Covering Theory of Buildings and their Quotients

William Norledge

Thesis presented for the degree of
Doctor of Philosophy

School of Mathematics and Statistics
Newcastle University
United Kingdom

April 2018

Abstract

We introduce structures which model the quotients of Bruhat-Tits buildings by type-preserving group actions. These structures, which we call Weyl graphs, generalize chamber systems of type M by allowing 2-residues to be quotients of generalized polygons. Weyl graphs also generalize Tits amalgams with a trivial chamber stabilizer group by allowing for group actions which are not chamber-transitive. We develop covering theory of Weyl graphs, and characterize buildings as connected, simply connected Weyl graphs. We describe a procedure for obtaining a group presentation of the fundamental group of a Weyl graph $\mathcal{W}$, which acts naturally on the universal cover of $\mathcal{W}$. We present an application of the theory of Weyl graphs to Singer lattices. We construct the Singer cyclic lattices of type M, where $m_{st} \in \{2, 3, \infty\}$ for all $s, t \in S$. In particular, by taking the Davis realization of a building, we obtain new examples of lattices in polyhedral complexes.

Acknowledgements

Firstly, I would like to thank my supervisors Alina Vdovina and Evgenios Kakariadis for all their help and encouragement. Secondly, I would like the thank the staff and postgraduate students in mathematics at Newcastle University for the useful discussions whilst producing this thesis. Additional thanks are due to Anne Thomas, with whom it was a pleasure to collaborate during my time at Newcastles.

Contents

List of Figures

Chapter 1

Introduction

In geometric group theory, one classically studies discrete groups by considering their actions on non-positively curved metric spaces. The subject began with the pioneering work of Gromov, who promoted the study of finitely generated groups as coarse geometric objects (see [Gro84]). A modern perspective on this idea is that every locally compact, second countable, compactly generated group has a canonical coarse structure, obtained by taking the bounded coarse structure of the word-metric associated to a compact generating set. Then, since compactly generated discrete groups are exactly finitely generated groups, one can study properties of finitely generated groups which are invariants of the underlying coarse structure. Group actions are an important tool, since a finitely generated group must have the same coarse structure as a geodesic metric space it can act on properly and cocompactly. One often studies CAT(0) groups, which are groups that can act properly and cocompactly on metric spaces which are non-positively curved, in the sense that their geodesics grow apart as fast as geodesics in Euclidean space. The CAT(0) property puts a bound on the torsion in a group; a CAT(0) group has finitely many conjugacy classes of finite subgroups.

One would like to find classes of non-positively curved metric spaces which are 'QI rigid', i.e. equivalent coarse geometry implies isometric. Common examples include Tits buildings, which, from a geometric point of view, are highly symmetrical cell complexes. Classically, (spherical) buildings are viewed as simplicial complexes which realize the symmetries described by Lie groups and groups of Lie type. An abstract definition of a building, which is motivated by the discovery of twin buildings, is that of a metric space whose metric, instead of taking values in a totally ordered abelian group (e.g. $\mathbb{R}$), takes values in a Coxeter group W ordered by the Bruhat order. Just as $\mathbb{R}$ is trivially a metric space, W is trivially a building called an apartment. One can construct a building by gluing together lots of apartments in a compatible way. If W is a lattice in either Euclidean or hyperbolic space, then apartments can be realized as tessellations of either Euclidean or hyperbolic space. The result of gluing together apartments is then a piecewise Euclidean or hyperbolic polyhedral complex. More generally, one realizes apartments as so-called Davis complexes, producing the

CAT(0) geometric realization of a building (see [Dav94].

The theory of lattices in Lie groups was extended by Bruhat-Tits, Ihara, Serre and others to algebraic groups over discretely valued fields by equipping such a group with an action on its associated Bruhat-Tits building. Similarly, lattices in Kac-Moody groups have been studied by constructing actions on Kac-Moody buildings (see [CG03], [RR06]). In general, the automorphism group of a locally finite building is a locally compact group, which, assuming the group is non-discrete, has a non-trivial theory of lattices (see [FHT11]). Therefore in the study of lattices in locally compact groups, the automorphism groups of buildings are natural examples to study after Lie groups and algebraic groups. Indeed, many aspects of lattice theory for algebraic groups have been extended to locally finite trees, which are the non-classical buildings of type $W = \widetilde{A}_1$ generalizing the Bruhat-Tits building for SL_2 (see the remarkably rich theory of tree lattices by Bass-Lubotzky [BL01]). One would like to generalize the theory of tree lattices to higher dimensional buildings, however few constructions of lattices in non-classical buildings are known (see e.g. [CMSZ93], [Bou00], [HP03], [Tho07], [NTV16]).

Important in the work of Bass-Lubotzky is the theory of graphs of groups, which are the 'stacky' quotients trees (see [Ser80], [Bas93], [Noo05]). However higher dimensional graphs of groups, called complexes of groups (see [Hae91]), do not take advantage of the combinatorial aspects of building. In this thesis, we develop a theory of quotients of buildings which makes use of the combinatorial W-metric structure enjoyed by buildings. We then use of this theory to construct and classify certain Singer lattices, which are lattices that act regularly (simply-transitively) on the panels of a building.

A Coxeter group W is a group which admits a certain geometric description as a group generated by a set of reflections $S \subset W$. Each Coxeter group W acts on an associated simplicial complex, called the Coxeter complex of W. The generators S and their conjugates act by reflections in 'walls' of the complex, and the action is regular on maximal simplices. In addition, each Coxeter group W acts geometrically on an associated CAT(0) regular cell complex (regular in the sense of [Bjö84]), called the Davis complex of W. However, unlike the Coxeter complex, the walls of the Davis complex cut through cells. For example, the Davis complex of the Coxeter group $\widetilde{A}_2$ is a tessellation of the Euclidean plane by hexagons, whose fundamental domain, called the *Davis chamber*, is an equilateral triangle.

A building Δ of type W was originally defined by Tits to be a simplicial complex which can be expressed as a union of copies of the Coxeter complex of W in a way which satisfies certain axioms (see [Tit74] and [AB08, Chapter 4]). In this simplicial approach, *chambers* are maximal simplices, *galleries* are sequences of adjacent maximal simplices, *apartments* are sub-complexes which are isomorphic to the Coxeter complex of W, and *residues* are links. If W is the dihedral group of order $2m$, then a building Δ of type W is exactly the same structure as a generalized m-gon Π by taking Δ to be the incidence graph of Π. In particular, if W is the dihedral group of order 6, then a building of type W is exactly the same structure as a projective plane.

Beginning with [Tit81], the notion of a building was abstracted. A building of type W is equivalently a so-called W-metric space $(\mathcal{C}, \delta)$, which is a set of points $\mathcal{C}$ equipped with a W-distance function $\delta : \mathcal{C} \times \mathcal{C} \to W$ which satisfies certain axioms (see [Tit92] and [AB08, Chapter 5]). In the W-metric approach, *chambers* are the points of the set $\mathcal{C}$, *galleries* are sequences of points such that the distance between consecutive points is a generator $s \in S \subset W$, *apartments* are subspaces 'isometric' to W, and *residues* are 'balls', i.e. subspaces of the form $\{C \in \mathcal{C} : \delta(C, D) \in W_J\}$, where $D \in \mathcal{C}$ is any chamber, and W_J is the subgroup of W generated by a subset $J \subseteq S$. The maximal simplices of the Coxeter complex of W come with a canonical W-metric, which is isometric to W. Simplicial buildings are obtained from W-metric spaces by modeling the apartments as the Coxeter complex of W. The rather obscure axioms of a simplicial building ensure that the W-metrics on each apartment induce a global W-metric. One can reasonably define geodesics in W-metric spaces to be so-called 'galleries of reduced type', which connect each pair of chambers. A consequence of geodesics is that the metric can be recovered from a structure associated to the W-metric space, called a chamber system, which only remembers when chambers are separated by distance a generator.

Every building of type W has a geometric realization which is a CAT(0) cell complex, obtained by modeling the apartments as the Davis complex of W (see [Dav94] and [AB08, Chapter 12]). In particular, the chambers are modeled as Davis chambers. Although this cell complex is not regular in general, if the Davis chamber of a Coxeter group W is a polytope, then the Davis realization of a building of type W is a (regular) polyhedral complex; for irreducible Euclidean buildings, Davis chambers are Euclidean simplices, and for hyperbolic buildings, Davis chambers are hyperbolic polytopes.

By the 'quotient' of a building, we mean a structure associated to the type-preserving action of a group on a building which is obtained by identifying chambers in the same orbit, and from which one can recover the action. The theory of complexes of groups is the theory of quotients of cell complexes (see [BH99, Chapter III.$\mathcal{C}$]). Since buildings are naturally cell complexes, either as a simplicial complex or by taking the Davis realization, complexes of groups can be used to model the quotients of buildings (e.g. [Bou97], [NTV17]). If one uses the Davis realization, and restricts to the action of torsion-free groups, then the corresponding complexes of groups have trivial local groups, and so cell complexes are sufficient to model quotients (e.g. [CMSZ93], [Vdo02]). In [Tit85] and [Tit86], Tits introduced a way of constructing buildings by amalgamating groups. Tits' amalgams model the quotients of buildings by chamber-transitive actions. From a modern point of view, Tits' amalgams are also complexes of groups (see [GP01]).

In this thesis we introduce the notion of a Weyl graph, which is an edge labeled quiver together with a rule for composing adjacent edges of the same label and a collection of relations between edges of different labels. We develop the theory of Weyl graphs and show that they are naturally the quotients of buildings by type-preserving and chamber-free group actions. We develop their covering theory, allowing for the construction of buildings by taking the universal cover of a Weyl graph. We show that one can associate to any type-preserving and chamber-free group action on a building its quotient Weyl graph, from which one can recover the group action.

The theory of Weyl graphs further develops Tits' local approach to buildings in [Tit81] by going 'beyond' 2-residues. Tits' notion of a chamber system is an indexed family of equivalence relations on a set of chambers. A chamber system of type M is a chamber system whose generalized polygons are buildings. By modeling certain quotients of buildings as chamber systems of type M, covering theory of buildings was developed by Tits in [Tit81], described by Kantor in [Kan86], and by Ronan in [Ron89] and [Ron92]. The theory of Weyl graphs reduces to the theory of chamber systems of type M if one assumes that coverings are injective on 2-residues.

By a generalization of Tits' local-to-global result concerning spherical 3-residues (see [Tit81, Corollary 3]), Weyl graphs can be constructed by amalgamating quotients of generalized polygons by flag-free group actions. Thus, one obtains a powerful way of constructing lattices in buildings. Weyl graphs generalize chamber system of type M by allowing 2-residues to be quotients of generalized polygons.

There are two main advantages of the Weyl graph approach. Firstly, the theory of Weyl graphs is tailored to buildings, unlike the theory of complexes of groups. Secondly, we take advantage of the fact that covering theory can be 'classical' (i.e. 'non-orbi') if we make the assumption that actions are chamber-free. Modeling buildings as CAT(0) cell complexes will force an 'orbi' approach whenever spherical residues have non-trivial isotropy. Weyl graphs also provide a framework in which quotients of generalized polygons can be glued together to form (quotients of) buildings. This notion of gluing is encapsulated in the defining graph of a Weyl graph (see Section 3.1.5).

In our final chapter, we present an application of the theory of Weyl graphs to so-called Singer lattices. A Singer lattice is a discrete subgroup of the locally compact automorphism group of a locally finite building whose associated quotient is finite, and which acts regularly on panels. A Singer cyclic lattice is a Singer lattice whose isotropy of spherical 2-residues is cyclic. We construct the Singer cyclic lattices of type M, where $m_{st} \in \{2, 3, \infty\}$ for all $s, t \in S$, and the defining graph of M is connected. We achieve this by first describing the 2-residues which can exist in the quotient of a Singer cyclic lattice of type M, and then determining all the ways in which these 2-residues can be glued together. Our lattices generalize those in [Ess13], in which the Singer cyclic lattices of type $\widetilde{A}_2$ are constructed using complexes of groups. We obtain simple presentations of these lattices which are, roughly speaking, free products of collections of Singer cycles, quotient out a set of relations which are read off the defining graph of M by going around cycles (see Theorem 5.11).

This thesis is structured as follows: Chapter 2 contains preliminary material, Chapter 3 and Chapter 4 develop the theory of Weyl graphs, and Chapter 5 features an application of the theory of Weyl graphs. Finally, Appendix A and Appendix B contain some extra material on Coxeter groups, in particular the solution to the word problem and the Bruhat order. The definitions used in the appendix are introduced in the sections up to Section 3.2.1. We only begin to make use of the results in the appendix in Section 3.2.2.

Chapter 2

Preliminaries

This chapter contains the basic definitions and results which are relevant to this thesis. We define graphs and galleries, Coxeter groups, and groupoids. We also develop covering theory of groupoids which does not use base-points, but instead represents coverings with so-called outer embeddings of groups.

2.1 Graphs and Galleries

In this section, we describe the notation and terminology we will be using for graphs. We also introduce our notion of a gallery.

2.1.1 Graphs

Definition of Graphs. For us, the term 'graph' will refer to a directed graph, possibly with loops and multiple edges. Formally, we define a **graph** $\Gamma = (\Gamma_0, \Gamma_1)$ to be a set of **vertices** Γ_0 together with a set of **edges** Γ_1 which is equipped with a function,

$$\Gamma_1 \to \Gamma_0 \times \Gamma_0, \quad i \mapsto (\iota(i), \tau(i)).$$

We call $\iota(i)$ and $\tau(i)$ the **extremities** of i. In particular, we call $\iota(i)$ the **initial vertex** of i, and we call $\tau(i)$ the **terminal vertex** of i. For an edge $i \in \Gamma_1$, we denote the ordered pair $(\iota(i), \tau(i))$ by e_i. A **loop** is an edge $i \in \Gamma_1$ with $\iota(i) = \tau(i)$, and **multiple edges** are edges $i, i' \in \Gamma_1$ with $i \neq i'$ and $e_i = e_{i'}$. In graphs without multiple edges, $i \in \Gamma_1$ can be identified with e_i.

Provided there is no ambiguity, we let Γ denote both Γ_0 and Γ_1. For example, we may speak of an edge $i \in \Gamma$.

Morphisms of Graphs. Let $\Gamma = (\Gamma_0, \Gamma_1)$ and $\Gamma' = (\Gamma'_0, \Gamma'_1)$ be graphs. A **morphism** of graphs $\omega : \Gamma \to \Gamma'$ is a pair $\omega = (\omega_0, \omega_1)$ consisting of a function of the vertices $\omega_0 : \Gamma_0 \to \Gamma'_0$, and an auxiliary function of the edges $\omega_1 : \Gamma_1 \to \Gamma'_1$ which preserves the extremities of Γ. Thus, for all $i \in \Gamma_1$, we have,

$$\omega_0(\iota(i)) = \iota(\omega_1(i)), \quad \omega_0(\tau(i)) = \tau(\omega_1(i)).$$

Provided there is no ambiguity, we let ω denote both ω_0 and ω_1. For example, given an edge $i \in \Gamma$, we may denote $\omega_1(i)$ by $\omega(i)$. Let $\omega : \Gamma \to \Gamma'$ and $\omega' : \Gamma' \to \Gamma''$ be morphisms of graphs. The **composition** $\omega' \circ \omega : \Gamma \to \Gamma''$ of ω with ω' is the morphism of graphs whose function of the edges is $\omega'_0 \circ \omega_0$, and whose auxiliary function of the edges is $\omega'_1 \circ \omega_1$.

Definition of Labeled Graphs. Let S be a set of labels. A **graph labeled over** S is a graph $\Gamma = (\Gamma_0, \Gamma_1)$ which is equipped with a **type function** on its edges into S,

$$\Gamma_1 \to S, \quad i \mapsto \upsilon(i).$$

The label $\upsilon(i) \in S$ is called the **type** of i. We call a labeled graph **slim** if $e_i = e_{i'}$ implies that $\upsilon(i) \neq \upsilon(i')$.

Morphisms of Labeled Graphs. Let Γ and Γ' be graphs labeled over S and S' respectively, and let $\sigma : S \to S'$ be a function of sets. A **morphism** $\omega : \Gamma \to \Gamma'$ of labeled graphs over σ is a morphism of the underlying unlabeled graphs such that for all edges $i \in \Gamma$, we have,

$$\upsilon(\omega(i)) = \sigma(\upsilon(i)).$$

Let $\omega : \Gamma \to \Gamma'$ and $\omega' : \Gamma' \to \Gamma''$ be morphisms of labeled graphs over σ and σ' respectively. The **composition** $\omega' \circ \omega : \Gamma \to \Gamma''$ of ω with ω' is just their composition as morphisms of graphs. Then $\omega' \circ \omega$ is a morphism of labeled graphs over $\sigma' \circ \sigma$. If Γ and Γ' are labeled over the same set S, then we assume that a morphism $\omega : \Gamma \to \Gamma'$ takes place over the identity $S \to S$. In this case, a morphism of labeled graphs is just an ordinary morphism of graphs which preserves the type function.

Morphisms of Slim Graphs. Let $\omega : \Gamma \to \Gamma'$ be a morphism of labeled graphs over σ. If Γ' is slim, then ω_1 is uniquely determined by ω_0. Therefore in this case, a morphism of labeled graphs is equivalently a single function between the vertices $\omega : \Gamma_0 \to \Gamma'_0$ such that for each edge $i \in \Gamma$, there exists an edge $i' \in \Gamma'$ such that,

$$\iota(i') = \omega(\iota(i)), \quad \tau(i') = \omega(\tau(i)), \quad \upsilon(i') = \sigma(\upsilon(i)).$$

2.1.2 Galleries

Roughly speaking, galleries are paths in labeled graphs which are determined by a sequence of adjacent edges. It will be useful to model galleries as certain morphisms.

Definition of Galleries. Let $a, b \in \mathbb{Z}$ be integers with $a \leq b$. The **finite line** $\lfloor a, b \rfloor$ is the graph whose set of vertices is the interval $[a, b] \subset \mathbb{Z}$, with a single edge i_k traveling from $k-1$ to k for each $k \in \{a+1, \ldots, b\}$. Let Γ be a graph labeled over S, and let $\lfloor a, b \rfloor$ be a finite line also labeled over S. A **gallery** β in Γ is a labeled graph morphism $\beta : \lfloor a, b \rfloor \to \Gamma$.

For a gallery $\beta : \lfloor a, b \rfloor \to \Gamma$, we denote by $\iota(\beta)$ the vertex $\beta(a)$, and we denote by $\tau(\beta)$ the vertex $\beta(b)$. We call the vertices $\iota(\beta)$ and $\tau(\beta)$ the **extremities** of β. In particular, we call $\iota(\beta)$ the **initial vertex** of β, and we call $\tau(\beta)$ the **terminal vertex** of β. The **length** $|\beta|$ of β is the number of edges of $\lfloor a, b \rfloor$. Putting $s_k = \upsilon(i_k)$ for $k \in \{a+1, \ldots, b\}$, then the **type** β_S of β is the sequence of labels,

$$\beta_S = s_{a+1}, \ldots, s_b.$$

A **trivial** gallery is a gallery β with $|\beta| = 0$, in which case β_S is empty. A **cycle** is a gallery β with $\iota(\beta) = \tau(\beta)$. A **minimal gallery** is a gallery β whose length is minimal amongst all the galleries from $\iota(\beta)$ to $\tau(\beta)$. The **sequence of edges of** β is the sequence of edges,

$$\beta(i_{a+1}), \ldots, \beta(i_b).$$

Conversely, a finite sequence of edges $j_1, \ldots, j_n$ of Γ such that $\tau(j_k) = \iota(j_{k+1})$ for all $k \in \{1, \ldots, n-1\}$ determines a gallery $\beta : \lfloor 0, n \rfloor \to \Gamma$ by putting $\beta(i_k) = j_k$.

Remark 2.1. Let $x, y \in \Gamma$ be vertices in a labeled graph. We have defined galleries such that there may be a gallery from x to y, but no gallery from y to x. However, in the graphs which will concern us, each directed edge i will have an associated inverse, which has the same extremities as i, but points in the opposite direction. Thus, a gallery from x to y will naturally induce an inverse gallery from y to x.

A **subgallery** of a gallery β is a gallery whose sequence of edges is a consecutive subsequence of the sequence of edges of β.

Let β and β' be galleries in Γ such that $\tau(\beta) = \iota(\beta')$. The **concatenation** $\beta\beta'$ of β with β' is a gallery whose sequence of edges is the sequence of edges of β followed by the sequence of edges of β'.

2.2 Coxeter Groups

Most of the material of this section is standard, although our terminology concerning homotopy of words is slightly different to that adopted by many authors (see Section 2.2.3). Some references for the material of this section are [Bou02], [Dav08], and [BB06].

2.2.1 Groups Generated by Involutions

A **marked group** (G, S) is a group G which is equipped with a choice of a finite generating set $S \subseteq G$. A **group generated by involutions** (W, S) is a marked group whose generators $S \subseteq W$ are involutions; thus $s^2 = 1$ for all $s \in S$. By abuse of notation, we let W denote both (W, S) and the underlying group of (W, S); the meaning will always be clear from the context.

Recall that the **order** of a group element $g \in G$ is the smallest positive integer n such that $g^n = 1$. For (W, S) a group generated by involutions and $s, t \in S$, we denote by m_{st} the order of $w(st) \in W$. In particular, $m_{ss} = 1$ for all $s \in S$.

Words over S. A **word** $f = s_1 \dots s_n$ over S is a finite sequence of elements of S. The **length** $|f|$ of a word f is just the length of f as a sequence. If $f = s_1 \dots s_n$, then we denote by f^{-1} the word $s_n \dots s_1$. A **subword** of a word f is a consecutive subsequence of f. A **substring** of a word f is a (perhaps non-consecutive) subsequence of f. For example, if $f = sttustu$, then $sttus$ and $tust$ are subwords, whereas $stsu$ is the substring obtained by skipping every second letter.

Equivalence of Words and Decompositions. Let $f = s_1 \dots s_n$ and $f' = s'_1 \dots s'_n$ be words over S. The **concatenation** ff' of f with f' is the word,

$$ff' = s_1 \dots s_n s'_1 \dots s'_n.$$

Let $M(S)$ denote the free monoid on S. Thus, $M(S)$ is the set of words over S equipped with the binary operation of the concatenation of words. Let $w : M(S) \to W$ be the unique monoid homomorphism such that $s \mapsto s$. In general the word,

$$f = s_1 \dots s_n \in M(S)$$

is mapped to the product,

$$w(f) = s_1 \dots s_n \in W.$$

We say that two words f and f' are **equivalent** if $w(f) = w(f')$. Notice that the map w is surjective because S is a generating set of involutions. When there is no risk of ambiguity, we may identify f with $w(f)$.

The word f is called a **decomposition** of $w(f)$. We call a word f **reduced** if there are no words equivalent to f which have a strictly shorter length. If f is reduced, then f is called a **reduced decomposition** of $w(f)$.

Word Length. For $w \in W$, the **word length** $|w|$ of w is length of the reduced decomposition(s) of w. It is straight forward to check that for all $w, w' \in W$, we have:

(i) $|w| = 0$ if and only if $w = 1$

(ii) $|w| = |w^{-1}|$

(iii) $|ww'| \leq |w| + |w'|$.

It then follows that the function,

$$d : W \times W \to \mathbb{Z}, \quad d(w, w') \mapsto |w^{-1}w'|$$

is a left-invariant metric on W, called the **word metric** of W.

Cayley Graphs. Let $W = (W, S)$ be a group generated by involutions. The **Cayley graph** of W is the graph labeled over S whose set of vertices is W, whose set of edges is $W \times S$, with,

$$\iota(w, s) = w, \quad \tau(w, s) = ws, \quad \upsilon(w, s) = s.$$

Notice that $\mathcal{C}(W)$ is an example of a graph whose edges have a natural structure of inverses, mentioned in Remark 2.1, since we can put $(w, s)^{-1} = (ws, s)$. This relies on the fact that S is a set of involutions.

We conclude this section with two easy observations:

Proposition 2.1. Let (W, S) be a group generated by involutions. Let $s_1 \dots s_n$ be a word over S which is a decomposition of $w \in W$. Then $w^{-1} = w(s_n \dots s_1)$.

Proof. If $w = s_1 \dots s_n$ as a product in W, then $w^{-1} = s_n^{-1} \dots s_1^{-1} = s_n \dots s_1$. □

Proposition 2.2. Let (W, S) be a group generated by involutions. For all $s, t \in S$, the order of $w(st)$ is equal to the order of $w(ts)$, i.e. $m_{st} = m_{ts}$.

Proof. We have,

$$w(st)^n = 1 \iff (w(st)^{-1})^n = 1 \iff w(ts)^n = 1$$

where the second 'if and only if' follows from Proposition 2.1. □

2.2.2 Coxeter Groups

In this section, we introduce Coxeter groups. There are several ways to characterize Coxeter groups amongst groups generated by involutions. The following characterization can be found in [Bou02, p. 4].

Definition of Coxeter Groups. A **Coxeter group** $W = (W, S)$ is a group generated by involutions such that for any group G and any function $F : S \to G$ such that $(F(s)F(t))^{m_{st}} = 1$ for all $s, t \in S$, then F extends to a unique homomorphism $\bar{F} : W \to G$. It follows that,

$$W = \big\langle S \mid (st)^{m_{st}} = 1 : s, t \in S \big\rangle.$$

We call this presentation the **canonical presentation** of W. The **rank** of a Coxeter group (W, S) is the cardinality of S. The data of a Coxeter group, together with its choice of generators, is sometimes called a 'Coxeter system'. This term is redundant for us since by 'Coxeter group' we mean a certain marked group.

Standard Subgroups. The marked subgroups of a Coxeter group (W, S) which are generated by subsets $J \subseteq S$ of the generators are called **standard subgroups**. We denote by $W_J = (W_J, J)$ the standard subgroup which is generated by J. We will see that standard subgroups W_J are themselves Coxeter groups in a natural way. A subset $J \subseteq S$ is called **spherical** if W_J is a finite group.

Definition of Coxeter Matrices. A **Coxeter matrix** M on a set S is a symmetric matrix,

$$M : S \times S \to \mathbb{Z}_{\geq 1} \cup \{\infty\}, \quad (s,t) \mapsto M_{st}$$

such that $M_{ss} = 1$ for all $s \in S$, and $M_{st} \neq 1$ for all distinct $s, t \in S$. If M is a Coxeter matrix on S, and $J \subseteq S$, then we denote by M_J the Coxeter matrix which is the restriction of M to $J \times J$. A Coxeter matrix M is called **universal** if $M_{st} = \infty$ for all distinct $s, t \in S$.

The Defining Graph of a Coxeter Matrix. A **simplicial graph** $L = (V(L), E(L))$ is an undirected graph without loops or multiple edges in which the **edges** $E(L)$ are modeled as 2-element subsets of the **vertices** $V(L)$. Let M be a Coxeter matrix. The **defining graph** $L = L(M)$ of M is the edge labeled simplicial graph with,

$$V(L) = S, \qquad E(L) = \big\{\{s,t\} : s, t \in S,\ s \neq t,\ m_{st} < \infty\big\}$$

where the edge $\{s,t\} \in E(L)$ is labeled by m_{st}. Notice that L is defined differently to the so-called Coxeter-Dynkin diagram of M.

Coxeter Groups vs Coxeter Matrices. A Coxeter group W determines a Coxeter matrix $M = M(W)$ by putting $M_{st} = m_{st}$. Conversely, given a Coxeter matrix M on S, the **Coxeter group of type** M is the group,

$$W(M) = \big\langle S \mid (st)^{M_{st}} = 1 : s, t \in S \big\rangle.$$

Notice that this is indeed a Coxeter group. So we have a map $W \mapsto M(W)$, which takes Coxeter groups to the matrices which encode their canonical presentations, and a right-inverse of this map $M \mapsto W(M)$, which takes matrices to Coxeter groups. The following result shows that $M(W(M)) = M$, and so $M \mapsto W(M)$ is also a left-inverse of $W \mapsto M(W)$.

Theorem 2.3. Let M be a Coxeter matrix on S. For all $s, t \in S$, the order of $st \in W(M)$ is equal to M_{st}, i.e. $m_{st} = M_{st}$.

The following proof is taken from [Ron89, (2.1) Lemma (i)]:

Proof. We construct a linear representation of $W(M)$. Let X be a vector space over $\mathbb{R}$ with the basis $(x_s)_{s \in S}$. We define a symmetric bilinear form $(-,-)$ on X by putting,

$$(x_s, x_t) = -\cos(\pi / M_{st}), \qquad \text{for all } s, t \in S.$$

Let $s \in S$ act on X by the involution,

$$x \mapsto x - 2(x, x_s), \quad \text{for all } x \in X.$$

Let $X_{st} = \text{span}\{x_s, x_t\} \subseteq X$. Then st acts with order M_{st} on X_{st}, and st acts trivially on $X_{st}^{\perp}$. Hence, st has order M_{st} on X, and so st must have order M_{st} in $W(M)$. □

Thus, Coxeter groups are essentially in bijection with Coxeter matrices (up to suitable notions of equivalence). However, there exist inequivalent Coxeter groups with isomorphic underlying groups.

From now on, given a Coxeter matrix M, we denote M_{st} by m_{st}.

We will need the following for the proof of an important result in the appendix:

Theorem 2.4. Let M be a Coxeter matrix on S, and let $J \subseteq S$. For each $s \in S$, if $s \in W_J$, then $s \in J$.

The following proof is taken from [Ron89, (2.1) Lemma (ii)]:

Proof. Let W act on the real vector space X with the basis $(x_s)_{s \in S}$ as in Theorem 2.3. Let $\varphi : W \to GL(X)$ be the corresponding homomorphism. Let $X_J \subseteq X$ denote the span of $\{x_t : t \in J\}$. If $s \in W_J$, then $\varphi(s) \in \varphi(W_J)$. Therefore $s \cdot x \in x + X_J$ for all $x \in X$. In particular $-x_t = s \cdot x_t \in x_t + X_J$, and so $x_t \in X_J$. Thus, $s \in J$. □

Morphisms of Coxeter Matrices. Let M and M' be Coxeter matrices on S and S' respectively. A **morphism** of Coxeter matrices $\sigma : M \to M'$ is a function $\sigma : S \to S'$ such that $m_{\sigma(s)\sigma(t)}$ is a factor of m_{st} for all $s, t \in S$. This is exactly the property that $\sigma : S \to S'$ extends to a homomorphism $\overline{\sigma} : W(M) \to W(M')$. An **embedding** of Coxeter matrices is an injective function $\sigma : S \hookrightarrow S'$ such that $m_{\sigma(s)\sigma(t)} = m_{st}$ for all $s, t \in S$. If $J \subseteq J' \subseteq S$, then we have the natural embedding,

$$\iota_{JJ'} : M_J \hookrightarrow M_{J'}, \quad s \mapsto s.$$

We denote ι_{JS} by ι_J. The fact that the corresponding extension,

$$\bar{\iota}_{JJ'} : W(M_J) \hookrightarrow W(M_{J'})$$

is an embedding of groups will follow from some classical facts about Coxeter groups (see Proposition 2.10).

2.2.3 Homotopy of Words

Our terminology concerning homotopy of words is slightly different to that adopted by many authors. We fix some notation for this section; let $W = (W, S)$ be a Coxeter group whose associated Coxeter matrix is M, let f, f' and f'' be (possibly empty) words over S, and let $s, t \in S$.

Words in Coxeter Groups. Let $s \neq t$. An (s,t)**-word**, or just **alternating word**, is a word over $\{s,t\}$ which begins with the letter s, and contains no consecutive letters. For $n \in \mathbb{Z}_{\geq 0}$, we denote by $p_n(s,t)$ the unique (s,t)-word which has length n. For example, we have $p_5(s,t) = ststs$ and $p_1(t,s) = t$. If $m_{st} < \infty$, we denote by $p(s,t)$ the word $p_{m_{st}}(s,t)$; thus,

$$p(s,t) = \underbrace{stst\ldots}_{m_{st}}.$$

Notice that in W, the element $w(p(s,t))$ is an involution. Therefore, by Proposition 2.1, we have $w(p(s,t)) = w(p(t,s))$. We denote by $p^{-1}(s,t)$ the word obtained from $p(s,t)$ by reversing the order. For example, if $m_{st} = 4$, then,

$$p(s,t) = stst, \quad p^{-1}(s,t) = tsts.$$

In particular, if m_{st} is odd, then $p(s,t) = p^{-1}(t,s)$. Note that authors such as Tits and Ronan take $p^{-1}(s,t)$ as their definition of $p(s,t)$.

Contractions and Expansions. A **contraction** is an alteration from a word of the form $fssf'$ to the word ff'. An **expansion** is the inverse of a contraction, that is an alteration from a word of the form ff' to the word $fssf'$. Notice that contractions and expansions of a word f produce words which are equivalent to f.

Strict Homotopy of Words. An **elementary strict homotopy** is an alteration from a word of the form $fp(s,t)f'$ to the word $fp(t,s)f'$. Since we have $w(p(s,t)) = w(p(t,s))$, an elementary strict homotopy of a word f produces a word which is equivalent to f. A **strict homotopy** is an alternation of a word which is a composition of elementary strict homotopies. If a word f can be altered via a strict homotopy to give the word $\hat{f}$, we say f is **strictly homotopic** to $\hat{f}$, and we write $f \simeq \hat{f}$. The relation '$\simeq$' is an equivalence relation on words over S.

Our notion of strict homotopy of words is what Tits and Ronan call homotopy of words in [Tit81] and [Ron89].

Example 2.1. Let $S = \{s,t,u\}$, and let (W,S) be the Coxeter group known as $\widetilde{A}_2$, i.e. $m_{st} = m_{tu} = m_{us} = 3$. The alteration $sttu \mapsto su$ is a contraction, and the alteration $stutu \mapsto sutuu$ is a strict elementary homotopy. The alteration $stusu \mapsto tstus$ is a strict homotopy since it is the composition of the strict elementary homotopy $stusu \mapsto stsus$ with the strict elementary homotopy $stsus \mapsto tstus$.

Homotopy of Words. A **homotopy** of words is any composition of contractions, expansions, and elementary strict homotopies. If a word f can be altered via a homotopy to give the word $\hat{f}$, we say f is **homotopic** to $\hat{f}$, and write $f \sim \hat{f}$. The relation '$\sim$' is an equivalence relation on words over S. One can easily see that if two words are strictly homotopic, then they are homotopic. A partial converse to this holds (see Theorem 2.8).

Lemma 2.5. Let (W, S) be a Coxeter group. Two words over S are equivalent if and only if they are homotopic.

Proof. This is a straightforward consequence of what it means for a Coxeter group to have its canonical presentation. □

Corollary 2.5.1. Let (W, S) be a Coxeter group, and fix $w \in W$. Then the words over S which are decompositions of w has the same length modulo 2.

Proof. Contractions and expansions change the length of a word by ± 2. Elementary strict homotopies do not change the length of a word. Then, since a homotopy is a composition of contractions, expansions, and elementary strict homotopies, the result follows from Lemma 2.5. □

M-Reduced Words. In light of Lemma 2.5, a word f over S is reduced if and only if there are no words homotopic to f which are strictly shorter than f. We say a word f is **M-reduced** if there are no words strictly homotopic to f which are of the form $f'ssf''$. One can easily see that reduced implies M-reduced. In fact, the converse also holds (see Theorem 2.8).

2.2.4 Properties of Coxeter Groups

In this section, we collect some classical results on word manipulation in Coxeter groups. We continue to denote by $W = (W, S)$ a Coxeter group with Coxeter matrix M.

Proposition 2.6. For all $w \in W$ and $s \in S$, we have the dichotomies,

$$|ws| = |w| + 1 \quad \text{or} \quad |ws| = |w| - 1$$

and,

$$|sw| = |w| + 1 \quad \text{or} \quad |sw| = |w| - 1.$$

Proof. We have,

$$|ws| \leq |w| + |s| = |w| + 1 \quad \text{and} \quad |w| \leq |ws| + |s| = |ws| + 1$$

from the triangle inequality of word length (see Section 2.2.1). Thus,

$$|w| - 1 \leq |ws| \leq |w| + 1.$$

We cannot have $|ws| = |w|$ by Corollary 2.5.1. The first dichotomy follows. The second dichotomy follows by a symmetric argument. □

Once we have defined the Bruhat order, we' will tend to think of these dichotomies in terms of the Bruhat order (see Remark 2.2). An alternative approach to the theory we will develop minimizes the roll of the Bruhat order, and refers only to the dichotomies as they are stated in Proposition 2.6.

Proposition 2.7. For all $w \in W$ and $s \in S$, if $|ws| = |w| - 1$ (alternatively $|sw| = |w| - 1$), then there exists a reduced decomposition f of w which ends (alternatively starts) with s.

Proof. Let f' be a reduced decomposition of ws. Then $|f'| = |ws| = |w| - 1$. Put $f = f's$. Firstly, f is a decomposition of w since,

$$w = (ws)s = w(f')s = w(f's) = w(f).$$

Secondly, f is reduced since,

$$|f| = |f's| = |f'| + 1 = |w|.$$

The result for when $|sw| = |w| - 1$ follows by a symmetric argument. □

We give a geometric proof of the following two theorems in Appendix A. Our approach is essentially a translation of [Ron89, Chapter 2] into the language of Cayley graphs of Coxeter groups. Such Cayley graphs are prototypical examples of pre-Weyl graphs, which are introduced in Section 3.2.1. The first theorem is a key result of Tits from [Tit69], which gives a solution to the word problem in Coxeter groups:

Theorem 2.8. (Main Theorem) For any Coxeter group W:

(MT1) M-reduced words are reduced

(MT2) homotopic reduced words are strictly homotopic.

Proof. See Section A.3. □

We also have the following classical result on word manipulation in Coxeter groups, called the **deletion condition**:

Theorem 2.9. Let (W, S) be a Coxeter group. If a word f over S is not reduced, then there exists a substring of f obtained by deleting two letters which is homotopic to f.

Proof. See Section A.3. □

Example 2.2. Let W be $\widetilde{A}_2$ with $S = \{s, t, u\}$, as in Example 2.1. Then $ustst u$ is not reduced since,

$$uststu \simeq utsttu \sim utsu.$$

However, we can obtain $utsu$ from $u\hat{s}ts\hat{t}u$ by deleting the two highlighted letters.

Notice that for a reduced word f, the statement 'fs is not reduced' is equivalent to the statement '$|w(fs)| = |w| - 1$'. Thus, the following consequence of the deletion condition, called the **exchange condition**, can be viewed as a strengthening of Proposition 2.7 above:

Corollary 2.9.1. Let f be a reduced word. If fs (alternatively sf) is not reduced, then there exists a substring f' of f, obtained by deleting one letter, which is homotopic to fs (alternatively sf).

Proof. By the deletion condition, fs is homotopic to a word f' which is obtained from fs by deleting two of its letters. Suppose neither of these letters is the last letter. Then $f's$ is homotopic to a word of length less than f by a contraction, since the last two letters of $f's$ are s. Since $f's \sim f$, this contradicts the fact that f is reduced. Thus, exactly one letter of f is deleted to obtain f'. The case where fs is not reduced follows by a symmetric argument. □

In fact, Tits proved more. The deletion condition and exchange condition have obvious generalizations to groups generated by involutions. Tits proved that if W is a group generated by involutions, then the property of being a Coxeter group, the deletion condition, and the exchange condition are all equivalent.

Corollary 2.9.2. Let f be a word, and let f' and f'' be reduced words. If $f'f \sim f''f$ (or $ff' \sim ff''$), then $f' \simeq f''$.

Proof. We have $w(f'f) = w(f''f)$, so $w(f')w(f) = w(f'')w(f)$, which implies that $w(f') = w(f'')$. Thus $f' \sim f''$, and so $f' \simeq f''$ by (MT2). The result for when $ff' \sim ff''$ follows by a symmetric argument. □

Proposition 2.10. Let $W = (W, S)$ be a Coxeter group with Coxeter matrix M, and let $J \subseteq S$. The extension $\bar{\iota}_J : W(M_J) \to W$ of the embedding $\iota_J : M_J \hookrightarrow M$ is an embedding of groups. In particular, the standard subgroup $W_J \leq W$ is naturally the Coxeter group $W(M_J)$.

Proof. Let f and f' be words over J which are homotopic with respect to M. It suffices to show that this homotopy is a composition of contractions, expansions, and elementary strict homotopies between words over J, since this shows that f and f' are also homotopic with respect to M_J. The result then follows by Lemma 2.5.

Indeed, (MT1) tells us we can homotope f and f' to reduced words, say $\hat{f}$ and $\hat{f}'$ respectively, using only strict homotopies and contractions. Then, by (MT2), we have $\hat{f} \simeq \hat{f}'$. □

Corollary 2.10.1. Let M be a Coxeter matrix on S, and let $J \subseteq J' \subseteq S$. Then,

$$\bar{\iota}_{JJ'} : W(M_J) \to W(M'_J)$$

is an embedding of groups.

2.2.5 The Bruhat Order

We now briefly describe the Bruhat order, which is a way of ordering the elements of a Coxeter group. We treat the Bruhat order in more detail in Appendix B. A good reference for the Bruhat order is [BB06].

To give a quick definition of the Bruhat order, we need the following:

Proposition 2.11. Let W be a Coxeter group, and let $w, w' \in W$. If a decomposition of w' is a substring of a reduced decomposition of w, then every reduced[1] decomposition of w contains a substring which is a decomposition of w'.

Proof. See Proposition B.5. □

Definition of the Bruhat Order. Let W be a Coxeter group. The **Bruhat order** of W is the binary relation '$\leq$' on the elements of W such that $w' \leq w$ if a reduced decomposition of w contains a substring which is a decomposition of w'. By Proposition 2.11, we have $w' \leq w$ if and only if every reduced decomposition of w contains a substring which is a decomposition of w'.

In general, we have $1 < s$, for $s \in S$. By multiplying both sides of the inequality by s on the left or right, we see that the Bruhat order is neither left- nor right-invariant in any non-trivial Coxeter group.

Proposition 2.12. Let W be a Coxeter group, and let '$\leq$' be the Bruhat order on W. Then '$\leq$' is a partial ordering of the elements of W.

Proof. Irreflexivity is clear. For antisymmetry, suppose that $w' \leq w$ and $w \leq w'$. Let f be a reduced decomposition of w. By hypothesis (and the deletion condition), there exists a substring f' of f which is a reduced decomposition of w'. Similarly, there exists a substring of f' which is a reduced decomposition of w. But $|w| = |f|$, therefore $f' = f$, and so $w = w'$. For transitivity, suppose that $w'' \leq w' \leq w$. Let f be a reduced decomposition of w, let f' be a substring of f which is a reduced decomposition of w' (here we use the deletion condition), and let f'' be a substring of f' which is a decomposition of w''. Then f'' is a substring of f, and so $w'' \leq w$. □

Remark 2.2. Let $w \in W$ and $s \in S$. Notice that $ws > w$ if and only if $|ws| = |w|+1$, and, by the exchange condition, $ws < w$ if and only if $|ws| = |w| - 1$. Similarly, $sw > w$ if and only if $|sw| = |w| + 1$, and $sw < w$ if and only if $|sw| = |w| - 1$.

Example 2.3. Let W be $\widetilde{A}_2$ with $S = \{s, t, u\}$, as in Example 2.1. If $w = stus$, then $ws < w$ and $sw < w$. If $w = tus$, then $ws < w$ and $sw > w$. If $w = tut$, then $ws > w$ and $sw > w$.

[1] by the deletion condition, we can remove the word 'reduced' here without changing the statement of the proposition

2.3 Groupoids

In this section, we introduce groupoids and collect some of their basic properties. We show that the local groups of a connected groupoid are naturally 'outer isomorphic', we define the fundamental group of a connected groupoid in a way which does not require the choice of a base-point, and we associate to every homomorphism of connected groupoids an 'outer homomorphism' of their fundamental groups.

2.3.1 Introducing Groupoids

Recall from Section 2.1.1 that by 'graph' we mean a directed multigraph, i.e. a quiver.

Definition of Groupoids. A **groupoid** $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ is a non-empty graph, with vertices $\mathcal{G}_0$ and edges $\mathcal{G}_1$, which is equipped with the following additional data:

(1) a function $\mathrm{id} : \mathcal{G}_0 \to \mathcal{G}_1$, which assigns to each vertex $x \in \mathcal{G}_0$ the **identity edge** 1_x of x

(2) a function $\mathrm{inv} : \mathcal{G}_1 \to \mathcal{G}_1$, which assigns to each edge $g \in \mathcal{G}_1$ the **inverse edge** g^{-1} of g

(3) a partial function $\mathcal{G}_1 \times \mathcal{G}_1 \to \mathcal{G}_1$, which assigns to each pair of edges (g, h) such that $\tau(g) = \iota(h)$, their **composition**, which is denoted either by gh, or by $g; h$ to avoid ambiguity

which satisfies the following compatibility:

(i) for all $g, h \in \mathcal{G}_1$ such that gh is defined, we have $\iota(gh) = \iota(g)$ and $\tau(gh) = \tau(h)$

(ii) for all $x \in \mathcal{G}_0$, we have $\iota(1_x) = x$ and $\tau(1_x) = x$

(iii) for all $g, h, k \in \mathcal{G}_1$ such that $\tau(g) = \iota(h)$ and $\tau(h) = \iota(k)$, we have $g(hk) = (gh)k$

(iv) for all $g \in \mathcal{G}_1$, if $\iota(g) = x$ and $\tau(g) = y$, then $1_x g = g = g 1_y$

(v) for all $g \in \mathcal{G}_1$, we have $gg^{-1} = 1_{\iota(g)}$ and $g^{-1}g = 1_{\tau(g)}$.

