Composition of permutation representations of triangle groups

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Abstract

A triangle group is denoted by $\varDelta(p, q, r)$ and has finite presentation

$$\varDelta(p, q, r) = \langle x, y | x^p = y^q = (xy)^r = 1\rangle.$$

In the 1960's Higman conjectured that almost every triangle group has among its homomorphic images all but finitely many of the alternating groups. This was proved by Everitt in [6].

In this thesis, we combine permutation representations using the methods used in the proof of Higman's conjecture. We do some experiments by using GAP code and then we examine the situations where the composition of a number of coset diagrams for a triangle group is imprimitive. Chapter 1 provides the introduction of the thesis. Chapter 2 contains some basic results from group theory and definitions. In Chapter 3 we describe our construction that builds compositions of coset diagrams. In Chapter 4 we describe three situations that make the composition of coset diagrams imprimitive and prove some results about the structure of the permutation groups we construct. We conduct experiments based on the theorems we proved and analyse the experiments. In Chapter 5 we prove that if a triangle group G has an alternating group as a finite quotient of degree deg > 6 containing at least one handle, then G has a quotient $C_p^{\mathrm{deg}-1} \rtimes \mathcal{A}_{\mathrm{deg}}$. We also prove that if, for an integer $m \neq \mathrm{deg} - 1$ such that $m > 4$ and the alternating group $\mathcal{A}_m$ can be generated by two product of disjoint p-cycles, and a triangle group G has a quotient $\mathcal{A}_{\mathrm{deg}}$ containing two disjoint handles, then G also has a quotient $\mathcal{A}_m \wr \mathcal{A}_{\mathrm{deg}}$.

Dedicated to my beloved parents.

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Chapter 1

Introduction

The intent of this chapteris to explain the objective of the thesis. Section 1.1 is precisely about the groups of isometries of the hyperbolic plane with some basic definitions. In Section 1.2 we define Fuchsian groups and its fundamental regions for the action of it on the hyperbolic plane. In Section 1.3 we describe some of the examples of Fuchsian group. Section 1.4 is about the outline of the thesis and Section 1.5 describes a history of Higman's conjecture on which the problem of this thesis is based upon.

1.1 Hyperbolic plane

Suppose $\mathbb{C}$ is a complex plane. Define $\mathbb{H}^2 = \{z = x + iy \in \mathbb{C} \mid y > 0\} = \{z \in \mathbb{C} \mid \mathrm{Im(z)} > 0\}$ as the upper half plane. When $\mathbb{H}^2$ is equipped with the hyperbolic metric $ds = \sqrt{dx^2 + dy^2}/y = |dz|/\mathrm{Im(z)}$, it becomes a model of the hyperbolic plane. The boundary $\partial\mathbb{H}^2$ defined by

$$\partial\mathbb{H}^2 = \mathbb{R} \cup \infty \subset \mathbb{C} \cup \infty$$

where ∞ means a point at infinity.

The above hyperbolic metric can be used to find the hyperbolic length of a differentiable path $\gamma : I \to \mathbb{H}^2$ defined by $\gamma(t) = x(t) + iy(t) = z(t)$ by integrating over its domain. This length is given by

$$L(\gamma) = \int_0^1 \frac{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}}{y(t)} dt = \int_0^1 \frac{|\frac{dz}{dt}|}{y(t)} dt$$

Definition 1.1. *Let $a, b \in \mathbb{H}^2$. The hyperbolic distance $\rho(a, b)$ between $a, b \in \mathbb{H}^2$ is defined by*

$$d(a, b) = inf \, \{L(\sigma) \;\; | \;\; \sigma \text{ is a differentiable path with end points } a, b\}$$

and this gives a metric on $\mathbb{H}^2$.

Definition 1.2. *Let d be the metric on $\mathbb{H}^2$ given by 1.1. We define a topology on $\mathbb{H}^2$ determinded by the metric d. A set $U \subset \mathbb{H}^2$ is open in this topology if for all $u \in U$ there is a $\delta > 0$ such that*

$$B_\delta(u) = \{v : d(v, u) < \delta\} \subset U.$$

Then these U form a topology on $\mathbb{H}^2$.

Definition 1.3. *A geodesic in $\mathbb{H}^2$ (the path of shortest hyperbolic length) is defined by a set of straight lines l_1 and l_2 which are either semi-circles orthogonal to $\mathbb{R}$ or vertical lines, as in Fig. 1.1.*

Definition 1.4. *A triangle in $\mathbb{H}^2$ is defined as the region bounded by three geodesics such that not all lines meet at one point and if two lines intersect then they meet in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$.*

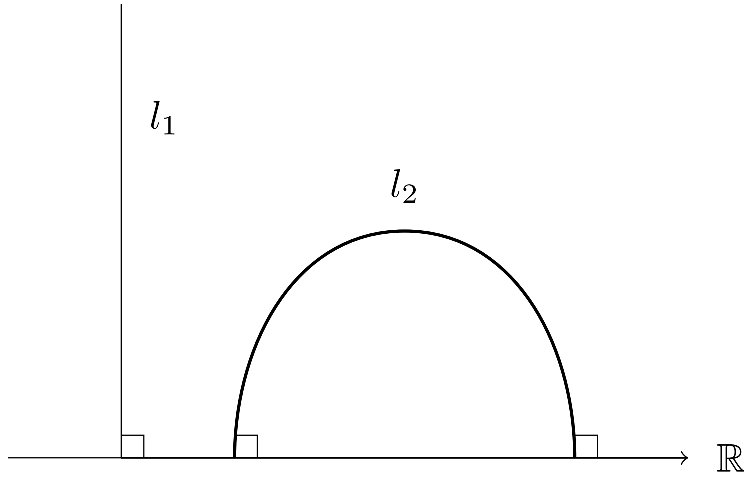


Figure 1.1: Geodesics

The following Fig. 1.2 illustrate the four types of triangle in $\mathbb{H}^2$.

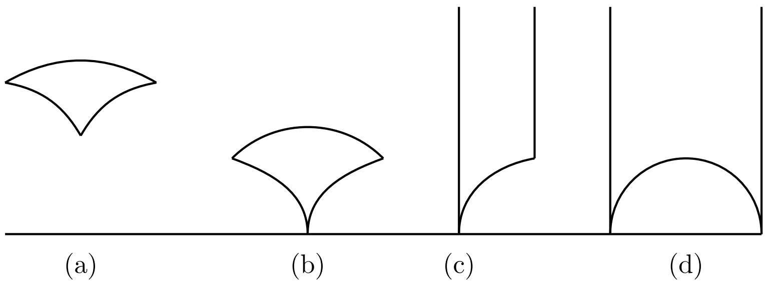


Figure 1.2: Types of triangles

Definition 1.5. *An isometry of hyperbolic plane is a function $f : \mathbb{H}^2 \to \mathbb{H}^2$ such that for any $a, b \in \mathbb{H}^2$, we have $\rho(a, b) = \rho(f(a), f(b))$. In other words, a transformation of $\mathbb{H}^2$ onto itself is called an isometry if it preserves the hyperbolic distance on $\mathbb{H}^2$.*

The set of isometries of $\mathbb{H}^2$

$$\text{Isom}(\mathbb{H}^2) = \{f : \mathbb{H}^2 \to \mathbb{H}^2 : f \text{ is an isometry}\}$$

forms a group under the operation of composition. The topology on $\text{Isom}(\mathbb{H}^2)$ called the compact-open topology. The multiplication and inverses of their elements are homeomorphisms, that is, continous functions. Thus, the set of isometries of the hyperbolic plane is a topological group. It turns out that there are reflections in geodesics of $\mathbb{H}^2$. Let $\mathrm{Isom}^+(\mathbb{H}^2)$ be those isometries that preserve orientation.

1.2 Fuchsian groups

A discrete subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ is called a **Fuchsian group** if it consists of orientation preserving transformations. Equivalently, a Fuchsian group is a group of isometries acting discontinously on $\mathbb{H}^2$. Let $G \subset \mathrm{Isom}(\mathbb{H}^2)$ be a subgroup. It acts discontinously on $\mathbb{H}^2$ if and only if each $x \in \mathbb{H}^2$ has a neighbour N such that $f(N) \cap N = \phi$ for all $f \in G$.

Fuchsian groups can also be envisioned by their fundamental regions. Let Γ be a Fuchsian group acting on the hyperbolic plane $\mathbb{H}^2$. Then F is a fundamental region for Γ if F is a closed set such that $\cup_{T\in\Gamma} T(F)$ is the entire hyperbolic plane and $F^\circ \cap T(F^\circ) = \emptyset$, for all $T \in \Gamma$, where F° is the interior of F [9, p. 240]. The set $\{T(F) : T \in \Gamma\}$ is called a tessellation of $\mathbb{H}^2$.

Every Fuchsian group possess a nice (connected and convex) fundamental region.

Example 1.6. *As described in [9], let τ be a hyperbolic triangle, with vertices v_1, v_2, v_3, angles $\pi/m_1, \pi/m_2, \pi/m_3$ at these vertices and sides M_1, M_2, M_3 opposite these vertices, as illustrated in the Fig. 1.3. Let R_i be the hyperbolic refection in the hyperbolic line containing M_i, $(i = 1, 2, 3)$, and let Γ^* be the group generated by the reflections R_1, R_2, R_3. Since R_1 does not preserve orientation, Γ^* is not a Fuchsian group. However, we consider $\Gamma = \Gamma^* \cap \mathrm{Isom}^+(\mathbb{H}^2)$. Then Γ^* is the union of two Γ- cosets, for example $\Gamma^* = \Gamma \cup \Gamma R_1$, for if $S \in \Gamma^* \setminus \Gamma$ then SR_1 is the composition of two orientation-reversing isometries, so it is orientation-preserving and therefore $SR_1 \in \mathrm{Isom}^+(\mathbb{H}^2)$.*

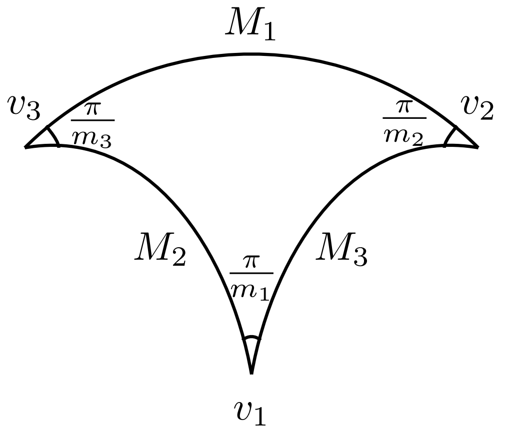


Figure 1.3

It can be shown that $\{T(\tau)|T \in \Gamma^*\}$ *form a tessellation of the hyperbolic plane, that is, no two* Γ^**-images of* τ *overlap and every point of* $\mathbb{H}^2$ *belongs to some* Γ^**-image of* τ*. It follows that* τ *is a fundamental region for* Γ^* *and* $\tau \cup R_1\tau$ *is a fundamental region of* Γ*. Because* τ *is a triangle, we call* Γ *a triangle group.*

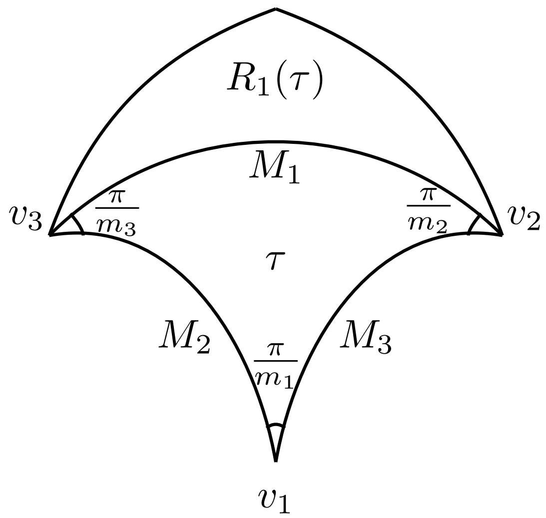


Figure 1.4

It turns out that the general form of a fundamental region is as shown in Fig. 1.5 which has signature $(g; m_1, m_2, \ldots, m_r)$.

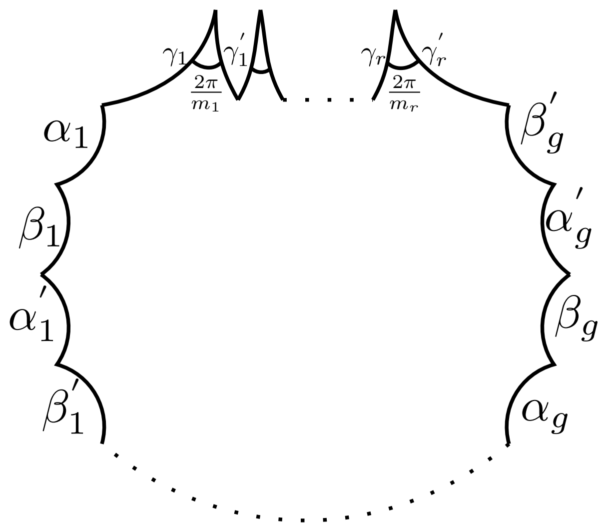


Figure 1.5: Fundamental region

The area of this region is

$$\mu(F) = 2\pi\{(2g-2) + \sum_{i=1}^{r}(1 - \frac{1}{m_i})\}. \tag{1.1}$$

A finite presentation of Fuchsian group Γ with signature $(g; m_1, \ldots, m_r)$ is given in terms of r generators $x_1, \ldots, x_r$ and $2g$ generators $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ with the relations

$$x_1^{m_1} = x_2^{m_2} = \cdots = x_r^{m_r} = [a_1, b_1] \ldots [a_g, b_g]x_1 \ldots x_r = 1.$$

1.3 Examples of Fuchsian groups

1.3.1 Triangle groups

A triangle group has signature $(0; m_1, m_2, m_3)$ and has presentation

$$\langle X_1, X_2, X_3 | X_1^{m_1} = X_2^{m_2} = X_3^{m_3} = X_1X_2X_3 = 1\rangle.$$

Let $x = X_1$, $y = X_2$ and $(xy)^{-1} = X_1X_2$, it gives the following

$$\langle x, y | x^{m_1} = y^{m_2} = (xy)^{m_3} = 1\rangle,$$

where m_1, m_2, m_3 are integers greater than one.

The area μ in equation 1.1 must be positive for Fuchsian groups. For $(0; m_1, m_2, m_3)$ we must have $1 - (\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}) > 0$ to get a Fuchsian group. For example $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$, $(2, 3, 6)$, $(3, 3, 3)$ are not Fuchsian groups. Throughout this thesis we use the following definition of a triangle group that is equivalent to the above definition.

Definition 1.7. *Triangle groups are denoted by $\Delta(p, q, r)$ with $2 \leq p \leq q \leq r$. The group $\Delta(p, q, r)$ has finite presentation:*

$$\Delta(p, q, r) = \langle x, y : x^p = y^q = (xy)^r = 1\rangle.$$

In this thesis, p will usually be prime. However, we shall not assume that in general. In our main work of this thesis r will be finite, however, in some contexts it can be infinite.

1.3.2 Modular groups

In the notation of defnition 1.7, a modular group is a triangle group with signature $(0; 2, 3, \infty)$ and has presentation

$$\langle X_1, X_2, X_3 | X_1^2 = X_2^3 = X_1X_2X_3 = 1\rangle$$

$$\cong \langle x, y | x^2 = y^3 = 1\rangle \cong \mathbb{Z}/2 * \mathbb{Z}/3.$$

It turns out that the modular group is isomorphic to $\mathrm{PSL}_2(\mathbb{Z})$. A fundamental region of the modular group on the upper half plane is shown in Fig. 1.6.

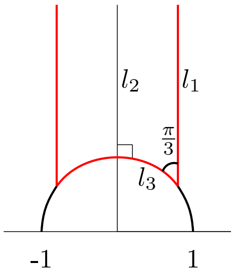


Figure 1.6: Fundamental region of modular group

1.3.3 Free groups of rank 2

Free group of rank 2 with signature $(0; \infty, \infty, \infty)$ is a Fuchsian group and has presentation

$$\langle X_1, X_2, X_3 | X_1X_2X_3 = 1\rangle$$

$$\cong \mathbb{Z} * \mathbb{Z}.$$

Fig. 1.7 shows the action of free group of rank 2 on the upper half plane.

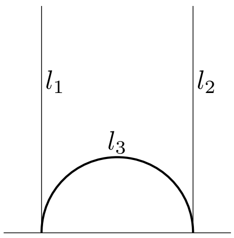


Figure 1.7: Fundamental region of free group of rank 2

1.4 Outline

This thesis develops a technique to combine permutation representations out of methods used in the proof of Higman's conjecture. By Lemma 2.1 of Everitt's paper

[6], to prove Higman's conjecture it is enough to prove if for all hyperbolic triangle groups $\Delta(p, q, r)$ with either

1. $2 \leq p < q < r$ distinct primes.
2. the triangle groups $(2, 4, r)$ for $r \geq 5$ a prime.
3. the groups $(2, 3, 8)$, $(2, 3, 9)$, $(2, 3, 10)$, $(2, 3, 12)$, $(2, 3, 15)$, $(2, 3, 25)$, $(2, 4, 6)$, $(2, 4, 8)$, $(2, 4, 9)$, $(2, 5, 6)$, $(2, 5, 9)$ and $(3, 4, 5)$.
4. non-triangle groups parametrised as $(2, 3, 3, 3)$ and $(3, 3, 3, 3)$.

In fact, Everitt's theorem proves that most hyperbolic triangle groups $\Delta(p, q, r)$ map onto all but finitely many alternating groups. This thesis specifically examines the situations where the composition of a number of coset diagrams for a triangle group $\Delta(p, q, r)$ is an imprimitive group, and analyses the structure of this imprimitive group.

Chapter 1 provides the introduction of the thesis. It explains some of the historical background, the research objective and lists the main results of this work.

Chapter 2 contains some definitions and basic results from group theory in particular permutation group theory which are required to prove the main results of the thesis. It also contains the definitions of coset diagrams and linear representation.

Chapter 3 defines the techniques of composition of t coset diagrams where $t \leq p$, illustrating with some examples and experiments that are shown in the Table 3.1. These experiments show us that very often the composition of distinct coset diagrams is primitive. We have been unable to decide in general whether or not the composition of permutation representations is primitive. However, we can find certain situations where the composition of a number of coset diagrams for a $\Delta(p, q, r)$ is imprimitive.

In chapter 4, three different types of composition of coset diagrams are defined that produce imprimitive permutation images of triangle groups. We give some partial analysis of those permutation groups in Theorems 4.1, 4.3 and 4.5. Theorem 4.3 is special but less interesting, giving an imprimitive permutation representation whenever we start with an imprimitive representation. However, Theorem 4.1 and 4.5 are interesting because they always give us imprimitive representations as a quotient whatever we start with. We use GAP programmes to perform experiments based on the construction of these theorems that helps us to find the structure of a group N that is the kernel of the permutation images of a triangle group and lead us to prove a lemma. Some interesting observations are found from these experiments, in particular what kind of groups we get as a quotient when we start with a representation that is the alternating group.

Chapter 5 concludes this thesis by analysing the constructions of Theorem 4.1 and 4.5 and the observations we investigated from experiments in Chapter 4, particularly in the cases where they are built out of images that are alternating groups. Theorem 5.1 and Theorem 5.3 describe the structure of the groups built using the constructions of Theorem 4.1 and Theorem 4.5. In the case where G has a finite quotient that is an alternating group of deg > 6 containing at least one handle, then G has a quotient $C_p^{\deg -1} \rtimes \mathcal{A}_{\deg}$. Moreover, for an integer $m \neq \deg - 1$ such that $m > 4$ and if the alternating group $\mathcal{A}_m$ can be generated by two products of disjoint p-cycles, and a triangle group G has a quotient $\mathcal{A}_{\deg}$ containing two disjoint handles, then G also has a quotient $\mathcal{A}_m \wr \mathcal{A}_{\deg}$. We conjecture in the end that for most $G = \Delta(p, q, r)$

$$G \twoheadrightarrow C_p^{\deg -1} \rtimes \mathcal{A}_{\deg}$$

and

$$G \twoheadrightarrow \mathcal{A}_m \wr \mathcal{A}_{\deg}$$

where $m \in \mathbb{Z}$ such that $\mathcal{A}_m$ can be generated by two elements of order p and $m \neq \deg - 1$. The appendices contain GAP code that we used to compute examples.

1.5 History of Higman's Conjecture

In the 1960's Higman conjectured that "Every Fuchsian group has among its homomorphic images all but finitely many of the alternating groups". In particular, the conjecture asserts that every hyperbolic triangle group surjects onto almost all alternating groups.

Two techniques are used to prove this conjecture and allow us to build higher degree coset diagrams for $\Delta(p, q, r)$ as follows.

1. Composition: combining coset tables for a given group $\Delta(p, q, r)$, in order to get coset tables for that group in arbitrarily many degrees.
2. Boosting: converting coset diagrams for a given $\Delta(p, q, r)$ into coset diagrams for various $\Delta(p, q, r')$ with $r' > r$.

Higman proposed the method of composition of coset diagrams to prove his conjecture and proved his result for the triangle groups $\Delta(2, 3, 7)$ and $\Delta(2, 4, 5)$ by using coset diagrams and their composition. In 1981, Conder used the method of composition and proved his conjecture for the group $(2, 3, k)$, for all $k \geq 7$ [2]. Following the same technique Mushtaq and Rota proved the result for all $\Delta(2, l, m)$, for all even $l \geq 6$. Later Mushtaq and Servatius proved the result for $\Delta(2, q, r)$ for all $5 \leq q \leq r$ [12]. In 1994 Everitt found the same result for $\Delta(2, 4, r)$ for all $r \geq 6$ [5]. In 2000 Everitt proved the conjecture of Higman [6], later it was reproved by Liebeck and Shalev [11] and see also a paper of Dunfield and Thurston [4]. In 2013 his student Kousar improved this result by working on Non-Euclidean crystallographic groups [10].

A recent article by Nebe, Parker and Rees [13] examines a general method for composition.

Chapter 2

Background from group theory

This chapter contains background material from group theory that we shall need in the remainder of the thesis [15, 16].

We shall use the following notation. $\mathcal{S}(\Omega)$ will denote the group of all permutations on a set Ω and $\mathcal{A}(\Omega)$ the subgroup of all even permutations (acting on the right). Further, $\mathcal{S}_n$ and $\mathcal{A}_n$ will denote the groups of all permutations and even permutations, respectively, of the set $\{1, 2, \ldots, n\}$. We use the notation x^y to denote the conjugate $y^{-1}xy$, as is consistent for right actions.

2.1 General results

Theorem 2.1 (Cayley's Theorem)**.** *Every group G can be embedded as a subgroup of $\mathcal{S}(G)$. In particular, if $|G| = n$, then G can be embedded in $\mathcal{S}_n$.*

Theorem 2.2 (Isomorphism Theorem)**.** *(a) Let $f : G \to H$ be a homomorphism with kernel K. Then K is a normal subgroup of G and $G/K \cong \mathsf{Im} f$.*

(b) Let N and T be subgroups of G with N normal in G. Then $N \cap T$ is normal in T and $T/(N \cap T) \cong NT/N$.

