Understanding Idempotents in Diagram Semigroups

By Nicholas J. Loughlin

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Nicholas J. Loughlin
School of Mathematics and Statistics
Newcastle University

To the memory of James and Elizabeth McCormick. Your family owes you for everything we've been able to do. You will never be forgotten.

Declaration

The contents of thesis constitute the product of my own work towards the degree of Doctor of Philosophy. Its contents and composition are entirely of my own device except where citation is given to other work or to the folklore of the theory, in accordance with the University's rules on academic misconduct; acknowledgement of non-academic contributions to the thesis, as a whole or in part, may be found in the relevant section (see page xiii).

My signature below affirms the latter facts where required.

Nicholas James Loughlin

Abstract

The principal concern of this document is to develop and expose methodology for enumerating idempotents in certain semigroups of diagrams in the sense of [76]. These semigroups are known to be significant in the representation theory of associated algebras. In particular these algebras are shown in many cases to be semisimple, giving certain idempotents (and in particular those of the monoids of concern) a prominent role in understanding certain features of the representation theory in this situation.

The results developed here are mostly theoretical in nature. We propose two viewpoints leading to some combinatorial understanding of the idempotents in the Motzkin (respectively Jones and partial Jones) monoid. In the first instance, we construct a cell complex, whose connected components partition the set of all idempotents into small, manageable chunks that can be analysed uniformly starting from those of particularly low rank. The structure of this complex captures some intricate combinatorics in the semigroup in a fairly simple, uniform way, and reduces our problem to finding and characterising idempotents of particularly low rank.

The latter viewpoint takes us closer to pure combinatorics; a family of parameters attached to the elements of the monoids in question. These are examined in the context of ordinary generating functions, counting the elements with various parameter profiles. In particular, important algebraic features of Motzkin pictures, such as degree, rank, idempotency, and membership in the Jones and partial Jones monoids, can be tested against parameter profiles, reducing the problem of understanding all three to that of a parametric under-

standing of only the Motzkin monoid.

We can then amalgamate these families of techniques into the development of fast linear-space algorithms for counting elements of various parameter profiles by examining certain "convex" elements. In particular, the general problem of enumeration by parameter profile is reduced greatly to enumerating convex elements by parameter profile.

As a corollary to this study of convexity, we observe that the sequence of numbers of idempotents (in each semigroup) of some fixed rank-deficiency $\delta = (n - r)$ is equal (apart from the first couple of values) to some polynomial of degree δ; for particularly low rank-deficiency, we calculate these polynomials.

Finally, we can show that the problem of understanding these idempotents in this way reduces to the classical open problem in combinatorics of counting meanders, witnessing the fact that significant progress on the former problem would necessitate some development of a better understanding of the latter.

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Acknowledgements

For the better part of a decade, I've had some sort of attachment to Newcastle. Never one to do things in any sort of traditional manner, I took up a studentship which was never quite part-time and never quite full-time, and extended over five years of funding. I arrived a semester late, of course, not moving to Newcastle until about six months after my intended start date, and three months after I matriculated as a student there.

All these years later, I submitted this thesis, in an unrecognisable form and was examined on it, given a year to correct and then an extension of six months after that when I had to spend much of that time selling my labour rather than correcting. In other words, I've been around for a long time and have many long-overdue acknowledgements to make. To keep things simple, I'll go roughly in chronological order.

Going back to my early life, it'd be thoughtless in the extreme of me not to mention my partents, Billy and Gina, who made sure that my youth was filled with stimulation. Dad brought home parts to let me build my first computer, taught me how to count in binary and was very supportive in several early coding projects, including helping me learn Microsoft DOS and IBM OS/2. Mum was equally supportive — she let me study electronics and algebra with her when studying for her engineering diploma. Both worked hard and instilled in me a healthy respect for a work ethic that I've never quite attained.

My mum's parents, Betty and Jim were almost always around during our childhood. They looked after us when mum and dad needed to work, and provided us with a second home to spend part of our childhood in free from the (admittedly relatively relaxed) discipline expected at home. In later years, we all moved in together in a large old house that mum and dad still live in today. Even over years of failing health, they still provided childcare every day for us and sometimes friends, and I don't remember a single complaint apart from gran occasionally threatening me with moderate to severe physical harm.