The function $\mathrm{id} : \mathcal{G}_0 \to \mathcal{G}_1$ is injective since if $x, y \in \mathcal{G}_0$ and $1_x = 1_y$, then $x = \iota(1_x) = \iota(1_y) = y$. We call an edge **trivial** if it is an identity edge, and **non-trivial** otherwise. Often, we will let 1 denote an arbitrary trivial edge, that is $1 = 1_x$ for some $x \in \mathcal{G}_0$.

For vertices $x, y \in \mathcal{G}$, we denote by $\mathcal{G}(x, y)$ the set of edges $i \in \mathcal{G}$ such that $\iota(i) = x$ and $\tau(i) = y$. Since $(g^{-1})^{-1} = g$ for all $g \in \mathcal{G}_1$, the function,

$$\mathcal{G}(x, y) \to \mathcal{G}(y, x), \quad g \mapsto g^{-1}$$

is a bijection. The **local group** $\mathcal{G}_x$ at x is the group whose set of elements is $\mathcal{G}(x,x)$, and whose binary operation is the restriction of the composition of $\mathcal{G}$.

For a vertex $x \in \mathcal{G}$, we denote by $\mathcal{G}(x,-)$ the set of edges $i \in \mathcal{G}$ such that $\iota(i) = x$, and we denote by $\mathcal{G}(-,x)$ the set of edges $i \in \mathcal{G}$ such that $\tau(i) = x$. Thus,

$$\mathcal{G}(x,-) = \bigsqcup_{y \in \mathcal{G}_0} \mathcal{G}(x,y), \quad \mathcal{G}(-,x) = \bigsqcup_{y \in \mathcal{G}_0} \mathcal{G}(y,x).$$

A **setoid** is a groupoid $\mathcal{G}$ such that $\mathcal{G}(x,y)$ contains at most one edge for all vertices $x,y \in \mathcal{G}$. A groupoid $\mathcal{G}$ is called **connected** if $\mathcal{G}(x,y)$ is non-empty for all vertices $x,y \in \mathcal{G}$. A groupoid $\mathcal{G}$ is called a **bundle of groups** if $\mathcal{G}(x,y)$ is empty for all vertices $x,y \in \mathcal{G}$ such that $x \neq y$. A groupoid is called **finite** if it has a finite number of edges.

Example 2.4 (Setoids vs Equivalence Relations)**.** The notion of a setoid is the same as that of an equivalence relation. Given a setoid $\mathcal{G}$, one can equip the vertices of $\mathcal{G}$ with the equivalence relation,

$$x \sim y \iff \mathcal{G}(x,y) \text{ is non-empty.}$$

Conversely, if $(X,\sim)$ is a set equipped with an equivalence relation, let,

$$R = \big\{(x,y) \in X \times X : x \sim y\big\}.$$

Then one can form the setoid $\mathcal{G} = (X,R)$, where,

$$\iota(x,y) = x, \quad \tau(x,y) = y, \quad (x,y);(y,z) = (x,z).$$

These two constructions are mutually inverse (up to isomorphism).

Later on, we will see that (the isomorphism classes of) connected groupoids are naturally in bijection with pairs (G,κ), where G is an isomorphism class of groups, and κ is a cardinal. We will denote the groupoid corresponding to the pair (G,κ) by $G \times \kappa$.

Subgroupoids, Subgroups and Cosets. Let $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ be a groupoid. A **subgroupoid** $\mathcal{G}' = (\mathcal{G}'_0, \mathcal{G}'_1)$ of $\mathcal{G}$ consists of a two subsets $\mathcal{G}'_0 \subseteq \mathcal{G}_0$, $\mathcal{G}'_1 \subseteq \mathcal{G}_1$, equipped with the restriction of the structure of $\mathcal{G}$, with the requirement that $\mathcal{G}'$ is itself a groupoid. A **subgroup** of a groupoid $\mathcal{G}$ is a subgroup of a local group of $\mathcal{G}$. Let $H \leq \mathcal{G}_x$ be a subgroup of $\mathcal{G}$. We let $H\backslash\mathcal{G}$ denote the set,

$$H\backslash\mathcal{G} = \big\{Hg : g \in \mathcal{G}(x,-)\big\}.$$

We call the elements of $H\backslash\mathcal{G}$ **right cosets** of $\mathcal{G}$. There is also the symmetrical notion of **left cosets** $\mathcal{G}/H$.

Homomorphisms of Groupoids. Let $\mathcal{G}$ and $\mathcal{G}'$ be groupoids. A **homomorphism** of groupoids $\varphi : \mathcal{G} \to \mathcal{G}'$ is a morphism of graphs $\varphi = (\varphi_0, \varphi_1)$, with vertex function φ_0 and edge function φ_1, which satisfies the following two properties:

(i) for all vertices $x \in \mathcal{G}$, we have,

$$\varphi_1(1_x) = 1_{\varphi_0(x)}$$

(ii) for all edges $g, h \in \mathcal{G}$ such that gh is defined, we have,

$$\varphi_1(gh) = \varphi_1(g)\varphi_1(h).$$

In order to define a homomorphism of groupoids $\varphi : \mathcal{G} \to \mathcal{G}'$, it suffices to define a function on the edges $\varphi_1 : \mathcal{G}_1 \to \mathcal{G}'_1$ such that for all edges $g, h \in \mathcal{G}$ such that gh is defined, $\varphi_1(g)\varphi_1(h)$ is defined with,

$$\varphi_1(gh) = \varphi_1(g)\varphi_1(h).$$

If we let $\varphi_0 : \mathcal{G}_0 \to \mathcal{G}'_0$ be the map which sends $x \in \mathcal{G}_0$ to the vertex of $\mathcal{G}'$ whose identity is $\varphi_1(1_x)$, then the pair $\varphi = (\varphi_0, \varphi_1)$ is a homomorphism of groupoids $\varphi : \mathcal{G} \to \mathcal{G}'$.

Let $\varphi : \mathcal{G} \to \mathcal{G}'$ and $\varphi' : \mathcal{G}' \to \mathcal{G}''$ be homomorphisms of groupoids. The **composition** $\varphi' \circ \varphi : \mathcal{G} \to \mathcal{G}''$ of φ with φ' is just their composition as morphisms of graphs. Then $\varphi' \circ \varphi : \mathcal{G} \to \mathcal{G}''$ is itself a homomorphism of groupoids.

We call a groupoid homomorphism $\varphi : \mathcal{G} \to \mathcal{G}'$ **faithful** if the restriction of φ to each local group of $\mathcal{G}$ is injective. We call a groupoid homomorphism φ an **embedding** if φ_1 is injective, **surjective** if φ_1 is surjective, and an **isomorphism** if φ_1 is bijective. It is easy to see that a groupoid homomorphism is an isomorphism if and only if it has an inverse. An **automorphism** of a groupoid $\mathcal{G}$ is an isomorphism from $\mathcal{G}$ to itself. We denote by $\mathrm{Aut}(\mathcal{G})$ the group whose elements are automorphisms of $\mathcal{G}$, and whose binary operation is the composition of homomorphisms.

Groupoidizing Graphs. Suppose we have a graph Γ which we want to become the underlying graph of a groupoid $\mathcal{G}$. When we equip Γ with the structure of a groupoid, we make the convention that every edge of Γ becomes a *non-trivial* edge in $\mathcal{G}$, so that the trivial edges of $\mathcal{G}$ are extra edges which can be identified with the vertices of Γ. For example, the graph consisting of one vertex and one loop has a unique 'groupoidization' as the groupoid with one vertex whose local group is the group of order 2.

Conjugation in Groupoids. Let $\mathcal{G}$ be a groupoid. Let $x, x', y, y' \in \mathcal{G}$ be vertices such that $\mathcal{G}(x, y)$ and $\mathcal{G}(x', y')$ are non-empty. Associated to each choice of $g \in \mathcal{G}(x, y)$ and $h \in \mathcal{G}(x', y')$ is the bijection,

$$\chi_{gh} : \mathcal{G}(x, x') \to \mathcal{G}(y, y'), \quad k \mapsto g^{-1}kh.$$

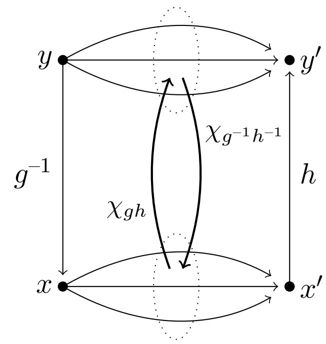

Figure 2.1: The mutually inverse maps χ_{gh} and $\chi_{g^{-1}h^{-1}}$

This is a bijection since χ_{gh} has the inverse,

$$\chi_{g^{-1}h^{-1}} : \mathcal{G}(y, y') \to \mathcal{G}(x, x'), \quad k \mapsto gkh^{-1}.$$

See Figure 2.1. If $h = g$, so that $x = x'$ and $y = y'$, then it is easy to see that χ_{gg} is an isomorphism of local groups. Let us denote χ_{gg} by χ_g. Thus,

$$\chi_g : \mathcal{G}_x \to \mathcal{G}_y, \quad k \mapsto g^{-1}kg.$$

This shows that if $\mathcal{G}(x, y)$ is non-empty, then $\mathcal{G}_x$ is isomorphic to $\mathcal{G}_y$. However, there is not a natural isomorphism from $\mathcal{G}_x$ to $\mathcal{G}_y$ in general; different choices of $g \in \mathcal{G}(x, y)$ may result in different isomorphisms χ_g.

2.3.2 Outer Homomorphisms

Let G and H be groups. Let φ be a homomorphism $\varphi : G \to H$, and let φ^h denote the conjugate of φ by h,

$$\varphi^h : G \to H, \quad g \mapsto h\varphi(g)h^{-1}.$$

An **outer homomorphism** $\Phi : G \to H$ is a conjugacy class of homomorphisms,

$$\Phi = [\varphi] = \{\varphi^h, h \in H\}.$$

In other words, an outer homomorphism is a group homomorphism defined only up to conjugacy. Outer automorphisms are classical examples of outer homomorphisms. For outer homomorphisms $\Phi : G \to H$ and $\Phi' : H \to K$, we put,

$$\Phi' \circ \Phi = [\varphi'] \circ [\varphi] = [\varphi' \circ \varphi].$$

This is well defined since $\varphi^k \circ \varphi^h = (\varphi \circ \varphi)^{k\varphi(h)}$. If $\psi : G \to H$ is an isomorphism of groups, we denote $[\psi^{-1}]$ by Ψ^{-1}. Let 1_G and 1_H be the identity homomorphisms of G and H respectively. We have,

$$\Psi \circ \Psi^{-1} = [1_G], \quad \Psi^{-1} \circ \Psi = [1_H].$$

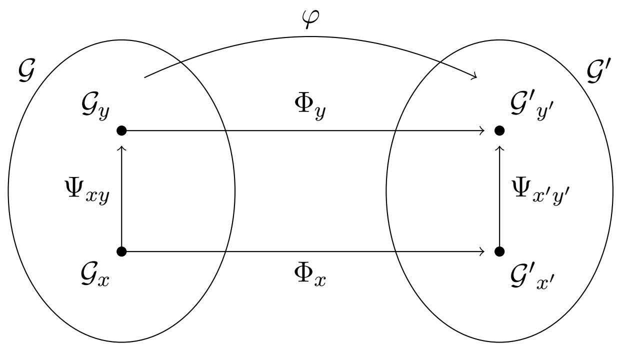

Figure 2.2: Proposition 2.13

Also, for any outer homomorphism $\Phi : G \to H$, we have,

$$[1_G] \circ \Phi = \Phi \circ [1_G] = \Phi.$$

An outer homomorphism Φ is called an **outer embedding** if one (and therefore every) homomorphism in Φ is injective. An outer homomorphism Φ is called an **outer isomorphism** if one (and therefore every) homomorphism in Φ is an isomorphism. We say two outer embeddings $\Phi : H \to G$ and $\Phi' : K \to G$ are **isomorphic** if there exists an outer isomorphism $\Psi : H \to K$ with $\Phi' \circ \Psi = \Phi$. One can identify isomorphism classes of outer embeddings into a group G with conjugacy classes of subgroups of G.

The Internal Outer Isomorphism Ψ_{xy}. Let $\mathcal{G}$ be a groupoid, and let $x, y \in \mathcal{G}$ be vertices. Let $g, g' \in \mathcal{G}(x, y)$, and let $g'' = g'^{-1}g$. Then $\chi_{g'} = {\chi_g}^{g''}$, and so $[\chi_{g'}] = [\chi_g]$. That is, different choices of $g \in \mathcal{G}(x, y)$ produce conjugate isomorphisms χ_g. Therefore, if $\mathcal{G}(x, y)$ is non-empty, there is a natural outer isomorphism,

$$\Psi_{xy} = [\chi_g] : \mathcal{G}_x \to \mathcal{G}_y$$

where $g \in \mathcal{G}(x, y)$. The outer isomorphism Ψ_{xy} is called the **internal** outer isomorphism from x to y. Notice that,

$$\Psi_{yz} \circ \Psi_{xy} = \Psi_{xz}, \quad \Psi_{xy}^{-1} = \Psi_{yx}, \quad \Psi_{xx} = [1].$$

The Outer Homomorphism Φ_x. Let $\varphi : \mathcal{G} \to \mathcal{G}'$ be a homomorphism of groupoids. Let $x \in \mathcal{G}$ be a vertex, and let $x' = \varphi(x)$. Let $\varphi \upharpoonright_{\mathcal{G}_x}$ denote the restriction of φ to the local group $\mathcal{G}_x$. Then we denote by Φ_x the outer homomorphism,

$$\Phi_x = [\varphi \upharpoonright_{\mathcal{G}_x}] : \mathcal{G}_x \to \mathcal{G}'_{x'}.$$

The internal outer isomorphisms Ψ_{xy} are compatible with the Φ_x in the following sense:

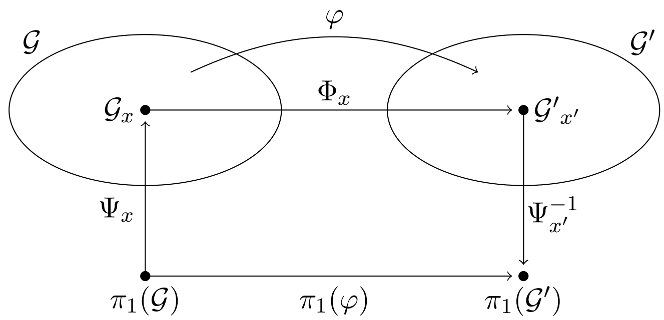

Figure 2.3: Defining $\pi_1(\varphi)$

Proposition 2.13. Let $\varphi : \mathcal{G} \to \mathcal{G}'$ be a homomorphism of groupoids. Let $x, y \in \mathcal{G}$ be vertices, and let $x' = \varphi(x)$ and $y' = \varphi(y)$. Then,

$$\Phi_y \circ \Psi_{xy} = \Psi_{x'y'} \circ \Phi_x.$$

See Figure 2.2.

Proof. Let $g \in \mathcal{G}(x, y)$. Then,

$$\Phi_y \circ \Psi_{xy} = [\varphi \restriction_{\mathcal{G}_y} \circ \chi_g] = [\chi_{\varphi(g)} \circ \varphi \restriction_{\mathcal{G}_x}] = \Psi_{x'y'} \circ \Phi_x$$

where the middle equality follows from the fact that φ preserves the composition of $\mathcal{G}$. $\square$

2.3.3 The Fundamental Group of a Groupoid

Usually, fundamental groups depend upon the choice of a base-point. Our notion of fundamental group does not require the choice of a base-point, but rather associates a group to a connected groupoid $\mathcal{G}$ which is naturally outer isomorphic to the local groups of $\mathcal{G}$.

Definition of the Fundamental Group. Let $\mathcal{G}$ be a connected groupoid. We define the **fundamental group** $\pi_1(\mathcal{G})$ of $\mathcal{G}$ to be the abstract group which is isomorphic to the local groups of $\mathcal{G}$, and which is equipped with an outer isomorphism $\Psi_x : \pi_1(\mathcal{G}) \to \mathcal{G}_x$ to each of the local groups of $\mathcal{G}$ such that $\Psi_{xx'} \circ \Psi_x = \Psi_{x'}$ for all vertices $x, x' \in \mathcal{G}$.

We associate to a homomorphism of connected groupoids $\varphi : \mathcal{G} \to \mathcal{G}'$ an outer homomorphism $\pi_1(\varphi) : \pi_1(\mathcal{G}) \to \pi_1(\mathcal{G}')$, called the outer homomorphism **induced** by φ, which is defined as follows; pick a vertex $x \in \mathcal{G}$, and let $x' = \varphi(x)$, then put,

$$\pi_1(\varphi) = \Psi_{x'}^{-1} \circ \Phi_x \circ \Psi_x.$$

See Figure 2.3.

Proposition 2.14. The definition of $\pi_1(\varphi)$ does not depend on the choice of x.

Proof. Suppose we make a different choice for x, say y. Let $y' = \varphi(y)$. Then, by Proposition 2.13, we have $\Phi_y \circ \Psi_{xy} = \Psi_{x'y'} \circ \Phi_x$. Thus, $\Phi_y = \Psi_{x'y'} \circ \Phi_x \circ \Psi_{yx}$, and,

$$\begin{aligned}\Psi_{y'}^{-1} \circ \Phi_y \circ \Psi_y &= \Psi_{y'}^{-1} \circ (\Psi_{x'y'} \circ \Phi_x \circ \Psi_{yx}) \circ \Psi_y \\ &= \Psi_{x'}^{-1} \circ \Phi_x \circ \Psi_x \\ &= \pi_1(\varphi). \qquad \square\end{aligned}$$

Proposition 2.15. The map $\varphi \mapsto \pi_1(\varphi)$ is functorial; for homomorphisms $\varphi : \mathcal{G} \to \mathcal{G}'$ and $\varphi' : \mathcal{G}' \to \mathcal{G}''$ of connected groupoids, we have,

$$\pi_1(\varphi' \circ \varphi) = \pi_1(\varphi') \circ \pi_1(\varphi).$$

Proof. Let $\varphi'' = \varphi' \circ \varphi$, let $x \in \mathcal{G}$ be a vertex, and let $x' = \varphi(x)$ and $x'' = \varphi''(x) = \varphi'(x')$. Since $\varphi'' \restriction_{\mathcal{G}_x} = \varphi' \restriction_{\mathcal{G}_{x'}} \circ \varphi \restriction_{\mathcal{G}_x}$, we have $\Phi''_x = \Phi'_{x'} \circ \Phi_x$. Then,

$$\begin{aligned}\pi_1(\varphi' \circ \varphi) &= \Psi_{x''}^{-1} \circ \Phi''_x \circ \Psi_x \\ &= \Psi_{x''}^{-1} \circ \Phi'_{x'} \circ \Phi_x \circ \Psi_x \\ &= (\Psi_{x''}^{-1} \circ \Phi'_{x'} \circ \Psi_{x'}) \circ (\Psi_{x'}^{-1} \circ \Phi_x \circ \Psi_x) \\ &= \pi_1(\varphi') \circ \pi_1(\varphi). \qquad \square\end{aligned}$$

We have the following characterization of isomorphisms of connected groupoids in terms of the induced outer homomorphism.

Proposition 2.16. A homomorphism of connected groupoids $\varphi : \mathcal{G} \to \mathcal{G}'$ is an isomorphism if and only if $\pi_1(\varphi)$ is an outer isomorphism and φ is bijective on vertices.

Proof. One can easily see that if φ is an isomorphism then $\pi_1(\varphi)$ is an outer isomorphism and φ is bijective on vertices. Conversely, suppose that $\pi_1(\varphi)$ is an outer isomorphism and φ is bijective on vertices. First we show surjectivity. Let $g' \in \mathcal{G}'(x', y')$ be an edge of $\mathcal{G}'$, and let $x = \varphi^{-1}(x')$ and $y = \varphi^{-1}(y')$. Let $g \in \mathcal{G}(x, y)$ and $h = \varphi^{-1}(g'\varphi(g)^{-1})$. Then $\varphi(hg) = g'$. For injectivity, let $g, h \in \mathcal{G}$ be edges such that $\varphi(g) = \varphi(h)$. We must have $\iota(g) = \iota(h) = x$ and $\tau(g) = \tau(h) = y$ because φ is injective on vertices. Then, $\varphi(gh^{-1}) = \varphi(g)\varphi(h)^{-1} = 1$. But $\pi_1(\varphi)$ is an outer isomorphism, and so $gh^{-1} = 1$. Therefore $g = h$. $\square$

2.3.4 Generating Sets and the Classification of Groupoids

In this section, we develop the idea of generating sets of groupoids, and show that the classification of groupoids easily reduces to the classification of groups.

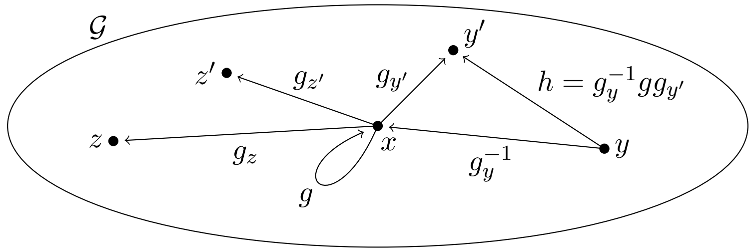

Figure 2.4: The unique expression for h

Generating Sets. Let $\mathcal{G}$ be a connected groupoid. Pick a base vertex $x \in \mathcal{G}$, and for each vertex $y \in \mathcal{G}$, pick $g_y \in \mathcal{G}(x, y)$. We make the convention that $g_x = 1_x$. Then we call $\mathcal{G}_x \cup \{g_y : y \in \mathcal{G}_0\}$ a **generating set** of $\mathcal{G}$. For every edge $h \in \mathcal{G}$, putting $y = \iota(h)$ and $y' = \tau(h)$, then there exists a unique edge $g \in \mathcal{G}_x$ such that $h = g_y^{-1} g g_{y'}$. See Figure 2.4. More generally, we say a subset $\mathcal{S} \subseteq \mathcal{G}_1$ **generates** $\mathcal{G}$ if every edge in $\mathcal{G}$ is a product of edges in $\mathcal{S} \cup \mathcal{S}^{-1}$.

Defining Homomorphisms. We can define a homomorphism of connected groupoids $\varphi : \mathcal{G} \to \mathcal{G}'$ by picking a generating set $\mathcal{G}_x \cup \{g_y : y \in \mathcal{G}_0\}$ of $\mathcal{G}$, picking the value of $\varphi(x)$, picking a group homomorphism $\bar{\varphi} : \mathcal{G}_x \to \mathcal{G}'_{\varphi(x)}$, and picking the values of $\varphi(g_y)$, with the requirement that $\iota(\varphi(g_y)) = \varphi(x)$. Any such collection of choices will determine φ; for $y \in \mathcal{G}_0$ and $h \in \mathcal{G}_1$, we must have,

$$\varphi(y) = \tau(\varphi(g_y)), \quad \varphi(h) = \varphi(g_y)^{-1} \bar{\varphi}(g) \varphi(g_{y'})$$

where $h = g_y^{-1} g g_{y'}$ is the unique expression for h.

The Classification of Groupoids. The isomorphism classes of connected groupoids are in bijection with pairs (G, κ), where G is (the isomorphism type of) a group, and κ is a cardinal. To see this, let $G \times \kappa$ be the following groupoid; the set of vertices is κ, for $x, y \in \kappa$ the set of edges from x to y is a copy of G, and the composition of edges is just their composition as elements of G. Now, suppose that $\mathcal{G}$ is a connected groupoid whose local groups are isomorphic to G, and whose set of vertices has cardinality κ. An isomorphism $\varphi : \mathcal{G} \to G \times \kappa$ can be constructed as follows. Let $\mathcal{G}_x \cup \{g_y : y \in \mathcal{G}_0\}$ be a generating set of $\mathcal{G}$, and let $F : \mathcal{G}_0 \to \kappa$ be any bijection. Let $G_{F(x)}$ denote the local group of $G \times \kappa$ at $F(x)$. Let $\bar{\varphi} : \mathcal{G}_x \to G_{F(x)}$ be any isomorphism, and let $\varphi(g_y)$ be the copy of the identity of G whose initial vertex is $F(x)$ and whose terminal vertex is $F(y)$. This defines an isomorphism φ by Proposition 2.16. Groupoids $\mathcal{G}$ which are not connected are disjoint unions $\mathcal{G} = \mathcal{G}_1 \sqcup \cdots \sqcup \mathcal{G}_n$ of connected groupoids. The groupoids in the disjoint union are the **connected components** of $\mathcal{G}$. We denote by $n\mathcal{G}$ the n-fold disjoint union,

$$\underbrace{\mathcal{G} \sqcup \cdots \sqcup \mathcal{G}}_{n}.$$

Thickness of Groupoids. Let us introduce some terminology which comes from the theory of buildings. The **thickness** of a connected groupoid $G \times \kappa$ is the cardinal $\big||G| \times \kappa\big| - 1$. A connected groupoid is called:

(i) **thin** if $\big||G| \times \kappa\big| = 2$, that is thickness equal to 1

(ii) **weak** if $\big||G| \times \kappa\big| \geq 2$, that is thickness at least 1

(iii) **thick** if $\big||G| \times \kappa\big| \geq 3$, that is weak but not thin.

A groupoid $\mathcal{G}$ is called **thin**, **weak**, or **thick** if $\mathcal{G}$ is a disjoint union of thin, weak, or thick connected components respectively. In particular, every thin groupoid is of the form $n(Z_2 \times 1) \sqcup m(1 \times 2)$, where Z_2 denotes the cyclic group of order 2.

2.4 Covering Theory of Groupoids

In this section, we describe covering theory of groupoids. Our approach models coverings of groupoids with outer embeddings of the fundamental groups. In general, information is lost when one replaces a homomorphism of connected groupoids with the outer homomorphism which it induces, however a covering of connected groupoids can be recovered from the outer embedding it induces. A good reference for the material of this section, which uses base-points, is [Bro06].

2.4.1 Coverings of Groupoids

In this section, we define coverings and morphisms of coverings, and collect some basic properties.

Definition of Coverings of Groupoids. A **covering** of groupoids is a surjective groupoid homomorphism $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ such that for all vertices $\tilde{x} \in \widetilde{\mathcal{G}}$, the restriction of p to $\widetilde{\mathcal{G}}(\tilde{x}, -)$ is a bijection into $\mathcal{G}(p(\tilde{x}), -)$. Notice that if $\mathcal{G}$ is connected, then surjectivity automatically follows since any edge $g \in \mathcal{G}$ can be written as $g = h^{-1}k$, where $h, k \in \mathcal{G}(p(\tilde{x}), -)$. We say that a covering $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ is **connected** if $\widetilde{\mathcal{G}}$ is connected, which implies that $\mathcal{G}$ is also connected.

It is easy to see that a covering of groupoids is equivalently a surjective groupoid homomorphism $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ such that, for all vertices $\tilde{x} \in \widetilde{\mathcal{G}}$, the restriction of p to $\widetilde{\mathcal{G}}(-, \tilde{x})$ is a bijection into $\mathcal{G}(-, p(\tilde{x}))$. We have the following equivalent definition in the case of connected groupoids:

Proposition 2.17. A covering of connected groupoids is equivalently a groupoid homomorphism $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ such that there exists a vertex $\tilde{x} \in \widetilde{\mathcal{G}}$ such that the restriction of p to $\widetilde{\mathcal{G}}(\tilde{x}, -)$ is a bijection into $\mathcal{G}(p(\tilde{x}), -)$.

Proof. Let $\tilde{x}, \tilde{y} \in \widetilde{\mathcal{G}}_0$, and let $g \in \widetilde{\mathcal{G}}(\tilde{x}, \tilde{y})$. Let φ_g be the function,

$$\varphi_g : \mathcal{G}(\tilde{x}, -) \to \mathcal{G}(\tilde{y}, -), \quad h \mapsto g^{-1}h$$

and let $\varphi_{p(g)^{-1}}$ be the function,

$$\varphi_{p(g)^{-1}} : \mathcal{G}(p(\tilde{y}), -) \to \mathcal{G}(p(\tilde{x}), -), \quad h \mapsto p(g)h.$$

Then,

$$p \restriction_{\widetilde{\mathcal{G}}(\tilde{x},-)} = \varphi_{p(g)^{-1}} \circ p \restriction_{\widetilde{\mathcal{G}}(\tilde{y},-)} \circ \varphi_g.$$

But φ_g and $\varphi_{p(g)^{-1}}$ are bijections since they have inverses $h \mapsto gh$ and $h \mapsto p(g)^{-1}h$ respectively. Thus, the restriction of p to $\widetilde{\mathcal{G}}(\tilde{x}, -)$ is a bijection if and only if the restriction of p to $\widetilde{\mathcal{G}}(\tilde{y}, -)$ is a bijection. The result follows. □

We have the following characterization of isomorphisms amongst coverings:

Proposition 2.18. A covering $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ is an isomorphism if and only if p is injective on the vertices of $\widetilde{\mathcal{G}}$.

Proof. One can easily see that if p is an isomorphism, then p is injective on vertices. Conversely, suppose that p is injective on vertices, and let $g, g' \in \widetilde{\mathcal{G}}$ be edges with $p(g) = p(g')$. Notice that we must have $\iota(g) = \iota(g')$ and $\tau(g) = \tau(g')$ by the fact that p is injective on the vertices of $\widetilde{\mathcal{G}}$. Let $x = \iota(g)$ and $y = \tau(g)$. Since p is a covering, its restriction to $\widetilde{\mathcal{G}}(x, y)$ is injective, and so we must have $g = g'$. Thus, p is an embedding. Finally, p is surjective by the fact it is a covering. □

Morphisms of Coverings. Let $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ and $p' : \widetilde{\mathcal{G}}' \to \mathcal{G}$ be coverings of a groupoid $\mathcal{G}$. A **morphism** of coverings $\lambda : p \to p'$ is a groupoid homomorphism $\lambda : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}'$ such that $p = p' \circ \lambda$. We call two coverings **isomorphic** if there exists a morphism between them which is a groupoid isomorphism. The **composition** of morphisms of coverings is just their composition as groupoid homomorphisms.

The following result shows that in particular, a morphism of coverings of connected groupoids is itself a covering:

Proposition 2.19. Let $p : \widetilde{\mathcal{G}} \to \mathcal{G}$, $p' : \widetilde{\mathcal{G}}' \to \mathcal{G}$, and $\lambda : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}'$ be homomorphisms of connected groupoids with $p = p' \circ \lambda$. If p and p' are coverings, then λ is a covering, and if p and λ are coverings, then p' is a covering.

Proof. Pick a vertex $\tilde{x} \in \widetilde{\mathcal{G}}$, and let $\tilde{x}' = \lambda(\tilde{x})$, and $x = p(\tilde{x}) = p'(\tilde{x}')$. Then,

$$p \restriction_{\widetilde{\mathcal{G}}(\tilde{x},-)} = p' \restriction_{\widetilde{\mathcal{G}}'(\tilde{x}',-)} \circ \lambda \restriction_{\widetilde{\mathcal{G}}(\tilde{x},-)} .$$

Therefore if $p \restriction_{\widetilde{\mathcal{G}}(\tilde{x},-)}$ and $p' \restriction_{\widetilde{\mathcal{G}}'(\tilde{x}',-)}$ are bijections, then so is $\lambda \restriction_{\widetilde{\mathcal{G}}(\tilde{x},-)}$, and if $p \restriction_{\widetilde{\mathcal{G}}(\tilde{x},-)}$ and $\lambda \restriction_{\widetilde{\mathcal{G}}(\tilde{x},-)}$ are bijections, then so is $p' \restriction_{\widetilde{\mathcal{G}}'(\tilde{x}',-)}$. The result then follows by Proposition 2.17. □

2.4.2 Coverings and Fundamental Groups

We now give an exposition of the relationship between coverings of connected groupoids and their induced outer homomorphisms.

Proposition 2.20. If $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ is a covering of connected groupoids, then $\pi_1(p)$ is an outer embedding.

Proof. If p is a covering, then for each vertex $\tilde{x} \in \widetilde{\mathcal{G}}$, the restriction $p \restriction_{\widetilde{\mathcal{G}}_{\tilde{x}}}$ is injective, i.e p is faithful. It then follows from the definition that $\pi_1(p)$ is an outer embedding. □

Thus, a covering $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ of connected groupoids induces a conjugacy class of subgroups of $\pi_1(\mathcal{G})$. We have the following characterization of isomorphisms amongst coverings of connected groupoids:

Proposition 2.21. A covering $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ of connected groupoids is an isomorphism if and only if $\pi_1(p)$ is an outer isomorphism.

Proof. If p is an isomorphism, then it follows from the definition that $\pi_1(p)$ is an outer isomorphism. Conversely, suppose that $\pi_1(p)$ is an outer isomorphism, and let $\tilde{x}, \tilde{y} \in \widetilde{\mathcal{G}}$ be vertices such that $p(\tilde{x}) = p(\tilde{y}) = x$. Then for each $g \in \widetilde{\mathcal{G}}(\tilde{x}, \tilde{y})$, we have $p(g) \in \mathcal{G}_x$. But $p \restriction_{\widetilde{\mathcal{G}}_{\tilde{x}}}$ is a bijection into $\mathcal{G}_x$ because $\pi_1(p)$ is an outer isomorphism. Therefore, since p is a covering, we must have $\tilde{x} = \tilde{y}$, and the result follows by Proposition 2.18 (or indeed Proposition 2.16). □

2.4.3 Lifting Outer Homomorphisms

In this section, we show that one can recover a covering of connected groupoids from the outer embedding which it induces. This result relies on the fact that certain outer homomorphisms can be 'lifted' to groupoid homomorphisms.

Theorem 2.22 (General Lifting)**.** Let $\mathcal{G}$, $\widetilde{\mathcal{G}}$, and $\mathcal{G}'$ be connected groupoids. Let $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ be a covering, and let $\varphi : \mathcal{G}' \to \mathcal{G}$ be a homomorphism. Let $\Phi : \pi_1(\mathcal{G}') \to \pi_1(\widetilde{\mathcal{G}})$ be an outer homomorphism with $\pi_1(p) \circ \Phi = \pi_1(\varphi)$. Then there exists a homomorphism $\varphi' : \mathcal{G}' \to \widetilde{\mathcal{G}}$ such that $p \circ \varphi' = \varphi$ and $\pi_1(\varphi') = \Phi$.

Proof. Pick a vertex $x' \in \mathcal{G}'$ and a generating set $\mathcal{G}'_{x'} \cup \{g_{y'} : y' \in \mathcal{G}'_0\}$ based at x'. Recall that we can construct $\varphi' : \mathcal{G}' \to \widetilde{\mathcal{G}}$ by giving $\varphi' \restriction_{\mathcal{G}'_{x'}}$ and the images of the $g_{y'}$.

Let $x = \varphi(x')$, and let $\tilde{x} \in \widetilde{\mathcal{G}}$ be any vertex such that $p(\tilde{x}) = x$. Let $\varphi_{x'} : \mathcal{G}'_{x'} \to \widetilde{\mathcal{G}}_{\tilde{x}}$ denote a homomorphism such that $[\varphi_{x'}] = \Psi_{\tilde{x}} \circ \Phi \circ \Psi_{x'}^{-1}$. Then,

$$\begin{aligned}
[p \restriction_{\widetilde{\mathcal{G}}_{\tilde{x}}}] \circ [\varphi_{x'}] &= (\Psi_x \circ \pi_1(p) \circ \Psi_{\tilde{x}}^{-1}) \circ (\Psi_{\tilde{x}} \circ \Phi \circ \Psi_{x'}^{-1}) \\
&= \Psi_x \circ \pi_1(p) \circ \Phi \circ \Psi_{x'}^{-1} \\
&= \Psi_x \circ \pi_1(\varphi) \circ \Psi_{x'}^{-1} \\
&= [\varphi \restriction_{\mathcal{G}'_{x'}}].
\end{aligned}$$

So there exists $g \in \mathcal{G}_x$ with,

$$\chi_g \circ p \restriction_{\widetilde{\mathcal{G}}_{\tilde{x}}} \circ \varphi_{x'} = \varphi \restriction_{\mathcal{G}'_{x'}} . \qquad (\spadesuit)$$

Let $\tilde{g} \in \widetilde{\mathcal{G}}(\tilde{x}, -)$ be the unique edge such that $p(\tilde{g}) = g$. Let $\tilde{y} = \tau(\tilde{g})$, and begin defining φ' by putting $\varphi'(x') = \tilde{y}$ and $\varphi' \restriction_{\mathcal{G}'_{x'}} = \chi_{\tilde{g}} \circ \varphi_{x'}$. Then,

$$\begin{aligned} p \restriction_{\widetilde{\mathcal{G}}(\tilde{y})} \circ \varphi' \restriction_{\mathcal{G}'_{x'}} &= p \restriction_{\widetilde{\mathcal{G}}_{\tilde{y}}} \circ \chi_{\tilde{g}} \circ \varphi_{x'} && \text{by the definition of } \varphi' \restriction_{\mathcal{G}'_{x'}} \\ &= \chi_g \circ p \restriction_{\widetilde{\mathcal{G}}_{\tilde{x}}} \circ \varphi_{x'} && \text{since } p \text{ is a homomorphism} \\ &= \varphi \restriction_{\mathcal{G}'_{x'}} && \text{by } (\spadesuit) \end{aligned}$$

as required (since we want $p \circ \varphi' = \varphi$). We finish defining φ' by letting $\varphi'(g_{y'})$ be the unique edge of $\widetilde{\mathcal{G}}(\tilde{y}, -)$ such that $p(\varphi'(g_{y'})) = \varphi(g_{y'})$. Then we have $p \circ \varphi' = \varphi$ since $p \circ \varphi'$ agrees with φ on the generating set $\mathcal{G}'_{x'} \cup \{g_{y'} : y' \in \mathcal{G}'_0\}$. Finally, we have $\pi_1(\varphi') = \Phi$ since,

$$\begin{aligned} \Phi &= \Psi_{\tilde{x}}^{-1} \circ [\varphi_{x'}] \circ \Psi_{x'} && \text{by the definition of } \varphi_{x'} \\ &= \Psi_{\tilde{y}}^{-1} \circ \Psi_{\tilde{x}\tilde{y}} \circ [\varphi_{x'}] \circ \Psi_{x'} \\ &= \Psi_{\tilde{y}}^{-1} \circ [\chi_{\tilde{g}}] \circ [\varphi_{x'}] \circ \Psi_{x'} \\ &= \Psi_{\tilde{y}}^{-1} \circ [\varphi' \restriction_{\mathcal{G}'_{x'}}] \circ \Psi_{x'} && \text{by the definition of } \varphi' \restriction_{\mathcal{G}'_{x'}} \\ &= \pi_1(\varphi') && \text{by the definition of } \pi_1(\varphi'). \end{aligned}$$

□

Corollary 2.22.1. Let $\mathcal{G}$, $\widetilde{\mathcal{G}}$, and $\widetilde{\mathcal{G}}'$ be connected groupoids. Let $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ and $p' : \widetilde{\mathcal{G}}' \to \mathcal{G}$ be coverings. Let $\Phi : \pi_1(\widetilde{\mathcal{G}}) \to \pi_1(\widetilde{\mathcal{G}}')$ be an outer embedding with $\pi_1(p') \circ \Phi = \pi_1(p)$. Then there exists a morphism of coverings $\lambda : p \to p'$ with $\pi_1(\lambda) = \Phi$. Moreover, if Φ is an outer isomorphism, then any such λ is an isomorphism.[2]

Proof. A homomorphism $\lambda : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}'$ such that $p' \circ \lambda = p$ and $\pi_1(\lambda) = \Phi$ exists by Theorem 2.22. Then λ is a covering of groupoids by Proposition 2.19, and so λ is an isomorphism if Φ is an outer isomorphism by Proposition 2.21. □

The following tells us that a covering is determined by the outer embedding which it induces:

Corollary 2.22.2. Let $\mathcal{G}$, $\widetilde{\mathcal{G}}$, and $\widetilde{\mathcal{G}}'$ be connected groupoids. Let $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ and $p' : \widetilde{\mathcal{G}}' \to \mathcal{G}$ be coverings. If $\pi_1(p)$ and $\pi_1(p')$ are isomorphic outer embeddings, then p and p' are isomorphic coverings.

[2] This can be expressed in the language of category theory as follows; let $\mathrm{cov}(\mathcal{G})$ denote the slice category of coverings of $\mathcal{G}$, and let $\mathrm{cov}(\pi_1(\mathcal{G}))$ denote the slice category of outer embeddings into $\pi_1(\mathcal{G})$, then the functor $\mathrm{cov}(\mathcal{G}) \to \mathrm{cov}(\pi_1(\mathcal{G}))$ induced by π_1 is full and conservative.

Proof. By hypothesis, there exists an outer isomorphism $\Psi : \pi_1(p) \to \pi_1(p')$ with $\pi_1(p') \circ \Psi = \pi_1(p)$. There exists an isomorphism of coverings $\lambda : p \to p'$ with $\pi_1(\lambda) = \Psi$ by Corollary 2.22.1. □

Therefore, the (isomorphism classes of) connected coverings of a connected groupoid $\mathcal{G}$ naturally inject into the conjugacy classes of subgroups of $\pi_1(\mathcal{G})$. Of course, we still do not know if, given a conjugacy class of subgroups, a covering exists which induces it.

2.4.4 Existence of Coverings

Let $\mathcal{G}$ be a connected groupoid. We begin by describing a construction of connected coverings of $\mathcal{G}$, and then show that this constructs coverings for each conjugacy class of subgroups of $\pi_1(\mathcal{G})$.

The Covering Based at H. Let $H \leq \mathcal{G}_x$ be a subgroup of a connected groupoid $\mathcal{G}$. For each coset Hg, pick a representative $g^* \in Hg$. We make the convention that $h^* = 1_x$ for $h \in H$. We construct a connected groupoid, denoted $\widetilde{\mathcal{G}}^H$, by letting the vertices of $\widetilde{\mathcal{G}}^H$ be the set of cosets $H\backslash\mathcal{G}$, and letting the edges of $\widetilde{\mathcal{G}}^H$ be the set,

$$\widetilde{\mathcal{G}}^H_1 = \big\{(h, Hg, Hg') : h \in H;\ g, g' \in \mathcal{G}(x, -)\big\}.$$

For the extremities, put,

$$\iota(h, Hg, Hg') = Hg, \quad \tau(h, Hg, Hg') = Hg'$$

and for the composition, put,

$$(h, Hg, Hg')(h', Hg', Hg'') = (hh', Hg, Hg'').$$

It is easy to check that this defines a groupoid $\widetilde{\mathcal{G}}^H$. Let $p^H : \widetilde{\mathcal{G}}^H \to \mathcal{G}$ be the homomorphism whose map on edges is,

$$(h, Hg, Hg') \mapsto (g^*)^{-1} h g'^*.$$

This implies that for vertices we have $p^H(Hg) = \tau(g)$, and in particular $p^H(H) = x$. Checking that p^H is a homomorphism, we have,

$$\begin{aligned} p^H(hh', Hg, Hg'') &= (g^*)^{-1} hh' g''^* \\ &= (g^*)^{-1} h g'^* (g'^*)^{-1} h' g''^* \\ &= p^H(h, Hg, Hg') p^H(h', Hg', Hg''). \end{aligned}$$

Proposition 2.23. Let $H \leq \mathcal{G}_x$ be a subgroup of a connected groupoid $\mathcal{G}$. Then $p^H : \widetilde{\mathcal{G}}^H \to \mathcal{G}$ is a covering of groupoids.

Proof. Consider the restriction of p^H to the edges (h, H, Hg) which issue from the vertex $H \in \widetilde{\mathcal{G}}^H$. For injectivity, if $p^H(h, H, Hg) = p^H(h', H, Hg')$, then $hg^* = h'g'^*$, and so $h = h'$ and $g = g'$. Thus, $(h, H, Hg) = (h', H, Hg')$. For surjectivity, let $g \in \mathcal{G}(x, -)$, and let $h \in H$ such that $g = hg^*$. Then we have $p^H(h, H, Hg) = g$. The fact that p^H is a covering then follows by Proposition 2.17. □

We call $p^H : \widetilde{\mathcal{G}}^H \to \mathcal{G}$ **the covering based at** H. Notice that the local group $\widetilde{\mathcal{G}}^H_H$ of $\widetilde{\mathcal{G}}^H$ at H is naturally isomorphic to H.

Proposition 2.24. Let $\mathcal{G}$ be a connected groupoid, and let $\Phi : K \to \pi_1(\mathcal{G})$ be an outer embedding. Then there exists a covering $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ such that $\pi_1(p)$ is isomorphic to Φ.[3]

Proof. Pick a subgroup $H \leq \mathcal{G}_x$ of $\mathcal{G}$ such that $\Psi_x^{-1} \restriction_H$ is isomorphic to Φ. Put $p = p^H : \widetilde{\mathcal{G}}^H \to \mathcal{G}$. Let $\varphi_H : \widetilde{\mathcal{G}}^H_H \to H \leq \mathcal{G}_x$ be the identity map and put $\Phi_H = [p \restriction_{\widetilde{\mathcal{G}}^H_H}]$. Notice that φ_H is just the embedding $p \restriction_{\widetilde{\mathcal{G}}^H_H}$ restricted to its image. Pick representatives $\psi_x^{-1} \in \Psi_x^{-1}$ and $\psi_H \in \Psi_H$. Then $[\varphi_H] \circ \Psi_H$ is an outer isomorphism, and,

$$\begin{aligned}
\pi_1(p) &= \Psi_x^{-1} \circ \Phi_H \circ \Psi_H \\
&= [\psi_x^{-1} \circ \ p \restriction_{\widetilde{\mathcal{G}}^H_H} \circ \psi_H] \\
&= [\psi_x^{-1} \restriction_H \circ \varphi_H \circ \psi_H] \\
&= [\psi_x^{-1} \restriction_H] \circ \ [\varphi_H] \circ [\psi_H] \\
&= \Psi_x^{-1} \restriction_H \circ [\varphi_H] \circ \Psi_H.
\end{aligned}$$

Therefore $\pi_1(p)$ and $\Psi_x^{-1} \restriction_H$ are isomorphic via $[\varphi_H] \circ \Psi_H$. Then, since $\Psi_x^{-1} \restriction_H$ is isomorphic to Φ, we have that $\pi_1(p)$ is also isomorphic to Φ. □

This shows that the connected coverings of a connected groupoid $\mathcal{G}$ are naturally in bijection with outer embeddings in $\pi_1(\mathcal{G})$ (up to isomorphism), which in turn are naturally in bijection with the conjugacy classes of subgroups of $\pi_1(\mathcal{G})$.