(c) Let $K \leq H \leq G$, where both K and H are normal subgroups of G. Then H/K is a normal subgroup of G/K and

$$(G/K)/(H/K) \cong G/H.$$

Theorem 2.3. *(Jordan-Hölder) Suppose G is a finite group with two composition series say*

$$1 = G_0 \lhd G_1 \lhd G_2 \lhd \ldots \lhd G_m = G$$

and

$$1 = K_0 \lhd K_1 \lhd K_2 \lhd \ldots \lhd K_n = G$$

such that G_{i+1}/G_i and K_{i+1}/K_i are simple for each i. Then $m = n$, and there is a permutation $f : \{1, ..., m\} \to \{1, ..., m\}$ such that for each i,

$$G_i/G_{i-1} \cong K_{f(i)}/K_{f(i)-1}.$$

Definition 2.4. *If $a, b \in G$, the **commutator** of a and b, denoted by $[a, b]$, is*

$$[a, b] = a^{-1}b^{-1}ab.$$

Note that some authors define the commutator as $aba^{-1}b^{-1}$. Our notation is consistent with a right action.

*The **commutator subgroup** (or derived subgroup) of G, denoted by G', is the subgroup of G generated by all commutators.*

The commutator subgroup G' is a normal subgroup of G. The quotient G/G' is called the derived quotient. It is abelian and it is the maximal abelian quotient of G.

Theorem 2.5. *If $K \lhd G$, then G/K is abelian if and only if $G' \leq K$.*

Example 2.6. *Consider a triangle group $G = \Delta(2,3,8)$. Then $G/G' = \langle X, Y | X^2 = 1, Y^3 = 1, (XY)^8 = 1, XY = YX\rangle$ with $X = xG'$ and $Y = yG'$. We get $1 = (XY)^8 = X^8Y^8 = Y^{-1}$, which implies $Y = 1$. Hence $G/G' = \langle X | X^2 = 1\rangle = C_2$.*

Example 2.7. *Consider a triangle group $G = \Delta(2,4,9)$. Then $G/G' = \langle X, Y | X^2 = Y^4 = (XY)^9 = 1, XY = YX\rangle$ with $X = xG'$ and $Y = yG'$. We get $1 = (XY)^9$, by squaring $1 = (XY)^{18} = X^{18}Y^{18} = Y^2$, which implies $Y^2 = 1$. Also, $1 = (XY)^9 = X^9Y^9 = XY$, so $X = Y$. Hence*

$$G/G' = \langle X | X^2 = 1\rangle = C_2.$$

Example 2.8. *Consider a triangle group $G = \Delta(2,3,9)$. Then $G/G' = \langle X, Y | X^2 = Y^3 = (XY)^9 = 1, XY = YX\rangle$ with $X = xG'$ and $Y = yG'$. We get $1 = (XY)^9 = X^9$, which implies $X = 1$. So we have*

$$G/G' = \langle Y | Y^3 = 1\rangle = C_3$$

Lemma 2.9. *Let G be the triangle group $\Delta(p,q,r)$ with $2 \leq p \leq q \leq r$ with p prime. Then if $p|r$, $G/G' \cong C_p \times C_{\mathtt{gcd}(q,r)}$ and if $p \nmid r$, $G/G' \cong C_b$, where $b = gcd(q, rp)$.*

Proof. The derived quotient of the triangle group $\Delta(p,q,r)$ denoted by G/G' is the abelian group with finite presentation

$$\langle X, Y | X^p = Y^q = (XY)^r = 1, XY = YX\rangle. \tag{2.1}$$

where $X = xG'$ and $Y = yG'$.

If $p|r$ then $(XY)^r = Y^r$ and then $Y^r = 1$ and $Y^q = 1$. This implies that $Y^{\mathtt{gcd}(q,r)} = 1$.

In that case

$$G/G' = \langle X, Y | X^p = Y^{\mathtt{gcd}(q,r)} = 1, XY = YX\rangle \cong C_p \times C_{\mathtt{gcd}(q,r)}$$

If $p \nmid r$, there exists $s \in \mathbb{Z}$ such that $r = sp + d$, where $0 < d < p$.

We have $1 = (XY)^r = X^r Y^r = X^d Y^r$. Also $\mathtt{gcd}(p, d) = 1$, so there exist $u, v \in \mathbb{Z}$ such that $1 = up + vd$. So $X = X^{up+vd} = X^{vd} = Y^{-rv}$. Hence X can be written as some power of Y.

We also have $(XY)^r = 1$. So $(XY)^{rp} = 1$, which implies $Y^{rp} = 1$ and $Y^q = 1$. By substituting $X = Y^{-rv}$ in presentation (2.1), we see that $X^p = Y^{-rvp} = (Y^{rp})^{-v} = 1$, also $(XY)^r = Y^{r(1-rv)} = 1$. We know that $\mathtt{gcd}(pr, r(1-rv)) = pr$, because $p \mid 1-rv$ due to $1 = up+vd = up+v(r-sp)$, which implies $1-rv = p(u-s)$. Therefore, 2.1 can be reduced to the following presentation $G/G' = \langle Y \mid Y^q = Y^{rp} = 1\rangle$. Hence, Y has order $b = \mathtt{gcd}(q, rp)$.

This gives $G/G' = \langle Y \mid Y^b = 1\rangle$. □

Corollary 2.10. *If $p \leq q \leq r$ and p is a prime integer, then $\Delta(p, q, r)$ has C_p as a quotient if and only if $p|qr$.*

2.2 Combining and decomposing groups

2.2.1 Direct products

Definition 2.11 (Direct product)**.** *Where $H_1, H_2, \ldots, H_n$ are groups then the (external) direct product $H_1 \times H_2 \times \cdots \times H_n$ of $H_1, H_2, \ldots, H_n$ is the group of all n-tuples $(h_1, \ldots, h_n)$, multiplied componentwise.*

We note that the direct product has subgroups isomorphic to each of the groups H_i.

If a group G is isomorphic to the direct product $H_1 \times H_2 \cdots \times H_n$ then we say that G has a decomposition as an (internal) direct product. In that case G has (normal) subgroups isomorphic to $H_1, \ldots, H_n$, and is generated by those subgroups. In fact we have the following result.

Theorem 2.12 (Internal Direct Product Theorem)**.** *Let G be a group whose identity is $\{e\}$. Let $H_1, H_2, \ldots H_n$ be a sequence of subgroups of G. Then G is the internal direct product of $\{H_i\}_{1 \le i \le n}$ if and only if*

1. $G = H_1 H_2 \ldots H_n$
2. *For each* $i = 1, 2, \ldots, n$ *we have* $H_i \cap (H_1 \ldots H_{i-1} H_{i+1} \ldots H_n) = \{e\}$.
3. $H_i \lhd G$, $\forall i \in \{1, \ldots, n\}$

Example 2.13. *The Klein four-group*

$$V = \{(),\ (1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3)\}$$

has a decomposition as a direct product $C_2 \times C_2$.

2.2.2 Subgroups of direct products

Note that a subgroup of a direct product $S = T_1 \times \cdots \times T_k$ need not itself be a direct product of k subgroups of $T_1, \ldots, T_k$. For instance, when $T_1 = T_2 = \cdots T_k = T$, then for any subgroup $K \subseteq T$, the group $\{(x, x, \cdots, x) : x \in K\}$ is a subgroup of S, that is isomorphic to K. It is certainly not a direct product of k subgroups of $T_1, \ldots, T_k$.

We call a subgroup L of S a *subdirect product* if for each i, the natural projection from L to T_i maps L onto T_i In particular, when $T_1 = T_2 = \cdots = T_k = T$, the subdirect product of S of the form $\{(x, x, \cdots, x) : x \in T\}$ is called the *full diagonal subgroup* of S.

The following result is [7] [Lemma 1.4.1(ii)] (with some changes in notation; e.g. we have not used Fawcett's notation 'subdirect subgroup', which seems unnecessary in our situation). We shall need the result later.

Lemma 2.14 (Fawcett's Lemma). *Let $S = T_1 \times T_2 \times \cdots \times T_k$ be a direct product of isomorphic non-abelian, simple groups $T_1, \ldots, T_k$ ($k \geq 1$). Let M be a subgroup of S and $I := \{1, \ldots, k\}$. If M is a subdirect product of S, then M is a direct product $H_{j_1} \times \cdots \times H_{j_r}$, where each H_j (with $j \in J = \{j_1, \ldots, j_r\}$) is a full diagonal subgroup of some subproduct $T_{i_1} \times \cdots \times T_{i_s}$ with $I_j = \{i_1, \ldots, i_s\}$, and I is partitioned by the I_j.*

2.2.3 Semidirect products

Definition 2.15. *Given groups H, K, and a map $\phi : K \to \mathtt{Aut}(H)$ (defining a right action of K on H), we define the* ***semidirect product*** *of H by K, denoted by $H \rtimes_\phi K$ (or just $H \rtimes K$), to be the set $\{(h, k) : h \in H, k \in K\}$ equipped with the product*

$$(h_1, k_1)(h_2, k_2) = (h_1 h_2^{\phi(k_1^{-1})}, k_1 k_2)$$

Often we may omit the ϕ and write simply $h_2^{k_1^{-1}}$; we note that within the semidirect product this corresponds to the product $k_1 h_2 k_1^{-1}$, that is the conjugate of h_2 by k_1^{-1}.

We note that $H \rtimes K$ has subgroups $\{(h, e) : h \in H\}$ and $\{(e, k) : k \in K\}$ isomorphic to H, K respectively, and that the first of these is a normal subgroup. If the action of K on H is trivial, then $H \rtimes K$ is isomorphic to $H \times K$.

If a group G is isomorphic to a semidirect product $H \rtimes K$, then we say that G

has a decomposition as a semidirect product. In that case G has a normal subgroup $H_0 := \{(h, e) : h \in H\}$ isomorphic to H and a subgroup $K_0 := \{(e, k) : k \in K\}$ isomorphic to K. Then K_0 is called a complement of H_0, that is $G = H_0K_0$ and $H_0 \cap K_0 = \{e\}$. In particular $K_0 \cong G/H_0$. Often we shall simply use the names H, K for H_0, K_0.

Example 2.16. *The symmetric group $\mathcal{S}_4$ can be decomposed as the semidirect product of the Klein four-group V and $\mathcal{S}_3$, alternatively as the semidirect product of $\mathcal{A}_4$ and C_2.*

2.2.4 Extension of one group by another

The semidirect product is also called a split extension. In general we say that a group E is an extension of N by Q if $N \lhd E$ and $E/N \cong Q$ and we write $E = N.Q$. Note that Q need not be a subgroup of E. If Q is a subgroup of E such that $Q \cap N = 1$, then we say that E is a split extension of N by Q, in that case E is isomorphic to a semidirect product $N \rtimes Q$. Otherwise we say that E is a non-split extension of N by Q.

Note that some authors use the notation $Q.N$ rather than $N.Q$ for the same extension; our notation is consistent for a right action. The term cover of Q is also used, instead of extension by Q.

Example 2.17. *C_4 is a non-split extension $C_2.C_2$. It's clear that the extension can't split, because C_4 only has one subgroup of order 2 (which is normal).*

Example 2.18. *The Mathieu group M_{10} is a non-split extension $\mathcal{A}_6.C_2$. It turns out that the extension can not split because all of the elements of order 2 in M_{10} are within its commutator subgroup, which is $\mathcal{A}_6$.*

Example 2.19. *$SL(2,3)$ is a non-split extension $2.\mathcal{A}_4$, $SL(2,5)$ is a non-split extension $2.\mathcal{A}_5$ and $SL(2,9)$ is a non-split extension $2.\mathcal{A}_6$. It is clear that none of these extensions can split since none of $SL(2,3)$, $SL(2,5)$ or $SL(2,9)$ can have a subgroup of index 2, since their derived subgroups have indices 3,1 and 1, respectively. (See Theorem 2.5)*

If G is an extension of N by Q for which N is abelian, then the conjugation action of G on N induces an action of Q on N. Hence, in particular, if N is elementary abelian, N is a module for Q under this action.

Note that a subgroup of a split extension need not itself be split.

2.2.5 Schur multipliers

The Schur multiplier of a group Q gives information about which non-split extensions of the form $N.Q$ can exist; The Schur multiplier is defined to be the kernel of a homorphism to Q from a Schur cover. A Schur cover is defined to be any extension $E = N.Q$ that is maximal subject to $N \subseteq Z(E) \cap [E,E]$; the conditions on N ensure in particular that E is non-split. There may be more than one Schur cover, but the Schur multiplier is uniquely defined. The Schur multipliers of the alternating groups were computed by Schur (1911)[17]; for $n \geq 4$, the Schur multiplier for $\mathcal{A}_n$ is C_2 except when $n = 6, 7$, when it is C_6. As a consequence of this, we can deduce that any extension of C_p by $\mathcal{A}_n$ with p an odd prime and n not equal to 5 or 6 must split.

The following description is taken from [22].

Let $\hat{S}_n$ be a non-split extension of C_2 by S_n. Where $(a_1, \ldots, a_k)$ is a k-cycle in S_n, we denote by $\pm[a_1, \ldots, a_k]$ the two elements of $\hat{S}_n$ that map to $(a_1, \ldots, a_k) \in S_n$ under the natural homomorphism, $\nu : \hat{S}_n \to S_n$. Every element of $\hat{S}_n$ can be represented as a product $\pm[a_1, \ldots, a_{k_1}][b_1, \ldots, b_{k_2}][c_1, \ldots, c_{k_3}]\ldots$, where $(a_1, \ldots, a_{k_1})$, $(b_1, \ldots, b_{k_2})$, $(c_1, \ldots, c_{k_3}), \ldots$ are disjoint cycles. However, notice that the elements $[a_1, \ldots, a_k]$

and $[a_2, a_3, \ldots, a_k, a_1]$ of $\hat{S}_n$ are not necessarily equal in $\hat{S}_n$, even though they map to the same permutation.

In order to understand multiplication in $\hat{S}_n$, we need more information. We define, for each odd permutation $\pi \in S_n$,

$$[i, j]^{\pm\pi} = -[i^\pi, j^\pi].$$

We also define

$$[a_1, a_2, \ldots, a_k] = [a_1, a_2][a_1, a_3] \cdots [a_1, a_k].$$

Finally, in order for all products to be defined, we need to define the products $(\pm[i, j])^2$. Either these are equal to e or they are all equal to $-e$.

The two possible choices we make here determine which of the two possible non-split extensions $C_2.\mathcal{S}_n$ we have. There is only one non-split extension $C_2.\mathcal{A}_n$, and it is a subgroup of each of the extensions $C_2.\mathcal{S}_n$.

So let's suppose that we have $(\pm[i, j])^2 = e$ for all i, j. Then multiplication in $\hat{S}_n$ is completely defined. Notice that for all i, j we have

$$[i, j]^{[i,j]} = [i, j].$$

But also, by definition

$$\begin{aligned} [i, j]^{[i,j]} &= -[i^{(i,j)}, j^{(i,j)}] \\ &= -[j, i] \end{aligned}$$

So in fact $[i, j] = -[j, i]$.

We illustrate the rules of multiplication with some examples:

$$\begin{aligned}[2,1,3][3,1,4] &= [2,1][2,3][3,1][3,4] \\ &= [2,1][2,3][3,1][2,3][2,3][3,4] \\ &= [2,1][3,1]^{[2,3]}[2,3][3,4] \\ &= -[2,1][2,1][2,3][3,4] \\ &= -[2,3][3,4] = [3,2][3,4] \\ &= [3,2,4]\end{aligned}$$

We also have

$$\begin{aligned}[2,1,3]^3 &= [2,1][2,3][2,1][2,3][2,1][2,3] \\ &= [2,3]^{[2,1]}[2,1]^{[2,3]} \\ &= -[1,3] \times -[3,1] \\ &= -[1,3] \times [1,3] = -[1,3]^2 = -e\end{aligned}$$

So $[2,1,3]$ has order 6 in $\hat{S}_n$. We see that $\hat{S}_n$ has $\hat{\mathcal{A}}_n = C_2.\mathcal{A}_n$ as a subgroup. That consists of all elements $\pm\pi$ for which π is an even permutation.

Lemma 2.20. *For $n \geq 4$,*

1. *The group $\hat{\mathcal{A}}_n$ does not contain any subgroup isomorphic to $\mathcal{A}_n$.*
2. *The subgroup of $\hat{\mathcal{A}}_n$ consisting of all elements $\pm[\pi]$ for which $\pi \in \mathcal{A}_n$ fixes the point n is isomorphic to $\hat{\mathcal{A}}_{n-1}$.*

Proof. Let $\nu : \hat{\mathcal{S}}_n \to \mathcal{S}_n$ be the natural map. Then $\nu(\hat{\mathcal{A}}_n) = \mathcal{A}_n$. The kernel of ν has order 2. Suppose that K is a subgroup of $\hat{\mathcal{A}}_n$ isomorphic to $\mathcal{A}_n$. Then $\nu(K)$ is a subgroup of $\mathcal{A}_n$ and is a quotient of K by a normal subgroup of K of order 1 or 2 which is $\texttt{ker}(\nu) \cap K$. Since $K \cong \mathcal{A}_n$ and $\mathcal{A}_n$ has no normal subgroup of order 2, we must have $\texttt{ker}(\nu) \cap K = \{e\}$ and so in fact $\nu(K) \cong K \cong \mathcal{A}_n$. So K must contain exactly one of $+[\pi]$ and $-[\pi]$ for each $\pi \in \mathcal{A}_n$, since $\nu|_K$ is injective.

In particular K must contain elements of the form $\pm[a_1, a_2, a_3]$ that map to 3-cycles (a_1, a_2, a_3) and they must have the same order as their images in $\nu(K)$. Since $+[a_1, a_2, a_3]$ has order 6, and (a_1, a_2, a_3) has order 3, we cannot have $[a_1, a_2, a_3]$ in K, but must have instead $-[a_1, a_2, a_3]$ (which has order 3).

So in particular $-[1, 2, 3], -[1, 2, 4] \in K$. But now

$$\begin{aligned}
-[1,2,3] \times -[1,2,4] &= [1,2,3][1,2,4] \\
&= [1,2][1,3][1,2][1,4] \\
&= [1,2]^{-1}[1,3][1,2][1,4] \\
&= [1,3]^{[1,2]}[1,4] \\
&= -[1^{(1,2)}, 3^{(1,2)}][1,4] \\
&= -[2,3][1,4].
\end{aligned}$$

We also see that

$$\begin{aligned}
(-[2,3][1,4])^2 &= [2,3][1,4][2,3][1,4] \\
&= [1,4]^{[2,3]}[1,4] \\
&= -[1^{(2,3)}, 4^{(2,3)}][1,4] \\
&= -[1,4]^2 = -1,
\end{aligned}$$

so $-[2, 3][1, 4]$ has order 4, not 2. But if $K \cong \mathcal{A}_n$, then the product of $-[1, 2, 3]$ and $-[1, 2, 4]$, which map to $(1, 2, 3)$ and $(1, 2, 4)$, must have order 2. So we have a contradiction.

(2) is immediate. The set $\{\pm[\pi] : \pi \in \mathcal{A}_n \ \text{ fixing } \ n\}$ defines the group $\hat{\mathcal{A}}_{n-1}$ by exactly the same construction we just described for $\hat{\mathcal{A}}_n$.

□

2.2.6 Wreath product

Definition 2.21 (Wreath product)**.** *Let $S \subseteq \mathcal{S}(\Omega)$ be a permutation group on a set $\Omega = \{1, 2, \ldots, \deg\}$ and Q a group. (For ease of notation, we write $i\sigma$ rather that i^σ for the image of i under σ.) Then the* ***wreath product*** *$Q \wr S$ of Q by S is defined as follows.*

First we define $Q^{\deg}$ to be the (external) direct product of $\deg$ copies of Q, the group of all $\deg$-tuples $(p_1, \ldots, p_{\deg})$, $p_i \in Q$. We can define $R_i \subseteq Q^{\deg}$ to be the subgroup of all elements with $p_j = 1$ for $j \neq i$. Then $R_i \cong Q$, and we see that $Q^{\deg} = R_1 \cdots R_{\deg}$ is the internal direct product of the subgroups R_i. Where $r = (p_1, \ldots, p_{\deg}) \in Q^{\deg}$, we call $r_i = (1, \ldots, 1, p_i, 1, \ldots, 1) \in R_i$ the component of r in R_i.

We can define an action of S on $Q^{\deg}$ as follows. For $r \in Q^{\deg}, \sigma \in S$,

$$r = (p_1, \ldots, p_{\deg}) \quad \mapsto^{\sigma} \quad r^\sigma := (p_{1\sigma^{-1}} \ldots p_{\deg\sigma^{-1}}).$$

We observe that

$$(\overbrace{1, \ldots, 1, p, 1, \ldots\ldots, 1}^{p \textit{ in position } i}) \quad \mapsto^{\sigma} \quad (\overbrace{1, \ldots\ldots, 1, p, \ldots, 1}^{p \textit{ in position } j})$$

iff $j\sigma^{-1} = i$, that is iff $j = i\sigma$. Hence we see that σ permutes the subgroups R_i, with $R_i^\sigma = R_{i\sigma}$, and, for $r \in Q^{\deg}$, the j-th component $(r^\sigma)_j$ of r^σ is $r_{j\sigma^{-1}}$.

Using the action of S on $Q^{\deg}$, we can define the semidirect product $Q^{\deg} \rtimes S$ as *the set of all pairs (r, σ) multiplied by the rule*

$$(r, \sigma)(r', \sigma') = (rr'^{\sigma^{-1}}, \sigma\sigma')$$

We define the wreath product $Q \wr S$ to be this semidirect product.

We see that the set of all pairs $(r, 1)$ with $r \in Q^{\deg}$ is a normal subgroup of the wreath product isomorphic to $Q^{\deg}$, and that the set C of all pairs $(1, \sigma)$ with $\sigma \in S$ is a complement of this.

If Q is a permutation group, of a finite set B, that is $Q \subseteq \mathcal{S}(B)$, then $Q \wr S$ is a subgroup of $\mathcal{S}(B \times \Omega)$, as follows. Each element of $Q \wr S$ has a unique representation as a pair (r, σ) with $r \in Q^{\deg}$, $\sigma \in S$, and as above, we write $r = r_1 \ldots r_{\deg}$, with $r_i \in R_i$. Under the action of $(r, \sigma) = (r_1 \ldots r_{\deg}, \sigma)$,

$$(b, i) \mapsto (b^{r_i}, i\sigma)$$

To see that this is a permutation representation of $Q \wr S$, we need to check that if we apply first (r, σ) and then (r', σ') to (b, i) we get the same result as when we apply the product $(r, \sigma)(r', \sigma')$. But now,

$$\begin{aligned}(b, i) \mapsto^{(r,\sigma)} (b^{r_i}, i\sigma) &\mapsto^{(r',\sigma')} ((b^{r_i})^{r'_{i\sigma}}, i\sigma\sigma') \\ &= (b^{r_i r'_{i\sigma}}, i\sigma\sigma') = (b^{r_i (r'^{\sigma^{-1}})_i}, i\sigma\sigma') = (b^{(rr'^{\sigma^{-1}})_i}, i\sigma\sigma') \\ &= (b, i)^{(rr'^{\sigma^{-1}}, \sigma\sigma')} = (b, i)^{(r,\sigma)(r',\sigma')},\end{aligned}$$

as we need.