I should also give an early mention to my supervisor, Sarah Rees, who has been a constant source of support in all my endeavours, academic or otherwise, and a good friend throughout my time in Newcastle. Her supervisory presence of constant positivity and direction — something I still sorely lack — has been useful in helping me to focus my efforts and to see that something has come of them even when that wasn't obvious to me from my viewpoint in the grass. I'd probably have spent my time wallowing in trivialities left to my own devices, and the whole process has at several points been in danger of being all for nought. Thankfully, she's helped me right the ship more times than I can remember. I'm deeply grateful.

I feel I should also thank a few people from St Andrews who were influential in my development from there. First, and most obviously, is James Hyde, who has been a close friend since we met at a Burns' supper in 2007, and a frequent collaborator and confidante. He's shared in some of my most valued achievements and experiences as an adult and helped me through some of my darkest days. Alistair, Zoe and Iain have also been important people in my life, during and since my days as an undergraduate, and I'm always happy to have some time to see them, little and seldom though it comes. I also feel indebted to Nik Ruškuc, James Mitchell, Martyn Quick, Lars Olson and Ken Falconer, who were some of my favourite lecturers as an undergraduate, who all contributed in numerous and important ways to my early development as a mathematician, and who all inspired me to learn.

Lastly, I've had a large number of friends come and go — I've been at Newcastle long enough to meet a couple of generations of undergraduates and half a dozen generations of postgraduates, as well as being around to watch the school of maths and stats grow enormously since my arrival. Without naming too many names, I'd like to acknowledge my fellow musicians from the jazz orchestra, j'Avisons, the blueswater, the sonic recreation, the guys at artistik records and a number of other unnamed collectives and ensembles — Chloe, Paul, Jordan, Andy, Saif, Cameron, Will, Charlie, Tom, Louise, Graeme, Alex, Michael, Jess, Harry, Ed, Jamie, Ifede, Joe, Dom, Ellie, Dan, Andy, Luc, Felipe. There were more than a few others. Having talented, fun, interesting people to share a stage, a recording studio, several houses, and many great evenings with, has been my profound and continual delight.

I'd also like to shout out to my officemates, and to Tia Maria and the postgraduates who helped keep my life inside and outside the department interesting, who have been there to drink with, to think with, to dance with and to keep me distracted when the prospect of actually finishing this threatened to rear its ugly head.

Lastly, I'd like to acknowledge my examiners, who have shaped the final form of this thesis more than they ought to have needed to. Their useful comments have helped direct me to turn this thesis from a relatively incoherent cobbling-together of various background materials accrued over a period of years, to something that hopefully makes sense as a companion to, and exposition of, the two papers on which it was based. In particular, the external examiner, James, who I've already acknowledged as an early influence from my undergraduate days, wrote the software I've been using to run the algorithms in and contributed several vast improvements to my code, including an independent implementation from my own which vastly outperforms my earlier efforts.

How To Stay Most Happy While Using This Document

This document is organised into five parts; there's this preamble section, three parts in the main body of the thesis, and then a set of appendices.

The preamble contains an abstract, the table of contents, lists of figures, tables and notations. It also contains a statement of good academic conduct, a little welcome blurb, and this guide, which may be of less use to the average reader.

The first part comprises two chapters and broadly aims to "set the scene". The introduction chapter motivates the research undertaken, and explains in very broad terms what sort of results the reader can expect in the sequel. The second chapter lays some foundations and establishes some linguistic and notational conventions of a foundational nature.

The second and third parts respectively develop the theory and its application to developing fast algorithms for counting idempotents and computing various statistics.

Part II comprises two chapters. In Chapter 4 we characterise idempotents in terms of statistics attached to related objects built from graphs, and organise the idempotents into a cell complex whose structure is intimately related to some combinatorics in the semigroup. In the latter, Appendix A, we develop an approach to indexing the $\mathscr{H}$-classes in these semigroups by words from certain context-free languages. This is of only peripheral interest to theorists, but builds a foundation for the design of algorithms with a tiny memory overhead, and provides a compact computational representation for elements from which it's easy to calculate various statistics about elements, and perform multiplications.

Part III comprises two more chapters. In Chapter 5 we assemble a toolbox of combinatorial techniques that allow us to view the idempotents in the three monoids of concern in a uniform manner, as well as developing a formalism that allows us to extract more data from the cell complex we built earlier. We also expose some surprising polynomial bounds for the numbers of idempotents of extreme rank. Chapter 6, closing the main body of work, discusses the development in GAP of tools for quickly computing these statistics.