The Universal Cover of a Groupoid. Let $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ be a covering of connected groupoids. Then p is called a **universal cover** if for any covering $p' : \widetilde{\mathcal{G}}' \to \mathcal{G}$ such that $\widetilde{\mathcal{G}}'$ is connected, there exists a covering morphism $\lambda : p \to p'$. Given the 1-1 correspondence between coverings and conjugacy classes of subgroups, it is easy to see that a covering $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ is universal if and only if $\widetilde{\mathcal{G}}$ is a connected setoid. Thus, each connected groupoid $\mathcal{G}$ has a unique universal cover (up to isomorphism).

2.4.5 Coverings and Group Actions

In this section, we expose the relationship between groups acting on groupoids and coverings.

[3] This says that the functor $\mathrm{cov}(\mathcal{G}) \to \mathrm{cov}(\pi_1(\mathcal{G}))$ induced by π_1 is essentially surjective. To summarize, we have shown $\mathrm{cov}(\mathcal{G}) \to \mathrm{cov}(\pi_1(\mathcal{G}))$ is conservative, full, and essentially surjective.

Groups Acting on Groupoids. Groups act by automorphisms on groupoids. We say a group G acts **freely** on a groupoid $\mathcal{G}$ if the action of G restricted to $\mathcal{G}_1$, or equivalently to $\mathcal{G}_0$, is free.

The Deck Transformation Group. Let $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ be a covering. An **automorphism** of p is a covering isomorphism from p to itself. The **deck transformation group** of p is the group $\mathrm{Aut}(p)$ whose elements are automorphisms of p, and whose binary operation is the composition of homomorphisms. Notice that $\mathrm{Aut}(p) \leq \mathrm{Aut}(\widetilde{\mathcal{G}})$. Therefore a covering $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ determines a faithful action of $\mathrm{Aut}(p)$ on the left of $\widetilde{\mathcal{G}}$.

Proposition 2.25. Let $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ be connected groupoids, and let $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ be a covering. Then $\mathrm{Aut}(p)$ acts freely on $\widetilde{\mathcal{G}}$.

Proof. We show that $\mathrm{Aut}(p)$ acts freely on vertices. Let $\gamma \in \mathrm{Aut}(p)$, and suppose there exists $x \in \widetilde{\mathcal{G}}_0$ with $\gamma \cdot x = x$. Since p is injective on $\widetilde{\mathcal{G}}(x, -)$, we have $\gamma \cdot g = g$ for all $g \in \widetilde{\mathcal{G}}(x, -)$. But $\widetilde{\mathcal{G}}(x, -)$ generates $\widetilde{\mathcal{G}}$, and so $\gamma = 1$. □

We will see that conversely, if a group G acts freely on a connected groupoid $\mathcal{G}'$, then G is naturally the deck transformation group of a covering $\mathcal{G}' \to \mathcal{G}$.

Regular Coverings. A covering $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ of connected groupoids is called **regular** if its associated conjugacy class of subgroups is a single normal subgroup; equivalently if $\pi_1(p)$ is a singleton. If p is regular, then we identify $\pi_1(p)$ with the embedding it contains, and we identify $\pi_1(\widetilde{\mathcal{G}})$ with its $\pi_1(p)$ image in $\pi_1(\mathcal{G})$.

Proposition 2.26. Let $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ be connected groupoids, and let $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ be a regular covering. Then the action of $\mathrm{Aut}(p)$ restricted to the p-preimage of a vertex or an edge is regular.

Proof. We know that these actions are free by Proposition 2.25. First, we show that the action is transitive in the case of a vertex. Let $\tilde{x}, \tilde{y} \in \widetilde{\mathcal{G}}$ be vertices with $p(\tilde{x}) = p(\tilde{y})$. We construct a deck transformation $\gamma \in \mathrm{Aut}(p)$, with $\gamma \cdot \tilde{x} = \tilde{y}$, by defining γ on a generating set $\widetilde{\mathcal{G}}_{\tilde{x}} \cup \{g_y : y \in \widetilde{\mathcal{G}}_0\}$. Let $\gamma \restriction_{\widetilde{\mathcal{G}}_{\tilde{x}}} : \widetilde{\mathcal{G}}_{\tilde{x}} \to \widetilde{\mathcal{G}}_{\tilde{y}}$ be defined by $\gamma \cdot g = g_{\tilde{y}}^{-1} g g_{\tilde{y}}$, and let $\gamma \cdot g_y$ be the unique edge of $\widetilde{\mathcal{G}}(\tilde{y}, -)$ such that $p(\gamma \cdot g_y) = p(g_y)$. Notice that γ is a covering morphism $\gamma : p \to p$ because, for $g \in \widetilde{\mathcal{G}}_{\tilde{x}}$, we have $p(\gamma \cdot g) = p(g_{\tilde{y}}^{-1} g g_{\tilde{y}}) = p(g)$ since p is regular. Then $\pi_1(\gamma)$ is an outer isomorphism, and so γ is an automorphism by Corollary 2.22.1.

In the case of an edge, let $g, g' \in \widetilde{\mathcal{G}}$ be edges with $p(g) = p(g')$. Then we have just shown that there exists $\gamma \in \mathrm{Aut}(p)$ such that $\gamma \cdot \iota(g) = \iota(g')$. Thus $\gamma \cdot g = g'$, since p is a covering. □

Theorem 2.27. Let $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ be connected groupoids, and let $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ be a regular covering. Then there exists a natural outer isomorphism,

$$\Psi : \pi_1(\mathcal{G}) / \pi_1(\widetilde{\mathcal{G}}) \to \mathrm{Aut}(p).$$

Proof. Pick vertices $x \in \mathcal{G}$ and $\tilde{x} \in \widetilde{\mathcal{G}}$ such that $p(\tilde{x}) = x$. For $g \in \mathcal{G}(x, -)$, let $\tilde{g} \in \mathcal{G}(\tilde{x}, -)$ be the unique edge such that $p(\tilde{g}) = g$. Let $\varphi : \mathcal{G}_x \to \mathrm{Aut}(p)$ be the surjective homomorphism such that $\varphi(g) \cdot \tilde{x} = \tau(\tilde{g})$. This is well defined by Proposition 2.26. To see that φ is a homomorphism, let $g, h \in \mathcal{G}_x$, and put $k = gh$. Then $\tau(\widetilde{k}) = \varphi(g) \cdot \tau(\tilde{h})$ since $\varphi(g) \cdot \tilde{h}$ must be in the p-preimage of h. Thus,

$$\varphi(gh) \cdot \tilde{x} = \varphi(k) \cdot \tilde{x} = \tau(\widetilde{k}) = \varphi(g) \cdot \tau(\tilde{h}) = \varphi(g) \cdot \varphi(h) \cdot \tilde{x}.$$

To see that φ is surjective, let $\gamma \in \mathrm{Aut}(p)$ and pick $g_\gamma \in \widetilde{\mathcal{G}}(\tilde{x}, \gamma \cdot \tilde{x})$. Then $p(g_\gamma) \in \mathcal{G}_x$, and $\varphi(p(g_\gamma)) = \gamma$.

Let $\Phi : \pi_1(\mathcal{G}) \to \mathrm{Aut}(p)$ be the outer homomorphism $\Phi = [\varphi] \circ \Psi_x$. We now show that Φ does not depend on the choice of x and $\tilde{x}$. Suppose that we make a different choice of vertices $y \in \mathcal{G}$ and $\tilde{y} \in \widetilde{\mathcal{G}}$ such that $p(\tilde{y}) = y$. Let $\varphi' : \mathcal{G}_y \to \mathrm{Aut}(p)$ be the new homomorphism. Pick $\tilde{g}' \in \widetilde{\mathcal{G}}(\tilde{y}, \tilde{x})$ and let $g' = p(\tilde{g}')$. Let $\chi_{g'} : \mathcal{G}_y \to \mathcal{G}_x$ be the usual isomorphism $g \mapsto g'^{-1} g g'$. Then $\varphi' = \varphi \circ \chi_{g'}$, and so,

$$[\varphi'] \circ \Psi_y = [\varphi \circ \chi_{g'}] \circ \Psi_y = [\varphi] \circ \Psi_{yx} \circ \Psi_y = [\varphi] \circ \Psi_x.$$

For $g \in \mathcal{G}_x$, we have $\varphi(g) = 1$ if and only if $\tilde{g}$ is a loop. Therefore the kernel of φ is $p(\mathcal{G}_{\tilde{x}}) \leq \mathcal{G}_x$, and so the kernel of each group homomorphism in Φ is $\pi_1(\widetilde{\mathcal{G}})$. Let Ψ be the set of isomorphisms obtained by factoring out the kernels of the homomorphisms in Φ. Then Ψ is an outer isomorphism $\Psi : \pi_1(\mathcal{G})/\pi_1(\widetilde{\mathcal{G}}) \to \mathrm{Aut}(p)$. □

We now show that if a group G acts freely on a connected groupoid $\mathcal{G}$, then there exists a groupoid $\mathcal{G}'$ and a regular covering $\mathcal{G} \to \mathcal{G}'$ of which G is naturally the automorphism group.

The Quotient by an Action. We associate to the free action of a group G on a groupoid $\mathcal{G}$ the **quotient groupoid** $G \backslash \mathcal{G}$, which is the groupoid defined as follows; the set of vertices of $G \backslash \mathcal{G}$ is the set of orbits of vertices $G \backslash \mathcal{G}_0 = \{[x] : x \in \mathcal{G}_0\}$, the set of edges of $G \backslash \mathcal{G}$ is the set of orbits of edges $G \backslash \mathcal{G}_1 = \{[g] : g \in \mathcal{G}_1\}$, and for the extremities of edges, we have,

$$\iota([g]) = [\iota(g)], \quad \tau([g]) = [\tau(g)].$$

The extremities are well defined since for all $\gamma \in G$ and all edges $g \in \mathcal{G}$, we have $\iota(\gamma \cdot g) = \gamma \cdot \iota(g)$ and $\tau(\gamma \cdot g) = \gamma \cdot \tau(g)$. The groupoid structure is as follows:

(1) for identities, let $1_{[x]} = [1_x]$

(2) for inverses, let $[g]^{-1} = [g^{-1}]$

(3) the composition $[g][g']$ is defined if there exists an edge $g'' \in [g']$ such that gg'' is defined, in which case we put,

$$[g][g'] = [gg''].$$

Notice that the identities and inverses are well defined since for all $\gamma \in G$, we have $1_{\gamma \cdot x} = \gamma \cdot 1_x$ and $(\gamma \cdot g)^{-1} = \gamma \cdot g^{-1}$. To see that the composition is well defined, first notice that if there exists an edge $g'' \in [g']$ such that gg'' is defined, then,

$$\tau([g]) = [\tau(g)] = [\iota(g'')] = \iota([g'']) = \iota([g']).$$

Conversely, if $\tau([g]) = \iota([g'])$, let $\gamma \in G$ be the unique element such that $\gamma \cdot \iota(g') = \tau(g)$. Then by putting $g'' = \gamma \cdot g'$, we see that $[g][g']$ is defined. Finally, if $g; g'$ is defined and so is $\gamma \cdot g; \gamma' \cdot g'$, then $\gamma = \gamma'$ because G acts freely, and so,

$$[\gamma \cdot g; \gamma' \cdot g'] = [\gamma \cdot g; g'] = [gg'].$$

From now on, whenever we write the composition $[g][g']$ we will assume that g' has been chosen such that gg' is defined, and so $[g][g'] = [gg']$. The **quotient map** $\pi : \mathcal{G} \to G\backslash\mathcal{G}$ is the graph morphism such that $x \mapsto [x]$ for $x \in \mathcal{G}_0$, and $g \mapsto [g]$ for $g \in \mathcal{G}_1$, which clearly preserves extremities.

Theorem 2.28. Let G be a group which acts freely on a groupoid $\mathcal{G}$. Then $G\backslash\mathcal{G}$ is a groupoid and $\pi : \mathcal{G} \to G\backslash\mathcal{G}$ is a covering of groupoids. Moreover, if $\mathcal{G}$ is connected, then G is naturally isomorphic to $\mathrm{Aut}(\pi)$.

Proof. First, we show that $\mathcal{G}/G$ is a groupoid. For the initial vertices we have,

$$\iota([g][g']) = \iota([gg']) = [\iota(gg')] = [\iota(g)] = \iota([g])$$

and for the terminal vertices we have,

$$\tau([g][g']) = \tau([gg']) = [\tau(gg')] = [\tau(g')] = \tau([g']).$$

Also,

$$\iota(1_{[x]}) = \iota([1_x]) = [\iota(1_x)] = [x]$$

and,

$$\tau(1_{[x]}) = \tau([1_x]) = [\tau(1_x)] = [x].$$

The fact that composition in $\mathcal{G}/G$ is associative follows from the fact that the composition in $\mathcal{G}$ is associative. Also, the edges $[1_x]$ clearly act as identities, and for the inverses have,

$$[g][g^{-1}] = [gg^{-1}] = [1], \quad [g^{-1}][g] = [g^{-1}g] = [1].$$

It follows that $\mathcal{G}/G$ is a groupoid. The fact that $\pi : \mathcal{G} \to G\backslash\mathcal{G}$ is a homomorphism follows directly from the definition of the composition of edges in $G\backslash\mathcal{G}$. To see that π is a covering, let $x \in \mathcal{G}$ be a vertex, and let $[g]$ be an edge which issues from $[x]$. Let $\gamma \in G$ be the element such that $\gamma \cdot \iota(g) = x$. Then $\gamma \cdot g$ is an edge which issues from x with $\pi(\gamma \cdot g) = [g]$. Suppose that g' is an edge which also issues from x with $\pi(g') = [g]$. Then there exists $\gamma' \in G$ with $\gamma' \cdot g = g'$. Then $\gamma\gamma'^{-1} \cdot x = x$, and so

$\gamma = \gamma'$ since G acts freely on $\mathcal{G}$. Therefore $g' = \gamma \cdot g$. Finally, π is clearly surjective on vertices. This proves that π is a covering.

We have a natural embedding $\varphi : G \hookrightarrow \mathrm{Aut}(\pi)$. To see that φ is surjective in the case where $\mathcal{G}$ is connected, let $a \in \mathrm{Aut}(\pi)$, and for any edge $g \in \mathcal{G}$, let $\gamma \in G$ such that $\gamma \cdot g = a \cdot g$. Then $\varphi(g) = \gamma$ by Proposition 2.25. $\square$

Finally, we show that given a regular covering p, the quotient map associated to the action of $\mathrm{Aut}(p)$ is p (up to isomorphism):

Proposition 2.29. Let $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ be connected groupoids, let $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ be a regular covering, and let $\pi : \widetilde{\mathcal{G}} \to \mathrm{Aut}(p)\backslash\widetilde{\mathcal{G}}$ be the quotient map associated to the action of $\mathrm{Aut}(p)$. Then there exists a unique isomorphism $\psi : \mathrm{Aut}(p)\backslash\widetilde{\mathcal{G}} \to \mathcal{G}$ such that $p = \psi \circ \pi$.

Proof. Since we want $p = \psi \circ \pi$, we have no choice but to let $\psi : \mathrm{Aut}(p)\backslash\widetilde{\mathcal{G}} \to \mathcal{G}$ be the homomorphism whose map on edges is,

$$[g] \mapsto p(g), \qquad \text{for } g \in \widetilde{\mathcal{G}}.$$

This is well defined since for $\gamma \in \mathrm{Aut}(p)$, we have $p(\gamma \cdot g) = p(g)$. Checking that ψ is a homomorphism, we have,

$$[g][g'] = [gg'] \mapsto p(gg') = p(g)p(g').$$

The restriction of ψ to edges of $\mathrm{Aut}(p)\backslash\widetilde{\mathcal{G}}$ is a bijection because it has the inverse $g \mapsto \pi(p^{-1}(g))$. This inverse is well defined by Proposition 2.26. Thus, ψ is an isomorphism. $\square$

Chapter 3

Pre-Weyl Graphs

In this chapter we introduce pre-Weyl graphs, which are structures that are 'almost' the quotients of buildings. A pre-Weyl graph which is in fact the quotient of a building will be called a Weyl graph, and will be studied in Chapter 4. Equivalently, a pre-Weyl graph is a Weyl graph if its universal cover is a building.

It will be convenient to introduce the additional notion of a 2-Weyl graph, which is a pre-Weyl that satisfies an extra local condition, and yet is not necessarily the quotient of a building. An equivalent definition is that a 2-Weyl graph is a pre-Weyl graph whose 2-residues are Weyl graphs. We will see in Chapter 4 that by a generalization of the famous local-to-global result of Tits in [Tit81], the universal cover of a 2-Weyl graph is a building if (and only if) its spherical 3-residues are covered by buildings. Thus, the pathological phenomenon of homotopic geodesics not having the same length arises locally in bad ≤ 3-residues. The notion of a 2-Weyl graph generalizes Tits' 'chamber systems of type M' by allowing the 2-residues to be quotients of generalized polygons.

We start in Section 3.1 by collecting several definitions and basic notions. For M a Coxeter matrix, we introduce graphs of type M which are directed multigraphs whose edges are labeled over the generators of the Coxeter group of type M. Then, a generalized chamber system of type M is a graph of type M whose adjacent edges of the same type have a well defined composition. A generalized chamber system can also be viewed as a family of groupoids indexed by the generators; one recovers Tits' notion of a chamber system when all these groupoids are simply connected. We define Weyl data to be a generalized chamber system which is equipped with a collection of 'suites' that tell us what (strict) homotopies of galleries are permitted. In Section 3.2, we define a pre-Weyl graph to be Weyl data which satisfies a thickness condition, and a property which implies the existence of 'geodesics', which is our name for galleries whose type is a reduced word. We study the property of being a 2-Weyl graph amongst pre-Weyl graphs. The 'W-length' of a geodesic is the element of the Coxeter group of type M corresponding to the sequence of types of its edges. The axioms of a pre-Weyl graph are strengthened to those of a Weyl graph by requiring that homotopic geodesics have the same W-length. In particular, a pre-Weyl graph is a 2-Weyl graph if the geodesics which are contained within 2-residues have a well defined W-length up to homotopy. Finally, in Section 3.3, we develop covering theory of 2-Weyl graphs, which is closely related to covering theory of groupoids.

3.1 Weyl Data

In this section, which consists mainly of definitions, we introduce Weyl data and some associated notions. In particular, we describe how the groupoid data of generalized chamber systems and the 'suites' of Weyl data induce homotopies of galleries.

3.1.1 Graphs of Type M

We begin by introducing graphs of type M, where M is a Coxeter matrix. From now on we denote graphs by '$\mathcal{W}$' for 'Weyl', instead of 'Γ' as in Chapter 2, and we say 'chambers' instead of 'vertices'.

Definition of Graphs of Type M. Recall from Section 2.1.1 that a graph labeled over S is a directed multigraph whose edges are equipped with a type function into S. For M a Coxeter matrix on S, a **graph of type** M is a graph $\mathcal{W} = (\mathcal{W}_0, \mathcal{W}_1)$ labeled over S, with **chambers** (vertices) $\mathcal{W}_0$ and edges $\mathcal{W}_1$.

As usual, we denote by W the Coxeter group whose Coxeter matrix is M. The Cayley graph $\mathcal{C}(W)$ of W is an example of a graph of type M.

W-Length of Galleries. Recall from Section 2.1.1 that a gallery is a path in a labeled graph which is determined by a sequence of adjacent edges. Let β be a gallery in a graph of type M. Recall that β_S denotes the word over S which is the sequence of the types of edges of β. The element $w(\beta_S) \in W$ for which β_S is a decomposition is called the W**-length** of β. We denote the W-length of β by β_W.

Geodesics. A **geodesic** γ is a gallery in a graph of type M whose type γ_S is a reduced word (with respect to M). In particular, trivial galleries are geodesics. In the theory of buildings, geodesics are usually called 'galleries of reduced type'.

Alternating Geodesics. Let $s, t \in S$, $s \neq t$, and $m_{st} < \infty$. An (s,t)**-geodesic**, or **alternating geodesic**, is a geodesic whose type is an (s,t)-word; that is a word of the form,

$$p_m(s,t) = \underbrace{stst\dots}_{m} \qquad \text{for} \ \ 0 \leq m \leq m_{st}.$$

An **maximal** (s,t)**-geodesic**, or **maximal alternating geodesic**, is a gallery whose type is the word,

$$p(s,t) = \underbrace{stst\dots}_{m_{st}} .$$

That is, a maximal alternating geodesic is an alternating geodesic of maximum length. We denote alternating geodesics with type $p_m(s,t)$ by $\rho_m(s,t)$, and maximal alternating geodesics with type $p(s,t)$ by $\rho(s,t)$, or sometimes just ρ.

(s,t)-Cycles. Let $s,t \in S$, $s \neq t$, and $m_{st} < \infty$. An **(s,t)-cycle** is a gallery in a graph of type M which is a cycle, and whose type is the word,

$$p_{2m_{st}}(s,t) = \underbrace{stst\ldots}_{2m_{st}} .$$

We denote (s,t)-cycles by $\theta(s,t)$, or sometimes just θ.

Remark 3.1. Let us collect some easy facts:

(i) a gallery β is a geodesic if and only if $|\beta| = |\beta_W|$

(ii) an (s,t)-geodesic $\rho_m(s,t)$ will have a different W-length to a (t,s)-geodesic $\rho_{m'}(t,s)$ unless $m = m' = m_{st}$, in which case the W-lengths are both equal to $w(p(s,t)) = w(p(t,s))$

(iii) an (s,t)-cycle $\theta(s,t)$ has length $2m_{st}$, and W-length 1.

Restrictions of Graphs of Type M. Let M be a Coxeter matrix on S, and let $\mathcal{W} = (\mathcal{W}_0, \mathcal{W}_1)$ be a graph of type M. For $J \subseteq S$, the **J-restriction** $\mathcal{W}_J$ of $\mathcal{W}$ is the graph of type M_J with chambers $\mathcal{W}_0$, and edges,

$$(\mathcal{W}_J)_1 = \big\{ i \in \mathcal{W}_1 : v(i) \in J \big\} \subseteq \mathcal{W}_1.$$

The extremities and type function of $\mathcal{W}_J$ are the restrictions to $(\mathcal{W}_J)_1$ of the corresponding functions of $\mathcal{W}$. For $J = \{s\}$, we write $\mathcal{W}_s$.

3.1.2 Generalized Chamber Systems and Weyl Data

In this section, we introduce generalized chamber systems and Weyl data. Classically, a chamber system is an indexed family of equivalence relations on a set. Recall that equivalence relations are equivalent to simply connected groupoids. Generalized chamber systems generalize chamber systems by moving from equivalence relations to groupoids. In the following definition, see Section 2.3.1 for our convention regarding the groupoidization of graphs.

Definition of Generalized Chamber Systems and Weyl Data. A **generalized chamber system** $\mathcal{W} = (\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_s)$ of type M is a graph $\mathcal{W} = (\mathcal{W}_0, \mathcal{W}_1)$ of type M, with additional data,

(1) for each $s \in S$, a groupoidization of $\mathcal{W}_s$, called the **panel groupoid** of type s.

If in addition $\mathcal{W}$ has data,

(2) for each pair $(s,t) \in S \times S$ such that $s \neq t$ and $m_{st} < \infty$, a set $\mathcal{W}(s,t)$ of (s,t)-cycles, called **defining** (s,t)**-suites**, or **defining suites**

then $\mathcal{W} = (\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_s, \mathcal{W}(s,t))$ is called **Weyl data**. An s**-panel** is a connected component of the panel groupoid of type s. More generally, a **panel** is an s-panel for some $s \in S$.

The defining suites can be viewed as analogs of defining relators in combinatorial group theory. They will tell us what (strict) homotopies of galleries are permitted.

The Panel Groupoids. Recall that the edges of the graph $\mathcal{W}_s$ are (in bijection with) the *non-trivial* edges of its groupoidization, and the trivial edges of its groupoidization can be thought of as an extra set of edges, in bijection with the set of chambers $\mathcal{W}_0$.

From now on, we let $\mathcal{W}_s$ denote its groupoidization, i.e. $\mathcal{W}_s$ denotes the panel groupoid of type s.

However when we speak of an edge $i \in \mathcal{W}_s$, we make the convention that we mean a non-trivial edge; that is an edge which is also an edge of the graph $\mathcal{W}$.

Chamber Systems. A generalized chamber system of type M is equivalent to a set of chambers $\mathcal{W}_0$ which is equipped with an indexed family of groupoids $(\mathcal{W}_s)_{s\in S}$, where the set of chambers of each groupoid $\mathcal{W}_s$ is $\mathcal{W}_0$. If all the indexed groupoids $\mathcal{W}_s$ are setoids, then they can be viewed as equivalence relations $\sim_s$ on $\mathcal{W}_0$, and we recover Tits' notion of a chamber system (see [Tit81]). Thus, we define a **chamber system** to be a generalized chamber system such that each panel groupoid is a setoid.

The **rank** of a generalized chamber system is the cardinality of S. Notice that in the rank 1 case, and the case where M is universal, generalized chamber systems are equivalent to Weyl data. A generalized chamber system is called **locally finite** if each of its panels is a finite groupoid.

Weyl data $\mathcal{W}$ is called **simple** if every (s,t)-cycle of the underlying graph of $\mathcal{W}$ is a defining suite. Thus, a generalized chamber system canonically induces simple Weyl data. A generalized chamber system is called **thin**, **weak**, or **thick** if each of its panels is correspondingly thin, weak, or thick in the sense of Section 2.3.4.

Inverses and Compositions of Edges. Let $\mathcal{W}$ be a generalized chamber system. For an edge $i \in \mathcal{W}$, with $\upsilon(i) = s$, we denote by i^{-1} the inverse of i in the panel groupoid $\mathcal{W}_s$. Thus, i^{-1} has the same set of extremities and type as i, but points in the opposite direction. We may have $i = i^{-1}$ if i is a loop. For edges $i, i' \in \mathcal{W}$ with $\upsilon(i) = \upsilon(i') = s$ and $\tau(i) = \iota(i')$, we denote by $i; i'$ their composition in $\mathcal{W}_s$. Thus, as long as $i' \neq i^{-1}$, then $i;i'$ is an edge of $\mathcal{W}$ with $\iota(i;i') = \iota(i)$, $\tau(i;i') = \tau(i')$, and $\upsilon(i;i') = s$.

For edges $i, i' \in \mathcal{W}$ with $\tau(i) = \iota(i')$, we let ii' denote the gallery which is the concatenation of i with i'.

The Inverse of a Gallery. Let β be a gallery in a generalized chamber system $\mathcal{W}$, and let $i_1, \ldots, i_n$ be the sequence of edges of β. The **inverse** β^{-1} of β is the gallery in $\mathcal{W}$ whose sequence of edges is $i_n^{-1}, \ldots, i_1^{-1}$.

Backtracks and Detours. A **backtrack** β is gallery which consists of an edge followed by its inverse; $\beta = ii^{-1}$. A **detour** is a gallery β which consists of two edges of the same type which are not mutually inverse; $\beta = ii'$, where $\upsilon(i) = \upsilon(i')$, and $i' \neq i^{-1}$. In particular, if ii' is a detour, then $i;i'$ is an edge of $\mathcal{W}$. If a gallery β is of type ss for some $s \in S$, then β is either a backtrack or a detour.

We now introduce morphisms of generalized chamber systems only. We postpone the definition of morphisms of Weyl data until we have developed the notion of homotopy of galleries (see Section 3.1.3).

Morphisms of Generalized Chamber Systems. Let $\sigma : M \to M'$ be a morphism of Coxeter matrices, and let $\mathcal{W}$ and $\mathcal{W}'$ be generalized chamber systems of type M and M' respectively. A **morphism** $\omega : \mathcal{W} \to \mathcal{W}'$ of a generalized chamber systems over σ is a labeled graph morphism over σ (in the sense of Section 2.1.1), which satisfies the following two properties:

(i) for all edges $i \in \mathcal{W}$, we have,

$$\omega(i^{-1}) = (\omega(i))^{-1}$$

(ii) for all detours ii' in $\mathcal{W}$, we have,

$$\omega(i;i') = \omega(i); \omega(i').$$

Let $\omega : \mathcal{W} \to \mathcal{W}'$ and $\omega' : \mathcal{W}' \to \mathcal{W}''$ be morphisms of generalized chamber systems over σ and σ' respectively. The **composition** $\omega' \circ \omega$ of ω with ω' is their composition as graph morphisms. It is easy to check that $\omega' \circ \omega : \mathcal{W} \to \mathcal{W}''$ is itself a morphism of generalized chamber systems over $\sigma' \circ \sigma$. If $\mathcal{W}$ and $\mathcal{W}'$ are generalized chamber systems of the same type M, then we assume that a morphism $\omega : \mathcal{W} \to \mathcal{W}'$ takes place over the identity $M \to M$.

Remark 3.2. Since a chamber system is slim as a labeled graph, a morphism of generalized chamber systems whose target is a chamber system does not need the auxiliary function ω_1 of the edges (see **??**). In particular, a morphism of chamber systems is just a function on the chambers which preserves s-equivalence $\sim_s$, for all $s \in S$, and such that equivalent but unequal chambers do not get mapped to the same chamber.

Remark 3.3. The composition of a backtrack with a morphism is a backtrack, and the composition of a detour with a morphism is a detour. The first statement follows from the first property of morphisms. To see the second statement, suppose that the composition of a detour ii' with a morphism ω is a backtrack. Then $\omega(i') = \omega(i)^{-1}$. But $\omega(i; i') = \omega(i); \omega(i')$ must be non-trivial, a contradiction.

We say a homomorphism of groupoids $\varphi : \mathcal{G} \to \mathcal{G}'$ has a **trivial kernel** if for each edge $g \in \mathcal{G}$, $\varphi(g) = 1$ implies that $g = 1$.

s-Homomorphisms. If we think of a generalized chamber system as an indexed collection of groupoids $(\mathcal{W}_0, (\mathcal{W}_s)_{s\in S})$, then a morphism of generalized chamber systems is equivalently an indexed collection of groupoid morphisms $\varphi_s : \mathcal{W}_s \to \mathcal{W}_{\sigma(s)}$, $s \in S$, each of which has a trivial kernel and consists of the same function on the chambers.

To see this, suppose we have a morphism $\omega : \mathcal{W} \to \mathcal{W}'$ of generalized chamber systems over σ. Let $\omega_s : \mathcal{W}_s \to \mathcal{W}_{\sigma(s)}$ be the graph morphism whose map on chambers is ω_0, and whose map on edges is the restriction of ω_1 to the edges labeled by s, union the map on trivial edges determined by ω_0. Then ω_s is a groupoid homomorphism by the properties ω has as a generalized chamber system morphism. It has a trivial kernel because non-trivial edges get mapped to non-trivial edges by ω. We call the groupoid homomorphism ω_s the **s-homomorphism** of ω. Conversely, suppose we have an indexed collection of groupoid morphisms $\varphi_s : \mathcal{W}_s \to \mathcal{W}_{\sigma(s)}$, $s \in S$, each of which has a trivial kernel and consists of the same function $(\varphi_s)_0 : \mathcal{W}_0 \to \mathcal{W}'_0$ on the chambers. Let $\omega_0 = (\varphi_s)_0$ for any $s \in S$, and let $\omega_1(i) = \varphi_{v(i)}(i)$. Then its easy to check that $\omega = (\omega_0, \omega_1)$ is a generalized chamber system morphism $\omega : \mathcal{W} \to \mathcal{W}'$ over σ.

Notice that the morphisms we have defined are like isometries in that they preserve the W-length of galleries. One could define a more general kind of morphism, which would be like so-called metric maps from metric geometry. These would allow non-trivial edges to be mapped to trivial edges. When making analogies with metric geometry, one should have the Bruhat order in mind.

3.1.3 Homotopy of Galleries

Throughout this section, we assume that $\mathcal{W}$ denotes Weyl data; that is $\mathcal{W}$ is a generalized chamber system equipped with a choice of defining suites. Concerning our description of gallery homotopy, the analogy one should have in mind is that of groups presented by generators and relations.

Elementary Homotopy. Let $i \in \mathcal{W}$ be any edge. Let $j, j' \in \mathcal{W}$ be edges such that jj' is a detour in $\mathcal{W}$, and let $k = j; j'$. Let $\theta(s,t)$ be a defining suite of $\mathcal{W}$. Then a **contraction** of a gallery in $\mathcal{W}$ is any of the following:

(i) delete a backtrack; an alternation from a gallery of the form $\beta i i^{-1} \beta'$ to the gallery $\beta\beta'$

(ii) take a shortcut; an alternation from a gallery of the form $\beta j j' \beta'$ to the gallery $\beta k \beta'$

(iii) delete a defining suite; an alternation from a gallery of the form $\beta\theta(s,t)\beta'$ to the gallery $\beta\beta'$.

An **expansion** is an alteration of a gallery which is the inverse of a contraction. An **elementary homotopy** is an expansion or a contraction. A 1**-elementary homotopy** is an elementary homotopy of type (i) or (ii). A 2**-elementary homotopy** is an elementary homotopy of type (iii).

Notice that 1-elementary homotopies of type (i) change the type of a gallery by adding or deleting the subword ss, and 1-elementary homotopies of type (ii) change the type of a gallery by moving between subwords s and ss. Finally, 2-elementary homotopies change the type of a gallery by adding or deleting the subword $p_{2m_{st}}(s,t)$.

Remark 3.4. A 1-elementary homotopy of type (i) does not preserve length, but it does preserve W-length. An 1-elementary homotopy of type (ii) does not preserve length or W-length. A 1-elementary homotopy of either kind will not preserve the property of being a minimal gallery or a geodesic. A 2-elementary homotopy does not preserve length, but it does preserve W-length. A 2-elementary homotopy also will not preserve the property of being a minimal gallery or a geodesic.

Homotopy. A **homotopy** is an alteration of a gallery which is a composition of elementary homotopies. If a gallery β can be altered via a homotopy to give the gallery $\hat{\beta}$, we say β is **homotopic** to $\hat{\beta}$, and write $\beta \sim \hat{\beta}$. Since the inverse of an elementary homotopy is an elementary homotopy, '$\sim$' is an equivalence relation on galleries. We denote by $[\beta]$ the homotopy equivalence class of the gallery β. We say a gallery β is **null-homotopic** if β is homotopic to a trivial gallery, i.e. a gallery of length 0.

Remark 3.5. Let β be a gallery in Weyl data $\mathcal{W}$. Then the concatenations $\beta\beta^{-1}$ and $\beta^{-1}\beta$ are null-homotopic since these galleries can be altered to give a trivial gallery via a composition of contractions of type (i).

Remark 3.6. If we have some homotopy altering β to give $\hat{\beta}$, then this homotopy can be applied to any gallery of the form $\beta'\beta\beta''$, to give $\beta'\hat{\beta}\beta''$. Thus, if galleries differ only by homotopic subgalleries, then they are homotopic.

We now introduce a kind of homotopy which preserves both length and W-length.

Elementary Strict Homotopy. Let $\rho(s,t)$ be a maximal (s,t)-geodesic, and let $\rho(t,s)$ be a maximal (t,s)-geodesic such that $\rho(s,t) \sim \rho(t,s)$. Note that $\rho(s,t)$ being homotopic to $\rho(t,s)$ does not imply that there is a defining suite containing them, i.e. homotopies between them may be complicated compositions of many elementary homotopies. An **elementary strict homotopy** is an alteration from a gallery of the form $\beta\rho(s,t)\beta'$ to the gallery $\beta\rho(t,s)\beta'$. It follows from Remark 3.6 that $\beta\rho(s,t)\beta' \sim \beta\rho(t,s)\beta'$. Elementary strict homotopies change the type of a gallery by an elementary strict homotopy of words.

Strict Homotopy. A **strict homotopy** is an alteration of a gallery which is a composition of elementary strict homotopies. If a gallery β can be altered via a strict homotopy to give the gallery $\hat{\beta}$, we say β is **strictly homotopic** to $\hat{\beta}$, and write $\beta \simeq \hat{\beta}$. Since the inverse of an elementary strict homotopy is an elementary strict homotopy, '$\simeq$' is an equivalence relation on galleries. We denote by $[\beta]_{\simeq}$ the strict homotopy equivalence class of the gallery β. One can easily see that if two galleries are strictly homotopic, then they are homotopic. Strict homotopies change the type of a gallery by a strict homotopy of words.

Remark 3.7. Strict homotopies preserve both the length and the W-length of a gallery, and therefore the property of being a minimal gallery or a geodesic. Thus, if γ is geodesic, then so is every gallery in $[\gamma]_{\simeq}$.

The W-Length of $[\beta]_{\simeq}$. We define the **W-length** of a strict homotopy class of galleries $[\beta]_{\simeq}$ (but usually geodesics) to be β_W. That is, the W-length of $[\beta]_{\simeq}$ is the W-length of the galleries which it contains.

The γ-Gallery Map F_γ. Let γ be a geodesic in $\mathcal{W}$. The **γ-gallery map** F_γ is the function whose domain is $[\gamma]_{\simeq}$, and which sends a gallery to its type. Thus,

$$F_\gamma : [\gamma]_{\simeq} \to M(S), \quad \hat{\gamma} \mapsto \hat{\gamma}_S.$$

By Remark 3.7, each $\hat{\gamma} \in [\gamma]_{\simeq}$ is a geodesic, and so the image of F_γ is a subset of the words which are reduced decompositions of γ_W.

Suites. An **(s,t)-suite**, or **suite**, is an (s,t)-cycle which is null-homotopic. Since a 2-elementary homotopy can be applied to a defining suite to produce a trivial gallery, we see that defining suites are suites. An (s,t)-cycle $\theta(s,t)$ can be presented as the concatenation of a maximal (s,t)-geodesic $\rho(s,t)$ with the inverse of a maximal (t,s)-geodesic $\rho(t,s)$ thus,

$$\theta(s,t) = \rho(s,t)\rho(t,s)^{-1}.$$

The geodesics $\rho(s,t)$ and $\rho(t,s)$ are uniquely determined. Notice that if m_{st} is even, then $\rho(t,s)^{-1}$ is a maximal (s,t)-geodesic, and if m_{st} is odd, then $\rho(t,s)^{-1}$ is a maximal (t,s)-geodesic. We now show that an elementary strict homotopy always take place 'within' a suite, and conversely, suites give rise to elementary strict homotopies.

Proposition 3.1. Let $\theta(s,t)$ be an (s,t)-cycle in Weyl data $\mathcal{W}$. Let $\theta(s,t) = \rho(s,t)\rho(t,s)^{-1}$ be the presentation of $\theta(s,t)$ as a concatenation of maximal alternating geodesics. Then $\theta(s,t)$ $\mathcal{W}$ if and only if $\rho(s,t) \sim \rho(t,s)$.

Proof. We have,

$$\begin{aligned} &\theta(s,t) \text{ is a suite} \\ \Longleftrightarrow\ &\rho(s,t)\rho(t,s)^{-1} \text{ is null-homotopic} \\ \Longleftrightarrow\ &\rho(s,t)\rho(t,s)^{-1}\rho(t,s) \sim \rho(t,s) \\ \Longleftrightarrow\ &\rho(s,t) \sim \rho(t,s) \end{aligned}$$

□

Let θ be a cycle in $\mathcal{W}$, and let $i_1, \ldots, i_n$ be its sequence of edges. A **cyclic permutation** of θ is a cycle whose sequence of edges is a cyclic permutation $i_m, \ldots, i_n, i_1, \ldots, i_{m-1}$ of the sequence of edges of θ.

Proposition 3.2. Let θ be a cycle in Weyl data $\mathcal{W}$. If θ is a suite of $\mathcal{W}$, then so is θ^{-1}, and so is any cyclic permutation of θ.

Proof. Since θ is null-homotopic, we have $\theta \sim \theta\theta^{-1}$ by Remark 3.5, and so $\theta^{-1}\theta \sim \theta^{-1}\theta\theta^{-1}$. Thus, $\theta^{-1}\theta\theta^{-1}$ is null-homotopic, and so θ^{-1} must also be null-homotopic.

Let $\theta = i_1, \ldots, i_n$, let $\theta' = i_m, \ldots, i_n, i_1, \ldots, i_{m-1}$ be a cyclic permutation of θ, and let $\beta = i_1, \ldots, i_{m-1}$. Then $\theta' \sim \beta^{-1}\theta\beta$. But θ is null-homotopic by hypothesis. Therefore $\theta' \sim \beta^{-1}\beta$, and so θ' is null-homotopic. □

3.1.4 Morphisms of Weyl Data

We are now able to define morphisms of Weyl data. Roughly speaking, a morphism of Weyl data is a morphism of the underlying generalized chamber systems which additionally sends defining suites to suites:

Morphisms of Weyl Data. Let $\sigma : M \to M'$ be a morphism of Coxeter matrices, and let $\mathcal{W}$ and $\mathcal{W}'$ be Weyl data of type M and M' respectively. A **morphism** $\omega : \mathcal{W} \to \mathcal{W}'$ of Weyl data over σ is a morphism of the underlying generalized chamber systems over σ (see Section 3.1.2), which additionally satisfies the following property

(iii) for each pair $(s,t) \in S \times S$ with $s \neq t$ and $m_{st} < \infty$, and each defining suite $\theta(s,t) \in \mathcal{W}(s,t)$, we have,

$$\omega \circ \theta(s,t) \text{ is a suite of } \mathcal{W}'.$$

As with generalized chamber system morphisms, if $\mathcal{W}$ and $\mathcal{W}'$ are Weyl data of type M, then we assume that a morphism $\omega : \mathcal{W} \to \mathcal{W}'$ takes place over the identity $M \to M$.

We need to establish a result before we mention the composition of morphisms of Weyl data. First, we show that morphisms of Weyl data preserve homotopies:

Lemma 3.3. Let $\omega : \mathcal{W} \to \mathcal{W}'$ be a morphism of Weyl data. If β and $\hat{\beta}$ are homotopic galleries in $\mathcal{W}$, then $\omega \circ \beta$ and $\omega \circ \hat{\beta}$ are homotopic galleries in $\mathcal{W}'$.

Proof. Suppose that $\beta \sim \hat{\beta}$ via an elementary homotopy. If it is a 1-elementary homotopy, then $\omega \circ \beta \sim \omega \circ \hat{\beta}$ by Remark 3.3. If it is a 2-elementary homotopy, then $\omega \circ \beta \sim \omega \circ \hat{\beta}$ by the fact that morphisms send defining suites to suites. The result then follows, since a homotopy is a composition of elementary homotopies. □

By definition, morphisms send defining suites to suites, however Lemma 3.3 shows that morphisms also send suites to suites:

Corollary 3.3.1. Let $\omega : \mathcal{W} \to \mathcal{W}'$ be a morphism of Weyl data and let θ be a suite of $\mathcal{W}$, then $\omega \circ \theta$ is a suite of $\mathcal{W}'$.

Proof. If θ is a suite, then $\theta \sim \beta_0$, where β_0 is a trivial gallery. Then, by Lemma 3.3, $\omega \circ \theta \sim \omega \circ \beta_0$. But $\omega \circ \beta_0$ is also a trivial gallery. Thus, $\omega \circ \theta$ is null-homotopic, and therefore is a suite. □

The **composition** of morphisms of Weyl data is just their composition as morphisms of generalized chamber systems. By Corollary 3.3.1, this composition is again a morphism of Weyl data.

Isomorphisms of Weyl Data. Let $\mathcal{W}$ and $\mathcal{W}'$ be Weyl data of type M. An **isomorphism** $\omega : \mathcal{W} \to \mathcal{W}'$ is a morphism which has an inverse. An **automorphism** of Weyl data $\mathcal{W}$ is an isomorphism from $\mathcal{W}$ to itself. We denote by $\mathrm{Aut}(\mathcal{W})$ the group whose elements are automorphisms of $\mathcal{W}$, and whose binary operation is the composition of morphisms.

We have the following characterization of isomorphisms:

Proposition 3.4. Let $\mathcal{W}$ and $\mathcal{W}'$ be Weyl data of type M. A morphism $\omega : \mathcal{W} \to \mathcal{W}'$ is an isomorphism if and only if ω is bijective on chambers and edges, and for all galleries β in $\mathcal{W}$ such that $\omega \circ \beta$ is a defining suite of $\mathcal{W}'$, we have that β is a suite of $\mathcal{W}$.

Proof. Suppose that $\omega : \mathcal{W} \to \mathcal{W}'$ is an isomorphism, and let ω^{-1} be the inverse of ω. Then ω_0 and ω_1 must be bijective since they have inverses ω_0^{-1} and ω_1^{-1} respectively as functions of sets. Let β be a gallery in $\mathcal{W}$ such that $\omega \circ \beta$ is a defining suite of $\mathcal{W}'$. But as a morphism, ω^{-1} sends defining suites to suites, and so $\omega^{-1} \circ \omega \circ \beta = \beta$ must be a suite.