In particular we notice the actions of the normal subgroup $Q^{\deg}$ and its complement C, with

$$(b, i) \mapsto^{(r,1)} (b^{r_i}, i)$$
$$(b, i) \mapsto^{(1,\sigma)} (b, i\sigma)$$

Now suppose that $\mathcal{U}$ is the disjoint union of sets $B_1, \ldots, B_{\deg}$, such that each B_i for $i = \{1, \ldots, \deg\}$ are of equal size. Then there is a bijection between $\mathcal{U}$ and $B \times \Omega$, that maps b_i to the set $\{(b, i) : b \in B\}$. So $Q \wr S$ can be seen as a subgroup of $\mathcal{S}(\mathcal{U})$.

Example 2.22. *We can decompose the group D_8 as $C_2 \wr C_2$, because $D_8 \cong C_2^2 \rtimes C_2$. We know that $C_2 \cong S_2 = \{(), (1, 2)\}$ and $C_2 = \{e, a\}$.*
$C_2 \times C_2 = \{(e, e), (e, a), (a, e), (a, a)\}$. *The elements of $(C_2 \times C_2) \rtimes C_2$ are*
$\{((e, e), ()), (e, a), ()), ((a, e), ()), ((a, a), ()), ((e, e)(1, 2)), ((e, a), (1, 2)),$
$((a, e), (1, 2)), ((a, a), (1, 2))\}$.

2.3 Permutation representations

Let G be a group. Then a **permutation representation** of G on a set Ω is a homomorphism π from G to a subgroup of $\mathcal{S}(\Omega)$; its **image** is the subgroup $\pi(G)$.

When $\omega \in \Omega$, we'll write $\omega^{\pi(g)}$ for the image of ω under the permutation $\pi(g)$.

Definition 2.23. *When a group G acts on a set Ω via π, a typical point $\alpha \in \Omega$ is moved by elements of G to various other points. The set of these images is called the* ***orbit*** *of α under G; we denote it by*

$$\alpha^{\pi}(G) := \{\alpha^{\pi(x)} \mid x \in G\}.$$

Definition 2.24. *A group G acting on a set Ω is said to be **transitive** on Ω if it has only one orbit, and so $\alpha^{\pi(G)} = \Omega$ for all $\alpha \in \Omega$.*

Definition 2.25. *A group G is acting on a set Ω and $\alpha \in \Omega$, the **stabilizer** of α, denoted by G_α or $\texttt{stab}(\alpha)$ is defined to be*

$$G_\alpha := \{x \in G \mid \alpha^{\pi(x)} = \alpha\}.$$

Note that the kernel of the action is the intersection of all point stabilisers, or also the core of the subgroup of $H = G_\alpha$ in G if the action is transitive, and is given by

$$\bigcap_{\alpha \in \Omega} \texttt{stab}(\alpha)$$

Definition 2.26. *Permutation representations π, π' of G on Ω, Ω' are **equivalent** if there exists a bijection $f : \Omega \to \Omega'$ such that, for all $g \in G$ and all $\omega \in \Omega$,*

$$f(\omega^{\pi(g)}) = f(\omega)^{\pi'(g)}.$$

Theorem 2.27. *[15] Let G be a group, acting transitively on a set Ω. Let α be an element of Ω, and let $H = \texttt{stab}(\alpha)$. Then the action of G on Ω is equivalent to the action of G on the right cosets of $\texttt{stab}(\alpha)$ by right multiplication.*

2.4 Coset diagrams

A coset diagram is a generalised form of a Cayley graph. It is in fact the graph whose vertices represents the cosets of a subgroup of finite index of the finitely presented groups, where the number of vertices is the degree or index of the subgroup. Suppose $G = \langle S \mid R \rangle$ is a group with $S = s_1, s_2, ...$ as the set of generators and R is the set of relations. Two vertices, U and V in the coset diagram are joined by an edge s_k

directed from U to V when $Us_k = V$, where s_k is one of the generators of the group. This definition allow loops in the graph that is defined by fixed elements i.e. vertex V is joined to itself by the generator s_k, satisfying $Vs_k = V$. These fixed elements are represented by heavy dots in the coset diagram of a triangle group, where it applies to fixed points of one of the generators of a given group.

In the following example we have used heavy dots for a point that is fixed by a generator y.

Example 2.28. *Take a triangle group*

$$\langle x, y : x^2 = y^3 = (xy)^7 = 1\rangle$$

and consider a transitive permutation representation (on 14 points) given by assigning permutations
x acts as $(3,4)(5,7)(6,8)(9,12)(10,13)(11,14)$
y acts as $(1,2,3)(4,5,6)(7,9,10)(8,11,12)$
This can be represented by the coset diagram in Fig. 2.1

Note that all points in each cycle of y are permuted anticlockwise.

2.5 Primitive and imprimitive permutation groups

Definition 2.29. *Let $\pi(G)$ be a permutation group acting on a set Ω via a homomorphism π. The subset X of Ω is said to be a **block** of $\pi(G)$ if for every $g \in G$ either $X^{\pi(g)} = X$ or $X^{\pi(g)} \cap X = \emptyset$.*
Here, $X^{\pi(g)} = \{x^{\pi(g)} : x \in X\}$ is the set. The sets Ω, $\emptyset$ and the singletons $\{x\}$ are the trivial blocks of $\pi(G)$ acting on a set Ω.[14]

Definition 2.30. *Suppose that a group $\pi(G)$ be a transitive action of a group G on set Ω. Then $\pi(G)$ is said to be **imprimitive** or act imprimitively, if there is at least*

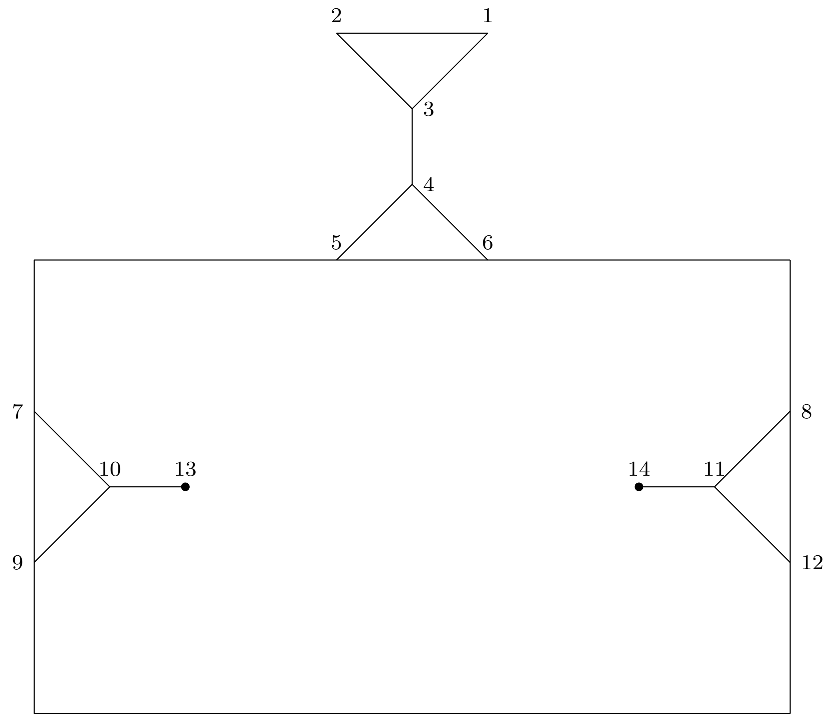


Figure 2.1: Coset diagram

one non-trivial block X. The images of a block form a partition of the set Ω into disjoint sets of equal size. Otherwise $\pi(G)$ is said to be primitive, or act primitively.

Example 2.31. *The triangle group $\Delta(2,3,6)$ acts on a set $\Omega = \{1, 2, .., 6\}$ with transitive permutation representation given by $\pi(x) = (3, 4)$, $\pi(y) = (1, 2, 3)(4, 5, 6)$, and $\pi(xy) = (1, 2, 3, 5, 6, 4)$ that is imprimitive with blocks*

$[[1, 5], [2, 6], [3, 4]]$.

Lemma 2.32. *(Jordan's Theorem [21]). Let $\pi(G)$ be a primitive permutation group of degree* deg *containing a prime cycle for some prime $q \leq n - 3$. Then $\pi(G)$ is either the alternating group $\mathcal{A}_n$ or the symmetric group $\mathcal{S}_n$.*

Lemma 2.33. *(Everitt's Lemma [6]). Let $H = \langle x_1, x_2, \ldots, x_k \rangle$ be a transitive action of a group of degree* deg *containing a prime cycle μ. For each x_i, suppose there is a point in the support of μ whose image under x_i is also in the support of μ. Then H*

is primitive.

Lemma 2.34. *Suppose that $H \subseteq S(\Omega)$ acts imprimitively with blocks $B_1, ..., B_k$. Then the subgroup $N = \{h \in H : {B_i}^h = B_i \ \forall i\}$ is normal in H.*

Proof. N is the kernel of the induced action of H on $\{B_1, \dots, B_k\}$. □

The following result is essentially from [1, Theorem 1.8], which refers to [18] for a proof. The proof we give is due to [8]

Proposition 2.35. *Let $H \subseteq \mathcal{S}(\mathcal{U})$ act transitively and imprimitively on a set $\mathcal{U}$ with block system $\mathcal{B} = \{B_1, \dots, B_{\mathrm{deg}}\}$, and let $\Omega = \{1, \dots, \mathrm{deg}\}$.*

Let $J_1, \dots, J_{\mathrm{deg}} \subseteq H$ be the setwise stabilisers of the blocks $B_1, \dots, B_{\mathrm{deg}}$ and let $K_1, \dots, K_{\mathrm{deg}}$ be the pointwise stabilisers of those blocks. Let $\psi : H \to \mathcal{S}_{\mathrm{deg}}$ define the action of H on the block system $\mathcal{B}$ and let $N = \cap_{i=1}^{\mathrm{deg}} J_i$ be the kernel of ψ.

Let $Q_i \subseteq \mathcal{S}(\mathcal{U})$ be the group of permutations of B_i defined by the action of J_i on B_i (so $Q_i \cong J_i/K_i$), and fixing all points of $\mathcal{U} \setminus B_i$, and let $P_i \subseteq Q_i \subseteq \mathcal{S}(\mathcal{U})$ be the group of permutations of B_i defined by the action of N on B_i (so $P_i \cong N/N \cap K_i$).

Then the groups Q_i are all isomorphic to a single group Q, and pairwise commute. Further H is isomorphic to a subgroup of

$$Q_1 Q_2 \cdots Q_{\mathrm{deg}} \rtimes \psi(H) \cong Q^{\mathrm{deg}} \rtimes \psi(H),$$

that is, H is isomorphic to a subgroup of $H \subseteq Q \wr \psi(H)$, and N is isomorphic to a subgroup of $P_1 \cdots P_{\mathrm{deg}}$.

Proof. Note that, for $h \in H$ and $i \in \Omega$, we have $J_i^h = J_{i^{\psi(h)}}$, and hence $Q_i^h = Q_{i^{\psi(h)}}$.

Choose $t_1 = 1$ and, for each $i \in \Omega$ with $i > 1$, choose $t_i \in H$ with $B_1^{t_i} = B_i$; or, equivalently, $1^{\psi(t_i)} = i$. So $\{t_i : i \in \Omega\}$ is a right transversal of J_1 in H and, for each $i \in \Omega$, we have $Q_1^{t_i} = Q_i$.

Let $m := |B_1|$, and label the points of B_1 as $(1,1),(2,1),\ldots,(m,1)$. Then for each $i \in \Omega$, we can use t_i to label the points of B_i as $(1,i),(2,i),\ldots,(m,i)$, where $(b,i) := (b,1)^{t_i}$ for $1 \le b \le m$. So we have now identified $\mathcal{U}$ with $B \times \Omega$.

For each $q_1 \in Q_1$, we can define a corresponding permutation $q \in \mathcal{S}(B)$, with $B := \{1,2,\ldots,\deg\})$ by $b^q = b'$ where $(b,1)^{q_1} = (b',1)$. Let $Q = \{q : q_1 \in Q_1\}$ and define the isomorphism $\tau_1 : Q \to Q_1$ by $\tau_1(q) = q_1$. Then, for each $i \in \Omega$, define the isomorphism $\tau_i : Q \to Q_i$ by $\tau_i(q) = \tau_1(q)^{t_i}$, and define an isomorphism $\tau : Q^{\deg} \to R$ by $\tau(r) = \prod_{i\in\Omega} \tau_i(r_i)$, where $r_i \in Q$ is the i-th component of $r \in Q^{\deg}$.

Now, for $\tau_i(q) \in Q_i$, and $b \in B$, we have

$$(b,i)^{\tau_i(q)} = (b,i)^{t_i^{-1}\tau_1(q)t_i} = (b,1)^{\tau_1(q)t_i} = (b^q,1)^{t_i} = (b^q,i).$$

So, for $r \in R$ with components $r_i \in Q$ and $\tau(r) = \prod_{i\in\Omega} \tau_i(r_i)$, we have $(b,i)^{\tau(r)} = (b,i)^{\tau_i(r_i)} = (b^{r_i},i)$ for all $b \in B$, $i \in \Omega$. In other words, by using the isomorphism τ to identify $Q^{\deg}$ with R, we see that the action of R on $\mathcal{U}$ is the same as in the permutation wreath product $Q \wr S$ defined above.

We also need to consider the action of the complement in the wreath product. Define a monomorphism $\mathrm{c} : \psi(H) = S \to \mathcal{S}(\mathcal{U})$ as follows. For each $b \in B$, $\mathrm{c}(\psi(g))$ acts on the set $\{(b,i) : i \in \Omega\}$ in the same way as $\psi(g)$ acts on Ω; that is, $(b,i)^{\mathrm{c}(\psi(g))} = (b, i^{\psi(g)})$. Define $C = \{c(\sigma) : \sigma \in S\}$. Now we see that the action of C on Ω is also the same as in the permutation wreath product $Q \wr S$ defined above, so the subgroup $\langle R, C\rangle$ of $\mathcal{S}(\mathcal{U})$ is the semidirect product $R \rtimes C$, and can be identified with $Q \wr S$.

To finish, we need to see that $H \subseteq R \rtimes C$.

Let $h \in H$ and $\sigma = \psi(h) \in S$. Then, for each $i \in \Omega$, we have $B_i^h = B_{i\sigma}$ and, since $B_1^{t_i} = B_i$, we have $B_1^{t_i h t_{i\sigma}^{-1}} = B_1$; that is, $h_i := t_i h t_{i\sigma}^{-1} \in J_1$. Let $\bar{h}_i \in Q_1$ be the induced action of $t_i h t_{i\sigma}^{-1}$ on B_1 and $r_i := \tau_1^{-1}(\bar{h}_i) \in Q$. Let $r = \prod_{i\in Q} r_i$.

We claim that $h = \tau(r)\mathrm{c}(\sigma) \in RC$, which will prove the result. Note that $\tau(r) = \prod_{i\in Q} \tau_i(r_i)$. Let $(b, i) \in B \times \Omega$. So $(b, i)^h = (b', i\sigma)$ for some $b' \in B$. Then, from the definition of the elements t_i, we have

$$(b, 1)^{h_i} = (b, 1)^{t_i h t_{i\sigma}^{-1}} = (b, i)^{h t_{i\sigma}^{-1}} = (b', i\sigma)^{t_{i\sigma}^{-1}} = (b', 1)$$

and hence $(b, i)^{\tau_i(r_i)} = (b, i)^{t_i^{-1} h_i t_i} = (b', i)$. Since $(b', i)^{\mathrm{c}(\sigma)} = (b', i\sigma) = (b, i)^h$, we see that (b, i) has the same image under h as under $\tau(r)\mathrm{c}(\sigma)$, which proves the claim. □

Notation : For the rest of this thesis, whenever we have a group $H \subseteq \mathcal{S}(\mathcal{U})$, acting transitively and imprimitively on $\mathcal{U}$ with block system $\mathcal{B} = \{B_1, \ldots, B_{\deg}\}$, we shall use the notation of this lemma. So $\psi : H \to \mathcal{S}(\mathcal{B})$ will define the action of H on $\mathcal{B}$, and N its kernel. We denote $J_1, \ldots, J_{\deg}$ as the setwise stabilisers of $B_1, \ldots, B_{\deg}$ in H and $K_1, \ldots, K_{\deg}$ will be the pointwise stabilisers of $B_1, \ldots, B_{\deg}$ in H. Moreover, $P_i \cong N/N\cap K_i$ and $Q_i \cong J_i/K_i$ such that $P_i \subseteq Q_i$ for $i = \{1, \ldots, \deg\}$.

2.6 Linear representations

An n-dimensional linear representation of a group G over a field K is a homomorphism $\rho : G \to GL_n(K)$, that is a homomorphism from G into the group of all $n \times n$ invertible matrices over K. For v a row vector of length n and $g \in G$, the image of v under $\rho(g)$ is then the matrix product $v\rho(g)$.

Definition 2.36. *A representation $\rho : G \to GL_n(K)$ is said to be faithful if $\mathtt{Ker}\rho = \{1\}$; that is, if the identity element of G is the only element g for which $\rho(g) = I_n$.*

Proposition 2.37. *Every permutation group $G \subseteq S_{\deg}$ has a faithful* deg*-dimensional linear representation, over any field K.*

Proof. If K is a field, let $e_1, \ldots, e_{\deg}$ denote the standard basis of the deg dimensional vector space V over K (that is, $e_1, \ldots, e_{\deg}$ form the rows of a $\deg \times \deg$ identity matrix $I_{\deg}$). Given a permutation $\alpha \in G$, we form the associated permutation matrix P_α over the field K by permutating the rows of $I_{\deg}$; i.e, the rows of P_α are $e_{1\alpha}, \ldots, e_{\deg\alpha}$. We can easily check that the map $\alpha \mapsto P_\alpha$ is an injective homomorphism from G to $GL_{\deg}(K)$. For each i, the basis vector e_i of V is mapped by P_α to $e_{i\alpha}$. □

It follows that the set of all permutation matrices over the field K denoted by $P(\deg, K)$ is a group isomorphic to $\mathcal{S}_{\deg}$.

We call ρ as defined the permutation representation of G, and we call the associated module the permutation module of G denoted by $W_{\deg}$ or W.

Suppose that K is the field of the integers mod p, where p is prime.

When $G = \mathcal{S}_{\deg}$ or $\mathcal{A}_{\deg}$, the permutation module W has just two non-trivial, proper submodules. Where $e_1, \ldots, e_{\deg}$ is the standard basis as above, then $W_1 := \langle \sum_{i=1}^{\deg} e_i \rangle$ is a one-dimensional submodule and $W_{\deg-1} := \{v = \sum_{i=1}^{\deg} \lambda_i e_i : \sum_{i=1}^{\deg} \lambda_i = 0\}$ is a $(\deg - 1)$-dimensional submodule.

When p does not divide deg, we can write

$$\begin{aligned} e_j &= \frac{1}{\deg} \sum_{i=1}^{\deg} e_i + \frac{1}{\deg} \sum_{i \neq j} (e_j - e_i) \\ &\in W_1 + W_{\deg-1}. \end{aligned}$$

So, $$W = W_1 + W_{\deg-1}.$$

If $p \nmid \deg$ then $\sum_{i=1}^{\deg} e_i \notin W_{\deg-1}$. So clearly $W_1 \cap W_{\deg-1} = \{0\}$, and so $W = W_1 \oplus W_{\deg-1}$.

But if p divides deg, then W_1 is a submodule of $W_{\deg-1}$. In that case, the quotient module $W_{\deg-1}/W_1$ is an irreducible module of dimension $p - 2$ for G.

For the group $\mathcal{A}_{\text{deg}}$ we also have the following result.

Lemma 2.38 (Wagner's lemma)**.** *[19, 20] The minimal dimension of a non-trivial faithful representation of $\mathcal{A}_{\text{deg}}$ over F_p is*

$$\begin{aligned} \textit{either} \quad & \deg - 1, \text{ if } \ p \nmid \deg \quad \textit{and} \quad (\deg > 8 \text{ or } (p = 2 \textit{ and } \deg > 6)) \\ \textit{or} \quad & \deg - 2, \text{ if } \ p \mid \deg \quad \textit{and} \quad (\deg > 8 \text{ or } (p = 2 \textit{ and } \deg > 6)). \end{aligned}$$

Chapter 3

Composition

This chapter illustrates the idea of composition and the algorithms we developed to compose coset diagrams for $p = 2$ or *odd*, which is described in [6]. In Section 3.1, we describe how we compose $t \leq p$ coset diagrams to get a transitive diagram. We illustrate this with some detailed examples. We also provide a table of further examples.

3.1 Composition of up to p coset diagrams

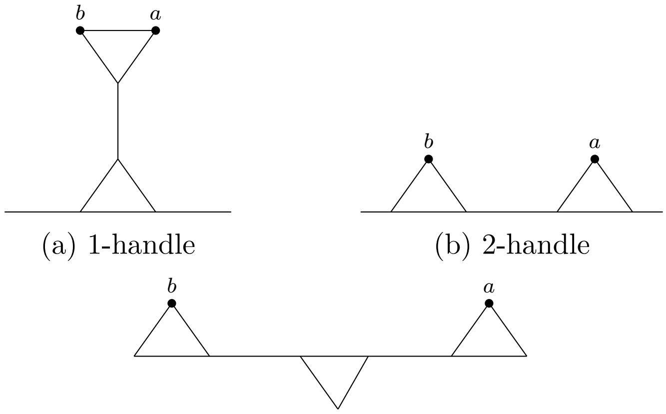
Here we use a number of coset tables for the triangle groups,

$$G = \Delta(p, q, r) = \langle x, y : x^p = y^q = (xy)^r \rangle$$

where x and y are generators of the triangle group of order p and q respectively.

Definition 3.1. *In an arbitrary permutation representation π of G, two points a and b, which are fixed by $\pi(x)$ such that $(\pi(x)\pi(y))^k$ map a to b, form a k-**handle** and are denoted by $[a, b]_k$ [3].*

The following picture of handles is for the triangle group when $p = 2$ and $q = 3$.



(a) 1-handle (b) 2-handle

(c) 3-handle

The idea for composition is the following: if we have p coset diagrams of degrees $\deg_1, \deg_2, .., \deg_p$ for the triangle group G, each having k-handles then we can join them together by using these handles to find a coset diagram of G of some larger degree.

