_n$. That is,

$$f_n = g_n + s_{n-1} \circ d^C_n + d^D_{n+1} \circ s_n.$$

Now, by Definition 1.5.11, $H_n(f)(c_n + \operatorname{Im} d^C_{n+1}) = f_n(c_n) + \operatorname{Im} d^D_{n+1}$, but by the above this also equals $g_n(c_n) + s_{n-1}(d^C_n(c_n)) + d^D_{n+1}(s_n(c_n)) + \operatorname{Im} d^D_{n+1}$. And since $c_n \in \operatorname{Ker} d^C_n$ and $d^D_{n+1}(s_n(c_n)) \in \operatorname{Im} d^D_{n+1}$, we have the required result that $f_n(c_n) + \operatorname{Im} d^D_{n+1} = g_n(c_n) + \operatorname{Im} d^D_{n+1}$; that is, $H_n(f) = H_n(g)$. $\square$

Remark 1.6.6. Proposition 1.6.5 proves that H_n is a functor $K(\mathscr{A}) \to \mathsf{Ab}$. $\diamond$

Lemma 1.6.7. *Let C and D be a pair of chain complexes over an abelian category $\mathscr{A}$. Let $\sigma_i : C_i \to D_{i+1}$ for $i \in \mathbb{Z}$ be a family of $\mathscr{A}$-morphisms. Set $\gamma_i = \sigma_{i-1} \circ d^C_i + d^D_{i+1} \circ \sigma_i$. Then the collection $\gamma = \{\gamma_i \mid i \in \mathbb{Z}\}$ is a chain map (which is null-homotopic by virtue of the σ_i maps).*

Proof. Consider the following diagram.

$$\begin{tikzcd}
C_i \arrow[r, "d^C_i"] \arrow[d, "\sigma_{i-1} \circ d^C_i + d^D_{i+1} \circ \sigma_i"'] & C_{i-1} \arrow[d, "\sigma_{i-2} \circ d^C_{i-1} + d^D_i \circ \sigma_{i-1}"] \\
D_i \arrow[r, "d^D_i"'] & D_{i-1}
\end{tikzcd}$$

On the one hand, $(\sigma_{i-2} \circ d^C_{i-1} + d^D_i \circ \sigma_{i-1}) \circ d^C_i = \sigma_{i-2} \circ d^C_{i-1} \circ d^C_i + d^D_i \circ \sigma_{i-1} \circ d^C_i = 0 + d^D_i \circ \sigma_{i-1} \circ d^C_i$, and on the other, $d^D_i \circ (\sigma_{i-1} \circ d^C_i + d^D_{i+1} \circ \sigma_i) = d^D_i \circ \sigma_{i-1} \circ d^C_i + d^D_i \circ d^D_{i+1} \circ \sigma_i = d^D_i \circ \sigma_{i-1} \circ d^C_i + 0$. These are equal, and so this is a chain map. $\square$

Lemma 1.6.8. *Let C and D be a pair of chain complexes over an abelian category $\mathscr{A}$. Let $f, g : C \to D$ be homotopic chain maps. Then the mapping cones $E(f), E(g)$ are isomorphic in $C(\mathscr{A})$.*

Proof. Let $s : f \to g$ be the homotopy, so that $f_n - g_n = d^D_{n+1} \circ s_n + s_{n-1} \circ d^C_n$. Then there is a chain map $S_{f,g}$ from $E(f)$ to $E(g)$ as follows.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\cdots \arrow[r] & C_{i+1} \oplus D_{i+2} \arrow[r, "\begin{pmatrix} -d^C_{i+1} & 0 \\ f_{i+1} & d^D_{i+2} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1_{C_{i+1}} & 0 \\ s_{i+1} & 1_{D_{i+2}} \end{pmatrix}"'] & C_i \oplus D_{i+1} \arrow[r, "\begin{pmatrix} -d^C_i & 0 \\ f_i & d^D_{i+1} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1_{C_i} & 0 \\ s_i & 1_{D_{i+1}} \end{pmatrix}"'] & C_{i-1} \oplus D_i \arrow[r] \arrow[d, "\begin{pmatrix} 1_{C_{i-1}} & 0 \\ s_{i-1} & 1_{D_i} \end{pmatrix}"] & \cdots \\
\cdots \arrow[r] & C_{i+1} \oplus D_{i+2} \arrow[r, "\begin{pmatrix} -d^C_{i+1} & 0 \\ g_{i+1} & d^D_{i+2} \end{pmatrix}"'] & C_i \oplus D_{i+1} \arrow[r, "\begin{pmatrix} -d^C_i & 0 \\ g_i & d^D_{i+1} \end{pmatrix}"'] & C_{i-1} \oplus D_i \arrow[r] & \cdots
\end{tikzcd}$$

The above diagram commutes, for

$$\begin{pmatrix} -d^C_{i+1} & 0 \\ g_{i+1} & d^D_{i+2} \end{pmatrix} \circ \begin{pmatrix} 1_{C_{i+1}} & 0 \\ s_{i+1} & 1_{D_{i+2}} \end{pmatrix} = \begin{pmatrix} -d^C_{i+1} & 0 \\ g_{i+1} + d^D_{i+2} \circ s_{i+1} & d^D_{i+2} \end{pmatrix}$$

and

$$\begin{pmatrix} 1_{C_i} & 0 \\ s_i & 1_{D_{i+1}} \end{pmatrix} \circ \begin{pmatrix} -d^C_{i+1} & 0 \\ f_{i+1} & d^D_{i+2} \end{pmatrix} = \begin{pmatrix} -d^C_{i+1} & 0 \\ -s_i \circ d^C_{i+1} + f_{i+1} & d^D_{i+2} \end{pmatrix}.$$

These matrices are equal, for $g_{i+1} + d^D_{i+2} \circ s_{i+1} = -s_i \circ d^C_{i+1} + f_{i+1}$ is equivalent to $f_n - g_n = d^D_{n+1} \circ s_n + s_{n-1} \circ d^C_n$ when $n = i + 1$. Further, there is clearly a chain map $S_{g,f} : E(g) \to E(f)$ with degree i part equal to the matrix

$$\begin{pmatrix} 1_{C_{i-1}} & 0 \\ -s_{i-1} & 1_{D_i} \end{pmatrix}.$$

The compositions

$$\begin{pmatrix} 1_{C_{i-1}} & 0 \\ -s_{i-1} & 1_{D_i} \end{pmatrix} \circ \begin{pmatrix} 1_{C_{i-1}} & 0 \\ s_{i-1} & 1_{D_i} \end{pmatrix}$$

and
$$\begin{pmatrix} 1_{C_{i-1}} & 0 \\ s_{i-1} & 1_{D_i} \end{pmatrix} \circ \begin{pmatrix} 1_{C_{i-1}} & 0 \\ -s_{i-1} & 1_{D_i} \end{pmatrix}$$
are equal to the degree i part of the identity on $E(f)$ and $E(g)$ respectively, so $S_{f,g}$ and $S_{g,f}$ are isomorphisms (and are each other's inverse), as required. □

Lemma 1.6.9. *Let $\mathscr{A}$ be an abelian category and suppose we have a commutative diagram of chain maps*
$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"] \\ C' \arrow[r, "\varphi'"'] & D' \end{tikzcd}$$
in $C(\mathscr{A})$. Then
$$\left\{ \begin{pmatrix} \gamma_{i-1} & 0 \\ 0 & \delta_i \end{pmatrix} : E(\varphi) \to E(\varphi') \right\}_{i\in\mathbb{Z}}$$
is a chain map which makes the following diagrams
$$\begin{tikzcd} D_i \arrow[r, hook] \arrow[d, "\delta_i"'] & C_{i-1} \oplus D_i \arrow[d, "\begin{pmatrix} \gamma_{i-1} & 0 \\ 0 & \delta_i \end{pmatrix}"'] \\ D'_i \arrow[r, hook] & C'_{i-1} \oplus D'_i \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} C_i \oplus D_{i+1} \arrow[r, "\begin{pmatrix} -d_i^C & 0 \\ \varphi_i & d_{i+1}^D \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \gamma_i & 0 \\ 0 & \delta_{i+1} \end{pmatrix}"] & C_{i-1} \oplus D_i \arrow[d, "\begin{pmatrix} \gamma_{i-1} & 0 \\ 0 & \delta_i \end{pmatrix}"] \\ C'_i \oplus D'_{i+1} \arrow[r, "\begin{pmatrix} -d_i^{C'} & 0 \\ \varphi'_i & d_{i+1}^{D'} \end{pmatrix}"'] & C'_{i-1} \oplus D'_i, \end{tikzcd}$$
commutative for all i.

Proof. That the left-hand diagram is commutative is trivial. For the second diagram, compute the compositions
$$\begin{pmatrix} \gamma_{i-1} & 0 \\ 0 & \delta_i \end{pmatrix} \circ \begin{pmatrix} -d_i^C & 0 \\ \varphi_i & d_{i+1}^D \end{pmatrix} = \begin{pmatrix} -\gamma_{i-1} \circ d_i^C & 0 \\ \delta_i \circ \varphi_i & \delta_i \circ d_{i+1}^D \end{pmatrix}$$
and
$$\begin{pmatrix} -d_i^{C'} & 0 \\ \varphi'_i & d_{i+1}^{D'} \end{pmatrix} \circ \begin{pmatrix} \gamma_i & 0 \\ 0 & \delta_{i+1} \end{pmatrix} = \begin{pmatrix} -d_i^{C'} \circ \gamma_i & 0 \\ \varphi'_i \circ \gamma_i & d_{i+1}^{D'} \circ \delta_{i+1} \end{pmatrix}.$$

These matrices are equal. To see why, consider the commutative diagram of chain maps

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"] \\ C' \arrow[r, "\varphi'"'] & D' \end{tikzcd}$$

given in the setup. Each of the top-right, top-left, bottom-right and bottom-left entries of the matrices are all seen to be (respectively) equal. $\square$

Definition 1.6.10. Let $\mathscr{C}$ be a category. Define the *morphism category of* $\mathscr{C}$, denoted $\mathscr{C}^{\to}$, in the following way. The objects of $\mathscr{C}^{\to}$ are the morphisms in $\mathscr{C}$. If $f : C \to D$ and $f' : C' \to D'$ are objects of $\mathscr{C}^{\to}$, then a $\mathscr{C}^{\to}$-morphism from f to f' is a pair of $\mathscr{C}$-morphisms (g_1, g_2) such that $g_2 \circ f = f' \circ g_2$ (that is, a commutative square in $\mathscr{C}$). The pair $(1_C, 1_D)$ is the identity on $f : C \to D$ and composition, whenever it is defined, is given straightforwardly by $(h_1, h_2) \circ (g_1, g_2) = (h_1 \circ g_1, h_2 \circ g_2)$. $\diamond$

Remark 1.6.11. Let $\mathscr{A}$ be an abelian category and consider the morphism category $C(\mathscr{A})^{\to}$. By virtue of the right-hand part of Lemma 1.6.9, there is a functor, "Cone", from $C(\mathscr{A})^{\to}$ to $C(\mathscr{A})$ given by $E(\varphi : C \to D) = E(\varphi)$. $\diamond$

By, for example, [19, Chapter I, Section 2], the homotopy category $K(\mathscr{A})$ of an abelian category $\mathscr{A}$ is a triangulated category. Recall Definition 1.5.18.

Definition 1.6.12. Let $\mathscr{A}$ be an abelian category and consider the homotopy category $K(\mathscr{A})$. Define the suspension functor on $K(\mathscr{A})$ in the way suggested by Definition 1.5.18; i.e., if C is a chain complex in $K(\mathscr{A})$, then ΣC is that same chain complex but shifted one degree to the left, together with a sign flip on the differentials. Further, given a chain map $f : C \to D$ in $C(\mathscr{A})$, complete f to a diagram

$$C \xrightarrow{f} D \longrightarrow E(f) \longrightarrow \Sigma C.$$

Then, by Lemma 1.6.8, after passing to $K(\mathscr{A})$, the mapping cone $E(f)$ is not dependent on the choice of representative from the (homotopy) equivalence class $[f]$. Such a diagram in $K(\mathscr{A})$ is known as a *standard triangle*. These standard triangles, together with the triangles isomorphic (in $K(\mathscr{A})$) to these, are the distinguished triangles in $K(\mathscr{A})$. $\diamond$

The full proof of the fact that $K(\mathscr{A})$ is triangulated with the structure described above is quite involved. Next we will explore some properties of $K(\mathscr{A})$ as a triangulated category. For instance, the following lemma verifies the fact that for each object $A \in K(\mathscr{A})$, we have that $A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0 \longrightarrow \Sigma A$ is a distinguished triangle.

Lemma 1.6.13. *Let $\mathscr{A}$ be an abelian category and let A be a chain complex in $C(\mathscr{A})$. Consider the identity on A inside $C(\mathscr{A})$. Then $E(1_A) \cong 0$ in $K(\mathscr{A})$.*

Proof. If $\mathrm{Hom}_{K(\mathscr{A})}(E(1_A), E(1_A)) \cong 0$, then $E(1_A) \cong 0$ in $K(\mathscr{A})$. It is enough to show that $1_{E(1_A)} : E(1_A) \to E(1_A)$ is null-homotopic. Consider the following diagram.

$$\begin{tikzcd}
E(1_A) = \cdots \arrow[r] & A_n \oplus A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, equal] & A_{n-1} \oplus A_n \arrow[r, "d_n"] \arrow[d, equal] & A_{n-2} \oplus A_{n-1} \arrow[r] \arrow[d, equal] & \cdots \\
E(1_A) = \cdots \arrow[r] & A_n \oplus A_{n+1} \arrow[r, "d_{n+1}"'] & A_{n-1} \oplus A_n \arrow[r, "d_n"'] & A_{n-2} \oplus A_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Define $s_n : A_{n-1} \oplus A_n \to A_n \oplus A_{n+1}$ by $s_n(a_{n-1}, a_n) = (a_n, 0)$. A diagram chase verifies that $s_{n-1} \circ d_n + d_{n+1} \circ s_n = 1$. Thus, $1_{E(1_A)}$ is null-homotopic, as required. $\square$

1.7 Derived functors

Definition 1.7.1. Let $\mathscr{C}$ be a category.

(i) An object P of $\mathscr{C}$ is a *projective object* if given an epimorphism $\pi : X \to Y$ in $\mathscr{C}$, and any $\eta \in (P, Y)$, there exists $\xi \in (P, X)$ such that $\eta = \pi \circ \xi$. That is, given $\eta : P \to Y$, we can always find a dashed arrow

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\xi"'] \arrow[d, "\eta"] \\
X \arrow[r, two heads, "\pi"'] & Y
\end{tikzcd}$$

which completes the above to a commutative diagram.

(ii) An *injective object* is defined dually. That is, an object N is injective if given a monomorphism $\iota : X \to Y$, and any $\eta \in (X, N)$, there exists $\xi \in (Y, N)$ such that $\eta = \xi \circ \iota$. The dashed arrow

$$\begin{tikzcd}
 & N \\
Y \arrow[ur, dashed, "\xi"] & X \arrow[l, hook', "\iota"] \arrow[u, "\eta"']
\end{tikzcd}$$

in the above commutative diagram can always be found if N is injective. $\diamond$

Remark 1.7.2. Let X be an object of an abelian category $\mathscr{C}$. Then the following are equivalent:

(i) X is projective.

(ii) Given a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{C}$ the induced sequence

$$0 \longrightarrow (X, A) \longrightarrow (X, B) \longrightarrow (X, C) \longrightarrow 0$$

is also exact.

Thus, X is projective if and only if the functor $(X, —) : \mathscr{C} \to \mathsf{Ab}$ is exact. Dually, X is injective if and only if $(—, X) : \mathscr{C} \to \mathsf{Ab}$ is exact. ◇

Definition 1.7.3. Let $\mathscr{C}$ be an abelian category. Let $A \in \operatorname{obj}\mathscr{C}$. Then a *projective presentation* of A is a short exact sequence in $\mathscr{C}$

$$0 \to Q \to P \to A \to 0,$$

with P projective. The dual notion is that of an *injective presentation*. An injective presentation of A is a short exact sequence in $\mathscr{C}$

$$0 \to A \to I \to J \to 0,$$

with I injective. The abelian category $\mathscr{C}$ *has enough projectives* if every object in $\mathscr{C}$ has a projective presentation. Analogously, $\mathscr{C}$ *has enough injectives* if every object in $\mathscr{C}$ has an injective presentation. ◇

Definition 1.7.4. Let $\mathscr{C}$ be an abelian category. A chain complex over $\mathscr{C}$

$$C = \cdots \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \longrightarrow 0$$

is called *acyclic* if $H_n(C) = 0$ for all $n \geq 1$. The chain complex C is said to *consist of projectives* if C_i is a projective object for all $i \in \mathbb{N}_0$. Similarly, if C is a cochain complex, then C is called *acyclic* if $H^n(C) = 0$ for all $n \geq 1$ and is said to *consist of injectives* if C^i is injective for all $i \in \mathbb{N}_0$. ◇

Definition 1.7.5. Let $\mathscr{C}$ be an abelian category and let $A \in \operatorname{obj}\mathscr{C}$.

(i) A *projective resolution* of A is a projective and acyclic chain complex

$$P = \cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

such that $H_0(P) \cong A$.

(ii) An *injective resolution* of A is an injective and acyclic cochain complex

$$I = 0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^{n-1} \longrightarrow I^n \longrightarrow \cdots$$

such that $A \cong H^0(I)$. ◇

Remark 1.7.6. Let $\mathscr{C}$ be an abelian category with enough projectives. Given an object $A \in \operatorname{obj}\mathscr{C}$, there is a projective presentation

$$0 \to Q_0 \to P_0 \to A \to 0.$$

Because $\mathscr{C}$ has enough projectives, there is a projective presentation of Q_0 given by

$$0 \to Q_1 \to P_1 \to Q_0 \to 0.$$

This gives a sequence

$$0 \to Q_1 \to P_1 \to P_0 \to A \to 0$$

and continuing in this manner gives a projective resolution of A. The upshot is that, if an abelian category $\mathscr{C}$ has enough projectives, then each object in $\mathscr{C}$ has a projective resolution (and similarly if $\mathscr{C}$ has enough injectives, then each object in $\mathscr{C}$ has an injective resolution). ◇

Definition 1.7.7. Let R be a ring with identity and let M be an R-module. Then M is said to have a *basis* if there is a subset $E \subseteq M$ such that:

(i) every element of M is a finite sum of the form $r_1e_1 + r_2e_2 + \cdots r_ne_n$ with each of the $r_i \in R$ and each of the $e_i \in E$ (that is, E is a generating set for M),

(ii) if each of the e_i are distinct and $r_1e_1+r_2e_2+\cdots r_ne_n = 0$, then $r_1 = r_2 = \cdots = r_n = 0$ (that is, E is linearly independent).

If in this situation, then E is called a basis of M. ◇

Definition 1.7.8. Let R be a ring with identity. A *free module* over R is a module with a basis. Given a set E, it is possible to construct a free module over R with E as a basis: the module of *formal linear combinations* of E, denoted $R^{(E)}$. An element of $R^{(E)}$ is an expression of the form $r_1X_1 + r_2X_2 + \cdots + r_nX_n$ with $r_1, r_2, \ldots, r_n \in R$ and $\{X_i\}_{i=1}^n$ a finite subset of E. ◇

Remark 1.7.9. The inclusion map $\iota : E \to R^{(E)}$ satisfies the following universal property, which defines $E^{(R)}$ up to isomorphism. If $f : E \to M$ is a mapping (on the level of sets) from E to an R-module M, then there exists a unique morphism (on the level of modules)

$\lambda : R^{(E)} \to M$ which makes the following diagram

$$\begin{tikzcd} & R^{(E)} \arrow[d, dashed, "\lambda"] \\ E \arrow[ur, "\iota"] \arrow[r, "f"'] & M \end{tikzcd}$$

commutative. $\diamond$

Remark 1.7.10. Let R be a ring with identity and let $f : M \to N$ be an R-module homomorphism with M free. Then f is completely determined by its behaviour on the elements of a given basis E of M. $\diamond$

Lemma 1.7.11. *Let R be a ring with identity. Then*

(i) if M is a free R-module, then M is projective,

(ii) if P is a projective module, then P is a direct summand in a free module M. $\diamond$

Proof. Let E be a basis for M and let X, Y be R-modules. To prove (i), note by Remark 1.7.10, that $f : M \to Y$ is determined by its behaviour on the basis elements $e \in E$. Let $\pi : X \to Y$ be an epimorphism. By virtue of the surjectivity of π, there exists $x \in X$ such that $\pi(x) = f(e)$. Then define $\xi(e) = x$ and extend linearly from the basis E to all of M. Now, because $f = \pi \circ \xi$ on E, this holds for M as well. Part (ii) is proved in [42, Proposition 2.2.1]. $\square$

Remark 1.7.12. Let R be a ring with identity. Then ${}_R\mathsf{Mod}$ has enough projectives. For if M is an R-module, the free (and hence, projective) module F can be generated by elements f_m indexed by $m \in M$, and the canonical projection $\pi : F \to M$ given by $\pi(f_m) = m$ is then the necessary surjection. $\diamond$

Lemma 1.7.13. *Let R be a ring with identity and let $\mathscr{C} = {}_R\mathsf{Mod}$. Let $\mu : M \to M'$ be a module homomorphism in $\mathscr{C}$. Let P be a projective resolution of M and P' a projective resolution of M' (so P and P' are chain complexes in $C(R)$). Then $\mu : M \to M'$ lifts to a chain map $\xi : P \to P'$ such that $\{H_0(P) \longrightarrow H_0(P')\} = \{M \xrightarrow{\mu} M'\}$.*

Proof. Take projective presentations of M and M' and form the short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} p_0 \arrow[r] & P_0 \arrow[r, "p_0"] & M \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & \operatorname{Ker} p_0' \arrow[r] & P_0' \arrow[r, "p_0'"'] & M' \arrow[r] & 0
\end{tikzcd}$$

with P_0 and P_0' projective. Then first, by projectivity of P_0 there is a morphism $\xi_0 : P_0 \to P_0'$ which makes

$$\begin{tikzcd}
P_0 \arrow[d, dashed, "\xi_0"'] \arrow[dr, "\mu \circ p_0"] & \\
P_0' \arrow[r, "p_0'"'] & M'
\end{tikzcd}$$

commute. And then second, by the universal property of the kernel, there is a morphism $k_0 : \operatorname{Ker} p_0 \to \operatorname{Ker} p_0'$ which turns

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} p_0 \arrow[r] \arrow[d, dotted, "k_0"'] & P_0 \arrow[r, "p_0"] \arrow[d, dashed, "\xi_0"] & M \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & \operatorname{Ker} p_0' \arrow[r] & P_0' \arrow[r, "p_0'"'] & M' \arrow[r] & 0
\end{tikzcd}$$

into a commutative diagram. This procedure is then repeated on the modules $\operatorname{Ker} p_0$ and $\operatorname{Ker} p_0'$, like in the process described in Remark 1.7.6. The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} p_1 \arrow[r] \arrow[d, dotted, "k_1"'] & P_1 \arrow[r, "p_1"] \arrow[d, dashed, "\xi_1"] & \operatorname{Ker} p_0 \arrow[r] \arrow[d, dotted, "k_0"] & 0 \\
0 \arrow[r] & \operatorname{Ker} p_1' \arrow[r] & P_1' \arrow[r, "p_1'"'] & \operatorname{Ker} p_0' \arrow[r] & 0
\end{tikzcd}$$

is obtained after doing this. Iterating this entire procedure gives projective resolutions P

and P' of M and M' respectively, with a chain map ξ between them, illustrated below,

$$\begin{tikzcd}
P = \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "\xi_2"] & P_1 \arrow[r] \arrow[d, "\xi_1"] & P_0 \arrow[r] \arrow[d, "\xi_0"] & 0 \\
P' = \cdots \arrow[r] & P'_2 \arrow[r] & P'_1 \arrow[r] & P'_0 \arrow[r] & 0
\end{tikzcd}$$

as required. □

Lemma 1.7.14. *Let $\mu : M \to M'$ be an R-module homomorphism where R is a ring with identity. Let P and P' be projective resolutions of M and M' respectively. Then the chain map ξ (seen in Lemma 1.7.13) : $P \to P'$, which was lifted from μ, is unique up to homotopy.*

Proof. This can be found in, for example, [20, Theorem IV.4.1]. □

Corollary 1.7.15. *Let P, P' be two different projective resolutions of an R-module M. Then P and P' are homotopy equivalent (i.e. isomorphic in $K(R)$).*

Proof. Consider $1_M : M \to M$. Let $\xi : P \to P'$ be the chain map lifted from 1_M (in one direction) and let $\xi' : P' \to P$ be the chain map lifted from 1_M (in the other direction). By Lemma 1.7.14, $\xi' \circ \xi \cong 1_P$ and $\xi \circ \xi' \cong 1_{P'}$, hence $P \cong P'$ in $K(R)$. □

Now we introduce the notion of derived functors.

Definition 1.7.16. Let $F : \mathscr{A} \to \mathscr{B}$ be a covariant, additive functor between abelian categories $\mathscr{A}$ and $\mathscr{B}$. Then F is called *left exact* if given a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$, it is the case that $0 \to F(A) \to F(B) \to F(C)$ is an exact sequence in $\mathscr{B}$. Correspondingly, F is called *right exact* if given a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$, it is the case that $F(A) \to F(B) \to F(C) \to 0$ is an exact sequence in $\mathscr{B}$. If instead F is contravariant, then F is left exact if the sequence $0 \to F(C) \to F(B) \to F(A)$ is exact in $\mathscr{B}$, and right exact if the sequence $F(C) \to F(B) \to F(A) \to 0$ is exact in $\mathscr{B}$. If F is both left exact and right exact, then it is exact. ◇

Remark 1.7.17. Let $\mathscr{A}$ be an abelian category and let $X \in \operatorname{obj} \mathscr{A}$. By [20, Theorems I.2.1 and I.2.2], both the covariant Hom functor, $\operatorname{Hom}_{\mathscr{A}}(X, -)$, and the contravariant Hom functor, $\operatorname{Hom}_{\mathscr{A}}(-, X)$, are left exact functors. ◇

Definition 1.7.18. Let R be a ring and let $M \in {}_R\mathsf{Mod}$. Define the *projective dimension over R of M*, denoted $\operatorname{projdim}_R M$, to be the minimum $n \in \mathbb{N}_0$ such that there exists a projective resolution of M of the form

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0.$$

If no finite resolution exists, then $\operatorname{projdim}_R M = \infty$. $\diamond$

Definition 1.7.19. Let R be a ring. Define the *left global dimension of* R to be

$$\operatorname{l.gldim} R = \sup\{\operatorname{projdim}_R M \mid M \in {}_R\mathsf{Mod}\}.$$

The *right global dimension*, $\operatorname{r.gldim} R$, is defined analogously. $\diamond$

Definition 1.7.20. Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories and let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor between them. Suppose that $\mathscr{A}$ has enough projectives. Let A be an object of $\mathscr{A}$ and construct a projective resolution

$$P = \cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0$$

of A. Apply F to the chain complex P. If F is covariant (respectively, contravariant), then define $L_nF(A) = H_n(F(P))$ (respectively, $R^nF(A) = H^n(F(P))$), the *left-derived functors of* F (respectively, the *right-derived functors of* F). In the covariant case, if F is right exact, then $L_0F(A) \cong F(A)$, and in the contravariant case, if F is left exact, then $R^0F(A) \cong F(A)$. Note, by [42, Lemma 2.4.1], which projective resolution is chosen does not matter, so long as it is fixed. $\diamond$

Definition 1.7.21. Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories and let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor between them. Suppose that $\mathscr{A}$ has enough injectives. Let A be an object of $\mathscr{A}$ and construct an injective resolution

$$I = 0 \to I^0 \to I^1 \to I^2 \to \cdots \to I^{n-1} \to I^n \to \cdots$$

of A. Apply F to the cochain complex I. If F is covariant (respectively, contravariant), then define $R^nF(A) = H^n(F(I))$ (respectively, $L_nF(A) = H_n(F(I))$), the *right-derived functors of* F (respectively, the *left-derived functors of* F). In the covariant case, if F is left exact, then $R^0F(A) \cong F(A)$, and in the contravariant case, if F is right exact, then $L_0F(A) \cong F(A)$. Again (see Definition 1.7.20), which injective resolution is chosen does not matter, so long as it is fixed. $\diamond$

The following theorem is [20, Theorem IV.6.1] and can be seen via applying the Snake Lemma.

Theorem 1.7.22. *Let $\mathscr{C}$ be an abelian category with enough projectives. Suppose that $F : \mathscr{C} \to \mathsf{Ab}$ is an additive functor and that $0 \longrightarrow X' \xrightarrow{f'} X \xrightarrow{f} X'' \longrightarrow 0$ is a short exact sequence in $\mathscr{C}$. Then there are connecting morphisms*

$$\omega_n : L_nF(X'') \to L_{n-1}F(X') \text{ for all } n \in \mathbb{N}$$

such that the following

$$\cdots \longrightarrow L_nF(X') \xrightarrow{f'_*} L_nF(X) \xrightarrow{f_*} L_nF(X'') \xrightarrow{\omega_n} L_{n-1}F(X') \longrightarrow \cdots \longrightarrow 0$$

is a long exact sequence. ◇

One of the main examples of a so-called classical derived functor is Ext.

Definition 1.7.23. Let $\mathscr{A}$ be a ring and $A, B \in {}_{\mathscr{A}}\mathsf{Mod}$. Define, by using a projective resolution, $\mathrm{Ext}^i_{\mathscr{A}}(-, B) = R^i\mathrm{Hom}_{\mathscr{A}}(-, B)$, the right derived functor of the contravariant, left exact, additive functor $\mathrm{Hom}_{\mathscr{A}}(-, B)$. Similarly, define, by using an injective resolution, $\mathrm{Ext}^i_{\mathscr{A}}(A, -) = R^i\mathrm{Hom}_{\mathscr{A}}(A, -)$, the right derived functor of the covariant, left exact, additive functor $\mathrm{Hom}_{\mathscr{A}}(A, -)$. ◇

Remark 1.7.24. The functor Ext is an example of a *bifunctor*, which is a functor of two variables. The two Ext functors defined above in Definition 1.7.23 are equivalent bifunctors. ◇

Remark 1.7.25. The right derived equivalent of Theorem 1.7.22 yields the *long exact Ext-sequence in the first (or second) variable*. ◇

Remark 1.7.26. Another classical derived functor is Tor. The Tor functors are the derived functors of the *tensor product* functor; see [20, Section IV.11] for details. ◇

1.8 The derived category

The following definition is found in [19, Chapter I, Section 3].

Definition 1.8.1. Let $\mathscr{C}$ be a category. Let S be a collection of morphisms in $\mathscr{C}$. Then S is called a *multiplicative system* if it satisfies the following (three) axioms.

- **[FR1].** If $f : A \to B$ and $g : B \to C$ are in S, then $g \circ f : A \to C$ is in S. Further, if $X \in \mathrm{obj}\,\mathscr{C}$, then id_X is in S.
- **[FR2].** If $Z \xrightarrow{s} Y$ is in S, then any diagram

$$\begin{tikzcd} & Z \arrow[d, "s"] \\ X \arrow[r, "v"'] & Y \end{tikzcd}$$

can be completed to a diagram

$$\begin{tikzcd} W \arrow[r, "u"] \arrow[d, "t"'] & Z \arrow[d, "s"] \\ X \arrow[r, "v"'] & Y \end{tikzcd}$$

which is commutative, with $W \xrightarrow{t} X$ in S. The dual statement must also hold.

- **[FR3].** If $f, g : A \to B$ are $\mathscr{C}$-morphisms, then the following are equivalent:
  - There exists a morphism $s : B \to B'$ in S such that $s \circ f = s \circ g$.
  - There exists a morphism $t : A' \to A$ in S such that $f \circ t = g \circ t$.

Suppose that $\mathscr{T}$ is a triangulated category with suspension functor Σ, and S is a collection of morphisms in $\mathscr{T}$ satisfying [FR1] to [FR3] (that is, S is a multiplicative system). If S also satisfies the following two axioms, then S is said to be *compatible with the triangulation*.

- **[FR4].** A morphism s is in S if and only if its suspension $\Sigma(s)$ is in S.
- **[FR5].** If f and g in [TR3] is in S, then the corresponding h of [TR3] is in S as well. ◇

The following is [41, Section II, 2.1.8].

Theorem 1.8.2. *Let $\mathscr{T}' \subseteq \mathscr{T}$ be a triangulated subcategory of a triangulated category $\mathscr{T}$. Then the set*

$$Q = \{\{t_2 \xrightarrow{q} t_1\} \in \mathscr{T} \mid t_0 \in \mathscr{T}' \textit{ if } \{t_2 \xrightarrow{q} t_1 \longrightarrow t_0\} \textit{ is a distinguished triangle in } \mathscr{T}\}$$

is a multiplicative system (which is compatible with the triangulation). ◇

Lemma 1.8.3. *Let $\mathscr{T}'$ be the full triangulated subcategory of $K(\mathscr{A})$ with objects given by all exact complexes. Then by 1.5.20, the set Q given by all quasi-isomorphisms in $K(\mathscr{A})$ is a multiplicative system (compatible with the triangulation).* ◇

Definition 1.8.4. Let $\mathscr{C}$ be a category and let S be a multiplicative system in $\mathscr{C}$. Then the *localisation of $\mathscr{C}$ with respect to S* is a category, denoted $S^{-1}\mathscr{C}$, with objects the same as the objects of $\mathscr{C}$ and morphisms described as follows (see, for example, [19, Chapter I, Section 3]).

(i) A morphism in $S^{-1}\mathscr{C}$ can be represented by a diagram of the form

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y, \end{tikzcd}$$

where $Z \xrightarrow{s} X$ is a morphism in S and $Z \xrightarrow{f} Y$ is a $\mathscr{C}$-morphism. This can be written as fs^{-1}. Morphisms in $S^{-1}\mathscr{C}$ are equivalence classes of diagrams of this form.

(ii) *(Equivalence relation on diagrams).* The diagrams

$$\begin{tikzcd} & Z_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ X & & Y \end{tikzcd}$$

and

$$\begin{tikzcd} & Z_2 \arrow[dl, "s_2"'] \arrow[dr, "f_2"] & \\ X & & Y \end{tikzcd}$$

represent the same morphism (and are therefore in the same equivalence class) if and only if there are morphisms $g_1 : Z \to Z_1$ and $g_2 : Z \to Z_2$ in $\mathscr{C}$ such that the compositions $s_1 \circ g_1$ and $s_2 \circ g_2$ are equal (and are in S) and $f_1 \circ g_1 = f_2 \circ g_2$, as in the following diagram.

$$\begin{tikzcd} & & Z \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & & \\ & Z_1 \arrow[dl, "s_1"'] \arrow[drrr, "f_1"] & & Z_2 \arrow[dlll, "s_2"'] \arrow[dr, "f_2"] & \\ X & & & & Y \end{tikzcd}$$

(iii) *(Composition of morphisms).* Let $X \to Y$ and $Y \to Z$ be morphisms in $S^{-1}\mathscr{C}$ represented as follows:

$$\begin{tikzcd} & V \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

and

$$\begin{tikzcd} & W \arrow[dl, "t"'] \arrow[dr, "g"] & \\ Y & & Z \end{tikzcd}$$

with $s, t \in S$. To compose these into a $S^{-1}\mathscr{C}$-morphism $X \to Z$, form the following commutative diagram

$$\begin{tikzcd} & & U \arrow[dl, "r"'] \arrow[dr, "h"] & & \\ & V \arrow[dl, "s"'] \arrow[dr, "f"] & & W \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

with $r \in S$, which exists by [FR2]. Then the composition is represented by

$$\begin{tikzcd} & U \arrow[dl, "s\circ r"'] \arrow[dr, "g\circ h"] & \\ X & & Z. \end{tikzcd}$$

Note that this is well-defined; in particular, it is independent of the choice of diagram above.

(iv) *(The canonical functor).* There is a canonical functor $\pi : \mathscr{C} \to S^{-1}\mathscr{C}$ which maps an object to itself, and given a morphism $\{f : X \to Y\} \in \mathscr{C}$, the functor π maps f to the morphism represented by the diagram

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{id}_X"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

in $S^{-1}\mathscr{C}$. $\diamond$

Lemma 1.8.5. *Let $\mathscr{C}$ be a category and S a multiplicative system in $\mathscr{C}$. Consider $\pi : \mathscr{C} \to S^{-1}\mathscr{C}$, the canonical functor. Then*

(i) if s is in S, then the morphism $\pi(s)$ has an inverse,

(ii) if S consists only of isomorphisms, then $\pi : \mathscr{C} \to S^{-1}\mathscr{C}$ is an equivalence of categories. $\diamond$

Proof.

(i) Now, $\pi(s)$ is equal to the morphism with class representative

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{id}_X"'] \arrow[dr, "s"] & \\ X & & Y. \end{tikzcd}$$

Then $\pi(s)^{-1}$ is represented by

$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "\mathrm{id}_X"] & \\ Y & & X \end{tikzcd}$$

because the composition is the identity:

$$\begin{tikzcd} & & X \arrow[dl, "\mathrm{id}_X"'] \arrow[dr, "\mathrm{id}_X"] & & \\ & X \arrow[dl, "\mathrm{id}_X"'] \arrow[dr, "s"] & & X \arrow[dl, "s"'] \arrow[dr, "\mathrm{id}_X"] & \\ X & & Y & & X \end{tikzcd}$$

$$= \begin{tikzcd} & X \arrow[dl, "\mathrm{id}_X"'] \arrow[dr, "\mathrm{id}_X"] & \\ X & & X. \end{tikzcd}$$

(ii) Define the functor $\tau : S^{-1}\mathscr{C} \to \mathscr{C}$ by mapping the morphism represented by

$$\begin{tikzcd} & Z \arrow[dl, "t"'] \arrow[dr, "d"] & \\ X & & Y. \end{tikzcd}$$

to $\{d \circ t^{-1} : X \to Y\}$.

That τ is well-defined and is a quasi-inverse to π is omitted. □

Lemma 1.8.6. *Let $\mathscr{C}$ be a category and S a multiplicative system in $\mathscr{C}$. Let $\mathscr{D}$ be a category and consider a functor $F : \mathscr{C} \to \mathscr{D}$. Suppose that the collection of morphisms in $\mathscr{D}$ given by $F(S)$ consists only of isomorphisms. Then there is a unique functor, $\tilde{F} : S^{-1}\mathscr{C} \to \mathscr{D}$, such that*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\pi"] \arrow[d, "F"'] & S^{-1}\mathscr{C} \arrow[dl, dashed, "\tilde{F}"] \\ \mathscr{D} & \end{tikzcd}$$

commutes. This is the universal property of $S^{-1}\mathscr{C}$.

Proof. See [19, Chapter I, Section 3, page 29]. □

Definition 1.8.7. Let $\mathscr{A}$ be an abelian category and consider $K(\mathscr{A})$. Let Q be the multiplicative system of quasi-isomorphisms in $K(\mathscr{A})$. Then the localisation of $K(\mathscr{A})$ with respect to Q is called the *derived category of* $\mathscr{A}$ and is denoted $D(\mathscr{A})$. $\diamond$

It can be shown that the derived category is a triangulated category (see, for example, [19, Proposition I.3.2]), although a proof will not be given here. The triangles in the derived category are the diagrams isomorphic to images of $K(\mathscr{A})$-triangles under the canonical localisation functor.