Now suppose that $\omega : \mathcal{W} \to \mathcal{W}'$ is a morphism, ω_0 and ω_1 are bijective, and for all galleries β in $\mathcal{W}$ such that $\omega \circ \beta$ is a defining suite of $\mathcal{W}'$, we have that β is a suite of $\mathcal{W}$. Let $\omega^{-1} = (\omega_0^{-1}, \omega_1^{-1})$. This is clearly a morphism of generalized chamber systems. Moreover, ω^{-1} is a morphism of Weyl data by the hypothesis on ω, and so is an inverse for ω. $\square$

3.1.5 Restrictions, the Defining Graph, and Residues

In this section, we define various notions of 'subdata' of Weyl data, and introduce a way of encoding Weyl data in a labeled simplicial graph. We fix the following notation; M is a Coxeter matrix on S, and $\mathcal{W}$ is Weyl data of type M.

Full Subdata. Let $\mathcal{C}$ be a subset of the chambers of $\mathcal{W}$. The **full subdata** $\mathcal{W}_{\mathcal{C}}$ of $\mathcal{W}$ on $\mathcal{C}$ is the Weyl data with chambers $\mathcal{C}$, and edges,

$$(\mathcal{W}_{\mathcal{C}})_1 = \left\{ i \in \mathcal{W}_1 : e_i \cap \mathcal{C} = e_i \right\} \subseteq \mathcal{W}_1.$$

The extremities and type function of $\mathcal{W}_{\mathcal{C}}$ are the restrictions to $(\mathcal{W}_{\mathcal{C}})_1$ of the corresponding functions of $\mathcal{W}$. The panel groupoids are the obvious restrictions of the panel groupoids of $\mathcal{W}$, and the defining suites are the defining suites of $\mathcal{W}$ whose images are contained in $\mathcal{W}_X$. This gives $\mathcal{W}_{\mathcal{C}}$ the structure of Weyl data of type M.

***J*-Restrictions.** Let $J \subseteq S$. The ***J*-restriction** $\mathcal{W}_J$ of $\mathcal{W}$ is the Weyl data of type M_J whose underlying graph of type M_J is the J-restriction of $\mathcal{W}$ (in the sense of Section 3.1.1), whose panel groupoid of type s, for $s \in J$, is the panel groupoid of type s of $\mathcal{W}$, and whose set of defining (s,t)-suites, for $(s,t) \in J \times J$, is the set of defining (s,t)-suites $\mathcal{W}(s,t)$ of $\mathcal{W}$.

For $J \subseteq J' \subseteq S$, there is a natural embedding $\varepsilon_{JJ'} : \mathcal{W}_J \hookrightarrow \mathcal{W}_{J'}$ over the inclusion $J \hookrightarrow J'$, called the **internal embedding** from J to J'. We denote ε_{JS} by ε_J.

In the same way that the data which defines a Coxeter group can be encoded in an edge labeled simplicial graph, Weyl data can be encoded in a vertex and edge labeled simplicial graph, whose flags (adjacent vertex-edges pairs) are associated to embeddings of rank 1 Weyl data into rank 2 Weyl data.

The Defining Graph of Weyl Data. Let $\mathcal{W}$ be Weyl data of type W. Let L be the defining graph of W. The **defining graph** $\mathcal{L}$ of $\mathcal{W}$ is the graph L whose vertex $s \in S$ is labeled by $\mathcal{W}_s$, and whose edge $J = \{s,t\} \in E(L)$ is labeled by $\mathcal{W}_J$. Each ordered pair, or **flag**, $(s, J) \in V(L) \times E(L)$ such that $s \in J$, is equipped with the embedding $\varepsilon_{sJ} : \mathcal{W}_s \hookrightarrow \mathcal{W}_J$. The defining graph of $\mathcal{W}$ essentially reconstructs $\mathcal{W}$ as an amalgam of rank 2 Weyl data along the panel groupoids of $\mathcal{W}$.

Gluing Data. Data which encodes the embeddings along flags is called **gluing data**. Examples of gluing data include the four diagrams in [Ron89, p. 48], which determine the quotients of the four chamber-regular lattices of type $\widetilde{A}_2$ and order 2, and the 'based difference matrices' in [Ess13], which determine the quotients of the so-called Singer lattices of type $\widetilde{A}_2$. See Chapter 5 for an explicit description of the Weyl data associated to the lattices of Essert.

***J*-Residues.** Let $C \in \mathcal{W}$ be a chamber. The ***J*-residue** $R_J(C)$ at C is the connected component of $\mathcal{W}_J$ which contains C. Formally, $R_J(C)$ is the full subdata of $\mathcal{W}_J$ on the subset of $\mathcal{W}_0$ containing those vertices which are connected by galleries in $\mathcal{W}_J$ to C. Thus, $R_J(C)$ is connected Weyl data of type M_J. For $J = \{s\}$, we write $R_s(C)$. If $|J| = n$, then we call $R_J(C)$ an ***n*-residue**. The 1-resides are the panels, and the 0-residues are the chambers. A J-residue is called **spherical** if J is a spherical subset.

Lemma 3.5. Let $\mathcal{W}$ be Weyl data and let R be a residue of $\mathcal{W}$. Let $g \in \mathrm{Aut}(\mathcal{W})$ be an automorphism of $\mathcal{W}$. If there exists a chamber $C \in R$ with $g \cdot C \in R$, then $g \cdot R = R$.

Proof. Let J be the type of R. The automorphism g is an automorphism of $\mathcal{W}_J$, preserving the connectivity of J-residues. Therefore $g \cdot R$ is contained in R. But $g \cdot R$ must be a J-residue of $\mathcal{W}$, and therefore $g \cdot R = R$. □

Remark 3.8. Groups act by automorphisms on Weyl data. By Lemma 3.5, the action of a group G on Weyl data $\mathcal{W}$ induces an action of G on the set of J-residues of $\mathcal{W}$, for each $J \subseteq S$. The action on $\emptyset$-residues is the restriction of the action to chambers, and the action on S-residues is the action induced on the connected components of $\mathcal{W}$.

Local Properties. If p is a property of rank n Weyl data, we say Weyl data $\mathcal{W}$ is n-p if each n-residue of $\mathcal{W}$ has the property p. Let G be a group which acts on Weyl data $\mathcal{W}$. If p is a property of G-sets, we say this action is n-p if the action induced on n-residues has the property p. In particular, we have the notion of n-free actions. A 0-free action is just an action which is free on chambers. An n-free action is m-free for all $m < n$.

3.1.6 The Fundamental Groupoid of Weyl Data

Roughly speaking, the fundamental groupoid of Weyl data $\mathcal{W}$ is the groupoid that's generated by the panel groupoids $(\mathcal{W}_s)_{s\in S}$, subject to the defining suites $\mathcal{W}(s,t)$ being treated as relators.

Definition of the Fundamental Groupoid. The **fundamental groupoid** of Weyl data $\mathcal{W}$, denoted $\overline{\mathcal{W}}$, is the groupoid whose set of vertices is the set of chambers of $\mathcal{W}$, and whose set of edges is,

$$\overline{\mathcal{W}}_1 = \big\{[\beta] : \beta \text{ is a gallery in } \mathcal{W}\big\}.$$

Recall that $[\beta]$ denotes the set of galleries which are homotopic to β. The extremities of edges are,

$$\iota([\beta]) = \iota(\beta), \quad \tau([\beta]) = \tau(\beta).$$

Let $\mathrm{id} : \mathcal{W}_0 \to \overline{\mathcal{W}}_1$ be the map which sends a chamber $C \in \mathcal{W}$ to the class of the trivial gallery at C, let $\mathrm{inv} : \overline{\mathcal{W}}_1 \to \overline{\mathcal{W}}_1$ be the map $[\beta] \mapsto [\beta^{-1}]$, and let the composition be,

$$[\beta][\beta'] = [\beta\beta']. \tag{$\heartsuit$}$$

First, we need to check that these functions are well defined:

Proposition 3.6. Let $\mathcal{W}$ be Weyl data. The extremities, inverses and composition of $\overline{\mathcal{W}}$ are well defined.

Proof. For the extremities, just notice that an elementary homotopy of a gallery preserve its extremities. For the inverses, suppose that $\beta \sim \hat{\beta}$ via an elementary homotopy, then we need to show that $\beta^{-1} \sim \hat{\beta}^{-1}$. In the case of a 1-elementary homotopy of type (i), just notice that the inverse of a backtrack is a backtrack. In the case of a 1-elementary homotopy of type (ii), notice that if jj' is a detour with $k = j; j'$, then $j'^{-1}j^{-1}$ is a detour with $k^{-1} = j'^{-1}; j^{-1}$. The case of a 2-elementary homotopy follows from Proposition 3.2. For the composition, if $\beta \sim \hat{\beta}$, then $\beta\beta' \sim \hat{\beta}\beta'$ by Remark 3.6. Similarly, if $\beta \sim \hat{\beta}$, then $\beta'\beta \sim \beta'\hat{\beta}$. □

We now show that $\overline{\mathcal{W}}$ is indeed a groupoid:

Proposition 3.7. Let $\mathcal{W}$ be Weyl data. Then the fundamental groupoid $\overline{\mathcal{W}}$ of $\mathcal{W}$ is a groupoid.

Proof. The properties (i) and (ii) of groupoids are clearly satisfied, property (iii) follows from the associativity of the concatenation of galleries, property (iv) follows from the fact that the concatenation of a gallery β with a trivial gallery is β, and for property (v), we have that $[\beta][\beta]^{-1}$ is trivial by Remark 3.5. □

Notice that $\overline{\mathcal{W}}$ is connected if and only if $\mathcal{W}$ is connected. Weyl data $\mathcal{W}$ is called **simply connected** if $\overline{\mathcal{W}}$ is a setoid. We will tend to denote connected and simply connected Weyl data by Δ. The **fundamental group** $\pi_1(\mathcal{W})$ of connected Weyl data $\mathcal{W}$ is the fundamental group of $\overline{\mathcal{W}}$. Thus, connected Weyl data is simply connected if and only if its fundamental group is trivial.

The Homomorphism Induced by a Morphism of Weyl Data. Given a morphism of Weyl data $\omega : \mathcal{W} \to \mathcal{W}'$, let $\bar{\omega}$ denote the following homomorphism of groupoids,

$$\bar{\omega} : \overline{\mathcal{W}} \to \overline{\mathcal{W}}', \quad [\beta] \mapsto [\omega \circ \beta].$$

This is well defined because homotopies descend (see Lemma 3.3). To see that $\bar{\omega}$ is a groupoid homomorphism, we have,

$$[\beta][\beta'] = [\beta\beta'] \mapsto [\omega \circ \beta\beta'] = [\omega \circ \beta][\omega \circ \beta'].$$

We call $\bar{\omega}$ the groupoid homomorphism **induced** by ω. The map $\omega \mapsto \bar{\omega}$ is functorial; for morphisms $\omega : \mathcal{W} \to \mathcal{W}'$ and $\omega' : \mathcal{W}' \to \mathcal{W}''$, putting $\omega'' = \omega' \circ \omega$, we have $\bar{\omega}'' = \bar{\omega}' \circ \bar{\omega}$. This follows directly from the fact that $(\omega' \circ \omega) \circ \beta = \omega' \circ (\omega \circ \beta)$.

The J-Groupoids. We call the fundamental groupoid $\overline{\mathcal{W}}_J$ of $\mathcal{W}_J$ the ***J*-groupoid** of $\mathcal{W}$. For $J \subseteq J' \subseteq S$, we have a homomorphism $\bar{\varepsilon}_{JJ'} : \overline{\mathcal{W}}_J \to \overline{\mathcal{W}}_{J'}$, called the **internal homomorphism** from J to J'. Later on, we will see that $\bar{\varepsilon}_{JJ'}$ is an embedding of groupoids when $\mathcal{W}$ is the quotient of a building.

3.1.7 The Weyl Data of Coxeter Groups

Coxeter groups and their Cayley graphs are prototypical examples of Weyl data. As usual, let W be the Coxeter group associated to the Coxeter matrix M on S. The 1-residues of the Cayley graph $\mathcal{C}(W)$ consist of pairs of edges pointing in opposite directions. We give these the structure of the groupoid 1×2. Then the defining suites of $\mathcal{C}(W)$ are all the possible ones. In particular, Cayley graphs are simply connected as Weyl data. Let us denote $\mathcal{C}(I_2(m))$ by $\mathcal{C}_m$, where $I_2(m)$ is the dihedral group of order $2m$.

A Coxeter group W is naturally Weyl data $\mathcal{W}(W)$ as follows. Take a single chamber and let the type function be a bijection between the edges of $\mathcal{W}(W)$ and S. The panel groupoids are all $Z_2 \times 1$. Again, the defining suites are all the possible ones. It is easy to see that the fundamental group of $\mathcal{W}(W)$ is isomorphic to W. Covering theory of Weyl graphs (see Section 3.3) will formalize and generalize the notion that $\mathcal{C}(W)$ is the universal cover of $\mathcal{W}(W)$, and that $\mathcal{W}(W)$ is the quotient of the action of W on $\mathcal{C}(W)$. See Figure 3.1, which shows $\mathcal{C}(W)$ and $\mathcal{W}(W)$ for $W = \widetilde{A}_2$.

3.2 Pre-Weyl Graphs and 2-Weyl Graphs

In this section, we introduce pre-Weyl and 2-Weyl graphs, and collect some of their basic properties. We will see that 2-Weyl graphs are exactly the quotients of chamber systems of type M, introduced in [Tit81], by chamber-free actions. Chamber systems of type M are also known as 'pre-buildings' in [TW02], and 'SCABs' in [Kan86].

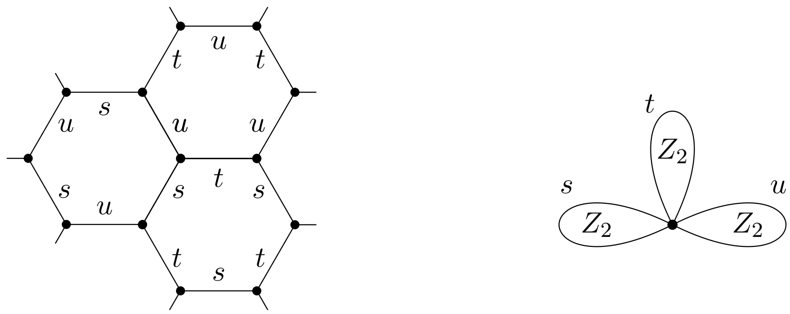

Figure 3.1: 13 chambers of $\mathcal{C}(\widetilde{A}_2)$, and $\mathcal{W}(\widetilde{A}_2)$

3.2.1 Definition of Pre-Weyl Graphs and 2-Weyl Graphs

See Section 3.1.2 for an introduction to Weyl data. We only briefly restate the definition here. Recall that by a 'geodesic', we just mean a gallery of reduced type.

Definition 3.1. Let M be a Coxeter matrix on S. A **pre-Weyl graph** $\mathcal{W}$ of type M is Weyl data $\mathcal{W} = (\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_s, \mathcal{W}(s,t))$ of type M, that is:

(1) a directed multigraph $\mathcal{W} = (\mathcal{W}_0, \mathcal{W}_1)$ whose edges are labeled by S

(2) for each $s \in S$, a groupoid $\mathcal{W}_s$ whose non-trivial edges are of those $\mathcal{W}$ which are labeled by s

(3) for each $(s,t) \in S \times S$ such that $s \neq t$ and $m_{st} < \infty$, a set $\mathcal{W}(s,t)$ of cycles in $\mathcal{W}$ of type $p_{2m_{st}}(s,t)$, called defining suites

which satisfies the following two properties:

(PW0) no panel is isomorphic to the trivial groupoid 1×1

(PW1) each maximal (s,t)-geodesic is homotopic to a maximal (t,s)-geodesic.

If, in addition, $\mathcal{W}$ satisfies the following property,

(2W) homotopic alternating geodesics have the same W-length

then $\mathcal{W}$ is called a **2-Weyl graph**, or we say that $\mathcal{W}$ is 2-Weyl.

Notice that property (PW0) just says that the underlying generalized chamber system of $\mathcal{W}$ is weak. Property (PW1) implies that given a gallery β and an elementary strict homotopy of words $\beta_S \mapsto f$, we can always find a gallery $\hat{\beta}$ such that $\hat{\beta}_S = f$ and $\hat{\beta} \sim \beta$.

These axioms should be compared with those in [Tit81, Section 3.2] (the axiom playing the role of (2W) is denoted ($\mathrm{CS}_M 2$) by Tits). A **morphism**, **isomorphism**, or **automorphism** of pre-Weyl or 2-Weyl graphs is a morphism, isomorphism, or automorphism, respectively, of the underlying Weyl data (see Section 3.1.3).

The Panel Groupoids as Subgroupoids of $\overline{\mathcal{W}}$. A gallery whose type is a one letter word is an alternating geodesic. Therefore property (2W) implies that a gallery consisting of a single edge cannot be null homotopic. Thus

$$\overline{\varepsilon}_s : \mathcal{W}_s \to \overline{\mathcal{W}}, \quad i \mapsto [i]$$

is injective for each $s \in S$. In the setting of 2-Weyl graphs, this allows us to make the convention of identifying $\mathcal{W}_s$ with the subgroupoid $\overline{\varepsilon}_s(\mathcal{W}_s) \leq \overline{\mathcal{W}}$. In other words, we think of the edges of a 2-Weyl graph $\mathcal{W}$ as also being edges of its fundamental groupoid $\overline{\mathcal{W}}$.

3.2.2 First Properties of Pre-Weyl Graphs

In this section, we collect some basic properties of pre-Weyl graphs. We now begin to make use of properties (MT1) and (MT2), the deletion condition, and the exchange condition of Coxeter groups; these are proven in the language of pre-Weyl graphs in Appendix A.

Proposition 3.8. Let $\mathcal{W}$ be Weyl data. Property (PW0) is equivalent to the property that for any chamber $C \in \mathcal{W}$, and any word f over S, there exists a gallery β in $\mathcal{W}$ with $\iota(\beta) = C$ and $\beta_S = f$.

Proof. By (PW0), connected components of panel groupoids cannot be the trivial groupoid 1×1. Thus, for each $s \in S$ and every chamber $C \in \mathcal{W}$, there exists an edge i with $\iota(i) = C$ and $\upsilon(i) = s$. The required gallery β can then be formed by concatenating edges. The converse is clear. □

Of course, β will rarely be unique for fixed choices of C and f. The lack of uniqueness is caused by thick panels. Thus, β will be unique in thin pre-Weyl graphs.

Proposition 3.9. Let $\mathcal{W}$ be Weyl data. Property (PW1) is equivalent to the property that for all geodesics γ in $\mathcal{W}$, the γ-gallery map F_γ is surjective into the words which are reduced decompositions of γ_W.

Proof. By (MT2), every reduced decomposition of γ_W is strictly homotopic to γ_S. By definition, this strict homotopy of words is a composition of elementary strict homotopies, which can be done at the gallery level by (PW1). The converse is clear. □

Proposition 3.10. Let $\mathcal{W}$ be a pre-Weyl graph. Then every gallery β of $\mathcal{W}$ is homotopic to a geodesic. Moreover, this homotopy can be chosen to be a composition of 1-elementary contractions, and strict homotopies.

Proof. If β_S is reduced, we are done. If not, then by (MT1), β_S is strictly homotopic to a word which repeats a letter. Therefore, by property (PW1), β is strictly homotopic

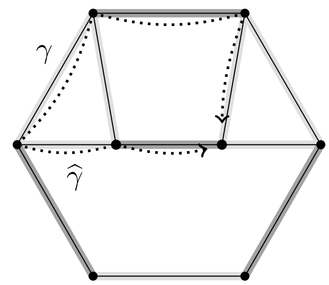

Figure 3.2: A pre-Weyl graph of type A_2

to a gallery β' which contains either a backtrack or a detour. Then, β' is homotopic via a 1-elementary contraction to a shorter gallery. We can keep carrying out this process of applying strict homotopies and then 1-elementary contractions until we obtain a geodesic, which will be homotopic to β. □

It follows that if $\mathcal{W}$ is a pre-Weyl graph, then the edges of its fundamental groupoid are homotopy classes of *geodesics*:

$$\overline{\mathcal{W}}_1 = \big\{[\beta] : \beta \text{ is a gallery in } \mathcal{W}\big\} = \big\{[\gamma] : \gamma \text{ is a geodesic in } \mathcal{W}\big\}.$$

Corollary 3.10.1. Let $\mathcal{W}$ be a pre-Weyl graph. Then the galleries consisting of a minimal number of edges between two fixed chambers of $\mathcal{W}$ are geodesics.

Proof. Let β be a minimal gallery. If β_S is not reduced, then we can obtain a gallery from β via a strict homotopy and a 1-elementary contraction which is homotopic to β, and yet is shorter than β. This contradicts the minimality of β. □

In buildings the converse holds; minimal galleries and geodesics coincide.

Example 3.1. Let $\mathcal{W}$ be the simple pre-Weyl graph of type $A_2 = \langle s, t\rangle$ shown in Figure 3.2, with s corresponding to the lighter gray, and t to the darker gray. Recall that simple just means that all suites are null homotopic. The panel groupoids are all setoids, with each edge in Figure 3.2 representing two mutually inverse edges of a groupoid. Notice that the geodesic γ is homotopic to the geodesic $\hat{\gamma}$. They are homotopic via a composition of two elementary homotopies and two elementary strict homotopies. They are; an expansion of type (ii), with type change $sts \mapsto stss$, two elementary strict homotopies with type changes $stss \mapsto tsts \mapsto stss$, and a contraction of the type (i) with type change $stss \mapsto st$. Therefore $\mathcal{W}$ does not satisfy property (2W).

3.2.3 The 2-Weyl Properties

Let us introduce three more properties which arbitrary Weyl data may satisfy. These properties, together with (2W), are the rank 2 cases of what we will call the 'Weyl properties' (see Section 4.1.1):

(2C) homotopic alternating geodesics are strictly homotopic.

(2SH) strictly homotopic alternating geodesics of the same type are equal.

(2H) homotopic alternating geodesics of the same type are equal.

Remark 3.9. Notice that alternating geodesics which are not maximal alternating geodesics are only strictly homotopic to themselves, since their type does not contain a subword of the form $p(s,t)$. Thus, (2SH) is equivalent to the property that homotopic maximal alternating geodesics of the same type are equal. Also, notice that (2SH) is equivalent to the property that F_γ is injective for all alternating geodesics γ.

We now show that for pre-Weyl graphs $\mathcal{W}$, we have the following,

$$(2\text{C}) \iff (2\text{W}) \implies (2\text{H}) \iff (2\text{SH}).$$

We begin with a simple observation:

Lemma 3.11. Let $\mathcal{W}$ be a pre-Weyl graph. If two homotopic maximal alternating geodesics of $\mathcal{W}$ have the same type, then they are strictly homotopic.

Proof. Let $\rho(s,t)$ and $\rho'(s,t)$ be two homotopic maximal (s,t)-geodesics of $\mathcal{W}$. By (PW1), $\rho(s,t)$ is homotopic to a maximal (t,s)-geodesic $\rho(t,s)$. Then $\rho'(s,t) \sim \rho(s,t) \simeq \rho(t,s)$, thus, $\rho'(s,t) \simeq \rho(t,s)$. Then $\rho(s,t) \simeq \rho(t,s) \simeq \rho'(s,t)$, and so, $\rho(s,t) \simeq \rho'(s,t)$. □

We now prove (2H) $\iff$ (2SH):

Proposition 3.12. Let $\mathcal{W}$ be a pre-Weyl graph. Then the following are equivalent,

(i) $\mathcal{W}$ has property (2SH)

(ii) $\mathcal{W}$ has property (2H)

(iii) each maximal (s,t)-geodesic is homotopic to at most one maximal (t,s)-geodesic.

Proof. We have (2SH) $\implies$ (2H) since two unequal homotopic alternating geodesics of the same type can be extended by property (PW0) to give two unequal homotopic maximal alternating geodesics of the same type, which will be strictly homotopic by Lemma 3.11. The converse (2H) $\implies$ (2SH) is clear.

To see (2SH) $\implies$ (iii), let $\rho(s,t)$ be a maximal (s,t)-geodesic which is homotopic to maximal (t,s)-geodesics ρ and ρ'. Then ρ and ρ' are strictly homotopic by Lemma 3.11, and so $\rho = \rho'$ by (2SH). For the converse (iii) $\implies$ (2SH), suppose that $\mathcal{W}$ does not have property (2SH). Then two unequal maximal (t,s)-geodesics ρ and ρ' are strictly homotopic. By (PW1), ρ is homotopic to a maximal (s,t)-geodesic $\rho(s,t)$, which by transitivity will also be homotopic to ρ'. Thus, $\mathcal{W}$ does not have property (iii). □

We now prove (2W) $\implies$ (2H):

Proposition 3.13. Let $\mathcal{W}$ be a 2-Weyl graph. Then $\mathcal{W}$ has property (2H).

Proof. Let ρ and ρ' be homotopic alternating geodesics in $\mathcal{W}$ with the same type. Let i be the last edge of ρ, and let i' be the last edge of ρ'. Towards a contradiction, suppose that $i \neq i'$. Let $j = i'i^{-1}$. Let α and α' be the subgalleries such that $\alpha i = \rho$ and $\alpha' i' = \rho'$. Then α and $\alpha' j$ are homotopic alternating geodesics with different W-lengths, a contradiction. Thus, $i = i'$. Then α and α' are also homotopic alternating geodesics, and so can apply the same argument to the penultimate edges of ρ and ρ'. Therefore, by induction, we may conclude that $\rho = \rho'$. $\square$

Remark 3.10. In particular, 2-Weyl graphs have property (iii) of Proposition 3.12. In the presence of (PW1), property (iii) implies that each maximal (s,t)-geodesic is homotopic to exactly one maximal (t,s)-geodesic. In light of Proposition 3.1, this just says that every maximal alternating geodesic is contained within exactly one suite. Therefore in 2-Weyl graphs, elementary strict homotopies of words induce unique elementary strict homotopies of galleries.

Finally, we prove (2C) $\iff$ (2W):

Proposition 3.14. Let $\mathcal{W}$ be a pre-Weyl graph. Then the following are equivalent,

(i) $\mathcal{W}$ is 2-Weyl

(ii) $\mathcal{W}$ has property (2C).

Proof. First we show that (2W) $\implies$ (2C). If (2W) holds, then distinct homotopic alternating geodesics must either be maximal with different types, or have the same type. If they have the same type, then this implies the existence of a null homotopic cycle of length $< 2m_{st}$. Such cycles are ruled out by (2W) since they can be cut in two to give distinct homotopic alternating geodesics with different W-lengths. Therefore in the presence of (2W), distinct homotopic alternating geodesics must be maximal and with different types, which means they are strictly homotopic. Thus, (2C) holds. The converse (2C) $\implies$ (2W) is clear. $\square$

3.3 Covering Theory of 2-Weyl Graphs

In this section, we develop covering theory of 2-Weyl graphs. In particular, we show that one can model coverings of 2-Weyl graphs with coverings of groupoids. We show that the (isomorphism classes of) coverings of a connected 2-Weyl graph $\mathcal{W}$ are naturally in bijection with the conjugacy classes of subgroups of the fundamental group of $\mathcal{W}$.

3.3.1 Coverings of Weyl Data

We begin by defining étale morphisms and coverings of Weyl data. All such morphisms will take place over the identity morphism of types $M \to M$, which is just the identity function on S.

Let $\mathcal{W}$ be Weyl data, and let $C \in \mathcal{W}$ be a chamber. We denote by $\mathcal{W}(C, -)$ the set of edges $i \in \mathcal{W}$ such that $\iota(i) = C$. Let $\omega : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a morphism of Weyl data, then we denote by $\omega \restriction_{\widetilde{\mathcal{W}}(C,-)}$ the restriction,

$$\omega : \widetilde{\mathcal{W}}(C, -) \to \mathcal{W}(\omega(C), -).$$

Definition of Étale Morphisms. A morphism $\omega : \widetilde{\mathcal{W}} \to \mathcal{W}$ of generalized chamber systems of type M is called **surjective-étale** if for each chamber $C \in \widetilde{\mathcal{W}}$, the restriction $\omega \restriction_{\widetilde{\mathcal{W}}(C,-)}$ of ω to $\widetilde{\mathcal{W}}(C, -)$ is a bijection into $\mathcal{W}(\omega(C), -)$. If ω is additionally surjective on chambers, then ω is called **étale**.

Given a morphism of generalized chamber systems $\omega : \widetilde{\mathcal{W}} \to \mathcal{W}$ of type M, recall that $\omega_s : \widetilde{\mathcal{W}}_s \to \mathcal{W}_s$ denotes the induced s-homomorphism (see Section 3.1.2). It follows directly from the definitions that ω is étale if and only if ω_s is a covering of groupoids for each $s \in S$.

Let $\omega : \mathcal{W}' \to \mathcal{W}$ be a morphism of generalized chamber systems. We say a gallery β in $\mathcal{W}$ **lifts** with respect to ω to the gallery β' if $\beta = \omega \circ \beta'$.

Proposition 3.15. Let $\omega : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a surjective-étale morphism of generalized chamber systems. Then for each chamber $\widetilde{C} \in \widetilde{\mathcal{W}}$, every gallery β in $\mathcal{W}$ which issues from $\omega(\widetilde{C})$ has a unique lifting with respect to ω to a gallery $\tilde{\beta}$ which issues from $\widetilde{C}$.

Proof. Let $C = \omega(\widetilde{C})$. First we prove existence. Let $i_1, \ldots, i_n$ be the sequence of edges of β. For $k \in \{1, \ldots, n\}$, let $\tilde{i}_k \in \widetilde{\mathcal{W}}$ be an edge with $\omega(\tilde{i}_k) = i_k$, $\iota(\tilde{i}_1) = C$, and $\tau(\tilde{i}_k) = \iota(\tilde{i}_{k+1})$. Notice that such $\tilde{i}_k$ exist by the fact that β is surjective-étale. Then, letting $\tilde{\beta}$ be the gallery whose sequence of edges is $\tilde{i}_1, \ldots, \tilde{i}_n$, we have $\beta = \omega \circ \tilde{\beta}$ as required. For uniqueness, let $\tilde{\beta} = \tilde{i}_1, \ldots, \tilde{i}_n$ and $\tilde{\beta}' = \tilde{i}'_1, \ldots, \tilde{i}'_n$ be galleries in $\widetilde{\mathcal{W}}$ issuing from $\widetilde{C}$, with $\omega \circ \tilde{\beta} = \omega \circ \tilde{\beta}' = \beta$. In particular $\omega(\tilde{i}_1) = \omega(\tilde{i}'_1)$, with $\tilde{i}_1$ and $\tilde{i}'_1$ issuing from the same chamber $\widetilde{C}$. Thus, $\tilde{i}_1 = \tilde{i}'_1$ since ω is surjective-étale. The same argument shows that $\tilde{i}_2 = \tilde{i}'_2$, and so on along the sequences of edges. We conclude that $\tilde{i}_1 \ldots \tilde{i}_n = \tilde{i}'_1 \ldots \tilde{i}'_n$. ☐

Proposition 3.16. Let $\omega : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a surjective-étale morphism of generalized chamber systems. If $\mathcal{W}$ is connected, then ω is étale.

Proof. Suppose that $\mathcal{W}$ is connected. To see that ω is surjective on chambers, let $C \in \mathcal{W}$ be any chamber. Pick a chamber $D \in \widetilde{\mathcal{W}}$, and let β be a gallery in $\mathcal{W}$ which goes from $\omega(D)$ to C. Lift β to a gallery $\tilde{\beta}$ of $\widetilde{\mathcal{W}}$ which issues from D. Then we have $\omega(\tau(\tilde{\beta})) = C$. ☐

Definition of Coverings. A **pre-covering** $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ from a generalized chamber system $\widetilde{\mathcal{W}}$ to Weyl data $\mathcal{W}$ is an étale morphism such that for all galleries β in $\widetilde{\mathcal{W}}$, if $p \circ \beta$ is a defining suite of $\mathcal{W}$, then β is a cycle. A **covering** $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ of Weyl data is an étale morphism such that for all galleries β in $\widetilde{\mathcal{W}}$, if $p \circ \beta$ is a defining suite of $\mathcal{W}$, then β is a suite of $\widetilde{\mathcal{W}}$. A covering $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ is called **connected** if $\widetilde{\mathcal{W}}$ is connected.

Completions of Generalized Chamber Systems. Let $\widetilde{\mathcal{W}}$ be a generalized chamber system, let $\mathcal{W}$ be Weyl data, and let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a pre-covering. The **completion** of $\widetilde{\mathcal{W}}$ with respect to p is the Weyl data whose underlying generalized chamber system is $\widetilde{\mathcal{W}}$, and whose suites are defined as follows; for any gallery β of $\widetilde{\mathcal{W}}$, if β descends to a defining suite of $\mathcal{W}$, then let β be a defining suite of $\widetilde{\mathcal{W}}$. With $\widetilde{\mathcal{W}}$ redefined to be its completion, then $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ is a covering of Weyl data.

We now show that homotopies lift with respect to coverings:

Lemma 3.17. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a covering of Weyl data. Let β and β' be galleries in $\mathcal{W}$ with $\beta \sim \beta'$. Let $C = \iota(\beta) = \iota(\beta')$, and pick any $\widetilde{C} \in \widetilde{\mathcal{W}}$ such that $p(\widetilde{C}) = C$. Let $\tilde{\beta}$ and $\tilde{\beta}'$ be the unique lifts of β and β', respectively, issuing from $\widetilde{C}$. Then $\tilde{\beta} \sim \tilde{\beta}'$.

Proof. By hypothesis, there exists a sequence of galleries,

$$\beta = \beta_1, \ldots, \beta_n = \beta'$$

such that consecutive galleries differ only by an elementary homotopy. Let,

$$\tilde{\beta} = \tilde{\beta}_1, \ldots, \tilde{\beta}_n = \tilde{\beta}'$$

be the sequence of galleries obtained by lifting each β_m, for $m \in \{1, \ldots, n\}$, to a gallery issuing from $\widetilde{C}$. Suppose that β_m and β_{m+1} differ by a 1-elementary homotopy of type (i). Since p is étale, for edges $i, i' \in \widetilde{\mathcal{W}}$ with $v(i) = v(i')$, if $p(i; i') = p(i); p(i') = 1$ then $i; i' = 1$. Thus, backtracks lift to backtracks, and so $\tilde{\beta}_m \sim \tilde{\beta}_{m+1}$. Suppose that β_m and β_{m+1} differ by a 1-elementary homotopy of type (ii). But for edges $j, j', k \in \widetilde{\mathcal{W}}$ with $v(j) = v(j') = v(k)$, if $p(j); p(j') = p(k)$ then $j; j' = k$ by the fact that p is étale. Thus, detours lift to detours, and so $\tilde{\beta}_m \sim \tilde{\beta}_{m+1}$. If β_m and β_{m+1} differ by a 2-elementary homotopy, then $\tilde{\beta}_m \sim \tilde{\beta}_{m+1}$ by the fact that p lifts defining suites to suites. Therefore, by transitivity, we have $\tilde{\beta} \sim \tilde{\beta}'$. □

By definition, coverings lift defining suites to suites, however Lemma 3.17 shows that coverings also lift suites to suites:

Corollary 3.17.1. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a covering of Weyl data. Then for all galleries β in $\widetilde{\mathcal{W}}$, if $p \circ \beta$ is a suite of $\mathcal{W}$, then β is a suite of $\widetilde{\mathcal{W}}$.

Proof. We have $p \circ \beta \sim \beta_0$, where β_0 is a trivial gallery. Then, by Lemma 3.17, β is homotopic to the lifting of β_0, which is trivial. Thus, β is null-homotopic. $\square$

We have the following characterization of isomorphisms amongst coverings:

Proposition 3.18. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a covering of Weyl data. Then p is an isomorphism if and only if p is injective on chambers.

Proof. One can easily see that if p is an isomorphism then p is injective on chambers. Conversely, suppose that p is injective on chambers. By Proposition 3.4, it suffices to show that p is bijective on chambers and edges. This follows from that fact that each p_s is an isomorphism by Proposition 2.18. $\square$

We now show that coverings preserve and reflect the property of being 2-Weyl amongst Weyl data.

Lemma 3.19. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be an étale morphism of Weyl data. Then $\widetilde{\mathcal{W}}$ has property (PW0) if and only if $\mathcal{W}$ has property (PW0).

Proof. This follows from the fact that each s-homomorphism $p_s : \widetilde{\mathcal{W}}_s \to \mathcal{W}_s$ is a covering of groupoids. $\square$

Lemma 3.20. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be an étale morphism of Weyl data. If $\widetilde{\mathcal{W}}$ has property (PW1), then $\mathcal{W}$ has property (PW1).

Proof. Let ρ be a maximal (s,t)-geodesic of $\mathcal{W}$. Lift ρ to a maximal (s,t)-geodesic $\tilde{\rho}$ of $\widetilde{\mathcal{W}}$. Let $\tilde{\rho}'$ be a maximal (t,s)-geodesic which is homotopic to $\tilde{\rho}$. Then $p \circ \tilde{\rho}'$ is a maximal (t,s)-geodesic which is homotopic to ρ. $\square$

Lemma 3.21. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a covering of Weyl data. If $\mathcal{W}$ has property (PW1), then $\widetilde{\mathcal{W}}$ has property (PW1).

Proof. Let $\tilde{\rho}$ be a maximal (s,t)-geodesic of $\widetilde{\mathcal{W}}$. Let $\rho = p \circ \tilde{\rho}$, and let ρ' be a maximal (t,s)-geodesic which is homotopic to ρ. Lift ρ' to the gallery $\tilde{\rho}'$ which issues from the same chamber as $\tilde{\rho}$. Then, since homotopies lift with respect to coverings, $\tilde{\rho}$ and $\tilde{\rho}'$ are homotopic. $\square$

Lemma 3.22. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a morphism of Weyl data. If $\mathcal{W}$ has property (2W), then $\widetilde{\mathcal{W}}$ has property (2W).

Proof. Let $\tilde{\gamma}$ and $\tilde{\gamma}'$ be two homotopic alternating geodesics in $\widetilde{\mathcal{W}}$. Let $\gamma = p \circ \tilde{\gamma}$ and $\gamma' = p \circ \tilde{\gamma}'$. Then γ and γ' are homotopic alternating geodesics in $\mathcal{W}$, and $\tilde{\gamma}_W = \gamma_W = \gamma'_W = \tilde{\gamma}'_W$, since $\mathcal{W}$ has property (2W). $\square$

Lemma 3.23. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be an covering of Weyl data. If $\widetilde{\mathcal{W}}$ has property (2W), then $\mathcal{W}$ has property (2W).

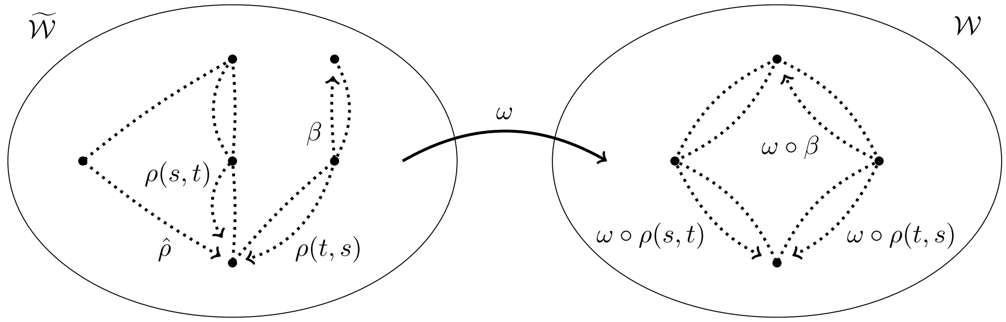

Figure 3.3: Proposition 3.26 with $m_{st} = 2$

Proof. Let γ and γ' be two homotopic alternating geodesics in $\mathcal{W}$. Lift these geodesics to homotopic alternating geodesics $\tilde{\gamma}$ and $\tilde{\gamma}'$ in $\widetilde{\mathcal{W}}$. Then $\gamma_W = \tilde{\gamma}_W = \tilde{\gamma}'_W = \gamma'_W$, since $\widetilde{\mathcal{W}}$ has property (2W). □

Theorem 3.24. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a covering of Weyl data. Then $\widetilde{\mathcal{W}}$ is 2-Weyl if and only if $\mathcal{W}$ is 2-Weyl.

Proof. Recall that a 2-Weyl graph is Weyl data with properties (PW0), (PW1), and (2W). The result then follows directly from Lemma 3.19, Lemma 3.20, Lemma 3.21, Lemma 3.22, and Lemma 3.23. □

A covering of Weyl data induces local coverings of the residues:

Proposition 3.25. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a covering of Weyl data. Let $\widetilde{R}$ be a J-residue of $\widetilde{\mathcal{W}}$, and let R be the J-residue of $\mathcal{W}$ which contains the p-image of $\widetilde{R}$. Then the restriction of p to $\widetilde{R}$ is a covering of R.

Proof. Let $p_{\widetilde{R}} : \widetilde{R} \to R$ be the morphism of Weyl data which is the restriction of p to $\widetilde{R}$. Clearly $p_{\widetilde{R}} : \widetilde{R} \to R$ is surjective-étale, and so is étale by Proposition 3.16. The fact that $p_{\widetilde{R}}$ is a covering then follows directly from the fact that $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ is a covering. □

We call $p_{\widetilde{R}} : \widetilde{R} \to R$ the **local covering at** $\widetilde{R}$. We finish this section with a result which shows that, as long as one establishes the 2-Weyl property, étale morphisms suffice when it comes to coverings:

Proposition 3.26. Let $\mathcal{W}$ be a 2-Weyl graph, and let $\widetilde{\mathcal{W}}$ be a pre-Weyl graph. Then an étale morphism $\omega : \widetilde{\mathcal{W}} \to \mathcal{W}$ is a covering.

Proof. Let β be a gallery in $\widetilde{\mathcal{W}}$ which descends to a defining suite. Towards a contradiction, suppose that β is not a suite. Since β has the type of a suite, we may write $\beta = \rho(s,t)\rho(t,s)^{-1}$. Let $\hat{\rho}$ be a gallery which is strictly homotopic to $\rho(s,t)$, which must exist by (PW1). We have $\hat{\rho} \neq \rho(t,s)$ by hypothesis. Since ω is étale, ω preserves the distinction of galleries, thus $\omega \circ \hat{\rho} \neq \omega \circ \rho(t,s)$. But $\omega \circ \hat{\rho} \sim \omega \circ \rho(s,t)$

since $\hat{\rho} \sim \rho(s,t)$ by our choice of $\hat{\rho}$, and homotopies descend through morphisms. Also,

$$\omega \circ \rho(s,t) \sim \omega \circ \rho(t,s)$$

since $\omega \circ \beta$ is a suite and,

$$(\omega \circ \rho(s,t))^{-1}(\omega \circ \rho(t,s)) = \omega \circ \beta.$$

Therefore $\omega \circ \hat{\rho} \sim \omega \circ \rho(t,s)$ by transitivity. This is a contradiction of the fact that $\mathcal{W}$ is 2-Weyl, since both $\omega \circ \hat{\rho}$ and $\omega \circ \rho(t,s)$ have the same type (see Proposition 3.13). □

Corollary 3.26.1. An étale morphism between 2-Weyl graphs is a covering.

3.3.2 The Classification of Coverings of 2-Weyl Graphs

We now move to the setting where Weyl data $\mathcal{W}$ is 2-Weyl. Recall that in this setting, the panel groupoids of $\mathcal{W}$ are naturally subgroupoids of the fundamental groupoid of $\mathcal{W}$; that is $\mathcal{W}_s \leq \overline{\mathcal{W}}$, for $s \in S$.

In this section, we develop a bijective correspondence between connected coverings of 2-Weyl graphs $\mathcal{W}$ and connected coverings of the fundamental groupoid $\overline{\mathcal{W}}$. By results in Section 2.4, this gives a bijective correspondence between connected coverings of $\mathcal{W}$ and conjugacy classes of subgroups of the fundamental group of $\mathcal{W}$.

We begin with the following simple observation:

Proposition 3.27. Let $\mathcal{W}$ be a 2-Weyl graph. Then the edges of the panel groupoids of $\mathcal{W}$ generate $\overline{\mathcal{W}}$.

Proof. Let $[\beta] \in \overline{\mathcal{W}}$ be a homotopy class of galleries. Let $i_1, \ldots, i_n$ be the sequence of edges of β. Then in $\overline{\mathcal{W}}$, we have,

$$[\beta] = [i_1 \ldots i_n] = i_1; \ldots; i_n.$$

□

Proposition 3.28. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a covering of 2-Weyl graphs. Then the induced homomorphism $\bar{p}$ of the fundamental groupoids is a covering of groupoids.

Proof. Firstly, $\bar{p}$ is surjective on vertices since p is surjective on chambers. Let $\widetilde{C} \in \widetilde{\mathcal{W}}$ be a chamber and let $C = p(\widetilde{C})$. Let β be a gallery of $\mathcal{W}$ which issues from C, and let $\tilde{\beta}$ be the lifting of β to a gallery which issues from $\widetilde{C}$. Then,

$$\bar{p}([\tilde{\beta}]) = [p \circ \tilde{\beta}] = [\beta].$$

Therefore the restriction of $\bar{p}$ to the homotopy classes which issue from $\widetilde{C}$ is surjective into the homotopy classes which issue from C. Finally, towards a contradiction, suppose that the restriction of $\bar{p}$ to the homotopy classes which issue from $\widetilde{C}$ is not injective. This implies that there exist non-homotopic galleries in $\widetilde{\mathcal{W}}$ issuing from $\widetilde{C}$ whose p-images are homotopic. This contradicts Lemma 3.17. □

Morphisms of Coverings. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ and $p' : \widetilde{\mathcal{W}}' \to \mathcal{W}$ be coverings of a 2-Weyl graph $\mathcal{W}$. A **morphism** of coverings $\mu : p \to p'$ is a morphism $\mu : \widetilde{\mathcal{W}} \to \widetilde{\mathcal{W}}'$ such that $p = p' \circ \mu$. We call two coverings **isomorphic** if there exists a morphism between them which is an isomorphism of 2-Weyl graphs. The **composition** of morphisms of coverings is just their composition as morphisms of 2-Weyl graphs.