Suppose that t is an integer with $t \leq p$. Then we define $\mathcal{U}$ (or $\mathcal{U}_t$ if it is necessary to specify t) to be the disjoint union of $\Omega_1, \Omega_2, \ldots, \Omega_t$. We shall use this notation throughout the remainder of the thesis. The following proof of the theorem is a new (algebraic) proof of Proposition 3.1 in [6].

Theorem 3.2. *Suppose that for some $t \leq p$. If $G = \Delta(p, q, r)$ has transitive permutation representations $\pi_1, \ldots, \pi_t$ on distinct, disjoint, finite sets $\Omega_1, \ldots, \Omega_t$, and that $[a_1, b_1], \ldots, [a_p, b_p]$ are all k-handles, with $[a_i, b_i]$ a handle in Ω_{j_i}. If $j_i = j_{i'}$ for $i \neq i'$, then suppose that the handles $[a_i, b_i]$ and $[a_{i'}, b_{i'}]$ are disjoint. Now define permutations ϕ_x, ϕ_y of $\mathcal{U} = \mathcal{U}_t$ via*

$$\begin{aligned} \phi_x &= \pi_1(x) \cdots \pi_t(x) \circ (a_1, \cdots, a_p)(b_b, b_{p-1}, \cdots, b_1), \\ \phi_y &= \pi_1(y) \cdots \pi_t(y). \end{aligned}$$

Then ϕ_x, ϕ_y are the images of x and y for a transitive permutation representation ϕ of G on $\mathcal{U}$.

Proof. We need to show that $\phi_x^p = 1, \phi_y^q = 1$ and $(\phi_x\phi_y)^r = 1$. We know that $\pi_1(x)^p = 1, \pi_2(x)^p = 1, ..., \pi_t(x)^p = 1$ and $(a_1, ..., a_p)^p = (b_p, ..., b_1)^p = 1$, so clearly $\phi_x^p = 1$. Similarly $\pi_1(y)^q = \pi_2(y)^q = ... = \pi_t(y)^q = 1$, therefore $(\phi_y)^q = 1$. To finish we need to verify that $(\phi_x\phi_y)^r = 1$. Now consider the cycle of $\phi_x\phi_y$ that contains a_i. Suppose that the cycle of $\pi_{j_i}(x)\pi_{j_i}(y)$ that contains a_i has length s such that $s|r$. Then it also contains b_i and satisfies the following equations $a_i^{\pi_{j_i}(x)} = a_i$, $b_i^{\pi_{j_i}(x)} = b_i$, $a_i^{\pi_{j_i}(xy)^k} = b_i$ and $b_i^{\pi_{j_i}(xy)^{s-k}} = a_i$. Now we have

$$a_i \xrightarrow{\phi_x\phi_y} a_{i+1}^{\phi_y} \xrightarrow{(\phi_x\phi_y)^{k-1}} b_{i+1} \xrightarrow{\phi_x\phi_y} b_i^{\phi_y} \xrightarrow{(\phi_x\phi_y)^{s-(k+1)}} a_i. \tag{3.1}$$

This implies

$$a_i \xrightarrow{(\phi_x\phi_y)^{1+k-1+1+s-k-1}} a_i, \tag{3.2}$$

so,

$$a_i \xrightarrow{(\phi_x\phi_y)^s} a_i, \tag{3.3}$$

and we see that a_i and b_{i+1} are together in a cycle of length s for $\phi_x\phi_y$ that contains some points from the cycle of $\pi_{j_i}(x)\pi_{j_i}(y)$ containing a_i and b_i and some points from the cycle of $\pi_{j_{i+1}}(x)\pi_{j_{i+1}}(y)$ containing a_{i+1} and b_{i+1}. We see also that a cycle of $\phi_x\phi_y$ that contains no a_i has the same length as a cycle of $\pi_j(x)\pi_j(y)$ for some j. So $(\phi_x\phi_y)^r = 1$.

Now define $\phi : G \to \mathcal{S}(\mathcal{U})$ by $\phi(x) = \phi_x$, $\phi(y) = \phi_y$ and extending multiplicatively. We have proved that ϕ is a homomorphism, defining an action of G on $\mathcal{U}$.

To prove that the action of G on the set $\mathcal{U}$ defined by ϕ is transitive, we need to check that for all $z, z' \in \mathcal{U}$, $\exists g \in G$ with $z^{\phi(g)} = z'$. We have the following cases

Case 1: If $z, z' \in \Omega_i$ are in the same subset, then $\exists g \in G$ such that $z^{\pi_i(g)} = z'$, because G is transitive on the set Ω_i. We can write

$$g = x^{i_1}y^{j_1}x^{i_2}y^{j_2}...x^{i_k}y^{j_k}$$

it is possible that $i_1 = 0$ or $j_k = 0$. Now we define $z_0 = z$ and

$$\begin{aligned} z_1 &= w_1^{\pi_i(y^{j_1})} & w_1 &= z_0^{\pi_i(x^{i_1})} \\ z_2 &= w_2^{\pi_i(y^{j_2})} & w_2 &= z_1^{\pi_i(x^{i_2})} \\ &\vdots & &\vdots \\ z_k &= z' = w_k^{\pi_i(y^{j_k})} & w_k &= z_{k-1}^{\pi_i(x^{i_k})}. \end{aligned}$$

We want to find $g' \in G$ such that $\phi(g')$ maps z to z' through the same points of $z_0 = z, w_1, z_1, ...w_k, z_k = z'$ as $\pi_i(g)$ does. For this, we know that $\pi_i(y)$ acts on Ω_i just as $\phi(y)$ does, however, $\pi_i(x)$ and $\phi(x)$ do not act the same on the set Ω_i. In fact, for each ℓ, $w_\ell^{\pi_i(y^{j_\ell})} = w_\ell^{\phi(y^{j_\ell})}$, however, $z_\ell^{\pi_i(x^{i_\ell})} = z_\ell^{\phi(x^{i_\ell})}$ unless $z = a_i$ or $z = b_i$. If $z = a_i$ or b_i then $z_\ell^{\pi_i(x^{i_\ell})} = z_\ell$. So in that case $w_{\ell+1} = z_\ell$. We form g' from g by deleting from g all those x^{i_ℓ} for which $z_\ell = a_i$ or b_i. Then we have $z^{\phi(g')} = z'$.

Case 2: If $z \in \Omega_i$ and $z' \in \Omega_j$, where $i \neq j$ then $\phi(x^{j-i})$ maps a_i to a_j. Then $\exists g_1, g_2, g_3 \in G$ such that $z^{\phi(g_1)} = a_i$, $a_i^{\phi(g_2)} = a_j$, $a_j^{\phi(g_3)} = z'$. This implies $z^{\phi(g_1)\phi(g_2)\phi(g_3)} = z'$. i.e $z^{\phi(g_1g_2g_3)} = z'$. □

Example 3.3. *Consider an action π of the triangle group $\Delta(5, 7, 11)$ on a set of 14* *points. We can find* π *with*

$$\pi(x) = (3, 5, 9, 11, 6)$$
$$\pi(y) = (1, 2, 4, 8, 13, 7, 3)(5, 10, 14, 12, 6, 11, 9)$$
$$\pi(xy) = (1, 2, 4, 8, 13, 7, 3, 10, 14, 12, 6)$$

Let $\pi_1, \pi_2, \pi_3, \pi_4, \pi_5$ *all equivalent to* π, *define five coset diagrams, say* $D_1, D_2, ..., D_5$ *corresponding to the triangle group* $\Delta(5, 7, 11)$ *each of degree 14. The cycles of the images xy for each of the diagrams are;*

D_1 : *(1,2=*a_1*,4,6,11=*b_1*,13,14,12,9,5,3);*

D_2 : *(15,16=*a_2*,18,20,25=*b_2*,27,28,26,23,19,17);*

D_3 : *(29,30=*a_3*,32,34,39=*b_3*,41,42,40,37,33,31);*

D_4 : *(43,44=*a_4*,46,48,53=*b_4*,55,56,54,51,47,45);*

D_5 : *(57,58=*a_5*,60,62,67=*b_5*,69,70,68,65,61,59),*

where the diagrams in figure 3.2 are on disjoint domains $\{1, \ldots, 14\}$, $\{15, \ldots, 28\}$ *etc. Here* a_i *and* b_i *for* $i = 1, 2, 3, 4, 5$ *satisfies* $a_i^{\pi_i(x)} = a_i$, $b_i^{\pi_i(x)} = b_i$ *and* $a_i^{\pi_i(xy)^3} = b_i$ *i.e.* $[a_i, b_i]$ *are all 3-handles. We can compose the above coset diagrams by using handles,* a_i, b_i $(i = 1 \ldots 5)$ *which gives us two 5-cycles* $(a_1, a_2, a_3, a_4, a_5)$ *and* $(b_5, b_4, b_3, b_2, b_1)$ *of* x. *We then have the cycles of* $\phi(xy)$ *in G:*

(1,58=a_5*,4,6,11=*b_1*,13,14,12,9,5,3)*

(2=a_1*,18,20,25=*b_2*,27,28,26,23,19,17,15)*

(16=a_2*,32,34,39=*b_3*,41,42,40,37,33,31,29)*

(30=a_3*,46,48,53=*b_4*,55,56,54,51,47,45,43)*

(44=a_4*,60,62,67=*b_5*,69,70,68,65,61,59,57).*

Now we see that each cycle for $\phi(xy)$ *contains* a_i, b_{i+1} *and has order 11.*

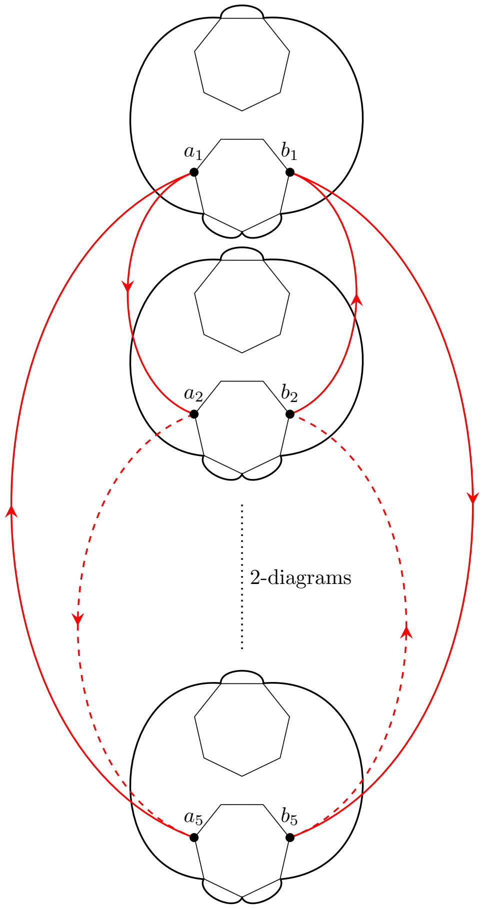


Figure 3.2: Composition of five coset diagrams

The following example illustrates the argument we used to prove transitivity.

Example 3.4. *Consider a triangle group $\Delta(2, 3, 7)$. Two equivalent representation of degree 7, π_1 and π_2 are defined by*

$\pi_1(x) = (3, 4)(6, 7)$

$\pi_1(y) = (1, 2, 3)(4, 5, 6)$

$\pi_1(xy) = (1, 2, 3, 5, 6, 7, 4)$, *with handles $a_1 = 1, b_1 = 2$ and*

$\pi_2(x) = (10, 11)(13, 14)$

$\pi_2(y) = (8, 9, 10)(11, 12, 13)$

$\pi_2(xy) = (8, 9, 10, 12, 13, 14, 11)$, *with handles $a_2 = 8, b_2 = 9$. The permutations*

$\phi(x), \phi(y)$ of Theorem 3.2 are:

$$\phi(x) = (3,4)(6,7)(10,11)(13,14)(1,8)(2,9)$$
$$\phi(y) = (1,2,3)(4,5,6)(8,9,10)(11,12,13)$$
$$\phi(xy) = (1,9,3,5,6,7,4)(2,10,12,13,14,11,8)$$

We illustrate case 1 of the transitivity proof of Theorem 3.2. For instance we examine points 1 and 7 in $\Omega_1 = \{1,\dots,7\}$. The word $g = (yx)^4$ sends 1 to 7 under the representation π_1, because

$$1 \xrightarrow{\pi_1(y)} 2 \xrightarrow{\pi_1(x)} 2 \xrightarrow{\pi_1(y)} 3 \xrightarrow{\pi_1(x)} 4 \xrightarrow{\pi_1(y)} 5 \xrightarrow{\pi_1(x)} 5 \xrightarrow{\pi_1(y)} 6 \xrightarrow{\pi(x)} 7$$

We remove the letters from g which fix points of Ω_1 in the above calculation. The resulting word is $g' = y^2xy^2x$. Each prefix of g' sends $1 \in \Omega_1$ to a different image under π_1, as the calculation

$$1 \xrightarrow{\pi_1(y)} 2 \xrightarrow{\pi_1(y)} 3 \xrightarrow{\pi_1(x)} 4 \xrightarrow{\pi_1(y)} 5 \xrightarrow{\pi(x)} 5 \xrightarrow{\pi_1(y)} 6 \xrightarrow{\pi_1(x)} 7$$

The same is true of g' under the representation ϕ, because

$$1 \xrightarrow{\phi(y)} 2 \xrightarrow{\phi(y)} 3 \xrightarrow{\phi(x)} 4 \xrightarrow{\phi(y)} 5 \xrightarrow{\phi(x)} 5 \xrightarrow{\phi(y)} 6 \xrightarrow{\phi(x)} 7$$

So, $\phi(g') = \phi(y)^2(\phi(x)\phi(y))^2\phi(x)$. Using GAP *we see that $\phi(G) \cong C_2^3.\mathtt{PSL}(3,2)$ has order 1344.*

Example 3.5. *The triangle group $\Delta(3,5,7)$ acts on a set $\Omega = \{1,2,\dots,14\}$ with* *transitive permutation representation* π *such that*

$$\pi(x) = (3,5,6)(9,11,12)(10,13,14)$$
$$\pi(y) = (1,2,4,7,3)(5,8,6,10,9)$$
$$\pi(xy) = (1,2,4,7,3,8,6)(5,10,13,14,9,11,12)$$

has handles $[1,2],[4,7]$. *We set* $t = 2$*, and use both handles in the first copy of the diagram,* $[1,2]$ *in the second copy. After composition we have the permutations* $\phi(x), \phi(y)$*:*

$$\phi(x) = (1,15,4)(2,7,16)(3,5,6)(9,11,12)(10,13,14)(17,19,20)(23,25,26)(24,27,28)$$
$$\phi(y) = (1,2,4,7,3)(5,8,6,10,9)(15,16,18,21,17)(19,22,20,24,23)$$
$$\phi(xy) = (1,16,4,2,3,8,6)(5,10,13,14,9,11,12)(7,18,21,17,22,20,15)(19,24,27,28,23,25,26)$$

The cycles (a_1,a_2,a_3) *and* (b_3,b_2,b_1) *are* $(1,15,4)$ *and* $(2,7,16)$*. Using* GAP *we see that* $\phi(G) \cong \mathcal{A}_{28}$.

Example 3.6. *Consider two inequivalent representation* π_1 *and* π_2 *by using low index sybgroup algorithm in* GAP *of the triangle group* $\Delta(2,3,7)$ *of degree 7, where* π_1 *has permutation representations* $\pi_1(x) = (3,4)(6,7)$, $\pi_1(y) = (1,2,3)(4,5,6)$, *has handle* $a_1 = 1, b_1 = 2$*, and* π_2 *is defined by* $\pi_2(x) = (3,4)(5,7)$, $\pi_2(y) = (1,2,3)(4,5,6)$, *has handle* $a_2 = 1, b_2 = 2$*. Suppose that these two permutation representations are equivalent, then there exist a map* $f : \Omega \to \Omega$ *where* $\Omega = \{1,\dots,7\}$, *so that for all* $\omega \in \Omega$ *and for all* $g \in G$*, it satisfies* $f(\omega^{\pi_1(g)}) = f(\omega)^{\pi_2(g)}$. *In particular,* f *would have to map the fixed point of* y *in the first representation* $\pi_1(g)$ *to the fixed points of* y *in the second representation* $\pi_2(g)$*. So we see that*

$f(7) = 7$. But then since $\pi_1(xy^2) = (3,4)(6,7)(1,3,2)(4,6,5) = (1,3,6,7,5,4,2)$ and $\pi_2(xy^2) = (3,4)(5,7)(1,3,2)(4,6,5) = (1,3,6,5,7,4,2)$, applying the rule with $g = xy^2$ and $\omega = 7$, we must have $f(5) = 4$, $f(4) = 2$. This gives us a contradiction because 2 is fixed by x but 4 is not. Hence these two permutation representations are inequivalent because of two distinct quotients that are isomorphic to $\mathtt{PSL}(3,2)$. *Using* GAP *we see that both have structure description* $\mathtt{PSL}(3,2)$. *Using the construction of Theorem 3.2 the permutations $\phi(x), \phi(y)$ on $\{1,\ldots,14\}$ are:*

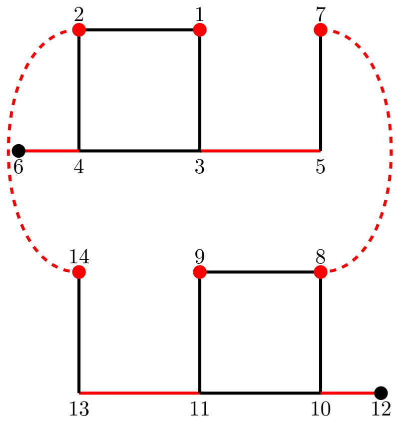
$$\phi(x) = (1,8)(2,9)(3,4)(6,7)(10,11)(12,14)$$
$$\phi(y) = (1,2,3)(4,5,6)(8,9,10)(11,12,13)$$

We find the group generated by $\phi(x), \phi(y)$ is primitive and has structure description $\mathtt{PSL}(2,13)$ *by using* GAP *of order 1092.*

Example 3.7. *Consider two inequivalent representation of the triangle group $\Delta(2,4,7)$ of degree 7 π_1, defined by $\pi_1(x) = (3,5)(4,6)$, $\pi_1(y) = (1,2,4,3)(5,7)$, has handle $a_1 = 7, b_1 = 2$ and π_2, defined by $\pi_2(x) = (3,5)(4,6)$, $\pi_2(y) = (1,2,4,3)(6,7)$, has handle $a_2 = 1, b_2 = 7$. Using* GAP *we can see, both have structure description* $\mathtt{PSL}(3,2)$. *Using the construction of Theorem 3.2 the permutations $\phi(x), \phi(y)$ are:*

$$\phi(x) = (2,14)(3,5)(4,6)(7,8)(10,12)(11,13)$$
$$\phi(y) = (1,2,4,3)(5,7)(8,9,11,10)(13,14)$$

In the following diagram red edges represents the composition with generator $\pi(x)$ and black edges represents the composition with generator $\pi(y)$. Fixed points of $\pi_i(x)$ for $i = \{1,2\}$ have shown by heavy red dots and fixed points of $\pi_i(y)$ for $i = \{1,2\}$ have shown by heavy black dots in the diagram.



Using GAP *we find the group generated by $\phi(x), \phi(y)$ is imprimitive and has structure description $(\prod_{i=1}^{i=6} C_2 : \mathcal{A}(7)) : C_2 = C_2 \wr \mathcal{A}_7$ of order 322560.*

The following table illustrates the construction of Theorem 3.2. We abbreviate Structure Description as S.D. We used GAP to find the structure description for the groups $\pi(G)$ and $\phi(G)$.

Group	Deg	S.D of $\pi(G)$	Handles	Chosen Handles	Deg$\phi(G)$	Info about $\phi(G)$.
$(2,3,6)$	6 6	$C_2 \times A_4$ $C_2 \times A_4$	$[[1,2],[5,6]]$ $[[1,2],[5,6]]$	$[[1,2],[1,2]]_1$	12	$((c_4 \times c_4) \rtimes c_3) \rtimes c_2$
$(2,3,7)$	7 7	$PSL(3,2)$ $PSL(3,2)$	$[[1,2]]_1$ $[[1,2]]_1$	$[[1,2],[1,2]]_1$	14	$(c_2^3 \times PSL(3,2)$
$(2,3,7)$	7_1 7_2	$PSL(3,2)$ $PSL(3,2)$	$[[1,2]]_1$ $[[1,2]]_1$	$[[1,2],[1,2]]_1$	14	$PSL(2,13)$
$(2,3,7)$	7_1 7_2	$PSL(3,2)$ $PSL(3,2)$	$[[2,5]]_2$ $[[6,1]]_2$	$[[2,5],[6,1]]_2$	14	$PSL(2,13)$
$(2,3,7)$	7_1 7_2	$PSL(3,2)$ $PSL(3,2)$	$[[1,5]]_3$ $[[6,2]]_3$	$[[1,5],[6,2]]_3$	14	$PSL(2,13)$
$(2,3,7)$	14 14	$PSL(2,13)$ $PSL(2,13)$	$[[1,2]]_1$ $[[1,2]]_1$	$[[1,2],[1,2]]_1$	28	$PSL(2,13)$
$(2,3,7)$	15 42	$Alt(42)$ $Alt(15)$	$[[1,2]]_1$ $[[1,2],[41,42]]_1$	$[[1,2],[1,2]]_1$	57	$Alt(57)$
$(2,3,7)$	15 42	$Alt(42)$ $Alt(15)$	$[[1,2]]_1$ $[[1,2],[41,42]]_1$	$[[1,2],[41,42]]_1$	57	$Alt(57)$
$(2,3,7)$	15 63	$Alt(15)$ $Alt(63)$	$[[1,2]]_1$ $[[1,2]]_1$	$[[1,2],[1,2]]_1$	78	$Alt(78)$