The appendices assemble "apocryphal" materials which are not suitably-formatted for inclusion in the main text. This includes tables of results stratified by various parameters (notably rank) and examinations of contributions of various parts of the monoids to the whole. There is also a dictionary of small-degree convex idempotents, from which we calculated the low-order approximation to the universal generating function listed in Chapter 5, a listing of code used to generate and verify results, and a readout of the generating function for the Motzkin idempotents in small degree.

Now that we've got the layout of the document down, I'll say a few words about the choices I've made in laying things out the way that I have.

Any notation that is not completely standard to my knowledge is listed in the preamble with an explanation and a page number of the first usage and/or definition in the text. These are listed in alphabetical order how they first appear in the main text.

This document is littered with pictures. Some of them are in captioned figures, and those are listed in the list of figures in the preamble alongside the page on which they appear, some are displayed between paragraphs and numbered, and some are unnumbered. The huge number of graphics required for good exposition here means I've made no attempt to compile a list (or even a count) of all graphics here; only those which are featured in a captioned figure are listed.

I've taken pains to make the text flow smoothly, although there are areas where many results are developed in a short space of time, and in these situations I've opted for a definition-example-result layout in these situations, with small observations or intuitions for objects or results interspersed throughout.

Notational Conventions

I never use any bracketings apart from parentheses to group objects under an algebraic operation. So, all square-bracketings will be either closed intervals or integer ranges ($[\![n]\!] = \{1, 2, \ldots, n\}$), and all bracings $\{\ldots\}$ will be sets.

Throughout, calligraphic letters will always be monoids: $\mathcal{M}_5, \mathcal{PB}r_n, \ldots$; the Greek lowercase α and β, and occasionally γ and η, will be elements of one of these monoids (usually $\mathcal{M}_n$). By contrast, p and q and variations will often denote partitioned binary relations. Boldface lower-case Latin letters will be tuples or sequences: $\mathbf{a} = a_0, a_1, \ldots$. Script-uppercase letters such as will be cell complexes ($\mathscr{E}, \ldots$) or Green's relations on a semigroup ($\mathscr{H}$, $\mathscr{L}$, $\mathscr{R}$, $\mathscr{D}$, $\mathscr{J}$).

Because $\mathscr{J}$ and $\mathcal{J}$ look similar, and because $\mathscr{D}$ and $\mathscr{J}$ coincide for finite semigroups (Lemma 1.2.24), we'll talk about $\mathscr{D}$ rather than $\mathscr{J}$, even though conceptually we'll often talk about the $\mathscr{D}$-classes in terms of two-sided ideals.

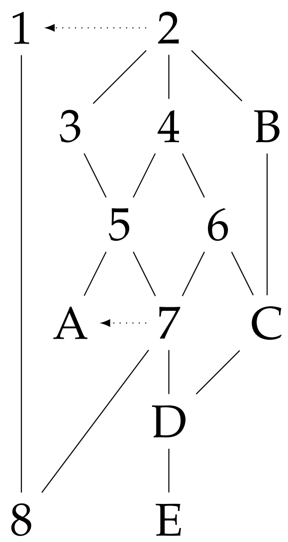
Upper-case Latin-alphabet letters in italic may mean one of several things: S and T may be sets or nonspecific semigroups; L will often be a language and C a prefix code; U is a generating function. Upper-case typewriter bold letters ($\mathtt{S}$, $\mathtt{T}$) are nonterminals letters in a grammar; $\mathtt{S}$ is always the start symbol.

Likewise, lower-case Latin-alphabet letters may mean several things. Upright sans-serif letters are statistical data: $\mathsf{p}(\alpha)$. Italic serif letters are often numbers (n,r,k,...), but may be words (u,v,w), or elements of a set or semigroup (s, t).

Upper-case Greek may be alphabets, and Γ is usually an interface graph. If there is a $\vee$ or $\wedge$ attached to Γ, that specifies the side of the interface; the orientation of the edges of interest are: $\vee$ pointing down, for instance, represents all edges and half-edges/stubs pointing down.

Document Map

The diagram below describes the chapters and how they relate to one another in terms of prerequisites.



Part I

First Steps

Chapter 1

Preliminaries

1.1 Sets, Combinatorial Structures

1.1.1 Sets and Number Systems

The symbol ":=" will be taken as assignment, so that $s := 2$ will mean "let s be 2."

We will make frequent references to the standard numeral systems in this thesis. Note that $0 \notin \mathbb{N}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Numbers henceforth will almost always be positive integers, occasionally zero, seldom negative and never non-integer unless specified. For instance $k \leq n$ will usually mean k is a positive integer between 1 and n, inclusive.