Definition 1.8.8. Let S be a multiplicative system in a category $\mathscr{C}$ and let $P \in \operatorname{obj}\mathscr{C}$. If $\{s : X \to Y\} \in S$ implies that the map $(P, s) : (P, X) \to (P, Y)$ is bijective, then P is called *K-projective*. Note that K-projectivity is a notion dependent on the choice of S in the category $\mathscr{C}$. An object may be K-projective with one choice of multiplicative system but not K-projective with the choice of another. $\diamond$

Lemma 1.8.9. *Let S be a multiplicative system in a category $\mathscr{C}$ and let $P \in \operatorname{obj}\mathscr{C}$ be K-projective. Then for any $X \in \operatorname{obj}\mathscr{C}$, we have that* $\operatorname{Hom}_{S^{-1}\mathscr{C}}(P, X) \cong \operatorname{Hom}_{\mathscr{C}}(P, X)$.

Proof. Define $\varphi : \operatorname{Hom}_{\mathscr{C}}(P, X) \to \operatorname{Hom}_{S^{-1}\mathscr{C}}(P, X)$ in the following way. Given a morphism $f \in \operatorname{Hom}_{\mathscr{C}}(P, X)$, define $\varphi(f)$ to be the morphism

$$\left[\begin{tikzcd} & P \arrow[dl, "\mathrm{id}_P"'] \arrow[dr, "f"] & \\ P & & X \end{tikzcd}\right],$$

where the [] notation is representing an equivalence class of diagrams. Now, given a morphism

$$\mu = \left[\begin{tikzcd} & W \arrow[dl, "s"'] \arrow[dr, "g"] & \\ P & & X \end{tikzcd}\right]$$

in $S^{-1}\mathscr{C}$, there is a bijection $(P, s) : (P, W) \xrightarrow{\cong} (P, P)$ by virtue of s being in S and P being K-projective. This means that there is a unique α in $\operatorname{Hom}_{\mathscr{C}}(P, W)$ such that $s \circ \alpha = \mathrm{id}_P$. Thus, define $\psi : \operatorname{Hom}_{S^{-1}\mathscr{C}}(P, X) \to \operatorname{Hom}_{\mathscr{C}}(P, X)$ by $\psi(\mu) = g \circ \alpha \in \operatorname{Hom}_{\mathscr{C}}(P, X)$. That φ and ψ are well-defined, and are each other's inverses, is omitted. $\square$

Remark 1.8.10. There is a well-defined functor $D(\mathscr{A}) \to \mathsf{Ab}$ given by

$$H_i\left(\left[\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}\right]\right) = H_i(f) \circ H_i(s)^{-1}.$$

This is by virtue of the homology functor sending quasi-isomorphisms to isomorphisms. $\diamond$

Henceforth in this section, $\mathscr{A}$ will denote an abelian category and Q will denote the multiplicative system inside $K(\mathscr{A})$ which consists of quasi-isomorphisms. An object of $K(\mathscr{A})$ will be said to be K-projective if it is K-projective with respect to this multiplicative system.

Lemma 1.8.11. *Let P be a complex over $\mathscr{A}$ which is right-bounded, that is, there exists $N \in \mathbb{Z}$ such that $P_n = 0$ for all $n \leq N$. Furthermore, let P consist of projective objects. Let X be a complex over $\mathscr{A}$ which is exact. Let $f : P \to X$ be a chain map. Then f is null-homotopic.*

Proof. Without loss of generality, suppose that $N = -1$. Further, write $Z_m X = \operatorname{Ker} d_m^X$. For each $m \in \mathbb{Z}$, factor in the canonical way d_m^X through $\operatorname{Im} d_m^X$ (see, for example, Remark 1.2.26), which, by exactness of X, is $\operatorname{Ker} d_{m-1}^X = Z_{m-1}X$. Thus, let the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[rr, "d_2^P"] \arrow[d, "f_2"] & & P_1 \arrow[rr, "d_1^P"] \arrow[d, "f_1"] & & P_0 \arrow[rr] \arrow[d, "f_0"] & & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & X_2 \arrow[rr, "d_2^X"'] \arrow[dr, two heads, "\pi_2"'] & & X_1 \arrow[rr, "d_1^X"'] \arrow[dr, two heads, "\pi_1"'] & & X_0 \arrow[rr, "d_0^X"'] \arrow[dr, two heads, "\pi_0"'] & & X_{-1} \arrow[r] & \cdots \\
& & Z_1X \arrow[ur, hook, "\iota_1"'] & & Z_0X \arrow[ur, hook, "\iota_0"'] & & Z_{-1}X \arrow[ur, hook, "\iota_{-1}"'] & &
\end{tikzcd}$$

be given. Now, because $d_0^X \circ f_0 = 0$, it must be that f_0 factors through the kernel of d_0^X. Hence, there exists a map $\rho_0 : P_0 \to Z_0X$ such that $\iota_0 \circ \rho_0 = f_0$. Then, by projectivity of P_0, there must exist a map $s_0 : P_0 \to X_1$ such that $\pi_1 \circ s_0 = \rho_0$. There is also clearly a map $s_{-1} : 0 \to X_0$ (the zero map); filling all of this in on the diagram yields the following commutative diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[rr, "d_2^P"] \arrow[d, "f_2"] & & P_1 \arrow[rr, "d_1^P"] \arrow[d, "f_1"] & & P_0 \arrow[rr] \arrow[d, "f_0"] \arrow[dll, bend right, "s_0"'] \arrow[ddl, bend left, "\rho_0"'] & & 0 \arrow[r] \arrow[d] \arrow[dll, "s_{-1}"'] & \cdots \\
\cdots \arrow[r] & X_2 \arrow[rr, "d_2^X"'] \arrow[dr, two heads, "\pi_2"'] & & X_1 \arrow[rr, "d_1^X"'] \arrow[dr, two heads, "\pi_1"'] & & X_0 \arrow[rr, "d_0^X"'] \arrow[dr, two heads, "\pi_0"'] & & X_{-1} \arrow[r] & \cdots \\
& & Z_1X \arrow[ur, hook, "\iota_1"'] & & Z_0X \arrow[ur, hook, "\iota_0"'] & & Z_{-1}X \arrow[ur, hook, "\iota_{-1}"'] & &
\end{tikzcd}$$

Now, since $\iota_0 \circ \rho_0 = f_0$ and $\pi_1 \circ s_0 = \rho_0$, it is the case that $\iota_0 \circ \pi_1 \circ s_0 = d_1^X \circ s_0 = f_0$. Thus, $f_0 = s_{-1} \circ d_0^P + d_1^X \circ s_0$.

Now, consider the composition $d_1^X \circ (f_1 - s_0 \circ d_1^P)$. This equals $d_1^X \circ f_1 - d_1^X \circ s_0 \circ d_1^P = d_1^X \circ f_1 - f_0 \circ d_1^P = 0$. Hence, $f_1 - s_0 \circ d_1^P$ maps into the kernel of d_1^X, and therefore there is a map $\rho_1 : P_1 \to Z_1X$ such that $\iota_1 \circ \rho_1 = f_1 - s_0 \circ d_1^P$. Thus, by projectivity of P_1, there is a map $s_1 : P_1 \to X_2$ such that $\pi_2 \circ s_1 = \rho_1$. The following diagram summarises this.

$$\begin{tikzcd}
 & & P_1 \arrow[dll, bend right, "s_1"'] \arrow[dl, bend right, "\rho_1"'] \arrow[d, "f_1 - s_0 \circ d_1^P"] \\
X_2 \arrow[rr, "d_2^X"] \arrow[dr, two heads, "\pi_2"'] & & X_1 \\
 & Z_1X \arrow[ur, hook, "\iota_1"'] &
\end{tikzcd}$$

Now, since $\iota_1 \circ \rho_1 = f_1 - s_0 \circ d_1^P$ and $\pi_2 \circ s_1 = \rho_1$, it is the case that $\iota_1 \circ \pi_2 \circ s_1 = d_2^X \circ s_1 = f_1 - s_0 \circ d_1^P$. That is, $f_1 = s_0 \circ d_1^P + d_2^X \circ s_1$.

Continuing in this manner yields a homotopy $s = \{s_n : P_n \to X_{n+1}\}$ such that $f_n = s_{n-1} \circ d_n^P + d_{n+1}^X \circ s_n$, as required. $\square$

Lemma 1.8.12. *Let P be a projective object of $\mathscr{A}$. Let A be a chain complex in $C(\mathscr{A})$. Then there is an isomorphism $\mathrm{Hom}_{\mathscr{A}}(P, H_0(A)) \cong H_0(\mathrm{Hom}_{\mathscr{A}}(P, A))$ which is natural in P and A.*

Proof. Let $A = \cdots \longrightarrow A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A_{-1} \longrightarrow \cdots$ and let (P, A) be the chain complex $\cdots \longrightarrow (P, A_1) \xrightarrow{\delta_1} (P, A_0) \xrightarrow{\delta_0} (P, A_{-1}) \longrightarrow \cdots$ with δ_i given by (P, d_i). Let $f : P \to H_0(A)$ be a morphism. Consider the canonical surjection $\pi_A : \mathrm{Ker}\, d_0 \to H_0(A)$. By virtue of P being projective, there exists a morphism $j : P \to \mathrm{Ker}\, d_0$ such that

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "j"'] \arrow[d, "f"] \\
\mathrm{Ker}\, d_0 \arrow[r, two heads, "\pi_0"'] & H_0(A)
\end{tikzcd}$$

is commutative. Let $\overline{j} : P \to A_0$ be given by $\overline{j}(p) = j(p)$. Now, $\overline{j} \in \mathrm{Ker}\, \delta_0$, because $d_0 \circ \overline{j} = 0$. Hence, define $\Theta(f) = \overline{j} + \mathrm{Im}\, \delta_1 \in H_0(\mathrm{Hom}_{\mathscr{A}}(P, A))$. Conversely, consider $\overline{j} + \mathrm{Im}\, \delta_1 \in H_0(\mathrm{Hom}_{\mathscr{A}}(P, A))$ where $\overline{j} \in (P, A_0)$ and $d_0 \circ \overline{j} = 0$. Given $p \in P$, define $\Theta'(\overline{j} + \mathrm{Im}\, \delta_1) = f$, where $f(p) = \overline{j}(p) + \mathrm{Im}\, d_1 \in \mathrm{Hom}_{\mathscr{A}}(P, H_0(A))$. It can be readily verified (via projectivity of P) that Θ and Θ' are both well-defined. Further, it is clear that $\Theta \circ \Theta'$ and $\Theta' \circ \Theta$ are the (respective) identities on $H_0(\mathrm{Hom}_{\mathscr{A}}(P, A))$ and $\mathrm{Hom}_{\mathscr{A}}(P, H_0(A))$.

That this isomorphism is natural in P and A is shown by showing that the following two squares commute. First, let $P \xrightarrow{f} Q$ be a morphism between projective objects in $\mathscr{A}$. That

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{A}}(P, H_0(A)) \arrow[d, "\Theta_{P,A}"'] & \mathrm{Hom}_{\mathscr{A}}(Q, H_0(A)) \arrow[l, "f^*"'] \arrow[d, "\Theta_{Q,A}"] \\
H_0(\mathrm{Hom}_{\mathscr{A}}(P, A)) & H_0(\mathrm{Hom}_{\mathscr{A}}(Q, A)) \arrow[l, "H_0(f^*)"]
\end{tikzcd}$$

commutes is shown by diagram chasing. Let $m \in (Q, H_0(A))$ and let j and $j \circ f$ be the morphisms completing the following diagram

$$\begin{tikzcd}
 & P \arrow[d, "f"] \arrow[ddl, dashed, "j \circ f"'] \\
 & Q \arrow[d, "m"] \arrow[dl, dashed, "j"] \\
\mathrm{Ker}\, d_0^A \arrow[r, two heads, "\pi_0^A"'] & H_0(A)
\end{tikzcd}$$

and note that by the well-definedness of Θ, the choice $j \circ f : P \to \mathrm{Ker}\, d_0^A$ is justified. Chasing m through the diagram, we get

$$\begin{tikzcd}
\{P \xrightarrow{m \circ f} H_0(A)\} \arrow[d, mapsto, "\Theta_{P,A}"'] & \{Q \xrightarrow{m} H_0(A)\} \arrow[l, mapsto, "f^*"'] \arrow[d, mapsto, "\Theta_{Q,A}"] \\
\{\overline{j} \circ f + \mathrm{Im}\, \delta_1^P\} & \{\overline{j} + \mathrm{Im}\, \delta_1^Q\} \arrow[l, mapsto, "H_0(f^*)"]
\end{tikzcd}$$

which is commutative. A similar argument shows that, when $A \xrightarrow{g} B$ is a chain map,

$$\begin{CD}
\mathrm{Hom}_{\mathscr{A}}(P, H_0(A)) @>{(H_0(g))_*}>> \mathrm{Hom}_{\mathscr{A}}(P, H_0(B)) \\
@V{\Theta_{P,A}}VV @VV{\Theta_{P,B}}V \\
H_0(\mathrm{Hom}_{\mathscr{A}}(P, A)) @>>{H_0(g_*)}> H_0(\mathrm{Hom}_{\mathscr{A}}(P, B))
\end{CD}$$

is also commutative, as required. ☐

Remark 1.8.13. Let P be an object of $\mathscr{A}$. Then P can be realised as an object of $C(\mathscr{A})$ by placing it in degree zero of a chain complex with zero in every other degree (together with zero differentials). In a slight abuse of notation, this allows for the following setup. If P is an object of $\mathscr{A}$, then in $C(\mathscr{A})$ we may write

$$P = \cdots \to 0 \to 0 \to P \to 0 \to 0 \to \cdots$$

with P in degree zero. ◇

Lemma 1.8.14. *Let $\mathscr{A}$ be an abelian category and let P be a projective object of $\mathscr{A}$. Further, let $X \in K(\mathscr{A})$. Then there is an isomorphism $\mathrm{Hom}_{K(\mathscr{A})}(P, X) \cong \mathrm{Hom}_{\mathscr{A}}(P, H_0(X))$ which is natural in P and X.*

Proof. The proof is very similar to the proof of Lemma 1.8.12 and is therefore omitted. ☐

Corollary 1.8.15. *Let P be a projective object of an abelian category $\mathscr{A}$. Let $f : A \to B$ be a quasi-isomorphism in $K(\mathscr{A})$. Then*

$$(P, A) \overset{f_*}{\cong} (P, B)$$

in $K(\mathscr{A})$.

Proof. This follows directly from the isomorphism in Lemma 1.8.14 being natural in X and the fact that $(H_0(f))_* : \mathrm{Hom}_{\mathscr{A}}(P, H_0(A)) \to \mathrm{Hom}_{\mathscr{A}}(P, H_0(B))$ is an isomorphism. ☐

Recall Lemma 1.5.20. Rephrasing this in terms of $K(\mathscr{A})$, we have the following Corollary.

Corollary 1.8.16. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be a distinguished triangle in $K(\mathscr{A})$. Then f is a quasi-isomorphism if and only if Z is an exact complex.* ◇

Lemma 1.8.17. *Let $N \in \mathrm{obj}\,\mathscr{A}$ and let X be a complex over $\mathscr{A}$. Then there is an isomorphism $\mathrm{Hom}_{K(\mathscr{A})}(X, \Sigma^i N) \cong H^i \,\mathrm{Hom}_{\mathscr{A}}(X, N)$ which is natural in X and N.*

Proof. Let

$$\begin{tikzcd}
X = \cdots \arrow[r] & X_{i+1} \arrow[r, "d_{i+1}^X"] \arrow[d, "0"'] & X_i \arrow[r, "d_i^X"] \arrow[d, "\xi"] & X_{i-1} \arrow[r] \arrow[d, "0"] & \cdots \\
\Sigma^i N = \cdots \arrow[r] & 0 \arrow[r, "0"'] & N \arrow[r, "0"'] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

be a chain map from X to $\Sigma^i N$. It is clear that

$$\operatorname{Hom}_{C(\mathscr{A})}(X, \Sigma^i N) = \{X_i \xrightarrow{\xi} N \mid X_{i+1} \xrightarrow{d_{i+1}^X} X_i \xrightarrow{\xi} N \text{ is } 0\}$$

which is equal to the i-cycles (see Definition 1.5.8), $Z^i \operatorname{Hom}_{\mathscr{A}}(X, N)$. The set of null-homotopic chain maps is equal to

$$\{X_i \xrightarrow{\xi} N \mid \text{There exists a factorisation } X_i \xrightarrow{d_i^X} X_{i-1} \longrightarrow N \text{ of } X_i \xrightarrow{\xi} N\}$$

which is equal to the i-boundaries, $B^i \operatorname{Hom}_{\mathscr{A}}(X, N)$. Now,

$$\operatorname{Hom}_{K(\mathscr{A})}(X, \Sigma^i N) = \frac{\operatorname{Hom}_{C(\mathscr{A})}(X, \Sigma^i N)}{\{\text{Null-homotopic chain maps}\}}$$

$$= \frac{Z^i \operatorname{Hom}_{\mathscr{A}}(X, N)}{B^i \operatorname{Hom}_{\mathscr{A}}(X, N)} = H^i \operatorname{Hom}_{\mathscr{A}}(X, N),$$

as required. That the isomorphism is natural follows from how the isomorphism is defined. □

Proposition 1.8.18. *Let P be a chain complex over $\mathscr{A}$ which is right-bounded and consists of projective modules. Then P is K-projective; in particular, a projective resolution of an object in $\mathscr{A}$ is K-projective.*

Proof. Let $q : X \to Y$ be a quasi-isomorphism. Extend this to a distinguished triangle $X \xrightarrow{q} Y \longrightarrow Z \longrightarrow \Sigma X$ in $K(\mathscr{A})$ and roll to obtain $\Sigma^{-1} Z \longrightarrow X \xrightarrow{q} Y \longrightarrow Z$. By Corollary 1.8.16, Z is exact. Apply the functor $(P, -)$ to obtain

$$(P, \Sigma^{-1} Z) \longrightarrow (P, X) \xrightarrow{q_*} (P, Y) \longrightarrow (P, Z),$$

which is exact by Theorem 1.4.10. Lemma 1.8.11 now says that this sequence of morphisms is equal to

$$0 \longrightarrow (P, X) \xrightarrow{q_*} (P, Y) \longrightarrow 0$$

because, in $K(\mathscr{A})$, any morphism from (a right-bounded complex consisting of projectives) P to (an exact complex) Z is zero. By exactness, q_* is a bijection, as required. □

Lemma 1.8.19. *Let $A \to B \to C \to \Sigma A$ be a distinguished triangle in the homotopy category $K(\mathscr{A})$. Then if A and B are K-projective, then C is K-projective.*

Proof. Let $q : X \to Y$ be in Q. The following commutative diagrams exists,

$$\begin{tikzcd}
(\Sigma B, X) \arrow[r] \arrow[d, "\cong"'] & (\Sigma A, X) \arrow[r] \arrow[d, "\cong"'] & (C, X) \arrow[r] \arrow[d] & (B, X) \arrow[r] \arrow[d, "\cong"] & (A, X) \arrow[d, "\cong"] \\
(\Sigma B, Y) \arrow[r] & (\Sigma A, Y) \arrow[r] & (C, Y) \arrow[r] & (B, Y) \arrow[r] & (A, Y)
\end{tikzcd}$$

for ΣA and ΣB are also K-projective. The rows are exact by Theorem 1.4.10. Therefore, by Lemma 1.3.6, $(C, X) \to (C, Y)$ is an isomorphism and C is K-projective. $\square$

Lemma 1.8.20. *Let $q : A \to B$ be a quasi-isomorphism between K-projective objects A and B in $K(\mathscr{A})$. Then $q : A \to B$ is an isomorphism in $K(\mathscr{A})$.*

Proof. Complete $A \xrightarrow{q} B$ to a distinguished triangle $A \xrightarrow{q} B \longrightarrow C \longrightarrow \Sigma A$. By Lemma 1.8.19, C is K-projective. The long exact sequence

$$\cdots \longrightarrow (C, A) \xrightarrow{\cong} (C, B) \longrightarrow (C, C) \longrightarrow (C, \Sigma A) \xrightarrow{\cong} (C, \Sigma B) \longrightarrow \cdots$$

is induced by this distinguished triangle, and therefore $(C, C) = 0$ which implies that $C \cong 0$. Hence, by Lemma 1.4.11(ii), it is the case that q is an isomorphism. $\square$

Definition 1.8.21. Let S be a multiplicative system in $\mathscr{A}$ and let $A \in \operatorname{obj} \mathscr{A}$. A *$K$-projective resolution* of A is K-projective object B in $\mathscr{A}$, together with an S-morphism $s : B \to A$. Note, the notion of K-projectivity is dependent on $\mathscr{A}$ and S. $\diamond$

Remark 1.8.22. Let R be a ring with identity. Let M be an R-module and consider M as the $K(R)$-object

$$M = \cdots \to 0 \to 0 \to M \to 0 \to 0 \to \cdots$$

with M placed in degree zero. Take a projective resolution of M

$$P = \cdots \to P_2 \to P_1 \to P_0 \to 0 \to 0 \to \cdots .$$

By Proposition 1.8.18, P is K-projective. It is clear that there exists a quasi-isomorphism $f : P \to M$ (with f_0 given by the canonical $P_0 \to M$ morphism and the rest of the f_i given by 0), hence P is a K-projective resolution of M. It is therefore the case that $K(R)$ has enough K-projectives (and, similarly, K-injectives). $\diamond$

Lemma 1.8.23. *Let $A, B \in D(\mathscr{A})$ and let $q : P \to A$ be a K-projective resolution of A. Then* $\operatorname{Hom}_{D(\mathscr{A})}(A, \Sigma^i B) \cong \operatorname{Hom}_{K(\mathscr{A})}(P, \Sigma^i B)$.

Proof. This follows from Lemma 1.8.9 and the fact that P is K-projective (with respect to quasi-isomorphisms). In particular, q becomes an isomorphism in $D(\mathscr{A})$. $\square$

Corollary 1.8.24. *Let $A, B \in \mathsf{Mod}(R)$ and let P be a projective resolution of A. Then* $\mathrm{Hom}_{D(R)}(A, \Sigma^i B) \cong \mathrm{Hom}_{K(R)}(P, \Sigma^i B)$. $\diamond$

The following relates Ext^i_R to the Hom-space in the derived category.

Lemma 1.8.25. *Let $A, B \in \mathsf{Mod}(R)$. Then* $\mathrm{Hom}_{D(R)}(A, \Sigma^i B) \cong \mathrm{Ext}^i_R(A, B)$.

Proof. Let P be a projective resolution of A. Then

$$\begin{aligned}\mathrm{Hom}_{D(R)}(A, \Sigma^i B) &\cong \mathrm{Hom}_{K(R)}(P, \Sigma^i B), \text{ by Corollary 1.8.24,}\\ &\cong H^i \,\mathrm{Hom}_R(P, B), \text{ by Lemma 1.8.17,}\\ &\cong \mathrm{Ext}^i_R(A, B),\end{aligned}$$

as required. $\square$

The following theorem has been proved by both Spaltenstein [40] and Bökstedt and Neeman [7]. It concerns the existence of K-projective (and K-injective) resolutions in $K(R)$, where R is a ring.

Theorem 1.8.26. *Let R be a ring. Any given complex in $K(R)$ has a K-projective resolution.* $\diamond$

Remark 1.8.27. Let X and Y be complexes over a ring R. Then, by Lemma 1.8.23, the Hom-space $\mathrm{Hom}_{D(R)}(X, Y)$ reduces to $\mathrm{Hom}_{K(R)}(P_X, Y)$ where $P_X \to X$ is a K-projective resolution of X, and by Theorem 1.8.26, such a K-projective resolution will always exist. $\diamond$

1.9 Differential graded algebras

Definition 1.9.1. A $\mathbb{Z}$-*graded ring* is a ring R that decomposes as a direct sum of abelian groups $\{R_i\}_{i\in\mathbb{Z}}$ such that if $r_i \in R_i$ and $r_j \in R_j$, then $r_i r_j \in R_{i+j}$. An algebra A over a ring R is said to be a $\mathbb{Z}$-*graded algebra* if, when considered as a ring, it is $\mathbb{Z}$-graded. $\diamond$

Remark 1.9.2. A graded ring need not be $\mathbb{Z}$-graded, although this is all that shall be considered in this thesis and, henceforth, "graded" shall mean "$\mathbb{Z}$-graded". $\diamond$

Definition 1.9.3. Let K be a commutative ring. A *differential graded algebra, R*, over K is a graded algebra $\bigoplus_{i\in\mathbb{Z}} R_i$, over K together with a K-linear differential $d^R : R \to R$.

The differential d^R is a collection of K-linear maps $d_i : R_i \to R_{i-1}$ such that $d_{i-1} \circ d_i = 0$ which also satisfies the *Leibniz rule*

$$d_{i+j}(rs) = d_i(r)s + (-1)^i r d_j(s)$$

where $r \in R_i$ and $s \in R_j$. $\diamond$

Remark 1.9.4. It is typical to think of a DGA R as a complex

$$\cdots \longrightarrow R_{i+1} \xrightarrow{d^R_{i+1}} R_i \xrightarrow{d^R_i} R_{i-1} \longrightarrow \cdots$$

of K-modules. Studying DGAs is therefore naturally associated with the study of homological algebra. $\diamond$

Henceforth in this section, the commutative ground ring will be denoted by K.

Definition 1.9.5. Let R and S be be DGAs over K. A *morphism of DGAs* $\Psi : R \to S$ is a collection of morphisms of K-modules $\Psi_i : R_i \to S_i$ such that:

(i) $\Psi_{i+j}(r_i r_j) = \Psi_i(r_i)\Psi_j(r_j)$ for $r_i \in R_i$ and $r_j \in R_j$,

(ii) $d^S_i \circ \Psi_i = \Psi_{i-1} \circ d^R_i$ for all $i \in \mathbb{Z}$,

(iii) $\Psi(1_R) = 1_S$. $\diamond$

Definition 1.9.6. The *forgetful functor on DGAs* is the functor $(-)^\natural$ which sends a DGA, R, to its underlying graded algebra $R^\natural = \bigoplus_{i\in\mathbb{Z}} R_i$. $\diamond$

Definition 1.9.7. The homology of a DGA R over K can be computed in the usual way (as if R is considered as a complex). Set $Z_n(R) = \operatorname{Ker} d^R_n$ (the n-cycles) and $B_n(R) = \operatorname{Im} d^R_{n+1}$ (the n-boundaries). Then $H_n(R) = \frac{Z_n(R)}{B_n(R)}$ is the n^{th} homology group of R. In the same way as in Remark 1.6.6, H_n is a functor. $\diamond$

Henceforth in this section, let R and S stand for DGAs over K.

Definition 1.9.8. A *differential graded left R-module* is a graded left $R^\natural$-module

$$M = \bigoplus_{i\in\mathbb{Z}} M_i$$

(which means that if $r_i \in R_i$ and $m_j \in M_j$, then $r_i m_j \in M_{i+j}$), together with a differential d^M. The differential is a collection of maps $d_i : M_i \to M_{i-1}$ which satisfies $d^M_{i-1} \circ d^M_i = 0$ and the *Leibniz rule for left-DG-R-modules*

$$d^M_{i+j}(rm) = d^R_i(r)m + (-1)^i r d^M_j(m)$$

where $r \in R_i$ and $m \in M_j$.

A *differential graded right R-module* is a graded right $R^\natural$-module

$$M = \bigoplus_{i \in \mathbb{Z}} M_i$$

(which means that if $m_i \in M_i$ and $r_j \in R_j$, then $m_i r_j \in M_{i+j}$), together with a differential d^M. The differential is a collection of maps $d_i : M_i \to M_{i-1}$ which satisfies $d^M_{i-1} \circ d^M_i = 0$ and the *Leibniz rule for right-DG-R-modules*

$$d^M_{i+j}(mr) = d^M_j(m)r + (-1)^j m d^R_i(r)$$

where $r \in R_i$ and $m \in M_j$. ◇

Definition 1.9.9. Let M and N be left-DG-R-modules. A *morphism of left-DG-R-modules* $\Psi : M \to N$ is a collection of morphisms of abelian groups $\Psi_i : M_i \to N_i$ satisfying:

(i) $\Psi_{i+j}(r_i m_j) = r_i \Psi_j(m_j)$ for $r_i \in R_i$ and $m_j \in M_j$,

(ii) $d^N_i \circ \Psi_i = \Psi_{i-1} \circ d^M_i$ for all $i \in \mathbb{Z}$. ◇

Remark 1.9.10. Let A be an algebra over K. Then A can be considered a DGA

$$\cdots \to 0 \to 0 \to A \to 0 \to 0 \to \cdots$$

concentrated in degree zero. It can readily be seen that left-DG-modules over this DGA are the same thing as complexes over A. ◇

Remark 1.9.11. Like DGAs, left-DG-R-modules can also be interpreted as chain complexes over K. This is accomplished by forgetting the R structure. A morphism $M \to N$ of left-DG-R-modules is then simply a chain map $\varphi : M \to N$ which satisfies $\varphi(rm) = r\varphi(m)$. ◇

Definition 1.9.12. The following conventions will be used throughout the rest of this thesis.

(i) Let

$$R = \bigoplus_{i \in \mathbb{Z}} R_i.$$

Then the elements of R_k are said to be *homogeneous of degree* k. If r is an element of R which is homogeneous of degree k, then the *degree of* r is denoted by $|r|$ and in this case, equals k.

(ii) The DGA R is said to be commutative if for $r_i \in R_i$ and $r_j \in R_j$, we have $r_j r_i = (-1)^{ij} r_i r_j$. $\diamond$

Henceforth, only left-DG-R-modules will be considered in this thesis, and will be referred to as DG-R-modules, or if the DGA R is known by context, simply DG-modules.

Definition 1.9.13. The *category of DG-R-modules* is the category $\mathsf{DMod}(R)$ whose objects are DG-R-modules and morphisms are morphisms of DG-R-modules. $\diamond$

Definition 1.9.14. Define the *suspension functor* of DG-R-modules to be

$$\Sigma : \mathsf{DMod}(R) \to \mathsf{DMod}(R), M \mapsto \Sigma M$$

where $(\Sigma M)_i = M_{i-1}$ and $d_i^{\Sigma M} = -d_{i-1}^M$. $\diamond$

Remark 1.9.15. Let M be a DG-R-module. We have that M_i appears in both M and ΣM. If $m \in M_i$, then "m" viewed in $(\Sigma M)_{i+1}$ is denoted by Σm. It is thus the case that $\Sigma m \in (\Sigma M)_{i+1}$, and if $r \in R$, then $\Sigma(rm) = (-1)^{|r|} r \Sigma m$. $\diamond$

Definition 1.9.16. Let X, Y be objects in $\mathsf{DMod}(R)$ and $f, g : X \to Y$ be two morphisms between them. Then f and g are *homotopic* (or, $f - g$ is null-homotopic), if there is a morphism of graded $R^\natural$-modules $s : X^\natural \to \Sigma^{-1} Y^\natural$ such that

$$f_n - g_n = s_{n-1} \circ d_n^X + d_{n+1}^Y \circ s_n.$$

If f and g are homotopic, then this is denoted $f \sim g$. $\diamond$

Like in the category of chain complexes over an abelian category, homotopy is an equivalence relation on $\mathrm{Hom}_{\mathsf{DMod}(R)}(X, Y)$ for any pair of DG-R-modules X and Y, and composition of equivalence classes is well-defined. This leads to the following definition.

Definition 1.9.17. The *homotopy category of* R, denoted $K(R)$, is the category whose objects are DG-R-modules and whose morphisms are cosets of morphisms of DG-R-modules modulo homotopy. Thus a morphism in $K(R)$ is an equivalence class of morphisms of DG-R-modules. $\diamond$

Definition 1.9.18. The *cone of a morphism* $M \xrightarrow{f} N$ in $\mathsf{DMod}(R)$ is the object $E(f)$ defined in the usual way, as in Definition 1.5.16. That is, $E(f)^\natural = (\Sigma M \oplus N)^\natural$ and $E(f)$ is equipped with the differential defined by $d_n^E = (-d_{n-1}^M, f_{n-1} + d_n^N)$. A *standard triangle* in $K(R)$ is given by

$$M \xrightarrow{f} N \to E(f) \to \Sigma C$$

and the distinguished triangles in $K(R)$ are diagrams isomorphic to these. By [19, Section I.2], $K(R)$ is a triangulated category with suspension functor Σ and distinguished triangles defined above. $\diamond$

Definition 1.9.19. Let $M \xrightarrow{f} N$ be a morphism of DG-R-modules. Then f is called a *quasi-isomorphism* if $H_*(f)$ is an isomorphism. By [19, Proposition 1.4.2], the quasi-isomorphisms form a multiplicative system (compatible with the triangulation) in $K(R)$. ◇

Definition 1.9.20. Let Q be the multiplicative system of quasi-isomorphisms in $K(R)$. Then the localisation of $K(R)$ with respect to Q is the *derived category of* R and is denoted $D(R)$. ◇

Remark 1.9.21. The derived category $D(R)$ inherits the triangulated structure of $K(R)$. The proof is similar to the one for complexes; a triangle in $D(R)$ is isomorphic to the image of a $K(R)$-triangle after localising. ◇

Remark 1.9.22. If R is a DGA concentrated in degree 0, then all of the theory in this section reduces to the theory of the previous sections associated with chain complexes. ◇

1.10 The category of functors

In this section, let k denote an algebraically closed field.

Definition 1.10.1. Let $\mathscr{R}$ be a category. Define $(\mathscr{R}, \mathsf{Sets})$ to be the category whose objects are contravariant functors from $\mathscr{R}$ to Sets and whose morphisms are, correspondingly, natural transformations. ◇

Lemma 1.10.2. ***[Yoneda].*** *Let $\mathscr{R}$ and $(\mathscr{R}, \mathsf{Sets})$ be as in Definition 1.10.1 and let $X \in \operatorname{obj}\mathscr{R}$. Then the following*

$$\Theta : \operatorname{Hom}_{(\mathscr{R},\mathsf{Sets})}(\mathscr{R}(-, X), F) \to F(X),\ \theta \mapsto \theta_X(\mathrm{id}_X)$$

is a bijective map.

Proof. First, it is clear that the map is well-defined, for if θ is a natural transformation, then $\theta_X(\mathrm{id}_X)$ is in $F(X)$. Now, for each $W \in \mathrm{obj}\,\mathscr{R}$ and $\phi \in \mathscr{R}(W, X)$, there is a diagram

$$\begin{tikzcd}
\mathrm{id}_X \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \theta_X(\mathrm{id}_X) \arrow[ddd, mapsto] \\
& (X,X) \arrow[r, "\theta_X"] \arrow[d, "\phi^*"'] & F(X) \arrow[d, "F\phi"] & \\
& (W,X) \arrow[r, "\theta_W"'] & F(W) & \\
\phi^*(\mathrm{id}_X) \arrow[rrr, mapsto] & & & (\dagger)
\end{tikzcd}$$

where on the one hand, $(\dagger) = (F\phi)\theta_X(\mathrm{id}_X)$, and on the other,

$$(\dagger) = \theta_W \phi^*(\mathrm{id}_X) = \theta_W(\mathrm{id}_X \circ \phi) = \theta_W(\phi).$$

Because the diagram is commutative, we have that $(F\phi)\theta_X(\mathrm{id}_X) = \theta_W(\phi)$. Now consider any other natural transformation $\theta' : (-, X) \to F$. Then we have that $\theta'_W(\phi) = (F\phi)\theta'_X(\mathrm{id}_X)$ for all $W \in \mathrm{obj}\,\mathscr{R}$. Hence, if $\Theta(\theta) = \Theta(\theta')$, then $\theta = \theta'$. That is, Θ is injective. To see that Θ is surjective, take $x \in F(X)$ and define a natural transformation $\kappa : \mathscr{R}(-, X) \to F$ by $\kappa_W(\phi) = (F\phi)(x)$ for $W \in \mathrm{obj}\,\mathscr{R}$ and $\phi \in \mathscr{R}(W, X)$ and note that, indeed, $(F\phi)(x) \in F(W)$ as required. Now, κ is a natural transformation, because F is a functor. Now $\Theta(\kappa) = \kappa_X(\mathrm{id}_x) = F(\mathrm{id}_X)(x) = x$, so Θ is bijective as required. □

Definition 1.10.3. Let $\mathscr{C}$ be a category with hom-spaces being k-vector spaces and composition being k-linear. Then $\mathsf{Mod}\,\mathscr{C}$ is defined to be the category of k-linear, contravariant functors from $\mathscr{C}$ to Vect_k, with morphisms given by natural transformations. ◇

Lemma 1.10.4. ***[Yoneda (k-linear version)].*** *Let k be a field and let $\mathscr{R}$ be a k-linear category (see Definition 1.2.37), $X \in \mathrm{obj}\,\mathscr{R}$ and let $\mathsf{Mod}\,\mathscr{R}$ be the category of k-linear, additive contravariant functors from $\mathscr{R}$ to Vect_k. Then the following is a k-linear bijective map.*

$$\Theta : \mathrm{Hom}_{\mathsf{Mod}\,\mathscr{R}}(\mathscr{R}(-, X), F) \to F(X),\ \theta \mapsto \theta_X(\mathrm{id}_X)$$

Proof. See [3, item c) on page 185]. □

Definition 1.10.5. Let $\mathscr{R}$ be an additive category. Then $\mathscr{R}$ is said to be *additively generated* by $\mathscr{R}$-objects $R_1, R_2, \ldots, R_n$, and is denoted $\mathrm{add}(R_1, R_2, \ldots, R_n)$, if all objects of $\mathscr{R}$ are direct summands of direct sums of finitely many copies of the R_i. If $\mathscr{R}$ is additively generated by only one $\mathscr{R}$-object, R, then $\mathscr{R}$ is said to have *additive generator* R. ◇

Definition 1.10.6. Let $\mathscr{R}$ be a k-linear category with additive generator R. Let $B = \mathscr{R}(R, R)$ be the endomorphism algebra of R. Define $\mathsf{Mod}\,\mathscr{R}$ as the category of contravariant functors from $\mathscr{R}$ to Vect_k and define $\mathsf{Mod}\,B^{\mathsf{op}}$ to be the category of left-B^{op}-modules (or equivalently, the category of right-B-modules). $\diamond$

Theorem 1.10.7. *There is a functor*

$$\Psi : \mathsf{Mod}\,\mathscr{R} \to \mathsf{Mod}\,B^{\mathsf{op}},\ F \mapsto F(R).$$

Proof. By definition, $F(R)$ is a k-vector space. Let $m \in F(R)$ and let $\{\phi : R \to R\} \in B$. Then $F(R)$ carries a right-B-module structure via setting $m \cdot \phi$ equal to $(F(\phi))(m)$, that is, the image of m under the linear transformation $F(\phi)$. Thus, to each object in $\mathsf{Mod}\,\mathscr{R}$, there is an associated object in $\mathsf{Mod}\,B^{\mathsf{op}}$. Let $\eta : F \to G$ be a morphism in $\mathsf{Mod}\,\mathscr{R}$, that is, a natural transformation, and let $\phi \in \mathscr{R}(R, R)$. By naturality of η, we have $(\eta_R \circ F(\phi))(m) = (G(\phi) \circ \eta_R)(m)$, which gives $\eta_R(m \cdot \phi) = \eta_R((F(\phi))(m)) = (G(\phi))(\eta_R(m)) = \eta_R(m) \cdot \phi$, so natural transformations indeed map to module homomorphisms under Ψ, with $\Psi(\eta)$ given by η_R. Clearly, then, Ψ is a functor, for $\mathrm{id}_{F(R)} = \Psi(\mathrm{id}_F) = \mathrm{id}_{\Psi(F)} = \mathrm{id}_{F(R)}$, and $\Psi(\xi \circ \eta) = (\xi \circ \eta)_R = \xi_R \circ \eta_R = \Psi(\xi) \circ \Psi(\eta)$. $\square$

Theorem 1.10.8. *The functor*

$$\Psi : \mathsf{Mod}\,\mathscr{R} \to \mathsf{Mod}\,B^{op},\ F \mapsto F(R)$$

is an equivalence of categories.