The following shows that in particular, a morphism of connected coverings of 2-Weyl graphs is itself a covering:

Proposition 3.29. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$, $p' : \widetilde{\mathcal{W}}' \to \mathcal{W}$, and $\mu : \widetilde{\mathcal{W}} \to \widetilde{\mathcal{W}}'$ be morphisms of connected 2-Weyl graphs with $p = p' \circ \mu$. If p and p' are coverings, then μ is a covering, and if p and μ are coverings, then p' is a covering.

Proof. Suppose that p and p' are coverings. For any chamber $C \in \widetilde{\mathcal{W}}$, we have,

$$p \restriction_{\mathcal{W}(C,-)} = p' \restriction_{\mathcal{W}(\mu(C),-)} \circ \mu \restriction_{\mathcal{W}(C,-)} .$$

Then, since $p \restriction_{\mathcal{W}(C,-)}$ and $p' \restriction_{\mathcal{W}(\mu(C),-)}$ are bijections, we must have that $\mu \restriction_{\mathcal{W}(C,-)}$ is a bijection for all $C \in \widetilde{\mathcal{W}}$. Therefore μ is surjective-étale, and so is étale because $\widetilde{\mathcal{W}}'$ is connected. To see that μ is a covering, let β be a gallery in $\widetilde{\mathcal{W}}$ such that $\mu \circ \beta$ is a defining suite of $\widetilde{\mathcal{W}}'$. Therefore $p' \circ (\mu \circ \beta)$ is a suite of $\mathcal{W}$, since p' is a morphism. But $p \circ \beta = p' \circ \mu \circ \beta$, and so β is a suite since p is a covering and coverings lift suites to suites by Corollary 3.17.1.

Now suppose that p and μ are coverings. Let $D \in \widetilde{\mathcal{W}}'$ be any chamber. Pick a chamber $C \in \widetilde{\mathcal{W}}$ such that $\mu(C) = D$ (here we use the fact that μ is surjective on chambers). Then we have,

$$p \restriction_{\mathcal{W}(C,-)} = p' \restriction_{\mathcal{W}(D,-)} \circ \mu \restriction_{\mathcal{W}(C,-)} .$$

Then, since $p \restriction_{\mathcal{W}(C,-)}$ and $\mu \restriction_{\mathcal{W}(C,-)}$ are bijections, we must have that $p' \restriction_{\mathcal{W}(D,-)}$ is a bijection. Therefore p' is surjective-étale, and so is étale because $\mathcal{W}$ is connected. To see that p' is a covering, let β be a gallery in $\widetilde{\mathcal{W}}'$ such that $p' \circ \beta$ is a defining suite of $\mathcal{W}$. Lift β with respect to μ to a gallery $\tilde{\beta}$ in $\widetilde{\mathcal{W}}$. Then $p \circ \tilde{\beta} = p' \circ \beta$. Since p is a covering, $p \circ \tilde{\beta}$ must be a suite of $\mathcal{W}$. Then, since μ is a morphism, β must be a suite of $\widetilde{\mathcal{W}}'$. □

The following result shows that morphisms of 2-Weyl graphs can be recovered from the groupoid homomorphisms they induce:

Proposition 3.30. Let $\omega, \omega' : \mathcal{W} \to \mathcal{W}'$ be two morphisms between the same 2-Weyl graphs. If $\bar{\omega} = \bar{\omega}'$, then $\omega = \omega'$.[1]

[1] In the language of category theory, this says that the functor on 2-Weyl graphs into groupoids is faithful.

Proof. Towards a contradiction, suppose that $\omega \neq \omega'$. Since $\mathcal{W}$ is weak, there exists an edge $i \in \mathcal{W}$ with $\omega(i) \neq \omega'(i)$. We cannot have $\omega(i) \sim \omega'(i)$ as galleries because $\mathcal{W}'_{v(i)} \leq \overline{\mathcal{W}}'$. Therefore,

$$\bar{\omega}([i]) = [\omega(i)] \neq [\omega'(i)] = \bar{\omega}'([i])$$

and so $\bar{\omega} \neq \bar{\omega}'$, a contradiction. □

We now show how coverings of groupoids can be 'lifted' to coverings of 2-Weyl graphs:

Proposition 3.31. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ and $p' : \widetilde{\mathcal{W}}' \to \mathcal{W}$ be coverings of a 2-Weyl graph $\mathcal{W}$. Let $\lambda : \bar{p} \to \bar{p}'$ be a covering morphism of the groupoid coverings $\bar{p}$ and $\bar{p}'$. Then there exists a unique covering morphism $\mu : p \to p'$ such that $\bar{\mu} = \lambda$.[2]

Proof. Let $\mu : \widetilde{\mathcal{W}} \to \widetilde{\mathcal{W}}'$ be the generalized chamber system morphism whose s-homomorphism μ_s is the restriction of λ to $\widetilde{\mathcal{W}}_s$. This is well defined because the restriction of λ preserves types by the fact its a covering morphism. Then μ is a morphism of Weyl data since a suite, as a sequence of edges whose composition the fundamental groupoid of $\widetilde{\mathcal{W}}$ is trivial, must get mapped by λ to a sequence of edges whose composition is also trivial. Then $\bar{\mu} = \lambda$ since they agree on a generating set (see Proposition 3.27). Finally, μ is unique by Proposition 3.30. □

Corollary 3.31.1. Let $\lambda : \bar{p} \to \bar{p}'$ be an isomorphism, and let $\mu : p \to p'$ be the unique morphism such that $\bar{\mu} = \lambda$. Then μ is an isomorphism. Thus, if two coverings of a 2-Weyl graph induce isomorphic groupoid coverings, then they are isomorphic.

Proof. Let $\mu' : p' \to p$ be the unique morphism with $\bar{\mu}' = \lambda^{-1}$. Put $\mu'' = \mu' \circ \mu$. Then $\bar{\mu}'' = \lambda^{-1} \circ \lambda = 1$. Thus, $\mu'' = 1$ by Proposition 3.30. By a symmetric argument, $\mu \circ \mu' = 1$, and so μ is an isomorphism $\mu : p \to p'$. □

This shows that coverings of 2-Weyl graphs can be modeled in a non-forgetful way by coverings of groupoids. We now show that the injective correspondence we have just established on coverings of a 2-Weyl graph $\mathcal{W}$ into groupoid coverings of $\overline{\mathcal{W}}$ is bijective.

Theorem 3.32. Let $\mathcal{W}$ be a 2-Weyl graph. For every groupoid covering $\varphi : \mathcal{G} \to \overline{\mathcal{W}}$, there exists a covering $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ with $\bar{p} \cong \varphi$.[3]

Proof. Let the chambers of $\widetilde{\mathcal{W}}$ be the vertices of $\mathcal{G}$. Let the panel groupoid $\widetilde{\mathcal{W}}_s$ be the subgroupoid $\mathcal{G}_s \leq \mathcal{G}$ which is the φ-preimage of $\mathcal{W}_s \leq \overline{\mathcal{W}}$. This gives $\widetilde{\mathcal{W}}$ the structure of a generalized chamber system. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be the generalized chamber system morphism such that the s-homomorphism p_s is the restriction of φ to $\mathcal{G}_s$. The map

[2] This says that the functor from the slice category $\mathrm{cov}(\mathcal{W})$ of coverings of $\mathcal{W}$ to the slice category $\mathrm{cov}(\overline{\mathcal{W}})$ of groupoid coverings of $\overline{\mathcal{W}}$ is full.

[3] This shows that the functor $\mathrm{cov}(\mathcal{W}) \to \mathrm{cov}(\overline{\mathcal{W}})$ is an equivalence of categories.

$p : \widetilde{\mathcal{W}} \to \mathcal{W}$ is étale because φ is a groupoid covering. It is a pre-covering since a suite in $\mathcal{W}$ lifts to a loop in $\mathcal{G}$ because the composition of the sequence of edges of a suite is trivial. Finally, redefine $\widetilde{\mathcal{W}}$ to be its completion with respect to p. Thus, we obtain a covering $p : \widetilde{\mathcal{W}} \to \mathcal{W}$. We have $\bar{p} = \varphi$ by Proposition 3.27. □

The Universal Cover of a 2-Weyl Graph. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a connected covering of 2-Weyl graphs. Then p is called a **universal cover** if for any connected covering $p' : \widetilde{\mathcal{W}}' \to \mathcal{W}$, there exists a covering morphism $\mu : p \to p'$.

Proposition 3.33. A connected covering $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ of 2-Weyl graphs is a universal cover if and only if $\widetilde{\mathcal{W}}$ is simply connected.

Proof. Suppose that $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ is a universal cover. Then $\bar{p}$ is a universal cover of groupoids. Therefore, the universal groupoid of $\widetilde{\mathcal{W}}$ is a setoid, and so $\widetilde{\mathcal{W}}$ is simply connected. Conversely, suppose that $\widetilde{\mathcal{W}}$ is simply connected. Again, this means that $\bar{p}$ is a universal cover of groupoids. Then p is a universal cover by Proposition 3.31. □

Recall that we denote simply connected Weyl data by Δ.

Corollary 3.33.1. Universal covers are unique up to isomorphism, and every connected 2-Weyl graph $\mathcal{W}$ has a universal cover $p : \Delta \to \mathcal{W}$.

Proof. Universal covers are unique up to isomorphism by Proposition 3.33 and the 1-1 correspondence between 2-Weyl graph coverings and groupoid coverings. The existence of universal covers follows from Proposition 3.33 and Theorem 3.32. □

Corollary 3.33.2. Let $p : \Delta \to \mathcal{W}$ and $p' : \Delta' \to \mathcal{W}$ be universal covers of a connected 2-Weyl graph $\mathcal{W}$, and let $\mu : p \to p'$ be a covering morphism. Then μ is an isomorphism.

Proof. Notice that $\pi_1(\bar{\mu})$ is an outer isomorphism between trivial groups. Then $\bar{\mu}$ is an isomorphism by Proposition 2.21, and μ is an isomorphism by Corollary 3.31.1. □

3.3.3 Coverings and Group Actions

In this section, we expose the relationship between groups acting on 2-Weyl graphs and coverings.

Groups Acting on 2-Weyl Graphs. Groups act on 2-Weyl graphs by automorphisms. We say a group G acts **chamber-freely** on a 2-Weyl graph $\mathcal{W}$ if the restriction of the action of G to the set of chambers of $\mathcal{W}$ is free; equivalently if its 0-free in the sense of Section 3.1.5.

The Deck Transformation Group. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a covering of 2-Weyl graphs. An **automorphism** of p is a covering isomorphism from p to itself. The **deck transformation group** of p is the group $\mathrm{Aut}(p)$ whose elements are the automorphisms of p, equipped with the composition of morphisms. We have $\mathrm{Aut}(p) \leq \mathrm{Aut}(\widetilde{\mathcal{W}})$. Thus, a covering $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ determines a faithful action of $\mathrm{Aut}(p)$ on the left of $\widetilde{\mathcal{W}}$. Notice that the homomorphism $\mathrm{Aut}(p) \to \mathrm{Aut}(\bar{p})$, $g \mapsto \bar{g}$, is injective by Proposition 3.30, and surjective by Proposition 3.31.

Proposition 3.34. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a connected covering. Then $\mathrm{Aut}(p)$ acts chamber-freely on $\widetilde{\mathcal{W}}$.

Proof. Given the natural isomorphism $\mathrm{Aut}(p) \to \mathrm{Aut}(\bar{p})$, $g \mapsto \bar{g}$, this follows from Proposition 2.25. □

We will see that conversely, if a group acts chamber-freely on a connected 2-Weyl graph $\mathcal{W}'$, then it is naturally the deck transformation group of a covering $\mathcal{W}' \to \mathcal{W}$.

Regular Coverings. A connected covering $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ of 2-Weyl graphs is called **regular** if the induced groupoid covering $\bar{p}$ is regular. Thus, if p is regular, then we can identify $\pi_1(\widetilde{\mathcal{W}})$ with its $\pi_1(\bar{p})$ image in $\pi_1(\mathcal{W})$.

Proposition 3.35. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a regular covering of 2-Weyl graphs. Then the action of $\mathrm{Aut}(p)$ restricted to the p-preimage of a chamber or an edge is regular.

Proof. Given the natural isomorphism $\mathrm{Aut}(p) \to \mathrm{Aut}(\bar{p})$, $g \mapsto \bar{g}$, the case of a chamber follows from Proposition 2.26. In the case of an edge, let $i, i' \in \widetilde{\mathcal{W}}$ be edges with $p(g) = p(g')$. Let $g \in \mathrm{Aut}(p)$ such that $g \cdot \iota(i) = \iota(i')$. Then $g \cdot i = i'$, since p is a covering. □

Proposition 3.36. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a regular covering of 2-Weyl graphs. Then there exists a natural outer isomorphism,

$$\pi_1(\mathcal{W})/\pi_1(\widetilde{\mathcal{W}}) \to \mathrm{Aut}(p).$$

Proof. Given the natural isomorphism $\mathrm{Aut}(p) \to \mathrm{Aut}(\bar{p})$, $g \mapsto \bar{g}$, this follows from Theorem 2.27. □

We now show that, conversely, if a group G acts chamber-freely on a connected 2-Weyl graph $\mathcal{W}$, then there exists a 2-Weyl graph $\mathcal{W}'$ and a regular covering $\mathcal{W} \to \mathcal{W}'$ of which G is naturally the automorphism group.

The Quotient of an Action. We associate to the chamber-free action of a group G on a connected 2-Weyl graph $\mathcal{W}$ the **quotient** 2-Weyl graph $G\backslash\mathcal{W}$, which is the 2-Weyl graph defined as follows.

First, we define a generalized chamber system $G\backslash\mathcal{W}$, by letting the set of chambers be the set of orbits $G\backslash\mathcal{W}_0$, and letting the panel groupoid of type s be the quotient groupoid $G\backslash\mathcal{W}_s$ (see Section 2.4.5). The **quotient map** $\pi : \mathcal{W} \to G\backslash\mathcal{W}$ is the morphism of generalized chamber systems such that the s-homomorphism π_s is the groupoid quotient map $\pi_s : \mathcal{W}_s \to G\backslash\mathcal{W}_s$. Notice that $\pi : \mathcal{W} \to G\backslash\mathcal{W}$ is étale since the π_s are coverings by Theorem 2.28.

We give $G\backslash\mathcal{W}$ the structure of Weyl data by letting the suites be the π-images of suites in $\mathcal{W}$; that is, θ is a defining suite of $G\backslash\mathcal{W}$ if there exists a defining suite θ' of $\mathcal{W}$ with $\theta = \pi \circ \theta'$. This ensures that π is a morphism of Weyl data.

Proposition 3.37. If a group G acts chamber-freely on a 2-Weyl graph $\mathcal{W}$, then $G\backslash\mathcal{W}$ is a 2-Weyl graph, and the quotient map $\pi : \mathcal{W} \to G\backslash\mathcal{W}$ is a covering.

Proof. Notice that $\pi : \mathcal{W} \to G\backslash\mathcal{W}$ is a covering since all the defining suites of $G\backslash\mathcal{W}$ are π-images of defining suites of $\mathcal{W}$. Then $G\backslash\mathcal{W}$ is 2-Weyl by Theorem 3.24. □

If a group G acts chamber-freely on a 2-Weyl graph $\mathcal{W}$, then this induces a free action of G on $\overline{\mathcal{W}}$. The following result shows that our 2-Weyl graph quotient construction is compatible with our groupoid quotient construction:

Proposition 3.38. Let G be a group which acts chamber-freely on a 2-Weyl graph $\mathcal{W}$ with quotient map $\pi : \mathcal{W} \to G\backslash\mathcal{W}$. Let $\varphi : \overline{\mathcal{W}} \to G\backslash\overline{\mathcal{W}}$ be the quotient map of the associated action of G on $\overline{\mathcal{W}}$. Then there exists a unique isomorphism $\psi : \overline{G\backslash\mathcal{W}} \to G\backslash\overline{\mathcal{W}}$ such that $\varphi = \psi \circ \bar{\pi}$.

Proof. Since $\varphi = \psi \circ \bar{\pi}$, then $\psi : \overline{G\backslash\mathcal{W}} \to G\backslash\overline{\mathcal{W}}$ must be the homomorphism whose map on edges is,

$$\bar{\pi}(g) \mapsto \varphi(g), \qquad \text{for } g \in \overline{\mathcal{W}}_1.$$

This is well defined because φ is constant on G-orbits. Checking that ψ is a homomorphism, we have,

$$\bar{\pi}(g)\bar{\pi}(g') = \bar{\pi}(gg') \mapsto \varphi(gg') = \varphi(g)\varphi(g').$$

Then ψ is a covering by Proposition 2.19. Notice that ψ is injective on chambers since both φ and $\bar{\pi}$ identify chambers if and only if they are in the same G-orbit. Therefore ψ is an isomorphism by Proposition 2.18. □

Corollary 3.38.1. Let G be a group which acts chamber-freely on a connected 2-Weyl graph $\mathcal{W}$, and let $\pi : \mathcal{W} \to G\backslash\mathcal{W}$ be the associated quotient map. Then G is naturally $\mathrm{Aut}(\pi)$.

Proof. We have a natural embedding $G \hookrightarrow \mathrm{Aut}(\pi)$. The composition of this embedding with the isomorphism $\mathrm{Aut}(\pi) \to \mathrm{Aut}(\bar{\pi})$ is an isomorphism by Theorem 2.28 and Proposition 3.38. Therefore $G \hookrightarrow \mathrm{Aut}(\pi)$ is an isomorphism. □

Proposition 3.39. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be a regular covering, and let $\pi : \widetilde{\mathcal{W}} \to \mathrm{Aut}(p)\backslash\widetilde{\mathcal{W}}$ be the quotient associated to the action of $\mathrm{Aut}(p)$. Then there exists a unique isomorphism $\omega : \mathrm{Aut}(p)\backslash\widetilde{\mathcal{W}} \to \mathcal{W}$ such that $p = \omega \circ \pi$.

Proof. Since $p = \omega \circ \pi$, then $\omega : \mathrm{Aut}(p)\backslash\widetilde{\mathcal{W}} \to \mathcal{W}$ must be the map such that for chambers,

$$\pi(C) \mapsto p(C), \quad \text{for } C \in \widetilde{\mathcal{W}}_0$$

and for edges,

$$\pi(i) \mapsto p(i), \quad \text{for } i \in \widetilde{\mathcal{W}}_1.$$

This is well defined since p is constant on $\mathrm{Aut}(p)$-orbits. Checking that ω is a graph morphism, for the extremities we have,

$$\iota(\pi(i)) = \pi(\iota(i)) \mapsto p(\iota(i)) = \iota(p(i)), \quad \tau(\pi(i)) = \pi(\tau(i)) \mapsto p(\tau(i)) = \tau(p(i))$$

and for the type function we have,

$$\upsilon(\pi(i)) = \upsilon(i) = \upsilon(p(i)).$$

Checking that ω is a morphism of generalized chamber systems, we have,

$$\pi(i^{-1}) \mapsto p(i^{-1}) = p(i)^{-1}, \quad \pi(i); \pi(i') = \pi(i; i') \mapsto p(i; i') = p(i); p(i').$$

Then ω is a bijection on chambers since it has the inverse $C \mapsto \pi(p^{-1}(C))$, for $C \in \mathcal{W}_0$, and ω is a bijection on edges since it has the inverse $i \mapsto \pi(p^{-1}(i))$, for $i \in \mathcal{W}_1$. Notice that these inverses are well defined by Proposition 3.35. Therefore ω is an isomorphism of generalized chamber systems.

Let θ be a suite of $\mathrm{Aut}(p)\backslash\widetilde{\mathcal{W}}$. Lift θ to a gallery $\tilde{\theta}$ in $\widetilde{\mathcal{W}}$. Then $\tilde{\theta}$ is a suite since π is a covering. Then $p \circ \tilde{\theta} = \theta$ is also a suite since p is a morphism of Weyl data. Therefore ω is an morphism of Weyl data. Then ω is a covering by Proposition 3.29, and so ω is an isomorphism of Weyl data. □

Let G be a group which acts chamber-freely on a 2-Weyl graph $\mathcal{W}$. Let R be a residue of $\mathcal{W}$. Then the **isotropy** H_R of R is the subgroup,

$$H_R = \{g \in G : g \cdot R = R\} \leq G.$$

We now show that the local covering at R of the quotient map $\mathcal{W} \to G\backslash\mathcal{W}$ is naturally the quotient map $R \to H_R\backslash R$:

Proposition 3.40. Let G be a group which acts chamber-freely on a 2-Weyl graph $\mathcal{W}$, and let $\pi : \mathcal{W} \to G\backslash\mathcal{W}$ be the associated quotient. Let R be a residue of $\mathcal{W}$, and let π_R denote the local covering at R. Let $H = H_R \leq G$ be the isotropy of R. Let $\pi_H : R \to H\backslash R$ be the quotient map associated to the action of H on R. Then there exists a unique isomorphism ω such that $\pi_R = \omega \circ \pi_H$.

Proof. Since $\pi_R = \omega \circ \pi_H$, then ω must be the map such that for chambers,

$$\pi_H(C) \mapsto \pi_R(C), \qquad \text{for } C \in R_0$$

and for edges,

$$\pi_H(i) \mapsto \pi_R(i), \qquad \text{for } i \in R_1.$$

This is well defined because π_R is constant on H-orbits. Also, ω is bijective on chambers and edges by the definition of H. Then ω is an isomorphism of Weyl data by the same arguments as those in the proof of Proposition 3.39. $\square$

Chapter 4

Weyl Graphs and Buildings

In this chapter, we introduce Weyl graphs, which model the quotients of buildings by type preserving chamber-free group actions. Weyl graphs are pre-Weyl graphs whose homotopy classes of galleries have a well defined W-length. Weyl graphs are examples of 2-Weyl graphs, but not all 2-Weyl graphs are Weyl graphs. More precisely, a 2-Weyl graph is a pre-Weyl graph which is a Weyl graph only locally at the level of 2-residues. There are several ways to characterize Weyl graphs amongst 2-Weyl graphs. These different characterizations are encapsulated in the Weyl properties (see Section 4.1.1), which are global versions of the 2-Weyl properties of Section 3.2.3.

In Section 4.1, we show that a pre-Weyl graph is a Weyl graph if and only if its universal cover is a building. In particular, every Weyl graph determines a building by taking its universal cover. Conversely, the quotient of a building by a type preserving chamber-free group action is naturally a Weyl graph. Thus, we show that Weyl graphs are essentially equivalent to buildings equipped with a type preserving and chamber-free group action. In this way, we obtain an equivalent definition of a building, that of a simply connected Weyl graph. Heuristically, one can say that Weyl graphs are buildings which are not necessarily simply connected.

We prove a Weyl graph version of Tits' local-to-global theorem of [Tit81]. This shows that 2-Weyl graphs are often Weyl graphs. Precisely, a 2-Weyl graph is a Weyl graph if (and only if) its spherical 3-residues of type C_3 and H_3 are Weyl graphs. Importantly, this result enables the construction of many Weyl graphs by gluing together 2-Weyl graphs. Examples of 2-Weyl graphs are easy to come by, since they are equivalent to generalized polygons equipped with a chamber-free (flag-free) group action.

Finally, in Section 4.2, we give a method to obtain a presentation of the fundamental group of a connected Weyl graph. This group then acts naturally on the covering building of the Weyl graph.

4.1 Weyl Graphs

In this section, we define Weyl graphs, and collect some of their basic properties. We also describe the various characterizations of Weyl graphs amongst 2-Weyl graphs.

4.1.1 The Weyl Properties

We begin by introducing the following stronger versions of the 2-Weyl properties:

(W) homotopic geodesics have the same W-length.

(C) homotopic geodesics are strictly homotopic.

(SH) strictly homotopic geodesics of the same type are equal.

(H) homotopic geodesics of the same type are equal.

We call these four properties the **Weyl properties**. Properties similar to these feature in [Tit81] and [Ron89]. Notice that the 2-Weyl properties (2W), (2C), (2SH), and (2H) are the properties that 2-residues satisfy (W), (C), (SH), and (H) respectively. Notice that (C) is the property which we proved that Cayley graphs of Coxeter groups have in Appendix A. Stated another way, (SH) says that the γ-gallery map F_γ is injective for all geodesics γ.

One can easily see that (H) $\implies$ (SH). In fact, we will show that for 2-Weyl graphs, we have the following,

$$\text{(C)} \implies \text{(W)} \implies \text{(H)} \implies \text{(SH)} \implies \text{(C)}.$$

Thus, for 2-Weyl graphs, the Weyl properties are all equivalent. The only implication which is not straight forward is (SH) $\implies$ (C). Finally, we have a property which will characterize buildings amongst connected 2-Weyl graphs,

(B) geodesics with the same extremities have the same W-length.

If Weyl data has property (B), then the geodesics can be used to give us a well defined notion of 'distance' between chambers, whose value is an element of W.

4.1.2 The Universal Cover of a 2-Weyl Graph with Property (SH)

In this section, we show that the universal cover of a connected 2-Weyl graph $\mathcal{W}$ with property (SH) can be constructed by representing the chambers as strict homotopy classes of geodesics issuing from a fixed chamber $C \in \mathcal{W}$. By using a method similar to the proof of [Ron89, Proposition 4.8], this construction will be used to prove that (SH) $\implies$ (C) in the setting of 2-Weyl graphs.

We begin with two important consequences of property (SH).

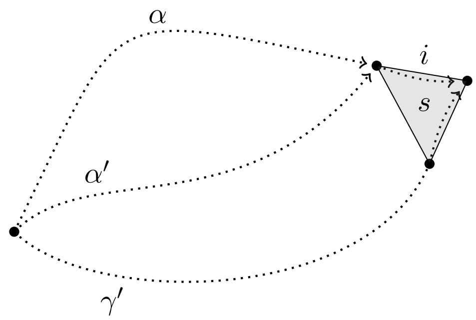

Figure 4.1: Lemma 4.1

Lemma 4.1. Let $\mathcal{W}$ be a pre-Weyl graph with property (SH), and let γ and γ' be strictly homotopic geodesics in $\mathcal{W}$ whose types end (begin) with the same letter $s \in S$. Then the final (initial) edges of γ and γ' must be equal.

Proof. Let $\gamma_S = fs$, and $\gamma'_S = f's$. Let i be the final edge of γ, and let α be the subgallery such that $\gamma = \alpha i$. In particular, $\alpha_S = f$. Since $\gamma \simeq \gamma'$ as galleries, we have $fs \simeq f's$ as words, and so $f \simeq f'$ by Corollary 2.9.2. Therefore there exists a gallery α' with $\alpha'_S = f'$ and $\alpha \simeq \alpha'$. By transitivity, $\alpha' i$ is strictly homotopic to γ', but they also have the same type. Thus, $\alpha' i = \gamma'$ by (SH), and so the final edge of γ' is also i. The case of beginning with the same letter follows by a symmetric argument. $\square$

Lemma 4.2. Let $\mathcal{W}$ be a pre-Weyl graph with property (SH), and let γ, γ', and γ'' be geodesics in $\mathcal{W}$. If $\gamma'\gamma \simeq \gamma''\gamma$ (or $\gamma\gamma' \simeq \gamma\gamma''$), then $\gamma' \simeq \gamma''$.

Proof. Suppose that $\gamma'\gamma \simeq \gamma''\gamma$. Since $\gamma'_S\gamma_S \simeq \gamma''_S\gamma_S$, we have $\gamma'_S \simeq \gamma''_S$ by Corollary 2.9.2. Therefore there exists a gallery $\hat{\gamma}$ with $\hat{\gamma}_S = \gamma''_S$ and $\hat{\gamma} \simeq \gamma'$. Then $\hat{\gamma}\gamma \simeq \gamma''\gamma$, and so $\hat{\gamma} = \gamma''$ by (SH). Thus $\gamma' \simeq \hat{\gamma} = \gamma''$ as required. The case where $\gamma\gamma' \simeq \gamma\gamma''$ follows by a symmetric argument. $\square$

For the remainder of this section, $\mathcal{W}$ is a connected 2-Weyl graph with property (SH), $C \in \mathcal{W}$ is a fixed chamber, and for geodesics γ in $\mathcal{W}$, we will denote by $[\gamma]$ the strict homotopy class $[\gamma]_{\simeq}$.

We now describe a certain representation of the universal cover of $\mathcal{W}$, which we will denote by $p : \widetilde{\mathcal{W}}^C \to \mathcal{W}$.

The Chambers and Edges. Let the chambers of $\widetilde{\mathcal{W}}^C$ be the strict homotopy classes of the geodesics which issue from C. Thus,

$$\widetilde{\mathcal{W}}^C_0 = \big\{[\gamma] : \gamma \text{ is geodesic in } \mathcal{W} \text{ such that } \iota(\gamma) = C\big\}.$$

We denote the class of the trivial geodesic at C by $\widetilde{C}$. Since we are trying to construct (at least) an étale morphism $\widetilde{\mathcal{W}}^C \to \mathcal{W}$, which will turn out to send the chamber $[\gamma]$

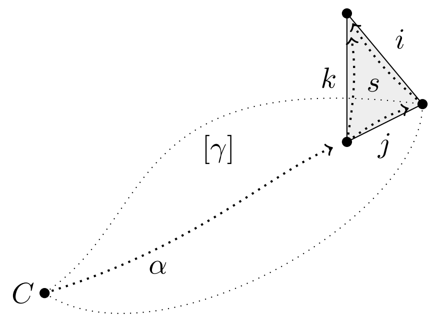

Figure 4.2: Defining $\tau([\gamma], i)$

to $\tau([\gamma])$, we let the edges be pairs $([\gamma], i)$, where i is an edge of $\mathcal{W}$ which issues from the chamber at which $[\gamma]$ terminates. Thus,

$$\widetilde{\mathcal{W}}_1^C = \big\{([\gamma], i) : [\gamma] \in \widetilde{\mathcal{W}}_0^C,\ \iota(i) = \tau(\gamma)\big\}.$$

The Extremities. We now define the extremities of $\widetilde{\mathcal{W}}^C$. Fix an edge $([\gamma], i)$. The value of ι is easy, and is suggested by how we have modeled the edges; we always have,

$$\iota([\gamma], i) = [\gamma].$$

Let $s = \upsilon(i)$, and $w = \gamma_W$. If $ws > w$ (in the Bruhat order), then γi is a geodesic, and we put,

$$\tau([\gamma], i) = [\gamma i].$$

This is clearly well defined. If $ws < w$, then γi is not a geodesic, however by the exchange condition, there exist geodesics in $[\gamma]$ whose types end with s. By Lemma 4.1, all these geodesics end with same s-labeled edge. Call this edge j, and let $k = j; i$. We may have $j = i^{-1}$, and $k = 1$, in which case we let k as a gallery denote the corresponding trivial gallery. Pick any geodesic α such that $\alpha j \in [\gamma]$ (see Figure 4.2). We put,

$$\tau([\gamma], i) = [\alpha k].$$

To see that this is well defined, suppose that α' is another geodesic used instead of α. Then, by Lemma 4.2, we have $\alpha \simeq \alpha'$, thus $[\alpha k] = [\alpha' k]$. Finally, we put $\upsilon([\gamma], i) = \upsilon(i)$. This gives $\widetilde{\mathcal{W}}^C$ the structure of a graph of type M.

The Panel Groupoids. We now show that $\widetilde{\mathcal{W}}^C$ is naturally a chamber system. Let s, w, j, k, and α be as before for a fixed edge $([\gamma], i)$. Firstly, notice that $\widetilde{\mathcal{W}}^C$ does not have loops, for this would imply either:

- $\gamma \simeq \gamma i$ in the case where $ws < w$, which is not possible since these galleries have different lengths

- $\gamma \simeq \alpha k$ in the case where $ws > w$, which also is not possible; if k is trivial, then these galleries have different lengths, and if k is not trivial, then $k = j$ by Lemma 4.1, which contradicts the fact that $k = j; i$.

Secondly, $\widetilde{\mathcal{W}}^C$ is slim. To see this, suppose that $([\gamma], i)$ and $([\gamma], i')$ are edges which issue from $[\gamma]$ and terminate at the same chamber D, with $\upsilon(i) = \upsilon(i')$. If $ws > w$, then $[\gamma i] = [\gamma i'] = D$. But by Lemma 4.1, we must have $i = i'$, and so $([\gamma], i) = ([\gamma], i')$. If $ws < w$, then $[\alpha k] = [\alpha k'] = D$, where $k' = j; i'$, and so:

- if both k and k' are trivial, then $i = i' = j^{-1}$, and $([\gamma], i) = ([\gamma], i')$
- if exactly one of $\{k, k'\}$ is trivial, then αk cannot be strictly homotopic to $\alpha k'$ since they have different lengths, a contradiction
- if neither are trivial, then $k = k'$ by Lemma 4.1, so again we must have $i = i'$, and so $([\gamma], i) = ([\gamma], i')$.

We now claim that if $[\gamma] \xrightarrow{([\gamma],i)} [\gamma']$ and $[\gamma'] \xrightarrow{([\gamma'],i')} D$ are two edges with $\upsilon(i) = \upsilon(i')$ and $i \neq i'$, then $([\gamma], i; i')$ is an edge which terminates at D. If $ws > w$, then $[\gamma'] = [\gamma i]$. Therefore $D = [\gamma i'']$, where $i'' = i; i'$. The result follows. If $ws < w$, then $([\gamma], i; i')$ terminates at $[\alpha i''']$, where $i''' = j; i; i'$. But $[\gamma'] = [\alpha i'']$, and so $D = [\alpha i''']$. Therefore, for edges $i, i' \in \mathcal{W}$ with $\upsilon(i) = \upsilon(i')$ and $i \neq i'$, we can define the composition,

$$([\gamma], i); ([\gamma'], i') = ([\gamma], i; i').$$

If $ws > w$, the inverse of an edge $([\gamma], i)$ is $([\gamma i], i^{-1})$. If $ws < w$, the inverse of an edge $([\gamma], i)$ is $([\alpha k], i^{-1})$. This gives $\widetilde{\mathcal{W}}^C$ the structure of a chamber system.

The Covering. Define a map $p : \widetilde{\mathcal{W}}^C \to \mathcal{W}$ by putting $p([\gamma]) = \tau(\gamma)$ for chambers, and $p([\gamma], i) = i$ for edges. In particular we have $p(\widetilde{C}) = C$. This map preserves extremities; for ι, this follows from the fact that $\tau(\gamma) = \iota(i)$, and for τ, this follows from the fact that $p([\gamma i]) = \tau(i)$ if $ws > w$, and $p([\alpha k]) = \tau(k) = \tau(i)$ if $ws < w$. Moreover, p is an surjective-étale morphism of generalized chamber systems; this follows directly from the definition of composition in $\widetilde{\mathcal{W}}^C$ and the definition of edges in $\widetilde{\mathcal{W}}^C$. Then p is étale since $\mathcal{W}$ is connected. We now show that $p : \widetilde{\mathcal{W}}^C \to \mathcal{W}$ is also a pre-covering. First, we need the following observation:

Lemma 4.3. Let β be a gallery in $\mathcal{W}$ which issues from C, and let $\tilde{\beta}$ be the lifting of β to a gallery which issues from $\widetilde{C}$. Then β is homotopic to the geodesics in $\tau(\tilde{\beta})$.

Proof. We prove by induction on the length of β. If β consists of one edge, then the result is trivial since $\tau(\tilde{\beta})$ is the set containing β. Suppose that the result holds for galleries of length n, and that $|\beta| = n + 1$. Let i be the last edge of β, and let β' be the subgallery such that $\beta' i = \beta$. Let $\tilde{\beta}'$ be the lifting of β' to a gallery which issues from $\widetilde{C}$, and let $\gamma \in \tau(\tilde{\beta}')$. Notice that $\beta' \sim \gamma$ by the induction hypothesis. Now, either $\tau(\tilde{\beta}) = [\gamma i]$, or $\tau(\tilde{\beta}) = [\alpha k]$. In either case, γi is homotopic to the geodesics in $\tau(\tilde{\beta})$ since $\alpha k \sim \gamma i$. Then $\beta = \beta' i \sim \gamma i$ by the induction hypothesis, and so β is homotopic to the geodesics in $\tau(\tilde{\beta})$. □

Lemma 4.4. The étale morphism $p : \widetilde{\mathcal{W}}^C \to \mathcal{W}$ is a pre-covering.

Proof. Let β be a gallery in $\widetilde{\mathcal{W}}^C$ such that $p \circ \beta$ is a suite of $\mathcal{W}$. Let ρ and $\hat{\rho}$ be a maximal (s, t)-geodesic and a maximal (t, s)-geodesic respectively such that $\beta = \rho^{-1}\hat{\rho}$. We now prove that $\tau(\rho) = \tau(\hat{\rho})$, which implies that β is a cycle. Let $[\gamma] = \iota(\rho) = \iota(\hat{\rho})$, and put $w = \gamma_W$. Let $\rho_{\mathcal{W}} = p \circ \rho$, and $\hat{\rho}_{\mathcal{W}} = p \circ \hat{\rho}$.

Let $J = \{s, t\}$, and let w' be the unique representative of the coset wW_J with minimal word length (see Theorem A.11). Then $w = w'w_J$, for some $w_J \in W_J$. We may assume that γ is of the form $\gamma = \gamma'\gamma_J$, where $\gamma'_W = w'$ and $\gamma_{JW} = w_J$. Let ρ' and $\hat{\rho}'$ be geodesics which are homotopic to $\gamma_J \rho_{\mathcal{W}}$ and $\gamma_J \hat{\rho}_{\mathcal{W}}$ respectively. It follows from Theorem A.11 that $\gamma'\rho'$ and $\gamma'\hat{\rho}'$ are geodesics. Since $p \circ \beta$ is a suite of $\mathcal{W}$, we have $\rho_{\mathcal{W}} \sim \hat{\rho}_{\mathcal{W}}$, and so,

$$\rho' \sim \gamma_J \rho_{\mathcal{W}} \sim \gamma_J \hat{\rho}_{\mathcal{W}} \sim \hat{\rho}'.$$

Then $\rho' \simeq \hat{\rho}'$ since $\mathcal{W}$ has property (2C), and so $\gamma'\rho' \simeq \gamma'\hat{\rho}'$. The geodesics in $\tau(\rho)$ are homotopic to $\gamma\rho_{\mathcal{W}}$, and therefore to $\gamma'\rho'$, by Lemma 4.3. Similarly, the geodesics in $\tau(\hat{\rho})$ are homotopic to $\gamma\hat{\rho}_{\mathcal{W}}$, and therefore to $\gamma'\hat{\rho}'$. Thus, $\gamma'\rho' \in \tau(\rho)$ and $\gamma'\hat{\rho}' \in \tau(\hat{\rho})$, and so $\tau(\rho) = \tau(\hat{\rho})$ as required. □

Redefine $\widetilde{\mathcal{W}}^C$ to be its completion with respect to p. Thus, we obtain a covering $p : \widetilde{\mathcal{W}}^C \to \mathcal{W}$.

Theorem 4.5. The covering $p : \widetilde{\mathcal{W}}^C \to \mathcal{W}$ is the universal cover of $\mathcal{W}$.

Proof. First, notice that p is a connected covering since for chambers $[\gamma], [\gamma'] \in \widetilde{\mathcal{W}}^C$, the lifting of γ and γ' to galleries in $\widetilde{\mathcal{W}}^C$ which issue from the class of the trivial gallery connect $[\gamma]$ and $[\gamma']$ respectively to the class of the trivial gallery.

Let $p' : \widetilde{\mathcal{W}} \to \mathcal{W}$ be the universal cover of $\mathcal{W}$. We now construct a morphism of coverings $\mu : p \to p'$, which proves that $\widetilde{\mathcal{W}}^C$ is also simply connected. Pick a chamber $\widetilde{C} \in \widetilde{\mathcal{W}}$ such that $p'(\widetilde{C}) = C$. Let $\mu : \widetilde{\mathcal{W}}^C \to \widetilde{\mathcal{W}}$ be the morphism whose map on chambers is $[\gamma] \mapsto \tau(\tilde{\gamma})$, where $\tilde{\gamma}$ is the lifting of γ to a gallery which issues from $\widetilde{C}$. To see that this is a morphism of chamber systems, let $[\gamma] \xrightarrow{([\gamma],i)} [\gamma']$ be an edge of $\widetilde{\mathcal{W}}^C$, and let $\tilde{i}$ be the unique edge of $\widetilde{\mathcal{W}}$ such that $p'(\tilde{i}) = i$ and $\iota(\tilde{i}) = \tau(\tilde{\gamma})$. Then $\tau(\tilde{i}) = \tau(\tilde{\gamma}')$, and so $[\gamma] \xrightarrow{([\gamma],i)} [\gamma']$ is mapped to $\tilde{i}$. We have $p = p' \circ \mu$ since $p'(\tau(\tilde{\gamma})) = \tau(\gamma)$. Finally, to see that μ is a morphism of Weyl data, let θ be a suite of $\widetilde{\mathcal{W}}^C$. Then $p \circ \theta$ must be a suite of $\widetilde{\mathcal{W}}$ because $\widetilde{\mathcal{W}}$ is simply connected. □

4.1.3 Introducing Weyl Graphs

In this section, we define Weyl graphs and collect some of their basic properties. We show that the notion of a building is equivalent to that of the universal cover of a connected Weyl graph. We also prove the equivalence of the Weyl properties for 2-Weyl graphs, which will give us several characterizations of Weyl graphs.

Definition of Buildings. Recall property (B) of pre-Weyl graphs,

(B) geodesics with the same extremities have the same W-length.

We define a **building** of type M to be a connected pre-Weyl graph of type M with property (B). In [Ron89], a building of type M is defined to be a weak chamber system $\mathcal{W}$ of type M which admits a function

$$\mathcal{W}_0 \times \mathcal{W}_0 \to W$$

such that if f is a reduced word over S, then $(C, D) \mapsto w(f)$ if and only if there exists a geodesic γ which travels from C to D with $\gamma_S = f$. One can easily see that the underlying chamber system of a pre-Weyl graph with property (B) admits such a function. Recall that we call Weyl data $\mathcal{W}$ simple if every (s,t)-cycle of the underlying graph of $\mathcal{W}$ is a defining suite. Conversely, given a building $\mathcal{W}$ in the sense of [Ron89], the simple Weyl data associated to the chamber system $\mathcal{W}$ will be a pre-Weyl graph with property (B). Thus, the definitions are equivalent.

We have the following characterization of buildings:

Proposition 4.6. A connected pre-Weyl graph satisfies property (B) if and only if it is simply connected and has property (W).

Proof. If a pre-Weyl graph with property (B) is simply connected, then all the geodesics with the same extremities will be homotopic, and therefore have the same W-length. Conversely, suppose that a connected pre-Weyl graph $\mathcal{W}$ satisfies property (B). Let γ be a geodesic in $\mathcal{W}$ with $\iota(\gamma) = \tau(\gamma) = C$. Then (B) implies that $\gamma_W = 1$, since the trivial gallery at C is a geodesic. Thus, γ is trivial, and so there is only one homotopy class of loops at C. Therefore $\mathcal{W}$ is simply connected. That $(B) \implies (W)$ is clear. □

Definition of Weyl Graphs. We now define a **Weyl graph** to be a pre-Weyl graph which has property (W). Thus, buildings are exactly connected and simply connected Weyl graphs. Notice that 2-Weyl graphs are exactly pre-Weyl graphs whose 2-residues are Weyl graphs. Notice also that the residues of Weyl graphs are again Weyl graphs, thus Weyl graphs are 2-Weyl. We will denote Weyl graphs which are buildings by Δ.

The Metrization of the Fundamental Groupoid. Property (W) allows us to define the W**-length** $[\gamma]_W$ of a homotopy class of galleries $[\gamma]$ in a Weyl graph to be the W-length of the geodesics which it contains. Thus, $[\gamma]_W = \gamma_W$. The function,

$$\overline{\mathcal{W}}_1 \to W, \quad [\gamma] \mapsto [\gamma]_W$$

is called the **metrization** of the fundamental groupoid $\overline{\mathcal{W}}$ of $\mathcal{W}$.

Simplicial Buildings. Associated to every building Δ is the simplicial complex whose poset of cells is the poset of residues of Δ, where the ordering is reverse inclusion. Thus, the smallest residues, i.e. the chambers, correspond to the maximal simplices. We call this simplicial complex the **simplicial building** of Δ. The building Δ can be recovered from its simplicial building, and the simplicial complexes which are isomorphic to the simplicial building of some building Δ were characterized in Tits' original definition of a building as an amalgam of Coxeter complexes (see [Tit74]).

Weyl Polygons and Generalized Polygons. A **Weyl polygon** is a rank 2 connected Weyl graph. A **generalized polygon** is a rank 2 building, i.e. a simply connected Weyl polygon (thus, our convention is that generalized polygons can be weak). A generalized polygon of type $I_2(m)$ is called a generalized m**-gon**. The simplicial building of a generalized m-gon, $m < \infty$, is a bipartite graph of girth m and diameter $2m$. Conversely, any such graph determines a generalized polygon. Generalized polygons can also be modelled as certain incidence geometries (see [VM12]). In light of covering theory of Weyl graphs, Weyl polygons are essentially equivalent to generalized polygons which are equipped with a chamber-free (i.e. flag-free) action of a group.