We'll write

$$[\![n]\!] := \{1, 2, \ldots, n\}.$$

For $x \in \mathbb{R}$, we write $\lfloor x \rfloor$ for the *floor* of x, which is the greatest integer not exceeding x. For a set $X \subseteq \mathbb{R}$, we write $\inf X$ (resp. $\sup X$) for the largest (smallest) number not exceeding (less than) any member of X, if it exists.

We conventionally write

$$f : A \to B : a \mapsto b_a$$

to say that f is a function from A to B, mapping a to b_a. We use the term *domain* for the set A, *codomain* for the set B, *range* for the subset of B of values attained by the function, and (in the case where f is partially-defined) the *preimage* for the set of values in A on which f is defined.

Definition 1.1.1. A function $f : X \to Y$ is *injective* if $f(x) = f(y)$ implies $x = y$ for all $x, y \in X$; f is *surjective* if for all $z \in Y$ there is $x \in X$ with $f(x) = z$. A *bijective* function is one which is injective and surjective. We may use the respective terms *injection* (also *embedding*, particularly when the map preserves additional structure), *surjection* (also *quotient* in the presence of additional preserved structure) or *bijection* (also *isomorphism* in the presence of additional structure).

In other words, injective maps separate elements and surjective maps fill their codomains.

Definition 1.1.2. An *endofunction* of X is a function $f : X \to X$. If an endofunction is bijective, we use the term *permutation*.

Given $X \subseteq A$, $Y \subseteq B$, we write

$$f(X) := \{b \in B : f(x) = b \text{ for some } x \in X\},$$
$$f^{-1}(Y) := \{a \in A : f(a) \text{ is defined}\},$$

for the image and preimage.

Overwhelmingly, we'll write only the correspondence between the sets in this notation, suppressing the specification of how elements map, but occasionally will only specify how elements map. We also will use the notation $a \mapsto b_a$ to describe mappings anonymously, where the domain is clear.

If $f : X \to X$ and $U \subseteq X$, we say that f *fixes* U if $f(u) \in U$ for any $u \in U$, that $x \in X$ is a *fixed point* of f if $f(x) = x$, and that U is *fixed pointwise* by f if every $u \in U$ is a fixed point.

Given a set $\mathcal{I}$, and a set A_i for each $i \in \mathcal{I}$, we talk about the *family* $(A_i)_{i \in \mathcal{I}}$ *of sets* (indexed by the *indexing set* $\mathcal{I}$); we use the notation $(A_i)_i$ where $\mathcal{I}$ is understood, in addition suppressing the index i where convenient. The intersection and union of a family of sets is defined:

$$\bigcup_{i \in \mathcal{I}} A_i := \{a : a \in A_i \text{ for some } i \in \mathcal{I}\}$$
$$\bigcap_{i \in \mathcal{I}} A_i := \{a : a \in A_i \text{ for each } i \in \mathcal{I}\}.$$

For convenience, we'll usually work with $\mathcal{I} = \mathbb{N}_0$ or subsets thereof; we don't need larger indexing sets.

Definition 1.1.3. A family $(A_i)_i$ of sets is *pairwise disjoint* is $i \neq j$ implies $A_i \cap A_j = \varnothing$. A *partition* of X is a family of sets whose union is X and which are pairwise disjoint. The *quotient* of X by a partition is simply the set of sets $\{A_i\}$ in the partition; we conventionally identify the set A_i with some element as convenient. These sets are referred to as *equivalence classes*.

Definition 1.1.4. The Cartesian product $\prod_{i \in \mathcal{I}} A_i$ of an $\mathcal{I}$-indexed family (A_i) of sets is the set of maps f from $\mathcal{I}$ into the union $\bigcup_i A_i$ such that each $f(i) \in A_i$.

Where $\mathcal{I} = \mathbb{N}_0$ or $\mathcal{I} = [\![n]\!]$, we'll write mappings as sequences $(a_1, a_2, \ldots)$ or tuples $(a_1, \ldots, a_n)$ where in each case $a_i := f(i)$. Where $\mathcal{I} = [\![n]\!]$ is finite, we'll often write

$$\prod_{i \in \mathcal{I}} A_i = A_1 \times A_2 \times \cdots \times A_n.$$

Indeed, if $A := A_1 = A_2 = \cdots A_n$, then its convenient to write A^n.

1.1.2 Graphs

Definition 1.1.5 (Graph)**.** A *graph* is a set $\Gamma = (V, E)$ with two kinds of elements:

- *vertices* in V, which could be anything at all, but will often correspond to some labelling or enumeration;
- *edges* in E, which are ordered pairs of elements.