Proof. See [3, Proposition 2.7 c)]. $\square$

Remark 1.10.9. The category $\mathsf{Mod}\,\mathscr{R}$ is abelian, in general. Kernels and cokernels are given objectwise. For example, if $F \xrightarrow{\varphi} G$, then the kernel is given by $K \to F \xrightarrow{\varphi} G$ with $K(X) = \mathrm{Ker}(F(X) \xrightarrow{\varphi_X} G(X))$, which makes sense because in the situation of Theorem 1.10.8, $\mathsf{Mod}\,B^{\mathsf{op}}$ is abelian. $\diamond$

1.11 Auslander-Reiten triangles

Definition 1.11.1. Let R be a ring. Then R is said to be a *local ring* if, given a sum $r_1 + r_2 = s$ in R, if s is invertible, then at least one of r_1 and r_2 is invertible. $\diamond$

Definition 1.11.2. Let $\mathscr{C}$ be an additive category. An *idempotent* is, for a given object X, a morphism $f : X \to X$ such that $f \circ f = f$. An idempotent $f : X \to X$ is called a *split idempotent* if there is a biproduct diagram in $\mathscr{C}$,

$$\begin{tikzcd} X_1 \arrow[r, shift right, "\iota_1"'] & X \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & X_2, \arrow[l, shift left, "\iota_2"] \end{tikzcd}$$

such that $f = \iota_1 \circ \pi_1$. A category is said to *have split idempotents* if every idempotent splits. $\diamond$

Definition 1.11.3. Let $\mathscr{C}$ be an additive category with split idempotents. An object X in $\mathscr{C}$ is said to be *indecomposable* if $X \cong X_1 \oplus X_2$ implies that either X_1 or X_2 is the zero object. If every object in $\mathscr{C}$ is isomorphic to a finite coproduct of objects with local endomorphism ring, then $\mathscr{C}$ is said to be *Krull-Schmidt*. $\diamond$

Remark 1.11.4. In any additive category $\mathscr{C}$, if an object $X \in \operatorname{obj}\mathscr{C}$ has local endomorphism ring, then X is indecomposable. In a Krull-Schmidt category, by [4, Thm. I.3.6], the indecomposable summands of a given object are unique up to isomorphism. $\diamond$

Recall Definition 1.2.35, which encompasses the notion of sections (split monomorphisms) and retractions (split epimorphisms). The following definition is taken from [32, Def. 2.1].

Definition 1.11.5. Let $\mathscr{C}$ be a triangulated category. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be a distinguished triangle in $\mathscr{C}$. Then $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is said to be an *Auslander-Reiten triangle* if

(i) every morphism (which is not a section) $X \to Y'$ factors through f,

(ii) every morphism (which is not a retraction) $Y' \to Z$ factors through g,

(iii) $h \neq 0$. $\diamond$

If $\mathscr{C}$ is Krull-Schmidt, then this definition is equivalent (see [32, Section 2]) to the following definition, found in [17, 3.1].

Definition 1.11.6. Let $\mathscr{C}$ be a triangulated category. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be a distinguished triangle in $\mathscr{C}$. Then $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is said to be an *Auslander-Reiten triangle* if

(i) X and Z are indecomposable,

(ii) every morphism (which is not a retraction) $Y' \to Z$ factors through g,

(iii) $h \neq 0$. $\diamond$

By [17, Prop. 3.5(i)], an Auslander-Reiten triangle is determined by either end term up to isomorphism of triangles.

Definition 1.11.7. Let $\mathscr{C}$ be a category. A morphism $f : X \to Y$ in $\mathscr{C}$ is called *left almost split* in $\mathscr{C}$ if

(i) f is not a section,

(ii) if $g : X \to X'$ in $\mathscr{C}$ is not a section, then there exists $g' : Y \to X'$ such that $g' \circ f = g$.

There is a dual notion of right almost split morphism. ◇

Remark 1.11.8. In an Auslander-Reiten triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, the morphism f is left almost split, and the morphism g is right almost split. ◇

Definition 1.11.9. Let $\mathscr{C}$ be a category. A morphism $f : A \to B$ in $\mathscr{C}$ is said to be *irreducible* if f is neither a section nor a retraction, and satisfies that if $f = g \circ h$ then either g is a retraction or h is a section. That is, f cannot be factorised "non-trivially". ◇

Definition 1.11.10. Let $\mathscr{C}$ be a triangulated category and suppose that $Z \in \operatorname{obj} \mathscr{C}$ fits into an Auslander-Reiten triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$. Then the object X is denoted τZ, where τ is said to be the *Auslander-Reiten translation* of $\mathscr{C}$. Note that by [30, prop. 3.5(i)], τZ is defined up to isomorphism. ◇

Definition 1.11.11. Let $\mathscr{C}$ be a triangulated category. If for each $\mathscr{C}$-object Z with local endomorphism ring, there exists an Auslander-Reiten triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, and for each $\mathscr{C}$-object X with local endomorphism ring, there exists an Auslander-Reiten triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, then $\mathscr{C}$ is said to *have Auslander-Reiten triangles.* ◇

Definition 1.11.12. A *quiver* Q consists of a set V_Q of *vertices* of Q and a set E_Q of ordered pairs of vertices of Q, called *arrows*. An arrow $(v_1, v_2) \in E_Q$ has *source* v_1 and *target* v_2. An example of a quiver is illustrated below. The quiver

$$A_n = 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \xrightarrow{a_{n-1}} n$$

has vertex set $\{1, 2, \ldots, n\}$ and arrow set $\{a_i = (i, i+1) \mid i = 1, 2, \ldots, n-1\}$. ◇

Remark 1.11.13. Technically, the quiver A_n given in the example above is the so-called "equioriented A_n". Forgetting the direction of the arrows yields the "true" Dynkin graph of type A_n. Henceforth, this mild abuse of notation will continue and A_n will refer to the quiver above. ◇

Definition 1.11.14. Let $\mathscr{C}$ be an additive category. The vertex set of the *Auslander-Reiten quiver* of $\mathscr{C}$ is the set of (isomorphism classes of) indecomposable objects X of $\mathscr{C}$. If X and Y are vertices of the Auslander-Reiten quiver of $\mathscr{C}$, then there is an arrow from X to Y if and only if there exists an irreducible morphism $X \to Y$. ◇

Henceforth in this section, k will denote an algebraically closed field and $\mathscr{C}$ will denote a k-linear triangulated category which is Krull-Schmidt and has the property that each of its Hom spaces is finite-dimensional over k. The following lemma follows from [17, Proposition 3.5].

Lemma 1.11.15. *Let $X \to Y \to Z \to \Sigma X$ be an Auslander-Reiten triangle in $\mathscr{C}$. Let $Y \cong \coprod_i Y_i$ be a decomposition of Y into indecomposable objects. Let Y' be any indecomposable object in $\mathscr{C}$. Then the following are equivalent:*

(i) There is a j such that $Y_j \cong Y'$.

(ii) There is an arrow in the Auslander-Reiten quiver of $\mathscr{C}$ from X to Y'.

(iii) There is an arrow in the Auslander-Reiten quiver of $\mathscr{C}$ from Y' to Z. $\diamond$

Remark 1.11.16. The upshot of Lemma 1.11.15 is that if $\mathscr{C}$ has Auslander-Reiten triangles, then knowledge of these provides complete knowledge of the Auslander-Reiten quiver of $\mathscr{C}$. $\diamond$

The following is described in [37, Section I.1].

Definition 1.11.17. A *Serre functor* is an additive automorphism $S : \mathscr{C} \to \mathscr{C}$ which satisfies, for any pair of $\mathscr{C}$-objects A and B, an isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \cong (\mathrm{Hom}_{\mathscr{C}}(B, S(A)))^*$$

which is natural in A and B, where $(-)^*$ is the dual space; that is, $(-)^* = \mathrm{Hom}_k(-, k)$. $\diamond$

The following theorem is [37, Thm. I.2.4].

Theorem 1.11.18. *The following are equivalent:*

(i) There is a Serre functor $S : \mathscr{C} \to \mathscr{C}$.

(ii) The category $\mathscr{C}$ has Auslander-Reiten triangles. $\diamond$

Remark 1.11.19. An Auslander-Reiten triangle in $\mathscr{C}$ is of the form $S\Sigma^{-1}Z \to Y \to Z \to SZ$, where Σ is the suspension functor of $\mathscr{C}$ and S is the Serre functor. This shows that the Auslander-Reiten translation of $\mathscr{C}$ is given by $\tau = S\Sigma^{-1}$; see [37, Section I.2] for details. $\diamond$

1.12 The Cluster Category of Dynkin Type A_n

Definition 1.12.1. Let Q be a quiver. A *path* in Q is a sequence of arrows

$$a_n a_{n-1} a_{n-2} \dots a_3 a_2 a_1$$

such that the source of the arrow a_i is the target of the arrow a_{i-1}. If $p = a_n a_{n-1} \dots a_1$ and $q = b_m b_{m-1} \dots b_1$ are paths such that the source of a_1 is the target of b_m, then *the concatenation* pq of p and q can be defined as $pq = a_n a_{n-1} \dots a_1 b_m b_{m-1} \dots b_1$. $\diamond$

Remark 1.12.2. Let Q be a quiver. It is typical to take for V_Q a set of numbers and for E_Q a set of letters. For example, the quiver

$$A_3 = 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

has vertex set $\{1, 2, 3\}$ and arrow set $\{a, b\}$, where $a = (1, 2)$ and $b = (2, 3)$. The possible paths are e_1, e_2, e_3, a, b and ba, where e_i denotes the so-called *empty path* at vertex i, which has source and target equal to the vertex i. $\diamond$

Henceforth, in this section, let k denote an algebraically closed field.

Definition 1.12.3. Let Q be a quiver. The *path algebra* kQ is the vector space with basis given by all paths in Q. Multiplication is given by linear extension of the rule that $p \cdot q = pq$ if p and q can be concatenated, and if not, then $p \cdot q = 0$. $\diamond$

Definition 1.12.4. Let Q be a quiver. A *representation* V *of* Q is a set

$$\{V_x \mid x \in V_Q\}$$

of finite-dimensional k-vector spaces together with a set

$$\{V_a : V_{s(a)} \to V_{t(a)} \mid a \in E_Q\}$$

of k-linear maps (one for each arrow in the quiver), where $t(a)$ is the target of a and $s(a)$ is the source of a. If V and W are quiver representations of a quiver Q, then a *morphism of quiver representations* $\varphi : V \to W$ is a collection

$$\{\varphi_x : V_x \to W_x \mid x \in V_Q\}$$

of k-linear maps such that for every arrow a in Q, the following square

$$\begin{tikzcd} V_{s(a)} \arrow[r, "V_a"] \arrow[d, "\varphi_{s(a)}"'] & V_{t(a)} \arrow[d, "\varphi_{t(a)}"] \\ W_{s(a)} \arrow[r, "W_a"'] & W_{t(a)} \end{tikzcd}$$

commutes. $\diamond$

It is well-known (see, for example, [2, Theorem III.1.6]) that the category $\mathsf{Mod}(kQ)$ and the category $\mathsf{Rep}_k(Q)$ of quiver representations are equivalent.

Definition 1.12.5. Let Q be a quiver and let V and W be representations of Q. Then W is said to be a *subrepresentation of* V if

(i) for every $x \in V_Q$, it is the case that W_x is a subspace of V_x, and

(ii) for each $a \in E_Q$, the restriction of the linear map V_a to $W_{s(a)}$ is equal to W_a. ◇

Definition 1.12.6. Let Q be a quiver. The *zero representation* of Q has at each vertex the zero vector space and each linear map equal to zero. ◇

Definition 1.12.7. Let Q be a quiver. Suppose that V is a nonzero representation. Then V is called *simple* if the only subrepresentations of V are V and 0. ◇

Definition 1.12.8. Let Q be a quiver and let V and W be two representations of Q. Then the *direct sum representation* $V \oplus W$ is defined by

$$(V \oplus W)_x = V_x \oplus W_x$$

for every x in V_Q and for every arrow a in E_Q, the linear map

$$(V \oplus W)_a : (V \oplus W)_{s(a)} \to (V \oplus W)_{t(a)}$$

is given by the matrix

$$\begin{pmatrix} V_a & 0 \\ 0 & W_a \end{pmatrix}.$$

A representation U of Q is called *decomposable* if U is isomorphic to a direct sum $V \oplus W$ with V and W each nonzero representations. The representation U is called *indecomposable* if, whenever it is written as a direct sum $V \oplus W$, either V or W is the zero representation. ◇

Remark 1.12.9. By [2, Cor. I.4.8 - Thm. I.4.10], the category $\mathsf{mod}(A)$ is Krull-Schmidt when A is a finite-dimensional algebra. ◇

Henceforth, Q will be a quiver with no loops or cycles. Let $A = kQ$ be the path algebra over Q and note that A is itself an A-module and that A is a finite-dimensional algebra.

Definition 1.12.10. Define $P_x = Ae_x \subset A$ for each $x \in V_Q$. Note that P_x is an A-module for every $x \in V_Q$. ◇

Lemma 1.12.11. *Let V be a representation of Q. Then* $\mathrm{Hom}_{kQ}(P_x, V) \cong V_x$.

Proof. Let $f : \mathrm{Hom}_{kQ}(P_x, V) \to V_x$ be given by $f(\varphi) = \varphi(e_x)$. By [2, Theorem III.2.11], this is an isomorphism. □

Proposition 1.12.12. *Up to isomorphism, the indecomposable projective modules over kQ are the P_x for $x \in V_Q$.*

Proof. This follows from [2, Theorem III.2.12]. $\square$

Recall the quiver introduced in Definition 1.11.12, the so-called "equioriented" A_n. In an abuse of notation, representations and modules will be thought of and referred to interchangeably. The following exemplifies this.

Example 1.12.13. Consider the category $\mathsf{mod}(kA_3)$. The indecomposable representations are $P_1 = k \to k \to k = I_3$, $P_2 = 0 \to k \to k$, $P_3 = 0 \to 0 \to k = S_3$, $I_1 = k \to 0 \to 0 = S_1$, $I_2 = k \to k \to 0$, and $S_2 = 0 \to k \to 0$, where S_1, S_2, S_3 are the simple indecomposable modules, P_1, P_2, P_3 are the projective indecomposable modules, and I_1, I_2, I_3 are the injective indecomposable modules. They are organised as follows in the Auslander-Reiten quiver of $\mathsf{mod}(kA_3)$.

$$\begin{tikzcd}
 & & P_1 = I_3 \arrow[dr] & & \\
 & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[dr] & \\
P_3 = S_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 = S_1
\end{tikzcd}$$

$\diamond$

In the preceding example, the S_i, P_i and I_i are given as representations, and thus *correspond* to modules and are not, themselves, modules.

The category $\mathsf{mod}(kA_n)$ is an abelian category, and therefore it has a derived category, see [19, Section I.4]. Consider the *bounded* derived category $D^b(\mathsf{mod}(kA_n))$. By [18, Section 1.1], $D^b(\mathsf{mod}(kA_n))$ is Krull-Schmidt, and by [18, the corollary on page 646], $D^b(\mathsf{mod}(kA_n))$ has Auslander-Reiten triangles.

Definition 1.12.14. Let Q be a quiver with no loops or cycles. The category $D^b(\mathsf{mod}(kQ))$ has a suspension functor Σ and a Serre functor S by [37, Theorem A], and Auslander-Reiten translation is, by definition, given by $\tau = S\Sigma^{-1}$. Define the *cluster category of Dynkin type* Q as

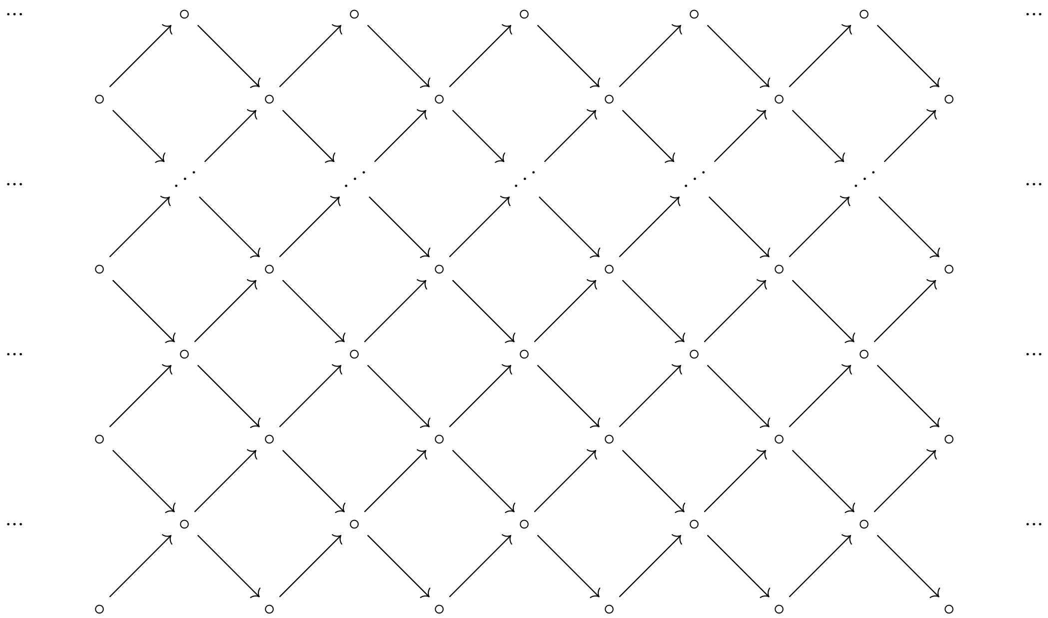
$$\mathsf{C}(Q) = \frac{D^b(\mathsf{mod}(kQ))}{\Sigma \circ \tau^{-1}};$$

i.e., the orbit category of the bounded derived category $D^b(\mathsf{mod}(kQ))$ with respect to the functor $\Sigma \circ \tau^{-1}$. This category has the same objects as $D^b(\mathsf{mod}(kQ))$ and its hom-spaces are given by

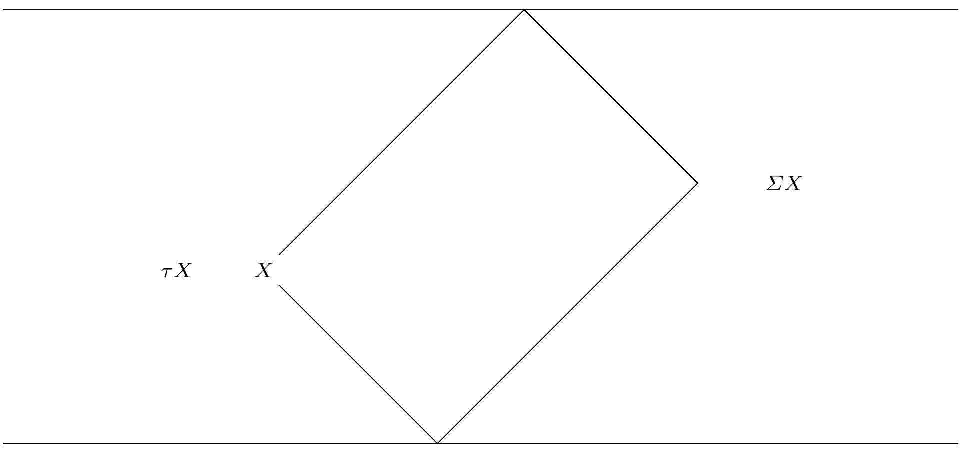
$$\big(D^b(\mathsf{mod}(kQ))/\Sigma \circ \tau^{-1}\big)(X, Y) = \coprod_{n \in \mathbb{Z}} D^b(\mathsf{mod}(kQ))\big(X, (\Sigma \circ \tau^{-1})^n Y\big),$$

see [28, Section 3]. $\diamond$

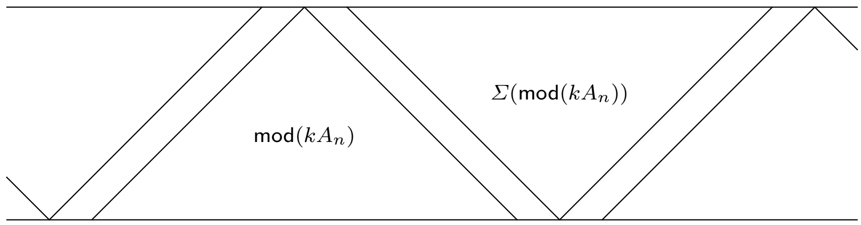
Remark 1.12.15. In [17, Corollary 4.5 part (i)], it is shown that the Auslander-Reiten quiver of $D^b(\mathsf{mod}(kA_n))$ is



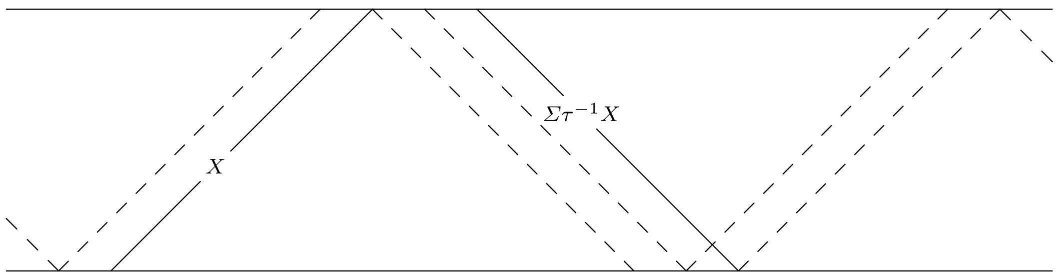
with n vertices in each diagonal slice. Let X be an indecomposable object represented by the vertex illustrated below.



The rectangle illustrates which indecomposables X has nonzero morphisms to. The action of τ is to shift X one vertex to the left, and the position of ΣX is illustrated similarly. A copy of the Auslander-Reiten quiver of $\mathsf{mod}(kA_n)$ is embedded in this quiver, as the following illustrates.



The objects of $\mathsf{C}(A_n)$ are the same as in $D^b(\mathsf{mod}(kA_n))$ (but there are more isomorphisms). The identification $X \sim \Sigma \circ \tau^{-1}(X)$ means that the following "slices" (illustrated below with solid lines) of the Auslander-Reiten quiver of $D^b(\mathsf{mod}(kA_n))$ become identified, with opposite orientation, in $\mathsf{C}(A_n)$:



This identification means that the Auslander-Reiten quiver of $\mathsf{C}(A_n)$ is finite, with a "Möbius twist". $\diamond$

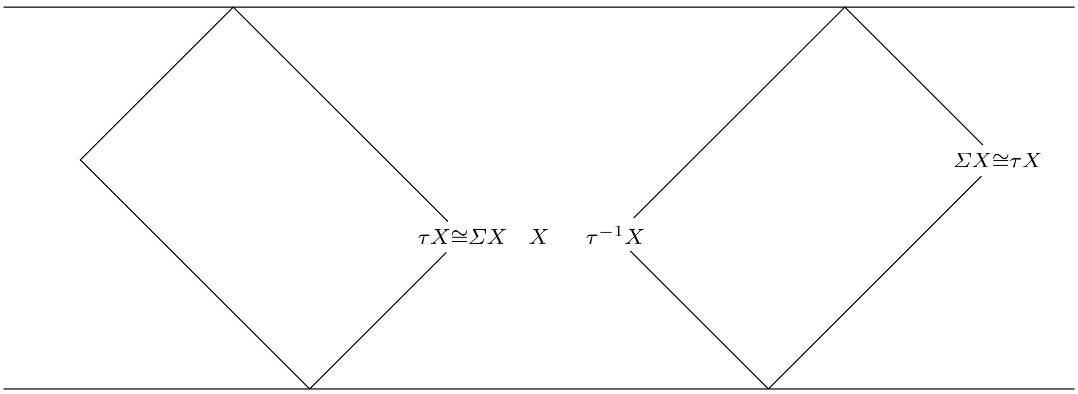
Definition 1.12.16. Let $\mathscr{C}$ be a triangulated category with suspension functor Σ, and let $\mathscr{C}$ have Auslander-Reiten triangles. If the Serre functor S on $\mathscr{C}$ is naturally equivalent to Σ^2, then $\mathscr{C}$ is said to be *2-Calabi-Yau*. $\diamond$

Lemma 1.12.17. *In* $\mathsf{C}(A_n)$, *the Serre functor* S *is naturally equivalent to* Σ^2 *(that is,* $\mathsf{C}(A_n)$ *is 2-Calabi-Yau), and hence* $\mathrm{Ext}^1(x,y) \cong (\mathrm{Ext}^1(y,x))^*$. *In particular, in this situation,* $\tau = \Sigma$.

Proof. See [9, Proposition 1.7 (b)]. $\square$

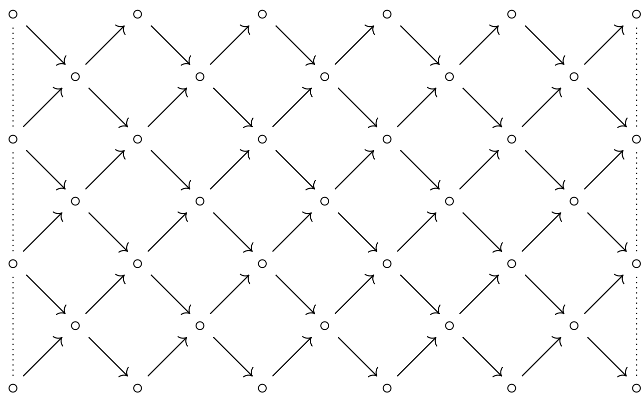
Remark 1.12.18. Suppose that $\mathrm{Ext}^1(x,y) \neq 0$. Then this implies that $\mathrm{Hom}_{\mathscr{C}}(x, \Sigma y) \neq 0 \iff \mathrm{Hom}_{\mathscr{C}}(x, \tau y) \neq 0$ which happens if and only if the vertex associated with y in the

Auslander-Reiten quiver of $\mathsf{C}(A_n)$ is located in one of the following rectangles.

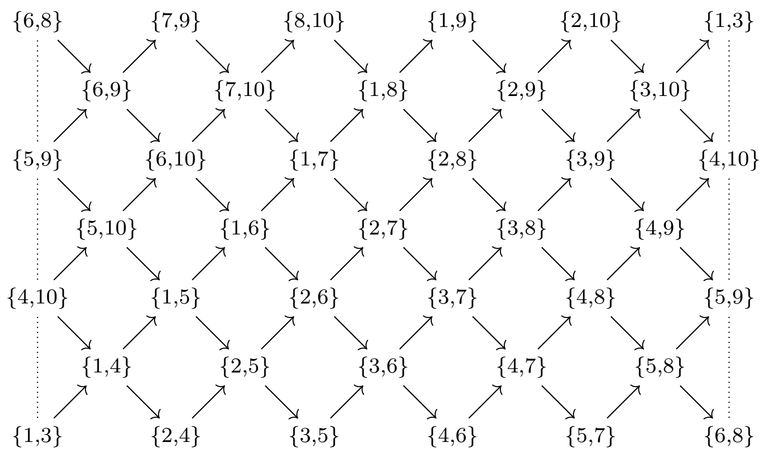


Indeed, the rectangles illustrated above are actually the same rectangle, as they are identified in $\mathsf{C}(A_n)$, but they have been illustrated like this for emphasis. $\diamond$

Remark 1.12.19. The Auslander-Reiten quiver of $\mathsf{C}(A_7)$ is illustrated below.



It has a coordinate system on it, detailed below.



In the general case of $\mathsf{C}(A_n)$, the coordinates are taken modulo $n+3$ and we have the following picture.

$$
\begin{tikzcd}[column sep=small]
{} \arrow[ddd, dotted, no head] & & & \{1,n{+}2\} \arrow[dr] & & \{2,n{+}3\} \arrow[dr] & & \{3,1\} \arrow[ddd, dotted, no head] \\
& & \iddots \arrow[ur] \arrow[dr] & & \iddots \arrow[ur] \arrow[dr] & & \iddots \arrow[ur] & \\
& \{1,4\} \arrow[ur] \arrow[dr] & & \{2,5\} \arrow[ur] \arrow[dr] & & \{3,6\} \arrow[ur] & & \\
\{1,3\} \arrow[ur] & & \{2,4\} \arrow[ur] & & \{3,5\} \arrow[ur] & & & {}
\end{tikzcd}
$$

The coordinate system used above in the Auslander-Reiten quiver has a very nice property. If we consider the $(n+3)$-gon P and number the vertices $1, 2, \ldots, n+3$, then a coordinate pair $\{i, j\}$, and hence an indecomposable object a, can be identified with a diagonal $\mathfrak{a}$ of P. If a and b are indecomposable objects with corresponding diagonals $\mathfrak{a}$ and $\mathfrak{b}$, then

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathscr{C}}(a, b) = \begin{cases} 1 & \text{if } \mathfrak{a} \text{ and } \mathfrak{b} \text{ cross,} \\ 0 & \text{if not.} \end{cases}$$

See, e.g. [11, Lemma 2.1, Thm. 4.4]. $\diamond$

Chapter 2

On the Enlargement of the Cluster Category of Type A_∞

2.1 Introduction

In this chapter, and in [21], k is an algebraically closed field and $R = k[T]$ is viewed as a DG algebra with zero differential, with T placed in cohomological degree -1. In [21], the category $\mathscr{D} = D^f(R)$, the derived category of (right) DG R-modules with finite-dimensional cohomology over k, is investigated.

It is a k-linear, Hom-finite, Krull-Schmidt, 2-Calabi-Yau triangulated category of algebraic origin. The indecomposables of $\mathscr{D}$ are the objects $\Sigma^i X_n$ for $i \in \mathbb{Z}$, $n \in \mathbb{N}_0$ where X_n is obtained via a distinguished triangle

$$\Sigma^{n+1} R \xrightarrow{\cdot T^{n+1}} R \longrightarrow X_n,$$

see, for example, [21, Section 1]. The Auslander-Reiten quiver of $\mathscr{D}$ is $\mathbb{Z}A_\infty$.

It was shown in [21] that one can think of $\mathscr{D}$ as a cluster category of type A_∞. In particular, $\mathscr{D}$ has a geometric model in terms of "arcs" between non-neighbouring integers on the number line, where a pair of integers (a, b) (with $a \leq b - 2$) corresponds to an indecomposable object, and the crossing of arcs corresponds to the non-vanishing of the Ext-space between the corresponding indecomposables.

It is in fact the sets of arcs which do not cross, which give rise to nicer constructions still: a maximal, non-crossing set of arcs is viewed as a triangulation of the ∞-gon, and the objects in the additive hull of the corresponding indecomposables will form a weakly cluster tilting subcategory. If the arcs are configured in a certain way it will turn out that the subcategory formed will be cluster tilting, i.e. it is also functorially finite.

Slices in $\mathbb{Z}A_\infty$ give rises to homotopy colimits - so-called "Prüfer objects", see, for example, [8, Introduction]. The purpose of this chapter is to study the category $\overline{\mathscr{D}}$, where these Prüfer objects have been adjoined to $\mathscr{D}$. Accordingly, we show that $\overline{\mathscr{D}}$ is a k-linear, Hom-finite, Krull-Schmidt, triangulated category and we compute homs, extend the geometric model with arcs to infinity (where the crossing of arcs still corresponds to the non-vanishing of Ext-spaces), and the chapter concludes by finding the cluster tilting subcategories of $\overline{\mathscr{D}}$.

Note that apart from [8], there are some other sources in the literature which study various incarnations of Prüfer objects, see for instance [1, Sec. 1] and [5, Sec. 3.3].

2.2 Introducing the category $\overline{\mathscr{D}}$

In this section we introduce the category $\overline{\mathscr{D}}$. We aim to define $\overline{\mathscr{D}}$ and conclude with proving it is triangulated.

Definition 2.2.1. The Auslander-Reiten quiver (see 1.11.14) of $\mathscr{D}$ has a vertex for each indecomposable object $\Sigma^i X_n$ and an arrow between vertices if there is an irreducible morphism between the corresponding objects. The Auslander-Reiten quiver of $\mathscr{D}$ is illustrated below, see [21, Remark 1.4].

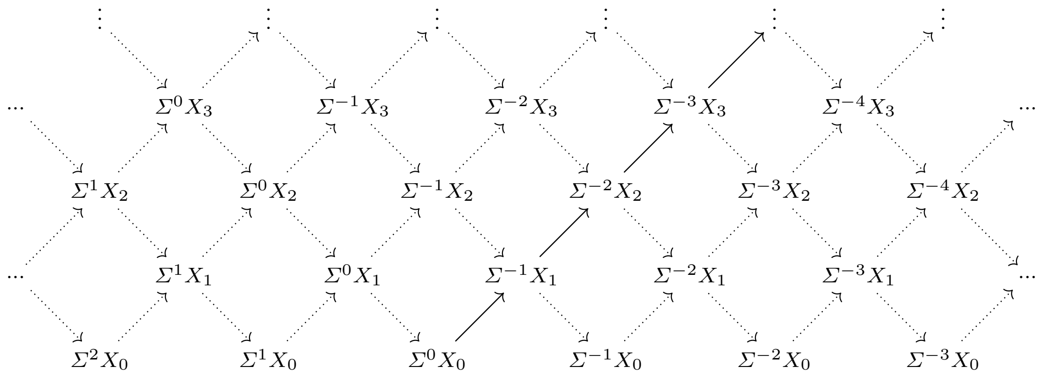
$$
\begin{tikzcd}[column sep=small]
 & \vdots \arrow[dr] & & \vdots \arrow[dr] & & \vdots \arrow[dr] & & \vdots \arrow[dr] & & \vdots \arrow[dr] & & \vdots & \\
\cdots \arrow[dr] & & \Sigma^0 X_3 \arrow[ur] \arrow[dr] & & \Sigma^{-1} X_3 \arrow[ur] \arrow[dr] & & \Sigma^{-2} X_3 \arrow[ur] \arrow[dr] & & \Sigma^{-3} X_3 \arrow[ur] \arrow[dr] & & \Sigma^{-4} X_3 \arrow[ur] \arrow[dr] & & \cdots \\
 & \Sigma^1 X_2 \arrow[ur] \arrow[dr] & & \Sigma^0 X_2 \arrow[ur] \arrow[dr] & & \Sigma^{-1} X_2 \arrow[ur] \arrow[dr] & & \Sigma^{-2} X_2 \arrow[ur] \arrow[dr] & & \Sigma^{-3} X_2 \arrow[ur] \arrow[dr] & & \Sigma^{-4} X_2 \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & \Sigma^1 X_1 \arrow[ur] \arrow[dr] & & \Sigma^0 X_1 \arrow[ur] \arrow[dr] & & \Sigma^{-1} X_1 \arrow[ur] \arrow[dr] & & \Sigma^{-2} X_1 \arrow[ur] \arrow[dr] & & \Sigma^{-3} X_1 \arrow[ur] \arrow[dr] & & \cdots \\
 & \Sigma^2 X_0 \arrow[ur] & & \Sigma^1 X_0 \arrow[ur] & & \Sigma^0 X_0 \arrow[ur] & & \Sigma^{-1} X_0 \arrow[ur] & & \Sigma^{-2} X_0 \arrow[ur] & & \Sigma^{-3} X_0 \arrow[ur] &
\end{tikzcd}
$$

$\diamond$

Definition 2.2.2. A *slice* in the Auslander-Reiten quiver of $\mathscr{D}$ is a collection of vertices and arrows associated with the direct system of irreducible morphisms

$$\Sigma^n X_0 \to \Sigma^{n-1} X_1 \to \cdots \to \Sigma^{n-i} X_i \to \cdots .$$

Note that by [21, Proposition 2.2] the hom-space between two neighbouring objects in the system is 1-dimensional, so the system is determined up to isomorphism. The integer n tells us where along the baseline the slice starts. For example, for $n = 0$, the corresponding slice is illustrated below.



The object $\Sigma^0 X_0$ is the start of the slice, and is located on the baseline. The associated direct system is

$$\Sigma^0 X_0 \to \Sigma^{-1} X_1 \to \cdots \to \Sigma^{-i} X_i \to \cdots .$$

$\diamond$

The homotopy colimit, or hocolimit for short, of such direct systems is defined on page 209 of [7]. The definition is reproduced here, in stages.