We now show that the image of a Weyl graph under a covering is a Weyl graph, and that a covering of a Weyl graph is again a Weyl graph.

Lemma 4.7. Let $\omega : \mathcal{W}' \to \mathcal{W}$ be a morphism of Weyl data. If $\mathcal{W}$ has property (W), then $\mathcal{W}'$ has property (W).

Proof. Let $\tilde{\gamma}$ and $\tilde{\gamma}'$ be two homotopic geodesics in $\mathcal{W}'$. Let $\gamma = \omega \circ \tilde{\gamma}$ and $\gamma' = \omega \circ \tilde{\gamma}'$. Then γ and γ' are homotopic geodesics in $\mathcal{W}$, and so $\tilde{\gamma}_W = \gamma_W = \gamma'_W = \tilde{\gamma}'_W$ since $\mathcal{W}$ has property (W). □

Lemma 4.8. Let $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ be an covering of Weyl data. If $\widetilde{\mathcal{W}}$ has property (W), then $\mathcal{W}$ has property (W).

Proof. Let γ and γ' be two homotopic geodesics in $\mathcal{W}$. Lift these geodesics to homotopic geodesics $\tilde{\gamma}$ and $\tilde{\gamma}'$ in $\widetilde{\mathcal{W}}$. Then $\gamma_W = \tilde{\gamma}_W = \tilde{\gamma}'_W = \gamma'_W$ since $\widetilde{\mathcal{W}}$ has property (W). □

The following is our main result:

Theorem 4.9. The universal cover of a connected Weyl graph is a building, and the image of a building under a covering is a connected Weyl graph. Thus, connected Weyl graphs are exactly the quotients of buildings by chamber-free actions, and buildings are exactly the universal covers of connected Weyl graphs.

Proof. Recall from Theorem 3.24 that coverings preserve and reflect the properties (PW0) and (PW1). Therefore it follows from Lemma 4.7 and Lemma 4.8 that if $p : \widetilde{\mathcal{W}} \to \mathcal{W}$ is a covering of Weyl data, then $\widetilde{\mathcal{W}}$ is a Weyl graph if and only if $\mathcal{W}$ is a Weyl graph. The result follows. □

We now show that the Weyl properties are equivalent in the setting of 2-Weyl graphs. The implications which we will prove are,

$$(\mathrm{C}) \implies (\mathrm{W}) \implies (\mathrm{H}) \implies (\mathrm{SH}) \implies (\mathrm{C}).$$

That (C) $\implies$ (W) is a simple observation:

Lemma 4.10. Let $\mathcal{W}$ be a 2-Weyl graph with property (C). Then $\mathcal{W}$ is a Weyl graph.

Proof. Recall that property (C) says homotopic geodesics are strictly homotopic. But strict homotopies preserve W-length. Therefore homotopic geodesics must have the same W-length. □

We now prove (W) $\implies$ (H):

Lemma 4.11. Let $\mathcal{W}$ be a Weyl graph. Then $\mathcal{W}$ has property (H).

Proof. The proof is essentially the same as the proof of Proposition 3.13, which was the rank 2 case. Let ρ and ρ' be homotopic geodesics in $\mathcal{W}$ with the same type. Let i be the last edge of ρ, and let i' be the last edge of ρ'. Towards a contradiction, suppose that $i \neq i'$. Let $j = i'i^{-1}$. Let α and α' be the subgalleries such that $\alpha i = \rho$ and $\alpha' i' = \rho'$. Then α and $\alpha' j$ are homotopic geodesics with different W-lengths, a contradiction. Thus, $i = i'$. Then α and α' are also homotopic geodesics, and so can apply the same argument to the penultimate edges of ρ and ρ'. Therefore, by induction, we may conclude that $\rho = \rho'$. □

Finally, we prove (SH) $\implies$ (C), using the construction of Section 4.1.2:

Lemma 4.12. Let $\mathcal{W}$ be a 2-Weyl graph with property (SH). Then $\mathcal{W}$ has property (C).

Proof. Let γ and γ' be homotopic geodesics in $\mathcal{W}$. Put $C = \iota(\gamma) = \iota(\gamma')$, and let $p : \widetilde{\mathcal{W}}_C \to \mathcal{W}$ be the covering constructed in Section 4.1.2. The map μ constructed in the proof of Theorem 4.5 is bijective on chambers by Corollary 3.33.2. Since $\gamma \sim \gamma'$, we have $\mu([\gamma]) = \mu([\gamma'])$, thus $[\gamma] = [\gamma']$, and so $\gamma \simeq \gamma'$. □

Theorem 4.13. Let $\mathcal{W}$ be a 2-Weyl graph. If $\mathcal{W}$ has any of the Weyl properties, then $\mathcal{W}$ is a Weyl graph. Conversely, if $\mathcal{W}$ is a Weyl graph, then $\mathcal{W}$ satisfies all the Weyl properties.

Proof. One can easily see that if $\mathcal{W}$ has property (H), then $\mathcal{W}$ has property (SH). The result then follows by Lemma 4.10, Lemma 4.11, and Lemma 4.12. □

Corollary 4.13.1. Let $\mathcal{W}$ be a Weyl graph which is labeled over S. For each $J \subseteq S$, the groupoid homomorphism,

$$\bar{\varepsilon}_J : \overline{\mathcal{W}}_J \to \overline{\mathcal{W}}$$

is an embedding. Thus, $\overline{\mathcal{W}}_J$ is naturally a subgroupoid of $\overline{\mathcal{W}}$.

Proof. Let γ and γ' be homotopic geodesics of $\mathcal{W}$ which are contained in a J-residue R. Then γ and γ' are strictly homotopic, and so a homotopy from γ to γ' can take place within R. Thus, γ and γ' are also homotopic in $\mathcal{W}_J$. □

Corollary 4.13.2. The residues of buildings are buildings.

Proof. Let Δ be a building. Then the fundamental groupoid $\overline{\Delta}$ is a connected setoid. For $J \subseteq S$, $\overline{\Delta}_J$ is naturally a subgroupoid of $\overline{\Delta}$. Thus, $\overline{\Delta}_J$ is also a setoid, and the result follows. □

4.1.4 Tits' Local to Global Results.

In this section, we show that 2-Weyl graphs are often Weyl graphs; in particular, the only thing which stops a 2-Weyl graph from being a Weyl graph are the spherical 3-residues of type C_3 and H_3. Given the preservation and reflection of property (W) under covering maps, and the fact that buildings are exactly the universal covers of connected Weyl graphs, this is an easy consequence of results in [Tit81]; however we provide a more direct proof of the first part of Tits' result, along the lines of [Ron89, Theorem 4.9]:

Lemma 4.14. Let $\mathcal{W}$ be a 2-Weyl graph. Then $\mathcal{W}$ is a Weyl graph if (and only if) the spherical 3-residues of $\mathcal{W}$ are Weyl graphs.

Proof. We claim that $\mathcal{W}$ has property (SH), which suffices by Theorem 4.13. To see this, let γ and γ' be strictly homotopic geodesics of $\mathcal{W}$ with the same type. A strict homotopy between γ and γ' induces a strict homotopy of words, which, by [Ron89, Theorem 2.17], decomposes into self strict homotopies which are either inessential, or else only alter a subword over J, where J is a 3-element spherical subset of S. A strict homotopy of galleries which induces an inessential self strict homotopy of words cannot change a gallery. It then follows that $\gamma = \gamma'$ by the hypothesis on $\mathcal{W}$. Thus, $\mathcal{W}$ has property (SH), and so is a Weyl graph. □

Lemma 4.15. If a 2-Weyl graph $\mathcal{W}$ of type M and of rank 3 is not a Weyl graph, then either $M = C_3$ or $M = H_3$.

Proof. Since $\mathcal{W}$ is not a Weyl graph, it must have a connected component which is not a Weyl graph. Let Δ be the universal cover of this connected component. By Theorem 4.9, Δ is a connected simply connected chamber system of type M which is not a building. Therefore $M = C_3$ or $M = H_3$ by the discussion in [Tit81, Section 2.2] and [Tit81, Theorem 1]. □

Then, combining Lemma 4.14 and Lemma 4.15 gives:

Theorem 4.16. Let $\mathcal{W}$ be a 2-Weyl graph. Then $\mathcal{W}$ is a Weyl graph if (and only if) the residues of $\mathcal{W}$ of type C_3 and H_3 are Weyl graphs.

4.2 The Fundamental Group of a Weyl Graph

In this section, we give a method for obtaining a group presentation of the fundamental group of a connected Weyl graph.

4.2.1 A Presentation of the Fundamental Group

Let M be a Coxeter matrix on S, and let $\mathcal{W}$ be a connected Weyl graph of type M.

Generating Sets of Weyl Graphs. For each $s \in S$, pick a base chamber in each connected component of $\mathcal{W}_s$. Let $B_s = \{B_{s,0}, B_{s,1}, \dots\}$ be the set of base chambers in $\mathcal{W}_s$, and let,

$$\mathcal{B}_s = \big\{i \in \mathcal{W}_1 : \upsilon(i) = s,\ \iota(i) = \tau(i) \in B_s\big\}.$$

Thus, $\mathcal{B}_s$ is the set of non-trivial loops in $\mathcal{W}_s$ at the base chambers. For each $s \in S$ and for each chamber $C \in \mathcal{W}_s$ with $C \notin B_s$, pick an edge $i_C \in \mathcal{W}_s$ with $\iota(i_C) \in B_s$ and $\tau(i_C) = C$. Let,

$$\mathcal{I}_s^+ = \{i_C : C \in \mathcal{W}_0 \setminus B_s\}, \quad \mathcal{I}_s^- = \{i_C^{-1} : C \in \mathcal{W}_0 \setminus B_s\}, \quad \mathcal{I}_s = \mathcal{I}_s^+ \sqcup \mathcal{I}_s^-.$$

Also put $\mathcal{S}_s = \mathcal{B}_s \sqcup \mathcal{I}_s$, and,

$$\mathcal{B} = \bigsqcup_{s\in S} \mathcal{B}_s, \quad \mathcal{I} = \bigsqcup_{s\in S} \mathcal{I}_s, \quad \mathcal{I}^+ = \bigsqcup_{s\in S} \mathcal{I}_s^+, \quad \mathcal{I}^- = \bigsqcup_{s\in S} \mathcal{I}_s^-, \quad \mathcal{S} = \bigsqcup_{s\in S} \mathcal{S}_s = \mathcal{B} \sqcup \mathcal{I}.$$

We call $\mathcal{S}$ a **generating set** of $\mathcal{W}$. Notice that $\mathcal{S}_s$ generates $\mathcal{W}_s$, and $\mathcal{S}$ generates $\overline{\mathcal{W}}$.

Expressions of Edges. For each edge $i \in \mathcal{W}$, putting $s = \upsilon(i)$, then there exists a unique 3-tuple $(i^-, i_{\mathcal{B}}, i^+)$ of edges of $\mathcal{W}_s$ such that:

(i) $i = i^-; i_{\mathcal{B}}; i^+$

(ii) either $i^- \in \mathcal{I}_s^-$ or i^- is trivial

(iii) either $i_{\mathcal{B}} \in \mathcal{B}_s$ or $i_{\mathcal{B}}$ is trivial

(iv) either $i^+ \in \mathcal{I}_s^+$ or i^+ is trivial.

The **expression** of i with respect to $\mathcal{S}$ is the word obtained from the sequence $i^-, i_{\mathcal{B}}, i^+$ by deleting any trivial edges. In particular, if $i \in \mathcal{S}$, then the expression of i is just i.

$\mathcal{S}$-Sequences and $\mathcal{S}$-Suites. Let $\mathcal{W}$ be a connected Weyl graph and let $\mathcal{S} \subseteq \mathcal{W}_1$ be a generating set of $\mathcal{W}$. Let $r = i_1, \ldots, i_n$ be a sequence of edges of $\mathcal{W}$. The **$\mathcal{S}$-sequence** of r is the sequence obtained from r by replacing each edge i_k, for $k \in \{1, \ldots n\}$, by the expression of i_k with respect to $\mathcal{S}$. Notice that if r is the sequence of edges of a gallery β, then the $\mathcal{S}$-sequence of r is the sequence of edges of a gallery which can be obtained from β by a composition of expansions.

We let $\mathcal{R}$ denote the set of sequences of edges of $\mathcal{W}$ which are obtained as the $\mathcal{S}$-sequences of the defining suites of $\mathcal{W}$. The elements of $\mathcal{R}$ are called **$\mathcal{S}$-suites**, and we think of them as words over $\mathcal{S}$.

The Universal Group of a Weyl Graph. Let $\mathcal{W}$ be a connected Weyl graph and let $\mathcal{S}$ be a generating set of $\mathcal{W}$. The **universal group** $\mathrm{FG}(\mathcal{W})$ of $\mathcal{W}$ with respect to $\mathcal{S}$ is the group generated by $\mathcal{S}$, subject to the relations of the local groups at each base chamber, and $\mathcal{R}$ (treated as a set of relators). Explicitly, let $\mathcal{R}_{s,k}$ be a set of defining relations for the local group of $\mathcal{W}_s$ at $B_{s,k} \in B_s$, and let $\mathcal{R}_{\mathcal{B}} = \bigsqcup_{s,k} \mathcal{R}_{s,k}$. Then,

$$\mathrm{FG}(\mathcal{W}) = \big\langle \mathcal{S} \mid \mathcal{R},\ \mathcal{R}_{\mathcal{B}},\ ij = 1 : i \in \mathcal{I}^+,\ j = i^{-1} \in \mathcal{I}^- \big\rangle.$$

This is just the smallest numbers of relations which makes the natural projection,

$$\pi : \overline{\mathcal{W}} \to \mathrm{FG}(\mathcal{W}), \quad i \mapsto i \quad \text{ for } i \in \mathcal{S}$$

a well defined homomorphism. To see that π is well defined, let $[\beta] \in \overline{\mathcal{W}}$. Let $r = i_1, \ldots, i_n$ be the sequence of edges of β. Then $\pi([\beta])$ is the product in $\mathrm{FG}(\mathcal{W})$ of the $\mathcal{S}$-sequence of r. Let $\hat{\beta}$ be a gallery obtained from β by a 1-elementary homotopy. Then $\pi([\hat{\beta}]) = \pi([\beta])$ by the inclusion of the relations $\mathcal{R}_{\mathcal{B}}$, and the relators of the form $ij = 1$. Let $\hat{\beta}$ be a gallery obtained from β by a 2-elementary homotopy. Then $\pi([\hat{\beta}]) = \pi([\beta])$ by the inclusion of the relators $\mathcal{R}$.

Spanning Trees. Let $\mathcal{W}$ be a connected Weyl graph and let $\mathcal{S}$ be a generating set of $\mathcal{W}$. Let Γ be the following undirected graph:

(i) the vertices of Γ are the chambers of $\mathcal{W}$

(ii) the edges of Γ are sets of the form $\{i, i^{-1}\}$, where $i \in \mathcal{I}^+$

(iii) the extremities of the undirected edge $\{i, i^{-1}\}$ are $\iota(i)$ and $\tau(i)$.

Notice that Γ is connected. Let Γ' be a spanning tree of Γ, and let $T \subseteq \mathcal{I}$ be the set of edges which are contained in some edge of Γ'. We call T a **spanning tree** of $\mathcal{W}$. Notice that if a panel groupoid $\mathcal{W}_s$ is connected, then we can take T to be $\mathcal{I}_s$.

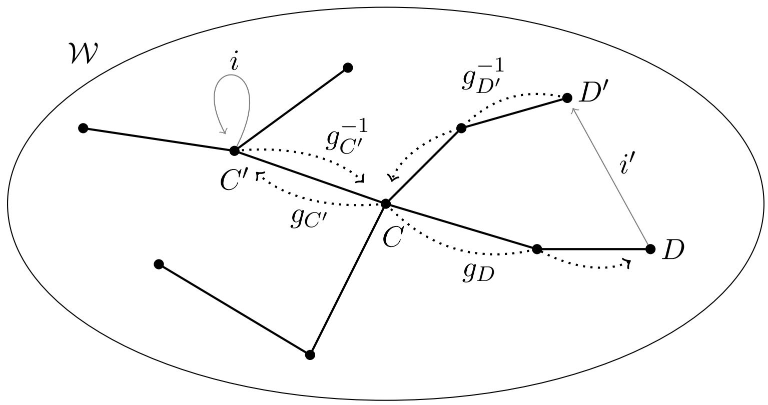

Figure 4.3: Defining $\psi : \mathrm{FG}(\mathcal{W})/T \to \overline{\mathcal{W}}_C$

The Fundamental Group at T. Let $\mathcal{W}$ be a connected Weyl graph. Let $\mathcal{S}$ be a generating set of $\mathcal{W}$ and let T be a spanning tree of $\mathcal{W}$. The **fundamental group** of $\mathcal{W}$ at T is the group,

$$\pi_1(\mathcal{W}, T) = \left\langle \mathcal{S} \mid \mathcal{R},\ \mathcal{R}_{\mathcal{B}},\ ij = 1,\ t = 1 : i \in \mathcal{I}^+,\ j = i^{-1} \in \mathcal{I}^-,\ t \in T \right\rangle.$$

The following is our main result:

Theorem 4.17. Let $\mathcal{W}$ be a connected Weyl graph. Let $\mathcal{S}$ be a generating set of $\mathcal{W}$ and let T be a spanning tree of $\mathcal{W}$. Then the local groups of $\overline{\mathcal{W}}$ are naturally isomorphic to $\pi_1(\mathcal{W}, T)$.

The fact that these isomorphisms are natural might seem surprising, but remember that a choice of T has been made.

Proof. Pick a chamber $C \in \mathcal{W}$, and let $\overline{\mathcal{W}}_C$ be the local group of $\overline{\mathcal{W}}$ at C. Let,

$$\varphi : \overline{\mathcal{W}}_C \to \pi_1(\mathcal{W}, T)$$

be the the restriction of $\pi : \overline{\mathcal{W}} \to \mathrm{FG}(\mathcal{W})$ to $\overline{\mathcal{W}}_C$, composed with the quotient map $\mathrm{FG}(\mathcal{W}) \to \pi_1(\mathcal{W}, T)$. For each chamber $D \in \mathcal{W}$, let $g_D \in \overline{\mathcal{W}}(C, D)$ be the homotopy class of the gallery corresponding to the unique path in T from C to D. Let,

$$\psi : \pi_1(\mathcal{W}, T) \to \overline{\mathcal{W}}_C$$

be the homomorphism mapping a generator $i \in \mathcal{S}$ to the composition $g_{\iota(i)}; i; g_{\tau(i)}^{-1} \in \overline{\mathcal{W}}$ (see Figure 4.3). This is a well defined homomorphism because the relations of $\pi_1(\mathcal{W}, T)$ are satisfied in $\overline{\mathcal{W}}_C$, since $\overline{\mathcal{W}}_C$ is a subgroup of $\overline{\mathcal{W}}$. Notice that $\varphi \circ \psi$ is the identity on $\mathcal{S}$ because $\pi(g_D)$ lies in the kernel of $\mathrm{FG}(\mathcal{W}) \to \pi_1(\mathcal{W}, T)$. Also, $\psi \circ \varphi$ is the identity because the g_D cancel via contractions, and one recovers the original gallery up to homotopy. Thus, φ and ψ are mutually inverse isomorphisms. $\square$

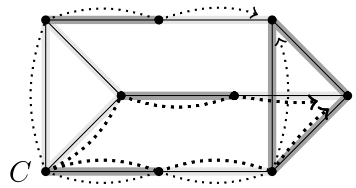

Figure 4.4: The flower $\mathcal{F}(C)$ in a weak generalized 3-gon

Therefore we have a natural isomorphism $\overline{\mathcal{W}}_C \cong \pi_1(\mathcal{W}, T)$. If we make the choice of a chamber in the universal cover Δ of $\mathcal{W}$, we get a well defined action of $\pi_1(\mathcal{W}, T)$ on Δ.

A method one can employ when calculating the fundamental group of a Weyl graph is to first calculate the universal groups of its Weyl polygons, take the union of each presentation, and then quotient out by a spanning tree.

4.2.2 Flowers and Petals

In this section, we show that in order to determine a Weyl polygon, i.e. a connected rank 2 Weyl graph, one only has to know the underlying generalized chamber system, and a set of homotopic maximal alternating geodesic pairs, called 'petals', issuing from a fixed chamber. This will simplify the task of calculating fundamental groups of Weyl graphs in many cases.

Flowers. Let C be a fixed chamber in a Weyl graph $\mathcal{W}$. Let $J = \{s,t\}$ be a 2-element spherical subset of S. The ***J*-flower** based at C, denoted $\mathcal{F}_J(C)$, is the set of maximal (s,t)-geodesics and maximal (t,s)-geodesics which issue from C. Thus, the maximal alternating geodesics of $\mathcal{F}_J(C)$ are contained in the residue $\mathcal{R}_J(C)$, which is a Weyl polygon. If $\mathcal{W}$ is a polygon, then we must have $J = S$, and we speak simply of the **flower** $\mathcal{F}(C)$ based at C.

Petals. A ***J*-petal** is a subset of a J-flower which contains a maximal (s,t)-geodesic $\rho(s,t)$, together with the unique maximal (t,s)-geodesic $\rho(t,s)$ such that $\rho(s,t) \sim \rho(t,s)$. The petals form a partition of the flower. A flower naturally induces a set of suites; for each petal $\{\rho(s,t), \rho(t,s)\}$, take the suite $\rho(s,t)\rho(t,s)^{-1}$.

Figure 4.4 shows a Weyl graph of type A_2, in fact a weak generalized 3-gon, with the flower at C drawn.

Theorem 4.18. Let $\mathcal{P}$ be a Weyl polygon, and let $C \in \mathcal{P}$ be a chamber. Let $\mathcal{P}_C$ denote the Weyl data whose underlying generalized chamber system is that of $\mathcal{P}$, and whose defining suites are those induced by the flower $\mathcal{F}(C)$ of $\mathcal{P}$ of C. Then $\mathcal{P}_C$ is isomorphic to $\mathcal{P}$.

Proof. Let $\omega : \mathcal{P}_C \to \mathcal{P}$ be the identity map, which is a morphism of Weyl data since the defining suites of $\mathcal{P}_C$ are suites of $\mathcal{P}$. Let θ be a suite of $\mathcal{P}$. We claim that θ is also a suite of $\mathcal{P}_C$, which proves that $\omega : \mathcal{P}_C \to \mathcal{P}$ is an isomorphism by Proposition 3.4.

To see this, let γ be a geodesic from C to $\iota(\theta) = \tau(\theta)$. Using 1-elementary homotopies and strict homotopies which take place in the suites induced by the flower $\mathcal{F}(C)$ at C, we can obtain a geodesic γ' which is homotopic to $\gamma\theta\gamma^{-1}$ in both $\mathcal{P}$ and $\mathcal{P}_C$. Since $\mathcal{P}$ has property (W) and $\gamma\theta\gamma^{-1}$ is null-homotopic in $\mathcal{P}$, any geodesic homotopic to $\gamma\theta\gamma^{-1}$ in $\mathcal{P}$ must be trivial. Thus, γ' is trivial. Therefore $\gamma\theta\gamma^{-1}$ is null-homotopic in $\mathcal{P}_C$, and so θ is also null-homotopic in $\mathcal{P}_C$. $\square$

Usually, the data of a Weyl polygon in the form of a rank 2 generalized chamber system equipped with a flower will come from a group acting freely on the chambers (flags) of a generalized polygon. The petals can be determined by inspecting the action. Weyl graphs of rank ≥ 3 can then be constructed by gluing together Weyl polygons.

Chapter 5

Singer Lattices

In this chapter, we present an application of the theory of Weyl graphs. We obtain presentations of all the so-called Singer cyclic lattices in buildings of type M, where $m_{st} \in \{2, 3, \infty\}$ for all $s, t \in S$, and the defining graph L associated to M is connected (recall that L is different to the Coxeter-Dynkin diagram of M). We achieve this by first describing the Weyl polygons which can exist as 2-residues in the quotient of a Singer cyclic lattice of type M, and then determining all the ways in which these polygons can be glued together to form a Singer graph. Our results generalize those of [Ess13], in which the Singer lattices of type $\widetilde{A}_2$ are constructed using complexes of groups.

5.1 Singer Cyclic Polygons

In this section, we obtain representations of those Weyl m-gons, for $m \in \{2, 3\}$, which can exist as 2-residues in the quotient of a Singer cyclic lattice. For 2-gons, the construction is straightforward. For 3-gons, we will use the method of difference sets from finite geometry.

5.1.1 Singer Graphs, Singer Polygons, and Singer Lattices

We begin with some definitions. Notice that n-transitive and n-free actions on buildings correspond to n-connected and n-simply connected quotients respectively. We call an action **panel-regular** if its 1-regular. The quotients of buildings which correspond to panel-regular actions will be called Singer graphs. Thus, a **Singer graph** is a Weyl graph whose underlying generalized chamber system consists of copies of a fixed connected setoid. The cardinal q such that $\mathcal{W}$ has $k = q + 1$ many chambers is called the **order** of $\mathcal{W}$. Since all the panel groupoids of $\mathcal{W}$ are then isomorphic to $1 \times k$, the order of a Singer graph determines its underlying generalized chamber system. We define a **Singer building** of order q to be a building which is the universal cover of a Singer graph of order q. If Δ is a Singer building of order

q, then each of its panels is isomorphic to $1 \times k$. Thus, a Singer building is locally finite if and only if it has finite order. The quotient of a building by a group acting panel-regularly is a Singer graph, and the deck transformation group of the universal cover of a Singer graph acts panel-regularly. Thus, Singer graphs are essentially equivalent to buildings which are equipped with a panel-regular action of a group.

We define a **Singer polygon** to be a rank 2 Singer graph.[1] Notice that a Weyl graph is a Singer graph if and only if its 2-residues are Singer polygons. A **Singer cyclic polygon** is a Singer polygon whose fundamental group is cyclic.

Example 5.1. For each $2 \leq m \leq \infty$, there exists a unique Singer m-gon of order 1. It is the quotient of $\mathcal{C}_m$ by the cyclic group of order m acting by rotations of $2\pi/m$. We denote this quotient by $m\backslash\mathcal{C}_m$. The Weyl graph $m\backslash\mathcal{C}_m$ can be characterized as the unique simple Weyl polygon with two chambers, and whose panel-groupoids are both 1×2.

We will see that for $q \geq 2$, there exists a unique Singer cyclic 2-gon of order q, and for $q \geq 2$ a prime power, there exists a Singer cyclic 3-gon of order q. The uniqueness of this Singer 3-gon is (equivalent to) a long standing conjecture.

We define a **Singer lattice** $\Gamma < \mathrm{Aut}(\Delta)$ of order q to be a subgroup of the automorphism group of a locally finite building Δ such that $\Gamma\backslash\Delta$ is a Singer graph of order q. Thus, a Singer lattice Γ acts panel-regularly on Δ. We define a **Singer cyclic lattice** to be a Singer lattice whose isotropy of each spherical 2-residue of Δ is (finite) cyclic. By Proposition 3.40, a Singer cyclic lattice is equivalently a Singer lattice whose quotient's 2-residues are Singer cyclic polygons. Our definitions of Singer lattices and Singer cyclic lattices generalize those in [Ess13] and [Wit16] to all types of building. Notice that covering theory of Weyl graphs reduces the construction of Singer lattices to the construction of Singer graphs of finite order.

In this Chapter, we construct the Singer cyclic lattices which act on buildings whose spherical 2-residues are either generalized 2-gons or generalized 3-gons. We do this by constructing the corresponding quotients by gluing together Weyl polygons. Thus, we need representations of the Singer cyclic 2-gons and the Singer cyclic 3-gons.

5.1.2 Weyl Digons

A **Weyl digon** is a Weyl 2-gon, or equivalently, a Weyl digon is a connected Weyl graph of type $I_2(2)$. Similarly, a **generalized digon** is a generalized 2-gon. It is well known that (the isomorphism classes of) finite generalized digons are in bijection with pairs (q_1, q_2), where $q_1, q_2 \in \mathbb{Z}_{\geq 1}$. The simplicial building of the digon corresponding

[1] This is more restrictive than the usual definition of a Singer polygon. Many authors define a Singer polygon to be a generalized polygon equipped with an action of a group which is point-regular. Our definition of a Singer polygon is equivalent to a generalized polygon equipped with an action of a group which is point-regular and line-regular.

to (q_1, q_2) is the complete bipartite graph on $q_1 + 1$ white vertices and $q_2 + 1$ black vertices. We denote by $\mathbf{D}(q_1, q_2)$ the Weyl digon which is the generalized digon corresponding to (q_1, q_2). For $q \in \mathbb{Z}_{\geq 1}$, we denote $\mathbf{D}(q, q)$ by $\mathbf{D}(q)$.

A Representation of $\mathbf{D}(q_1, q_2)$. Put $k_1 = q_1 + 1$ and $k_2 = q_2$. We use the fact that the simplicial building of $\mathbf{D}(q_1, q_2)$ is the complete bipartite graph on $k_1 + k_2$ many vertices to obtain the following representation of $\mathbf{D}(q_1, q_2)$. Let the chambers of $\mathbf{D}(q_1, q_2)$ be the set,

$$\mathbf{D}(q_1, q_2)_0 = \mathbb{Z}/k_1\mathbb{Z} \times \mathbb{Z}/k_2\mathbb{Z}.$$

Let $S = \{s, t\}$ be the set of labels of $\mathbf{D}(q_1, q_2)$. The panel groupoid of type s of $\mathbf{D}(q_1, q_2)$, which is a setoid, is (equivalent to) the equivalence relation,

$$(x, y) \sim_s (x', y') \quad \text{if} \quad x = x'.$$

The panel groupoid of type t of $\mathbf{D}(q_1, q_2)$, which is also a setoid, is the equivalence relation,

$$(x, y) \sim_t (x', y') \quad \text{if} \quad y = y'.$$

Thus, the generalized digon $\mathbf{D}(q_1, q_2)$ is a $k_1 \times k_2$ grid of chambers, with chambers in the same column being s-equivalent, and chambers in the same row being t-equivalent. See the left part of Figure 5.1, which shows $\mathbf{D}(2)$.

5.1.3 The Singer Cyclic Digons $k\backslash\mathbf{D}(q)$

A **Singer cyclic digon** is Singer cyclic 2-gon. In this section, we obtain representations of the Singer cyclic digons. If a group G acts panel-regularly on $\mathbf{D}(q_1, q_2)$, then $|G| = q_1 + 1 = q_2 + 1$. Put $q = q_1 = q_2$ and $k = q + 1$.

Proposition 5.1. Let G be the cyclic group of order k. Then G acts panel-regularly on $\mathbf{D}(q)$, and this action is unique up to equivariant automorphism.

Proof. The group G acts panel-regularly on $\mathbf{D}(q)$ as follows; pick a generator $g \in G$ and for $(x, y) \in \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$ a chamber of $\mathbf{D}(q)$, put,

$$g \cdot (x, y) = (x + 1, y + 1).$$

This is an automorphism of $\mathbf{D}(q)$ since it is a permutation of the chambers which preserves $\sim_s$ and $\sim_t$. Suppose that G acts in a second way on $\mathbf{D}(q)$, which we denote by '$\bullet$'. For $x \in \mathbb{Z}/k\mathbb{Z}$, let x_s be the s-panel which contains $g^x \bullet (0, 0)$, and for $y \in \mathbb{Z}/k\mathbb{Z}$, let y_t be the t-panel which contains $g^y \bullet (0, 0)$. Let (x_s, y_t) denote the unique chamber which is contained in both x_s and x_t. We claim that $(x, y) \mapsto (x_s, y_t)$ is a permutation of the chambers of $\mathbf{D}(q)$. To see this, suppose that,

$$(x, y) \mapsto (x_s, y_t), \qquad (x', y') \mapsto (x'_s, y'_t), \qquad \text{and} \qquad (x_s, y_t) = (x'_s, y'_t).$$

Then $x_s = x'_s$ and $y_t = y'_t$, and so $x = x'$ and $y = y'$ since the action '$\bullet$' is free on panels. This permutation is an automorphism since if $(x, y) \sim_s (x', y')$, then $x = x'$, and so $x_s = x'_s$ (likewise for t). To see that this automorphism is equivariant, we need to show that $h \cdot (x, y) = h \bullet (x_s, y_t)$, for each $h \in G$ and each chamber (x, y) of $\mathbf{D}(q)$. Let $h \cdot x$ and $h \cdot y$ be the integers such that $(h \cdot x, h \cdot y) = h \cdot (x, y)$, and let $h \bullet x_s$ and $h \bullet y_t$ be the integers such that $(h \bullet x_s, h \bullet y_t) = h \bullet (x_s, y_t)$. For $h = g^n$, we have,

$$(h \cdot x)_s = (x+n)_s = g^{x+n} \bullet (0,0) = g^n \bullet g^x \bullet (0,0) = g^n \bullet x_s = h \bullet x_s$$

$$(h \cdot y)_t = (y+n)_s = g^{y+n} \bullet (0,0) = g^n \bullet g^y \bullet (0,0) = g^n \bullet y_t = h \bullet y_s. \qquad \square$$

Corollary 5.1.1. For each $q \in \mathbb{Z}_{\geq 1}$, there is a unique Singer cyclic digon of order q (up to isomorphism).

Proof. Equivariant actions will produce isomorphic quotients. Therefore the result follows from Proposition 5.1. $\square$

We denote by $k \backslash \mathbf{D}(q)$ the unique Singer cyclic digon of order q. Notice that for $q = 1$, we have $2 \backslash \mathbf{D}(1) \cong 2 \backslash \mathcal{C}_2$ (see Example 5.1).

A Representation of $k \backslash \mathbf{D}(q)$. Whilst $k \backslash \mathbf{D}(q)$ is well defined up to isomorphism, it will be useful to have a canonical representation of $k \backslash \mathbf{D}(q)$. Let G be the cyclic group of order k, and let $g \in G$ be a generator. We represent $k \backslash \mathbf{D}(q)$ as the quotient of $\mathbf{D}(q)$ by the action of G given by $g \cdot (x, y) = (x+1, y+1)$, for (x, y) a chamber of $\mathbf{D}(q)$.

Recall that we have an associated covering $\pi : \mathbf{D}(q) \to G \backslash \mathbf{D}(q)$. Let the set of chambers of $k \backslash \mathbf{D}(q)$ be $\mathcal{C} = \mathbb{Z}/k\mathbb{Z}$, and identify $k \backslash \mathbf{D}(q)$ with $G \backslash \mathbf{D}(q)$ by letting a chamber $x \in \mathcal{C}$ be the π-image of $(0, x) \in \mathbf{D}(q)$. To complete our representation of $k \backslash \mathbf{D}(q)$, we need to specify a set of defining suites. We first determine what the flowers of $k \backslash \mathbf{D}(q)$ are by inspecting the covering $\pi : \mathbf{D}(q) \to k \backslash \mathbf{D}(q)$. We can do this because the suites of $k \backslash \mathbf{D}(q)$ are exactly the images of the suites of $\mathbf{D}(q)$.

Let us introduce some notation for galleries. Let x, y and z be chambers. Then we denote by,

$$[x \xrightarrow{s} y \xrightarrow{t} z]$$

the gallery whose sequence of edges is i, j, where $\upsilon(i) = s$, $\upsilon(j) = t$, i goes from x to y, and j goes from y to z.

Proposition 5.2. For each chamber $x \in \mathcal{C}$, the petals of the flower of $k \backslash \mathbf{D}(q)$ based at x are,

$$[x \xrightarrow{s} y \xrightarrow{t} z] \sim [x \xrightarrow{t} y' \xrightarrow{s} z]$$

where $y, z, y' \in \mathcal{C}$, $x \neq y \neq z$, and $y' = x - y + z$.

Proof. Let,

$$\rho(s,t) = [x \xrightarrow{s} y \xrightarrow{t} z]$$

be a maximal (s,t)-geodesic of $k\backslash\mathbf{D}(q)$ which issues from x. Lift $\rho(s,t)$ with respect to π to a gallery $\tilde{\rho}(s,t)$ of $\mathbf{D}(q)$. Let $(c,d) \in \mathbf{D}(q)$ be the initial chamber of $\tilde{\rho}(s,t)$. It follows from the construction of $\mathbf{D}(q)$ that,

$$\tilde{\rho}(s,t) = \big[(c,d) \xrightarrow{s} (c, d+y-x) \xrightarrow{t} (c+y-z, d+y-x)\big].$$

Let,

$$\rho(t,s) = [x \xrightarrow{t} y' \xrightarrow{s} z]$$

be the unique maximal (t,s)-geodesic of $k\backslash\mathbf{D}(q)$ with $\rho(s,t) \sim \rho(t,s)$. Let $\tilde{\rho}(t,s)$ be the unique lifting of $\rho(t,s)$ to a gallery which issues from (c,d). Then,

$$\tilde{\rho}(t,s) = \big[(c,d) \xrightarrow{t} (c+x-y', d) \xrightarrow{s} (c+x-y', d+z-y')\big].$$

We have $\tilde{\rho}(s,t) \sim \tilde{\rho}(t,s)$ since π is a covering, and so,

$$(c+y-z, d+y-x) = (c+x-y', d+z-y')$$

which occurs if and only if,

$$y' = x - y + z. \qquad \square$$

The Defining Suites of $k\backslash\mathbf{D}(q)$. We let the defining suites of $k\backslash\mathbf{D}(q)$ be those which are induced by the flower based at 0. Thus, by Proposition 5.2, the defining suites of $k\backslash\mathbf{D}(q)$ are the cycles,

$$[0 \xrightarrow{s} y \xrightarrow{t} z \xrightarrow{s} (z-y) \xrightarrow{t} 0] \qquad (\blacklozenge)$$

where $y, z \in \mathcal{C}$ and $0 \neq y \neq z$. Notice that there are q^2 many defining suites. These defining suites are sufficient by Theorem 4.18.

Example 5.2. Let $q = 2$. Let us calculate the $2^2 = 4$ defining suites of $3\backslash\mathbf{D}(2)$:

$$[0 \xrightarrow{s} 1 \xrightarrow{t} 0 \xrightarrow{s} 2 \xrightarrow{t} 0]$$

$$[0 \xrightarrow{s} 1 \xrightarrow{t} 2 \xrightarrow{s} 1 \xrightarrow{t} 0]$$

$$[0 \xrightarrow{s} 2 \xrightarrow{t} 0 \xrightarrow{s} 1 \xrightarrow{t} 0]$$

$$[0 \xrightarrow{s} 2 \xrightarrow{t} 1 \xrightarrow{s} 2 \xrightarrow{t} 0].$$

One can check that the lifting of these galleries in Figure 5.1 to galleries which issue from either $(0,0)$, $(1,1)$, or $(2,2)$ are cycles in $\mathbf{D}(2)$.

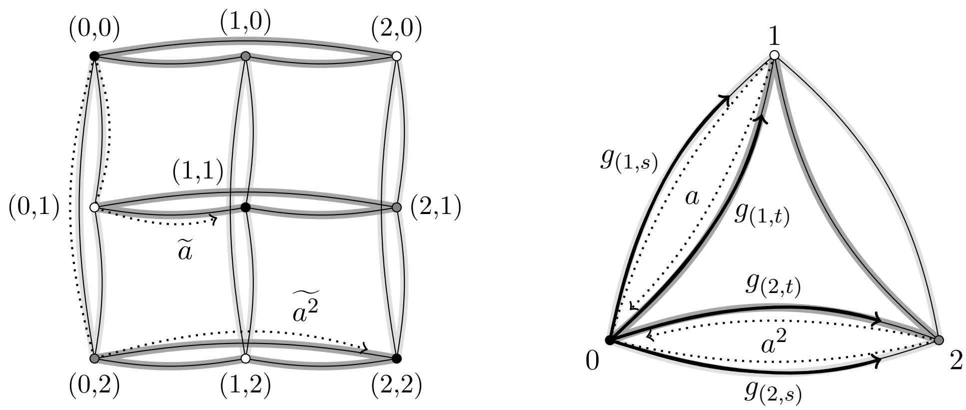

Figure 5.1: The generalized digon $\mathbf{D}(2)$ and the Singer cyclic digon $3\backslash\mathbf{D}(2)$

Lemma 5.3. Let $q \in \mathbb{Z}_{\geq 1}$, $k = q + 1$, and let $\mathcal{C} = \mathbb{Z}/k\mathbb{Z}$. Let $r \in \mathbb{Z}/k\mathbb{Z}$. Then the map,

$$\mathcal{C} \to \mathcal{C}, \quad x \mapsto x + r$$

is an automorphism of $k\backslash\mathbf{D}(q)$.

Proof. This map is clearly a chamber system automorphism. To check that it is also an automorphism of Weyl data, by the characterization of isomorphisms Proposition 3.4 and the properties of flowers Theorem 4.18, we just have to check the preservation of the flowers of $k\backslash\mathbf{D}(q)$. This follows from Proposition 5.2, and the fact that,

$$y' + r = (x + r) - (y + r) + (z + r). \qquad \square$$

Corollary 5.3.1. Let $\mathcal{P}$ be a Singer cyclic digon of order q, and let $C \in \mathcal{P}$ be any chamber. Then there exists an isomorphism $\omega : \mathcal{P} \to k\backslash\mathbf{D}(q)$ such that $\omega(C) = 0$.

Proof. By Corollary 5.1.1 there exists an isomorphism $\omega' : \mathcal{P} \to k\backslash\mathbf{D}(q)$. Let ω'' be the isomorphism,

$$\omega'' : k\backslash\mathbf{D}(q) \to k\backslash\mathbf{D}(q), \quad x \mapsto x - \omega'(C).$$

Then we can take $\omega = \omega'' \circ \omega'$. $\square$

5.1.4 The Universal Group of $k\backslash\mathbf{D}(q)$

To calculate the universal group of $k\backslash\mathbf{D}(q)$, we need to equip $k\backslash\mathbf{D}(q)$ with a generating set $\mathcal{S} = \mathcal{B} \sqcup \mathcal{I}$. Since each panel groupoid of $k\backslash\mathbf{D}(q)$ is a setoid, we have $\mathcal{B} = \emptyset$. Let $\mathcal{C}^* = \{1, \ldots, q\} \subset \mathcal{C}$. For $n \in \mathcal{C}^*$ and $\sigma \in \{s, t\}$, let $g_{(n,\sigma)}$ be the edge $0 \xrightarrow{\sigma} n$ of $k\backslash\mathbf{D}(q)$. Then put,

$$\mathcal{S} = \mathcal{I} = \left\{g_{(n,s)}, g_{(n,t)}, g^{-1}_{(n,s)}, g^{-1}_{(n,t)} : n \in \mathcal{C}^*\right\}.$$

For notational convenience, let $g_{(0,s)}$, $g_{(0,t)}$, $g_{(0,s)}^{-1}$, and $g_{(0,t)}^{-1}$ denote the empty word. Then it follows from ($\blacklozenge$) that the set of $\mathcal{S}$-suites of $k\backslash\mathbf{D}(q)$ is,

$$\mathcal{R} = \left\{ g_{(y,s)}g_{(y,t)}^{-1}g_{(z,t)}g_{(z,s)}^{-1}g_{(z-y,s)}g_{(z-y,t)}^{-1} :\ y, z \in \mathcal{C};\ 0 \neq y \neq z \right\}.$$

We now substitute $a_{(n)} = g_{(n,s)}g_{(n,t)}^{-1}$, for $n \in \mathcal{C}^*$. Thus, the new generating set is,

$$\mathcal{S}' = \left\{ g_{(n,s)}, g_{(n,t)}, g_{(n,s)}^{-1}, g_{(n,t)}^{-1}, a_{(n)} : n \in \mathcal{C}^* \right\}$$

and, letting $a_{(0)}$ and $a_{(0)}^{-1}$ denote the empty word, a new set of equivalent relations is,

$$\begin{aligned} \mathcal{R}' = \{\ & a_{(y)}a_{(z)}^{-1}a_{(z-y)} = 1 :\ y, z \in \mathcal{C};\ 0 \neq y \neq z \\ & a_{(n)} = g_{(n,s)}g_{(n,t)}^{-1} :\ n \in \mathcal{C}^*\ \}. \end{aligned}$$

By putting $z = 0$, we see that $a_{(y)}a_{(-y)} = 1$ for $y \in \mathcal{C}^*$. By putting $z = 1$, we see that $a_{(y)}a_{(1)}^{-1}a_{(1-y)} = 1$, or equivalently $a_{(y)} = a_{(y-1)}a_{(1)}$, for $y \in \mathcal{C}^*$, $y \neq 1$. Thus, by induction, we have $a_{(y)} = a_{(1)}^{y}$ for $y \in \mathcal{C}^*$. In particular, we have $a_{(1)}^{k} = 1$. If we put $a = a_{(1)}$ and include the consequence $a^k = 1$, then the relations $a_{(y)}a_{(z)}^{-1}a_{(z-y)} = 1$ are redundant. Therefore a new generating set is,

$$\mathcal{S}'' = \left\{ g_{(n,s)}, g_{(n,t)}, g_{(n,s)}^{-1}, g_{(n,t)}^{-1}, a : n \in \mathcal{C}^* \right\}$$

and a new set of equivalent relations is,

$$\mathcal{R}'' = \left\{ a^k = 1,\ a^n = g_{(n,s)}g_{(n,t)}^{-1} : n \in \mathcal{C}^* \right\}.$$

Thus, we obtain the following:

Lemma 5.4. Let $k\backslash\mathbf{D}(q)$ be the unique Singer cyclic digon of order q. Then the universal group of $k\backslash\mathbf{D}(q)$ is,

$$\mathrm{FG}(k\backslash\mathbf{D}(q)) = \left\langle g_{(n,\sigma)}, g_{(n,\sigma)}^{-1}, a \mid a^k = 1,\ a^n = g_{(n,s)}g_{(n,t)}^{-1},\ g_{(n,\sigma)}g_{(n,\sigma)}^{-1} = 1 \right\rangle$$

for $\sigma \in \{s, t\}$ and $n \in \mathcal{C}^*$.