Group	Deg	S.D of $\pi(G)$	Handles	Chosen Handles	Deg$\phi(G)$	Info about $\phi(G)$.
$(2,3,19)$	57_1 57_2	$PSL(3,7)$ $PSL(3,7)$	$[[20,32]]_1$ $[[24,42],[47,17]]_1$	$[[20,32],[24,42]]_1$	114	$Alt(114)$
$(2,3,19)$	57_1 57_2	$PSL(3,7)$ $PSL(3,7)$	$[[20,32]]_1$ $[[24,42],[47,17]]_1$	$[[20,32],[47,17]]_1$	114	$Alt(114)$
$(2,3,31)$	31_1 31_2	$PSL(3,5)$ $PSL(3,5)$	$[[28,31]]$ $[[26,25]]$	$[[28,31],[25,26]]_1$	62	$Alt(62)$
$(2,4,7)$	7_1 7_2	$PSL(3,2)$ $PSL(3,2)$	$[[1,2]]_1$ $[[1,2]]_1$	$[[1,2],[1,2]]_1$	14	$((\Pi_{i=1}^{i=6}c_2):PSL(3,2))$
$(2,4,7)$	7_1 7_2	$PSL(3,2)$ $PSL(3,2)$	$[[7,1]_2$ $[[2,7]]_2$	$[[7,1],[2,7]]_2$	14	$Alt(14)$
$(2,4,7)$	7_1 7_2	$PSL(3,2)$ $PSL(3,2)$	$[[7,2]]_3$ $[[1,7]]_3$	$[[7,2],[1,7]]_3$	14	$((\Pi_{i=1}^{i=6}c_2):A_7):c_2$
$(2,5,7)$	21_1 21_2	$PSL(3,4)$ $PSL(3,4)$	$[[11,13]]_1$ $[[16,15]]_1$	$[[11,13],[15,16]]_1$	42	$Alt(42)$
$(2,4,31)$	31_1 31_2	$PSL(3,5)$ $PSL(3,5)$	$[[28,15]]$ $[[15,18]]$	$[[28,15].[15,18]]_1$	62	$Alt(62)$
$(2,5,31)$	31_1 31_2	$PSL(3,5)$ $PSL(3,5)$	$[[1,6]]$ $[[1,6]]$	$[[1,6],[1,6]]_1$	62	$Alt(62)$
$(2,7,7)$	7_1 7_2	$PSL(3,2)$ $PSL(3,2)$	$[[1,2]]_1$ $[[1,2]]_1$	$[[1,2],[1,2]]_1$	14	$Alt(14)$
$(2,7,7)$	7_1 7_2	$PSL(3,2)$ $PSL(3,2)$	$[[7,1]_2$ $[[2,6]]_2$	$[[7,1],[2,6]]_2$	14	$PSL(2,13)$
$(2,7,7)$	7_1 7_2	$PSL(3,2)$ $PSL(3,2)$	$[[7,2]]_3$ $[[1,6]]_3$	$[[7,2],[1,6]]_3$	14	$Alt(14)$
$(2,7,7)$	21_1 21_2	$PSL(3,4)$ $PSL(3,4)$	$[[1,3]]_1$ $[[1,3]]_2$	$[[1,3],[1,3]]_1$	42	$Alt(42)$
$(2,7,19)$	57_1 57_2	$PSL(3,7)$ $PSL(3,7)$	$[[1,8],[1,9]]_1$ $[[1,8],[14,2]]_1$	$[[1,8],[1,8]]_1$	114	$Alt(114)$
$(3,3,5)$	15 15 15	$Alt(15)$ $Alt(15)$ $Alt(15)$	$[[1,2]]_1$ $[[1,2]]_1$ $[[1,2]]_1$	$[[1,2],[1,2],[1,2]]_1$	45	$NotAlternating$
$(3,3,5)$	15 15 20	$Alt(15)$ $Alt(15)$ $Alt(20)$	$[[1,2]]_1$ $[[1,2]]_1$ $[[1,2]]_1$	$[[1,2],[1,2],[1,2]]_1$	50	$Alt(50)$
$(3,3,7)$	14 14 14	$Alt(14)$ $Alt(14)$ $Alt(14)$	$[[1,2]]_1$ $[[1,2]]_1$ $[[1,2]]_1$	$[[1,2],[1,2],[1,2]]_1$	42	$NotAlternating$

Group	Deg	S.D of $\pi(G)$	Handles	Chosen Handles	Deg$\phi(G)$	Info about $\phi(G)$.
	13_1	$PSL(3,3)$	$[[1,2]]_1$			
$(3,3,13)$	13_1	$PSL(3,3)$	$[[1,2]]_1$	$[[1,2],[1,2],[1,2]]_1$	39	2984572656
	13_2	$PSL(3,3)$	$[[1,2]]_1$			
	13_1	$PSL(3,3)$	$[[1,2]]_1$			
$(3,3,13)$	13_2	$PSL(3,3)$	$[[1,2]]_1$	$[[1,2],[1,2],[3,4]]_1$	39	2984572656
	13_3	$PSL(3,3)$	$[[3,4]]_1$			
	14	$Alt(14)$	$[[1,2],[4,7]]_1$			
$(3,5,7)$	14	$Alt(14)$	$[[1,2],[4,7]]_1$	$[[1,2],[1,2],[1,2]]_1$	42	$NotAlternating$
	14	$Alt(14)$	$[[1,2],[4,7]]_1$			
	14	$Alt(14)$	$[[1,2],[4,7]]_1$			
$(3,5,7)$	14	$Alt(14)$	$[[1,2],[4,7]]_1$	$[[1,2],[4,7],[1,2]]_1$	42	$Alt(42)$
	14	$Alt(14)$	$[[1,2],[4,7]]_1$			
	14	$Alt(14)$	$[[1,2],[4,7]]_1$			
$(3,5,7)$	14	$Alt(14)$	$[[1,2],[4,7]]_1$	$[[1,2],[4,7],[4,7]]_1$	42	$Alt(42)$
	14	$Alt(14)$	$[[1,2],[4,7]]_1$			
	14	$Alt(14)$	$[[1,2],[4,7]]_1$			
$(3,5,7)$	14	$Alt(14)$	$[[1,2],[4,7]]_1$	$[[4,7],[4,7],[4,7]]_1$	42	$NotAlternating$
	14	$Alt(14)$	$[[1,2],[4,7]]_1$			
	14	$Alt(14)$	$[[1,2],[2,4],[4,7]]_1$			
$(3,5,7)$	14	$Alt(14)$	$[[1,2],[2,4],[4,7]]_1$	$[[1,2],[1,2],[2,4]]_1$	43	$Alt(43)$
	15	$Alt(15)$	$[[1,2],[2,4],[4,7]]_1$			
	14	$Alt(14)$	$[[1,2],[2,4],[4,7]]_1$			
$(3,5,7)$	14	$Alt(14)$	$[[1,2],[2,4],[4,7]]_1$	$[[1,2],[1,2],[1,2]]_1$	43	$Alt(43)$
	15	$Alt(15)$	$[[1,2],[2,4],[4,7]]_1$			
	14	$Alt(14)$	$[[1,4],[2,7],[7,8]]_2$			
$(3,5,7)$	14	$Alt(14)$	$[[1,4],[2,7],[7,8]]_2$	$[[1,4],[1,4],[1,4]_2$	43	$Alt(43)$
	15	$Alt(15)$	$[[1,4],[2,7],[11,12]]_2$			
	14	$Alt(14)$	$[[1,4],[2,7],[7,8]]_2$			
$(3,5,7)$	14	$Alt(14)$	$[[1,4],[2,7],[7,8]]_2$	$[[1,4],[2,7],[11,12]]_2$	43	$Alt(43)$
	15	$Alt(15)$	$[[1,4],[2,7],[11,12]]_2$			
	14	$Alt(14)$	$[[1,2],[4,7]]_1$			
$(3,5,9)$	14	$Alt(14)$	$[[1,2],[4,7]]_1$	$[[1,2],[1,2],[1,2]]_1$	42	$NotAlternating$
	14	$Alt(14)$	$[[1,2],[4,7]]_1$			
	14	$Alt(14)$	$[[1,2],[4,7]]_1$			
$(3,5,9)$	14	$Alt(14)$	$[[1,2],[4,7]]_1$	$[[1,2],[4,7],[1,2]]_1$	42	$Alt(42)$
	14	$Alt(14)$	$[[1,2],[4,7]]_1$			

Group	Deg	S.D of $\pi(G)$	Handles	Chosen Handles	Deg$\phi(G)$	Info about $\phi(G)$.
	14	$Alt(14)$	$[[1,2],[4,7]]_1$			
$(3,5,9)$	14	$Alt(14)$	$[[1,2],[4,7]]_1$	$[[1,2],[4,7],[4,7]]_1$	42	$Alt(42)$
	14	$Alt(14)$	$[[1,2],[4,7]]_1$			
	14	$Alt(14)$	$[[1,2],[4,7]]_1$			
$(3,5,9)$	14	$Alt(14)$	$[[1,2],[4,7]]_1$	$[[4,7],[4,7],[4,7]]_1$	42	$Alt(42)$
	14	$Alt(14)$	$[[1,2],[4,7]]_1$			
	14	$Alt(14)$	$[[1,4],[2,7]]_2$			
$(3,5,9)$	14	$Alt(14)$	$[[1,4],[2,7]]_2$	$[[1,4],[1,4],[1,4]_2$	42	$NotAlternating$
	14	$Alt(14)$	$[[1,4],[2,7]]_2$			
	14	$Alt(14)$	$[[1,4],[2,7]]_2$			
$(3,5,9)$	14	$Alt(14)$	$[[1,4],[2,7]]_2$	$[[1,4],[2,7],[1,4]_2$	42	$Alt(42)$
	14	$Alt(14)$	$[[1,4],[2,7]]_2$			
	11	$Alt(11)$	$[[1,2],[2,4],[4,7]]]_1$			
$(3,5,11)$	11	$Alt(11)$	$[[1,2],[2,4],[4,7]]]_1$	$[[4,7],[4,7],[4,7]]_1$	33	$NotAlternating$
	11	$Alt(11)$	$[[1,2],[2,4],[4,7]]]_1$			
	11	$Alt(11)$	$[[1,2],[2,4],[4,7]]]_1$			
$(3,5,11)$	11	$Alt(11)$	$[[1,2],[2,4],[4,7]]]_1$	$[[1,2],[4,7],[4,7]]_1$	33	$Alt(33)$
	11	$Alt(11)$	$[[1,2],[2,4],[4,7]]]_1$			
	14	$Alt(14)$	$[[1,2]]_1$			
$(3,5,11)$	14	$Alt(14)$	$[[1,2]]_1$	$[[1,2],[1,2],[1,2]]_1$	42	$NotAlternating$
	14	$Alt(14)$	$[[1,2]]_1$			
	14	$Alt(14)$	$[[1,2]]_1$			
$(3,5,11)$	14	$Alt(14)$	$[[1,2]]_1$	$[[1,2],[1,2],[3,1]]_1$	50	$Alt(50)$
	14	$Alt(14)$	$[[1,2]]_1$			
	21_1	$PSL(3,4)$	$[[1,2]]_1$			
$(3,7,7)$	21_1	$PSL(3,4)$	$[[1,2]]_1$	$[[1,2],[1,2],[1,2]]_1$	63	70293573524160
	21_2	$PSL(3,4)$	$[[1,2]]_1$			

Table 3.1: Composition

These experiments show that very often composition gives us a primitive group and most of the time that primitive group is alternating. When the group is not primitive, in fact it seems that almost always we are in one of the situations that we investigate in the next chapter.

Chapter 4

Imprimitive composition

In the first Section 4.1 of this chapter we use the method of composition that we described in Chapter 3 to construct particular representations of G that we will prove to be imprimitive. Then in the second Section we compute a number of examples, and display the results in a table. Our experiments lead us to the theorems in the following chapter.

4.1 Imprimitive constructions

Theorem 4.1. *Let $G, \pi_1, \ldots, \pi_p$ be as in Theorem 3.2, with $t = p$ where p is prime. Suppose that, for some finite set Ω of size* deg *and a permutation representation π of G on Ω, each π_i is equivalent to π, via a bijection f_i. Suppose further that each (a_i, b_i) is a handle of Ω_i, the image of a handle (a, b) of Ω.*

Now, for $\omega \in \Omega$, define $B_\omega \subseteq \mathcal{U}$ via

$$B_\omega = \{f_i(\omega) : i = 1, \ldots p\},$$

and let

$$\mathcal{B} = \{B_\omega : \omega \in \Omega\}.$$

Then the action ψ *of the permutation group* $H := \phi(G)$ *on* $\mathcal{B}$ *is equivalent to the action of* $\pi(G)$ *on* Ω. H *acts imprimitively on* $\mathcal{U}$ *with blocks of imprimitivity* B_ω.

Let $\psi, N, J_i, K_i, Q_i, B_i$ *be as defined in Section 2.5. Then* Q *is cyclic of order* p *and* N *is elementary abelian of order at most* p^{deg}. *Then* H *is isomorphic to a subgroup of* $C_p \wr \psi(H)$ *and the action of* H *on* N *by conjugation induces an action of* $\psi(H)$. *Under this action* N *is a submodule of the* deg-*dimensional permutation module over* F_p *for the subgroup* $\psi(H)$ *of* $\mathcal{S}(\Omega)$.

Proof. For $\omega \in \Omega$, for all $g \in G$, and for $i = 1, .., p$, we have

$$f_i(\omega^{\pi(g)}) = f_i(\omega)^{\pi_i(g)}. \tag{4.1}$$

Recall that for $t = p$, we have

$$\begin{aligned} \phi_x &= \pi_1(x) \cdots \pi_p(x) \circ (a_1, \ldots, a_p)(b_b, b_{p-1}, \cdots b_1) \\ \phi_y &= \pi_1(y) \cdots \pi_p(y). \end{aligned}$$

We want to prove that the action of $H = \phi(G)$ on $\mathcal{B}$ is equivalent to the action of $\pi(G)$ on Ω. We need a bijection $F : \Omega \to \mathcal{B}$ so that for all $\omega \in \Omega$, all $g \in G$

$$F(\omega^{\pi(g)}) = F(\omega)^{\phi(g)}.$$

We define $F : \Omega \to \mathcal{B}$ by $F(\omega) = B_\omega$. We need to check that F is a bijection.

Clearly it is surjective. Now,

$$F(\omega) = F(\omega') \Rightarrow \{f_1(\omega) \dots f_p(\omega)\} = \{f_1(\omega') \dots f_p(\omega')\}.$$

Since $f_i(\omega), f_i(\omega') \in \Omega_i$ and the sets Ω_i's are disjoint, this implies that $f_i(\omega) = f_i(\omega')$ for each i. So since each f_i is a bijection, we get $\omega = \omega'$.

It remains to check that $F(\omega^{\pi(g)}) = F(\omega)^{\phi(g)}$ i.e. $B_{\omega^{\pi(g)}} = (B_\omega)^{\phi(g)}$ for all $g \in G$, $\omega \in \Omega$. First suppose that $g = y$, for all ω, the image of B_ω under $\phi(y)$ is

$$\begin{aligned} B_\omega^{\phi(y)} &= \{f_1(\omega)^{\pi_1(y)}, f_2(\omega)^{\pi_2(y)}, ..., f_p(\omega)^{\pi_p(y)}\} \\ &= \{f_1(\omega^{\pi(y)}), f_2(\omega^{\pi(y)}), ..., f_p(\omega^{\pi(y)})\} \\ &= B_{\omega^{\pi(y)}}. \end{aligned}$$

Now suppose that $g = x$ if $\omega \neq a, b$ we have $B_\omega^{\phi(x)} = B_{\omega^{\pi(x)}}$. Finally

$$\begin{aligned} B_a^{\phi(x)} &= \{a_1, a_2, ..., a_p\}^{\phi(x)} \\ &= \{a_2, a_3, ..., a_p, a_1\} \\ &= B_a = B_{a^{\pi(x)}} \end{aligned}$$

and

$$B_b^{\phi(x)} = \{b_p, b_1, \dots, b_{p-1}\} = B_b = B_{b^{\pi(x)}}.$$

So for all ω, $B_\omega^{\phi(x)} = B_{\omega^{\pi(x)}}$. Since G is generated by x and y this proves that $B_\omega^{\phi(g)} = B_{\omega^{\pi(g)}}$, and hence F is an equivalence.

Now the sets B_ω are blocks of imprimitivity for ϕ if and only if for each $\omega \in \Omega$, $g \in G$ either $B_\omega = B_{\omega^{\phi(g)}}$ or $B_\omega \cap B_{\omega^{\phi(g)}} = \emptyset$.

So suppose that $B_\omega \cap B_\omega^{\phi(g)} \neq \emptyset$. Then, since $B_\omega^{\phi(g)} = B_{\omega^{\pi(g)}}$ we have $f_i(\omega) = f_j(\omega^{\pi(g)})$ for some i, j. Since $\Omega_i \cap \Omega_j = \emptyset$, we have $i = j$. Then since f_i is a bijection we have $\omega = \omega^{\pi(g)}$, and $B_\omega = B_{\omega^{\pi(g)}} = B_\omega^{\phi(g)}$.

Let J_i, K_i, P_i, Q_i be as defined in Section 2.5. By Proposition 2.35, H is isomorphic to a subgroup of $Q \wr \psi(H)$ and N to a subgroup of $P_1 P_2 \cdots P_{\text{deg}}$, where $Q \cong Q_i$ for each i.

In order to find N we need to identify the groups Q_i and P_i.

J_a is the subgroup of H that fixes B_a setwise. So $\phi(x) \in J_a$, since $\phi(x)$ fixes the block B_a. In fact $\phi(x)$ permutes the points of B_a in a p-cycle. We claim that for any $g \in G$, for any blocks $B_c = \{c_1, \ldots, c_p\}$, $B_{c'} = \{c'_1, \ldots, c'_p\}$, if $c_i^{\phi(g)} = c'_j$ then $c_{i+k}^{\phi(g)} = c'_{j+k}$.

We justify our claim by examining the actions of the generators $\phi(x), \phi(y)$ on the union $\mathcal{U}$ of the blocks. For $g = x$, $\phi(x)$ acts on B_a, B_b via $a_i^{\phi(x)} = a_{i+1}$ and $a_{i+1}^{\phi(x)} = a_{i+2}$ and $b_p^{\phi(x)} = b_{p-1}$ and $b_{p-1}^{\phi(x)} = b_{p-2}$. Otherwise it maps $c_i \in B_c$ to some $c'_i \in B_{c'}$. For $g = y$, $\phi(y)$ maps each c_i to some c'_i.

So if $\phi(g) \in J_a$ then it preserves the cyclic order of B_a. So $J_a/K_a \cong Q_a$ is contained in a cyclic group of order p. Then since $\phi(x)$ acts on B_a as an element of order p, we see that J_a/K_a contains the cyclic group of order p; hence $J_a/K_a \cong C_p$. It follows from transitivity that for each i, we have $Q_i \cong J_i/K_i \cong C_p$.

Now $P_i \subseteq Q_i$. So $N \subseteq P_1 \cdots P_{\text{deg}} \subseteq Q_1 \cdots Q_{\text{deg}} = Q^{\text{deg}}$. Hence N is at most C_p^{deg}. So it is elementary abelian of order at most p^{deg}.

Now we consider the action of H on N by conjugation. Since N is abelian, N is in the kernel of this action and so there is an induced action of $\psi(H)$ on N. It follows from the decription of the wreath product that Q^{deg} is the permutation module for $\psi(H)$. So $N \subseteq Q^{\text{deg}}$ must be a submodule of that permutation module.

□

Example 4.2. *Consider a triangle group* $(2, 3, 7)$ *and two equivalent representations of degree 7. Then* π_1*, defined by* $\pi_1(x) = (3, 4)(6, 7)$*,* $\pi_1(y) = (1, 2, 3)(4, 5, 6)$*, has handle* $a_1 = 1, b_1 = 2$ *and* π_2*, defined by* $\pi_2(x) = (10, 11)(13, 14)$ $\pi_2(y) = (8, 9, 10)(11, 12, 13)$ *has handle* $a_2 = 8, b_2 = 9$*. Using Theorem* 3.2*, the permutations* $\phi(x), \phi(y)$ *are:*

$$\phi(x) = (3, 4)(6, 7)(10, 11)(13, 14)(1, 8)(2, 9)$$
$$\phi(y) = (1, 2, 3)(4, 5, 6)(8, 9, 10)(11, 12, 13)$$

Using Theorem 4.1, we see that the group H *generated by* $\phi(x)$ *and* $\phi(y)$ *is imprimitive of degree 14, with blocks* $[[1, 8], [2, 9], [3, 10], [4, 11], [5, 12], [6, 13], [7, 14]]$*. Using* GAP*,* $\phi(G)$ *has structure description* $(\prod_{i=1}^{i=3} C_2).\mathtt{PSL}(3, 2)$*.*

Also, we see that J_1*, the subgroup of* H *that fixes the block* $B_1 = [1, 8]$ *setwise, is generated by*

$$(3, 4, 10, 11)(6, 14, 13, 7),\ (3, 13, 10, 6)(4, 14, 11, 7),$$
$$(2, 3, 6)(4, 7, 12)(5, 11, 14)(9, 10, 13),\ (2, 9)(3, 7)(4, 13)(5, 12)(6, 11)(10, 14),$$
$$(1, 8)(2, 9)(3, 4)(6, 7)(10, 11)(13, 14).$$

and N *(the intersection of the subgroups* J_i*) is generated by*

$$(1, 8)(4, 11)(5, 12)(7, 14),\ (3, 10)(4, 11)(6, 13)(7, 14),$$
$$(2, 9)(5, 12)(6, 13)(7, 14).$$

Now we compute generators for Q_1*, the permutation group defined by the action of* J_1 *on* B_1*, and* P_1*, the permutation group defined by the action of* N *on* B_1*, by deleting all cycles from the generators of* J_1, N *that involve points of* $\mathcal{U}$ *outside* B_1*.*

We see that

$$Q_1 = P_1 = \langle (1, 8) \rangle.$$

Theorem 4.3. *Suppose that $G = \Delta(p, q, r)$ has a permutation representation π on a finite set Ω, and let $h_1 = \iota, h_2, \ldots h_p : \Omega \to \Omega$ be permutations of Ω that commute with π, that is, they satisfy*

$$h_i(\omega^{\pi(g)}) = (h_i(\omega))^{\pi(g)}, \forall \omega \in \Omega, g \in G. \tag{4.2}$$

Now suppose that (a, b) is a k-handle for π, and for each i, define $a_i = h_i(a), b_i = h_i(b)$. Suppose that the points $a_1, b_1, a_2, b_2, ..., a_p, b_p$ are all distinct. Define $\phi : G \to \mathcal{S}(\Omega)$ as in Theorem 3.2.

Let

$$B_\omega = \{h_i(\omega) : i = 1, \ldots p\}$$

and let

$$\mathcal{B} = \{B_\omega : \omega \in \Omega\}.$$

Then the action of $\phi(G)$ on $\mathcal{B}$ is equivalent to the action of $\pi(G)$ on Ω. And the sets B_ω are blocks of imprimitivity for the action of $\phi(G)$ on Ω if and only if they are blocks of imprimitivity for the action of $\pi(G)$ on Ω.

Proof. We have $a^{\pi(xy)^k} = b$, so $a_i^{\pi(xy)^k} = h_i(a)^{\pi(xy)^k} = h_i(a^{\pi(xy)^k}) = h_i(b) = b_i$ and $a_i^{\pi(x)} = h_i(a)^{\pi(x)} = h_i(a^{\pi(x)}) = h_i(a) = a_i$. Similarly, $b_i^{\pi(x)} = b_i$. Therefore, a_i, b_i for $i = \{1, \ldots, p\}$ are all handles. Our conditions ensure that for $i \neq i'$, the handles $[a_i, b_i]$ and $[a_i', b_i']$ are always disjoint. As in the proof of Theorem 4.1, we see that the image of B_ω under $\phi(y)$ is

$$B_\omega{}^{\phi(y)} = \{h_i(\omega)^{\pi(y)} : i = 1 \ldots p\}$$
$$= \{h_i(\omega^{\pi(y)}) : i = 1 \ldots p\} = B_{\omega^{\pi(y)}} = B_\omega^{\pi(y)}.$$

Similarly

$$B_\omega^{\phi(x)} = B_{\omega^{\pi(x)}} = B_\omega^{\pi(x)},$$

so

$$B_\omega^{\phi(g)} = B_{\omega^{\pi(g)}} = B_\omega^{\pi(g)}.$$

The sets B_ω are blocks for $\phi(G)$ acting on Ω if and only if $B_\omega \cap B_\omega^{\phi(g)} = \emptyset$ whenever $B_\omega^{\phi(g)} \neq B_\omega$, and they are blocks for $\pi(G)$ acting on Ω if and only if $B_\omega \cap B_\omega^{\pi(g)} = \emptyset$ whenever $B_\omega^{\pi(g)} \neq B_\omega$. So since $B_\omega^{\pi(g)} = B_\omega^{\phi(g)}$, the B_ω are blocks of imprimitivity for $\phi(G)$ if and only if they are blocks of imprimitivity for $\pi(G)$.