Definition 2.2.3. Let

$$y_i \xrightarrow{\ \upsilon_i\ } y_{i+1} \xrightarrow{\ \upsilon_{i+1}\ } y_{i+2} \xrightarrow{\ \upsilon_{i+2}\ } \cdots$$

be a direct system of finite indecomposable objects in $\mathscr{D}$. Then we define the map

$$\coprod_{k=i}^{\infty} y_k \xrightarrow{\ \mathrm{id-shift}\ } \coprod_{k=i}^{\infty} y_k$$

by

$$\mathrm{id} - \mathrm{shift} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ -\upsilon_i & 1 & 0 & \cdots \\ 0 & -\upsilon_{i+1} & 1 & \cdots \\ 0 & 0 & -\upsilon_{i+2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that the map "id − shift" is dependent on the morphisms in its associated direct system, which is usually made clear by context. $\diamond$

Definition 2.2.4. Consider the direct system

$$y_i \xrightarrow{\ \upsilon_i\ } y_{i+1} \xrightarrow{\ \upsilon_{i+1}\ } y_{i+2} \xrightarrow{\ \upsilon_{i+2}\ } \cdots$$

and its associated "id − shift" map

$$\coprod_{k=i}^{\infty} y_k \xrightarrow{\text{id}-\text{shift}} \coprod_{k=i}^{\infty} y_k \ .$$

Then the *hocolimit* of this direct system is defined as the mapping cone of the "id − shift" map, which can be found by completing

$$\coprod_{k=i}^{\infty} y_k \xrightarrow{\text{id}-\text{shift}} \coprod_{k=i}^{\infty} y_k \overset{\Phi}{\dashrightarrow} \operatorname{hocolim} y_i \longrightarrow \tag{2.1}$$

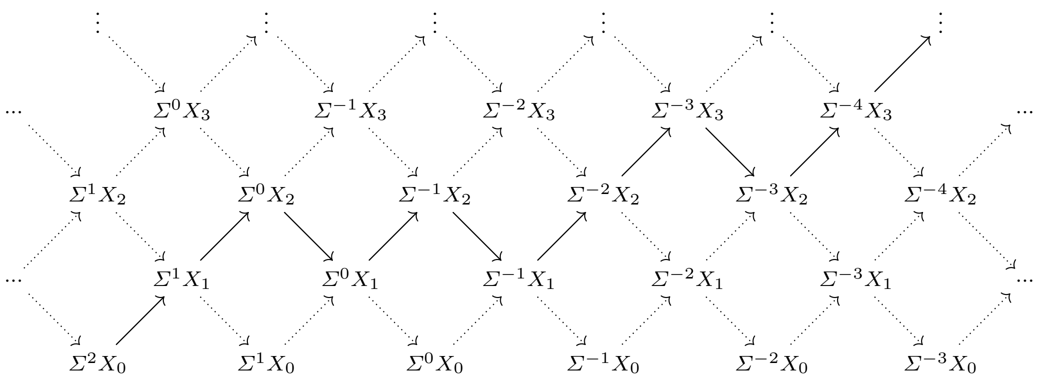
to a distinguished triangle. ◇

Notation 2.2.5. We will sometimes write $\operatorname{hocolim}_i(\Sigma^{n-i}X_i)$ as E_n. Note that E_n is in the derived category $D(R)$ but not in $\mathscr{D} = D^f(R)$. ◇

Remark 2.2.6. The only hocolimits which can be built from the indecomposables of $\mathscr{D}$ are the hocolimits of direct systems of the form

$$\Sigma^n X_0 \xrightarrow{\xi_0} \Sigma^{n-1}X_1 \xrightarrow{\xi_1} \Sigma^{n-2}X_2 \longrightarrow \cdots \ .$$

This is because the only other way to obtain a direct system in the quiver would be to move up (and down) in the quiver infinitely often, like in a zig-zag. This is illustrated below with an example.



Notice that if we stop moving down in the quiver, we necessarily end up on one of the direct systems of the form

$$\Sigma^n X_0 \xrightarrow{\xi_0} \Sigma^{n-1}X_1 \xrightarrow{\xi_1} \Sigma^{n-2}X_2 \longrightarrow \cdots$$

and the zig-zagging at the beginning has no influence on its hocolimit. If we have a direct system where zig-zagging occurs infinitely often, then the results of this chapter will show that the objects in the direct system will not have any nonzero maps to the hocolimit. This means that the hocolimit of such a direct system is zero. $\diamond$

Definition 2.2.7. We will let $\overline{\mathscr{D}}$ be the category "built" from $\mathscr{D}$ and all hocolimits of direct systems of the form

$$\Sigma^n X_0 \xrightarrow{\xi_0} \Sigma^{n-1} X_1 \xrightarrow{\xi_1} \Sigma^{n-2} X_2 \longrightarrow \cdots .$$

Formally, we have

$$\overline{\mathscr{D}} = \operatorname{add}\{\Sigma^i X_n, \mathsf{E}_m\}$$

for $n \in \mathbb{N}_0$ and $i, m \in \mathbb{Z}$, where add is taken inside $D(R)$. $\diamond$

Remark 2.2.8. Let E_n be a hocolimit of the form in Notation 2.2.5. Then $\Sigma^t \mathsf{E}_n = \mathsf{E}_{n+t}$, because it is obtained by applying the functor Σ^t to the direct system (2.4),

$$\Sigma^n X_0 \xrightarrow{\xi_0} \Sigma^{n-1} X_1 \xrightarrow{\xi_1} \Sigma^{n-2} X_2 \longrightarrow \cdots ,$$

resulting in

$$\Sigma^{n+t} X_0 \xrightarrow{\Sigma^t \xi_0} \Sigma^{n+t-1} X_1 \xrightarrow{\Sigma^t \xi_1} \Sigma^{n+t-2} X_2 \longrightarrow \cdots ,$$

which has hocolimit equal to E_{n+t} in the category $\overline{\mathscr{D}}$. $\diamond$

Before computing morphisms in the category $\overline{\mathscr{D}}$, we first prove that $\overline{\mathscr{D}}$ is triangulated.

Remark 2.2.9. There are two uses in this chapter of the notation $(-)^*$. One use is to denote the dual of a vector space. Another use is to represent a grading. We aim to make it very clear which one is being used by the context. $\diamond$

Remark 2.2.10. There are two uses in this chapter of the notation $(-,-)$. One use is to denote the hom-space between two objects, like in $\operatorname{Hom}(X, Y)$. Another use is to represent an arc between two non-neighbouring integers on the number line, which we will see later on. It is always clear from the context which one is being used. $\diamond$

Remark 2.2.11. There is a graded k-algebra, $A = (R, \Sigma^* R)$. The ith graded piece, A^i, is given by $(R, \Sigma^i R)$, with multiplication given by

$$\cdot : A^i \times A^j \to A^{i+j},\ (\alpha, \alpha') \mapsto \Sigma^j(\alpha) \circ \alpha'.$$

If $X \in D(R)$, then $M = (R, \Sigma^* X)$ is a graded right-A-module. Multiplication is given by $\cdot : M^i \times A^j \to M^{i+j}$, $(m, \alpha') \mapsto \Sigma^j(m) \circ \alpha'$. $\diamond$

Definition 2.2.12. Let Λ be a graded k-algebra. Then $\operatorname{Gr}(\Lambda)$ denotes the category of graded right-modules over Λ. $\diamond$

Remark 2.2.13. There is a functor

$$F(-) = (R, \Sigma^*(-)) : D(R) \to \mathrm{Gr}((R, \Sigma^* R)),\ X \mapsto (R, \Sigma^* X)$$

from $D(R)$ to the category of graded right-modules over the graded k-algebra $(R, \Sigma^* R)$. We know that $\mathrm{Hom}_{D(R)}(R, —) \cong H^0(—)$ and this gives the first of the following isomorphisms:

$$(R, \Sigma^* R) \cong H^*(R) \cong k[T].$$

The second holds since R is just $k[T]$ equipped with the zero differential. Hence we have $\mathrm{Gr}((R, \Sigma^* R)) \cong \mathrm{Gr}(k[T])$. $\diamond$

It will be convenient to abuse notation slightly and sometimes think of the functor F being a functor

$$F(-) : D(R) \to \mathsf{Gr}(k[T]),$$

due to the equivalence of categories $\mathrm{Gr}((R, \Sigma^* R)) \cong \mathrm{Gr}(k[T])$.

Lemma 2.2.14. *The functor F is homological; that is, if $X \to Y \to Z \to \Sigma X$ is a distinguished triangle in $D(R)$, then $F(X) \to F(Y) \to F(Z) \to F(\Sigma X)$ is an exact sequence in* $\mathrm{Gr}(k[T])$.

Proof. Apply F to the distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ to obtain

$$(R, \Sigma^* X) \xrightarrow{(R,\Sigma^* f)} (R, \Sigma^* Y) \xrightarrow{(R,\Sigma^* g)} (R, \Sigma^* Z) \xrightarrow{(R,\Sigma^* h)} (R, \Sigma^{*+1} X). \quad (2.2)$$

In degree i, this is

$$(R, \Sigma^i X) \xrightarrow{(R,\Sigma^i f)} (R, \Sigma^i Y) \xrightarrow{(R,\Sigma^i g)} (R, \Sigma^i Z) \xrightarrow{(R,\Sigma^i h)} (R, \Sigma^{i+1} X).$$

This sequence is exact, because $\Sigma^i X \to \Sigma^i Y \to \Sigma^i Z \to \Sigma^{i+1} X$ is again a distinguished triangle, by [TR2] of Definition 1.4.1, and $\mathrm{Hom}_{D(R)}(R, —)$ is homological, by Theorem 1.4.10. Therefore, the sequence (2.2) is an exact sequence, because it is exact at every degree. $\square$

If M is a graded module then ΣM denotes the shift, so $(\Sigma M)^i = M^{i+1}$.

Note that $F\Sigma(-) = (R, \Sigma^* \Sigma(-))$ can be identified naturally with $\Sigma F(-)$.

Lemma 2.2.15. *If $\mathscr{M} \subseteq \mathrm{Gr}(k[T])$ is a full subcategory closed under extensions, kernels, and cokernels, and $\Sigma \mathscr{M} = \mathscr{M}$, then the full subcategory*

$$\mathscr{J} = F^{-1}\mathscr{M} = \{X \in D(R) \mid FX \in \mathscr{M}\}$$

is a triangulated subcategory of $D(R)$.

Proof. Let $X \to Y$ be a morphism in $\mathscr{J}$. Then there exists an object Z in $D(R)$ such that $X \to Y \to Z \to \Sigma X$ is a distinguished triangle. Extend this to

$$X \to Y \to Z \to \Sigma X \to \Sigma Y$$

and apply F to obtain the exact sequence

$$FX \to FY \to FZ \to F\Sigma X \to F\Sigma Y.$$

Because $F\Sigma \cong \Sigma F$, this is isomorphic to

$$FX \to FY \to FZ \to \Sigma FX \to \Sigma FY.$$

It is clear that $FX \to FY$ is a morphism in $\mathscr{M}$ and $\Sigma FX \to \Sigma FY$ is a morphism in $\Sigma\mathscr{M}$. But because $\mathscr{M} = \Sigma\mathscr{M}$, both $FX \to FY$ and $\Sigma FX \to \Sigma FY$ are morphisms in $\mathscr{M}$. So let C be the cokernel of the map $FX \to FY$, and K the kernel of the map $\Sigma FX \to \Sigma FY$. Then

$$0 \to C \to FZ \to K \to 0$$

is a short exact sequence in $\mathrm{Gr}(k[T])$. Now, both C and K are in $\mathscr{M}$ because $\mathscr{M}$ is closed under kernels and cokernels. This means that FZ is in $\mathscr{M}$ as well, and hence $Z \in F^{-1}\mathscr{M} = \mathscr{J}$, because $\mathscr{M}$ is closed under extensions.

We also have that $\Sigma^{\pm 1}\mathscr{J} \subseteq \mathscr{J}$ because $F(\Sigma^{\pm 1}\mathscr{J}) = \Sigma^{\pm 1}F(\mathscr{J}) = \Sigma^{\pm 1}\mathscr{M} = \mathscr{M}$. Hence $\mathscr{J}$ is a triangulated subcategory of $D(R)$. □

Definition 2.2.16. If

$$M = \bigoplus_{n \in \mathbb{Z}} M_n$$

is a graded module, then we say that M is *locally finite* if $\dim_k M_n < \infty$ for each $n \in \mathbb{Z}$. ◇

Definition 2.2.17. We define the k-linear dual, $(-)^*$, of a graded $k[T]$-module M by setting $(M^*)_i = (M_{-i})^*$. If M is a graded left-module and $r \in R_j$, then this defines a linear map $(r\cdot) : M_i \to M_{i+j}$ and hence a linear map $(r\cdot)^* : (M_{i+j})^* \to (M_i)^*$ defined to be equal to $\cdot r : (M^*)_{-i-j} \to (M^*)_{-i}$. ◇

Let $\mathrm{gr}(k[T])$ denote the category of finitely generated graded $k[T]$-modules. Let $(\mathrm{gr}(k[T]))^*$ be the category of of k-linear duals of graded modules in $\mathrm{gr}(k[T])$. Note that $(\mathrm{gr}(k[T]))^*$ also consists of graded $k[T]$-modules.

Corollary 2.2.18. *We have that $F^{-1}((\mathrm{gr}(k[T]))^*)$ is a triangulated subcategory of $D(R)$.*

Proof. Let $\mathscr{M} = (\mathrm{gr}(k[T]))^*$ where gr denotes the category of finitely generated modules. We have that $\mathrm{gr}(k[T])$ is closed under subobjects, quotient objects and extensions (and hence, kernels and cokernels). This implies that $\mathscr{M}$ is also closed under subobjects, quotient objects and extensions (and hence, kernels and cokernels). This is because $(-)^*$ is a duality from locally finite modules to itself. Also, $\mathscr{M}$ is a full subcategory of $\mathrm{Gr}(k[T])$ and $\Sigma\mathscr{M} = \mathscr{M}$. Hence $F^{-1}\mathscr{M}$ is a triangulated subcategory of $D(R)$, by Lemma 2.2.15. $\square$

The following is a well-known property of the polynomial algebra in one variable.

Lemma 2.2.19. *We have that*

$$\mathrm{gr}(k[T]) = \mathrm{add}\{\Sigma^i k[T], \Sigma^j k[T]/(T^{n+1}) \mid i, j \in \mathbb{Z}, n \in \mathbb{N}_0\},$$

and hence $\mathscr{M} = \mathrm{add}\{(\Sigma^i k[T])^*, (\Sigma^j k[T]/(T^{n+1}))^* \mid i, j \in \mathbb{Z}, n \in \mathbb{N}_0\}$. $\diamond$

Lemma 2.2.20. *We have the following isomorphisms.*
(1) $F(\Sigma^i X_n) \cong \Sigma^i k[T]/(T^{n+1})$.
(2) $F(\mathsf{E}_m) \cong \Sigma^m k[T^{-1}]$.

Proof.
(1) Because $F\Sigma \cong \Sigma F$, it is enough to show that $F(X_n) \cong k[T]/(T^{n+1})$. By [21, Remark 1.3], we have a distinguished triangle

$$\Sigma^{n+1} R \xrightarrow{\cdot T^{n+1}} R \longrightarrow X_n \tag{2.3}$$

and by Lemma 2.2.14, we obtain the following long exact sequence after applying F.

$$\cdots \longrightarrow F(\Sigma^{n+1} R) \xrightarrow{F(\cdot T^{n+1})} F(R) \longrightarrow F(X_n) \longrightarrow \cdots$$

This becomes

$$\cdots \longrightarrow \Sigma^{n+1} A \xrightarrow{\cdot T^{n+1}} A \longrightarrow F(X_n) \longrightarrow \Sigma^{n+2} A \xrightarrow{\Sigma(\cdot T^{n+1})} \Sigma A \longrightarrow \cdots$$

where $A = (R, \Sigma^* R) \cong k[T]$. Now, $\cdot T^{n+1} : \Sigma^{n+1} A \to A$ is injective, whence $F(X_n) \to \Sigma^{n+2} A$ is the zero map because the sequence is exact and $\Sigma(\cdot T^{n+1}) : \Sigma^{n+2} A \to \Sigma A$ is injective. Therefore, $A \to F(X_n)$ is surjective, and by the first isomorphism theorem, $F(X_n) \cong A/\operatorname{Im}(\cdot T^{n+1}) \cong k[T]/(T^{n+1})$.

(2) Because $F(\mathsf{E}_m) = F(\Sigma^m \mathsf{E}_0) \cong \Sigma^m F(\mathsf{E}_0)$, it is enough to show that $F(\mathsf{E}_0) \cong k[T^{-1}]$. Now,

$$\mathrm{hocolim}(\Sigma^0 X_0 \to \Sigma^{-1} X_1 \to \cdots) =$$

$$\operatorname{hocolim}(R/(T) \xrightarrow{\cdot T} \Sigma^{-1}R/(T^2) \xrightarrow{\cdot T} \cdots) = \mathsf{E}_0.$$

Now, R is a compact object (see Definition 1.4.17) of the derived category $D(R)$, so by Lemma 2.8 of [35], applying the functor F gives

$$F(\mathsf{E}_0) = F(\operatorname{hocolim}(R/(T) \xrightarrow{\cdot T} \Sigma^{-1}R/(T^2) \xrightarrow{\cdot T} \cdots)) =$$

$$\operatorname{colim}(F(R/(T) \xrightarrow{\cdot T} \Sigma^{-1}R/(T^2) \xrightarrow{\cdot T} \cdots)) =$$

$$\operatorname{colim}(A/(T) \xrightarrow{\cdot T} \Sigma^{-1}A/(T^2) \xrightarrow{\cdot T} \cdots) = k[T^{-1}].$$

□

Lemma 2.2.21. *We have the following isomorphisms in* $\operatorname{Gr}(k[T])$.
(A) $(\Sigma^{-i-n}(k[T]/(T^{n+1})))^* \cong \Sigma^i(k[T]/(T^{n+1}))$.
(B) $(\Sigma^{-m}k[T])^* \cong \Sigma^m k[T^{-1}]$.

Proof.
(A) Because $(\Sigma^i M)^* \cong \Sigma^{-i}(M^*)$, it is enough to show $(k[T]/(T^{n+1}))^* \cong \Sigma^{-n}k[T]/(T^{n+1})$. Consider the following component of an A-module homomorphism,

$$\Psi^i : \operatorname{Hom}\Big((k[T]/(T^{n+1}))^{-i}, k\Big) \to \Big(k[T]/(T^{n+1})\Big)^{i-n}$$

defined on $0 \le i \le n$ by $\Psi^i(\alpha) = \alpha(T^i)\cdot T^{n-i}$ and by $\Psi^i(\alpha) = 0$ if i is not in this range. To see that this is indeed compatible with multiplication by A-elements, consider the following diagram.

$$\begin{tikzcd}
\operatorname{Hom}\Big((k[T]/(T^{n+1}))^{-i+1}, k\Big) \arrow[d, "\Psi^{i-1}"] & \operatorname{Hom}\Big((k[T]/(T^{n+1}))^{-i}, k\Big) \arrow[l, "\cdot T"'] \arrow[d, "\Psi^i"] \\
\Big(k[T]/(T^{n+1})\Big)^{i-n-1} & \Big(k[T]/(T^{n+1})\Big)^{i-n} \arrow[l, "\cdot T"]
\end{tikzcd}$$

If $0 \le i-1 < i \le n$ and $\alpha \in \operatorname{Hom}\Big((k[T]/(T^{n+1}))^{-i}, k\Big)$, then diagram-chasing yields

$$T\cdot\Psi^i(\alpha) = T\alpha(T^i)\cdot T^{n-i} = \alpha(T^i)\cdot T^{1+n-i} = \alpha(T^{i-1+1})\cdot T^{1+n-i} =$$

$$(\alpha T)(T^{i-1})\cdot T^{1+n-i} = \Psi^{i-1}(\alpha T).$$

If $i = 0$, then $T\cdot\Psi^0(\alpha) = T\alpha(T^0)\cdot T^n = \alpha(T^0)\cdot T^{n+1} = 0 = \Psi^{-1}(\alpha T)$ as required. Finally at each i we have a k-linear map which is nonzero for $0 \leq i \leq n$ between 1-dimensional k-vector spaces and hence each Ψ^i is an isomorphism of k-vector spaces (and for i outside $0, \ldots, n$, it is just the zero map). This, combined with compatibility with A-multiplication gives the result, namely that $(k[T]/(T^{n+1}))^* \cong \Sigma^{-n} k[T]/(T^{n+1})$.

(B) Because $(\Sigma^i M)^* \cong \Sigma^{-i}(M^*)$, it is enough to show that $(k[T])^* \cong k[T^{-1}]$. Consider the following component of an A-module homomorphism,

$$\Phi^i : \operatorname{Hom}\Big((k[T])^{-i}, k\Big) \to \Big(k[T^{-1}]\Big)^i : \alpha \mapsto \alpha(T^i) \cdot T^{-i}.$$

Again, we check compatibility with multiplication by A-elements. Consider the following diagram.

$$\begin{tikzcd}
\operatorname{Hom}\Big((k[T])^{-i+1}, k\Big) \arrow[d, "\Phi^{i-1}"] & \operatorname{Hom}\Big((k[T])^{-i}, k\Big) \arrow[l, "\cdot T"'] \arrow[d, "\Phi^i"] \\
\Big(k[T^{-1}]\Big)^{i-1} & \Big(k[T^{-1}]\Big)^i \arrow[l, "\cdot T"]
\end{tikzcd}$$

Diagram-chasing is then even simpler than in part (A), since $\alpha \in \operatorname{Hom}\Big((k[T])^{-i}, k\Big)$ can easily be seen to map to $\alpha(T^i) \cdot T^{1-i} \in \Big(k[T^{-1}]\Big)^{i-1}$ after following either direction in the diagram. For similar reasons as in part (A), we have that the collection $(\Phi^i)_{i \in \mathbb{Z}}$ is an isomorphism between $(k[T])^*$ and $k[T^{-1}]$. □

Corollary 2.2.22. *We have that*

$$\mathscr{M} = \operatorname{add}\{F(\Sigma^i X_n), F(\mathsf{E}_m) \mid i, m \in \mathbb{Z}, n \in \mathbb{N}_0\}.$$

Proof. This follows from Lemmas 2.2.19, 2.2.20 and 2.2.21. □

Lemma 2.2.23. *We have that $\overline{\mathscr{D}} = F^{-1}\mathscr{M}$.*

Proof. Corollary 2.2.22 implies

$$F(\overline{\mathscr{D}}) = F(\operatorname{add}\{\Sigma^i X_n, \mathsf{E}_m \mid i, m \in \mathbb{Z}, n \in \mathbb{N}_0\}) \subseteq$$

$$\operatorname{add}\{F(\Sigma^i X_n), F(\mathsf{E}_m) \mid i, m \in \mathbb{Z}, n \in \mathbb{N}_0\} = \mathscr{M}.$$

On the other hand, Lemmas 2.2.19 and 2.2.21 show that each object in $\mathscr{M}$ is a direct sum of finitely many objects from $\{\Sigma^i(k[X]/(X^{n+1})), \Sigma^m k[T^{-1}] \mid i, m \in \mathbb{Z}, n \in \mathbb{N}_0\}$, so

$\mathscr{M} \subseteq F(\mathrm{add}\{\Sigma^i X_n, \mathsf{E}_m \mid i, m \in \mathbb{Z}, n \in \mathbb{N}_0\}) = F(\overline{\mathscr{D}})$ by Lemma 2.2.20. We conclude $F(\overline{\mathscr{D}}) = \mathscr{M}$ which implies $\overline{\mathscr{D}} = F^{-1}\mathscr{M}$ by [29, thm 3.1]. □

Theorem 2.2.24. *The category $\overline{\mathscr{D}}$ is a triangulated subcategory of $D(R)$.*

Proof. By Lemmas 2.2.18 and 2.2.23, the result follows. □

2.3 Morphisms in the category $\overline{\mathscr{D}}$

In this section we compute homs between finite objects (i.e. the X_n and their shifts) and hocolimits (i.e. the E_m and their shifts) in the category $\overline{\mathscr{D}}$ and also homs between hocolimits.

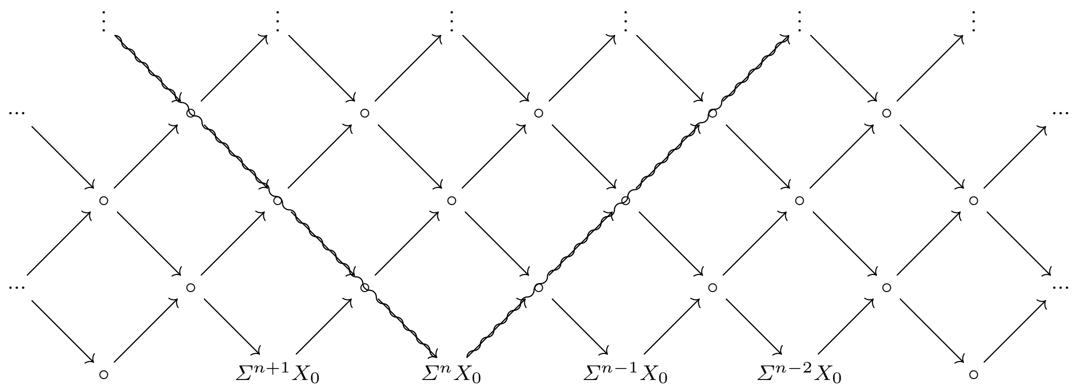
Definition 2.3.1. Let $X \in \mathrm{ind}\,\mathscr{D}$, i.e., X is an indecomposable object of $\mathscr{D}$, which means that X is represented by a vertex in the Auslander-Reiten quiver of $\mathscr{D}$. We write X uniquely as $\Sigma^{n-i}X_i$ for $n \in \mathbb{Z}$, $i \in \mathbb{N}_0$. Then there is a unique slice associated with the direct system

$$\Sigma^n X_0 \to \Sigma^{n-1}X_1 \to \cdots \to \Sigma^{n-i}X_i \to \cdots$$

containing $\Sigma^{n-i}X_i$. We define the *wedge associated with* $\Sigma^{n-i}X_i$ (or *wedge based at* $\Sigma^n X_0$ if $i = 0$) as

$$\mathbb{W}(\Sigma^{n-i}X_i) = \{\Sigma^{n-j}X_k : 0 \leq j \leq k\}.$$

Pictorially, we have the following, where the subset contains the edges.



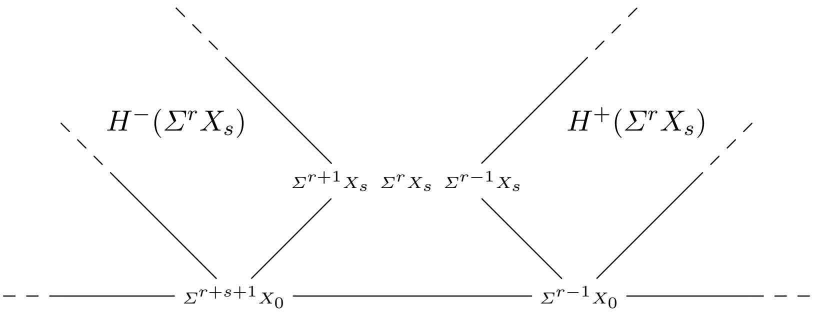
Notice that $\mathbb{W}(\Sigma^{n-i}X_i)$ is independent of the number i. In particular, a general indecomposable object $\Sigma^{n-i}X_i$ has wedge based at $\Sigma^n X_0$. Therefore, we may choose to write $\mathbb{W}(\Sigma^n X_0)$ instead of $\mathbb{W}(\Sigma^{n-i}X_i)$, because they are equal. ◇

Notation 2.3.2. The sets

$$H^-(\Sigma^r X_s) = \{\Sigma^{-n}X_{n-m-2} \mid m \leq -r-s-3, -r-s-1 \leq n \leq -r-1\},$$

$$H^+(\Sigma^r X_s) = \{\Sigma^{-n} X_{n-m-2} \mid -r-s-1 \leq m \leq -r-1, -r+1 \leq n\}$$

and $H(\Sigma^r X_s) = H^-(\Sigma^r X_s) \cup H^+(\Sigma^r X_s)$ were originally defined in [21, Definition 2.1]. These subsets are illustated below.



In this picture, the subsets include the edges. $\diamond$

Lemma 2.3.3. *Let*

$$\Sigma^n X_0 \xrightarrow{\xi_0} \Sigma^{n-1} X_1 \xrightarrow{\xi_1} \Sigma^{n-2} X_2 \longrightarrow \cdots \tag{2.4}$$

be a direct system associated with a slice in the Auslander-Reiten quiver of $\mathscr{D}$, and let Y be any indecomposable object of $\mathscr{D}$. Consider the direct system

$$(Y, \Sigma^n X_0) \xrightarrow{(Y,\xi_0)} (Y, \Sigma^{n-1} X_1) \xrightarrow{(Y,\xi_1)} (Y, \Sigma^{n-2} X_2) \longrightarrow \cdots, \tag{2.5}$$

which is obtained by applying the functor $(Y, -)$ to the direct system (2.4) above. Then the direct system (2.5) is naturally isomorphic to the direct system

$$(\Sigma^n X_0, \Sigma^2 Y)^* \xrightarrow{(\xi_0, \Sigma^2 Y)^*} (\Sigma^{n-1} X_1, \Sigma^2 Y)^* \xrightarrow{(\xi_1, \Sigma^2 Y)^*} (\Sigma^{n-2} X_2, \Sigma^2 Y)^* \longrightarrow \cdots. \tag{2.6}$$

Furthermore,

$$(\Sigma^{n-i} X_i, \Sigma^2 Y)^* \cong \begin{cases} k & \text{if } \Sigma^2 Y \in H(\Sigma^{n-i+1} X_i), \\ 0 & \text{if } \Sigma^2 Y \notin H(\Sigma^{n-i+1} X_i). \end{cases} \tag{2.7}$$

Proof. Consider an arbitrary map $(Y, \Sigma^{n-i} X_i) \xrightarrow{(Y,\xi_i)} (Y, \Sigma^{n-(i+1)} X_{i+1})$ in the direct system (2.5). Then this is a morphism of vector spaces over k, and there is a commutative diagram

$$\begin{array}{ccc} (Y, \Sigma^{n-i} X_i) & \xrightarrow{(Y,\xi_i)} & (Y, \Sigma^{n-(i+1)} X_{i+1}) \\ \downarrow \cong & & \downarrow \cong \\ (Y, \Sigma^{n-i} X_i)^{**} & \xrightarrow{(Y,\xi_i)^{**}} & (Y, \Sigma^{n-(i+1)} X_{i+1})^{**} \end{array} \tag{2.8}$$

where * is the contravariant functor from the category of k-vector spaces to itself, $^* : \mathsf{Vect}_k \to \mathsf{Vect}_k$, which maps a vector space over k to its dual space and maps a linear map to its transpose. We can rewrite the bottom morphism of (2.8) as

$$((Y, \Sigma^{n-i}X_i)^*)^* \; (\xleftarrow{(Y,\xi_i)^*})^* \; ((Y, \Sigma^{n-(i+1)}X_{i+1})^*)^* \; . \tag{2.9}$$

By [21, Remark 1.2], the category $\mathscr{D}$ is 2-Calabi-Yau and hence Serre duality states that $(Y, \Sigma^{n-i}X_i) \cong (\Sigma^{n-i}X_i, \Sigma^2 Y)^*$. Now, functors preserve isomorphisms, so we can apply * to this to obtain $(Y, \Sigma^{n-i}X_i)^* \cong (\Sigma^{n-i}X_i, \Sigma^2 Y)^{**}$, which is naturally isomorphic to $(\Sigma^{n-i}X_i, \Sigma^2 Y)$. Therefore, (2.9) is naturally isomorphic to

$$(\Sigma^{n-i}X_i, \Sigma^2 Y)^* \xrightarrow{(\xi_i, \Sigma^2 Y)^*} (\Sigma^{n-(i+1)}X_{i+1}, \Sigma^2 Y)^* \; . \tag{2.10}$$

Combining (2.10) with (2.8) shows that

$$(Y, \Sigma^{n-i}X_i) \xrightarrow{(Y,\xi_i)} (Y, \Sigma^{n-(i+1)}X_{i+1})$$

is naturally isomorphic to

$$(\Sigma^{n-i}X_i, \Sigma^2 Y)^* \xrightarrow{(\xi_i, \Sigma^2 Y)^*} (\Sigma^{n-(i+1)}X_{i+1}, \Sigma^2 Y)^* \; ,$$

and hence the direct system (2.5) is naturally isomorphic to the direct system (2.6)

$$(\Sigma^n X_0, \Sigma^2 Y)^* \xrightarrow{(\xi_0, \Sigma^2 Y)^*} (\Sigma^{n-1}X_1, \Sigma^2 Y)^* \xrightarrow{(\xi_1, \Sigma^2 Y)^*} (\Sigma^{n-2}X_2, \Sigma^2 Y)^* \longrightarrow \cdots \; ,$$

as required. We now prove claim (2.7) of the lemma. By [21, Proposition 2.2], we have that

$$(\Sigma^{n-i}X_i, \Sigma^2 Y) \cong \begin{cases} k \text{ if } \Sigma^2 Y \in H(\Sigma^{n-i+1}X_i), \\ 0 \text{ if } \Sigma^2 Y \notin H(\Sigma^{n-i+1}X_i). \end{cases}$$

This directly implies that

$$(\Sigma^{n-i}X_i, \Sigma^2 Y)^* \cong \begin{cases} k \text{ if } \Sigma^2 Y \in H(\Sigma^{n-i+1}X_i), \\ 0 \text{ if } \Sigma^2 Y \notin H(\Sigma^{n-i+1}X_i), \end{cases}$$

because a vector space which is finite-dimensional has the same dimension as its dual. $\square$

Lemma 2.3.4. *The colimit of the direct system (2.5),*

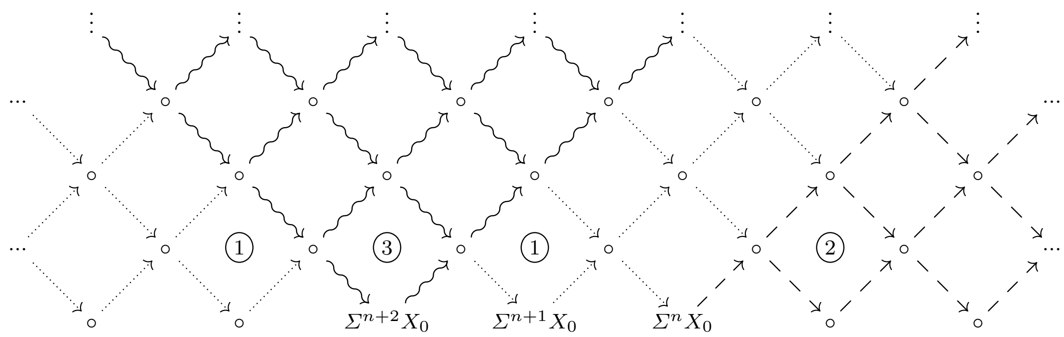
$$(Y, \Sigma^n X_0) \xrightarrow{(Y,\xi_0)} (Y, \Sigma^{n-1}X_1) \xrightarrow{(Y,\xi_1)} (Y, \Sigma^{n-2}X_2) \longrightarrow \cdots \; ,$$

is isomorphic to k *if* $Y \in \mathbb{W}(\Sigma^n X_0)$, *and* 0 *if* $Y \notin \mathbb{W}(\Sigma^n X_0)$.

Proof. Using Lemma 2.3.3, we can rewrite the direct system as

$$(\Sigma^n X_0, \Sigma^2 Y)^* \xrightarrow{(\xi_0, \Sigma^2 Y)^*} (\Sigma^{n-1} X_1, \Sigma^2 Y)^* \xrightarrow{(\xi_1, \Sigma^2 Y)^*} (\Sigma^{n-2} X_2, \Sigma^2 Y)^* \longrightarrow \cdots,$$

from Equation (2.6). Now consider the three numbered regions in the Auslander-Reiten quiver below.



Region 3 is highlighted with wavy arrows and contains all of the vertices on its boundary, so, in particular, equals $\mathbb{W}(\Sigma^{n+2} X_0)$. Region 2 is highlighted with dashed arrows and contains all of the vertices on its boundary. Region 1 contains none of the vertices which have wavy or dashed arrows coming in or going out. Recall the following useful fact from Lemma 2.3.3.

$$(\Sigma^{n-i} X_i, \Sigma^2 Y)^* \cong \begin{cases} k \text{ if } \Sigma^2 Y \in H(\Sigma^{n-i+1} X_i), \\ 0 \text{ if } \Sigma^2 Y \notin H(\Sigma^{n-i+1} X_i). \end{cases}$$

If the indecomposable object $\Sigma^2 Y$ is in region 1, then $\Sigma^2 Y$ is never in $H(\Sigma^{n-i+1} X_i)$, for any $i \in \mathbb{N}_0$. If $\Sigma^2 Y$ is located in region 2, then $\Sigma^2 Y$ will be in $H(\Sigma^{n-i+1} X_i)$ for only finitely many values of i. This is because, once i is large enough, $\Sigma^2 Y$ will be to the left of $H^+(\Sigma^{n-i+1} X_i)$. Therefore, if $\Sigma^2 Y$ is located in regions 1 or 2, the categorical colimit of the direct system (2.6) is trivially zero. What remains is to check what happens when $\Sigma^2 Y$ is in region 3. Note that this means $\Sigma^2 Y \in \mathbb{W}(\Sigma^{n+2} X_0)$, or equivalently, $Y \in \mathbb{W}(\Sigma^n X_0)$.

Suppose $\Sigma^2 Y$ is in region 3. Then there exists $N \in \mathbb{N}_0$ such that whenever $i \geq N$, $\Sigma^2 Y \in H^-(\Sigma^{n-i+1} X_i) \subset H(\Sigma^{n-i+1} X_i)$. Hence, the direct system (2.6) looks like

$$0 \to 0 \to \cdots \to 0 \to k \to k \to \cdots$$

with at most N zeroes. We claim that the colimit of this system is k. Consider one of the maps in the direct system (2.6) between two of the nonzero hom spaces. Denote this map

$$(\Sigma^{n-i}X_i, \Sigma^2 Y)^* \xrightarrow{(\xi_i, \Sigma^2 Y)^*} (\Sigma^{n-(i+1)}X_{i+1}, \Sigma^2 Y)^* \tag{2.11}$$

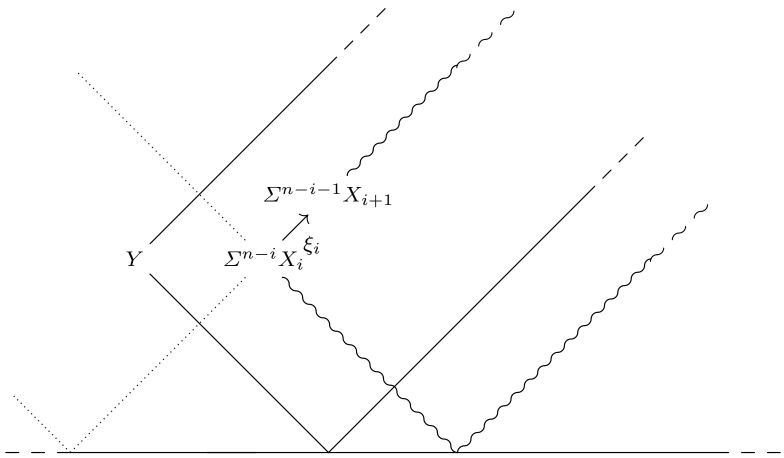
for $i \geq N$, so that $\Sigma^2 Y \in H^-(\Sigma^{n-i+1}X_i)$. It is enough to show that such a map is nonzero, because if that is the case, then it is an isomorphism; this is due to the fact that the hom spaces are one-dimensional. Observe that $\Sigma^2 Y \in H^-(\Sigma^{n-i+1}X_i)$ is equivalent to requiring that

$$Y \in H^-(\Sigma^{n-(i+1)}X_i). \tag{2.12}$$

By Lemma 2.3.3, (2.11) is naturally isomorphic to

$$(Y, \Sigma^{n-i}X_i) \xrightarrow{(Y,\xi_i)} (Y, \Sigma^{n-(i+1)}X_{i+1}). \tag{2.13}$$

Let $\varphi \in (Y, \Sigma^{n-i}X_i)$ be nonzero. Then the image of φ under (Y, ξ_i) is nonzero, by virtue of Lemma 2.5 of [21]. This is because Y, $\Sigma^{n-i}X_i$ and $\Sigma^{n-i-1}X_{i+1}$ are indecomposable objects of $\mathscr{D}$ such that $\Sigma^{n-i}X_i, \Sigma^{n-i-1}X_{i+1} \in H^+(\Sigma Y)$ and $\Sigma^{n-i-1}X_{i+1} \in H^+(\Sigma^{n-i+1}X_i)$, as shown in the following sketch.