The Fundamental Group of $k\backslash\mathbf{D}(q)$. Recall that to calculate the fundamental group of $k\backslash\mathbf{D}(q)$ we need to quotient out a spanning tree. Of course, we know the fundamental group should be cyclic of order k. Let T be the spanning tree,

$$T = \left\{ g_{(n,t)}, g_{(n,t)}^{-1} : n \in \mathcal{C}^* \right\}.$$

Then we recover the fundamental group of $k\backslash\mathbf{D}(q)$,

$$\pi_1(k\backslash\mathbf{D}(q), T) = \left\langle g_{(n,s)}, g_{(n,s)}^{-1}, a \mid a^k = 1,\ a^n = g_{(n,s)},\ g_{(n,s)}g_{(n,s)}^{-1} = 1 \right\rangle \sim \left\langle a | a^k = 1 \right\rangle$$

for $n \in \mathcal{C}^*$. Notice that the image of the gallery $[0 \xrightarrow{s} n \xrightarrow{t} 0]$ in the fundamental group at T is a^n (see Figure 5.1). Recall that one obtains an action of $\pi_1(k\backslash\mathbf{D}(q), T)$ on $\mathbf{D}(q)$ by picking a chamber in $\mathbf{D}(q)$. In fact, different choices of chamber will result in the same action since $\pi_1(k\backslash\mathbf{D}(q))$ is abelian.

5.1.5 Weyl Triangles

A **Weyl triangle** is a Weyl 3-gon, or equivalently, it is a connected Weyl graph of type A_2. A simply connected Weyl triangle is called a **generalized triangle**.

Thick Generalized Triangles vs Projective Planes. The simplicial building of a thick generalized triangle is (isomorphic to) the incidence graph of a projective plane; conversely, the incidence graph of a projective plane is (isomorphic to) a simplicial building (see [AB08, Section 4.2]). In this way, thick generalized triangles are equivalent to projective planes.

Finite Moufang Triangles. See [VM12, Section 2.2] and [TW02] for more details on the following. Let $q \geq 2$ be a prime power, and let $\mathbb{F}_q$ be the Galois field of order q. Associated to each $\mathbb{F}_q$ is the finite Desarguesian projective plane $\mathrm{PG}(2, q)$. We denote by $\mathcal{T}(q)$ the building whose associated simplicial building is the incidence graph of $\mathrm{PG}(2, q)$. Thus, the $\mathcal{T}(q)$ are exactly the thick generalized triangles which correspond to finite Desarguesian projective planes. The generalized triangles $\mathcal{T}(q)$ are exactly the so-called finite Moufang triangles. Let $k = q + 1$ and $\delta = q^2 + q + 1$. We have the following; $\mathcal{T}(q)$ has δ many panels of each type, $k\delta$ many chambers, and the panel groupoids are all $1 \times k$.

Finite Non-Moufang Triangles. As well as the $\mathcal{T}(q)$ are the weak but not thick finite generalized triangles, and the thick finite triangles which correspond to non-Desarguesian projective planes.

5.1.6 The Singer Cyclic Triangles $\delta\backslash\mathcal{T}(\mathcal{D})$

A **Singer cyclic triangle** is Singer cyclic 3-gon. In this section, we obtain representations of the Singer cyclic triangles using the method of difference sets. A good reference for the material of this section is [Dem68].

Difference Sets. In the language of Weyl graphs, the method of difference sets $\mathcal{D}$ in a group G provides a way of constructing a generalized triangle $\mathcal{T}(\mathcal{D})$ and a universal cover,

$$\pi : \mathcal{T}(\mathcal{D}) \to G\backslash\mathcal{T}(\mathcal{D})$$

where π is the quotient map associated to a panel-regular action of G on $\mathcal{T}(\mathcal{D})$. A good reference for general difference sets is [Dem68]. We focus on the case where G is cyclic, or equivalently, where $G\backslash\mathcal{T}(\mathcal{D})$ is a Singer cyclic triangle. Such difference sets are usually called cyclic difference sets, and are studied in [Ber53]. We define a **difference set** $\mathcal{D}$ of order q to be a subset $\mathcal{D} \subset \mathbb{Z}/\delta\mathbb{Z}$ such that the map,

$$\mathcal{D} \times \mathcal{D} \to \mathbb{Z}/\delta\mathbb{Z}, \quad (x, y) \mapsto x - y$$

when restricted to the off-diagonal elements is a bijection into $\{1, \ldots, \delta - 1\}$. Thus, for all non-zero $n \in \mathbb{Z}/\delta\mathbb{Z}$, there exists a unique pair $d, d' \in \mathcal{D}$ such that $d - d' = n$. Notice that $|\mathcal{D}| = q + 1 = k$. A difference set $\mathcal{D}$ is called **based** if $0 \in \mathcal{D}$, in which case we let $\mathcal{D}^* = \mathcal{D} \setminus \{0\}$.

Operations on Difference Sets. Let $\mathcal{D}$ be a difference set of order q, and let $x \in \mathbb{Z}/\delta\mathbb{Z}$ and $r \in \mathrm{Aut}(\mathbb{Z}/\delta\mathbb{Z}) = \mathbb{Z}/\delta\mathbb{Z}^*$. Then $r\mathcal{D} + x = \{rd + x : d \in \mathcal{D}\}$ is also a difference sets of order q. Two difference sets $\mathcal{D}$ and $\mathcal{D}'$ are called **equivalent** if there exists $x \in \mathbb{Z}/\delta\mathbb{Z}$ and $r \in \mathbb{Z}/\delta\mathbb{Z}^*$ such that $\mathcal{D}' = r\mathcal{D} + x$. Our notion is equivalence is different to that of [Ber53].

The Generalized Triangle $\mathcal{T}(\mathcal{D})$. Let $\mathcal{D}$ be a difference set of order q. By results of Singer [Sin38], we obtain a generalized triangle $\mathcal{T}(\mathcal{D})$ of order q from $\mathcal{D}$ as follows. Let the chambers of $\mathcal{T}(\mathcal{D})$ be the set,

$$\mathcal{T}(\mathcal{D})_0 = \{(x, x + d) : x \in \mathbb{Z}/\delta\mathbb{Z},\ d \in \mathcal{D}\}.$$

Let $S = \{s, t\}$ be the set of labels of $\mathcal{T}(\mathcal{D})$. The panel groupoid of type s of $\mathcal{T}(\mathcal{D})$ is (equivalent to) the equivalence relation,

$$(x, y) \sim_s (x', y') \quad \text{if} \quad x = x'.$$

The panel groupoid of type t of $\mathcal{T}(\mathcal{D})$ is the equivalence relation,

$$(x, y) \sim_t (x', y') \quad \text{if} \quad y = y'.$$

Figure 5.2 shows $\mathcal{T}(\mathcal{D})$ for $\mathcal{D} = \{0, 1, 3\}$, in which case $\mathcal{T}(\mathcal{D}) \cong \mathcal{T}(2) \sim \mathrm{PG}(2, 2)$. Let G be the cyclic group of order δ, and let $g \in G$ be a generator. Then G acts panel-regularly on $\mathcal{T}(\mathcal{D})$ via $g \cdot (x, y) = (x + 1, y + 1)$. Hence, we obtain a universal cover,

$$\pi : \mathcal{T}(\mathcal{D}) \to G\backslash\mathcal{T}(\mathcal{D})$$

where $G\backslash\mathcal{T}(\mathcal{D})$ is a Singer cyclic triangle of order q. Let us denote by $\delta\backslash\mathcal{T}(\mathcal{D})$ a Weyl graph which is isomorphic to $G\backslash\mathcal{T}(\mathcal{D})$.

A difference set $\mathcal{D}$ is called **Desarguesian** if $\mathcal{T}(\mathcal{D}) \cong \mathcal{T}(q)$. We have the following well known open question:

Conjecture 5.5. All difference sets are Desarguesian.

Equivalent Difference Sets. Let $\mathcal{D}$ and $\mathcal{D}'$ be equivalent difference sets of order q. Let $z \in \mathbb{Z}/\delta\mathbb{Z}$ and $r \in \mathbb{Z}/\delta\mathbb{Z}^*$ such that $\mathcal{D}' = r\mathcal{D} + z$. Then the bijection of chambers,

$$\omega_0 : \mathcal{T}(\mathcal{D})_0 \to \mathcal{T}(\mathcal{D}')_0, \quad (x, y) \mapsto (rx + z, ry + z)$$

clearly preserves $\sim_s$ and $\sim_t$. Thus, ω_0 determines an isomorphism $\omega : \mathcal{T}(\mathcal{D}) \to \mathcal{T}(\mathcal{D}')$. Let G and G' be the cyclic groups of order δ which act on $\mathcal{T}(\mathcal{D})$ and $\mathcal{T}(\mathcal{D}')$ respectively, with $g \in G$ and $g' \in G'$ being the chosen generators. Let $\psi : G \to G'$ be the isomorphism such that $g \mapsto (g')^r$. Then ω is ψ-equivariant. Thus, equivalent difference sets essentially construct the same universal cover.

Difference Sets Obtained from Actions. Let G be a cyclic group of order δ which acts panel-regularly on a generalized triangle $\mathcal{T}$ of order q. Pick a generator $g \in G$, an s-panel P of $\mathcal{T}$, and a t-panel L of $\mathcal{T}$. Then,

$$\mathcal{D} = \{d \in \mathbb{Z}/\delta\mathbb{Z} : P \cap g^d \cdot L \neq \emptyset\}$$

is a difference set of order q, called a difference **obtained** from the action of G. Different choices of g, P and L produce equivalent difference sets.

Let G also act on $\mathcal{T}(\mathcal{D})$ by $g \cdot (x, y) = (x+1, y+1)$. For $x \in \mathbb{Z}/\delta\mathbb{Z}$ and $d \in \mathcal{D}$, let (P_x, L_{x+d}) denote the unique chamber of $\mathcal{T}$ which is contained in $g^x \cdot P$ and $g^{x+d} \cdot L$. Then,

$$\mathcal{T}(\mathcal{D}) \to \mathcal{T}, \quad (x, x+d) \mapsto (P_x, L_{x+d})$$

is an equivariant isomorphism. Conversely, the difference sets obtained from the action of G on $\mathcal{T}(\mathcal{D})$ will be equivalent to $\mathcal{D}$. Thus, there is essentially a 1-1 correspondence between difference sets up to equivalence and panel-regular actions of cyclic groups on generalized triangles.

The following is a classical result of Singer, combined with the uniqueness result of Berman:

Theorem 5.6 (Singer-Berman)**.** For all q a prime power, there exists a Desarguesian difference set of order q, and this difference set is unique up to equivalence.

Proof. For existence see [Sin38], for uniqueness see [Ber53]. □

A Representation of $\delta\backslash\mathcal{T}(\mathcal{D})$. As with the Singer cyclic digons, it will be useful to have a canonical representation of $\delta\backslash\mathcal{T}(\mathcal{D})$. Let the set of chambers of $\delta\backslash\mathcal{T}(\mathcal{D})$ be $\mathcal{D}$, where $d \in \mathcal{D}$ is the π-image of $(x, x+d) \in \mathcal{T}(\mathcal{D})$. To complete our representation of $\delta\backslash\mathcal{T}(\mathcal{D})$, we need to specify a set of defining suites. Let us first determine what the flowers of $\delta\backslash\mathcal{T}(\mathcal{D})$ are by inspecting the covering $\pi : \mathcal{T}(\mathcal{D}) \to \delta\backslash\mathcal{T}(\mathcal{D})$:

Proposition 5.7. Let $\mathcal{D}$ be a difference set of order q. For each chamber $C \in \mathcal{D}$, the petals of the flower of $\delta\backslash\mathcal{T}(\mathcal{D})$ based at C are,

$$[C \xrightarrow{s} x \xrightarrow{t} y \xrightarrow{s} z] \sim [C \xrightarrow{t} x' \xrightarrow{s} y' \xrightarrow{t} z]$$

where $C, x, y, z, x', y' \in \mathcal{D}$, $C \neq x \neq y \neq z$, and $x' - y' = C - x + y - z$.

Proof. Let

$$\rho(s,t) = [C \xrightarrow{s} x \xrightarrow{t} y \xrightarrow{s} z]$$

be a maximal (s,t)-geodesic of $\delta\backslash\mathcal{T}(\mathcal{D})$ which issues from C. Lift $\rho(s,t)$ with respect to π to a gallery $\tilde{\rho}(s,t)$ of $\mathcal{T}(\mathcal{D})$. Let $(c, d) \in \mathbb{Z}/\delta\mathbb{Z} \times \mathbb{Z}/\delta\mathbb{Z}$ be the initial chamber of $\tilde{\rho}(s,t)$. It follows from the construction of $\mathcal{T}(\mathcal{D})$ that,

$$\tilde{\rho}(s,t) = \big[(c,d) \xrightarrow{s} (c, d+x-C) \xrightarrow{t} (c+x-y, d+x-C) \xrightarrow{s} (c+x-y, d+x-C+z-y)\big].$$

Let,

$$\rho(t,s) = [C \xrightarrow{t} x' \xrightarrow{s} y' \xrightarrow{t} z]$$

be the unique maximal (t,s)-geodesic of $\delta\backslash\mathcal{T}(\mathcal{D})$ with $\rho(s,t) \sim \rho(t,s)$. Let $\tilde{\rho}(t,s)$ be the lifting of $\rho(t,s)$ to a gallery which issues from (c,d). Then,

$$\tilde{\rho}(t,s) = \big[(c,d) \xrightarrow{t} (c{+}C{-}x', d) \xrightarrow{s} (c{+}C{-}x', d{+}y'{-}x') \xrightarrow{t} (c{+}C{-}x'{+}y'{-}z, d{+}y'{-}x')\big].$$

We have $\tilde{\rho}(s,t) \sim \tilde{\rho}(t,s)$, and so,

$$(c + x - y, d + x - C + z - y) = (c + C - x' + y' - z, d + y' - x')$$

which occurs if and only if,

$$x' - y' = C - x + y - z. \qquad \square$$

Let us assume that $\mathcal{D}$ is based. One can easily see that every difference set is equivalent to a based difference set, so this is no real restriction. We let the defining suites of $\delta\backslash\mathcal{T}(\mathcal{D})$ be those which are induced by the flower based at $0 \in \mathcal{D}$. Thus, by Proposition 5.7, the defining suites of $\delta\backslash\mathcal{T}(\mathcal{D})$ are the cycles,

$$[0 \xrightarrow{s} x \xrightarrow{t} y \xrightarrow{s} z \xrightarrow{t} y' \xrightarrow{s} x' \xrightarrow{t} 0]$$

where $x, y, z, x', y' \in \mathcal{D}$, $0 \neq x \neq y \neq z$, and,

$$y' - x' = x - y + z. \qquad (\clubsuit)$$

Notice that there will be q^3 many defining suites.

Example 5.3. Let $\mathcal{D} = \{0, 1, 3\}$. Then $\mathcal{D}$ is a difference set of order 2. Let us calculate the $2^3 = 8$ defining suites of $7\backslash\mathcal{T}(\mathcal{D})$:

$$[0 \xrightarrow{s} 1 \xrightarrow{t} 0 \xrightarrow{s} 1 \xrightarrow{t} 3 \xrightarrow{t} 1 \xrightarrow{s} 0]$$
$$[0 \xrightarrow{s} 1 \xrightarrow{t} 0 \xrightarrow{s} 3 \xrightarrow{s} 0 \xrightarrow{t} 3 \xrightarrow{t} 0]$$
$$[0 \xrightarrow{s} 1 \xrightarrow{t} 3 \xrightarrow{s} 0 \xrightarrow{t} 1 \xrightarrow{s} 3 \xrightarrow{t} 0]$$
$$[0 \xrightarrow{s} 1 \xrightarrow{t} 3 \xrightarrow{s} 1 \xrightarrow{s} 0 \xrightarrow{t} 1 \xrightarrow{t} 0]$$
$$[0 \xrightarrow{s} 3 \xrightarrow{t} 0 \xrightarrow{s} 1 \xrightarrow{s} 0 \xrightarrow{t} 3 \xrightarrow{t} 0]$$
$$[0 \xrightarrow{s} 3 \xrightarrow{t} 0 \xrightarrow{s} 3 \xrightarrow{s} 0 \xrightarrow{t} 1 \xrightarrow{t} 0]$$
$$[0 \xrightarrow{s} 3 \xrightarrow{t} 1 \xrightarrow{s} 0 \xrightarrow{s} 3 \xrightarrow{t} 1 \xrightarrow{t} 0]$$
$$[0 \xrightarrow{s} 3 \xrightarrow{t} 1 \xrightarrow{s} 3 \xrightarrow{s} 1 \xrightarrow{t} 3 \xrightarrow{t} 0].$$

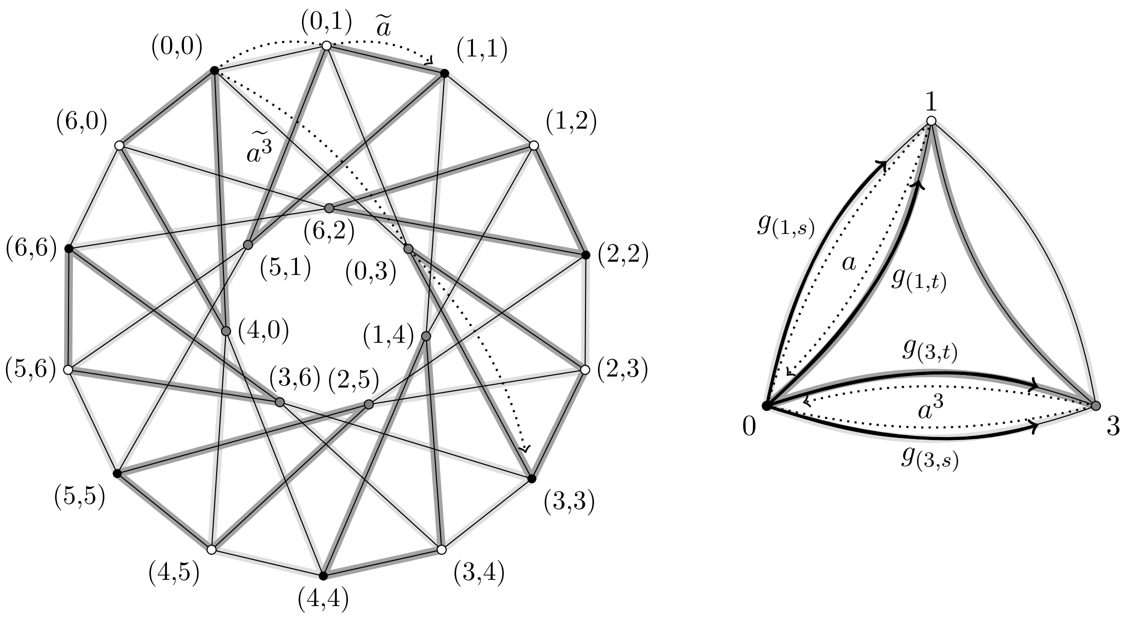

Figure 5.2: The generalized triangle $\mathcal{T}(\mathcal{D})$ and the Singer cyclic triangle $7\backslash\mathcal{T}(\mathcal{D})$

The Singer Cyclic Triangle $\delta\backslash\mathcal{T}(q)$. Let $\mathcal{D}$ and $\mathcal{D}'$ be equivalent difference sets of order q with $\mathcal{D}' = r\mathcal{D} + x$. Then it follows from Proposition 5.7 that,

$$\omega : \delta\backslash\mathcal{T}(\mathcal{D}) \to \delta\backslash\mathcal{T}(\mathcal{D}'), \quad d \mapsto rd + x$$

preserves flowers, and so is an isomorphism. By Theorem 5.6, this shows that if Conjecture 5.5 holds, then for each q a prime power, there is a unique Singer cyclic triangle of order q. We let $\delta\backslash\mathcal{T}(q)$ denote the unique (up to isomorphism) Singer cyclic triangle of order q whose universal cover is $\mathcal{T}(q)$. Thus, if $\mathcal{D}$ is Desarguesian, we have $\delta\backslash\mathcal{T}(\mathcal{D}) \cong \delta\backslash\mathcal{T}(q)$.

The following result shows that given any chamber C in a Singer cyclic triangle $\mathcal{P}$, we can represent $\mathcal{P}$ as $\delta\backslash\mathcal{T}(\mathcal{D})$, for some based difference set $\mathcal{D}$, such that C is identified with $0 \in \mathcal{D}$.

Lemma 5.8. Let $\mathcal{P}$ be a Singer cyclic triangle of order q, and let $C \in \mathcal{P}$ be any chamber. Then there exists a based difference set $\mathcal{D}$ of order q and an isomorphism $\omega : \mathcal{P} \to \delta\backslash\mathcal{T}(\mathcal{D})$ such that $\omega(C) = 0$.

Proof. Let g be a generator of the deck transformation group of the universal cover $\Delta \to \mathcal{P}$. Let $\mathcal{D}'$ be a difference set obtained from this action. Then there exists an isomorphism $\omega' : \mathcal{P} \to \delta\backslash\mathcal{T}(\mathcal{D}')$. Let $\mathcal{D} = \mathcal{D}' - \omega'(C)$, and let ω' be the isomorphism,

$$\omega' : \delta\backslash\mathcal{T}(\mathcal{D}') \to \delta\backslash\mathcal{T}(\mathcal{D}), \quad x \mapsto x - \omega'(C).$$

Then we can take $\omega = \omega'' \circ \omega'$. $\square$

5.1.7 The Universal Group of $\delta\backslash\mathcal{T}(\mathcal{D})$

In this section, we calculate the universal group of $\delta\backslash\mathcal{T}(\mathcal{D})$. We assume that $\mathcal{D}$ is a based difference set. Recall that $\mathcal{D}^* = \mathcal{D} \setminus \{0\}$.

First, we need to equip $\delta\backslash\mathcal{T}(\mathcal{D})$ with a generating set $\mathcal{S} = \mathcal{B} \cup \mathcal{I}$. Since each panel groupoid of $\delta\backslash\mathcal{T}(\mathcal{D})$ is a setoid, we have $\mathcal{B} = \emptyset$. For $n \in \mathcal{D}^*$ and $\sigma \in \{s, t\}$, let $g_{(n,\sigma)}$ be the edge $0 \xrightarrow{\sigma} n$. Then put,

$$\mathcal{S} = \mathcal{I} = \left\{ g_{(n,s)}, g_{(n,t)}, g^{-1}_{(n,s)}, g^{-1}_{(n,t)} : n \in \mathcal{D}^* \right\}.$$

For notational convenience, let $g_{(0,s)}$, $g_{(0,t)}$, $g^{-1}_{(0,s)}$, and $g^{-1}_{(0,t)}$ denote the empty word. It follows from (♣) that the set of $\mathcal{S}$-suites of $\delta\backslash\mathcal{T}(\mathcal{D})$ is,

$$\begin{aligned}\mathcal{R} = \big\{ & g_{(x,s)}g^{-1}_{(x,t)}g_{(y,t)}g^{-1}_{(y,s)}g_{(z,s)}g^{-1}_{(z,t)}g_{(y',t)}g^{-1}_{(y',s)}g_{(x',s)}g^{-1}_{(x',t)} : \\ & \qquad x, y, z, y', x' \in \mathcal{D};\ 0 \neq x \neq y \neq z;\ y' - x' = x - y + z \big\}.\end{aligned}$$

We now substitute $a_{(n)} = g_{(n,s)}g^{-1}_{(n,t)}$, for $n \in \mathcal{D}^*$. Thus, the new generating set is,

$$\mathcal{S}' = \left\{ g_{(n,s)}, g_{(n,t)}, g^{-1}_{(n,s)}, g^{-1}_{(n,t)}, a_{(n)} : n \in \mathcal{D}^* \right\}$$

and, letting $a_{(0)}$ and $a^{-1}_{(0)}$ denote the empty word, a new set of equivalent relations is,

$$\begin{aligned}\mathcal{R}' = \big\{ & a_{(x)}a^{-1}_{(y)}a_{(z)}a^{-1}_{(y')}a_{(x')} = 1 : x, y, z, y', x' \in \mathcal{D};\ 0 \neq x \neq y \neq z;\ y' - x' = x - y + z \\ & a_{(n)} = g_{(n,s)}g^{-1}_{(n,t)} : n \in \mathcal{D}^* \big\}.\end{aligned}$$

Let us rearrange $a_{(x)}a^{-1}_{(y)}a_{(z)}a^{-1}_{(y')}a_{(x')} = 1$ to give,

$$a_{(x)}a^{-1}_{(y)}a_{(z)} = a^{-1}_{(x')}a_{(y')} \qquad (♠)$$

for $x, y, z, y', x' \in \mathcal{D}$, $0 \neq x \neq y \neq z$, and $y' - x' = x - y + z$. Let $e, e' \in \mathcal{D}$ be the unique integers such that $e - e' = 1$, and put $a = a_{(e)}a^{-1}_{(e')}$. In particular, if $1 \in \mathcal{D}$, we just have $a = a_{(1)} = g_{(1,s)}g^{-1}_{(1,t)}$. Fix $n \in \{1, \ldots, \delta - 1\}$, and let $c, c' \in \mathcal{D}$ be the unique integers such that,

$$c - c' = n.$$

If $n \neq \delta - 1$, let $d, d' \in \mathcal{D}$ be the unique integers such that,

$$d - d' = n + 1.$$

If $n \neq -e$, let $f, f' \in \mathcal{D}$ be the unique integers such that,

$$f' - f = n + e.$$

Now, if $n = \delta - 1$, then $c' = e$ and $c = e'$, and so,

$$a_{(c)}a^{-1}_{(c')}a = a_{(c)}a^{-1}_{(c')}a_{(e)}a^{-1}_{(e')} = 1. \qquad (♥)$$

We now claim that if $n \neq \delta - 1$, then,

$$a_{(c)}a^{-1}_{(c')}a = a_{(d)}a^{-1}_{(d')}.$$

If $n \neq \delta - 1$ but $n = -e$, then $c = 0$, $c' = e$, $d = 0$, and $e' = d'$, and so,

$$a_{(c)}a^{-1}_{(c')}a = a_{(c)}a^{-1}_{(c')}a_{(e)}a^{-1}_{(e')} = a^{-1}_{(e')} = a_{(d)}a^{-1}_{(d')}.$$

If $n \neq \delta - 1$, $n \neq -e$, $c' \neq e$, and $f' \neq e'$, then by two applications of (♠), we have,

$$a_{(c)}a^{-1}_{(c')}a = a_{(c)}a^{-1}_{(c')}a_{(e)}a^{-1}_{(e')} = a^{-1}_{(f)}a_{(f')}a^{-1}_{(e')} = a_{(d)}a^{-1}_{(d')}.$$

If $n \neq \delta - 1$, $n \neq -e$, and $c' = e$, then $c = d$ and $e' - d'$, and so,

$$a_{(c)}a^{-1}_{(c')}a = a_{(c)}a^{-1}_{(c')}a_{(e)}a^{-1}_{(e')} = a_{(c)}a^{-1}_{(e')} = a_{(d)}a^{-1}_{(d')}.$$

Finally, if $n \neq \delta - 1$, $n \neq -e$, and $f' = e'$, then $d = 0$ and $f = d'$, and so,

$$a_{(c)}a^{-1}_{(c')}a = a_{(c)}a^{-1}_{(c')}a_{(e)}a^{-1}_{(e')} = a^{-1}_{(f)}a_{(f')}a^{-1}_{(e')} = a_{(d)}a^{-1}_{(d')}.$$

This proves the claim. Therefore, by induction, we have $a^n = a_{(c)}a^{-1}_{(c')}$, and in particular $a^n = a_{(n)}$ if $n \in \mathcal{D}$. The fact that $a^\delta = 1$ then follows from (♥). Notice that in the presence of $a^n = a_{(n)}$ and $a^\delta = 1$, the relations of the form $a_{(x)}a^{-1}_{(y)}a_{(z)}a^{-1}_{(y')}a_{(x')} = 1$ are redundant. Thus, similar to Section 5.1.4, a new generating set is,

$$\mathcal{S}'' = \left\{g_{(n,s)}, g_{(n,t)}, g^{-1}_{(n,s)}, g^{-1}_{(n,t)}, a : n \in \mathcal{D}^*\right\}$$

with relations,

$$\mathcal{R}'' = \left\{a^\delta = 1,\ a^n = g_{(n,s)}g^{-1}_{(n,t)} : n \in \mathcal{D}^*\right\}.$$

We obtain the following:

Lemma 5.9. Let $\mathcal{D}$ be a based difference set and let $\delta\backslash\mathcal{T}(\mathcal{D})$ be the Singer cyclic triangle constructed from $\mathcal{D}$. Then the universal group of $\delta\backslash\mathcal{T}(\mathcal{D})$ is,

$$\mathrm{FG}(\delta\backslash\mathcal{T}(\mathcal{D})) = \left\langle g_{(n,\sigma)}, g^{-1}_{(n,\sigma)}, a \mid a^\delta = 1,\ a^n = g_{(n,s)}g^{-1}_{(n,t)},\ g_{(n,\sigma)}g^{-1}_{(n,\sigma)} = 1\right\rangle$$

for $\sigma \in \{s, t\}$ and $n \in \mathcal{D}^*$.

The Fundamental Group of $\delta\backslash\mathcal{T}(\mathcal{D})$. Let T be the spanning tree,

$$T = \left\{g_{(n,t)}, g^{-1}_{(n,t)} : n \in \mathcal{D}^*\right\}.$$

Thus, we recover the fundamental group of $\delta\backslash\mathcal{T}(\mathcal{D})$,

$$\pi_1(\delta\backslash\mathcal{T}(\mathcal{D}), T) = \left\langle g_{(n,s)}, g^{-1}_{(n,s)}, a \mid a^\delta = 1,\ a^n = g_{(n,s)},\ g_{(n,s)}g^{-1}_{(n,s)} = 1\right\rangle \sim \left\langle a \mid a^\delta = 1\right\rangle.$$

Notice that the image of the gallery $[0 \xrightarrow{s} n \xrightarrow{t} 0]$ in the fundamental group at T is a^n (see Figure 5.2).

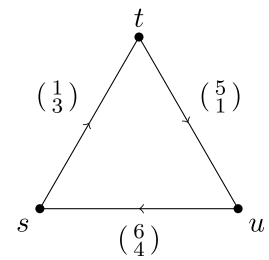

Figure 5.3: A gluing matrix of type $\widetilde{A}_2$ and order 2

5.2 Singer Cyclic Lattices of Type M

In this section, we construct the Singer cyclic lattices of type M, where $m_{st} \in \{2, 3, \infty\}$ for all $s, t \in S$, and the defining graph of M is connected. Modulo an equivalence relation on our construction, this classifies all such lattices. This equivalence is described in the $\widetilde{A}_2$ case in terms of 'based difference matrices' by Witzel [Wit16], which builds on the work of Essert [Ess13]. Based difference matrices correspond to what we will call gluing matrices. Gluing matrices are examples of gluing data, described in Section 3.1.5.

5.2.1 Gluing Matrices $\mathcal{M}$

The Graph L. Let M be a Coxeter matrix with $m_{st} \in \{2, 3, \infty\}$ for all $s, t \in S$, whose defining graph L is connected. Thus, L is a connected simplicial graph with $V(L) = S$, and whose edges are labeled over $\{2, 3\}$.

Gluing Matrices $\mathcal{M}$. Gluing matrices will play the same roll as based difference matrices in the work of Essert [Ess13]. Let $q \in \mathbb{Z}_{\geq 2}$ and $k = q + 1$. Let $\mathcal{C} = \mathbb{Z}/k\mathbb{Z}$ and $\mathcal{C}^* = \{1, \ldots, q\} \subset \mathcal{C}$. Let $\bar{E}$ be an orientation of L; formally, let $\bar{E} \subseteq S \times S$ such that for all $(s, t) \in \bar{E}$, we have $\{s, t\} \in E(L)$, and for all $\{s, t\} \in E(L)$, exactly one of $\{(s, t), (t, s)\}$ is contained in $\bar{E}$. We define a **gluing matrix** $\mathcal{M} = \mathcal{M}_{M,q}$ of type M and order q to be a matrix,

$$\mathcal{M} : \mathcal{C}^* \times \bar{E} \to \mathbb{Z}_{\geq 1}$$

such that,

(i) for $(s, t) \in \bar{E}$ with $m_{st} = 2$, each $\mathcal{C}^*$-tuple $\mathcal{M}(-, (s, t))$ is a permutation of $\mathcal{C}^*$

(ii) for $(s, t) \in \bar{E}$ with $m_{st} = 3$, each $\mathcal{C}^*$-tuple $\mathcal{M}(-, (s, t))$ is a permutation of some $\mathcal{D}^*$, where $\mathcal{D}$ is a based difference set of order q.

Let us denote $\mathcal{M}(n, (s, t))$ by $n(st)$. If $m_{st} = 2$, we think of the integer $n(st)$ as being an element of $\mathbb{Z}/k\mathbb{Z}$, and if $m_{st} = 3$, we think of the integer $n(st)$ as being an element of $\mathbb{Z}/\delta\mathbb{Z}$.

Example 5.4. Let L be the defining graph of $\widetilde{A}_2$. Figure 5.3 shows L decorated with the data of the following gluing matrix $\mathcal{M} = \mathcal{M}_{L,2}$:

	(s,t)	(t,u)	(u,s)
1	1	5	6
2	3	1	4

Notice that each column is of the form $\mathcal{D}^*$ for some based difference set $\mathcal{D}$ of order 2.

5.2.2 The Singer Graph $\mathcal{W}_{\mathcal{M}}$

The Weyl Data of $\mathcal{W}_{\mathcal{M}}$. Let L, $\bar{E}$, q, k, $\mathcal{C}$, and $\mathcal{C}^*$ be as above, and let $\mathcal{M}$ be a gluing matrix of order q. We associated a Singer graph $\mathcal{W} = \mathcal{W}_{\mathcal{M}}$ of type M to $\mathcal{M}$ as follows. The set of chambers is $\mathcal{W}_0 = \mathcal{C}$, and each panel groupoid is $\mathcal{W}_s \cong 1 \times k$, for $s \in S$. Let $(s,t) \in \bar{E}$, and let $J = \{s,t\}$. Notice that there is exactly one J-residue $\mathcal{W}_J$ of $\mathcal{W}$.

If $m_{st} = 2$, let $\Omega_{st} : \mathcal{W}_J \to k\backslash\mathbf{D}(q)$ be the chamber system morphism (over $\mathrm{id} : J \to J$) such that for $x \in \mathcal{W}_0$, we have,

$$\Omega_{st}(x) = \begin{cases} 0 & \text{if } x = 0 \\ x(st) & \text{otherwise.} \end{cases}$$

Then, let the defining suites of $\mathcal{W}_J$ be the Ω_{st}-preimages of the defining suites of $k\backslash\mathbf{D}(q)$. Thus, the defining suites are,

$$[0 \xrightarrow{s} y \xrightarrow{t} z \xrightarrow{s} y' \xrightarrow{t} 0]$$

where $y, z, y' \in \mathcal{C}$, $0 \neq y \neq z$, and $\Omega_{st}(y') = \Omega_{st}(z) - \Omega_{st}(y)$.

If $m_{st} = 3$, put $\mathcal{D} = \mathcal{M}(-,(s,t)) \cup \{0\}$, and let $\Omega_{st} : \mathcal{W}_J \to \delta\backslash\mathcal{T}(\mathcal{D})$ be the chamber system morphism (over $\mathrm{id} : J \to J$) such that for $x \in \mathcal{W}_0$, we have,

$$\Omega_{st}(x) = \begin{cases} 0 & \text{if } x = 0 \\ x(st) & \text{otherwise.} \end{cases}$$

Then, let the defining suites of $\mathcal{W}_J$ be the Ω_{st}-preimages of the defining suites of $\delta\backslash\mathcal{T}(\mathcal{D})$. Thus, the defining suites are,

$$[0 \xrightarrow{s} x \xrightarrow{t} y \xrightarrow{s} z \xrightarrow{t} y' \xrightarrow{s} x' \xrightarrow{t} 0]$$

where $y, z, y', x' \in \mathcal{C}$, $0 \neq x \neq y \neq z$, and $\Omega_{st}(y') - \Omega_{st}(x') = \Omega_{st}(x) - \Omega_{st}(y) + \Omega_{st}(z)$. This defines the Weyl data $\mathcal{W}$. Then, $\mathcal{W}$ is a Weyl graph by Theorem 4.16.

The Universal Group of $\mathcal{W}_{\mathcal{M}}$. Let $\mathcal{W} = \mathcal{W}_{\mathcal{M}}$ be as constructed above. We now calculate the universal group of $\mathcal{W}$. First, we need a generating set $\mathcal{S} = \mathcal{B} \sqcup \mathcal{I}$. Since each panel groupoid of $\mathcal{W}$ is a setoid, we have $\mathcal{B} = \emptyset$. For $n \in \mathcal{C}^*$ and $\sigma \in S$, let $g_{(n,\sigma)}$ be the edge $0 \xrightarrow{\sigma} n$. Then put,

$$\mathcal{S} = \mathcal{I} = \left\{ g_{(n,\sigma)}, g_{(n,\sigma)}^{-1} : n \in \mathcal{C}^*, \sigma \in S \right\}.$$

Then the set of $\mathcal{S}$-suites $\mathcal{R}$ of $\mathcal{W}$ is the union of the $\mathcal{S}$-suites in each of its spherical 2-residues,

$$\begin{aligned} \mathcal{R} = \{ \; & g_{(y,s)} g_{(y,t)}^{-1} g_{(z,t)} g_{(z,s)}^{-1} g_{(y',s)} g_{(z-y,t)}^{-1} \\ & g_{(x,u)} g_{(x,v)}^{-1} g_{(y,v)} g_{(y,u)}^{-1} g_{(z,u)} g_{(z,v)}^{-1} g_{(y',v)} g_{(y',u)}^{-1} g_{(x',u)} g_{(x',v)}^{-1} \; \} \end{aligned}$$

where $(s,t) \in \bar{E}$ with $m_{st} = 2$, $y, z \in \mathcal{C}$ with $0 \neq y \neq z$, and,

$$\Omega_{st}(y') = \Omega_{st}(z) - \Omega_{st}(y)$$

and $(u,v) \in \bar{E}$ with $m_{uv} = 3$, $x, y, z, y', x' \in \mathcal{C}$ with $0 \neq x \neq y \neq z$, and,

$$\Omega_{st}(y') - \Omega_{st}(x') = \Omega_{st}(x) - \Omega_{st}(y) + \Omega_{st}(z).$$

We now make the same substitutions in each spherical 2-residue of $\mathcal{W}$ that we made in order to obtain the presentations of Lemma 5.4 and Lemma 5.10. For $(s,t) \in \bar{E}$ with $m_{st} = 2$, let $n \in \mathcal{C}^*$ such that $\Omega_{st}(n) = 1$. Then put,

$$a_{st} = g_{(n,s)} g_{(n,t)}^{-1}.$$

For $(s,t) \in \bar{E}$ with $m_{st} = 3$, let $n, n' \in \mathcal{C}$ such that $\Omega_{st}(n) - \Omega_{st}(n') = 1$. For notational convenience, let $g_{(0,\sigma)}$ denote the empty word for all $\sigma \in S$, and put,

$$a_{st} = g_{(n,s)} g_{(n,t)}^{-1} \left(g_{(n',s)} g_{(n',t)}^{-1} \right)^{-1}.$$

Thus, for both $m_{st} = 2$ and $m_{st} = 3$, a_{st} is the 'a' from the calculation of the universal group of the target of Ω_{st}. Let us also stop including the $g_{(n,s)}^{-1}$ as generators for simplicity, which is obviously possible. Then the new set of generators is,

$$\mathcal{S}' = \left\{ g_{(n,\sigma)}, a_{st} : n \in \mathcal{C}^*; \; \sigma \in S; \; (s,t) \in \bar{E} \right\}$$

and it follows from Lemma 5.4 and Lemma 5.10 that a new set of equivalent relations is,

$$\mathcal{R}' = \left\{ (a_{st})^{\delta(st)} = 1, \; (a_{st})^{n(st)} = g_{(n,s)} g_{(n,t)}^{-1} : (s,t) \in \bar{E}; \; n \in \mathcal{C}^* \right\}$$

where for $(s,t) \in \bar{E}$, we have,

$$\delta(st) = \begin{cases} k = q + 1 & \text{if } m_{st} = 2 \\ \delta = q^2 + q + 1 & \text{if } m_{st} = 3. \end{cases}$$

Thus, we obtain the following:

Lemma 5.10. Let $\mathcal{M}$ be a gluing matrix and let $\mathcal{W} = \mathcal{W}_{\mathcal{M}}$ be the Weyl graph associated to $\mathcal{M}$. Then the universal group of $\mathcal{W}$ is,

$$\mathrm{FG}(\mathcal{W}) = \left\langle g_{(n,\sigma)}, a_{st} \mid (a_{st})^{\delta(st)} = 1,\ (a_{st})^{n(st)} = g_{(n,s)} g_{(n,t)}^{-1} \right\rangle$$

for $n \in \mathcal{C}^*$, $\sigma \in S$, and $(s,t) \in \bar{E}$.

L**-Cycles.** We define an L**-cycle** to be a sequence $\sigma_1, \ldots, \sigma_\mu$ of adjacent vertices of L with $\sigma_1 = \sigma_\mu$. We denote the set of L-cycles in L by $\mathrm{Cyc}(L)$. If $(s,t) \notin \bar{E}$, let $a_{st} = a_{ts}^{-1} \in \mathrm{FG}(\mathcal{W})$. Then for each L-cycle $\sigma_1, \ldots, \sigma_\mu$, and for each $n \in \mathcal{C}^*$, we have,

$$\begin{aligned}
&(a_{\sigma_1\sigma_2})^{n(\sigma_1\sigma_2)} (a_{\sigma_2\sigma_3})^{n(\sigma_2\sigma_3)} \ldots (a_{\sigma_{\mu-1}\sigma_\mu})^{n(\sigma_{\mu-1}\sigma_\mu)} \\
&= g_{(n,\sigma_1)} g_{(n,\sigma_2)}^{-1} g_{(n,\sigma_2)} g_{(n,\sigma_3)}^{-1} \cdots g_{(n,\sigma_{\mu-1})} g_{(n,\sigma_\mu)}^{-1} && \text{since } (a_{st})^{n(st)} = g_{(n,s)} g_{(n,t)}^{-1} \\
&= 1 && \text{since } \sigma_1 = \sigma_\mu.
\end{aligned}$$

Therefore $(a_{\sigma_1\sigma_2})^{n(\sigma_1\sigma_2)} (a_{\sigma_2\sigma_3})^{n(\sigma_2\sigma_3)} \ldots (a_{\sigma_{\mu-1}\sigma_\mu})^{n(\sigma_{\mu-1}\sigma_\mu)} = 1$ is a consequence of the relations of $\mathrm{FG}(\mathcal{W})$, which we call the relation **induced** by $\sigma_1, \ldots, \sigma_\mu$.

5.2.3 The Singer Lattice $\Gamma_{\mathcal{M}}$.

In this section, we obtain a presentation of the fundamental group of $\mathcal{W} = \mathcal{W}_{\mathcal{M}}$ whose generators are the a_{st}.

The Fundamental Group of $\mathcal{W}$. Fix $\xi \in S$, and let us assume that $\bar{E}$ has been chosen such that for all $u \in S$, if $\{u, \xi\} \in E(L)$, then $(u, \xi) \in \bar{E}$, i.e. the oriented edges at ξ terminate at ξ. Let T be the spanning tree,

$$T = \left\{ g_{(n,\xi)}, g_{(n,\xi)}^{-1} : n \in \mathcal{C}^* \right\}.$$

Then by quotienting out T in $\mathrm{FG}(\mathcal{W})$, we obtain,

$$\begin{aligned}
\pi_1(\mathcal{W}, T) = \big\langle g_{(n,\sigma)}, a_{st}, a_{u\xi} \mid \\
(a_{st})^{\delta(st)} = (a_{u\xi})^{\delta(u\xi)} = 1,\ (a_{st})^{n(st)} = g_{(n,s)} g_{(n,t)}^{-1},\ (a_{u\xi})^{n(u\xi)} = g_{(n,u)} \big\rangle
\end{aligned}$$

for $n \in \mathcal{C}^*$, $\sigma \in S \setminus \{\xi\}$, $(s,t) \in \bar{E}$ with $s, t \neq \xi$, and $(u, \xi) \in \bar{E}(L)$.