□

Example 4.4. *Consider a triangle group* $(2, 3, 6)$*, and the representation* π *of degree 6 defined by* $\pi(x) = (3, 4)$*,* $\pi(y) = (1, 2, 3)(4, 5, 6)$*. This has handles* $a_1 = 1, b_1 = 2$ *and* $a_2 = 5, b_2 = 6$*. The group* $\pi(G)$ *is imprimitive as shown in the figure 4.1 with blocks of imprimitiviy denoted by heavy coloured dots. Where* $h_1 = \iota$ *and* h_2 *must map* $[1, 5]$ *to* $[2, 6]$ *and commute with* $\pi(x)$ *and* $\pi(y)$*. Here,* $h_2 = (1, 5)(2, 6)(3, 4)$*.*

After composition we have the permutations $\phi(x), \phi(y)$ *are:*

$$\phi(x) = (1, 5)(2, 6)(3, 4)$$
$$\phi(y) = (1, 2, 3)(4, 5, 6)$$

Using Theorem 4.3 the group H *generated by* $\phi(x)$*,* $\phi(y)$ *is imprimitive with blocks* $B_1 = \{h_1(1), h_2(1)\} = \{1, 5\},\ B_2 = \{h_1(2), h_2(2)\} = \{2, 6\},\ B_3 = \{h_1(3), h_2(3)\} =$

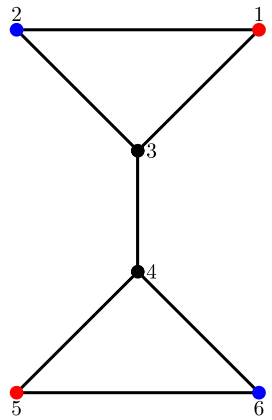


Figure 4.1: Blocks of imprimitivity

$\{3, 4\}$ *of degree 6 and has structure description* C_6*. Here* J_1*, the subgroup of* H *that fixes the block* $B_1 = [1, 5]$ *setwise is equal to* $\langle(1, 5)(2, 6)(3, 4)\rangle$*, and is also equal to* N*, the subgroup of* H *that fixes each of the blocks setwise. Now* Q_1 *that is induced by the action of* J_1 *on* B_1 *is the group generated by* $\langle(1, 5)\rangle$ *and is equal to* P_1*, that is induced by the action of* N *on* B_1*.*

Theorem 4.5. *Suppose that* $G = \Delta(p, q, r)$ *has a transitive permutation representation* $\pi : G \to \mathcal{S}(\Omega)$ *of degree* deg *with disjoint* k*-handles* (a, b) *and* (c, d)*, for some* k*.*

Let m *be an integer, and suppose that* $\alpha = \alpha_1 \cdots \alpha_k$ *and* $\beta = \beta_1 \cdots \beta_l$ *are two permutations, both products of disjoint* p*-cycles (the* α_i *and* β_j*), that generate a transitive subgroup of* $\mathcal{S}_m$*.*

Then we can make a transitive permutation representation ϕ *of* G *of degree* $m\mathrm{deg}$ *as follows.*

Suppose that $\pi_1 = \pi$ *and that* $\pi_2, \ldots, \pi_m$ *are representations equivalent to* π *on* $\{\mathrm{deg} + 1, \ldots, 2\mathrm{deg}\}$, $\{2\mathrm{deg} + 1, , \ldots, 3\mathrm{deg}\}$,$\ldots$, $\{(m - 1)\mathrm{deg} + 1, \ldots, m\mathrm{deg}\}$*, and let* (a_i, b_i)*,* (c_i, d_i) *be the copies of the handles* (a, b) *and* (c, d) *in* π_i*.*

Then for each of the cycles $\alpha_i = (i_1, \dots, i_p)$ *of* α*, we define*

$$\gamma_i = (a_{i_1}, \dots, a_{i_p})(b_{i_p}, \dots, b_{i_1}),$$

and for each of the cycles $\beta_j = (j_1, \dots, j_p)$ *of* β*, we define*

$$\delta_j = (c_{j_1}, \dots, c_{j_p})(d_{j_p}, \dots, d_{j_1}).$$

Then we define

$$\begin{aligned} \phi(x) &= \pi_1(x)\pi_2(x)\cdots\pi_m(x)\gamma_1\gamma_2\cdots\gamma_k\delta_1\delta_2\cdots\delta_l \\ \phi(y) &= \pi_1(y)\pi_2(y)\cdots\pi_m(y). \end{aligned}$$

Let

$$\mathcal{B} = \{B_\omega : \omega \in \Omega\}.$$

The action ψ *of* $H := \phi(G)$ *on* $\mathcal{B}$ *is equivalent to the action of* $\pi(G)$ *on* Ω*.*

The representation ϕ *is imprimitive with blocks*

$$B_\omega = \{\omega, \deg + \omega, 2\deg + \omega, \dots, (m-1)\deg + \omega\},$$

for each $\omega \in \Omega$*. Let* J_i, K_i, P_i, Q_i *be as defined in Section 2.5. Then* $Q_i \cong \langle \alpha, \beta \rangle$*. Hence* H *is isomorphic to a subgroup of* $\langle \alpha, \beta \rangle \wr \psi(H)$*.*

Proof. Since we could construct ϕ by repeating the construction of Theorem 3.2, it is clear we have a permutation representation ϕ of degree $m\deg$. To see that the sets B_ω are blocks, we consider their images under $\phi(y)$ and $\phi(x)$. We see that

$$B_\omega^{\phi(y)} = \{\omega^{\pi(y)}, \deg + \omega^{\pi(y)}, ..., (m-1)\deg + \omega^{\pi(y)}\} = B_{\omega^{\pi(y)}}$$

Similarly, if $\omega \neq a, b, c, d$ then image of B_ω under $\phi(x)$ is

$$B_\omega^{\phi(x)} = \{\omega^{\pi(x)}, \deg + \omega^{\pi(x)}, ..., (m-1)\deg + \omega^{\pi(x)}\} = B_{\omega^{\pi(x)}}.$$

Finally, for $\omega = a, b, c, d$ we have $B_a^{\phi(x)} = B_a = B_{a^{\pi(x)}}$, $B_b^{\phi(x)} = B_b = B_{b^{\pi(x)}}$, $B_c^{\phi(x)} = B_c = B_{c^{\pi(x)}}$ and $B_d^{\phi(x)} = B_d = B_{d^{\pi(x)}}$.

So for all ω, $g \in G$ $B_\omega^{\phi(g)} = B_{\omega^{\pi(g)}}$. This proves that the action of $\phi(G)$ of $\mathcal{B}$ is equivalent to the action of $\pi(G)$ on set Ω. Then just as in Theorem 4.1 we can see that B_ω are blocks of imprimitivity for the image of G under ϕ.

To prove that $Q_a \cong \langle\alpha, \beta\rangle$, we prove first that $Q_a \subseteq \langle\alpha, \beta\rangle$ and then that $Q_a \supseteq \langle\alpha, \beta\rangle$.

To prove that $Q_a \subseteq \langle\alpha, \beta\rangle$, we need to show that for any $g \in G$ with $\phi(g) \in J_a$, $\phi(g)$ acts on B_a as an element of $\langle\alpha, \beta\rangle$.

Let $g = x^{i_1}y^{j_1}\ldots x^{i_k}y^{j_k}$, where $\phi(g) = \phi(x)^{i_1}\phi(y)^{j_1}\ldots\phi(x)^{i_r}\phi(y)^{j_k} \in J_a$. Define $z_0, z_1, \ldots, z_k = a' \in \mathcal{U}$ with $a' \in B_a$, by

$$a = z_0 \xrightarrow{\phi(x)^{i_1}\phi(y)^{j_1}} z_1 \xrightarrow{\phi(x)^{i_2}\phi(y)^{j_2}} z_2 \ldots z_{k-1} \xrightarrow{\phi(x)^{i_r}\phi(y)^{j_r}} z_k = a.$$

Let $a_1, \ldots, a_{\deg}$ be the points of B_a. Where the points of Ω are labelled $1, 2, \ldots, \deg$, the construction ensures that the point a_ℓ of $B_a \subseteq \mathcal{U}$ is numbered $a + (\ell-1)\deg$. So now suppose that $a_\ell = a + (\ell-1)\deg$ is an arbitrary point of B_a for some $\ell \in \{1 \ldots m\}$ We have

$$(a + (\ell-1)\deg)^{\phi(g)} = a + (\ell^{\xi^{i_1}\xi^{i_2}\ldots\xi^{i_r}} - 1)\deg$$

where each $\xi_i = 1, \alpha, \alpha^{-1}, \beta, \beta^{-1}$ such that

$$\xi_i = \begin{cases} 1 & \text{if } z_i \notin B_a \cup B_b \cup B_c \cup B_d, \\ \alpha & \text{if } z_i \in B_a, \\ \alpha^{-1} & \text{if } z_i \in B_b, \\ \beta & \text{if } z_i \in B_c, \\ \beta^{-1} & \text{if } z_i \in B_d. \end{cases}$$

So $\phi(g)$ acts on B_a as $\xi^{i_1}\xi^{i_2}\ldots\xi^{i_r} \in \langle\alpha,\beta\rangle$. This shows that $Q_a \subseteq \langle\alpha,\beta\rangle$.

We see easily that $\phi(x) \in J_a \cap J_b \cap J_c \cap J_d$, and that the element $\phi(x)$ permutes the points of B_a in the same way that α permutes the points of $\{1,\ldots,m\}$.

To complete the proof that $\langle\alpha,\beta\rangle \subseteq Q_a$ we need to find a conjugate of $\phi(x)$, $\phi(g)\phi(x)\phi(g)^{-1}$, that acts on B_a as β. For this, we choose $g \in G$ to be a shortest possible word in x and y such that $\pi(g)$ (acting on $\Omega = \{1, 2, \ldots, \deg\}$) takes a to c. Define $z_0, z_1, \ldots, z_k \in \Omega$ such that

$$a = z_0 \xrightarrow{\pi(x^{i_1})\pi(y^{j_1})} z_1 \xrightarrow{\pi(x^{i_2})\pi(y^{j_2})} z_2 \ldots z_{k-1} \xrightarrow{\pi(x^{i_k})\pi(y^{j_k})} z_k = c.$$

Suppose $z_j = a, b, c, d$ for $j < k$. Then $z_j^{\pi(x)} = z_j$ and $z_j^{\pi(x^{i_{j+1}})} = z_j$, so we can leave out $x^{i_{j+1}}$ and get a shorter choice for g. This cannot happen, so we can assume that $z_j \notin \{a, b, c, d\}$ for $j < k$. We choose $i_1 = 0$ since $a^{\pi(x)} = a$. So $(a + (\ell-1)\deg)^{\phi(y)^{j_1}} = z_1 + (\ell-1)\deg$.

$z_1 \neq \{a, b, c, d\}$ by assumption as above. So $(z_1+(\ell-1)\deg)^{\phi(x)^{i_2}} = z_1'+(\ell-1)\deg$ for some $z_1' \neq z_1$, and $(z_1' + (\ell-a)\deg)^{\phi(y)^{j_2}} = z_2 + (\ell-1)\deg$.

$z_2 \notin \{a, b, c, d\}$ provided that $2 < k$, and we continue as above.

We end up with, for each ℓ,

$$\begin{aligned}(a+(\ell-1)\mathrm{deg})^{\phi(g)} &= c+(\ell-1)\mathrm{deg}\\ (a+(\ell^{\beta}-1)\mathrm{deg})^{\phi(g)} &= c+(\ell^{\beta}-1)\mathrm{deg}\\ (a+(\ell-1)\mathrm{deg})^{\phi(g)\phi(x)} &= c+(\ell^{\beta}-1)\mathrm{deg}\\ (a+(\ell-1)\mathrm{deg})^{\phi(g)\phi(x)\phi(g)^{-1}} &= a+(\ell^{\beta}-1)\mathrm{deg}\end{aligned}$$

So $\phi(g)\phi(x)\phi(g)^{-1}$ acts on B_a as β. This shows that $\langle\alpha,\beta\rangle \subseteq Q_a$. Hence, $Q_a \cong \langle\alpha,\beta\rangle$. To finish we apply Proposition 2.35.

□

Example 4.6. *Let G be the $\Delta(3,2,7)$ triangle group, consider a representation of degree 56 and let $p=3, m=6$. We can find π with*

$$\begin{aligned}\pi(x) &= (2,3,4)(5,7,8)(6,9,10)(11,13,14)(15,16,17)(18,19,20)(21,23,24)\\ &\quad (22,25,26)(27,29,30)(28,31,32)(34,36,37)(35,38,39)(40,41,42)\\ &\quad (43,44,45)(46,47,48)(49,51,52)(50,53,54),\\ \pi(y) &= (1,2)(3,5)(4,6)(7,9)(8,11)(10,12)(13,15)(14,16)(17,18)(19,21)\\ &\quad (20,22)(23,26)(24,27)(25,28)(29,33)(30,34)(31,35)(32,36)(37,38)\\ &\quad (39,40)(41,43)(42,44)(45,46)(47,49)(48,50)(51,54)(52,55)(53,56).\end{aligned}$$

Then $\pi(G)$ has structure description $\mathcal{A}_{56}$ and has 4-handles $(a,b)=(1,12)$ and $(c,d)=(55,56)$. Now let

$$\alpha=\alpha_1\alpha_2=(1,2,3)(4,5,6), \quad \beta=\beta_1=(2,3,4).$$

We can see that $\langle\alpha,\beta\rangle=\mathcal{A}_6$. We set $\pi_1=\pi$ and then make $\pi_1,\ldots,\pi_6$ by shifting *the domain of* π *by each of* $56, 112, 168, 224$ *and* 280. *In that case we have*

$$a_1 = 1,\ a_2 = 56 + 1 = 57,\ a_3 = 112 + 1 = 113,\ a_4 = 168 + 1 = 169,$$
$$a_5 = 224 + 1 = 225,\ a_6 = 280 + 1 = 281,$$
$$b_1 = 12,\ b_2 = 56 + 12 = 68,\ b_3 = 112 + 12 = 124,\ b_4 = 168 + 12 = 180,$$
$$b_5 = 224 + 12 = 236,\ b_6 = 280 + 12 = 292,$$
$$c_1 = 55,\ c_2 = 56 + 55 = 111,\ c_3 = 112 + 55 = 167,\ c_4 = 168 + 55 = 223,$$
$$c_5 = 224 + 55 = 279,\ c_6 = 280 + 55 = 335,$$
$$d_1 = 56,\ d_2 = 56 + 56 = 112,\ d_3 = 112 + 56 = 168,\ d_4 = 168 + 56 = 224,$$
$$d_5 = 224 + 56 = 280,\ d_6 = 280 + 56 = 336.$$

So

$$\begin{aligned}
\gamma_1 &= (1, 57, 113)(124, 68, 12),\\
\gamma_2 &= (169, 225, 281)(292, 236, 180),\\
\delta_1 &= (111, 167, 223)(224, 168, 112),
\end{aligned}$$

and so

$$\begin{aligned}
\phi(x) &= \pi_1(x)\pi_2(x)\pi_3(x)\pi_4(x)\pi_5(x)\pi_6(x)(1, 57, 113)(124, 68, 12)\\
&\quad (169, 225, 281)(292, 236, 180)(111, 167, 223)(224, 168, 112),\\
\phi(y) &= \pi_1(y)\pi_2(y)\pi_3(y)\pi_4(y)\pi_5(y)\pi_6(y).
\end{aligned}$$

According to GAP, $Q_1 \cong J_1/K_1 \cong \mathcal{A}_6$ *and* $P_1 \cong N/N \cap K_1 \cong \mathcal{A}_6$. *By Theorem 4.5* $H \subseteq \mathcal{A}_6 \wr \psi(H)$. *Here* $H = \mathcal{A}_6^{56} \rtimes \mathcal{A}_{56} = \mathcal{A}_6 \wr \mathcal{A}_{56}$.

Example 4.7. *Let* G *be the* $\Delta(2, 5, 5)$ *triangle group with a representation of degree*

10, *and let* $p = 2, m = 4$. *We can find* π *with*

$$\pi(x) = (3,5)(6,8)$$
$$\pi(y) = (1,2,4,6,3)(5,7,9,10,8)$$

Then $\pi(G)$ *has structure description* $C_2^4 \rtimes C_5 = C_2 \wr C_5$ *and has* 1*-handles* $(a,b) = (1,2)$ *and* $(c,d) = (7,9)$. *Now let*

$$\alpha = (1,2)(3,4) \text{ and } \beta = (2,3).$$

Then $\langle \alpha, \beta \rangle = D_8$. *We set* $\pi_1 = \pi$ *and then make* $\pi_1, \ldots, \pi_4$ *by shifting the domain of* π *by each of* 10, 20 *and* 30. *In that case we have*

$$a_1 = 1,\ a_2 = 10 + 1 = 11,\ a_3 = 20 + 1 = 21,\ a_4 = 30 + 1 = 31$$
$$b_1 = 2,\ b_2 = 10 + 2 = 12,\ b_3 = 20 + 2 = 22,\ b_4 = 30 + 2 = 32$$
$$c_1 = 7,\ c_2 = 10 + 7 = 17,\ c_3 = 20 + 7 = 27,\ c_4 = 30 + 7 = 37$$
$$d_1 = 9,\ d_2 = 10 + 9 = 19,\ d_3 = 20 + 9 = 29,\ d_4 = 30 + 9 = 39$$

So

$$\begin{aligned} \gamma_1 &= (1,11)(12,2), \\ \gamma_2 &= (21,31)(32,22), \\ \delta_1 &= (17,27)(29,19), \end{aligned}$$

and hence

$$\begin{aligned}\phi(x) &= \pi_1(x)\pi_2(x)\pi_3(x)\pi_4(x)(1,11)(12,2)(21,31)(32,22)(17,27)(29,19)\\ \phi(y) &= \pi_1(y)\pi_2(y)\pi_3(y)\pi_4(y).\end{aligned}$$

Applying Theorem 4.5, we see that $H = \langle\phi(x),\phi(y)\rangle$ *is imprimitive with blocks* $[1,11,21,31], [2,12,22,32], [3,13,23,33], [4,14,24,34], [5,15,25,35], [6,16,26,36], [7,17,27,37], [8,18,28,38], [9,19,29,39], [10,20,30,40]$ *each of size 4. The group has order* 81920. *As in the theorem , we define* J_1 *to be the subgroup of* H *that fixes the block* $B_1 = [1,11,21,31]$ *setwise, and* N *to be the group the fixes each of the blocks setwise.*

Using GAP *we see that*

$$\begin{aligned}J_1 = \langle &(6,16,36,26)(8,28,38,18),\quad (6,36)(8,38)(16,26)(18,28),\\ &(4,14,34,24)(10,30,40,20),\quad (4,34)(10,40)(14,24)(20,30),\\ &(3,5)(6,8)(7,17)(9,19)(11,21)(12,22)(13,15)(16,18)(23,25)(26,28)\\ &(27,37)(29,39)(33,35)(36,38),\quad (3,13,33,23)(5,25,35,15),\\ &(3,15,23,35,33,25,13,5)(4,10)(9,19)(12,22)(14,20)(24,30)(29,39)(34,40),\\ &(3,33)(5,35)(13,23)(15,25),\\ &(2,9,12,29,32,39,22,19)(3,5)(6,16)(13,15)(18,28)(23,25)(26,36)(33,35),\\ &(2,12,32,22)(9,29,39,19),\quad (2,32)(9,39)(12,22)(19,29),\\ &(1,11,31,21)(7,27,37,17),\quad (1,31)(7,37)(11,21)(17,27)\rangle,\end{aligned}$$

and

$$N = \langle (3,13,33,23)(5,25,35,15),$$
$$(3,13,33,23)(5,25,35,15)(6,26,36,16)(8,18,38,28),$$
$$(3,13,33,23)(4,14,34,24)(5,25,35,15)(10,30,40,20),$$
$$(1,21,31,11)(3,13,33,23)(5,25,35,15)(7,17,37,27),$$
$$(1,21,31,11)(2,12,32,22)(3,13,33,23)(5,25,35,15)(7,17,37,27)(9,29,39,19)\rangle.$$

Now (by suppressing in each case cycles involving points outside B_1*), we see that* $Q_1 = \langle (11,21),(1,11,31,21),(1,31)\rangle \cong D_8$ *and* $P_1 = \langle (1,21,31,11)\rangle \cong C_4$*.*

We have the blocks $B_1 = [1,11,21,31]$*,* $B_2 = [2,12,22,32]$*,* $B_7 = [7,17,21,37]$ *and* $B_9 = [9,19,29,39]$*. Here we can find a conjugate* $\phi(x)$ *which is* $\phi(y)^4\phi(x)\phi(y)$ *that acts on* B_1 *as* $\beta = (11,21)$ *such that*

$$\begin{aligned}
1 &\xrightarrow{\phi(y)^4\phi(x)\phi(y)} 7 \xrightarrow{\phi(x)} 7 \xrightarrow{\phi(y)^4\phi(x)\phi(y)} 1\\
11 &\xrightarrow{\phi(y)^4\phi(x)\phi(y)} 17 \xrightarrow{\phi(x)} 27 \xrightarrow{\phi(y)^4\phi(x)\phi(y)} 21\\
21 &\xrightarrow{\phi(y)^4\phi(x)\phi(y)} 27 \xrightarrow{\phi(x)} 17 \xrightarrow{\phi(y)^4\phi(x)\phi(y)} 11\\
31 &\xrightarrow{\phi(y)^4\phi(x)\phi(y)} 37 \xrightarrow{\phi(x)} 37 \xrightarrow{\phi(y)^4\phi(x)\phi(y)} 31
\end{aligned}$$

4.2 Experiments

In order to study the results of experiments based on Theorem 4.5 we need the following Lemma. It enables us to find the structure of N when the examples are too big for GAP to give us full information.