The region $H^-(\Sigma^{n-(i+1)}X_i)$ is illustrated by dotted lines; this is where Y is located. Notice that wherever Y is in this region, $H^+(\Sigma Y)$ contains $\xi_i : \Sigma^{n-i}X_i \to \Sigma^{n-i-1}X_{i+1}$. The wavy lines illustrate the region $H^+(\Sigma^{n-i+1}X_i)$. Lemma 2.5 of [21] tells us that in this situation, the composition of nonzero morphisms $Y \to \Sigma^{n-i}X_i$ and $\Sigma^{n-i}X_i \to \Sigma^{n-i-1}X_{i+1}$ is nonzero. We have therefore shown that the morphism (2.11) is nonzero when $\Sigma^2 Y \in \mathbb{W}(\Sigma^{n+2}X_0)$ and $i \geq N$, so that $\Sigma^2 Y \in H^-(\Sigma^{n-i+1}X_i)$ holds. Hence, the morphisms between nonzero hom spaces in the direct system (2.6) are isomorphisms, and the colimit of this direct system is isomorphic to k whenever $\Sigma^2 Y$ is in region 3. Now, the direct system (2.6),

$$(\Sigma^n X_0, \Sigma^2 Y)^* \xrightarrow{(\xi_0,\Sigma^2 Y)^*} (\Sigma^{n-1} X_1, \Sigma^2 Y)^* \xrightarrow{(\xi_1,\Sigma^2 Y)^*} (\Sigma^{n-2} X_2, \Sigma^2 Y)^* \longrightarrow \cdots,$$

and the direct system (2.5),

$$(Y, \Sigma^n X_0) \xrightarrow{(Y,\xi_0)} (Y, \Sigma^{n-1} X_1) \xrightarrow{(Y,\xi_1)} (Y, \Sigma^{n-2} X_2) \longrightarrow \cdots,$$

are naturally isomorphic to one another, and so the colimit of (2.5) is isomorphic to k if $Y \in \mathbb{W}(\Sigma^n X_0)$, and 0 if $Y \notin \mathbb{W}(\Sigma^n X_0)$, as required. □

Proposition 2.3.5. *Consider the direct system*

$$\Sigma^n X_0 \to \Sigma^{n-1} X_1 \to \cdots \to \Sigma^{n-i} X_i \to \cdots$$

associated with a slice in the Auslander-Reiten quiver of $\mathscr{D}$. If $Y \in \operatorname{ind} \mathscr{D}$ is any indecomposable object, then

$$\operatorname{Hom}_{\overline{\mathscr{D}}}(Y, \operatorname{hocolim}_i(\Sigma^{n-i} X_i)) \cong \begin{cases} k & \textit{if } Y \in \mathbb{W}(\Sigma^n X_0), \\ 0 & \textit{if } Y \notin \mathbb{W}(\Sigma^n X_0). \end{cases}$$

Proof. The indecomposable objects of $\mathscr{D}$ are compact in $D(R)$. This is because R and its associated shifts are compact, and the distinguished triangle (2.3) gives that X_n is compact. Hence by [35, Lemma 2.8], we have that

$$(Y, \operatorname{hocolim}_i(\Sigma^{n-i} X_i)) \cong \operatorname{colim}_i(Y, \Sigma^{n-i} X_i). \tag{2.14}$$

This is isomorphic to k if $Y \in \mathbb{W}(\Sigma^n X_0)$, and 0 if $Y \notin \mathbb{W}(\Sigma^n X_0)$ by Lemma 2.3.4. □

Corollary 2.3.6. *Consider the direct system*

$$\Sigma^n X_0 \to \Sigma^{n-1} X_1 \to \cdots \to \Sigma^{n-i} X_i \to \cdots$$

associated with a slice in the Auslander-Reiten quiver of $\mathscr{D}$. If $Y \in \operatorname{ind} \mathscr{D}$ is any indecomposable object, then

$$\operatorname{Hom}_{\overline{\mathscr{D}}}(\operatorname{hocolim}_i(\Sigma^{n-i} X_i), Y) \cong \begin{cases} k & \textit{if } Y \in \mathbb{W}(\Sigma^{n+2} X_0), \\ 0 & \textit{if } Y \notin \mathbb{W}(\Sigma^{n+2} X_0). \end{cases}$$

Proof. There is a short exact sequence

$$0 \to \lim^1_i(\Sigma^{1+n-i} X_i, \Sigma^2 Y) \to (\operatorname{hocolim}_i(\Sigma^{n-i} X_i), \Sigma^2 Y) \to \lim_i(\Sigma^{n-i} X_i, \Sigma^2 Y) \to 0$$

by [27, Lemma 1.13.1], where lim is defined in [33, Section III.4] and $\lim^1$ is the first right-derived functor of lim. Now, for each $i \in \mathbb{N}_0$, the space $(\Sigma^{1+n-i}X_i, \Sigma^2 Y)$ is either isomorphic to 0 or k, and therefore

$$\lim_i^1(\Sigma^{1+n-i}X_i, \Sigma^2 Y) = 0,$$

by Exercise 3.5.2 of [42]. Therefore,

$$(\mathrm{hocolim}_i(\Sigma^{n-i}X_i), \Sigma^2 Y) \cong \lim_i(\Sigma^{n-i}X_i, \Sigma^2 Y). \tag{2.15}$$

Now, apply the functor $(-, \Sigma^2 Y)$ to the direct system

$$\Sigma^n X_0 \xrightarrow{\xi_0} \Sigma^{n-1}X_1 \xrightarrow{\xi_1} \Sigma^{n-2}X_2 \longrightarrow \cdots$$

to obtain the inverse system

$$(\Sigma^n X_0, \Sigma^2 Y) \xleftarrow{(\xi_0,\Sigma^2 Y)} (\Sigma^{n-1}X_1, \Sigma^2 Y) \xleftarrow{(\xi_1,\Sigma^2 Y)} (\Sigma^{n-2}X_2, \Sigma^2 Y) \longleftarrow \cdots . \tag{2.16}$$

This is dual to the direct system (2.6),

$$(\Sigma^n X_0, \Sigma^2 Y)^* \xrightarrow{(\xi_0,\Sigma^2 Y)^*} (\Sigma^{n-1}X_1, \Sigma^2 Y)^* \xrightarrow{(\xi_1,\Sigma^2 Y)^*} (\Sigma^{n-2}X_2, \Sigma^2 Y)^* \longrightarrow \cdots ,$$

which is naturally isomorphic to the direct system (2.5), and so has colimit isomorphic to k when $Y \in \mathbb{W}(\Sigma^n X_0)$, and 0 when $Y \notin \mathbb{W}(\Sigma^n X_0)$, by Lemma 2.3.4. Therefore, the limit of the inverse system (2.16) is isomorphic to k when $Y \in \mathbb{W}(\Sigma^n X_0)$, and 0 when $Y \notin \mathbb{W}(\Sigma^n X_0)$, because $(\mathrm{colim}_i(U_i))^* \cong \lim_i(U_i^*)$. By (2.15), we have that $\lim(\Sigma^{n-i}X_i, \Sigma^2 Y)$ is isomorphic to $(\mathrm{hocolim}_i(\Sigma^{n-i}X_i), \Sigma^2 Y)$, so

$$(\mathrm{hocolim}_i(\Sigma^{n-i}X_i), \Sigma^2 Y) \cong \begin{cases} k \text{ if } Y \in \mathbb{W}(\Sigma^n X_0), \\ 0 \text{ if } Y \notin \mathbb{W}(\Sigma^n X_0). \end{cases}$$

A simple shift of vertices yields the result, namely that

$$(\mathrm{hocolim}_i(\Sigma^{n-i}X_i), Y) \cong \begin{cases} k \text{ if } Y \in \mathbb{W}(\Sigma^{n+2} X_0), \\ 0 \text{ if } Y \notin \mathbb{W}(\Sigma^{n+2} X_0). \end{cases}$$

□

After shifting vertices, Proposition 2.3.5 and Corollary 2.3.6 can be rewritten and summarised in the following way.

Theorem 2.3.7. *We have the following isomorphisms:*

$$(Y, \Sigma \mathsf{E}_n) \cong \begin{cases} k & \text{if } Y \in \mathbb{W}(\Sigma^{n+1} X_0), \\ 0 & \text{if } Y \notin \mathbb{W}(\Sigma^{n+1} X_0) \end{cases} \tag{2.17}$$

and

$$(\mathsf{E}_n, \Sigma Y) \cong \begin{cases} k & \text{if } Y \in \mathbb{W}(\Sigma^{n+1} X_0), \\ 0 & \text{if } Y \notin \mathbb{W}(\Sigma^{n+1} X_0). \end{cases} \tag{2.18}$$

◇

Theorem 2.3.8. *Consider the two direct systems*

$$\Sigma^m X_0 \to \Sigma^{m-1} X_1 \to \cdots \to \Sigma^{m-j} X_j \to \cdots$$

and

$$\Sigma^n X_0 \to \Sigma^{n-1} X_1 \to \cdots \to \Sigma^{n-i} X_i \to \cdots$$

associated with slices in the Auslander-Reiten quiver of $\mathscr{D}$, with respective hocolimits E_m and E_n in $\overline{\mathscr{D}}$. Then

$$\operatorname{Hom}_{\overline{\mathscr{D}}}(\mathsf{E}_m, \mathsf{E}_n) \cong \begin{cases} k & \text{if } n \leq m, \\ 0 & \text{if } n > m. \end{cases}$$

Proof. By [27, Lemma 1.13.1], there is a short exact sequence

$$0 \to \lim\nolimits_j^1(\Sigma^{m-j} X_j, \mathsf{E}_{n-1}) \to (\operatorname{hocolim}_j(\Sigma^{m-j} X_j), \mathsf{E}_n) \to \lim\nolimits_j(\Sigma^{m-j} X_j, \mathsf{E}_n) \to 0,$$

whose middle object is $\operatorname{Hom}_{\overline{\mathscr{D}}}(\mathsf{E}_m, \mathsf{E}_n)$. Now, for all $j \in \mathbb{N}_0$, we have that $(\Sigma^{m-j} X_j, \mathsf{E}_{n-1})$ is either isomorphic to 0 or k, by Proposition 2.3.5. Therefore

$$\lim\nolimits_j^1(\Sigma^{m-j} X_j, \mathsf{E}_{n-1}) = 0,$$

by Exercise 3.5.2 of [42]. Therefore,

$$(\operatorname{hocolim}_j(\Sigma^{m-j} X_j), \mathsf{E}_n) \cong \lim\nolimits_j(\Sigma^{m-j} X_j, \mathsf{E}_n). \tag{2.19}$$

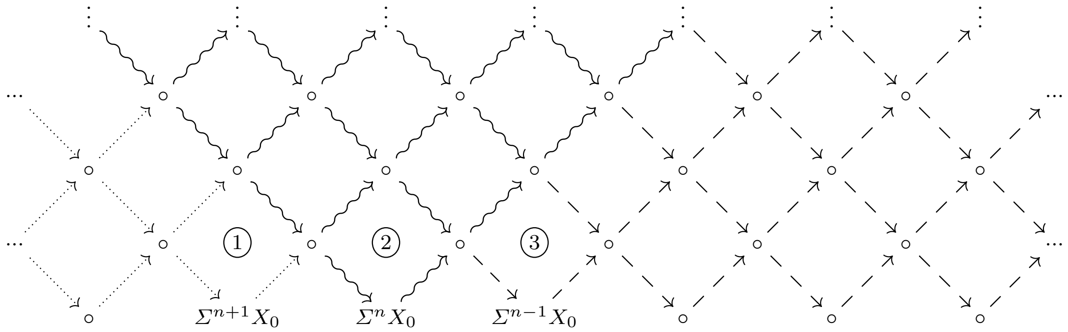
Now, apply the functor $(-, \mathsf{E}_n)$ to the direct system

$$\Sigma^m X_0 \xrightarrow{\eta_0} \Sigma^{m-1} X_1 \xrightarrow{\eta_1} \Sigma^{m-2} X_2 \longrightarrow \cdots \tag{2.20}$$

to obtain the inverse system

$$(\Sigma^m X_0, \mathsf{E}_n) \xleftarrow{(\eta_0, \mathsf{E}_n)} (\Sigma^{m-1} X_1, \mathsf{E}_n) \xleftarrow{(\eta_1, \mathsf{E}_n)} (\Sigma^{m-2} X_2, \mathsf{E}_n) \longleftarrow \cdots. \tag{2.21}$$

Two things can happen depending on where the slice associated with the direct system (2.20) starts. Consider the regions of the Auslander-Reiten quiver of $\mathscr{D}$ illustrated below.



Region 2 is highlighted with wavy arrows and contains all of the vertices on its boundary, so, in particular, equals $\mathbb{W}(\Sigma^n X_0)$. Neither region 1, illustrated with dotted arrows, nor region 3, illustrated with dashed arrows, contain the vertices on their respective shared boundaries with region 2.

If the slice associated with the direct system (2.20) starts in either region 1 or region 2, so $m = n + x$ with $x \geq 0$, then it will eventually meet the left boundary of region 2 and the inverse system (2.21) is of the form

$$0 \leftarrow 0 \leftarrow \cdots 0 \leftarrow k \leftarrow k \leftarrow \cdots \tag{2.22}$$

with x zeroes, by Proposition 2.3.5. Alternatively, suppose the slice associated with the direct system (2.20) starts in region 3, so $n > m$. Then $\Sigma^{m-j}X_j \notin \mathbb{W}(\Sigma^n X_0)$ for each $j \in \mathbb{N}_0$, so the inverse system (2.21) is of the form

$$0 \leftarrow 0 \leftarrow 0 \leftarrow \cdots$$

by Proposition 2.3.5. Therefore, when $n > m$, the limit of the inverse system (2.21) is trivially zero. So, suppose $n \leq m$. To establish the theorem, we must show that the inverse limit of (2.21) is k. Consider one of the nonzero maps in (2.21). Denote this map

$$(\Sigma^{m-j}X_j, \mathsf{E}_n) \xleftarrow{(\eta_j, \mathsf{E}_n)} (\Sigma^{m-j-1}X_{j+1}, \mathsf{E}_n) \tag{2.23}$$

for $j \geq x$. It is enough to show that each such map is nonzero, because this implies it is an isomorphism, as the hom-spaces are one-dimensional vector spaces. In order to do this, consider the direct system (2.4),

$$\Sigma^n X_0 \xrightarrow{\xi_0} \Sigma^{n-1}X_1 \xrightarrow{\xi_1} \Sigma^{n-2}X_2 \longrightarrow \cdots ,$$

and apply the functors $(\Sigma^{m-j-1}X_{j+1}, -)$ and $(\Sigma^{m-j}X_j, -)$ separately to obtain the respective direct systems

$$(\Sigma^{m-j-1}X_{j+1}, \Sigma^n X_0) \xrightarrow{(\Sigma^{m-j-1}X_{j+1},\xi_0)} (\Sigma^{m-j-1}X_{j+1}, \Sigma^{n-1}X_1) \longrightarrow \cdots \tag{2.24}$$

and

$$(\Sigma^{m-j}X_j, \Sigma^n X_0) \xrightarrow{(\Sigma^{m-j}X_j,\xi_0)} (\Sigma^{m-j}X_j, \Sigma^{n-1}X_1) \longrightarrow \cdots . \tag{2.25}$$

Both of these direct systems look like the direct system (2.5),

$$(Y, \Sigma^n X_0) \xrightarrow{(Y,\xi_0)} (Y, \Sigma^{n-1}X_1) \xrightarrow{(Y,\xi_1)} (Y, \Sigma^{n-2}X_2) \longrightarrow \cdots ,$$

with Y replaced with $\Sigma^{m-j-1}X_{j+1}$ and $\Sigma^{m-j}X_j$ respectively. Since both $\Sigma^{m-j-1}X_{j+1}$ and $\Sigma^{m-j}X_j$ are in $\mathbb{W}(\Sigma^n X_0)$, the direct systems (2.24) and (2.25) both have colimit isomorphic to k, by Proposition 2.3.5. We may write their colimits as $(\Sigma^{m-j-1}X_{j+1}, \mathsf{E}_n)$ and $(\Sigma^{m-j}X_j, \mathsf{E}_n)$, respectively. The direct systems (2.24) and (2.25) can be combined to yield the commutative ladder

$$\begin{tikzcd}
(\Sigma^{m-j-1}X_{j+1}, \Sigma^n X_0) \arrow[r, "(\Sigma^{m-j-1}X_{j+1},\xi_0)"] \arrow[d, "(\eta_j,\Sigma^n X_0)"] & (\Sigma^{m-j-1}X_{j+1}, \Sigma^{n-1}X_1) \arrow[r] \arrow[d, "(\eta_j,\Sigma^{n-1}X_1)"] & \cdots \\
(\Sigma^{m-j}X_j, \Sigma^n X_0) \arrow[r, "(\Sigma^{m-j}X_j,\xi_0)"] & (\Sigma^{m-j}X_j, \Sigma^{n-1}X_1) \arrow[r] & \cdots
\end{tikzcd} \tag{2.26}$$

which has colimit equal to (2.23). Now, note that the direct systems (2.24) and (2.25) will look like

$$0 \to 0 \to \cdots \to 0 \to k \to k \to \cdots$$

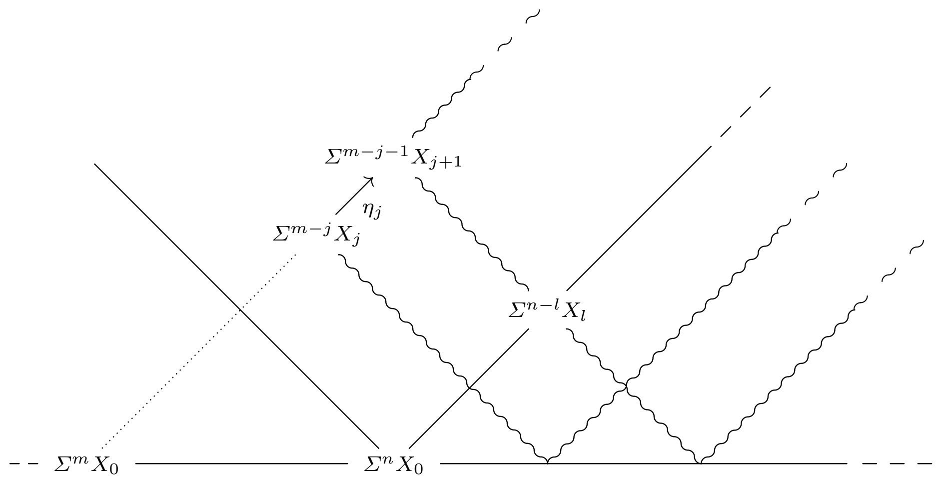
by Equation (2.22). Therefore, the ladder (2.26) looks like

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d] & k \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & k \arrow[r] & k \arrow[r] & k \arrow[r] & \cdots .
\end{tikzcd}$$

For $l \geq j-1$, the "step"

$$(\Sigma^{m-j}X_j, \Sigma^{n-l}X_l) \xleftarrow{(\eta_j,\Sigma^{n-l}X_l)} (\Sigma^{m-j-1}X_{j+1}, \Sigma^{n-l}X_l)$$

in the ladder (2.26) is nonzero by [21, Lemma 2.5]. For example, for $l = j - 1$, we can illustrate this on the Auslander-Reiten quiver as follows.



Hence the colimit of the ladder (2.26) is a nonzero map, which means the map (2.23),

$$(\Sigma^{m-j}X_j, \mathsf{E}_n) \xleftarrow{(\eta_j, \mathsf{E}_n)} (\Sigma^{m-j-1}X_{j+1}, \mathsf{E}_n),$$

is nonzero as desired. □

Corollary 2.3.9. *The hocolimits E_n of $\overline{\mathscr{D}}$ are indecomposable objects.*

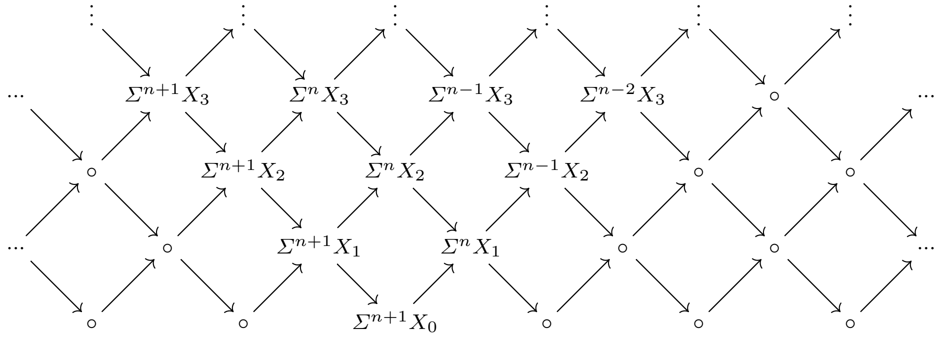
Proof. By Theorem 2.3.8, we have $\mathrm{Hom}_{\overline{\mathscr{D}}}(\mathsf{E}_n, \mathsf{E}_n) \cong k$. Hence, $\mathrm{Hom}_{\overline{\mathscr{D}}}(\mathsf{E}_n, \mathsf{E}_n)$ is a one-dimensional vector space over k, and is therefore a local ring. Thus, E_n is an indecomposable object, as required. □

2.4 The geometric model of $\overline{\mathscr{D}}$

Theorem 2.3.7 states that

$$\mathrm{Hom}_{\overline{\mathscr{D}}}(Y, \Sigma\mathsf{E}_n) \cong \mathrm{Hom}_{\overline{\mathscr{D}}}(\mathsf{E}_n, \Sigma Y) \cong \begin{cases} k \text{ if } Y \in \mathbb{W}(\Sigma^{n+1}X_0), \\ 0 \text{ if } Y \notin \mathbb{W}(\Sigma^{n+1}X_0). \end{cases} \tag{2.27}$$

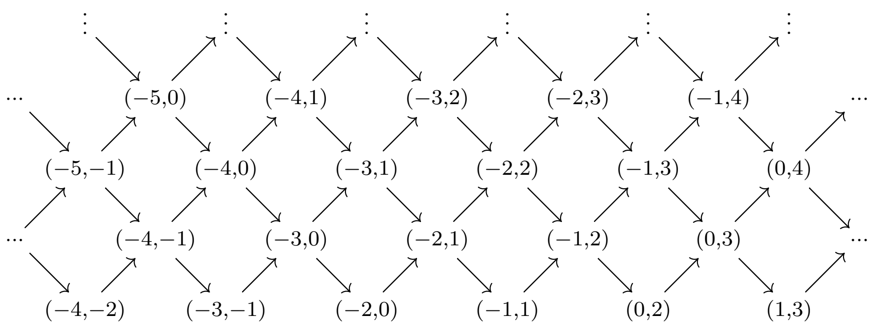
The indecomposables corresponding to $\mathbb{W}(\Sigma^{n+1}X_0)$ are highlighted in the Auslander-Reiten quiver of $\mathscr{D}$ below.



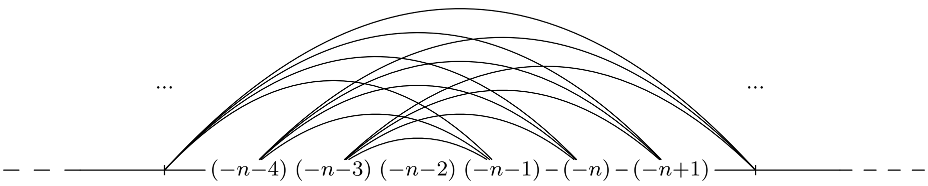
The formula

$$\Sigma^{n-l}X_{l-k-2} = (-n+k, -n+l)$$

defines the "standard" coordinate system on the Auslander-Reiten quiver of $\mathscr{D}$ given in Remark 1.4 of [21]. This is illustrated below.



Pairs (i,j) with $i \leq j-2$ can be viewed as both coordinate pairs and arcs between non-neighbouring integers on the number line. For example, the coordinate pair associated with $\Sigma^{n+1}X_0$ is $(-n-3, -n-1)$, and by Remark 3.4 of [21], this would be associated with an arc drawn between the integers $-n-3$ and $-n-1$ on the number line. We illustrate below the arcs which correspond to the indecomposable objects of $\mathbb{W}(\Sigma^{n+1}X_0)$.



(2.28)

They are, in fact, all the 'overarcs' of the integer $-n-2$.

Lemma 3.6 of [21] states that if $\mathfrak{a}, \mathfrak{b}$ are arcs with a and b the corresponding indecomposables of $\mathscr{D}$, then $(a, \Sigma b) \cong (b, \Sigma a) \cong k$ if and only if $\mathfrak{a}$ and $\mathfrak{b}$ cross; if $\mathfrak{a}$ and $\mathfrak{b}$ don't cross, then $(a, \Sigma b) \cong (b, \Sigma a) \cong 0$.

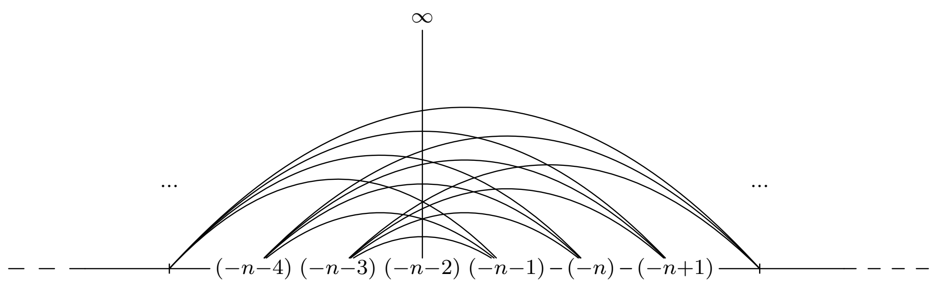
To extend this arc model to $\overline{\mathscr{D}}$, we formulate the following definition.

Definition 2.4.1. An arc is either (1) a pair of integers (i, j) with $i \leq j-2$, which will be referred to as a *finite arc*, or (2) a pair (i, ∞) with i an integer, which will be referred to as an *infinite arc*. Two finite arcs $(i, j), (r, s)$ cross if either $i < r < j < s$ or $r < i < s < j$. A finite arc (i, j) crosses an infinite arc (n, ∞) if $i < n < j$. There is no meaning associated with two infinite arcs crossing. $\diamond$

The sketches with arcs are simply visualisations of sets of arcs. The infinite arcs 'represent' the objects E_n by the following extension of [21, Lemma 3.6].

Proposition 2.4.2. *Let E_n be a hocolimit in $\overline{\mathscr{D}}$. Let $\mathfrak{Y}$ be an arc representing a finite indecomposable object Y of $\overline{\mathscr{D}}$ and consider the infinite arc $\mathfrak{E}_n = (-n-2, \infty)$. Then $\mathrm{Hom}_{\overline{\mathscr{D}}}(Y, \Sigma\mathsf{E}_n) \cong \mathrm{Hom}_{\overline{\mathscr{D}}}(\mathsf{E}_n, \Sigma Y) \cong k$ if and only if $\mathfrak{E}_n$ and $\mathfrak{Y}$ cross.*

Proof. The arc $\mathfrak{E}_n$ crosses precisely the arcs drawn in Figure (2.28); this is illustrated below.



(2.29)

Hence $\mathfrak{E}_n$ crosses precisely the arcs $\mathfrak{Y}$ which correspond to the indecomposable objects Y of $\mathbb{W}(\Sigma^{n+1}X_0)$, so Theorem 2.3.7 says that $\mathrm{Hom}_{\overline{\mathscr{D}}}(Y, \Sigma\mathsf{E}_n) \cong \mathrm{Hom}_{\overline{\mathscr{D}}}(\mathsf{E}_n, \Sigma Y) \cong k$ if and only if $\mathfrak{E}_n$ and $\mathfrak{Y}$ cross. $\square$

Remark 2.4.3. By Theorem 2.3.8, we have that

$$\mathrm{Hom}_{\overline{\mathscr{D}}}(\mathsf{E}_m, \Sigma\mathsf{E}_n) \cong \begin{cases} k \text{ if } m-1 \geq n, \\ 0 \text{ if } m-1 < n, \end{cases}$$

and

$$\mathrm{Hom}_{\overline{\mathscr{D}}}(\mathsf{E}_n, \Sigma\mathsf{E}_m) \cong \begin{cases} k \text{ if } n \geq m+1, \\ 0 \text{ if } n < m+1. \end{cases}$$

Together, these imply

$$\mathrm{Hom}_{\overline{\mathscr{D}}}(\mathsf{E}_m, \Sigma\mathsf{E}_n) \cong \mathrm{Hom}_{\overline{\mathscr{D}}}(\mathsf{E}_n, \Sigma\mathsf{E}_m) \cong 0$$

if $m-1<n<m+1$, which is only possible if $n=m$, and

$$\mathrm{Hom}_{\overline{\mathscr{D}}}(\mathsf{E}_m, \Sigma\mathsf{E}_n) \cong \mathrm{Hom}_{\overline{\mathscr{D}}}(\mathsf{E}_n, \Sigma\mathsf{E}_m) \cong k$$

if $m-1 \geq n \geq m+1$, which is impossible. Hence, it is impossible to devise a symmetric notion of crossing of the infinite arcs $\mathfrak{E}_m$ and $\mathfrak{E}_n$ which corresponds to non-vanishing of the Hom-spaces in the equations. Indeed, the isomorphisms in (2.27) display symmetry, whilst the isomorphism of Theorem 2.3.8 is inherently non-symmetrical. ◇

2.5 Weakly cluster tilting subcategories

A series of arc lemmas is necessary to prove the subsequent results about (weakly) cluster tilting subcategories of $\overline{\mathscr{D}}$. In this section we also look at subcategories of $\overline{\mathscr{D}}$ with special arc configurations and show when they correspond to weakly cluster tilting subcategories.

By [26], a weakly cluster tilting subcategory $\mathscr{T}$ of a triangulated category $\mathscr{S}$ is defined by the conditions

$$\begin{aligned}\mathscr{T} &= \{s \in \mathscr{S} \mid (\mathscr{T}, \Sigma s) = 0\} \\ &= \{s \in \mathscr{S} \mid (s, \Sigma\mathscr{T}) = 0\}.\end{aligned}$$

A weakly cluster tilting subcategory is closed under direct sums and summands, so if the ambient triangulated category is $\overline{\mathscr{D}}$, it is determined by its indecomposable objects, i.e. by the sets of arcs corresponding to its indecomposables.

Proposition 2.5.1. *A weakly cluster tilting subcategory $\mathscr{T}$ of $\overline{\mathscr{D}}$ may contain at most one of the objects E_n.*

Proof. If $\mathscr{T}$ is weakly cluster tilting, then $\mathsf{E}_n \in \mathscr{T}$ implies $(\mathscr{T}, \Sigma\mathsf{E}_n) \cong (\mathsf{E}_n, \Sigma\mathscr{T}) \cong 0$. If E_m is also in $\mathscr{T}$, then by Remark 2.4.3 we have that $(\mathsf{E}_m, \Sigma\mathsf{E}_n) \cong (\mathsf{E}_n, \Sigma\mathsf{E}_m) \cong 0$ implies $m = n$. □

Definition 2.5.2. Fountains are originally defined in [21, Definition 3.2]. We recall their definitions. Let $\mathfrak{T}$ be a set of finite arcs. If there are infinitely many arcs of the form $(m, —)$ in $\mathfrak{T}$, then m is called a *right-fountain* of $\mathfrak{T}$. Conversely if there are infinitely many arcs of the form $(—, n)$ in $\mathfrak{T}$, then n is called a *left-fountain* of $\mathfrak{T}$. If $m = n$ is both a left- and a right-fountain of $\mathfrak{T}$, then it is a fountain of $\mathfrak{T}$. ◇

Definition 2.5.3. Let $\mathfrak{T}$ be a set of finite arcs. If for all $n \in \mathbb{Z}$ there are only finitely many arcs of the form $(n, —)$ in $\mathfrak{T}$, and also only finitely many arcs of the form $(—, n)$ in $\mathfrak{T}$, then $\mathfrak{T}$ is called *locally finite*. ◇

Definition 2.5.4. Let $\mathfrak{T}$ be a maximal, non-crossing set of finite arcs. If $(p,q) \in \mathfrak{T}$, then a *strong overarc* of (p,q) in $\mathfrak{T}$ is a finite arc (x,y) in $\mathfrak{T}$ where $x < p < q < y$. We also define a strong overarc of an integer n. If $n \in \mathbb{Z}$ then a strong overarc (with respect to $\mathfrak{T}$) of n is an arc (x,y) in $\mathfrak{T}$ where $x < n < y$. ◇

Lemma 2.5.5. *Let $\mathfrak{T}'$ be a set of finite arcs and let $\mathfrak{T} = \{(m,\infty)\} \cup \mathfrak{T}'$ be a set of arcs which satisfies the following condition: if a finite arc $\mathfrak{a}$ crosses neither (m,∞) nor an arc in $\mathfrak{T}'$, then $\mathfrak{a} \in \mathfrak{T}$. Then*
(1) $\mathfrak{T}'$ has a left-fountain p and a right-fountain q (necessarily with $p \leq m \leq q$), and
(2) Either $p \in \{m-1,m\}$ or $(p,m) \in \mathfrak{T}$ (and symmetrically, either $q \in \{m,m+1\}$ or $(m,q) \in \mathfrak{T}$).

Proof.
By symmetry, it is enough to find a left-fountain p and show that either $p \in \{m-1,m\}$ or $(p,m) \in \mathfrak{T}$. There are seven cases to check.
Case (1). Suppose there are infinitely many arcs of the form $(—,m)$ in $\mathfrak{T}'$. Then we have that $p = m$ is a left-fountain.
Case (2). Suppose that there are only finitely many arcs of the form $(—,m)$ in $\mathfrak{T}'$ and that (l,m) is the longest such arc. Furthermore, suppose that there are infinitely many arcs in $\mathfrak{T}'$ of the form $(—,l)$. Then we have that $p = l$ is a left-fountain, and the finite arc (p,m) is in $\mathfrak{T}$, by assumption.
Case (3). Suppose that there are only finitely many arcs of the form $(—,m)$ in $\mathfrak{T}'$ and that (l,m) is the longest such arc. Furthermore, suppose that there are only finitely many arcs in $\mathfrak{T}'$ of the form $(—,l)$, and that (v,l) is the longest. But now the arc (v,m) crosses no arc in $\mathfrak{T}'$ nor the arc (m,∞), and is hence in $\mathfrak{T}$, contradicting the fact that (l,m) is the longest arc of the form $(—,m)$ in $\mathfrak{T}'$.
Case (4). Suppose that there are only finitely many arcs of the form $(—,m)$ in $\mathfrak{T}'$ and that (l,m) is the longest such arc. Furthermore, suppose that there is no arc in $\mathfrak{T}'$ of the form $(—,l)$. Then the arc $(l-1,m)$ crosses no arc in $\mathfrak{T}'$ nor the arc (m,∞), and is hence in $\mathfrak{T}$, contradicting the fact that (l,m) is the longest arc of the form $(—,m)$ in $\mathfrak{T}'$.
Case (5). Suppose that there is no arc in $\mathfrak{T}'$ of the form $(—,m)$, and there is also no arc in $\mathfrak{T}'$ of the form $(—,m-1)$. Then this is a contradiction, for the arc $(m-2,m)$ is now necessarily in $\mathfrak{T}$, for it crosses no finite arc in $\mathfrak{T}'$ nor the arc (m,∞).
Case (6). Suppose that there is no arc in $\mathfrak{T}'$ of the form $(—,m)$, and that there are infinitely many arcs in $\mathfrak{T}'$ of the form $(—,m-1)$. Then we have that $p = m-1$ is a left-fountain.
Case (7). Suppose that there is no arc in $\mathfrak{T}'$ of the form $(—,m)$, and that there are only finitely many arcs in $\mathfrak{T}'$ of the form $(—,m-1)$. Let $(l,m-1)$ be the longest such arc. Then (l,m) crosses no arc in $\mathfrak{T}'$ nor the arc (m,∞) and is hence in $\mathfrak{T}$. This is a contradiction,

because we assumed that there was no arc in $\mathfrak{T}'$ of the form $(—, m)$.

Note, in this proof, the arc (m, ∞) plays a vital "blocking" role. Take, for example, Case (5). The fact that there is no arc in $\mathfrak{T}$ of the form $(m-1, —)$ is what permits us to conclude that $(m-2, m)$ is in $\mathfrak{T}$ (for if there was a finite arc of the form $(m-1, —)$ in $\mathfrak{T}$, then it would necessarily cross (m, ∞), which would be a contradiction). $\square$

Lemma 2.5.6. *Let $\mathfrak{T}$ be a maximal, non-crossing, locally finite set of finite arcs. Let p be an arbitrary integer. Suppose $\nexists x \in \mathbb{Z}$ such that $(x, p) \in \mathfrak{T}$. Then $\exists y' \in \mathbb{Z}$ such that $(p-1, y') \in \mathfrak{T}$.*

Proof. First, suppose $\nexists\, x' \in \mathbb{Z}$ such that $(p, x') \in \mathfrak{T}$. Then there is neither an arc (x, p) nor an arc (p, x') in $\mathfrak{T}$. Because no arcs end in p, there is room for $(p-1, p+1)$ in the configuration. By maximality of $\mathfrak{T}$, we must therefore have $(p-1, p+1) \in \mathfrak{T}$. Therefore the claim is satisfied with $y' = p+1$. Now suppose $\exists x' \in \mathbb{Z}$ such that $(p, x') \in \mathfrak{T}$. Because $\mathfrak{T}$ is locally finite (and hence, in particular, does not contain a right fountain) there is a longest arc of the form $(p, r) : r \in \mathbb{Z}$, say, (p, x''). Now, there is room in the configuration for $(p-1, x'')$ because there are no arcs of the form $(x, p) : x \in \mathbb{Z}$ to "block" such an arc. And because $\mathfrak{T}$ is maximal, we must therefore have that $(p-1, x'') \in \mathfrak{T}$. Hence the claim is again satisfied with $y' = x''$. $\square$

By symmetry, we have the following corollary.

Corollary 2.5.7. *Let $\mathfrak{T}$ be a maximal, non-crossing, locally finite set of finite arcs. Suppose $\nexists x \in \mathbb{Z}$ such that $(q, x) \in \mathfrak{T}$. Then $\exists y' \in \mathbb{Z}$ such that $(y', q+1) \in \mathfrak{T}$.*

Lemma 2.5.8. *Let $\mathfrak{T}$ be a maximal, non-crossing set of finite arcs. Let (p, q) be the longest arc in $\mathfrak{T}$ of the form $(p, r) : r \in \mathbb{Z}$ and also the longest arc in $\mathfrak{T}$ of the form $(l, q) : l \in \mathbb{Z}$. Then $\mathfrak{T}$ has a right fountain.*

Proof. Suppose q is not a right fountain. Then either

(i) there is no arc of the form $(q, s) : s \in \mathbb{Z}$, or

(ii) there is an arc of the form $(q, s) : s \in \mathbb{Z}$ and hence a longest arc of this form, say, (q, s').