In an *undirected* graph, we disregard order on the edges; in a *directed* graph we care about order. The significance of this will become apparent later.

In practice, we may label edges by symbols or ordered pairs rather than unordered pairs. We will often say that the graph Γ is *on V* or similar. We also notationally conflate the graph Γ with the union where convenient, writing $e \in \Gamma$ for instance.

Definition 1.1.6. A*(n undirected) path* on a graph $\Gamma = (V, E)$ from u to v is a sequence of vertices of the form

$$u = v_0, v_1, v_2, \ldots, v_{l-1}, v_l = v,$$

such that for each i, (v_{i-1}, v_i) is an edge or the reverse of an edge.

A path is directed if every edge appears in the correct orientation. The number l is the length of the path. A path of either type is called a *cycle* if it starts and ends at the same vertex, that is $u = v$; we often say the path is a *cycle at u*.

Definition 1.1.7. Two vertices in an undirected graph are *connected* if there is a path from one to the other, and *adjacent* if connected by an edge.

In the directed setting, one has two symmetric notions of connectedness and one asymmetric notion. Two elements are *weakly connected* if there is an undirected path between. Two vertices are *strongly connected* if there is a cycle at one passing through the other. There is an asymmetric form of connectedness; we'll say that u is *upstream* of v (and v is *downstream* of u) if there is a directed path from u to v.

Definition 1.1.8. The *valency* of a vertex in a graph is the number of edges it meets. The *in-valency* (respectively *out-valency*) of a vertex v is the number of edges ending at (respectively starting at) v.

Definition 1.1.9 (Partition)**.** Let X be a set. A *partition* of the set X is a decomposition

$$X = P_1 \sqcup P_2 \sqcup \cdots \sqcup P_k$$

where the P_i are disjoint. The *size* of P is k, and the *parts* or *classes* of P are the sets P_i.

If the parts don't exceed two in size, we call P a *partial matching*. If the parts are all size two, we use the term *(perfect) matching*.

Let's assume X is ordered. We say that a partition is *noncrossing* if, whenever $i < j < k < l$ with i and k in the same class, j and l are in different classes unless the four elements share a class.

Partitions may be depicted as graphs, by drawing the vertex sets as points and adding edges to make the parts and connected components agree. If $X = [\![n]\!]$ and we write the numbers $1, \ldots, n$ around a circle in order, drawing edges through the interior between elements in the same part, then the graph can be drawn without edges crossing precisely if the partition is noncrossing, hence the terminology.

1.1.3 Relations

Any subset $\varrho \subseteq X \times X$ is called a *(binary) relation* on X. We will often write $x\varrho y$ as a convenient shorthand for $(x, y) \in \varrho$. We often suppress the adjective "binary," as all relations henceforth will be binary.

We say that a relation ϱ *preserves* some property $\mathscr{P}$ if x having $\mathscr{P}$ implies y does also for every $x \ \varrho \ y$. Dually, ϱ *reflects* property $\mathscr{P}$ if the above holds for every y satisfying $y\varrho x$.

Given two relations $\varrho, \varsigma \subseteq X \times X$, we can write $x \ \varrho \ y \ \varsigma \ z$ as a shorthand for $(x, y) \in \varrho$ and $(y, z) \in \varsigma$. An inductive form

$$x_0 \ \varrho_1 \ x_1 \ \varrho_2 \ x_2 \ \varrho_3 \cdots \varrho_{n-1} \ x_{n-1} \ \varrho_n \ x_n, \tag{1.1}$$

meaning $x_{i-1}\varrho_i x_i$ simultaneously for $i \in [\![n]\!]$ will be useful. We can form the composition

$$\varsigma \circ \varrho = \varrho\varsigma = \{(x, z) \in X \times X : \text{there is } y \in X \text{ with } x \ \varrho \ y \ \varsigma \ z\}. \tag{1.2}$$

Again, this composition can be extended to arbitrary lengths. The set of (x_0, x_n) for which a sequence of the form given in (1.1) exists is the composition of $\varrho_n, \ldots, \varrho_1$.

For example, the relationship "less than" can be composed with itself. The comparison $1 < 2 < 3 < 4$ is perfectly valid; and we might say that "$1 < \circ < \circ < 4$".