Lemma 4.8. *Suppose that* $\pi(G)$ *is a group acting on set* Ω *and* H *is the permutation group* $\phi(G)$ *acting on set* $\mathcal{U}$ *as defined in Theorems 4.1 or 4.5 and let* ψ *define the* *action of H on the set $\mathcal{B}$ of blocks. Let $N = Ker(\psi)$ be the normal subgroup of H that fixes each block B_i as a set. Now let M_i be the normal subgroup of N that fixes the block B_i pointwise .*

Choose $\{i_1, \ldots, i_k\}$ so that

$$M_{i_1} \cap M_{i_2} \cap \ldots \cap M_{i_k} = \{e\}$$

but suppose that each subgroup

$$N_{i_j} = M_{i_1} \cap M_{i_2} \cap \ldots M_{i_{j-1}} \cap M_{i_{j+1}} \cap \ldots M_{i_k}$$

is non trivial and isomorphic to a subgroup M. Then if $|N| = |M|^k$, we have N is a direct product of k copies of M.

Proof. We see that

$$\begin{aligned} N_{i_1} \cap N_{i_2} N_{i_3} \ldots N_{i_k} &\subset N_{i_1} \cap M_{i_1} \\ &= M_{i_2} \cap M_{i_3} \ldots \cap M_{i_k} \cap M_{i_1} \\ &= \{e\}. \end{aligned} \tag{4.3}$$

Now we claim that if $n_{i_l} \in N_{i_l}$ and $n_{j_m} \in N_{j_m}$ with $n_{i_l} \neq n_{j_m}$, then $n_{i_l} n_{i_m} = n_{i_m} n_{i_l}$. Since both N_{i_l}, N_{i_m} are normal subgroups of N, we see that the commutator of n_{i_l}, n_{i_m}

$$[n_{i_l}, n_{i_m}] = n_{i_l} n_{i_m} n_{i_l}^{-1} n_{i_m}^{-1} \in N_{i_l} \cap N_{i_m}.$$

Equation 4.3 clearly implies that $N_{i_l} \cap N_{i_m} = \{e\}$, and so it follows that $n_{i_l} n_{i_m} = n_{i_m} n_{i_l}$, as required. So the product $N_{i_1} \ldots N_{i_k}$ is a direct product, isomorphic to M^k. Since it has the same size as N, and is certainly a subgroup of N, we see that it is equal to N. □

Example 4.9. *Let G be a $(2,5,5)$ triangle group, and π a permutation representation of degree 10, defined by*

$$\pi(x) = (3,5)(6,8),$$
$$\pi(y) = (1,2,4,6,3)(5,7,9,10,8),$$

with 1-handles $[1,2],[7,9]$. For $m = 4$ and $\alpha = (1,2)(3,4)$, $\beta = (2,3)$, $\phi(G)$ acts imprimitively on the set $\mathcal{U} = \{1,\ldots,40\}$ with blocks $B_1, B_2, \ldots, B_{10}$ as $[1,11,21,31],\ldots,[10,21,31,41]$. We find

$$M = M_1 \cap M_2 \cap M_3 \cap M_4;$$

here M fixes B_1, B_2, B_3, B_4 pointwise and fixes B_6 as a set (it also fixes B_5, B_7, B_9, B_{10} pointwise and B_8 as a set). We find $M \cap M_6 = \{1\}$. Here $|M| = 4$ and $|N| = 1024 = 4^5$. We see that N is a direct product of groups isomorphic to M. i.e.

$$N = N_1 \times N_2 \times N_3 \times N_4 \times N_6$$

where $N_6 = M$.

$$N_6 = M_1 \cap M_2 \cap M_3 \cap M_4 = M$$
$$N_1 = M_2 \cap M_3 \cap M_4 \cap M_6$$
$$N_2 = M_1 \cap M_3 \cap M_4 \cap M_6$$
$$N_3 = M_1 \cap M_2 \cap M_4 \cap M_6$$
$$N_4 = M_1 \cap M_2 \cap M_3 \cap M_6$$

We know that $N_1, N_2, N_3, N_4 \subseteq M_6$, *so* $N_1N_2N_3N_4 \subseteq M_6$

$$\begin{aligned} & N_6 \cap N_1N_2N_3N_4 \\ & \quad \subseteq N_6 \cap M_6 \\ & \quad = M_1 \cap M_2 \cap M_3 \cap M_4 \cap M_6 \\ & \quad = \{1\}. \end{aligned}$$

Since

$$\begin{aligned} N_1 &= \langle (1, 21, 31, 11)(7, 17, 37, 27) \rangle \\ N_2 &= \langle (2, 12, 32, 22)(9, 29, 39, 19) \rangle \\ N_3 &= \langle (3, 13, 33, 23)(5, 25, 35, 15) \rangle \\ N_4 &= \langle (4, 14, 34, 24)(10, 30, 40, 20) \rangle \\ N_6 &= \langle (6, 26, 36, 16)(8, 18, 38, 28) \rangle, \end{aligned}$$

and each cycle is of order 4 and disjoint, the elements that generate the N_i*'s commute. The following diagram describes how* m *diagrams are joined by using the handles as defined in* α *and* β.

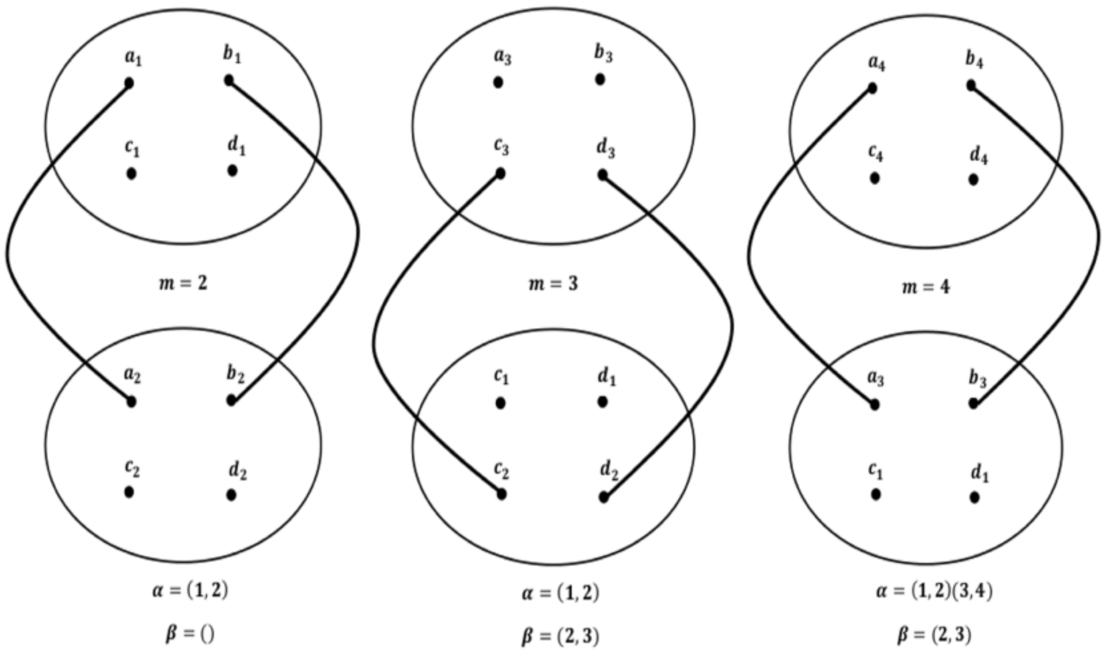


Figure 4.2: m-Composition

The following table illustrates the construction of Theorem 4.1.

G	$Deg(\pi(G))$	$\pi(G)$	$Handle$	$Deg(\phi(G)$	N	$k, \{i_1, \ldots, i_k\}$	M
$(2,3,6)$	6	$C_2 \times A_4$	$[1,2]_1$	12	$C_2 \times C_2$	$2, \{1,2\}$	C_2
$(2,3,7)$	7	$PSL(3,2)$	$[1,2]_1$	14	$\prod_{i=1}^{i=3} C_2$	$3, \{1,2,3\}$	C_2
$(2,3,7)$	14	$PSL(2,13)$	$[1,2]_1$	28	1	1	1
$(2,3,7)$	15	$Alt(15)$	$[1,2]_1$	30	$\prod_{i=1}^{i=14} C_2$	$14, \{1,2,...,14\}$	C_2
$(2,3,7)$	42	$Alt(42)$	$[1,2]_1$	84	$\prod_{i=1}^{i=41} C_2$	$41, \{1,2,...,,41\}$	C_2
$(2,3,8)$	10	$Alt(10)$	$[1,9]_3$	20	C_2	1	1
$(2,3,11)$	12	M_12	$[2,5]_2$	24	$\prod_{i=1}^{i=11} C_2$	$11, \{1,2,...,11\}$	C_2
$(2,3,11)$	22	$Alt(22)$	$[1,2]_1$	44	$\prod_{i=1}^{i=21} C_2$	$21, \{1,2,...,21\}$	C_2
$(2,3,13)$	13	$Alt(13)$	$[1,2]_1$	26	$\prod_{i=1}^{i=12} C_2$	$12, \{1,2,...,12\}$	C_2
$(2,4,7)$	7	$PSL(3,2)$	$[1,2]_1$	14	$\prod_{i=1}^{i=6} C_2$	$6, \{1,2,...,6\}$	C_2
$(2,5,5)$	10	$(\prod_{i=1}^{i=4} C_2 : C_5)$	$[1,2]_1$	20	$\prod_{i=1}^{i=5} C_2$	$5, \{1,2,...,5\}$	C_2
$(2,5,9)$	18	$Alt(18)$	$[1,2]_1$	36	$\prod_{i=1}^{i=17} C_2$	$17, \{1,2,...17\}$	C_2
$(2,5,9)$	20	$Alt(20)$	$[1,2]_1$	40	$\prod_{i=1}^{i=19} C_2$	$19, \{1,2,...19\}$	C_2
$(2,5,10)$	10	$C_2 \times ((\prod_{i=1}^{i=4} C_2) : C_5)$	$[1,2]_1$	20	$\prod_{i=1}^{i=9} C_2$	$9, \{1,2,...,9\}$	C_2
$(2,5,7)$	10	$Alt(10)$	$[1,2]_1$	20	$\prod_{i=1}^{i=9} C_2$	$9, \{1,2,...,9\}$	C_2
$(3,3,5)$	15	$Alt(15)$	$[1,2]_1$	45	$\prod_{i=1}^{i=14} C_3$	$14, \{1,2,...,14\}$	C_3
$(3,5,7)$	14	$Alt(14)$	$[1,2]_1$	42	$\prod_{i=1}^{i=13} C_3$	$13, \{1,2,...,13\}$	C_3
$(3,5,9)$	12	$Alt(12)$	$[1,2]_1$	36	$\prod_{i=1}^{i=11} C_3$	$11, \{1,2,...,11\}$	C_3
$(3,5,11)$	11	$Alt(11)$	$[1,2]_1$	33	$\prod_{i=1}^{i=10} C_3$	$10, \{1,2,...,10\}$	C_3
$(5,5,7)$	14	$Alt(14)$	$[1,2]_1$	70	$\prod_{i=1}^{i=13} C_5$	$13, \{1,2,...,13\}$	C_5
$(5,7,11)$	15	$Alt(15)$	$[1,2]_1$	75	$\prod_{i=1}^{i=14} C_5$	$14, \{1,2,...,14\}$	C_5

Table 4.1: Imprimitive Composition

Example 4.10. *Consider a triangle group* $(2, 3, 8)$*, and the representation* π *of degree 10 defined by* $\pi(x) = (2,4)(3,5)(6,7)(8,10)$*,* $\pi(y) = (1,2,3)(4,6,5)(7,8,9)$*. This has handles* $a = 1, b = 9$*. The group* $\pi(G)$ *is* $\mathcal{A}_{10}$ *acting primitively on* $\{1, 2, \ldots, 10\}$*. After composition we have the permutations* $\phi(x), \phi(y)$ *given by*

$$\phi(x) = (1,11)(2,4)(3,5)(6,7)(8,10)(9,19)(12,14)(13,15)(16,17)(18,20)$$
$$\phi(y) = (1,2,3)(4,6,5)(7,8,9)(11,12,13)(14,16,15)(17,18,19).$$

Using Theorem 4.3, the group H *generated by* $\phi(x)$*,* $\phi(y)$ *is imprimitive with blocks* $[1, 11], [2, 12], [3, 13], [4, 14], [5, 15], [6, 16], [7, 17], [8, 18], [9, 19], [10, 20]$ *of degree 20 and has structure description* $C_2 \times \mathcal{A}_{10}$*.*

The following tables illustrates the construction of Theorem 4.5.

G	$Deg(\pi(G))$	$\pi(G)$	$Handles$	m	α	β	$\langle\alpha,\beta\rangle$	N or $\|N\|$	k	M
$(2,3,6)$	6	$C_2 \times A_4$	$[[1,2],[5,6]]$	6	$(1,2)(3,4)(5,6)$	$(2,3)(4,5)$	D_{12}	$C_6 \times C_6$	$2,\{1,2\}$	C_6
$(2,3,6)$	6	$C_2 \times A_4$	$[[1,2],[5,6]]$	6	$(1,2)(3,4)(5,6)$	$(2,3)(4,5)(1,6)$	S_3	$C_3 \times C_3$	$2,\{1,2\}$	C_3
$(2,3,6)$	6	$C_2 \times A_4$	$[[1,2],[5,6]]$	7	$(1,2)(3,4)(5,6)$	$(2,3)(4,5)(6,7)$	D_{14}	$C_7 \times C_7$	$2,\{1,2\}$	C_7
$(2,3,6)$	6	$C_2 \times A_4$	$[[1,2],[5,6]]$	7	$(1,5)(2,4)(6,7)$	$(1,3)(4,6)(5,7)$	D_{14}	$C_7 \times C_7$	$2,\{1,2\}$	C_7
$(2,3,7)$	42	$Alt(42)$	$[[1,2],[41,42]]$	3	$(1,2)$	$(2,3)$	S_3	$2^{41}3^{42}$	$42,\{1,..,42\}$	C_3
$(2,3,7)$	42	$Alt(42)$	$[[1,2],[41,42]]$	3	$(1,2)$	$(1,3)$	S_3	$2^{41}3^{42}$	$42,\{1,..,42\}$	C_3
$(2,3,7)$	42	$Alt(42)$	$[[1,2],[41,42]]$	4	$(1,2)(3,4)$	$(2,3)$	D_8	2^{124}	$42,\{1,..,42\}$	C_2
$(2,3,7)$	42	$Alt(42)$	$[[1,2],[41,42]]$	4	$(1,3)(2,4)$	$(2,3)$	D_8	2^{124}	$42,\{1,..,42\}$	C_2
$(2,3,7)$	42	$Alt(42)$	$[[1,2],[41,42]]$	4	$(1,2)(3,4)$	$(1,4)(2,3)$	$C_2 \times C_2$	$\prod_{i=1}^{82} C_2$	$41,\{1,..,41\}$	$C_2 \times C_2$
$(2,5,5)$	10	$\prod_{i=1}^{i=4} C_2 : C_5$	$[[1,2],[7,9]]$	4	$(1,2)(3,4)$	$(2,3)$	D_8	$\prod_{i=1}^{5} C_4$	$5,\{1,2,3,4,6\}$	C_4
$(2,5,5)$	10	$\prod_{i=1}^{i=4} C_2 : C_5$	$[[1,2],[7,9]]$	4	$(1,2)(3,4)$	$(1,4)(2,3)$	$C_2 \times C_2$	$\prod_{i=1}^{5} C_2$	$5,\{1,2,3,4,6\}$	C_2
$(2,5,5)$	10	$\prod_{i=1}^{i=4} C_2 : C_5$	$[[1,2],[7,9]]$	5	$(1,2)(3,4)$	$(2,3)(4,5)$	D_{10}	$\prod_{i=1}^{5} C_5$	$5,\{1,2,3,4,6\}$	C_5
$(2,5,5)$	10	$\prod_{i=1}^{i=4} C_2 : C_5$	$[[1,2],[7,9]]$	6	$(1,2)(3,4)(5,6)$	$(2,3)(4,5)$	D_{12}	$\prod_{i=1}^{5} C_6$	$5,\{1,2,3,4,6\}$	C_6
$(2,5,5)$	10	$\prod_{i=1}^{i=4} C_2 : C_5$	$[[1,2],[7,9]]$	6	$(1,2)(3,4)(5,6)$	$(2,3)(4,5)(1,6)$	S_3	$\prod_{i=1}^{5} C_3$	$5,\{1,2,3,4,6\}$	C_3
$(2,5,6)$	12	$(\prod_{i=1}^{i=4} C_2 : A_5) : C_2$	$[[1,2],[9,8]]$	4	$(1,2)(3,4)$	$(2,3)$	D_8	$\prod_{i=1}^{4} C_4$	$4,\{1,2,3,4\}$	C_4
$(2,5,6)$	12	$(\prod_{i=1}^{i=4} C_2 : A_5) : C_2$	$[[1,2],[9,8]]$	4	$(1,2)(3,4)$	$(1,4)(2,3)$	$C_2 \times C_2$	$\prod_{i=1}^{4} C_2$	$4,\{1,2,3,4\}$	C_2
$(2,5,6)$	12	$(\prod_{i=1}^{i=4} C_2 : A_5) : C_2$	$[[1,2],[9,8]]$	5	$(1,2)(3,4)$	$(2,3)(4,5)$	D_{10}	$\prod_{i=1}^{4} C_5$	$4,\{1,2,3,4\}$	C_5
$(2,3,19)$	57	$PSL(3,7)$	$[[24,42],[47,17]]$	4	$(1,2)(3,4)$	$(2,3)$	D_8	2^{169}	$57,\{1,..,57\}$	C_2
$(2,3,19)$	57	$PSL(3,7)$	$[[24,42],[47,17]]$	4	$(1,2)(3,4)$	$(1,4)(2,3)$	$C_2 \times C_2$	$\prod_{i=1}^{112} C_2$	$56,\{1,..,56\}$	$C_2 \times C_2$
$(3,5,7)$	14	$Alt(14)$	$[[1,2],[4,7]]$	4	$(1,2,3)$	$(2,3,4)$	$Alt(4)$	$2^{28}.3^{13}$	$14,\{1,..,14\}$	$C_2 \times C_2$
$(3,5,7)$	14	$Alt(14)$	$[[1,2],[4,7]]$	6	$(1,4,5)(2,3,6)$	$(1,5,3)$	$Alt(6)$	$2^{42}.3^{28}.5^{14}$	$14,\{1,..,14\}$	$Alt(6)$
$(3,5,7)$	14	$Alt(14)$	$[[1,2],[4,7]]$	6	$(1,2,3)(4,5,6)$	$(1,3,5)(2,6,4)$	$Alt(5)$	$2^{28}.3^{14}.5^{14}$	$14,\{1,..,14\}$	$Alt(5)$
$(3,5,7)$	14	$Alt(14)$	$[[1,2],[4,7]]$	6	$(1,2,3)(4,5,6)$	$(1,3,5)(2,4,6)$	$Alt(4)$	$2^{28}.3^{13}$	$14,\{1,..,14\}$	$C_2 \times C_2$
$(3,5,9)$	12	$Alt(12)$	$[[1,2],[4,7]]$	4	$(1,2,3)$	$(2,3,4)$	$Alt(4)$	$2^{24}.3^{11}$	$12,\{1,..,12\}$	$C_2 \times C_2$
$(3,5,11)$	11	$Alt(11)$	$[[1,2],[4,7]]$	4	$(1,2,3)$	$(2,3,4)$	$Alt(4)$	$2^{22}.3^{10}$	$11,\{1,..,11\}$	$C_2 \times C_2$
$(3,5,11)$	11	$Alt(11)$	$[[1,2],[4,7]]$	6	$(1,2,3)(4,5,6)$	$(1,3,5)(2,4,6)$	$Alt(4)$	$2^{22}.3^{10}$	$11,\{1,..,11\}$	$C_2 \times C_2$
$(3,5,11)$	11	$Alt(11)$	$[[1,2],[4,7]]$	6	$(1,2,3)(4,5,6)$	$(1,3,5)(2,6,4)$	$Alt(5)$	$2^{22}.3^{11}.5^{11}$	$11,\{1,..,11\}$	$Alt(5)$
$(3,5,11)$	11	$Alt(11)$	$[[1,2],[4,7]]$	6	$(1,4,5)(2,3,6)$	$(1,5,3)$	$Alt(6)$	$2^{33}.3^{22}.5^{11}$	$11,\{1,..,11\}$	$Alt(6)$
$(3,5,11)$	11	$Alt(11)$	$[[1,2],[8,10]]$	4	$(1,2,3)$	$(2,3,4)$	$Alt(4)$	$2^{22}.3^{10}$	$11,\{1,..,11\}$	$C_2 \times C_2$
$(3,5,11)$	11	$Alt(11)$	$[[1,2],[8,10]]$	6	$(1,2,3)(4,5,6)$	$(1,3,5)(2,4,6)$	$Alt(4)$	$2^{22}.3^{10}$	$11,\{1,..,11\}$	$C_2 \times C_2$
$(3,5,11)$	11	$Alt(11)$	$[[1,2],[8,10]]$	6	$(1,2,3)(4,5,6)$	$(1,3,5)(2,6,4)$	$Alt(5)$	$2^{22}.3^{11}.5^{11}$	$11,\{1,..,11\}$	$Alt(5)$
$(3,5,11)$	11	$Alt(11)$	$[[1,2],[8,10]]$	6	$(1,4,5)(2,3,6)$	$(1,5,3)$	$Alt(6)$	$2^{33}.3^{22}.5^{11}$	$11,\{1,..,11\}$	$Alt(6)$

G	$Deg(\pi(G))$	$\pi(G)$	$Handles$	m	α	β	$\langle\alpha,\beta\rangle$	N or $\|N\|$	k	M
$(5,7,11)$	14	$Alt(14)$	$[[1,2],[4,8]]$	5	$(1,2,3,4,5)$	$(1,2,4,5,3)$	$Alt(5)$	$2^{28}.3^{14}.5^{14}$	$14, \{1,..,14\}$	$Alt(5)$
$(5,7,11)$	14	$Alt(14)$	$[[1,2],[4,8]]$	5	$(1,2,3,4,5)$	$(1,3,5,2,4)$	C_5	$\prod_{i=1}^{13} C_5$	$13, \{1,..,13\}$	C_5
$(5,7,11)$	14	$Alt(14)$	$[[1,2],[4,8]]$	6	$(1,3,2,4,6)$	$(1,2,3,4,5)$	$Alt(5)$	$2^{28}.3^{14}.5^{14}$	$14, \{1,..,14\}$	$Alt(5)$
$(5,7,11)$	14	$Alt(14)$	$[[1,2],[4,8]]$	6	$(1,3,2,4,6)$	$(2,1,3,4,5)$	$Alt(6)$	$2^{42}.3^{28}.5^{14}$	$14, \{1,..,14\}$	$Alt(6)$
$(5,7,11)$	15	$Alt(15)$	$[[1,2],[4,8]]$	5	$(1,2,3,4,5)$	$(1,2,4,5,3)$	$Alt(5)$	$2^{30}.3^{15}.5^{15}$	$15, \{1,..,15\}$	$Alt(5)$

Table 4.2: Imprimitive Composition

The following observation from the experiments above motivate us to prove the theorems in Chapter 5.