In case (i), there is an arc in $\mathfrak{T}$ of the form $(l', q+1) : l \in \mathbb{Z}$ by Corollary 2.5.7. By assumption, $l' \neq p$, else $(p, q+1)$ is an arc in $\mathfrak{T}$ longer than (p, q). So there are two arcs $(l', q+1)$ and (p, q) in $\mathfrak{T}$. Now, $(p, q+1) \notin \mathfrak{T}$ by assumption, whence $(l', q) \in \mathfrak{T}$ by maximality of $\mathfrak{T}$: a contradiction.

In the second case (ii), there is a longest arc $(q, s') \in \mathfrak{T}$. By assumption, $(p, s') \notin \mathfrak{T}$ and by maximality of $\mathfrak{T}$ the only way this is possible if there is an arc of the form (l', q) with

$l' < p$, because then the arc (p, s') would be "blocked". In this case, (l', q) is longer then (p, q): a contradiction.

Now suppose q is a right fountain. Then $\mathfrak{T}$ has a right fountain: q itself. □

Lemma 2.5.9. *Let $\mathfrak{T}$ be a maximal, non-crossing, locally finite set of finite arcs. Let $(p, q) \in \mathfrak{T}$. Then $\exists (x, y) \in \mathfrak{T}$ with $x < p < q < y$.*

Proof. By Lemma 2.5.8, if (p, q) is both the longest arc in $\mathfrak{T}$ of the form $(p, r) : r \in \mathbb{Z}$ and also the longest arc in $\mathfrak{T}$ of the form $(l, q) : l \in \mathbb{Z}$, then $\mathfrak{T}$ has a right fountain and in particular is not locally finite. So either

(i) there is an arc (p, q') in $\mathfrak{T}$ with $q' > q$, or

(ii) there is an arc (p', q) in $\mathfrak{T}$ with $p' < p$,

but not both, else (p', q) and (p, q') cross. Notice that, if the claim is proved assuming (i) holds, then by symmetry, the claim would be proved if it were the case that (ii) holds. So suppose it is (i) that holds. Note that p is not a right fountain, because $\mathfrak{T}$ is locally finite. So there is a longest arc of the form $(p, r) : r \in \mathbb{Z}$, say, (p, y). By assumption, $y \neq q$. Because (p, y) is the longest arc of the form $(p, r) : r \in \mathbb{Z}$, and $\mathfrak{T}$ is locally finite, (p, y) is not the longest arc of the form $(u, y) : u \in \mathbb{Z}$, and because y is not a left fountain, there is a longest arc $(x, y) \in \mathfrak{T}$ of this form. Hence, if (i) holds, we are done: $(x, y) \in \mathfrak{T}$ with $x < p < q < y$, and as noted before, if (i) proves the claim then so does (ii) by symmetry. □

Corollary 2.5.10. *Let $\mathfrak{T}$ be a maximal, non-crossing, locally finite set of finite arcs. Let $(p, q) \in \mathfrak{T}$. Then there exists a strong overarc $(p - \varepsilon, q + \delta)$ in $\mathfrak{T}$ with ε, δ arbitrarily large integers.*

Proof. By Lemma 2.5.9, (p, q) has a strong overarc in $\mathfrak{T}$. But any strong overarc of (p, q) will itself have a strong overarc in $\mathfrak{T}$ - which itself will be an (even longer) overarc of (p, q). This can be continued indefinitely with the overarcs of (p, q) becoming arbitrarily long, hence proving the claim. □

Corollary 2.5.11. *Let $\mathfrak{T}$ be a maximal, non-crossing, locally finite set of finite arcs. Let $h \in \mathbb{Z}$. Then $\exists (x, y) \in \mathfrak{T}$ with $x < h < y$.*

Proof. Let $(p, q) \in \mathfrak{T}$ be any arc in $\mathfrak{T}$. Then by Corollary 2.5.10, there is a (potentially very long) strong overarc (x, y) of (p, q) in $\mathfrak{T}$ with

$$h \in \{x + 1, \cdots, p, p + 1, \cdots, q, q + 1, \cdots, y - 1\},$$

which proves the claim. □

Theorem 2.5.12. *Let $\mathscr{T}$ be a a subcategory of $\overline{\mathscr{D}}$ which is closed under direct sums and direct summands, so it corresponds to a set $\mathfrak{T}$ of arcs (finite and infinite). Then $\mathscr{T}$ is weakly cluster tilting if and only if one of the following happens:*
(1) $\mathfrak{T}$ is a set of finite arcs which is maximal, non-crossing, and locally finite, or
(2) $\mathfrak{T} = \{(m,\infty)\} \cup \mathfrak{T}'$ where $\mathfrak{T}'$ is a set of finite arcs which is maximal non-crossing and has a fountain in the sense of [21, Section 3] (and the fountain is necessarily at m).

Proof.
"If": Suppose that $\mathscr{T}$ is a weakly cluster tilting subcategory of $\overline{\mathscr{D}}$, i.e., $\mathscr{T} = \{t \mid (t, \Sigma\mathscr{T}) = 0\}$ and $\mathscr{T} = \{t \mid (\mathscr{T}, \Sigma t) = 0\}$. By Proposition 2.5.1, $\mathscr{T}$ contains either no E_n or it contains precisely one E_n.

If $\mathscr{T}$ contains no E_n, then $\mathfrak{T}$ is a set of finite arcs. The set $\mathfrak{T}$ is clearly non-crossing, because if any two arcs in $\mathfrak{T}$ cross, say $\mathfrak{a}$ and $\mathfrak{b}$, then the objects associated with these arcs have a non-vanishing Ext, contradicting the fact that $\mathscr{T} = \{t \mid (t, \Sigma\mathscr{T}) = 0\}$. The set $\mathfrak{T}$ is also maximal, because if $\mathfrak{a}$ crosses no arc in $\mathfrak{T}$, then $(a, \Sigma\mathscr{T}) = 0$, whence $a \in \mathscr{T}$ so $\mathfrak{a} \in \mathfrak{T}$. Note that $\mathfrak{T}$ couldn't have a left fountain or a right fountain at $-m-2$, because if it did, then $(-m-2, \infty)$ would cross no arc in $\mathfrak{T}$ whence $(\mathsf{E}_m, \Sigma\mathscr{T}) = 0$, so $\mathsf{E}_m \in \mathscr{T}$ and $(-m-2, \infty) \in \mathfrak{T}$ which contradicts the fact that $\mathfrak{T}$ is a set of finite arcs. Altogether this gives that $\mathfrak{T}$ is a maximal, non-crossing, locally finite set of finite arcs.

Suppose that $\mathscr{T}$ contains a single E_n, so $\mathfrak{T}$ contains a single infinite arc, i.e. $\mathfrak{T} = \{(m,\infty)\} \cup \mathfrak{T}'$ where $\mathfrak{T}'$ consists of finite arcs. Define $\mathscr{T}'$ to be the category with objects in the additive hull of the corresponding (finite) indecomposables. We know that if a finite arc $\mathfrak{a}$ crosses neither (m,∞) nor an arc in $\mathfrak{T}'$, then $\mathfrak{a} \in \mathfrak{T}'$, because $\mathscr{T}$ is weakly cluster tilting. Hence, by the first part of Lemma 2.5.5, $\mathfrak{T}'$ has a left-fountain (call it p) and a right-fountain (call it q). In particular, since $\mathscr{T}$ is weakly cluster tilting, no two arcs in $\mathfrak{T}$ can cross, so $p \leq m \leq q$. Now suppose $p < m$. Then by the second part of Lemma 2.5.5, $p = m-1$ or $(p,m) \in \mathfrak{T}'$. Since no two arcs in $\mathfrak{T}$ cross, p has no strong overarc in $\mathfrak{T}'$. Now let $F = \mathsf{E}_{-p-2} \in \overline{\mathscr{D}}$ be the object associated with the arc (p,∞). Since p has no strong overarc in $\mathfrak{T}'$, we get (p,∞) crossing no arc in $\mathfrak{T}'$, so $(\mathscr{T}', \Sigma F) = (F, \Sigma\mathscr{T}') = 0$. However, we also have $(\mathsf{E}_{-m-2}, \Sigma F) = 0$ as $p < m$. Hence $(\mathscr{T}, \Sigma F) = 0$ so $F \in \{t \mid (\mathscr{T}, \Sigma t) = 0\} = \mathscr{T}$, which is a contradiction. So we conclude that $p = m$, and symmetrically, $q = m$. Observe that in this case, $\mathfrak{T}'$ is in fact a maximal, non-crossing set of finite arcs. For if a finite arc $\mathfrak{a}$ crosses no arc in $\mathfrak{T}'$, then $\mathfrak{a}$ does not overarc m, because that is where $\mathfrak{T}'$ has a fountain. Hence $\mathfrak{a}$ also doesn't cross (m,∞). Therefore $\mathfrak{a} \in \mathfrak{T}$ so $\mathfrak{a} \in \mathfrak{T}'$.

It is impossible for $\mathfrak{T}$ to contain two or more infinite arcs. For if it did, they would correspond to different hocolims $\mathsf{E}, \mathsf{E}' \in \mathscr{T}$, contradicting Proposition 2.5.1.

"Only if": There are two cases.

Case (1). Suppose that $\mathfrak{T}$ is a set of finite arcs which is maximal, non-crossing and locally finite in the sense of [21, Section 3]. Then we show that $\mathscr{T} = \{t \mid (t, \Sigma\mathscr{T}) = 0\}$ and $\mathscr{T} = \{t \mid (\Sigma\mathscr{T}, t) = 0\}$. In fact it is enough just to consider the first of these equations,

$$\mathscr{T} = \{t \mid (t, \Sigma\mathscr{T}) = 0\}, \tag{2.30}$$

as the second equation is symmetric. There are two inclusions to show in order to establish Equation (2.30). In each case it is enough to consider indecomposable objects.

The inclusion $\mathscr{T} \subseteq \{t \mid (t, \Sigma\mathscr{T}) = 0\}$ follows because the arcs in $\mathfrak{T}$ do not cross.

For the inclusion $\mathscr{T} \supseteq \{t \mid (t, \Sigma\mathscr{T}) = 0\}$, consider an indecomposable $s \in \{t \mid (t, \Sigma\mathscr{T}) = 0\}$. We aim to show that $s \in \mathscr{T}$. There are two possibilities: either $s \in \mathscr{D}$ or $s = \mathsf{E}_h$, for some $h \in \mathbb{Z}$. If $s \in \mathscr{D}$, then $s \in \{t \mid (t, \Sigma\mathscr{T}) = 0\}$ means that the arc of s crosses no arc in $\mathfrak{T}$. Then the arc of s is in $\mathfrak{T}$ and hence $s \in \mathscr{T}$. If, on the other hand, $s = \mathsf{E}_h$, then by Corollary 2.5.11, $s \in \{t \mid (t, \Sigma\mathscr{T}) = 0\}$ cannot happen, because there will always be an arc in $\mathfrak{T}$ which crosses the arc associated with E_h.
Case (2). Suppose that $\mathfrak{T} = \{(m, \infty)\} \cup \mathfrak{T}'$ where $\mathfrak{T}'$ is a set of finite arcs which is maximal non-crossing and has a fountain at m. Like before, we show that $\mathscr{T} = \{t \mid (t, \Sigma\mathscr{T}) = 0\}$ and $\mathscr{T} = \{t \mid (\mathscr{T}, \Sigma t) = 0\}$, and again it is enough just to consider the first of these equations as the second equation is symmetric. There are two inclusions to show in order to establish Equation (2.30). Again in each case it is enough to consider indecomposable objects.

The inclusion $\mathscr{T} \subseteq \{t \mid (t, \Sigma\mathscr{T}) = 0\}$ follows because the arcs in $\mathfrak{T}$ do not cross.

For the inclusion $\mathscr{T} \supseteq \{t \mid (t, \Sigma\mathscr{T}) = 0\}$, consider an indecomposable $s \in \{t \mid (t, \Sigma\mathscr{T}) = 0\}$. We aim to show that $s \in \mathscr{T}$. There are two possibilities: either $s \in \mathscr{D}$ or $s = \mathsf{E}_h$, for some $h \in \mathbb{Z}$. If $s \in \mathscr{D}$, then $s \in \{t \mid (t, \Sigma\mathscr{T}) = 0\}$ means that the arc of s crosses no arc in $\mathfrak{T}$. Then, since $\mathfrak{T}'$ is maximal non-crossing, the arc of s is in $\mathfrak{T}'$ and hence $s \in \mathscr{T}$. If $s = \mathsf{E}_h$, then $s \in \{t \mid (t, \Sigma\mathscr{T}) = 0\}$ means that the arc of E_h crosses no arc in $\mathfrak{T}'$ which has a fountain at m. Hence the arc of E_h must be (m, ∞), i.e., $h = -m-2$, so $s = \mathsf{E}_h \in \mathscr{T}$. $\square$

2.6 Cluster tilting subcategories

Leading on from the last section, this section concludes with a theorem stating when a subcategory of $\overline{\mathscr{D}}$ is cluster tilting. Right now we aim to show that $\mathscr{T}$ is functorially finite if and only if $\mathfrak{T}$ has a fountain.

Definition 2.6.1. Let $\mathscr{T}$ be a subcategory of $\overline{\mathscr{D}}$. We say that $\mathscr{T}$ is *right-approximating* if for any $d \in \overline{\mathscr{D}}$ there exists a $\mathscr{T}$-object t, and a right-$\mathscr{T}$-approximation $\tau : t \to d$. This means that for any $\tau' : t' \to d$, there exists a factorisation

$$\begin{tikzcd} & t' \arrow[dl, dashed, "\exists"'] \arrow[d, "\tau'"] \\ t \arrow[r, "\tau"'] & d \end{tikzcd} . \tag{2.31}$$

Analogously, $\mathscr{T}$ is left-approximating if for any $\overline{\mathscr{D}}$-object d there exists a left-approximation $\tau : d \to t$. This means that for any $\mathscr{T}$-object t' and a morphism $\tau' : d \to t'$, there exists a factorisation

$$\begin{tikzcd} & t' \\ t \arrow[ur, dashed, "\exists"] & d \arrow[l, "\tau"] \arrow[u, "\tau'"'] \end{tikzcd} . \tag{2.32}$$

We say that $\mathscr{T}$ is *functorially finite* if and only if it is both right-approximating and left-approximating. $\diamond$

Definition 2.6.2. We say that a property holds for almost all indecomposables in $\mathscr{T}$ if it holds for all but finitely many indecomposables. An *almost-right-$\mathscr{T}$-approximation* of $d \in \overline{\mathscr{D}}$ is a morphism $\tau : t \to d$ such that for almost all indecomposables t' in $\mathscr{T}$, each morphism $\tau' : t' \to d$ factors through τ. An almost-left-$\mathscr{T}$-approximation is defined dually. $\diamond$

Lemma 2.6.3. *An almost-right-$\mathscr{T}$-approximation of a $\overline{\mathscr{D}}$-object d exists if and only if a right-$\mathscr{T}$-approximation of d exists (and similarly for the case of left-approximations).*

Proof. Given an almost-right-$\mathscr{T}$-approximation $\widetilde{\tau} : \widetilde{t} \to d$, let $t_1, \dots, t_j$ be the finitely many indecomposables in $\mathscr{T}$ for which morphisms to d don't necessarily factor through $\widetilde{\tau}$. There is a right-$\mathscr{T}$-approximation

$$\tau : \widetilde{t} \oplus s_{1,1} \oplus \cdots \oplus s_{1,n_1} \oplus s_{2,1} \oplus \cdots \oplus s_{2,n_2} \oplus \cdots \oplus s_{j,1} \oplus \cdots \oplus s_{j,n_j} \to d$$

such that $s_{x,y}$ is a copy of t_x and the restrictions of τ to $s_{x,1}, \dots, s_{x,n_x}$ are a basis of the vector space $\mathrm{Hom}(t_x, d)$; that is, $\dim \mathrm{Hom}(t_x, d) = n_x$. The opposite implication is seen instantly: by definition, right-$\mathscr{T}$-approximations are themselves almost-right-$\mathscr{T}$-approximations. The proof for left-approximations is done analogously. $\square$

Remark 2.6.4. In Notation 2.2.5, we write hocolimits in terms of indecomposable objects; that is, E_n is written as $\mathrm{hocolim}_i(\Sigma^{n-i} X_i)$. We draw attention to the fact that we may use coordinate-pair notation, and may write, for example, $\mathrm{hocolim}(-n-2, —)$ instead of $\mathrm{hocolim}_i(\Sigma^{n-i} X_i)$. We like to be flexible and, to this end, may continue to write E_n

instead of $(-n-2,\infty)$. Before proving that $\mathscr{T}$ is functorially finite if and only if $\mathfrak{T}$ has a fountain, we prove two important factorisation lemmas. $\diamond$

Note that in the geometric arc model, $E_{n+k} = (-n-k-2,\infty)$.

Lemma 2.6.5. *Let $k \in \mathbb{N}$ and let $\mathsf{E}_{n+k} = \operatorname{hocolim}_i(\Sigma^{n+k-i}X_i)$. Let $z_1, z_2 \in \operatorname{ind}\mathscr{D}$ where*

$$z_1 = (y,-n), \text{ where we assume } y < -n-k-2,$$

$$z_2 = (x,-n+p), \text{ where we assume } y \leq x \leq -n-k-2 \text{ and } p \geq 0.$$

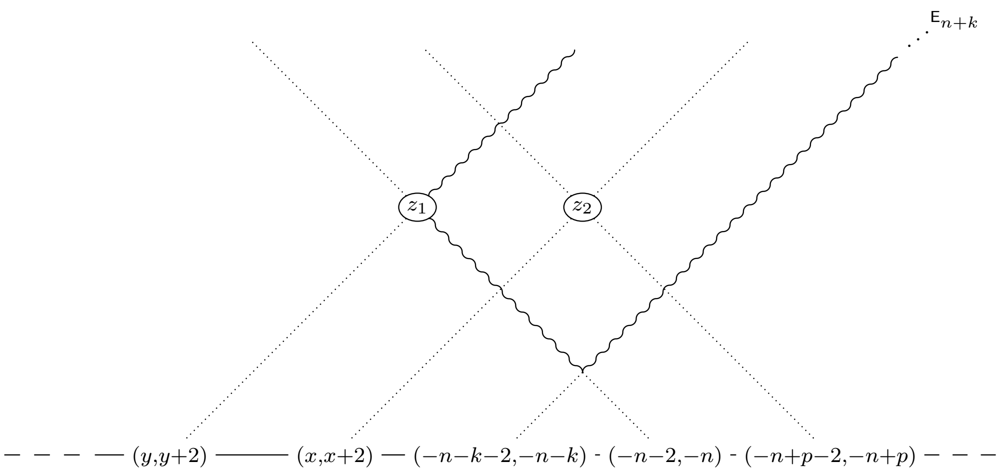
Then any nonzero morphism

$$\tau' : z_1 \to \mathsf{E}_{n+k}$$

factorises as

$$z_1 \xrightarrow{g} z_2 \xrightarrow{\tau} \mathsf{E}_{n+k}.$$

Proof. First, let us clarify the situation. The objects z_1 and z_2 are defined so as to satisfy $z_1, z_2 \in \mathbb{W}(\Sigma^n X_0)$ and $\operatorname{Hom}_{\overline{\mathscr{D}}}(z_1,z_2) \neq 0$. This is illustrated below.



Here, z_2 can be located anywhere within the region bounded by wavy lines. Now, the following diagram is natural in the variable z.

$$\begin{tikzcd}
\operatorname{colim}_i(z,\Sigma^{n+k-i}X_i) \arrow[rr, "\cong"] & & (z,\mathsf{E}_{n+k}) \\
& (z,\Sigma^{n+k-j}X_j) \arrow[ul] \arrow[ur] &
\end{tikzcd} \tag{2.33}$$

Hence, we may draw the following commutative diagram.

$$\begin{tikzcd}
\operatorname{colim}_i(z_2, \Sigma^{n+k-i}X_i) \arrow[rr, "\cong"] \arrow[dd] & & (z_2, \mathsf{E}_{n+k}) \arrow[dd] \\
& (z_2, \Sigma^{n+k-j}X_j) \arrow[ul] \arrow[ur] \arrow[dd] & \\
\operatorname{colim}_i(z_1, \Sigma^{n+k-i}X_i) \arrow[rr, "\cong"] & & (z_1, \mathsf{E}_{n+k}) \\
& (z_1, \Sigma^{n+k-j}X_j) \arrow[ul] \arrow[ur] &
\end{tikzcd} \tag{2.34}$$

The object $\Sigma^{n+k-j}X_j$ has coordinate $(-n-k-2, -n-k+j)$.

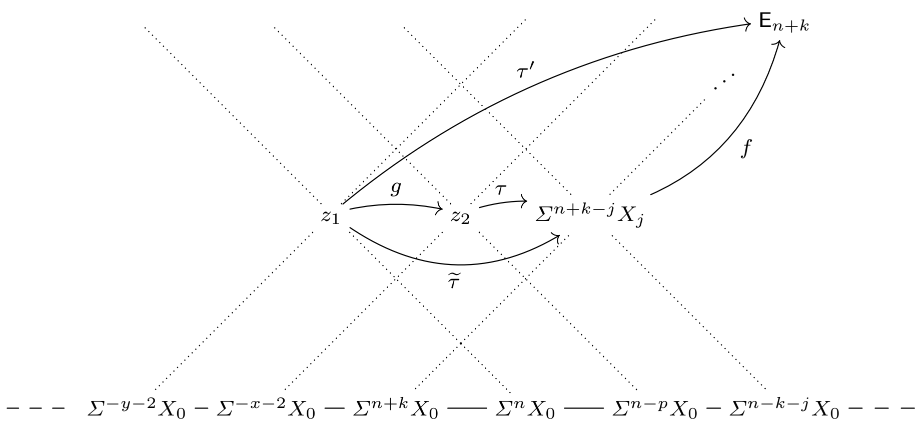
Now, let $\tau' \in (z_1, \mathsf{E}_{n+k})$. For j sufficiently big there is a $\widetilde{\tau} \in (z_1, \Sigma^{n+k-j}X_j)$ which maps to τ' (found by diagram-chasing in Figure (2.34)). That is, $\tau' = f \circ \widetilde{\tau}$ where $f : \Sigma^{n+k-j}X_j \to \mathsf{E}_{n+k}$ is the canonical morphism. By Lemma 2.5 of [21],

$$z_1 \xrightarrow{\widetilde{\tau}} \Sigma^{n+k-j}X_j$$

factors through

$$z_1 \xrightarrow{g} z_2.$$

Let the associated map from z_2 to $\Sigma^{n+k-j}X_j$ be called τ. Pictorially, we have the following.



The commutative diagram (2.34) contains the following subdiagram.

$$\begin{tikzcd}
(z_2, \mathsf{E}_{n+k}) \arrow[r, "{(g,\mathsf{E}_{n+k})}"] & (z_1, \mathsf{E}_{n+k}) \\
(z_2, \Sigma^{n+k-j} X_j) \arrow[u, "{(z_2,f)}"] \arrow[r, "{(g,\Sigma^{n+k-j}X_j)}"'] & (z_1, \Sigma^{n+k-j} X_j) \arrow[u, "{(z_1,f)}"']
\end{tikzcd}$$

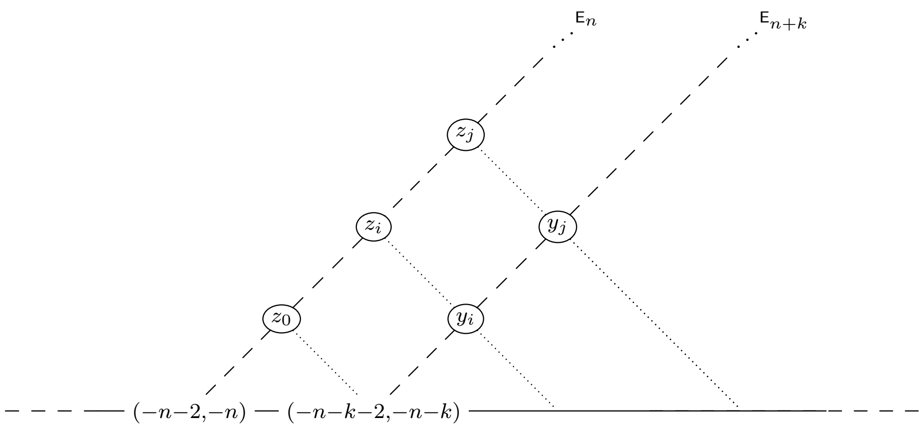
Diagram-chasing yields the following.

$$\begin{tikzcd}
f \circ \tau \arrow[r, mapsto, "{(g,\mathsf{E}_{n+k})}"] & f \circ \tau \circ g = \tau' \\
\tau \arrow[u, mapsto, "{(z_2,f)}"] \arrow[r, mapsto, "{(g,\Sigma^{n+k-j}X_j)}"'] & \widetilde{\tau} \arrow[u, mapsto, "{(z_1,f)}"']
\end{tikzcd}$$

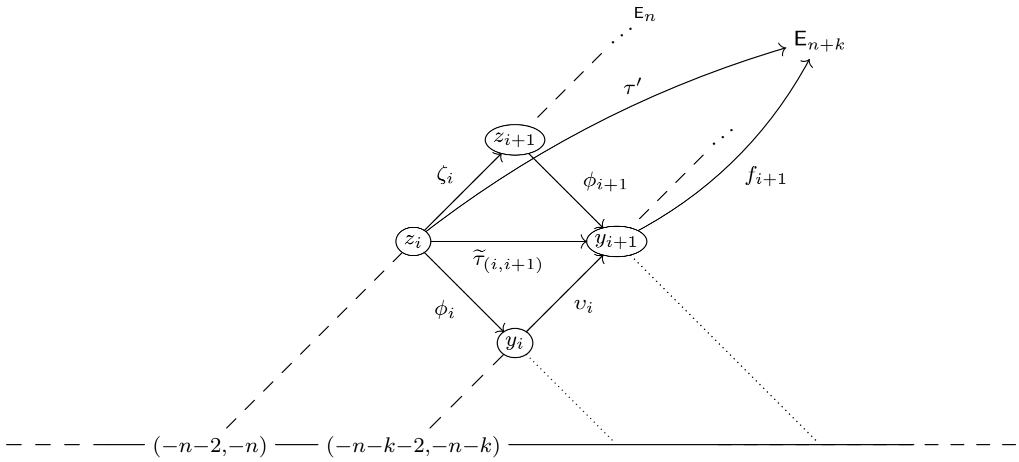
Hence, τ' has a preimage in (z_2, E_{n+k}), namely $f \circ \tau$. Therefore τ' factors through $g : z_1 \to z_2$, as required. $\square$

Lemma 2.6.6. *Let $k < 0 \leq i$ be integers. Let z_i be the indecomposable with coordinate $(-n-2, -n-k+i)$ in $\mathscr{D}$, and let $\mathsf{E}_{n+k} = \operatorname{hocolim}_j(\Sigma^{n+k-j} X_j) = (-n-k-2, \infty)$. Then any nonzero morphism $\tau' : z_i \to \mathsf{E}_{n+k}$ factorises as $z_i \xrightarrow{g_i} \mathsf{E}_n \xrightarrow{\tau} \mathsf{E}_{n+k}$ where E_n is the hocolimit associated with the slice $(-n-2, —)$.*

Proof. Let $j > i$ be an integer. Then $z_i, z_j \in \mathbb{W}(\Sigma^{n+k} X_0) \cap \mathbb{W}(\Sigma^n X_0)$, with z_i, z_j sitting on the slice $(-n-2, —)$, and $\operatorname{Hom}_{\overline{\mathscr{D}}}(z_i, z_j) \neq 0$. Let y_i be the "corresponding" indecomposable sat on the slice $(-n-k-2, —)$; that is, $y_i = (-n-k-2, -n-k+i)$. The following picture shows all of this.



Suppose $j = i+1$. Then, by Lemma 2.6.5, $\tau' : z_i \to \mathsf{E}_{n+k}$ can be factored through z_{i+1}, y_i, and y_{i+1}, to form a commutative diagram illustrated below.



Note that $\widetilde{\tau}_{(i,i+1)} = \phi_{i+1} \circ \zeta_i = \upsilon_i \circ \phi_i$. This method induces the following ladder.

$$\begin{tikzcd}
z_i \arrow[r, "\zeta_i"] \arrow[d, "\phi_i"'] \arrow[dr, bend left, "\widetilde{\tau}_{(i,i+1)}"] \arrow[drr, bend left, "\widetilde{\tau}_{(i,i+2)}"] & z_{i+1} \arrow[r, "\zeta_{i+1}"] \arrow[d, "\phi_{i+1}"] & z_{i+2} \arrow[r] \arrow[d, "\phi_{i+2}"] & \cdots \\
y_i \arrow[r, "\upsilon_i"'] & y_{i+1} \arrow[r, "\upsilon_{i+1}"'] & y_{i+2} \arrow[r] & \cdots
\end{tikzcd} \tag{2.35}$$

Recall that hocolimits are defined in [7] on page 209. Recall Definition 2.2.3 and Definition 2.2.4 which states that the hocolimit of the direct system

$$y_i \xrightarrow{\upsilon_i} y_{i+1} \xrightarrow{\upsilon_{i+1}} y_{i+2} \xrightarrow{\upsilon_{i+2}} \cdots$$

is defined as the mapping cone of the "id $-$ shift" map. There are coproduct inclusions

$$\iota_r : y_r \to \coprod_{k=i}^{\infty} y_k$$

for $r \geq i$. Let $r \geq i$ be an integer.

$$\begin{tikzcd}
\coprod_{k=i}^{\infty} y_k \arrow[r, "\mathrm{id}-\mathrm{shift}"] & \coprod_{k=i}^{\infty} y_k \arrow[r, dashed, "\varPhi"] & \operatorname{hocolim} y_i \arrow[r] & {} \\
y_r \arrow[u, hook, "\iota_r"] & y_r \arrow[u, hook, "\iota_r"] \arrow[ur, "f_r"'] & &
\end{tikzcd} \tag{2.36}$$

Now, because $\varPhi \circ (\mathrm{id} - \mathrm{shift}) = 0$, we see that $f_{r+1} \circ \upsilon_r = f_r$. Hence the ladder in Figure (2.35) can be extended in the following way.

$$\begin{tikzcd}
z_i \arrow[r, "\zeta_i"] \arrow[d, "\phi_i"'] \arrow[dr, bend left, "\widetilde{\tau}_{(i,i+1)}"'] \arrow[drr, bend left, "\widetilde{\tau}_{(i,i+2)}"] \arrow[rrrr, bend left=40, "g_i"] & z_{i+1} \arrow[r, "\zeta_{i+1}"] \arrow[d, "\phi_{i+1}"] \arrow[rrr, bend left=30, "g_{i+1}"] & z_{i+2} \arrow[r] \arrow[d, "\phi_{i+2}"] \arrow[rr, bend left=20, "g_{i+2}"] & \cdots & \mathsf{E}_n \\
y_i \arrow[r, "\upsilon_i"'] \arrow[rrrr, bend right=40, "f_i"'] & y_{i+1} \arrow[r, "\upsilon_{i+1}"'] \arrow[rrr, bend right=30, "f_{i+1}"'] & y_{i+2} \arrow[r] \arrow[rr, bend right=20, "f_{i+2}"'] & \cdots & \mathsf{E}_{n+k}
\end{tikzcd} \tag{2.37}$$

There are two distinguished triangles:

$$\coprod_{k=i}^{\infty} y_k \xrightarrow{\mathrm{id}-\mathrm{shift}} \coprod_{k=i}^{\infty} y_k \dashrightarrow \operatorname{hocolim} y_i \longrightarrow \varSigma \coprod_{k=i}^{\infty} y_k \tag{2.38}$$

and

$$\coprod_{k=i}^{\infty} z_k \xrightarrow{\mathrm{id}-\mathrm{shift}} \coprod_{k=i}^{\infty} z_k \dashrightarrow \operatorname{hocolim} z_i \longrightarrow \varSigma \coprod_{k=i}^{\infty} z_k, \tag{2.39}$$

which, by axiom (TR3) of triangulated categories (see, for example, [42, Definition 10.2.1]) can be connected with morphisms in the following way, creating a commutative diagram with rows distinguished triangles.

$$\begin{tikzcd}
\coprod_{k=i}^{\infty} z_k \arrow[r, "\mathrm{id-shift}"] \arrow[d, "\coprod \phi_k"'] & \coprod_{k=i}^{\infty} z_k \arrow[r, dashed] \arrow[d, "\coprod \phi_k"'] & \operatorname{hocolim} z_i \arrow[r] \arrow[d, dashed, "\tau"'] & \Sigma \coprod_{k=i}^{\infty} z_k \arrow[d, "\Sigma \coprod \phi_k"'] \\
\coprod_{k=i}^{\infty} y_k \arrow[r, "\mathrm{id-shift}"] & \coprod_{k=i}^{\infty} y_k \arrow[r, dashed] & \operatorname{hocolim} y_i \arrow[r] & \Sigma \coprod_{k=i}^{\infty} y_k
\end{tikzcd} \tag{2.40}$$

Hence, Figure (2.35) can be completed with a morphism from E_n to E_{n+k}.

$$\begin{tikzcd}
z_i \arrow[r, "\zeta_i"] \arrow[d, "\phi_i"'] \arrow[dr, bend left, "\widetilde{\tau}_{(i,i+1)}"'] \arrow[drr, bend left, "\widetilde{\tau}_{(i,i+2)}"'] \arrow[rrrr, bend left=40, "g_i"] & z_{i+1} \arrow[r, "\zeta_{i+1}"] \arrow[d, "\phi_{i+1}"] \arrow[rrr, bend left=30, "g_{i+1}"] & z_{i+2} \arrow[r] \arrow[d, "\phi_{i+2}"] \arrow[rr, bend left=20, "g_{i+2}"] & \cdots & \mathsf{E}_n \arrow[d, "\tau"'] \\
y_i \arrow[r, "\upsilon_i"'] \arrow[rrrr, bend right=40, "f_i"'] & y_{i+1} \arrow[r, "\upsilon_{i+1}"'] \arrow[rrr, bend right=30, "f_{i+1}"'] & y_{i+2} \arrow[r] \arrow[rr, bend right=20, "f_{i+2}"'] & \cdots & \mathsf{E}_{n+k}
\end{tikzcd} \tag{2.41}$$

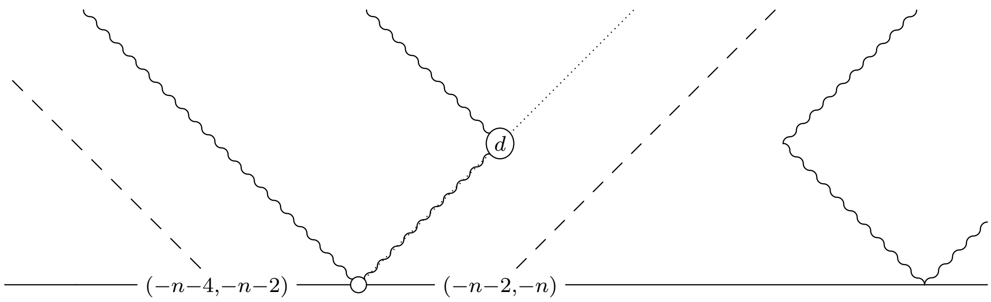
Thus, $\tau' = f_{i+1} \circ \widetilde{\tau}_{(i,i+1)} = \tau \circ g_i$, which means that $\tau' : z_i \to \mathsf{E}_{n+k}$ factors through $\tau : \mathsf{E}_n \to \mathsf{E}_{n+k}$, as required. $\square$

Definition 2.6.7. Let $\mathscr{T}$ be a weakly cluster tilting subcategory and let $X \in \operatorname{obj} \mathscr{T}$. Then, in this context, $H(X)$ (recall the notation set up in Notation 2.3.2) is called the *forbidden region* for X. The reason for this is that the indecomposables associated with $H(X)$ cannot be in $\mathscr{T}$. $\diamond$

Now we are in a position to prove that $\mathscr{T}$ is functorially finite if and only if $\mathfrak{T}$ has a fountain.

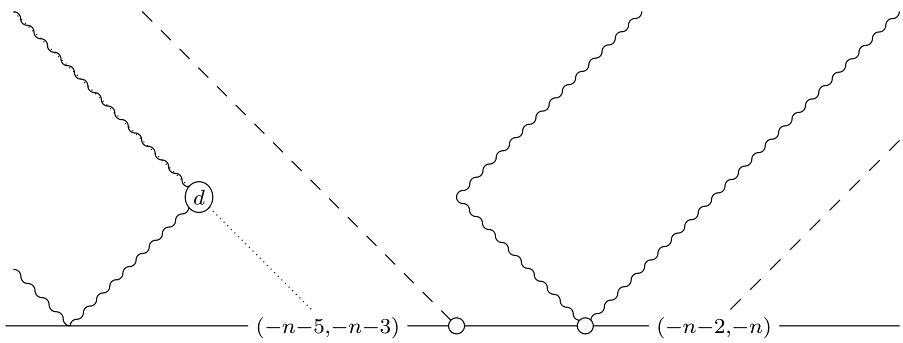
Proposition 2.6.8. *Let $\mathscr{T}$ be a weakly cluster tilting subcategory of the form in Theorem 2.5.12, which contains the hocolimit E_n, with corresponding set of arcs $\mathfrak{T}$. Then $\mathscr{T}$ is functorially finite.*

Proof. We begin by proving that if $\mathfrak{T}$ has a fountain at $-n-2$, then $\overline{\mathscr{D}}$ has almost-right-$\mathscr{T}$-approximations. If $\mathfrak{T}$ has a fountain at $-n-2$, then in the Auslander-Reiten quiver, the slice $(-n-2, —)$ and the coslice $(—, -n-2)$ have an infinite number of $\mathscr{T}$-objects on them, and no other slice or coslice has this property. Suppose we seek an almost-right-$\mathscr{T}$-approximation of d, where d sits on the slice $(-n-3, —)$, or is to the right of it in the quiver, as shown below.

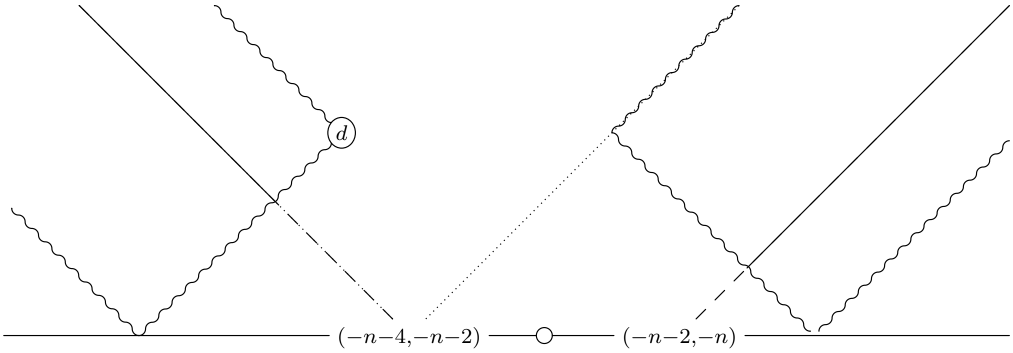


The region which has nonzero maps to d, illustrated with wavy lines, does not contain an infinite number of $\mathscr{T}$-objects, because it intersects only finitely many slices and coslices which all have only finitely many $\mathscr{T}$-indecomposables. This means that if d is on $(-n-3, —)$, or is to the right of it in the quiver, then only finitely many objects in ind $\mathscr{T}$ have nonzero morphisms to d. Hence, $0 \to d$ is an almost-right-$\mathscr{T}$-approximation of d.