Eliminating the $g_{(n,\sigma)}$. For $s, t \in S$, if $(s,t) \notin \bar{E}$, let $a_{st} = a_{ts}^{-1} \in \pi_1(\mathcal{W}, T)$. For $\sigma \in S$, let $\sigma, \sigma_1, \ldots, \sigma_\mu, \xi$ be a sequence of adjacent vertices of L (recall that L is connected). For $n \in \mathcal{C}^*$, we have $(a_{\sigma_\mu \xi})^{n(\sigma_\mu \xi)} = g_{(n,\sigma_\mu)}$, and so,

$$\begin{aligned}
g_{(n,\sigma)} &= g_{(n,\sigma)} g_{(n,\sigma_1)}^{-1} g_{(n,\sigma_1)} \cdots g_{(n,\sigma_\mu)}^{-1} g_{(n,\sigma_\mu)} \\
&= (a_{\sigma\sigma_1})^{n(\sigma\sigma_1)} (a_{\sigma_1\sigma_2})^{n(\sigma_1\sigma_2)} \ldots (a_{\sigma_{\mu-1}\sigma_\mu})^{n(\sigma_{\mu-1}\sigma_\mu)} (a_{\sigma_\mu\xi})^{n(\sigma_\mu\xi)}.
\end{aligned}$$

In the presence of relations induced by L-cycles, the relations of the form $(a_{st})^{n(st)} = g_{(n,s)}g^{-1}_{(n,t)}$ are redundant since, after substitution, the relation is induced by an L-cycle of the form,

$$\xi, \sigma_\mu, \ldots, \sigma_1, t, s, \sigma'_1, \ldots, \sigma'_\mu, \xi.$$

Therefore we can drop the $g_{(n,\sigma)}$ from the generating set of $\pi_1(\mathcal{W}, T)$ and include the relations induced by L-cycles to obtain an equivalent group presentation of $\pi_1(\mathcal{W}, T)$:

Theorem 5.11. Let $\mathcal{M}$ be a gluing matrix and let $\mathcal{W} = \mathcal{W}_{\mathcal{M}}$ be the Weyl graph associated to $\mathcal{M}$. Then the fundamental group $\Gamma = \Gamma_{\mathcal{M}}$ of $\mathcal{W}$ has the presentation,

$$\Gamma = \left\langle a_{st}, a_{ts} \mid a_{st}a_{ts} = (a_{st})^{\delta(st)} = (a_{\sigma_1\sigma_2})^{n(\sigma_1\sigma_2)} \ldots (a_{\sigma_{\mu-1}\sigma_\mu})^{n(\sigma_{\mu-1}\sigma_\mu)} = 1 \right\rangle$$

for $(s,t) \in \bar{E}$, $n \in \mathcal{C}^*$, and $\sigma_1, \ldots, \sigma_\mu \in \mathrm{Cyc}(L)$. Thus, Γ is a Singer cyclic lattice in a building $\Delta = \Delta_{\mathcal{M}}$ of type M, with $\Gamma\backslash\Delta \cong \mathcal{W}$.

The number of relations induced by L-cycles that need to be included in order to obtain a sufficient set of relations will depend upon the homotopy type of L

5.2.4 Classification of Singer Cyclic Lattices of type M

Let M be a Coxeter matrix with $m_{st} \in \{2, 3, \infty\}$ for all $s, t \in S$, whose defining graph L is connected. We now show that every Singer cyclic lattice $\Gamma < \mathrm{Aut}(\Delta)$ in a building Δ of type M is equivalent to a Singer cyclic lattice of the form $\Gamma_{\mathcal{M}} < \mathrm{Aut}(\Delta_{\mathcal{M}})$, for some gluing matrix $\mathcal{M}$ of type M. This generalizes a theorem of Essert, who did type $\widetilde{A}_2$.

The Quotient by a Singer Cyclic Lattice. Let Δ be a locally finite building of type M, and let $\Gamma < \mathrm{Aut}(\Delta)$ be a Singer cyclic lattice in Δ of order q. Then $\Gamma\backslash\Delta$ is a Singer graph of order q with spherical Weyl polygons either the digon $k\backslash\mathbf{D}(q)$, the triangle $\delta\backslash\mathcal{T}(q)$, or (if they exist) non-Desarguesian Singer cyclic triangles. Therefore the only choices $\Gamma\backslash\Delta$ has are however these polygons are glued together. We now show that gluing matrices encode all the possible choices, and so $\Gamma\backslash\Delta \cong \mathcal{W}_{\mathcal{M}}$ for some gluing matrix $\mathcal{M}$.

The Gluing Matrix $\mathcal{M}(\Gamma)$. Let $\mathcal{C} = \mathbb{Z}/k\mathbb{Z}$, and let $\bar{E}$ be an orientation of L. By choosing a bijection $(\Gamma\backslash\Delta)_0 \to \mathcal{C}$, let us identify the chambers of $\Gamma\backslash\Delta$ with $\mathcal{C}$. We now define a gluing matrix,

$$\mathcal{M} = \mathcal{M}(\Gamma) : \mathcal{C}^* \times \bar{E} \to \mathbb{Z}_{\geq 1}$$

of type M and order q as follows. For each $(s,t) \in \bar{E}$ such that $m_{st} = 2$, by Corollary 5.3.1, there exists a permutation $\mathcal{C} \to \mathcal{C}$ with $0 \mapsto 0$ which is an isomorphism on the $\{s,t\}$-residue of $\Gamma\backslash\Delta$ into $k\backslash\mathbf{D}(q)$. Let $\mathcal{M}(-, (s,t))$ be the restriction of the permutation $\mathcal{C} \to \mathcal{C}$ to $\mathcal{C}^*$. For each $(s,t) \in \bar{E}$ such that $m_{st} = 3$, by Lemma 5.8, there exists a based difference set $\mathcal{D}$ and a bijection $\mathcal{C} \to \mathcal{D}$ with $0 \mapsto 0$ which is an isomorphism on the $\{s,t\}$-residue of $\Gamma\backslash\Delta$ into $\delta\backslash\mathcal{T}(\mathcal{D})$. Let $\mathcal{M}(-,(s,t))$ be the restriction of $\mathcal{C} \to \mathcal{D}$ to $\mathcal{C}^*$. We call $\mathcal{M}$ the gluing matrix **associated** to Γ. Then we obtain the following:

Theorem 5.12. Let M be a Coxeter matrix with $m_{st} \in \{2,3,\infty\}$ for all $s,t \in S$, whose defining graph is connected. Let Δ be a locally finite building of type M, and let $\Gamma < \mathrm{Aut}(\Delta)$ be a Singer cyclic lattice in Δ. Let $\mathcal{M} = \mathcal{M}(\Gamma)$ be a gluing matrix associated to Γ. Then there exists an isomorphism $\omega : \Gamma\backslash\Delta \to \mathcal{W}_{\mathcal{M}}$.

Proof. Let $\omega : \Gamma\backslash\Delta \to \mathcal{W}_{\mathcal{M}}$ be the morphism whose map on chambers is the identity $\mathcal{C} \to \mathcal{C}$. The fact that ω is an isomorphism then follows directly from the definition of $\mathcal{M}$ and the construction of $\mathcal{W}_{\mathcal{M}}$. □

This gives a classification of Singer cyclic lattices of type M modulo an equivalence relation on gluing matrices, since different gluing matrices may construct the same lattice. This equivalence for $M = \widetilde{A}_2$ is described in [Wit16, Corollary 3.19] (in terms of difference matrices).

5.3 Examples of Singer Cyclic Lattices

We finish by constructing some examples of Singer cyclic lattices.

5.3.1 Singer Cyclic Lattices of Type $\widetilde{A}_2$

The Singer cyclic lattices of type $\widetilde{A}_2$ were constructed by Essert in [Ess13], and classified by Witzel in [Wit16]. In [Wit16], it is shown that there are two non-isomorphic Singer lattices of type $\widetilde{A}_2$ and order 2. We construct these lattices:

(1) Take the gluing matrix $\mathcal{M}$ of type $\widetilde{A}_2$ shown in Figure 5.4. Put $a = a_{st}$, $b = a_{tu}$, and $c = a_{us}$. Notice that we only need to include one L-cycle in the presentation. From Theorem 5.11, we obtain,

$$\Gamma_{\mathcal{M}} = \big\langle a,b,c \mid a^7 = b^7 = c^7 = abc = a^3b^3c^3 = 1\big\rangle.$$

(2) Take the gluing matrix $\mathcal{M}'$ of type $\widetilde{A}_2$ shown in Figure 5.5. Put $a = a_{st}$, $b = a_{tu}$, and $c = a_{us}$. Then,

$$\Gamma_{\mathcal{M}} = \big\langle a,b,c \mid a^7 = b^7 = c^7 = abc = a^3b^3c^5 = 1\big\rangle.$$

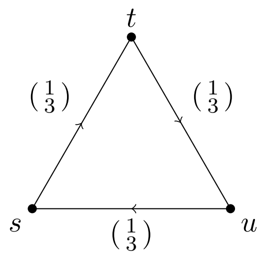

Figure 5.4: The gluing matrix $\mathcal{M}$

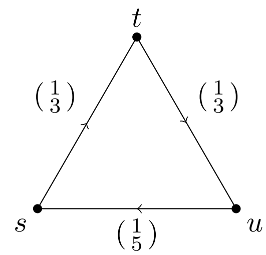

Figure 5.5: The gluing matrix $\mathcal{M}'$

5.3.2 Hyperbolic Singer Lattices

We define a **hyperbolic Coxeter group** to be a Coxeter group whose Davis chamber is a compact hyperbolic polytope. A **hyperbolic building** is a building whose type is a hyperbolic Coxeter group. A **Fuchsian building** is a locally finite hyperbolic building whose Davis chamber is a hyperbolic polygon.

Example 5.5. Let M be the Coxeter matrix on $S = \{s, t, u, v\}$ with $m_{st} = m_{tu} = m_{uv} = m_{vs} = 3$, and $m_{su} = m_{tv} = \infty$. The Coxeter group W of type M acts on the hyperbolic plane by tiling with squares whose internal angles are all $\pi/3$. Let $\mathcal{M}$ be the following gluing matrix of type M and order 3,

	(s,t)	(t,u)	(u,v)	(v,s)
1	1	1	1	1
2	3	3	3	3
3	9	9	9	9

Put $a = a_{st}$, $b = a_{tu}$, $c = a_{uv}$, and $d = a_{vs}$. Again we only need to include one L-cycle in the presentation. We obtain,

$$\Gamma_{\mathcal{M}} = \left\langle a, b, c, d \mid a^{13} = b^{13} = c^{13} = d^{13} = abcd = a^3b^3c^3d^3 = a^9b^9c^9d^9 = 1 \right\rangle.$$

Then $\Delta_{\mathcal{M}}$ is a Fuchsian building. In the language of polyhedral complexes (see [FHT11]), $\Gamma_{\mathcal{M}}$ is a uniform lattice in the Davis realization of $\Delta_{\mathcal{M}}$, which is a $(4, L)$-complex, where L is the simplicial building of the projective plane $\mathrm{PG}(2, 3)$.

Example 5.6. Let M be the Coxeter matrix on $S = \{s, t, u, v\}$ with $m_{st} = m_{vs} = 2$, $m_{tu} = m_{uv} = 3$, and $m_{su} = m_{tv} = \infty$. Let $\mathcal{M}$ be the following gluing matrix of type M,

	(s,t)	(t,u)	(u,v)	(v,s)
1	1	1	1	1
2	2	3	3	2
3	3	9	9	3

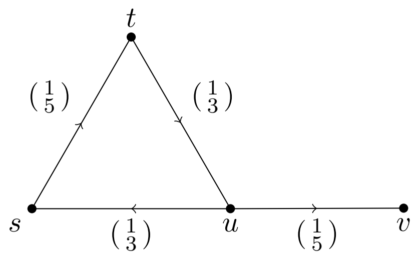

Figure 5.6: The gluing matrix $\mathcal{M}$ of Example 5.7

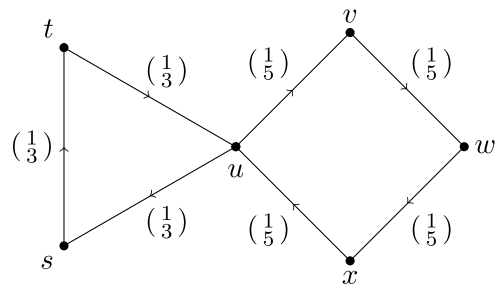

Figure 5.7: The gluing matrix $\mathcal{M}$ of Example 5.8

Put $a = a_{st}$, $b = a_{tu}$, $c = a_{uv}$, and $d = a_{vs}$. Then,

$$\Gamma_{\mathcal{M}} = \langle a, b, c, d \mid a^4 = b^{13} = c^{13} = d^4 = abcd = a^2b^3c^3d^2 = a^3b^9c^9d^3 = 1 \rangle.$$

This is a lattice in a Fuchsian building whose Davis realization models apartments as copies of the hyperbolic plane tessellated by Saccheri quadrangles. The links of the Davis realization are the complete bipartite graph on $4+4$ vertices, and the simplicial building of the projective plane $\mathrm{PG}(2,3)$.

5.3.3 Wild Singer Cyclic Lattices

We finish by constructing two exotic Singer cyclic lattices.

Example 5.7. Let M be the Coxeter matrix on $S = \{s, t, u, v\}$ with $m_{st} = m_{tu} = m_{us} = m_{uv} = 3$, and $m_{sv} = m_{tv} = \infty$. Take the gluing matrix $\mathcal{M}$ of type M and order 2 shown in Figure 5.6. Put $a = a_{st}$, $b = a_{tu}$, $c = a_{us}$, and $d = a_{uv}$. Then we only need to include one L-cycle in the presentation. We have,

$$\Gamma_{\mathcal{M}} = \langle a, b, c, d \mid a^7 = b^7 = c^7 = d^7 = abc = a^5b^3c^3 = 1 \rangle.$$

Example 5.8. Let M be the Coxeter matrix whose defining graph consists of a 3-cycle and a 4-cycle identified at a vertex, and whose labels are all 3. Take the gluing matrix $\mathcal{M}$ of type M and of order 2 shown in Figure 5.7. Put $a = a_{st}$, $b = a_{tu}$, $c = a_{us}$, $d = a_{uv}$, $e = a_{vw}$, $f = a_{wx}$, and $g = a_{xu}$. The 3-cycle and the 4-cycle of L are sufficient L-cycles for the presentation, thus,

$$\Gamma_{\mathcal{M}} = \langle a, b, c, d, e, f, g \mid a^7, b^7, c^7, d^7, e^7, f^7, g^7, abc, a^3b^3c^3, defg, d^5e^5f^5g^5 \rangle.$$

Appendix A

Strict Homotopy in Cayley Graphs

In this section, we establish several properties of those pre-Weyl graphs which are (isomorphic to) Cayley graphs of Coxeter groups. By Remark A.1, these results are equivalent to well known facts of word manipulation in Coxeter groups (e.g. the deletion condition). This section is essentially a translation of [Ron89, Chapter 2] into the language of pre-Weyl graphs.

Throughout this appendix, $W = (W, S)$ is a Coxeter group and $\mathcal{C}(W)$ is the pre-Weyl graph which is the Cayley graph of W.

Remark A.1. Since $\mathcal{C}(W)$ is thin, the map $\beta \mapsto \beta_S$ taking a gallery to its type is a bijection into words over S when restricted to the galleries which issue from a fixed chamber. The gallery starting at the chamber $w \in W$ and with type $s_1 \dots s_n$ will finish at the chamber $ww(s_1 \dots s_n)$. Moreover, for $w_1, w_2 \in W$, the map $\beta \mapsto \beta_S$ is a bijection into the homotopy class of words which are decompositions of $w_1^{-1}w_2$ when restricted to all the galleries going from w_1 to w_2. This bijection is compatible with the notions of homotopy that exist on words and galleries. In other words, this bijection takes an x homotopy of galleries to an x homotopy of words, where x can be any of the types of homotopy we defined (apart from a 1-elementary homotopy of type (ii), since $\mathcal{C}(W)$ is thin).

Minimal Galleries and Geodesics in $\mathcal{C}(W)$. Notice that β is a minimal gallery if and only if β_S is reduced. Thus, geodesics and minimal galleries coincide in $\mathcal{C}(W)$. Let us call a gallery in $\mathcal{C}(W)$ an M**-geodesic** if its type is an M-reduced word, i.e. an M-geodesic is a gallery which is not strictly homotopic to a gallery containing a backtrack. One can easily see that geodesics are M-geodesics. In this appendix we prove that in $\mathcal{C}(W)$, M-geodesics are geodesics, and homotopic geodesics are in fact strictly homotopic.

A.1 Reflections and Walls

We define a **reflection** $r \in W$ to be a conjugate of a generator $s \in S$. Let $T(W)$ denote the set of all reflections of W. The **wall** M_r of $r \in T(W)$ is the set of edges in $\mathcal{C}(W)$ which are mapped to their inverse by r. Thus, if we model the edges of $\mathcal{C}(W)$ as ordered pairs of chambers, then,

$$M_r = \big\{(w, w') \in W \times W : rw = w' \text{ and } ws = w' \text{ for some } s \in S\big\}.$$

Since s and r are involutions, this is equivalent to requiring that $w = rw'$ and $w = w's$. Hence, if $(w, w') \in M_r$, then $(w', w) \in M_r$. Notice that for $r = wsw^{-1}$, we always have $(w, ws) \in M_r$. Thus, walls are non-empty. Clearly r is recoverable from any edge in the wall, therefore,

$$M_r = M_{r'} \implies r = r'$$

and it follows that reflections are in bijection with walls.

Proposition A.1. The set of edges of $\mathcal{C}(W)$ is the disjoint union of its walls. In particular, each edge is contained in a unique wall.

Proof. The edges of $\mathcal{C}(W)$ are the union of the walls since an edge of the form (w, ws) is contained in the wall $M_{wsw^{-1}}$. This union is disjoint since if (w, ws) is contained in M_r, then we must have $r = wsw^{-1}$. □

Let $\beta : \lfloor 0, b \rfloor \to \mathcal{C}(W)$ be a gallery, and let $i_c \in \lfloor 0, b \rfloor$ be an edge for some $c \in \{1, \dots, b\}$. We say the gallery β **crosses** the wall M_r at i_c if $\beta(i_c) \in M_r$.

Lemma A.2. A geodesic in $\mathcal{C}(W)$ cannot cross a wall in two places or more.

Proof. Towards a contradiction, suppose that we have a geodesic,

$$\gamma : \lfloor 0, b \rfloor \to \mathcal{C}(W)$$

such that there exist distinct edges $i_c, i_d \in \lfloor 0, b \rfloor$ with $\gamma(i_c), \gamma(i_d) \in M_r$. Let α be the subgallery of γ which is the restriction of γ to $\lfloor c, d-1 \rfloor$, i.e. α is the part of γ between the crossings. Let $\alpha' = r \circ \alpha$ (here r denotes r's corresponding automorphism of $\mathcal{C}(W)$). It follows from our definition of walls that α' is a gallery from $\gamma(c-1)$ to $\gamma(d)$. Let β be the restriction of γ to $\lfloor 0, c-1 \rfloor$, and let β' be the restriction of γ to $\lfloor d, b \rfloor$. Then the concatenation $\beta\alpha'\beta'$ is a gallery with the same extremities as γ, but with $|\beta\alpha'\beta'| = |\gamma| - 2$. This contradicts the fact that γ is a geodesic, because geodesics are minimal galleries in $\mathcal{C}(W)$. □

A corollary of the result we are working towards is the converse of this; that is, if a gallery does not cross any wall twice, then its a geodesic.

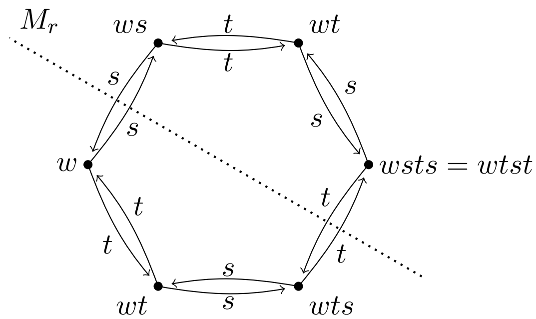

Figure A.1: The $\{s, t\}$-residue at w

Lemma A.3. Let $J = \{s, t\}$ be a 2-element spherical subset of S. Then the J-residues in $\mathcal{C}(W)$ are isomorphic to $\mathcal{C}_{m_{st}}$, and if a wall has a non-empty intersection with a J-residue, then it intersects in exactly two opposite pairs of edges.

Proof. Let $J = \{s, t\}$ be a 2-element spherical subset of S. It follows from the canonical presentation of a Coxeter group that the J-residues are cycles of girth $2m_{st}$, alternately labeled by s and t (see Figure A.1).

If a wall M_r intersects an $\{s, t\}$-residue R at an edge (w, ws) (without loss of generality), then r restricts to an involution of the $2m_{st}$-cycle R (see Lemma 3.5), which we know must invert an edge. It follows that r must be acting as a reflection on R, therefore r must also invert the edge $(wts, wtst)$, which is opposite to (w, ws) in R. □

Corollary A.3.1. The number of times a gallery crosses a given wall is an invariant of strict homotopy.

Proof. A strict homotopy takes place in a spherical 2-residue. The result then follows from Lemma A.3. □

Lemma A.4. Fix two chambers $w_1, w_2 \in W$, and a wall M_r. Then all of the galleries from w_1 to w_2 cross M_r the same number of times modulo 2.

Proof. Let β, $\hat{\beta}$ be two galleries from w_1 to w_2. Then β and $\hat{\beta}$ are homotopic via a composition of 1-elementary homotopies (of type (i)) and elementary strict homotopies, which determines a sequence of galleries,

$$\beta = \beta_1, \ldots, \beta_n = \hat{\beta}$$

such that any two consecutive galleries β_k, β_{k+1} differ by either a 1-elementary homotopy or an elementary strict homotopy. If suffices to show that β_k and β_{k+1} cross M_r the same number of times modulo 2. If they differ by a 1-elementary homotopy, the result is clear. If they differ by an elementary strict homotopy, the result follows from Corollary A.3.1. □

Corollary A.4.1. The number of times modulo 2 a gallery crosses a given wall is invariant of homotopy.

For a given wall M_r, for $w_1, w_2 \in W$ temporarily put $w_1 \sim w_2$ if the galleries between w_1 and w_2 cross M_r an even number of times. In light of Lemma A.2, this is equivalent to saying that the geodesics between w_1 and w_2 do not cross M_r. It follows from Lemma A.4 that '$\sim$' is an equivalence relation with two equivalence classes. Thus, each wall M_r partitions W into two parts called the **roots** of M_r.

If $w_1 \nsim w_2$, then we say M_r **separates** w_1 and w_2. Thus, two chambers are separated by a wall M_r if and only if the geodesics between those chambers cross M_r. The roots of a wall are always non-empty since, if $r = wsw^{-1}$, then M_r separates w and ws.

Notice that galleries between two chambers must cross all the walls separating those chambers at least once. On the other hand, a geodesic must cross all the walls separating those chambers exactly once, and it crosses no other walls than these. Hence, the distance between chambers is the number of walls separating them.

Lemma A.5. Let $w_1, w_2 \in W$ and let M_r be a wall of $\mathcal{C}(W)$. Then exactly one of the following must hold:

(i) M_r separates w_1 and w_2, which implies that $d(w_1, rw_2) < d(w_1, w_2)$

(ii) M_r separates w_1 and rw_2, which implies that $d(w_1, rw_2) > d(w_1, w_2)$.

Proof. Suppose that M_r separates w_1 and w_2. Let,

$$\gamma : \lfloor 0, b \rfloor \to \mathcal{C}(W)$$

be a geodesic from w_1 to w_2, and let i_c be the single edge of $\lfloor 0, b \rfloor$ such that $\beta(i_c) \in M_r$. Let α be the subgallery which is the restriction of γ to $\lfloor c, b \rfloor$, i.e. α is the part of γ after the crossing. Let $\alpha' = r \circ \alpha$. Then α' is a gallery from $\gamma(c-1)$ to rw_2, which does not cross M_r. Let β be the restriction of γ to $\lfloor 0, c-1 \rfloor$. Then the concatenation $\beta\alpha'$ is a gallery from w_1 to rw_2. Notice that M_r cannot separate w_1 and rw_2 because $\beta\alpha'$ does not cross M_r, and $d(w_1, rw_2) < d(w_1, w_2)$ because $|\beta\alpha'| = |\gamma| - 1$.

Now suppose that M_r does not separate w_1 and w_2. Let w be a chamber which is separated from w_1 by M_r (recall that the roots of M_r are non-empty). Now, M_r also separates w and w_2, so M_r does not separate w and rw_2. Thus, M_r does separate w_1 and rw_2, and the fact that $d(w_1, rw_2) > d(w_1, w_2)$ follows from the first part of the proof. ☐

Remark A.2. For $w_1, w_2 \in W$ and $r \in T(W)$, we have the following dichotomy; either $d(rw_1, w_2) < d(w_1, w_2)$ or $d(rw_1, w_2) > d(w_1, w_2)$. Letting $w_1 = w$ and $w_2 = 1$, we see that for all $w \in W$ and $r \in T(W)$, we have either $|rw| > |w|$ or $|rw| < |w|$.

Lemma A.6. A chamber w lies on a geodesic between w_1 and w_2 if and only if the set of walls separating w_1 and w_2 is the disjoint union of the walls separating w_1 and w, and the walls separating w and w_2.

Proof. If w lies on a geodesic γ from w_1 to w_2, then the part of γ before w crosses exactly once those walls separating w_1 and w, and the part of γ after w crosses exactly once those walls separating w and w_2. On the other hand, the whole of γ crosses exactly once those walls separating w_1 and w_2. The result follows.

Conversely, if the set of walls separating w_1 and w_2 is the disjoint union of the walls separating w_1 and w, and the walls separating w and w_2, then $d(w_1, w) + d(w, w_2) = d(w_1, w_2)$. So to get a geodesic from w_1 to w_2 via w, simply concatenate a geodesic from w_1 to w with a geodesic from w to w_2. □

A.2 Projections and the Gate Property

Lemma A.7. Let $w \in W$ be a chamber and let R be a residue of $\mathcal{C}(W)$. There exists a unique chamber of R which is at minimal distance from w.

Proof. Suppose that w_1 and w_2 are two chambers in R at minimal distance from w. Towards a contradiction, suppose that w_1 and w_2 are distinct, and take a wall M_r separating them. This wall intersects R since any gallery in R from w_1 to w_2 must cross M_r at least once (in particular an odd number of times). It follows that r is an automorphism of R (see Lemma 3.5), in particular $rw_2 \in R$. Without loss of generality, suppose this wall separates w_2 and w. Then, by Lemma A.5, rw_2 is nearer to w than w_1 or w_2, a contradiction of the hypothesis on w_1 and w_2. □

Let us denote the unique chamber of Lemma A.7 by $\operatorname{proj}_R w$.

Lemma A.8. Let R be a residue of $\mathcal{C}(W)$. A geodesic from any chamber $w \in W$ to any chamber $w_R \in R$ can be chosen to go via $\operatorname{proj}_R w$.

Proof. By Lemma A.6, it suffices to show that the set of walls separating w_R from w is the disjoint union of the walls separating w_R from $\operatorname{proj}_R w$ and $\operatorname{proj}_R w$ from w. Towards a contradiction, suppose this union is not disjoint. Thus, a wall M_r separates w_R from $\operatorname{proj}_R w$ *and* $\operatorname{proj}_R W$ from w. Then M_r intersects R, and r is an automorphism of R. Therefore $r \cdot \operatorname{proj}_R w$ is a chamber in R which is nearer to w than $\operatorname{proj}_R w$, a contradiction. □

Lemma A.9. The residues of $\mathcal{C}(W)$ are convex.

Proof. Take any residue R, and let $w_1, w_2 \in R$ be chambers. Suppose the chamber w lies on a geodesic γ from w_1 to w_2. Then $\operatorname{proj}_R w$ must coincide with w, otherwise a gallery shorter than γ would go via $\operatorname{proj}_R w$. Therefore $w \in R$. The fact that γ_S will be a word over the type of R follows from Theorem 2.4. Thus, γ is a gallery in R. □

A.3 The Main Theorem

Theorem A.10. In the Cayley graph of a Coxeter group, the following hold:

(i) homotopic geodesics are strictly homotopic

(ii) M-geodesics are geodesics.

Proof. Let $\mathcal{C}(W)$ be the Cayley graph of a Coxeter group W. We prove (i) by induction on the length of geodesics. Let $w_1, w_2 \in W$, and let γ and $\hat{\gamma}$ be (homotopic) geodesics going from w_1 to w_2. Let s be the final letter of γ_S, and let t be the final letter of $\hat{\gamma}_S$. If $s = t$, then the result follows by induction. So assume $s \neq t$.

Let $R = R_{\{s,t\}}(w_2)$, and let $z = \operatorname{proj}_R w_1$. Let $z_s = w_2 s$, and let $z_t = w_2 t$. Then $z_s, z_t \in R$. By Lemma A.8, there exists a geodesic γ^s from w_1, via z, to z_s, and a geodesic γ^t from w_1, via z, to z_t. Then, γ^s extends by one edge (with type s) to a geodesic β^s from w_1 to w_2. Similarly, γ^t extends by one edge (with type t), to a geodesic β^t from w_1 to w_2. By Lemma A.9, the subgalleries of β^s and β^t which go from z to w_2 are contained in R.

Suppose that $m_{st} = \infty$, so that R is a bi-infinite line. Then, without loss of generality, γ^s travels to z_s via w_2. But then β^s crosses a wall twice, a contradiction. Therefore $m_{st} < \infty$.

Now, the subgalleries of β^s and β^t which go from z to w_2 and are of types $p^{-1}(s,t)$ and $p^{-1}(t,s)$ respectively (since R is isomorphic to $\mathcal{C}_{m_{st}}$). Therefore, by the induction hypothesis, we have $\beta^s \simeq \beta^t$. Then, also by the induction hypothesis, we have $\gamma \simeq \beta^s$ and $\beta^t \simeq \hat{\gamma}$, since both pairs of galleries agree on their final edges. Thus, $\gamma \simeq \hat{\gamma}$ by transitivity.

We now prove (ii), also using induction. Let β be a M-geodesic from w_1 to w_2. The result clearly holds if β has length ≤ 2. So assume that $\beta_S = fst$, where f is a non-empty word over S, and $s, t, \in S$. We assume all M-geodesics of length less than the length of β are geodesics. In particular, the M-geodesic γ', obtained by removing the last edge from β, is a geodesic. Let w be the final chamber of γ'.

Towards a contraction, suppose that β is not a geodesic. Then, since $d(w_1, w) = |\beta| - 1$, we have $d(w_1, w_2) = |\beta| - 2$. So there exists a geodesic γ'' from w_1 to w, via w_2. Since $\gamma'_S = fs$ and $\gamma''_S = f't$ for some word f', by (i), β is not an M-geodesic, a contradiction. □

Corollary A.10.1. If a gallery is not a geodesic, then it crosses a wall twice. Thus, geodesics are characterized by the property that they only cross walls once.

Proof. If a gallery β is not a geodesic then it is not a M-geodesic, i.e. β is strictly homotopic to a gallery whose type repeats a letter. Therefore, by Corollary A.3.1, β must cross a wall twice. □

We now use the close relationship between words and galleries in $\mathcal{C}(W)$ (see Remark A.1) to give us some results on word manipulation in Coxeter groups.

Theorem 2.8. (Main Theorem) For any Coxeter group W:

(MT1) M-reduced words are reduced

(MT2) homotopic reduced words are strictly homotopic.

Proof. This follows from Theorem A.10 and the bijective correspondence between galleries issuing from a fixed chamber in $\mathcal{C}(W)$ and words over S. $\square$

Theorem 2.9. Let (W, S) be a Coxeter group. If a word f over S is not reduced, then there exists a substring of f obtained by deleting two letters which is homotopic to f.

Proof. Let $\beta : \lfloor 0, b \rfloor \to \mathcal{C}(W)$ be a gallery with type f. Then β crosses a wall M_r twice by Corollary A.10.1. So let $i_c, i_d \in \lfloor 0, b \rfloor$ be the edges such that $\beta(i_c), \beta(i_d) \in M_r$. Let α be the restriction of β to $\lfloor c-1, d \rfloor$. Let $\alpha' = r \circ \alpha$. Let β' be the restriction of β to $\lfloor 0, c-1 \rfloor$ and let β'' be the restriction of β to $\lfloor d, b \rfloor$. Then $\hat{\beta} = \beta'\alpha'\beta''$ is a gallery which is homotopic to β, thus $\hat{\beta}_S$ is homotopic to f. Also, $\hat{\beta}_S$ is obtained from f by deleting two letters, corresponding to the edges i_c and i_d. $\square$

Theorem A.11. Let $W = (W, S)$ be a Coxeter group and let $J \subseteq S$. Let $W_J \leq W$ be the standard subgroup generated by J. Then every coset wW_J has a unique representative w' of minimal word length. Moreover, we have,

$$|w'w_J| = |w'| + |w_J|$$

for all $w_J \in W_J$.

Proof. Let R be the J-residue of $\mathcal{C}(W)$ which contains w, and put $w' = \mathrm{proj}_J(1)$. Then w' is the unique representative of minimal word length of wW_J by Lemma A.7. By Lemma A.8, a geodesic $\gamma = \gamma'\gamma''$ exists such that γ' goes from 1 to w', and γ'' goes from w' to $w'w_J$. Then,

$$|w'w_J| = |\gamma| = |\gamma'| + |\gamma''| = |w'| + |w_J|. \qquad \square$$

Appendix B

The Bruhat Order

The set up for this section is that the Bruhat order is yet to be defined, and that the established facts about Coxeter groups are those which have been proven so far in this appendix. Throughout this section, W is a Coxeter group with generators S.

Recall that $T(W)$ denotes the set of reflections of W, where a reflection r is a conjugate $r = wsw^{-1}$ of a generator $s \in S$. For $w \in W$, let $T(w)$ be the set of reflections which correspond to the walls which separate w and 1.

For $w, w' \in W$, we write $w' \lessdot w$ if there exists $r \in T(w)$ such that $w' = rw$. Thus, $w' \lessdot w$ if and only if w' is the image of w by a reflection in a wall which separates w and 1. It follows directly from Lemma A.5 that,

$$T(w) = \big\{ r \in T(W) : |rw| < |w| \big\}.$$

Therefore, if $w' \lessdot w$, then $|w'| < |w|$.

Lemma B.1. Let W be a Coxeter group, and let $w, w' \in W$. If $w' \lessdot w$, then a decomposition of w' may be obtained from *any* decomposition of w by deleting one letter.

Proof. Let $w' = rw$, $r \in T(w)$. Let f be any decomposition of w, and let β be the gallery of type f from 1 to w. Apply r to the part of β after its final crossing of M_r. Then, in the same way as the proof of Lemma A.2, we obtain a gallery from 1 to w' whose type is f with one letter deleted. $\square$

Lemma B.2. Let W be a Coxeter group, and let $w, w' \in W$. If a decomposition of w' is obtained by deleting a letter in a *reduced* decomposition f of w, then $w' \lessdot w$.

Proof. Let γ be the geodesic with type f from 1 to w. Let $i \in \mathcal{C}(W)$ be the edge of γ which corresponds to the deleted letter of f, and let M_r be the wall which contains i. Then $r \in T(w)$ since γ is a geodesic which crosses M_r, and $w' = rw$. Thus, $w' \lessdot w$. $\square$

Combining these last two lemmas, we obtain:

Proposition B.3. Let W be a Coxeter group, and let $w, w' \in W$. The following are equivalent:

(i) $w' \lessdot w$

(ii) a decomposition of w' may be obtained from *any* decomposition of w by deleting one letter

(iii) a decomposition of w' may be obtained from a *reduced* decomposition of w by deleting one letter.

Proof. (i) $\Longrightarrow$ (ii) is Lemma B.1, (ii) $\Longrightarrow$ (iii) is clear, (iii) $\Longrightarrow$ (i) is Lemma B.2. $\square$

Definition of the Bruhat Order. Let us define a pre-order '$\leq$' on W, called the **Bruhat order**, by letting '$\leq$' be the transitive and reflexive closure of '$\lessdot$'. Thus, $w' \leq w$ if and only if there is a sequence of elements $w_0, \ldots, w_n$ of W such that,

$$w' = w_0 \lessdot \cdots \lessdot w_n = w.$$

The Bruhat order is antisymmetric since if $w' \lessdot w$, then $|w'| < |w|$. Thus, the Bruhat order is a partial order.

Let $w \in W$ and $s \in S$. If $|sw| = |w| - 1$, then $s \in T(w)$. Also, if $sw < w$, then $|sw| < |w|$. It then follows that,

$$sw < w \text{ if and only if } |sw| = |w| - 1$$

$$w < sw \text{ if and only if } |sw| = |w| + 1.$$

Notice that the symmetric results, which do in fact hold, are not clear at this point. They will follow from Proposition B.5 below.

Lemma B.4. Let W be a Coxeter group, and let $w, w' \in W$. If $w' < w$, then $sw' \leq \max\{sw, w\}$.

Proof. Let $w' = r_1 \ldots r_n w$, where $r_k \in T(W)$, and $r_{k+1} \ldots r_n w \lessdot r_k \ldots r_n w$. We prove by induction on n. If $n = 1$, then $sw' = sr_1 w$, or equivalently,

$$sw' = (sr_1 s)sw.$$

Notice that $sw' = w$ if and only if $s = r_1$. So assume that $s \neq r_1$. If $sw' < w'$ we are done, so assume further that $sw' > w'$. Notice that if $|sw'| < |sw|$, then we may conclude that $sw' < sw$ since $sr_1 s$ is a reflection. So, towards a contradiction, assume that $|sw'| \geq |sw|$. Then $sr_1 s \in T(sw')$ by Lemma A.5, and so,

$$sw = (sr_1 s)sw' < sw'.$$

Let f be a reduced decomposition of w'. Then sf is a reduced decomposition of sw'. By Proposition B.3, we can delete a letter from sf to obtain a decomposition of sw. This letter cannot be the first letter because $r_1 \neq s$. But then $|w| < |w'|$, which is a contradiction of the fact that $w' < w$. Thus, the result holds for $n = 1$. For $n > 1$, we have $r_1 \ldots r_n w < r_2 \ldots r_n w$, and so,

$$sw' = sr_1 \ldots r_n w \leq \max\{sr_2 \ldots r_n w, r_2 \ldots r_n w\}.$$

By the induction hypothesis, we have $sr_2 \ldots r_n w \leq \max\{sw, w\}$. We also have $r_2 \ldots r_n w < w$. Thus,

$$sw' \leq \max\{sw, w\}$$

as required. □

Proposition B.5. Let W be a Coxeter group, and let $w, w' \in W$. The following are equivalent:

(i) $w' \leq w$

(ii) *every* decomposition of w contains a substring which is a decomposition of w'

(iii) a *reduced* decomposition of w contains a substring which is a decomposition of w'

Proof. (i) $\Longrightarrow$ (ii) follows from Proposition B.3, and (ii) $\Longrightarrow$ (iii) is clear. We prove (iii) $\Longrightarrow$ (i) by induction on $|w|$. The result is clear for $|w| = 1$. For $|w| > 1$, let f be a reduced decomposition of w, and let s be the first letter of f. Let f' be a substring of f, and let $w' = w(f')$. If f' does not contain the first letter of f, then by applying the induction hypothesis to sw, we obtain $w' \leq sw < w$. If f' does contain the first letter of f, then the induction hypothesis tells us that $sw' \leq sw$. Then, by Lemma B.4, either $w' \leq sw < w$, or $w' \leq w$. □

Bibliography

[AB08] Peter Abramenko and Kenneth S Brown. *Buildings: Theory and Applications*, volume 248. Springer Science & Business Media, 2008.

[Bas93] Hyman Bass. Covering theory for graphs of groups. *J. Pure Appl. Algebra*, 89(1-2):3–47, 1993.

[BB06] Anders Bjorner and Francesco Brenti. *Combinatorics of Coxeter Groups*, volume 231. Springer Science & Business Media, 2006.

[Ber53] Gerald Berman. Finite projective plane geometries and difference sets. *Transactions of the American Mathematical Society*, 74(3):492–499, 1953.

[BH99] Martin R. Bridson and André Haefliger. *Metric Spaces of Non-Positive Curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, 1999.

[Bjö84] A. Björner. Posets, regular CW complexes and Bruhat order. *European Journal of Combinatorics*, 5(1):7–16, 1984.

[BL01] Hyman Bass and Alexander Lubotzky. *Tree lattices*. Springer, 2001.

[Bou97] M. Bourdon. Immeubles hyperboliques, dimension conforme et rigidité de Mostow. *Geometric and Functional Analysis*, 7(2):245–268, 1997.

[Bou00] Marc Bourdon. Sur les immeubles fuchsiens et leur type de quasi-isométrie. *Ergodic Theory and Dynamical Systems*, 20(2):343–364, 2000.

[Bou02] Nicolas Bourbaki. *Lie Groups and Lie Algebras: Chapters 4-6*. Elements of Mathematics. Springer-Verlag, 2002.

[Bro06] Ronald Brown. *Topology and Groupoids*. www.groupoids.org, 2006.

[CG03] Lisa Carbone and Howard Garland. Existence of lattices in Kac–Moody groups over finite fields. *Communications in Contemporary Mathematics*, 5(05):813–867, 2003.

[CMSZ93] Donald I Cartwright, Anna Maria Mantero, Tim Steger, and Anna Zappa. Groups acting simply transitively on the vertices of a building of type $\tilde{A}_2$, I. *Geometriae Dedicata*, 47(2):143–166, 1993.

[Dav94] Michael W Davis. Buildings are CAT(0). *Geometry and cohomology in group theory (Durham, 1994)*, 252:108–123, 1994.

[Dav08] Michael W. Davis. *The Geometry and Topology of Coxeter groups*, volume 32 of *London Mathematical Society Monographs Series*. Princeton University Press, 2008.

[Dem68] Peter Dembowski. *Finite Geometries*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 1968.

[Ess13] Jan Essert. A geometric construction of panel-regular lattices for buildings. *Algebraic & Geometric Topology*, 13(3):1531–1578, 2013.

[FHT11] Benson Farb, Chris Hruska, and Anne Thomas. Problems on automorphism groups of nonpositively curved polyhedral complexes and their lattices. In *Geometry, Rigidity, and Group Actions*, Chicago Lectures in Mathematics, pages 515–560. University of Chicago Press, 2011.

[GP01] Damien Gaboriau and Frédéric Paulin. Sur les immeubles hyperboliques. *Geometriae Dedicata*, 88(1-3):153–197, 2001.

[Gro84] Mikhael Gromov. Infinite groups as geometric objects. In *Proceedings of the International Congress of Mathematicians*, volume 1, page 2, 1984.

[Hae91] André Haefliger. Complexes of groups and orbihedra. In *Group theory from a geometrical viewpoint (Trieste, 1990)*, pages 504–540. World Sci. Publ., River Edge, NJ, 1991.

[HP03] Frédéric Haglund and Frédéric Paulin. Constructions arborescentes d'immeubles. *Mathematische Annalen*, 325(1):137–164, 2003.

[Kan86] William M Kantor. Generalized polygons, scabs and gabs. In *Buildings and the Geometry of Diagrams*, pages 79–158. Springer, 1986.

[Noo05] Behrang Noohi. Foundations of topological stacks I. *arXiv preprint math/0503247*, 2005.

[NTV16] William Norledge, Anne Thomas, and Alina Vdovina. Maximal torsion-free subgroups of certain lattices of hyperbolic buildings and Davis complexes. *Geometriae Dedicata*, pages 1–23, 2016.

[NTV17] William Norledge, Anne Thomas, and Alina Vdovina. Maximal torsion-free subgroups of certain lattices of hyperbolic buildings and Davis complexes. *Geometriae Dedicata*, 2017.

[Ron89] Mark Ronan. *Lectures on Buildings*, volume 7 of *Perspectives in Mathematics*. Academic Press Inc., 1989.

[Ron92] Mark Ronan. Buildings: Main ideas and applications I. Main ideas. *Bulletin of the London Mathematical Society*, 24(1):1–51, 1992.

[RR06] Bertrand Rémy and Mark Ronan. Topological groups of Kac–Moody type, right-angled twinnings and their lattices. *Commentarii Mathematici Helvetici*, 81(1):191–219, 2006.

[Ser80] Jean Pierre Serre. *Trees*. Springer, 1980.

[Sin38] James Singer. A theorem in finite projective geometry and some applications to number theory. *Transactions of the American Mathematical Society*, 43(3):377–385, 1938.

[Tho07] Anne Thomas. On the set of covolumes of lattices for Fuchsian buildings. *Comptes Rendus Mathematique*, 344(4):215–218, 2007.

[Tit69] Jacques Tits. Le problème des mots dans les groupes de Coxeter. In *Symposia Mathematica (INDAM, Rome, 1967/68)*, volume 1, pages 175–185. Academic Press, 1969.

[Tit74] Jacques Tits. Buildings of spherical type and finite BN-pairs. In *Lecture Notes in Mathematics*, volume 386. Springer-Verlag, 1974.

[Tit81] Jacques Tits. A local approach to buildings. In *The Geometric Vein*, pages 519–547. Springer-Verlag, 1981.

[Tit85] Jacques Tits. Buildings and group amalgamations. In *Proceedings of Groups— St. Andrews 1985*, volume 121 of *London Mathematical Society Lecture Notes Series*, pages 110–127. Cambridge University Press, 1985.

[Tit86] Jacques Tits. Ensembles ordonnés, immeubles et sommes amalgamées. *Bulletin de la Société Mathématique de Belgique Séries A*, 38:367–387, 1986.

[Tit92] Jacques Tits. Twin buildings and groups of Kac-Moody type. In *Groups, Combinatorics and Geometry*, volume 165 of *London Mathematical Society Lecture Note Series*, page 249–286. Cambridge University Press, 1992.

[TW02] Jacques Tits and Richard M Weiss. *Moufang Polygons*. Springer Monographs in Mathematics. Springer, 2002.

[Vdo02] Alina Vdovina. Combinatorial structure of some hyperbolic buildings. *Mathematische Zeitschrift*, 241(3):471–478, 2002.

[VM12] Hendrik Van Maldeghem. *Generalized Polygons*, volume 93 of *Monographs in Mathematics*. Birkhäuser Basel, 2012.

[Wit16] Stefan Witzel. On panel-regular $\tilde{\mathrm{A}}_2$ lattices. *arXiv preprint*, arXiv:1608.07141, 2016.