In Table 4.1 we have seen when $\pi(G) = \mathcal{A}_{\mathrm{deg}}$ then N is the direct product of $\mathrm{deg} - 1$ copies of C_p except when $G = \Delta(2,3,8)$ and $p = 2$ and $\pi(G) = \mathcal{A}_{10}$, in this case we have $N = C_2$. We also note that when $\pi(G) = M_{12}$ and for $G = (2,4,7)$ such that $\pi(G) = \mathtt{PSL}(3,2)$ then we have N is the direct product of $\mathrm{deg} - 1$ copies of C_p. When $\pi(G) = \mathtt{PSL}(2,13)$ then N is the trivial group.

In Table 4.2 we observed that when $\pi(G) = \mathcal{A}_{\mathrm{deg}}$ and $\langle \alpha, \beta \rangle = \mathcal{A}_m$ for $m \geq 5$ then $|N| = |\langle \alpha, \beta \rangle|^{\mathrm{deg}}$. We also found examples where $\pi(G) = \mathcal{A}_{\mathrm{deg}}$ for $G = (3,5,7)$, $(3,5,9)$ and $(3,5,11)$ and $m = 4$ for which we have $\langle \alpha, \beta \rangle = \mathcal{A}_4$ then $|N| \neq |\langle \alpha, \beta \rangle|^{\mathrm{deg}}$. Moreover, we can see that when $\pi(G) = \mathtt{PSL}(3,7)$ and $\langle \alpha, \beta \rangle = C_2 \times C_2$ then N is the direct product of deg copies of $\langle \alpha, \beta \rangle$.

Chapter 5

Imprimitive composition with alternating groups

In this chapter we prove the results that we analysed from the experiments in Chapter 4. Here we find the structures of the groups built out of the constructions of Theorem 4.1 and Theorem 4.5. We also find some conjectures in the end of the theorems. Section 5.1 illustrates the future work describing the approach to prove the conjectures we made.

Theorem 5.1. *Suppose that $G = \Delta(p, q, r)$ is a triangle group with p prime, $p \leq q \leq r$. Suppose that $\pi(G) \cong \mathcal{A}_{\deg}$ and $H = \phi(G)$ is constructed as in Theorem 4.1. Assume the notation of Theorem 4.1. Suppose that* $\deg > 6$*. Then either*

1. *$p|qr$, $p|\deg$ and $H \cong C_p \times \mathcal{A}_{\deg}$, or*
2. *$H \cong C_p^{\deg-1} \rtimes \mathcal{A}_{\deg}$.*

Note that the second case might occur even when $p|qr$ and $p|\deg$.

Proof. By Theorem 4.1, H is isomorphic to a subgroup of $C_p \wr \psi(H) \cong C_p^{\deg} \rtimes \psi(H)$. And since $\psi(H) \cong \pi(G) \cong \mathcal{A}_{\deg}$, we have $H \subseteq C_p^{\deg} \rtimes \mathcal{A}_{\deg}$. In fact $H \cong N.\mathcal{A}_{\deg}$.

In addition we know from Theorem 4.1 that $N \subseteq C_p^{\text{deg}}$ is a submodule of the deg-dimensional permutation module W_{deg} for $\mathcal{A}_{\text{deg}}$. It is possible that we have $N = 1$.

We hope to prove that, except in the situation which can only occur for the particular values of p, d covered in (1), the normal subgroup N is isomorphic to $C_p^{\text{deg}-1}$.

First we show that $N \neq 1$. If $N = 1$, then $H \cong \psi(H) = \mathcal{A}_{\text{deg}}$. In that case the group J_a, (which stabilises B_a as a set) is isomorphic to $\mathcal{A}_{\text{deg}-1}$. However, by Theorem 4.1 J_a acts on B_a as Q_a, which is cyclic of order p. So there is a homomorphism from J_a to C_p, i.e. there is a homomorphism from $\mathcal{A}_{\text{deg}-1}$ to C_p. Now the kernel of this homomorphism is a normal subgroup of $\mathcal{A}_{\text{deg}-1}$ of index p. Since if $\text{deg} - 1 \geq 5$, the group $\mathcal{A}_{\text{deg}-1}$ is simple, but this cannot happen.

Now we need to show that N is a submodule of $W_{\text{deg}-1}$. Examining the construction of $H = \phi(G)$ we see that H is generated by

$$\phi_x = \pi_1(x)\pi_2(x)\ldots\pi_p(x)\tau_a\tau_b^{-1}$$

where $\tau_\omega = (\omega_1, \omega_2, \ldots, \omega_p)$ and

$$\phi_y = \pi_1(y)\pi_2(y)\ldots\pi_p(y).$$

Now define

$$C = \{\pi_1(g)\pi_2(g)\ldots\pi_p(g) : g \in G\}$$

and

$$V = \langle \tau_\omega\tau_{\omega'}^{-1} : \omega, \omega' \in \Omega \rangle$$

both subgroups of $\mathcal{S}(\mathcal{U})$. We see that V is isomorphic to the $(\text{deg} - 1)$-dimensional submodule $W_{\deg-1}$ of $Q^{\deg}$ for $\mathcal{A}_{\deg}$, and C is isomorphic to $\pi(G) \cong \mathcal{A}_{\deg}$. Then ϕ_x, ϕ_y can both be written as products within CV, so H which is generated by ϕ_x and ϕ_y is a subgroup of $W_{\deg-1} \rtimes \mathcal{A}_{\deg}$. In particular $N \subseteq W_{\deg-1}$, so $|N| \leq p^{\deg-1}$. Hence N is a non-trivial submodule of $W_{\deg-1}$.

$W_{\deg-1}$ has no proper submodules unless p divides deg. If $p|\deg$ then the trivial permutation module W_1 is a submodule of $W_{\deg-1}$. So if p does not divide deg we see that we must have $N = W_{\deg-1}$.

So suppose now that $p|\deg$, that N is the trivial submodule W_1 of $W_{\deg-1}$, and so $H = N.\mathcal{A}_{\deg} = C_p.\mathcal{A}_{\deg}$. We look at J_a, the stabiliser of B_a. We have $\psi(J_a) = \mathcal{A}_{\deg-1}$. We also have $Q_a \cong C_p$. So J_a has a quotient isomorphic to C_p. So J_a maps onto C_p and also maps onto $\mathcal{A}_{\deg-1}$. By the Jordan-Hölder Theorem 2.3, $J_a \cong C_p.\mathcal{A}_{\deg-1}$ where C_p and $\mathcal{A}_{\deg-1}$ are simple and their intersection is the identity subgroup. So J_a must be split.

We see that H must also be a split extension in this case, because the non-split extensions are well known by [17], see Section 2.2.5, in particular Lemma 2.20. The non-split extension $C_p.\mathcal{A}_{\deg}$ can only exist when $p = 2$ and cannot admit such subgroups J_a as above.

So the extension $N.\mathcal{A}_{\deg}$ splits, then, since the action of $\mathcal{A}_{\deg}$ on N is trivial, we must have a direct product, $H = C_p \times \mathcal{A}_{\deg}$, and so are in case (1). In that case C_p is a homomorphic image of $G = \Delta(p, q, r)$, and hence an abelian quotient of it. But from Corollary 2.10, we know that G/G' can only map onto C_p when $p|qr$.

If $N = W_{\deg-1}$, then H is an extension of $C_p^{\deg-1}$ by $\mathcal{A}_{\deg}$. So now H is a subgroup of $W_{\deg-1} \rtimes \mathcal{A}_{\deg}$ of order $p^{\deg-1}.|\mathcal{A}_{\deg}|$. Hence H is the whole of $W_{\deg-1} \rtimes \mathcal{A}_{\deg}$ and the result is proved.

□

The following is an immediate corollary of the theorem.

Corollary 5.2. *Suppose that $G = \Delta(p,q,r)$ is a triangle group with p prime, $p \leq q \leq r$. Suppose that* $\deg > 6$ *and in addition $p \nmid qr$ and $p \nmid$* $\deg$*. Then provided that G has a quotient $\mathcal{A}_{\deg}$ containing at least one handle, G also has a quotient $C_p^{\deg-1} \rtimes \mathcal{A}_{\deg}$.*

Theorem 5.3. *Suppose that $G = \Delta(p,q,r)$ is a triangle group with p prime, $p \leq q \leq r$. Suppose that $\pi(G) \cong \mathcal{A}_{\deg}$ and $H = \phi(G)$ is constructed as in Theorem 4.5. Assume the notation of Theorem 4.5. Suppose that* $\deg > 6$ *and $\langle \alpha, \beta \rangle \cong \mathcal{A}_m$, where $m \neq \deg - 1$ and $m \geq 5$. Then $H \cong \mathcal{A}_m \wr \mathcal{A}_{\deg}$.*

Proof. By Theorem 4.5, H is isomorphic to a subgroup of $\mathcal{A}_m \wr \psi(H) \cong \mathcal{A}_m^{\deg} \rtimes \mathcal{A}_{\deg}$. Then N, the kernel of the map $\psi : H \to \mathcal{A}_{\deg}$ is a subgroup of $\mathcal{A}_m^{\deg}$ and $H = N.\mathcal{A}_{\deg}$. In order to prove the result, we need simply to prove that N is the whole of $\mathcal{A}_m^{\deg}$.

To prove this property for N, we need to use a result of [7], which we have described in Section 2.2.2.

We have $N \subseteq T_1 \times T_2 \times \cdots \times T_{\deg}$, where $T_i \cong \mathcal{A}_m$ for each i.

By Fawcett's Lemma 2.14, N is a direct product of groups $H_1 \cdots H_r$ where each H_i is a full diagonal subgroup of $\prod_{i \in I_j} T_i$, and $I_1, \ldots, I_r$ is a partition of $\{1, \ldots, \deg\}$. Now the partition must be preserved by $\psi(H) = \mathcal{A}_{\deg}$, in its action on N by conjugation. Since $\mathcal{A}_{\deg}$ acts primitively, so either we have $r = 1$ and $I_1 = \{1, \ldots, \deg\}$ or we have $r = \deg$ and $I_j = \{j\}$ for each j.

In the first case we have $N = A_m$ and in the second case we have

$$N = \underbrace{\mathcal{A}_m \times \mathcal{A}_m \times \cdots \times \mathcal{A}_m}_{\deg \quad \text{times}}.$$

When $N = \mathcal{A}_m$ then we have $H \cong \mathcal{A}_m.\mathcal{A}_{\deg}$.

Considering the subgroup J_a, which maps onto $\mathcal{A}_m$, we see that in this case H must be the direct product $\mathcal{A}_m \times \mathcal{A}_{\deg}$. So then H is the direct product of N ($\cong \mathcal{A}_m$)

and its complement

$$C = \{\pi_1(g), \pi_2(g), \ldots, \pi_m(g) : g \in G\}.$$

Then every element of H can be written as a product nc where $n \in N$, $c \in C$ and the elements n, c commute. Now $\phi(x) = c_1 n_1$, where

$$c_1 = \pi_1(x)\pi_2(x)\cdots\pi_m(x)$$
$$n_1 = \gamma_1 \cdots \gamma_k \delta_1 \cdots \delta_l$$

and $\phi(y) \in C$. Since H is generated by $\phi(x)$ and $\phi(y)$, we see that N must be cyclic, generated by $\gamma_1 \cdots \gamma_k \delta_1 \cdots \delta_l$, and hence N is cyclic of order p.

This contradicts the fact that $N \cong \mathcal{A}_m$ for $m \geq 3$ and so this case is excluded. □

The following is an immediate corollary.

Corollary 5.4. *Suppose that $G = \Delta(p, q, r)$ is a triangle group with p prime, $p \leq q \leq r$. Suppose that* $\deg > 6$*, and that for some m not equal to* $\deg - 1$ *the alternating group $\mathcal{A}_m$ can be generated by two p-cycles. Then provided that G has a quotient $\mathcal{A}_{\deg}$ containing two disjoint handles, G also has a quotient $\mathcal{A}_m \wr \mathcal{A}_{\deg}$.*

Our results suggest the following two conjectures.

Conjecture 5.5. *Suppose that $G = \Delta(p, q, r)$ is a triangle group with p prime, $p \leq q \leq r$. Then for all but finitely many integers* deg*, G maps on to $C_p^{\deg - 1} \rtimes \mathcal{A}_{\deg}$.*

Conjecture 5.6. *Suppose that $G = \Delta(p, q, r)$ is a triangle group with p prime $p \leq q \leq r$ and choose m such that the alternating group $\mathcal{A}_m$ can be generated by 2 p-cycles. Then for all but finitely many integers* deg*, G maps on to $\mathcal{A}_m \wr \mathcal{A}_{\deg}$.*

We note that it follows from Higman's conjecture that for almost all m and p, $\mathcal{A}_m$ is generated by two elements of order p.

5.1 Future work

In order to verify the conjectures we need to know not just that (by Everitt's theorem) almost all triangle groups map onto almost all $\mathcal{A}_{\text{deg}}$, but that there exist such images with appropriate handles. So a vertification of the conjectures requires us to look closely at Everitt's proof to see whether the coset diagrams constructed have the necessary handles. If they do not, it is possible that we can adapt the construction so that they do. We see this as future work.

Appendix A

Algorithm of Composition

We constructed a GAP procedure to compose $t \leq p$ coset diagrams of a triangle group G by finding all possible subgroups of a triangle group

$$G = \Delta(p, q, r) = \langle x, y, t : x^p = y^q = (xy)^r = 1 \rangle$$

upto a finite index say n. We named this function by FindCosetTablesTriangle-Group having input as parameters of p, q, r and n, where n is the degree of subgroup and p, q, r are the parameters of the triangle group.

```
FindCosetTablesTriangleGroup := function(p,q,r,n)

local x,t,y,f,g,hlist,h,permslist,perms;

f := FreeGroup(3);
g := f/[f.1^p,f.2^q,(f.1*f.2)^r];

hlist := Filtered(LowIndexSubgroupsFpGroup(g,n),i->Index(g,i)=n);
permslist := [];
```

```
for h in hlist do
  x := List(CosetTable(g,h){[1]},PermList)[1];
  y := List(CosetTable(g,h){[5]},PermList)[1];
  perms := [];
  Add(perms,x);
  Add(perms,y);
  Print("perms=",perms,"\n");
  Add(permslist,perms);
od;

return permslist;
end;
```

We construct an algorithm for the composition of p coset diagrams $[A_1, .., A_p]$ that represent transitive permutation representations of a triangle group $\Delta(p, q, r)$. For this we use p-composition and k-handles to join the coset diagrams. In an arbitrary permutation representation of G, two points a and b which are fixed by x such that both t and $(xy)^k$ map a to b form k-handle and are denoted by $[a, b]_k$.

FindHandles.g find k-handles of a given arbitrary permutation representation. FindHandles is a function that takes various input value of permutations of triangle group and a parameter "k" that is used to identify whether it is 1-handles, 2-handles and 3-handles of given permutations of triangle group $\Delta(p, q, r)$, where $k = 1, 2, 3, \ldots$ (depending on k-handles).

```
FindHandles := function(perms,k)
local x,t,y,i,j;
x := perms[1];
```

```
t := perms[2];
y := perms[3];
j := [];

for i in MovedPoints(t) do
 if i < i^t and i=i^x and i^t=i^((x*y)^k)
  then Add(j,[i,i^t]);
 fi;
od;

return j;
end;
```

ShiftPermutationDomain has a vital role in the composition of coset tables that changes the label of the permutation representations of $[2, \ldots, p]$ coset tables to join them together.

```
ShiftPermutationDomain:=function(perms,n1)
local l,m;
Print("perms␣=",perms,"\n");
l:=ListPerm(perms)+n1;
m:=PermList(Concatenation([1..n1],l));
Print("shifts␣to␣",m,"\n");
return m;
end;
```

CompositionByHandles takes input value of degreelist that illustrates list of all degrees (could be different from each other) subgroups of an extended triangle group, permslist shows a permutation list of each of the degree defined in the degreelist, pairlist is the list of all handles of permutation list for each of the degree in degreelist.

Output is the permutation list by the composition of p coset tables by using k type handles that represents transitive permutation representation of degree $n_1 + n_2 + \ldots + n_p$ of a group G, here $n_1, n_2, \ldots, n_p$ are the degrees of index subgroups of a group. Here we use CompositionByHandles that are used to compose p coset tables by using k-handle in each of the coset table.

```
CompositionByHandles := function(degreelist,permslist,pairlist)

local perms,x,y,xx,yy,n,a,b,c,d,i,pair,p,genericCycle1,genericCycle2,
aa,bb,cc,dd;
#Print("Entering CompositionBy1Handles with degreelist=",degreelist,"\n",
"permslist=",permslist,"\n","pairlist␣=",pairlist,"\n");

p := Length(degreelist);
x := permslist[1][1];
y := permslist[1][2];
n := degreelist[1];
a :=[];
cc := pairlist[1][1];
Add(a,cc);
b := [];
dd := pairlist[1][2];
Add(b,dd);

#Print("x=",x,"\n");
```

```
#Print("y=",y,"\n");
#Print("n=",n,"\n");
for i in [2..p] do
  c := pairlist[i][1] + n;
  d := pairlist[i][2] + n;
  Add(a,c); # this will find the list of joining handles
            # on the left side of the axis of symmetry like [a1,a2..,ap]
  Add(b,d); # this will find the list of joining handles
            # on the right side of the axis of symmetry like [b1,b2..,bp]

  xx := ShiftPermutationDomain(permslist[i][1],n);
  yy := ShiftPermutationDomain(permslist[i][2],n);

  #Print("xx=",xx,"\n");
  #Print("yy=",yy,"\n");
  x := x*xx;
  y := y*yy;
  #Print("x=",x,"\n");
  #Print("y=",y,"\n");

  n := n + degreelist[i];
  #Print("n=",n,"\n");
od;

a:=Reversed(a); # this will reverse the the list of joining handles
                # on left hand side of axis of symmetry because we want
```

```
                    # to make [ap,..,a1]
genericCycle1:=PermList(Concatenation([2..Size(a)],[1]));
aa:=MappingPermListList(Permuted(a,genericCycle1),a);
                     #this will convert the list of joining handles on the
                     # left side
                    # of axis of symmetry into permutations like (ap,..,a1)

genericCycle2:=PermList(Concatenation([2..Size(b)],[1]));
bb:=MappingPermListList(Permuted(b,genericCycle2),b);
                     # this will convert the list of joining handles on the
                     # right side of axis of symmetry into permutations
                     # like (b1,..,bp)

x := x*aa*bb;
#Print("Joining handles permutations =",p,"\n");
perms:=[];

Add(perms,x);
Add(perms,y);

#Print("x:=",perms[1],"\n","\n","y =",perms[2],"\n");
#Print("x*y =",x*y,"\n");
#Print("Order of x*y = ",Order(x*y),"\n");

return perms;
end;
```

```
CompositionByMultHandles := function(degreelist,permslist,handlelist)

local perms,x,y,xx,yy,k,n,m,t,a,b,c,d,i,j,pair,pairlist,p,
genericCycle1,genericCycle2,aa,bb,cc,dd;
# Print("Entering CompositionByMultHandles with degreelist=",degreelist,
"\n","permslist=",permslist,"\n","pairlist =",pairlist,"\n");
t := Length(degreelist);
p:=Length(handlelist);

n:=degreelist[1];
x:=permslist[1][1];
y:=permslist[1][2];

for i in [2..t] do

  xx := ShiftPermutationDomain(permslist[i][1],n);
  yy := ShiftPermutationDomain(permslist[i][2],n);

  x := x*xx;
  y := y*yy;

  n := n + degreelist[i];
od;

pairlist:=[];
for i in [1..p] do
```

```
  k :=handlelist[i][3];

  m:=0;
   for j in [1..k-1] do
      m:=m+degreelist[j];
   od;

  pairlist[i]:=handlelist[i]+m;
od;
Print("Pairlist␣=␣",pairlist,"\n");

a:=[];
b:=[];

for i in [1..p] do
 cc:=pairlist[i][1];
 Add(a,cc);
 dd:=pairlist[i][2];
 Add(b,dd);
od;


a:=Reversed(a); # this will reverse the the list of joining
                          # handles on left hand side of axis of
                          # symmetry because we want to make [ap,..,a1]
genericCycle1:=PermList(Concatenation([2..Size(a)],[1]));
```

```
aa:=MappingPermListList(Permuted(a,genericCycle1),a);
                              # this will convert the list of joining handles
                              # on the left side of axis of symmetry into
                              # permutations like (ap,..,a1)

genericCycle2:=PermList(Concatenation([2..Size(b)],[1]));
bb:=MappingPermListList(Permuted(b,genericCycle2),b);
                              # this will convert the list of joining handles
                              # on the right side of axis of symmetry into
                              # permutations like (b1,..,bp)

x := x*aa*bb;

perms:=[];
Add(perms,x);
Add(perms,y);

return perms;

end;
```

Appendix B

Algorithms of Imprimitive Composition

```
ImprimitiveComposition:=function(x,y,m,Hand,alpha,beta)

local n,nn,xx,yy,x1,y1,a,aa,b,bb,c,cc,d,dd,Cycle_alpha,Cycle_beta,i,j,
g,h,p,gamma,delta,cycle,cycle1,cycle2;

n:=LargestMovedPoint([x,y]);
p:=Order(x);

a:=[Hand[1][1]];
b:=[Hand[1][2]];
c:=[Hand[2][1]];
d:=[Hand[2][2]];

Cycle_alpha:=Cycles(alpha,[1..m]);
```

```
Cycle_alpha := Filtered(Cycle_alpha,i->Length(i)=p);
Cycle_beta:= Cycles(beta,[1..m]);
Cycle_beta := Filtered(Cycle_beta,i->Length(i)=p);

# Cycle_alpha and Cycle_beta are lists whose entries
# are the list of vectors in the cycles of alpha and beta

aa:=a;
bb:=b;
cc:=c;
dd:=d;

for i in [2..m] do #This loop find the handles by adding (m-1)n
  aa:=aa+n;
  bb:=bb+n;
  cc:=cc+n;
  dd:=dd+n;
  Append(a,aa);
  Append(b,bb);
  Append(c,cc);
  Append(d,dd);
od;

gamma:=();

for cycle in Cycle_alpha do
```

```
  cycle1 := ( a[cycle[1]] , a[cycle[2]] );
  cycle2 := ( b[cycle[1]] , b[cycle[2]]);
  for j in [3..p] do
    cycle1 := cycle1 * ( a[cycle[1]] , a[cycle[j]] );
    cycle2 := ( b[cycle[1]] , b[cycle[j]] ) * cycle2;
  od;
  gamma:=gamma*cycle1*cycle2;
od;

delta:=();

for cycle in Cycle_beta do
  cycle1 := ( c[cycle[1]] , c[cycle[2]] );
  cycle2 := ( d[cycle[1]] , d[cycle[2]] );
  for j in [3..p] do
    cycle1 := cycle1 * ( c[cycle[1]] , c[cycle[j]] );
    cycle2 := ( d[cycle[1]] , d[cycle[j]] ) * cycle2;
  od;
  delta:=delta*cycle1*cycle2;
od;
```

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