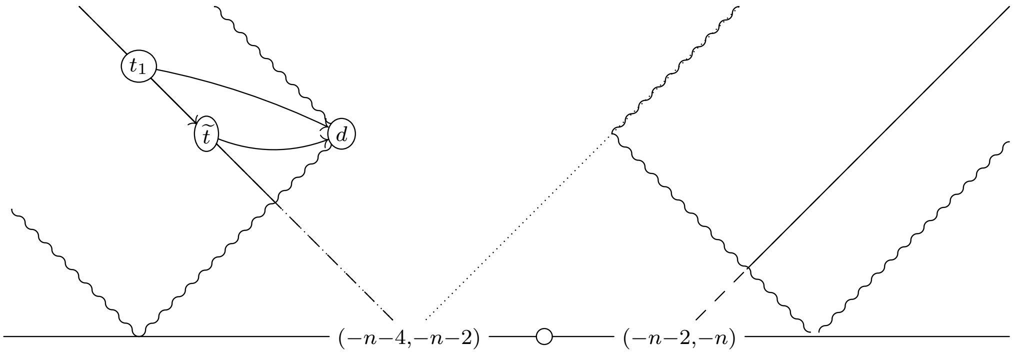
If d is on the coslice $(—, -n-3)$, or is left of it on the quiver, then d also has a right $\mathscr{T}$-approximation, for the same reasons as above. This is shown below.



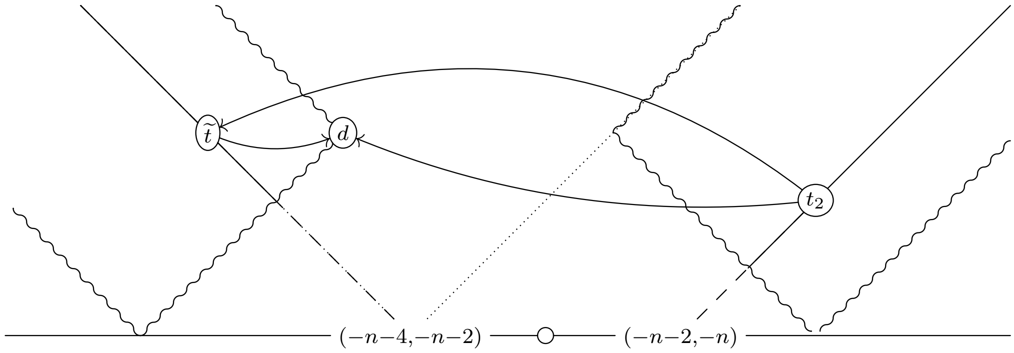
Now suppose that $d \in \mathbb{W}(\Sigma^{n+2}X_0)$. This is illustrated below.



The wavy lines indicate the region with nonzero maps to d. The dotted lines illustrate $\mathbb{W}(\Sigma^{n+2}X_0)$. The dashed lines are the slice/coslice which contain an infinite number of $\mathscr{T}$-objects. The solid lines illustrate the part of the slice/coslice which have an infinite number of $\mathscr{T}$-objects which also map to d. Pick $\widetilde{t} \in \mathscr{T} \cap (—, -n-2)$ and pick a nonzero morphism $\widetilde{t} \to d$. Then $\widetilde{t} = (-n-k, -n-2)$, where $k \geq 4$. By Lemma 2.5 of [21], each $t_1 \to d$, with $t_1 = (-n-s, -n-2)$ for $s \geq k$, factors through $\widetilde{t} \to d$. This is shown below.

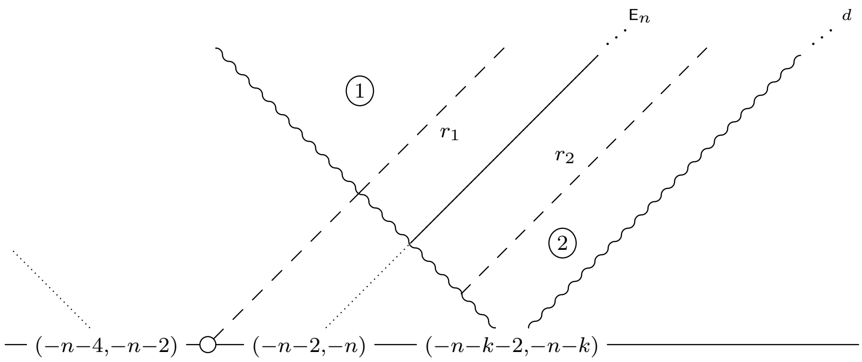


Similarly, by Lemma 2.7 of [21], each $t_2 \to d$, with $t_2 = (-n-2, -n+k)$ for $k \geq 0$ factors through $\widetilde{t} \to d$. This is shown below.



Hence, $\widetilde{t} \to d$ is an almost-right-$\mathscr{T}$-approximation of d, because $\widetilde{t}$ deals with all but finitely many indecomposables of $\mathscr{T}$. This is because the wavy region intersects finitely many slices and coslices, and except for $(-n-2, —)$ and $(—, -n-2)$ each has only finitely many indecomposable $\mathscr{T}$-objects. Therefore, by Lemma 2.6.3, d has a right-$\mathscr{T}$-approximation. We have now dealt with each $d \in \operatorname{ind} \mathscr{D}$.

Now, suppose d is a hocolimit object. First we consider $d = \mathsf{E}_{n+k}$ for $k \leq 0$. This is illustrated below.

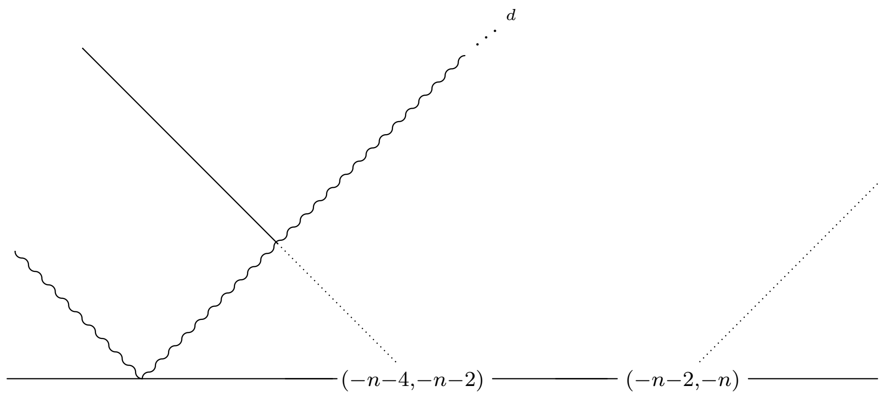


To clarify, region 1 is the V-shaped region with the wavy line on the left-hand-side and the dashed line labelled r_1 on the right-hand-side (note, $r_1 = (-n-3, —)$). There are no $\mathscr{T}$-objects in region 1 due to the forbidden region property of the $\mathscr{T}$-indecomposables which lie on the slice $(-n-2, —)$. Region 2 is bounded by the dashed line labelled r_2 and the wavy lines (note, $r_2 = (-n-1, —)$). There are only finitely many $\mathscr{T}$-objects in here, because region 2 intersects finitely many slices each with finitely many $\mathscr{T}$-indecomposables. The wavy lines illustrate the region of indecomposables which have nonzero morphisms to d. Note, E_n has nonzero morphisms to d, and is included in this region. In all cases, the boundary lines are part of their respective regions. The solid line indicates the part of the slice $(-n-2, —)$, together with its hocolimit E_n, which has an infinite number of

$\mathscr{T}$-objects which map to d. Let $\mathsf{E}_n \to d$ be a nonzero morphism. We claim $\mathsf{E}_n \to d$ is an almost-right-$\mathscr{T}$-approximation. Indeed, if $t \in \mathscr{T} \cap (-n-2, —)$ then $t \to d$ factorises as $t \to \mathsf{E}_n \to d$, by Lemma 2.6.6.

Now let $d = \mathsf{E}_{n+1}$. The right-$\mathscr{T}$-approximation is trivially zero, because $\mathrm{Hom}_{\overline{\mathscr{D}}}(\mathscr{T}, d) = 0$ since $d = \Sigma\mathsf{E}_n \in \Sigma\mathscr{T}$ and $\mathscr{T}$ is weakly cluster tilting.

Finally, let $d = \mathsf{E}_{n+k}$ for $k > 1$. The following illustrates this.



Pick any $\widetilde{t} \in \mathscr{T} \cap (—, -n-2)$, and a nonzero morphism $\widetilde{t} \to d$, where $\widetilde{t} = (-n-k, -n-2)$. Then this is an almost-right-$\mathscr{T}$-approximation of d, for if $t \in \mathscr{T} \cap (—, -n-2)$ is equal to $(-n-s, -n-2)$, for $s \geq k$, then $t \to d$ factors through $\widetilde{t} \to d$, by Lemma 2.6.5. Note, $\mathscr{T} \cap (-n-2, —)$ has only zero morphisms to d. Therefore, there exists a right-$\mathscr{T}$-approximation of d.

This covers all possibilities of what d can be if d is one of the hocolims. In every case, there exists a right-$\mathscr{T}$-approximation of d, so $\mathscr{T}$ is right-approximating. By Lemma 3.2 of [31], this is enough to show that $\mathscr{T}$ is also left-approximating. Hence, if $\mathfrak{T}$ has a fountain at $-n-2$, then $\mathscr{T}$ is functorially finite. $\square$

Lemma 2.6.9. *Let $\mathscr{T}$ be a subcategory of $\overline{\mathscr{D}}$ associated with the arc diagram $\mathfrak{T}$, where $\mathfrak{T}$ is a maximal, non-crossing, locally finite set of finite arcs. Then $\mathscr{T}$ is not cluster tilting.*

Proof. If a category is weakly cluster tilting, but fails to be cluster tilting, then that category must fail to be functorially finite. This in turn amounts to the category failing to be both right and left-approximating. So we prove that $\mathscr{T}$ is not right-approximating. Let (p, q) be an arc in $\mathfrak{T}$. By Lemma 2.5.9, there exists a strong overarc $(p-\delta_1, q+\varepsilon_1) \in \mathfrak{T}$ where $\varepsilon_1, \delta_1 > 0$. But this arc has a strong overarc $(p-\delta_2, q+\varepsilon_2)$, where $\varepsilon_2 > \varepsilon_1$, and $\delta_2 > \delta_1$. We are led to a sequence of arcs in $\mathfrak{T}$

$$(p, q), (p-\delta_1, q+\varepsilon_1), (p-\delta_2, q+\varepsilon_2), (p-\delta_3, q+\varepsilon_3), \cdots$$

where

$$0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \cdots$$

and

$$0 < \delta_1 < \delta_2 < \delta_3 < \cdots .$$

Now, if $(p,q) \in \mathfrak{T}$, then this corresponds to $\Sigma^{-q} X_{q-p-2} \in \mathscr{T}$, and this can be rewritten as $\Sigma^{-p-2-i} X_i$ where $i = q-p-2$ (and note that $i \geq 0$ since $q \geq p+2$). Hence (p,q) lies on the slice whose hocolimit is $\mathsf{E}_{-p-2} \in \overline{\mathscr{D}}$. We claim that E_{-p-2} has no right-$\mathscr{T}$-approximation. Let $(p-\delta, q+\varepsilon)$ be an arc in $\mathfrak{T}$, where $\varepsilon, \delta > 0$. Then the corresponding indecomposable in $\mathscr{T}$ is $\Sigma^{-q-\varepsilon} X_{q-p-2+\varepsilon+\delta}$ which can be rewritten as $\Sigma^{-p-2-j} X_k$, where $j = -p-2+q+\varepsilon$, and $k = -p-2+q+\varepsilon+\delta$. Now, it is clear that $j \geq 0$ and $k \geq j$. Forgiving the abuse of notation, this leads us to conclude that $(p-\delta, q+\varepsilon) \in \mathbb{W}((p,q))$, and therefore the indecomposable corresponding to the arc $(p-\delta, q+\varepsilon)$ is in the region of the Auslander-Reiten quiver which has nonzero maps to E_{-p-2}.

Now, let us again excuse the abuse of notation and consider

$$H^+((p-1, q-1)) = \{(a,b) \mid q \leq b \text{ and } p \leq a \leq q-2\}$$

and

$$H^-((p-1, q-1)) = \{(a,b) \mid a \leq p-2 \text{ and } p \leq b \leq q-2\}.$$

Clearly, $(p-\delta, q+\varepsilon) \notin H^+((p-1, q-1))$, since it fails the first condition, and also $(p-\delta, q+\varepsilon) \notin H^-((p-1, q-1))$ since it fails the second condition. Hence,

$$(p-\delta, q+\varepsilon) \notin H((p-1, q-1))$$

and so there are no nonzero morphisms from the indecomposable corresponding to the arc (p,q) to the indecomposable corresponding to the arc $(p-\delta, q+\varepsilon)$. Similarly, there are no nonzero morphisms in the other direction, either.

The sequence

$$(p,q), (p-\delta_1, q+\varepsilon_1), (p-\delta_2, q+\varepsilon_2), (p-\delta_3, q+\varepsilon_3), \cdots$$

gives an infinite sequence of indecomposables in $\mathscr{T}$ all of which have nonzero morphisms to E_{-p-2}. Let us group all of these together in a set T. We showed above that there are no nonzero morphisms from any object in the set T to any other object in the set T, except for that same object itself. A right-$\mathscr{T}$-approximation of E_{-p-2} must be of the form $\tau : \widetilde{t_1} \oplus \cdots \oplus \widetilde{t_n} \to \mathsf{E}_{-p-2}$. But for such an object $\widetilde{t_1} \oplus \cdots \oplus \widetilde{t_n}$ to exist, it must have an indecomposable summand $\widetilde{t_l}$ which allows nonzero morphisms from infinitely many indecomposables in T, but no such $\widetilde{t_l}$ exists. Hence, the $\overline{\mathscr{D}}$-object E_{-p-2} has no right $\mathscr{T}$-approximation, and therefore $\mathscr{T}$ is not cluster tilting. $\square$

Theorem 2.6.10. *If a weakly cluster tilting subcategory $\mathscr{T}$ of $\overline{\mathscr{D}}$ corresponds to a set of arcs $\mathfrak{T}$, then $\mathscr{T}$ is cluster tilting if and only if $\mathfrak{T}$ has an arc to infinity and a fountain.* $\diamond$

Proof. Theorem 2.5.12 says that $\mathscr{T}$ is weakly cluster tilting if and only if $\mathfrak{T}$ is either a set of finite arcs which is maximal, non-crossing and locally finite; or $\mathscr{T}$ has an arc to infinity and a fountain. By Theorem 2.6.8, in the second case where $\mathfrak{T}$ has an arc to infinity and a fountain, $\mathscr{T}$ is functorially finite (and is hence cluster tilting). By Theorem 2.6.9, in the first case where $\mathfrak{T}$ is a set of finite arcs which is maximal, non-crossing and locally finite, $\mathscr{T}$ is not functorially finite (and is hence not cluster tilting). This proves the theorem. $\square$

Chapter 3

A Caldero-Chapoton Map Depending on a Torsion Class

Frieze patterns of integers were studied by Conway and Coxeter, see [13] and [14]. Let $\mathscr{C}$ be the cluster category of Dynkin type A_n. Indecomposables in $\mathscr{C}$ correspond to diagonals in an $(n+3)$-gon. Work done by Caldero and Chapoton showed that the Caldero-Chapoton map (which is a map dependent on a fixed object R of a category, and which goes from the set of objects of that category to $\mathbb{Z}$), when applied to the objects of $\mathscr{C}$ can recover these friezes, see [10]. This happens precisely when R corresponds to a triangulation of the $(n+3)$-gon, i.e. when R is basic and cluster tilting. Later work (see [6], [22]) generalised this connection with friezes further, now to d-angulations of the $(n+3)$-gon with R basic and rigid. In this chapter, we extend these generalisations further still, to the case where the object R corresponds to a general Ptolemy diagram, i.e. R is basic and $\mathrm{add}(R)$ is the most general possible torsion class (where the previous efforts have focused on special cases of torsion classes).

3.1 Introduction

In [13], [15] and their direct sequel [14], frieze patterns of integers were considered by Conway and Coxeter. A frieze pattern of Dynkin type A_n consists of $n+2$ infinite, interlacing rows of positive integers (with both the top and bottom row consisting entirely of ones) satisfying the so-called *unimodular rule*, which says for all adjacent numbers forming a diamond in the frieze

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

we have $ad - bc = 1$. An example frieze lifted from [13] is illustrated below.

$$\begin{array}{cccccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\cdots & & 3 & & 1 & & 2 & & 2 & & 1 & & 3 & \\
 & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}$$

It was shown by Conway and Coxeter that frieze patterns arise from triangulations of (simple) polygons via applying the following algorithm.

(i) Triangulate the given polygon.

(ii) Choose a vertex and assign to it the value of 0.

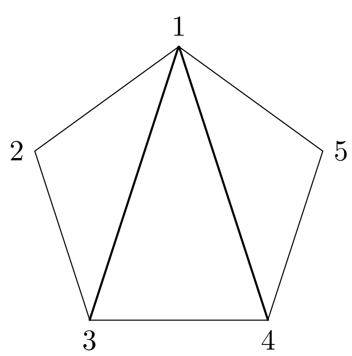
(iii) To each vertex which creates a triangle with this given vertex, assign the value 1.

(iv) For each pair of vertices which already have values assigned, assign their sum as the value of any missing vertices which form a triangle with the given pair. Do this inductively until all vertices have values assigned.

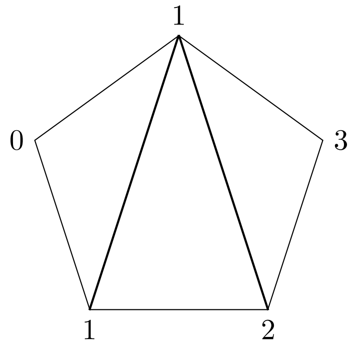
(v) Starting at the vertex labelled 0, read counter-clockwise the numbers until you come back to the vertex labelled 0. These numbers (not including the zeroes at the beginning and end) form a diagonal in the frieze pattern.

(vi) To get the next diagonal (reading left-to-right) in the frieze pattern, repeat this procedure moving the initial 0 to the next vertex in the polygon (moving counter-clockwise).

Example 3.1.1. Let



be a triangulation of the pentagon (with vertices labelled 1 to 5 for notational convenience). Suppose the vertex 2 is picked to begin with. Assign it the value 0. Then the vertices 1 and 3 will be assigned the value 1. Applying the algorithm, the vertex 4 will have the value $1+1=2$ assigned to it, and the vertex 5 will have value $1+2=3$ assigned to it. This is shown below.



This yields the frieze diagonal 1231. Repeating this procedure but now with vertex 3 being assigned the value 0 yields the frieze diagonal 1211. Continuing in this manner generates the example frieze lifted from [13], seen above. ◇

Definition 3.1.2. Let U be a vector space over $\mathbb{C}$. The *Grassmannian of n-dimensional subspaces* is, as a set, simply the set of all n-dimensional vector subspaces of U and is denoted $\mathsf{Gr}(U,n)$. It is an *algebraic variety* over the complex numbers, which means that locally, it looks like the zero set of a finite set of polynomials. See, for example, [25, p. 15]. An algebraic variety has a topology, the so-called *Zariski topology*. If U is a module over an algebra A, then the subset of $\mathsf{Gr}(U,n)$ which consists of A-submodules is known as the *module Grassmannian of U*. It is a closed subset in the Zariski topology, and is hence itself an algebraic variety. ◇

Let $\mathscr{C}$ be a category which is $\mathbb{C}$-linear and Hom-finite, and let $c, R \in \operatorname{obj}\mathscr{C}$. In [10, Section 5], Caldero and Chapoton discovered that the formula

$$\rho(c) = \chi(\mathsf{Gr}\, Gc)$$

in a very special case recovers Conway-Coxeter friezes. Here,

$$G(-) = \operatorname{Hom}_{\mathscr{C}}(R,-) : \mathscr{C} \to \mathsf{mod}(B)$$

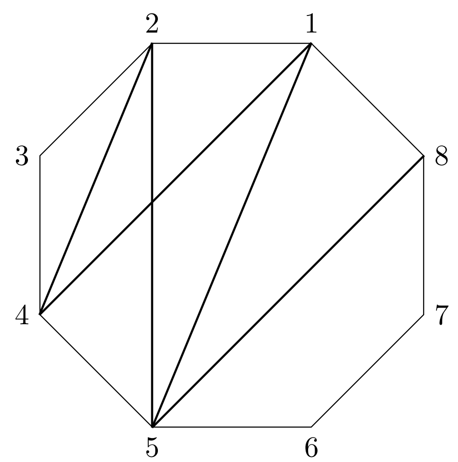
is a functor, where $B = \operatorname{End}_{\mathscr{C}}(R)$ is the endomorphism algebra and mod is the category of finitely generated right modules, $\mathsf{Gr}\, Gc$ is the module Grassmannian of submodules of Gc, and χ is the *Euler characteristic* (defined by cohomology with compact support), see [16, Page 93]. In particular, Caldero and Chapoton let the category $\mathscr{C}$ be the cluster category $\mathsf{C}(A_n)$ of Dynkin type A_n (see Definition 1.12.14, or [9, Section 1] and [11, Section 5], for more details). Now, indecomposables in $\mathsf{C}(A_n)$ correspond to diagonals in an $(n+3)$-gon. Caldero and Chapoton let the object R represent a triangulation; this is equivalent to $\operatorname{add}(R)$ being a cluster tilting subcategory of $\mathsf{C}(A_n)$, for clearly, $\mathsf{C}(A_n)$ is functorially finite (see Definition 2.6.1), and is weakly cluster tilting (see Section 2.5) because in a triangulation, the diagonals do not cross. The formula then recovered the combinatorial work of Conway and Coxeter, yielding friezes of type A_n, see [10, Section 5].

In [6], Bessenrodt, Holm and Jørgensen generalised the work done by Conway and Coxeter [13], [14] to d-angulations of polygons in a purely combinatorial way. In a later paper [22], Holm and Jørgensen proved that the Caldero-Chapoton formula gives the generalised friezes in [6] where now R corresponds to a d-angulation, not a triangulation. See [23] for a generalisation to polynomial values. We remark that in both [6] and [22], everything works for general polygon dissections, not just d-angulations, and that a polygon dissection corresponds to R being a rigid object.

Now note that in the cases considered so far, $\operatorname{add}(R)$ is a torsion class in $\mathsf{C}(A_n)$ (i.e. it is precovering and closed under extensions), albeit a rather special one. It is natural to consider the "ultimate generalisation" where $\operatorname{add}(R)$ is a general torsion class, and that is what we shall do in this chapter. By [24], this means that R corresponds to a so-called *Ptolemy diagram.*

Definition 3.1.3. Let $P = P_n$ be an n-gon with vertices labelled $1, 2, \ldots, n$. Let $\mathfrak{D} = \mathfrak{D}(P_n)$ be a set of diagonals of P_n. A Ptolemy diagram in P is a set of diagonals which satisfy the following rule: let $\{a_1, a_2\}$ and $\{b_1, b_2\}$ be two diagonals in $\mathfrak{D}$. If $\{a_1, a_2\}$ and $\{b_1, b_2\}$ cross, then all pairs $\{a_1, b_1\}$, $\{a_1, b_2\}$, $\{a_2, b_1\}$ and $\{a_2, b_2\}$ which are diagonals are also in $\mathfrak{D}$. $\diamond$

Example 3.1.4. The following is an example of a Ptolemy diagram with eight vertices.



Here, the diagonals can be given by the pairs of integers $\{2,4\},\{2,5\},\{1,4\},\{1,5\}$, and $\{5,8\}$. $\diamond$

We also remind the reader of the definition of a *rigid object*.

Definition 3.1.5. Let $\mathscr{C}$ be a $\mathbb{C}$-linear triangulated category with suspension functor Σ. An object R of $\mathscr{C}$ is said to be rigid if it satisfies $\mathrm{Hom}_{\mathscr{C}}(R, \Sigma R) = 0$. $\diamond$

3.2 Properties of the Caldero-Chapoton Map

Recall Remark 1.12.19, which describes the coordinate system on the Auslander-Reiten quiver of $\mathsf{C}(A_n)$. In this section, the requirement that $\mathscr{C} = \mathsf{C}(A_n)$ is relaxed, although, in particular, the results in this section apply to $\mathsf{C}(A_n)$.

Setup 3.2.1. Let $\mathscr{C}$ be a $\mathbb{C}$-linear, Hom-finite, triangulated category with an object R, and set $B = \mathrm{End}_{\mathscr{C}}(R)$. Define the functor

$$G(-) = \mathrm{Hom}_{\mathscr{C}}(R, \Sigma(-)) : \mathscr{C} \to \mathsf{mod}(B)$$

and define the (modified) Caldero-Chapoton map

$$\rho(c) = \chi(\mathsf{Gr}\, Gc)$$

on the objects c of $\mathscr{C}$. Here, $\mathsf{Gr}\, Gc$ is the module Grassmannian of submodules of Gc (see Definition 3.1.2), and χ (defined by cohomology with compact support) is the *Euler characteristic*, see [16, Page 93]. $\diamond$

Definition 3.2.2. An algebraic variety has a topology, called the Zariski topology. The Boolean algebra generated by the open sets consists, by definition, of the *constructible sets*. $\diamond$

Remark 3.2.3. A *constructible map* sends constructible sets to constructible sets. For complex algebraic varieties, the Euler characteristic is additive on disjoint unions of constructible sets, see [16, Page 92, item (3)]. $\diamond$

Lemma 3.2.4. *In the situation of Setup 3.2.1, we have for $x, y \in \mathrm{obj}\,\mathscr{C}$, that $\rho(x \oplus y) = \rho(x)\rho(y)$, see [10, Corollary 3.7].*

Proof. Consider the sequence of morphisms

$$x \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} x \oplus y \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} y$$

and apply G to obtain

$$Gx \xrightarrow{\iota} Gx \oplus Gy \xrightarrow{\pi} Gy$$

where ι and π are the natural injection and projection respectively. Apply Gr to obtain

$$\begin{array}{ccccc} \mathsf{Gr}\,Gx & \longleftarrow & \mathsf{Gr}\,G(x \oplus y) & \longrightarrow & \mathsf{Gr}\,Gy \\ \\ \iota^{-1}M & \longleftarrow\!\!\dashv & M & \longmapsto & \pi M \end{array}$$

where $M \in \mathsf{Gr}\,G(x \oplus y)$. This yields $\varGamma$, a surjective constructible map with affine fibres, defined by

$$\varGamma : \mathsf{Gr}\,G(x \oplus y) \to (\mathsf{Gr}\,Gx) \times (\mathsf{Gr}\,Gy),\ M \mapsto (\iota^{-1}M, \pi M),$$

see [10, Lemma 3.11] which is stated for Auslander-Reiten sequences but has a proof which works in our situation. Taking χ of this gives the result, namely that

$$\chi(\mathsf{Gr}\,G(x \oplus y)) = \chi((\mathsf{Gr}\,Gx) \times (\mathsf{Gr}\,Gy)) = \chi(\mathsf{Gr}\,Gx)\chi(\mathsf{Gr}\,Gy),$$

hence $\rho(x \oplus y) = \rho(x)\rho(y)$, see [16, p. 93, Exercise]. $\square$

Lemma 3.2.5. *If*

$$\tau r \longrightarrow y \xrightarrow{\upsilon} r \xrightarrow{\sigma} \varSigma\tau r$$

is an Auslander-Reiten triangle and

$$m \longrightarrow z \xrightarrow{\zeta} r \xrightarrow{\delta} \varSigma m$$

is a distinguished triangle with $\delta \neq 0$, then $\exists h : \varSigma m \to \varSigma\tau r$ such that $\sigma = h \circ \delta$.

Proof. Because $\delta \neq 0$, we have that ζ is not a split epimorphism, and because

$$\tau r \longrightarrow y \xrightarrow{\upsilon} r \xrightarrow{\sigma} \varSigma\tau r$$

is an Auslander-Reiten triangle, there exists $f : z \to y$ such that $\upsilon \circ f = \zeta$. The following diagram illustrates this.

$$\begin{tikzcd}
 & & m \arrow[d] & \\
 & & z \arrow[dl, "f"'] \arrow[d, "\zeta"] \arrow[dr, "0"] & \\
\tau r \arrow[r] & y \arrow[r, "\upsilon"'] & r \arrow[r, "\sigma"'] \arrow[d, "\delta"] & \Sigma\tau r \\
 & & \Sigma m &
\end{tikzcd}$$

Now, because $\sigma\circ\zeta = \sigma\circ\upsilon\circ f = 0\circ f = 0$, there exists $h : \Sigma m \to \Sigma\tau r$ such that $\sigma = h\circ\delta$, as shown below.

$$\begin{tikzcd}
 & & m \arrow[d] & \\
 & & z \arrow[dl, "f"'] \arrow[d, "\zeta"] \arrow[dr, "0"] & \\
\tau r \arrow[r] & y \arrow[r, "\upsilon"'] & r \arrow[r, "\sigma"'] \arrow[d, "\delta"] & \Sigma\tau r \\
 & & \Sigma m \arrow[ur, "h"'] &
\end{tikzcd}$$

□

Setup 3.2.6. Let $\mathscr{C}$ be a triangulated, $\mathbb{C}$-linear, 2-Calabi-Yau category. Let $R \in \mathscr{C}$ be an object, and let $B = \operatorname{End}_{\mathscr{C}}(R)$.

Moreover, let m, r be indecomposable objects of $\mathscr{C}$ satisfying

$$\dim_{\mathbb{C}} \operatorname{Ext}^1(r,m) = \dim_{\mathbb{C}} \operatorname{Ext}^1(m,r) = 1$$

and $\operatorname{Ext}^1(R,r) = 0$. ◇

Setup 3.2.7. We have the following distinguished triangles

$$m \xrightarrow{\mu} a \longrightarrow r \xrightarrow{\delta} \Sigma m,$$

$$r \longrightarrow b \xrightarrow{\beta} m \xrightarrow{\epsilon} \Sigma r$$

where δ, $\epsilon \neq 0$ and due to the one-dimensionality of the Ext^1-spaces, these are the unique non-split extensions between m and r up to isomorphism.

Upon applying G to the distinguished triangles and rolling, we obtain the following exact sequences in $\mathsf{mod}(B)$:

$$G(\Sigma^{-1}r) \xrightarrow{G(\Sigma^{-1}\delta)} Gm \xrightarrow{G\mu} Ga \longrightarrow 0,$$

$$0 \longrightarrow Gb \xrightarrow{G\beta} Gm \xrightarrow{G\epsilon} G(\Sigma r).$$

The zeroes result from the condition $\mathrm{Ext}^1(R, r) = 0$. ◇

Lemma 3.2.8. *Let M be a submodule of Gm. Either* $\mathrm{Ker}\, G\mu \subseteq M$ *or* $M \subseteq \mathrm{Im}\, G\beta$*, but not both.*

Proof. "Either/or": Suppose $M \not\subseteq \mathrm{Ker}\, G\epsilon = \mathrm{Im}\, G\beta$. Then we must show that $\mathrm{Im}\, G(\Sigma^{-1}\delta) = \mathrm{Ker}\, G\mu \subseteq M$.

Now, because $M \not\subseteq \mathrm{Ker}\, G\epsilon$, there exists $x \in M$ such that $(G\epsilon)(x) \neq 0$. Note, $(G\epsilon)(x) = (\Sigma\epsilon) \circ x$ because after applying G to the morphism $\epsilon : m \to \Sigma r$ we obtain $\mathrm{Hom}(R, \Sigma\epsilon) : (R, \Sigma m) \to (R, \Sigma^2 r)$.

So now consider

$$x^* = \mathrm{Hom}(x, \Sigma^2 r) : \mathrm{Hom}(\Sigma m, \Sigma^2 r) \to \mathrm{Hom}(R, \Sigma^2 r),\ \Sigma\epsilon \mapsto (\Sigma\epsilon) \circ x \neq 0$$

and consider the functor $D(-) = \mathrm{Hom}_{\mathbb{C}}(-, \mathbb{C})$ applied to this, shown as the upper horizontal morphism in the commutative square below,

$$\begin{tikzcd}
D\,\mathrm{Hom}(R, \Sigma^2 r) \arrow[r, "D(x^*)"] \arrow[d, "\cong"] & D\,\mathrm{Hom}(\Sigma m, \Sigma^2 r) \arrow[d, "\cong"] \\
\mathrm{Hom}(r, R) \arrow[r, "x_* = \mathrm{Hom}(r, x)"'] & \mathrm{Hom}(r, \Sigma m),
\end{tikzcd}$$

which exists by Serre duality.

We have that $x^* \neq 0 \Rightarrow D(x^*) \neq 0 \Rightarrow x_* \neq 0 \Rightarrow x_*$ is surjective due to the fact that m and r satisfy $\dim_{\mathbb{C}} \mathrm{Hom}(r, \Sigma m) = 1$.

Since x_* is onto and $\delta : r \to \Sigma m$ is given, there exists $z : r \to R$ such that

$$\begin{tikzcd}
 & r \arrow[dl, "z"'] \arrow[d, "\delta = x_*(z)"] \\
R \arrow[r, "x"'] & \Sigma m
\end{tikzcd}$$

is commutative.

Now, apply G to $\Sigma^{-1}\delta : \Sigma^{-1} r \to m$ to obtain $\delta_* = \mathrm{Hom}(R, \delta) : \mathrm{Hom}(R, r) \to \mathrm{Hom}(R, \Sigma m)$ and note that requiring $\mathrm{Im}\, G(\Sigma^{-1}\delta) \subseteq M$ is equivalent to requiring that $\mathrm{Im}\, \delta_* \subseteq M$.

So now let $w \in \operatorname{Hom}(R, r)$ and consider $\delta_*(w) = \delta \circ w$. The following diagram

$$\begin{tikzcd}
& R \arrow[d, "w"] \arrow[ddl, bend right, "z\circ w"'] \arrow[dd, bend left=60, "\delta_*(w)"] \\
& r \arrow[d, "\delta"] \arrow[dl, "z"'] \\
R \arrow[r, "x"'] & \Sigma m
\end{tikzcd}$$

where z exists by the above, helps illustrate that

$$\delta_*(w) = \delta \circ w = x \circ z \circ w = x \cdot (z \circ w)$$

which is in M, as required, because $x \in M$ and $z \circ w \in B$.

"Not both": There is an Auslander-Reiten triangle $\Sigma r \longrightarrow y \longrightarrow r \xrightarrow{\sigma} \Sigma^2 r$, and because $r \xrightarrow{\delta} \Sigma m \neq 0$, we have that σ factors as $r \xrightarrow{\delta} \Sigma m \xrightarrow{\Psi} \Sigma^2 r$ by Lemma 3.2.5. Since $\Psi\delta \neq 0$, we have that $\Psi \neq 0$, and since $\Psi \in \operatorname{Ext}^1(\Sigma m, \Sigma r) \cong \operatorname{Ext}^1(m, r)$, which is one-dimensional, Ψ is a nonzero scalar multiple of $\Sigma\epsilon$. This means that $\Sigma(\epsilon)\delta \neq 0$, which implies $G(\epsilon\Sigma^{-1}\delta) \neq 0$.

Now suppose $\operatorname{Im} G(\Sigma^{-1}\delta) \subseteq M$. Apply $G\epsilon$ to give $\operatorname{Im} G(\epsilon\Sigma^{-1}\delta) \subseteq (G\epsilon)M$. Since $\operatorname{Im} G(\epsilon\Sigma^{-1}\delta)$ is nonzero, $(G\epsilon)M \neq 0$, that is $M \nsubseteq \operatorname{Ker} G\epsilon$, as required. □

Proposition 3.2.9. *We have $\rho(m) = \rho(a) + \rho(b)$.*

Proof. Take the two exact sequences

$$0 \longrightarrow Gb \xrightarrow{G\beta} Gm \xrightarrow{G\epsilon} G(\Sigma r),$$

$$G(\Sigma^{-1} r) \xrightarrow{G(\Sigma^{-1}\delta)} Gm \xrightarrow{G\mu} Ga \longrightarrow 0$$

from Setup 3.2.7 and apply Gr to obtain the following morphisms of algebraic varieties.

$$\mathsf{Gr}\, Gb \hookrightarrow \mathsf{Gr}\, Gm \hookleftarrow \mathsf{Gr}\, Ga$$

$$B \longmapsto (G\beta)(B)$$

$$(G\mu)^{-1}(A) \longleftarrow\!\shortmid A$$

Lemma 3.2.8 implies that $\mathsf{Gr}\,Gm$ is the disjoint union of the images of $\mathsf{Gr}\,Ga$ and $\mathsf{Gr}\,Gb$. The morphisms

$$\mathsf{Gr}\,Gb \hookrightarrow \mathsf{Gr}\,Gm \qquad \text{and} \qquad \mathsf{Gr}\,Gm \hookleftarrow \mathsf{Gr}\,Ga$$

are constructible, and so their images inside $\mathsf{Gr}\,Gm$ are constructible subsets. Therefore, $\chi(\mathsf{Gr}\,Gm) = \chi(\mathsf{Gr}\,Ga)+\chi(\mathsf{Gr}\,Gb)$, by [16, p. 92, item (3)]. This proves the proposition. $\square$

3.3 Ptolemy Diagrams and an Inductive Procedure for computing ρ

Now and throughout the rest of this chapter, let $\mathscr{C} = \mathsf{C}(A_n)$ be the cluster category of Dynkin type A_n, and recall Remark 1.12.19 which describes some of its important properties, including its connection to the $(n+3)$-gon. Let R be a basic object of $\mathscr{C}$ and let $\mathscr{R} = \operatorname{add}(R)$. Recall the definition $B = \operatorname{End}_{\mathscr{C}}(R)$.

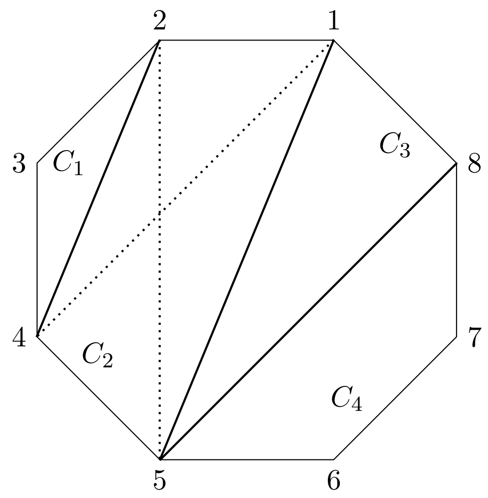
Setup 3.3.1. Let P be an $(n+3)$-gon. As explained in Remark 1.12.19, there is a bijection between the indecomposable objects of $\mathscr{C}$ and the diagonals of P. We will think of edges and diagonals in P in combinatorial terms by identifying an edge or a diagonal with its pair of endpoints, although it is worth keeping in mind the geometric interpretation of diagonals as line segments inside a geometric polygon. In this vein, let P have vertex set $V_P = \{1, 2, \ldots, n+3\}$ and edge set $E_P = \{\{a, b\} \mid a, b \in V_P,\ |b-a| \in \{1, n+2\}\}$. Let $D_P = \{\{a, b\} \mid a, b \in V_P,\ |b-a| \geq 2, \{1, n+3\} \notin D_P\}$ be the set of diagonals of P and D'_P be a subset of D_P consisting of diagonals which are pairwise non-crossing. If in a given context, P is fixed, then whilst working we drop the subscript notation and write V, E, D, D' respectively.

Suppose P and D' are fixed. The diagonals in D' are then referred to as *dissecting diagonals.* The diagonals in D' partition the $(n+3)$-gon into subpolygons called *cells*. Note, if D' has been chosen such that it is maximal, then the corresponding diagonals triangulate the $(n+3)$-gon, and each cell is a triangle. Conversely, if D' is the empty set, then the corresponding partition yields a single cell, namely P itself.

For a particular choice of D', we assign the corresponding cells labels $C_1, C_2, \ldots, C_m$. Then we assign each cell C_i a subset T_i of D as follows. Either T_i is empty, or it is the subset of D containing all interior diagonals of C_i. A nonempty cell is called a *clique* and the interior diagonals are referred to as *clique diagonals*. Note that when C_i is a triangle it is both empty and a clique. Then the collection

$$\mathfrak{P} = E \bigcup D' \bigcup_{i=1}^{m} T_i$$

is called a *Ptolemy diagram* (see [24, Definition 2.1]) and an example of one is illustrated below for the case $n = 5$.



Here, P is an octagon and $D' = \{\{2,4\},\{1,5\},\{5,8\}\}$ is the set of dissecting diagonals, illustrated above with solid lines. We also have that $T_i = \emptyset$ for $i \in \{1,3,4\}$ and for $i = 2$, T_i consists of two diagonals, $\{2,5\}$ and $\{1,4\}$, illustrated above with dotted lines.

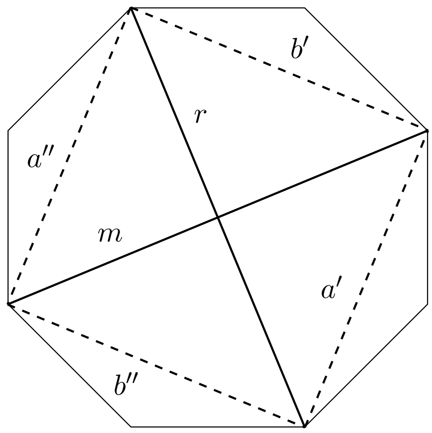
Any given cell can be uniquely identified by a subset of V. In the example above, C_1 is given by $\{2,3,4\}$, C_2 is given by $\{1,2,4,5\}$, C_3 is given by $\{1,5,8\}$ and C_4 is given by $\{5,6,7,8\}$. $\diamond$

Remark 3.3.2. Let $\mathfrak{P}$ be a Ptolemy diagram in the $(n+3)$-gon. Take the collection of all the diagonals in D' and T_i for each i and set the object $R \in \operatorname{obj}\mathscr{C}$ to be the direct sum of the corresponding indecomposables in $\mathscr{C}$. By [24, Theorem A], the subcategory $\operatorname{add}(R)$ is a torsion class of $\mathscr{C}$, and each torsion class has this form. We henceforth use this R in the Caldero-Chapoton map ρ from Setup 3.2.1.

Now let r be any indecomposable summand of R. If r corresponds to a dissecting diagonal, then because a dissecting diagonal crosses no other diagonal in $\mathfrak{P}$, we have that $\operatorname{Ext}^1(R, r) = 0$. If $m \in \operatorname{ind}(\mathscr{C})$ is such that

$$\dim_{\mathbb{C}} \operatorname{Ext}^1(r, m) = \dim_{\mathbb{C}} \operatorname{Ext}^1(m, r) = 1$$

then Proposition 3.2.9 and [22, Figure 5] together say that ρ, the Caldero-Chapoton map from Setup 3.2.1, satisfies $\rho(m) = \rho(a) + \rho(b)$ where $a = a' \oplus a''$ and $b = b' \oplus b''$ have diagonals corresponding to



where if any of a', a'', b', b'' are edges of the polygon, then they are interpreted as the zero object. Using Lemma 3.2.4, this means $\rho(m) = \rho(a')\rho(a'') + \rho(b')\rho(b'')$. $\diamond$

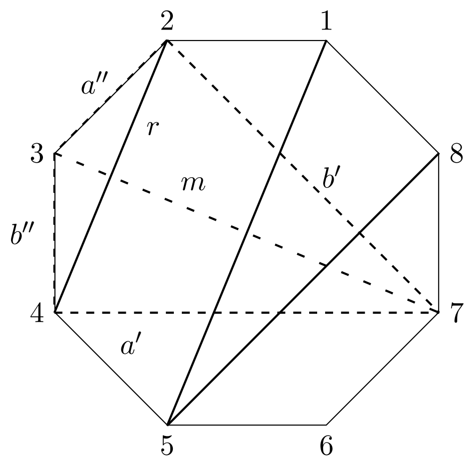
Remark 3.3.3. Remark 3.3.2 provides an inductive method to compute $\rho(m)$. Namely, because $\rho(m) = \rho(a) + \rho(b) = \rho(a' \oplus a'') + \rho(b' \oplus b'') = \rho(a')\rho(a'') + \rho(b')\rho(b'')$, the problem reduces to calculating $\rho(a'), \rho(a''), \rho(b')$ and $\rho(b'')$, each of which is simpler to calculate than $\rho(m)$.

To see why, recall that the R-indecomposables are associated with diagonals of two different types: *dissecting* and *clique*. Draw the diagonal associated with m in the same diagram as the dissecting diagonals of R. Then notice that, if we travel from left to right along the diagonal of m, two things can happen. We either stay inside a single cell or cross a dissecting diagonal.

If we stay inside a single cell and the cell is empty, then $Gm = 0$, so $\rho(m)$ will equal 1. If the cell is a clique, then the calculation of $\rho(m)$ is more involved and the details are found in the next section.

If m is not inside a cell, then travelling along the diagonal of m inevitably crosses a dissecting diagonal. We show an example.

The dissecting diagonals are solid, m is $\{3, 7\}$, and moving along m first encounters $r = \{2, 4\}$. We then utilise Remark 3.3.2 and fill in the diagonals a', a'', b', b'' as in the following figure.



We compute $\rho(m)$ as follows:

$$\rho(m) = \rho(a' \oplus a'') + \rho(b' \oplus b'') = \rho(a')\rho(a'') + \rho(b')\rho(b'')$$

$$\Rightarrow \rho(m) = \rho(a') + \rho(b')$$

where the implication on the second line is because $\rho(0) = 1$.

Now notice that the calculation for $\rho(a')$ and $\rho(b')$ will involve only two dissecting diagonals (instead of the original three for m). Indeed, it will always be the case that, as the inductive process unfolds, there are fewer and fewer dissecting diagonals involved. This is because (after fixing an orientation) a'' and b'' are always to the left of the first dissecting diagonal that m crosses, and so will not cross any dissecting diagonals themselves. Furthermore, each of a' and b' will cross fewer dissecting diagonals than m because they lie to the right of the first dissecting diagonal that m crosses. Indeed, in the example above, whereas m crosses the dissecting diagonals $\{2, 4\}, \{1, 5\}$ and $\{5, 8\}$, a' and b' both only cross $\{1, 5\}$ and $\{5, 8\}$.

We can then repeat the procedure on each of a' and b' (thereby reducing the number of diagonals again) and continue in this manner until we are finished. As there is a finite number of diagonals in a polygon, the algorithm will finish.

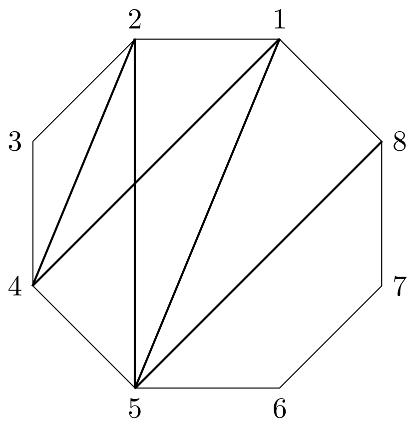
The matter is complicated when there are clique diagonals. When these are present, the same method as above is applied, but the computation of ρ on clique diagonals requires more work, detailed in the forthcoming section. $\diamond$

3.4 Computing ρ on Clique Diagonals

Recall the setup introduced at the beginning of section 3.3 and let the object R of $\mathscr{C}$ be given by a Ptolemy diagram $\mathfrak{P}$ in the polygon P. Throughout this section, let $\mathscr{R} = \mathrm{add}(R)$. By the inductive procedure of Section 3.3, in order to get an algorithm for computing ρ, we need to determine ρ on clique diagonals.

A clique S in the Ptolemy diagram corresponding to R may be given by the set of endpoints of its diagonals. This set is a subset of $\{1, 2, \ldots, n+2, n+3\}$. Consider the example below.

Example 3.4.1. Consider the following Ptolemy diagram $\mathfrak{P}$.



The corresponding set D' is $\{\{2,4\},\{1,5\},\{5,8\}\}$; these are the dissecting diagonals of $\mathfrak{P}$. There is one clique, given by the subset $\{1,2,4,5\}$ of $\{1,2,\ldots,7,8\}$. The diagonals of the clique are $S = \{\{1,4\},\{2,5\}\}$. Note $\{1,2\}$ and $\{4,5\}$ are not in the clique, as they are edges, and $\{2,4\}$ and $\{1,5\}$ are not in the clique, because they are dissecting diagonals. $\diamond$

Setup 3.4.2. The diagonals of a clique S correspond to a set of indecomposables in $\mathscr{C} = \mathsf{C}(A_n)$. Taking add of these gives $\mathscr{S}$, which is itself a subcategory of $\mathscr{R}$. Henceforth, forgiving our abuse of notation we may refer to $\mathscr{S}$ as simply being a clique, and as explained before in Example 3.4.1, this clique is given by certain vertices of the $(n+3)$-gon. This permits us to say things like $\mathscr{S}$ is a clique but $c \in \mathrm{ind}(\mathscr{S})$, i.e. c is an indecomposable object corresponding to one of the diagonals in S. $\diamond$

Remark 3.4.3. Recall that if $\mathscr{A}$ is a $\mathbb{C}$-linear category, then $\mathsf{Mod}(\mathscr{A})$ denotes the abelian $\mathbb{C}$-linear category of contravariant $\mathbb{C}$-linear functors $\mathscr{A} \to \mathsf{Vect}_{\mathbb{C}}$ (see Definition 1.10.3). $\diamond$

Recall that $\mathscr{R} = \mathrm{add}(R)$ and $B = \mathrm{End}_{\mathscr{C}}(R)$.

Remark 3.4.4. By 1.10.8, the functor

$$\Psi : \mathsf{Mod}(\mathscr{R}) \to \mathsf{Mod}(B)$$

$$F \mapsto F(R)$$

is an equivalence of categories. $\diamond$

Remark 3.4.5. Consider the functors

$$\Gamma : \mathscr{C} \to \mathsf{Mod}(\mathscr{R})$$

$$c \mapsto \mathrm{Hom}_{\mathscr{C}}(-, \Sigma c)|_{\mathscr{R}}$$

and

$$G : \mathscr{C} \to \mathsf{Mod}(B)$$

$$c \mapsto \mathrm{Hom}_{\mathscr{C}}(R, \Sigma c).$$

Note that the following diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\Gamma"] \arrow[dr, "G"'] & \mathsf{Mod}(\mathscr{R}) \arrow[d, "\Psi"] \\ & \mathsf{Mod}(B) \end{tikzcd}$$

commutes and due to the fact Ψ is an equivalence, $\rho(c) = \chi(\mathsf{Gr}\, Gc) = \chi(\mathsf{Gr}\, \Gamma c)$ where $\mathsf{Gr}\, \Gamma c = \{M' \subseteq \Gamma c \mid M' \text{ is a subfunctor}\}$. $\diamond$

Henceforth, we will use the notation $\mathscr{C}(-,-)$ to denote $\mathrm{Hom}_{\mathscr{C}}(-,-)$.

Definition 3.4.6. Define the *support* of an additive functor $F : \mathsf{C} \to \mathsf{D}$, denoted $\mathsf{Supp}\, F$, to be the set of objects of C which under F do not map to the zero object of D. $\diamond$

Proposition 3.4.7. *If $c \in \mathrm{ind}(\mathscr{C})$ then a subfunctor $M' \subseteq \Gamma c = \mathscr{C}(-, \Sigma c)|_{\mathscr{R}}$ is determined by $\mathsf{Supp}\, M'$.*

Proof. A subfunctor $M' \subseteq \mathscr{C}(-, \Sigma c)|_{\mathscr{R}}$ is determined by its values on indecomposable objects. On indecomposables, this functor is either 0 or $\mathbb{C}$ and hence M' is completely determined by its support. $\square$

The Grassmannian $\mathsf{Gr}\, \Gamma c$ consists of different subfunctors M' of Γc. If $c \in \mathrm{ind}(\mathscr{C})$ then these subfunctors have different supports by Proposition 3.4.7, so each is an isolated point in the Grassmannian. This gives us the following proposition.

Proposition 3.4.8. *If $c \in \mathrm{ind}(\mathscr{C})$, then each $M' \subseteq \Gamma c$ is an isolated point in $\mathsf{Gr}\, \Gamma c$. Therefore $\rho(c)$ is equal to the number of subfunctors of Γc.*

Proof. The Grassmannian is a finite collection of isolated points, therefore the Euler characteristic χ counts the number of points. Therefore ρ counts the number of points and hence the number of subfunctors. $\square$

In the rest of this section, $\mathscr{S}$ is a clique and $c \in \operatorname{ind}(\mathscr{S})$. We will compute $\rho(c)$. By Proposition 3.4.8 we must count the number of subfunctors of Γc. This is accomplished by Remarks 3.4.9, 3.4.10, 3.4.12, 3.4.14 and 3.4.16.

Remark 3.4.9. Because $\mathsf{Mod}(\mathscr{R})$ has enough projectives (see [3, item c) on page 185]), each subfunctor $M' \subseteq \Gamma c$ is the image of a morphism $P \to \Gamma c$ where P is projective. We can assume P is the direct sum of finitely many indecomposable projectives, each of the form $\mathscr{C}(-,r)|_{\mathscr{R}} = \mathscr{R}(-,r) = P_r$ for $r \in \operatorname{ind}(\mathscr{R})$ (see [3, Proposition 2.2 e)]). This means that M' is the image of a morphism $\bigoplus_{i=1}^{m} P_{r_i} \to \Gamma c$, i.e. M' is the sum of images of morphisms of the form $P_r \to \Gamma c$.

Let $a : \mathscr{R}(-,r) \to \mathscr{C}(-,\Sigma c)|_{\mathscr{R}}$ be such a morphism. By Yoneda's Lemma, it corresponds to an element $\alpha \in \mathscr{C}(r,\Sigma c)$. Note, a is a natural transformation, and on r' it evaluates to $\mathscr{R}(r',r) \xrightarrow{a_{r'}} \mathscr{C}(r',\Sigma c)$ which is the map given by composition (on the left) by α.

Now, there are two possibilities for r. Either it is not in the clique $\mathscr{S}$, which means the diagonals of r and c do not cross. This implies $\mathscr{C}(r,\Sigma c) = 0$, so $\alpha = 0$, and hence $a = 0$. The other possibility for r is that it is in the clique $\mathscr{S}$. In this case, $\operatorname{Im}(a)$ must be determined.

We have that $\mathscr{C}(-,\Sigma c)|_{\mathscr{R}}$ is supported on $\mathscr{S}$ because the clique diagonal corresponding to c only crosses those diagonals in the Ptolemy diagram $\mathfrak{P}$ which are in the clique $\mathscr{S}$. So since $\operatorname{Im}(a)$ is a subfunctor of $\mathscr{C}(-,\Sigma c)|_{\mathscr{R}}$, it is determined by its restriction to $\operatorname{ind}(\mathscr{S})$. Let us investigate the functor $\operatorname{Im}(a)$ on $s \in \operatorname{ind}(\mathscr{S})$. By objectwise computation,

$$(\operatorname{Im}(a))(s) = \operatorname{Im}(\{a_s : \mathscr{C}(s,r) \to \mathscr{C}(s,\Sigma c)\})$$

$$\text{hence } (\operatorname{Im}(a))(s) = \begin{cases} \mathbb{C}, & \text{if } \exists\ \sigma : s \to r \text{ s.t. } \alpha\sigma \neq 0, \\ 0, & \text{if no such } \sigma : s \to r \text{ exists.} \end{cases} \tag{3.1}$$

$\diamond$

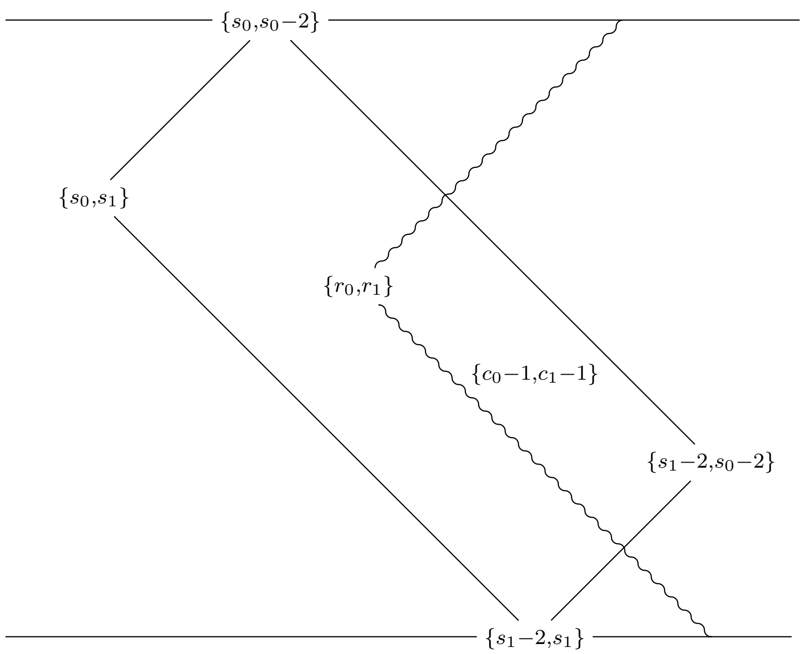
Remark 3.4.10. Consider again the indecomposable objects c, r, and s from Remark 3.4.9. Let s be the indecomposable associated with the diagonal $\{s_0, s_1\}$, r the indecomposable associated with the diagonal $\{r_0, r_1\}$, and Σc the indecomposable associated with $\{c_0 - 1, c_1 - 1\}$. The diagonals are in the $(n+3)$-gon P. Consider the Auslander-Reiten quiver of $\mathscr{C} = \mathsf{C}(A_n)$. Illustrated below with solid lines emanating from $\{s_0, s_1\}$ is the rectangle in the Auslander-Reiten quiver spanned from s; this is the region to which s

has nonzero morphisms. A similar rectangle can be drawn from r. There will be nonzero morphisms $s \to r \to \Sigma c$ with nonzero composition if and only if

(i) $\{r_0, r_1\}$ is in the rectangle spanned from $\{s_0, s_1\}$, and

(ii) $\{c_0 - 1, c_1 - 1\}$ is inside both the rectangle spanned from $\{s_0, s_1\}$ and the rectangle spanned from $\{r_0, r_1\}$.

This is illustrated below.



In other words, this will happen if and only if

$$s_0 \le r_0 \le c_0 - 1 \le s_1 - 2 \le s_1 \le r_1 \le c_1 - 1 \le s_0 - 2 \tag{3.2}$$

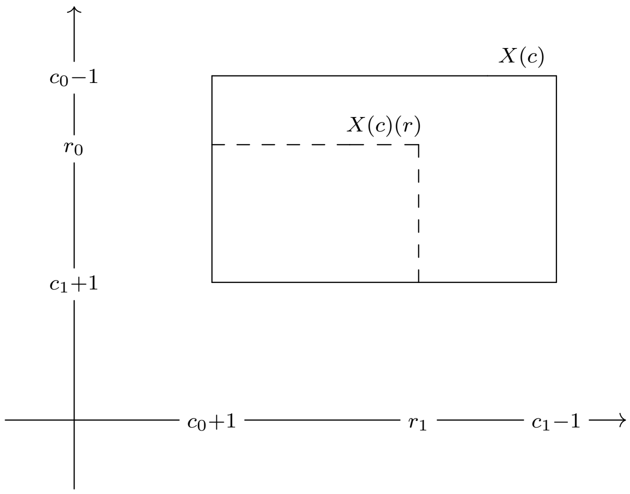
where the sequence of inequalities means the vertices must occur in this order when moving anti-clockwise around the polygon. We formulate this in a proposition. $\diamond$

Proposition 3.4.11. *Let $s \in \operatorname{ind}(\mathscr{S})$. There exist $s \xrightarrow{\sigma} r \xrightarrow{\alpha} \Sigma c$ with nonzero composition if and only if the endpoints of the corresponding diagonals satisfy the sequence of inequalities (3.2).* $\diamond$

Remark 3.4.12. The inequalities in the proposition imply that $c_0 + 1 \le r_1 \le c_1 - 1$ and $c_1 + 1 \le r_0 \le c_0 - 1$, hence the diagonals representing r and c cross. The inequalities also imply $c_0 + 1 \le s_1 \le r_1$ and $c_1 + 1 \le s_0 \le r_0$.

We now apply this to our clique $\mathscr{S}$ and object $c \in \operatorname{ind}(\mathscr{S})$.

Consider the rectangle of grid points $X(c)$ below (where the edges have been filled in for illustrative purposes).



On axes, put clique vertices in the indicated intervals. The corresponding set of grid points will be denoted $X(c)$. It corresponds to the diagonals in the clique $\mathscr{S}$ which cross $\{c_0, c_1\}$, hence to the support of Γc (inside $\operatorname{ind}(\mathscr{R})$). The subrectangle of grid points $X(c)(r)$ shows the support of $\operatorname{Im}(a)$, by a combination of the formula (3.1) and Proposition 3.4.11. ◇

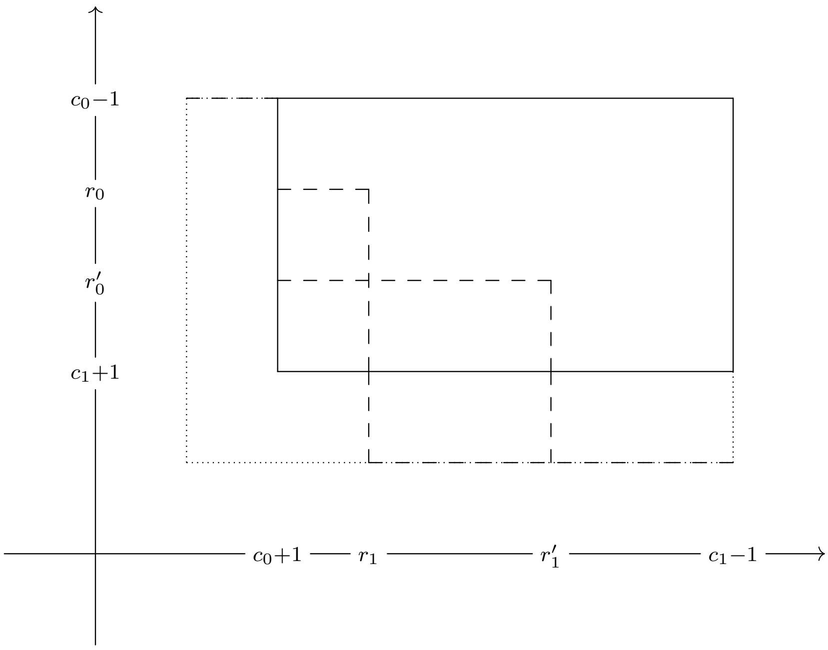
Definition 3.4.13. Let $c_0 < s_1 < \ldots < s_a < c_1 < t_1 < \ldots < t_b$ be the vertices of the clique. A solid rectangle of grid points is a set of the form

$$\{(s_i, t_j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$

The numbers of indecomposables along the edges of the rectangle are m and n. A *solid staircase of grid points* is a union of solid rectangles of grid points. ◇

Remark 3.4.14. We have now determined the subfunctors $\operatorname{Im}(a) \subseteq \Gamma c$. Recall Remark 3.4.9 which says that the sums of these give all subfunctors of Γc. So let us determine these sums.

For each $r \in \operatorname{ind}(\mathscr{S})$ we have a subfunctor $E_r = \operatorname{Im}(a) \subseteq \mathscr{C}(-, \Sigma c)|_{\mathscr{R}}$ where a comes from a nonzero map $\alpha : r \to \Sigma c$. The support of E_r is given by the subrectangle $X(c)(r)$ as indicated above in Remark 3.4.12. Now, consider a sum $E_r + E_{r'}$ inside $\mathscr{C}(-, \Sigma c)|_{\mathscr{R}}$ for $r, r' \in \operatorname{ind}(\mathscr{S})$. Evaluating at $s \in \mathsf{Supp}\, \mathscr{C}(-, \Sigma c)|_{\mathscr{R}}$ gives $E_r(s) + E_{r'}(s) \subseteq \mathscr{C}(s, \Sigma c)|_{\mathscr{R}} = \mathbb{C}$. This implies that $E_r(s) + E_{r'}(s)$ either equals zero (if both summands are zero) or $\mathbb{C}$ (if at least one of them is $\mathbb{C}$). The upshot is that $\mathsf{Supp}(E_r + E_{r'})$ is a solid staircase of grid points (illustrated below).



◇

Theorem 3.4.15. *A subfunctor $M' \subseteq \mathscr{C}(-,\Sigma c)|_{\mathscr{R}}$ has support* $\mathsf{Supp}\, M'$ *equal to a solid staircase of grid points inside the rectangle $X(c)$.*

Proof. Each $M' \subseteq \mathscr{C}(-,\Sigma c)|_{\mathscr{R}}$ is a sum of images of morphisms of the form $a : \mathscr{R}(-,r) \to \mathscr{C}(-,\Sigma c)|_{\mathscr{R}}$, by Remark 3.4.9. Each $\operatorname{Im} a$ has support equal to a solid rectangle of grid points, anchored at the bottom-left, by Remark 3.4.12. For each indecomposable r, we have that $(\Gamma c)(r)$ is 0 or $\mathbb{C}$, so each $M'(r) \subseteq (\Gamma c)(r)$ is equal to 0 or $\mathbb{C}$, by Remark 3.4.14. We therefore have that $\mathsf{Supp}\ M'$ is equal to the union of the supports of each of the images of the morphisms of the form $a : \mathscr{R}(-,r) \to \mathscr{C}(-,\Sigma c)|_{\mathscr{R}}$, and hence to overlapping rectangles; i.e. a solid staircase of grid points (inside the rectangle $X(c)$). □

Remark 3.4.16. We want to count the number of such "staircases". Instead of counting them directly, it turns out to be easier to count their upper edges. Accordingly, we extend the rectangle of grid points $X(c)$ one vertex to the left and one vertex down, yielding the rectangle $X^+(c)$. The upper edge of a staircase should be drawn in $X^+(c)$; for instance, the "empty staircase" corresponds to an upper edge going straight down, then straight right. Where there are a grid points along the horizontal axis of $X(c)$ and b grid points along the vertical axis of $X(c)$, the corresponding rectangle $X^+(c)$ *in which to embed the upper edge of a staircase* has side lengths a and b. ◇

Proposition 3.4.17. *There is a bijection between solid staircases of grid points inside $X(c)$ (illustrated above, representing the support of a given* $\mathrm{Im}(a)$*) and upper edges of staircases in $X^+(c)$.*

Proof. Consider the function from the set of solid staircases of grid points inside $X(c)$ to the set of upper edges of staircases in $X^+(c)$ which is described as follows.

(i) Extend the rectangle of grid points $X(c)$ one vertex to the left and one vertex down.

(ii) Start at the top-left of this extended rectangle, $X^+(c)$.

(iii) If the vertex immediately to the right is filled in (i.e., part of the solid staircase of grid points), draw a line 1-unit long going right to this vertex. Otherwise, draw a line 1-unit long downwards.

(iv) Repeat the third step. Continue repeating until you reach the bottom-right corner of the rectangle $X^+(c)$.

It is clear that this function is both injective and surjective, and hence bijective. □

Corollary 3.4.18. *The number of subfunctors of $\mathscr{C}(-, \Sigma c)|_{\mathscr{R}}$ is equal to the number of upper edges of staircases in the rectangle $X^+(c)$.*

Proof. By Proposition 3.4.7, subfunctors M' of Γc (for an indecomposable c) are determined by their supports. Theorem 3.4.15 says that each of these supports correspond to a solid staircase of grid points inside the rectangle $X(c)$. Proposition 3.4.17 says that the solid staircases of grid points inside $X(c)$ are in bijection with upper edges of staircases inside $X^+(c)$. Thus, the number of subfunctors of Γc is the number of supports, which is the number of upper edges of staircases in the rectangle $X^+(c)$. □

Lemma 3.4.19. *The number of upper edges of staircases in a rectangle with side lengths a and b is the multimomial coefficient*

$$\binom{a+b}{a, b} = \frac{(a+b)!}{a!b!}.$$

Proof. An upper edge of a staircase is given by a unique permutation of the word

$$rrr \dots rddd \dots d$$

which has a copies of the letter d ("d" for *down*) and b copies of the letter r ("r" for *right*). This is given by the multinomial coefficient

$$\binom{a+b}{a, b} = \frac{(a+b)!}{a!b!}.$$

□

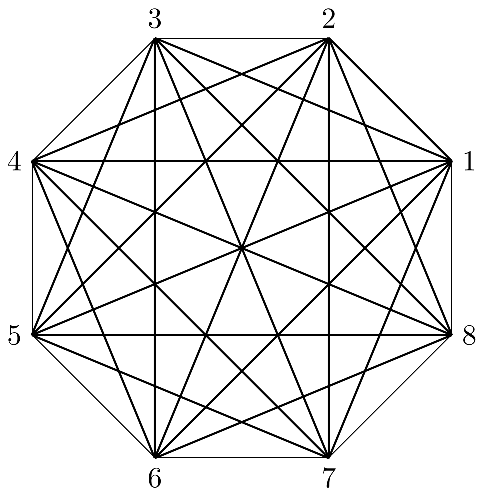
Corollary 3.4.20. *If c is an object corresponding to a clique diagonal, then*

$$\rho(c) = \#\text{ subfunctors of } \mathscr{C}(-, \Sigma c)|_{\mathscr{R}} = \#\text{ upper edges of staircases of } X^+(c).$$

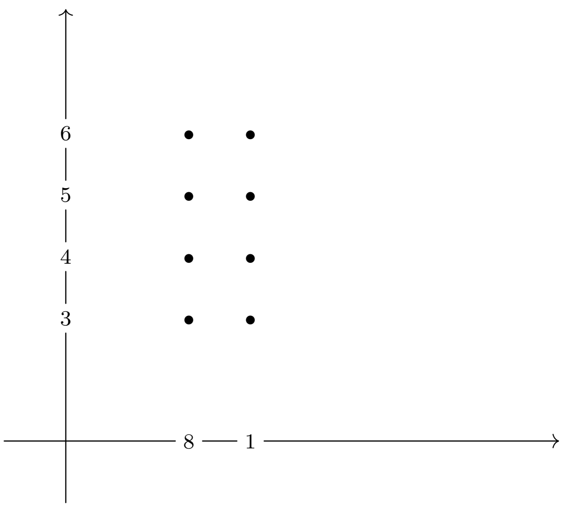
Proof. Proposition 3.4.8 together with the Corollary 3.4.18 gives the result. □

Given a clique diagonal c, to find the corresponding rectangle $X^+(c)$, we may start by drawing the clique, highlighted in an example below.

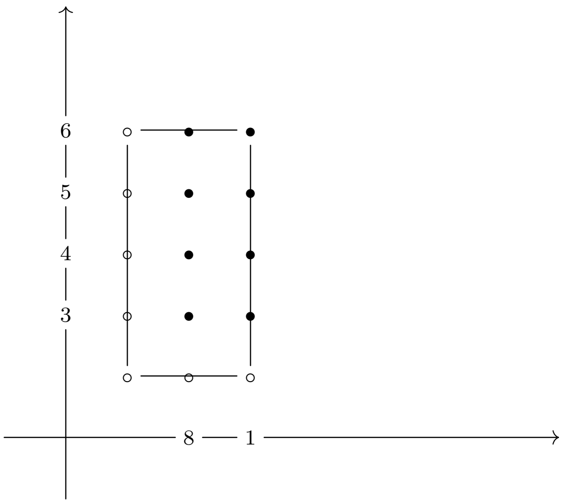
Example 3.4.21. Let us suppose we have a clique inside an octagon and are interested in $c = \{2, 7\}$.



The rectangle of grid points X associated with the diagonal $\{2, 7\}$ is the following.



To count upper edges of staircases in the rectangle $X^+(\{2, 7\})$, we extend one vertex in each direction, as follows.



The number of upper edges of staircases is $\dfrac{(4+2)!}{4!2!} = 15$. We therefore conclude that $\rho(\{2,7\}) = 15$. $\diamond$

Remark 3.4.22. In general, ρ of a clique diagonal is $\dfrac{(a+b)!}{a!b!}$ where a and b are the numbers of clique vertices on either side of the clique diagonal.

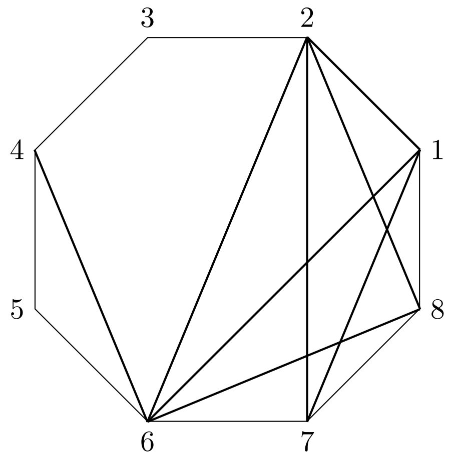
In an n-gon, with vertices numbered $1, \ldots, n$, and diagonals ordered $\{a, b\}$ with $a < b$, these side lengths are always given by $b-a-1$ and $n-b+a-1$. That is, ρ of any diagonal in a clique of size n is given by

$$\frac{(n-2)!}{(b-a-1)!(n-b+a-1)!} = \binom{n-2}{b-a-1},$$

a binomial coefficient. $\diamond$

We conclude by calculating ρ of diagonals in an example of a Ptolemy diagram.

Example 3.4.23. Consider the following Ptolemy diagram.



The diagonal $\{5,8\}$ crosses the dissecting diagonal $\{4,6\}$ so Remark 3.3.2 gives

$$\rho(\{5,8\}) = \rho(\{4,5\})\rho(\{6,8\}) + \rho(\{5,6\})\rho(\{4,8\}) = 1 \times \rho(\{6,8\}) + 1 \times \rho(\{4,8\}).$$

In turn, $\{4,8\}$ crosses the dissecting diagonal $\{2,6\}$ so Remark 3.3.2 gives

$$\rho(\{4,8\}) = \rho(\{4,6\})\rho(\{2,8\}) + \rho(\{2,4\})\rho(\{6,8\}) = 1 \times \rho(\{2,8\}) + 1 \times \rho(\{6,8\}).$$

Substituting this into the previous equation gives

$$\rho(\{5,8\}) = \rho(\{2,8\}) + 2 \times \rho(\{6,8\}).$$

Note that $\{4,6\}$ and $\{2,4\}$ cross none of the diagonals in the Ptolemy diagram, so G of these diagonals is 0 and ρ of them is 1. Finally, $\{2,8\}$ and $\{6,8\}$ are diagonals in the clique defined by the vertices $\{6,7,8,1,2\}$, and each has one of these vertices on one side, and two on the other, so $\rho(\{2,8\}) = \rho(\{6,8\}) = \dfrac{(2+1)!}{2!1!} = 3$, so $\rho(\{5,8\}) = 3 + 2 \times 3 = 9$. We can compute values of ρ for the remaining diagonals and from this Ptolemy diagram the following values are thus produced, as shown on the Auslander-Reiten quiver of $C(A_5)$.

$$
\begin{array}{ccccccccccccc}
\cdots & & 5 & & 3 & & 3 & & 3 & & 4 & & \cdots \\
 & 4 & & 9 & & 3 & & 3 & & 6 & & 4 & \\
\cdots & & 6 & & 7 & & 1 & & 4 & & 6 & & \cdots \\
 & 6 & & 4 & & 2 & & 1 & & 4 & & 9 & \\
\cdots & & 4 & & 1 & & 2 & & 1 & & 5 & & \cdots
\end{array}
$$

The determinants of four adjacent numbers

$$
\begin{array}{ccc}
 & b & \\
a & & d \\
 & c &
\end{array}
$$

forming a diamond are given by $ad - bc$ and in this example, they are either $0, 1$, or 6. $\diamond$

Remark 3.4.24. Given a Ptolemy diagram, there is a generalised Caldero-Chapoton map associated with that Ptolemy diagram. When that Ptolemy diagram is a single-cell clique, for different size polygons we arrive at the following values as shown on the relevant Auslander-Reiten quiver (the determinants of each diamond are written in the circles, and boundaries have ρ values of 1).

(1+3)-gon

$$
\begin{array}{ccc}
2 & 3 & 2
\end{array}
$$

(2+3)-gon

$$
\begin{array}{ccccccc}
 & 3 & 6 & 3 & 6 & 3 & \\
3 & 6 & 3 & 6 & 3 & 6 & 3
\end{array}
$$

(3+3)-gon

$$\begin{array}{ccccccc}
4 & 10 & 4 & 10 & 4 & 10 & 4 \\
 & 6 & 20 & 6 & 20 & 6 & \\
4 & 10 & 4 & 10 & 4 & 10 & 4
\end{array}$$

(4+3)-gon

$$\begin{array}{cccccccc}
 & 5 & 15 & 5 & 15 & 5 & 15 & 5 \\
10 & 50 & 10 & 50 & 10 & 50 & 10 & \\
 & 10 & 50 & 10 & 50 & 10 & 50 & 10 \\
5 & 15 & 5 & 15 & 5 & 15 & 5 &
\end{array}$$

As anticipated (see Remark 3.4.21), binomial coefficients appear. The determinants form part of the sequence known as the “Triangle of Narayana Numbers”, or the “Catalan Triangle”, see sequence A001263 in the OEIS [39]. This can be seen by noticing the determinants are of the form

$$\binom{n}{k}^2 - \binom{n}{k-1}\binom{n}{k+1}$$

and then utilising the following proposition.

Proposition 3.4.25. *Suppose $k, n \in \mathbb{N}$ and $0 < k < n$. Then*

$$\frac{1}{k+1}\binom{n}{k}\binom{n+1}{k} = \binom{n}{k}^2 - \binom{n}{k-1}\binom{n}{k+1}.$$

The proof of the proposition is given by straightforward computation. Comparing the expression on the left-hand-side with the formula given in the OEIS gives the result. $\diamond$

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