Where convenient, we will omit reference to an understood context. For example, where the set X is known in (1.2), we may write instead

$$\varrho\varsigma = \{(x,z) : \text{there is } y \text{ with } x\varrho y\varsigma z\}.$$

A relation $\varrho \subseteq X \times X$ is:

- *reflexive* if it contains the diagonal $\Delta_X = \{(x,x) : x \in X\}$;
- *irreflexive* or *strict* if it does not intersect the diagonal;
- *(a)symmetric* if it contains (respectively, does not intersect) its reverse $\varrho^{\mathrm{rev}} = \{(y,x) : x\varrho y\}$ (outside the diagonal);
- *transitive* if it contains its compositional square $\varrho \circ \varrho$;
- *total* if any two elements can be compared: at least one of $x\varrho y$ and $y\varrho x$ holds;
- a *preorder** if reflexive and transitive
- a *partial order (total order)* if a (total) antisymmetric preorder;
- an *equivalence relation* is a symmetric preorder;
- a *functional* † if $x\varrho z$ and $y\varrho z$ implies $x = y$;
- the *(co)restriction* of ς to $Y \subseteq X$ if for every pair $y, z \in X$, we have $y \ \varrho \ z$ precisely if $y \ \varsigma \ z$ and y (respectively z) is in Y);
- the *birestriction* of ς to Y if the corestriction and restriction;
- the *reverse* of ς if $x \ \varrho \ y$ whenever $y \ \varsigma \ x$.

Note that many functions of interest will not map from a set to itself. While this is not important in practice, the graphs of such functions are not relations of the

*Some others prefer the term quasi-order, with or without hyphenation.

†Functional relations are graphs of partial functions; total functional relations are functions.

restricted type discussed above; differing notation and treatment prevent this from becoming an issue.

The *support* of a relation $\varrho \subseteq X \times X$ is the set

$$\mathrm{supp}(\varrho) = \{x \in X : x \ \varrho \ y \text{ for some } y \in X\}.$$

Clearly ϱ is total if $\mathrm{supp}(\varrho) = X$.

Proposition 1.1.10. *Given an equivalence relation, one can consider the subsets of X of elements which are related to one another. These are the* equivalence classes*.*

A set P with respectively a preorder, partial order, or total order is said to be *preordered*, *partially ordered* or *(totally) ordered* by the relation. We refer to preordered sets, partially ordered sets (*posets*) and ordered sets (*chains*). In each case, we use the relation $\leq$ and its usual variations to describe the order, its reverse, its strict (irreflexive) counterpart unless stated otherwise.

Definition 1.1.11. Given a poset P, we say that a function $\mathrm{cl} : P \to P$ is a *closure operator* on P if

$$x \leq \mathrm{cl}(y) \Leftrightarrow \mathrm{cl}(x) \leq \mathrm{cl}(y)$$

The identity map on a poset is always a closure operator, but some posets admit no other *nontrivial* examples; any antichain has no nontrivial closure operators for example. For a slightly more instructive example, consider the set $\{1,2,3\}$ ordered by division. If $\mathrm{cl}(1) = 3$ then we have $2, 3 \leq \mathrm{cl}(2)$ in this order, which is impossible.

Proposition 1.1.12. *Closure operators are idempotent ($cl(cl(p)) = cl(p)$), nondecreasing and extensive $x \leq cl(x)$. Any such function acting on P is a closure operator.*

Given a set $S \subseteq P$, the *downset* $\downarrow S$ is the set of elements below some $s \in S$. The set $T \subseteq P$ is a downset if $T = \downarrow S$ for some S (T will suffice). A map is nondecreasing* if the preimage of a downset is a downset.

*The terms monotone and order-preserving are used unambiguously in the literature. The term increasing also features, but this for us means injective and nondecreasing.

A downset is principal if it equals $\downarrow \{s\}$ for some s; we write $\downarrow b$. A map from a poset P to Q is called *residuated* if preimages of principal downsets are principal downsets themselves.

Proposition 1.1.13. *An endofunction of a poset P is a closure operator precisely if it is residuated.*

Definition 1.1.14. Given a set $Q \subseteq P$ of elements, an element $p \in P$ is called an upper bound for Q (written $p \geq S$) if for all $q \in Q$, $q \leq p$. An element is called a *least upper bound* if no lower elements provide an upper bound for S. We call a least upper bound p the *join* of S (written $p = \vee S$) if every upper bound u for S.

A poset is called a (complete) join-semilattice is every pair (resp. every subset) has a join. (Greatest) lower bounds are defined dually, and we refer to *meets* as dual to joins. A *lattice* is simultaneously a meet- and join-semilattice, and is *complete* if complete both as a meet- and join-semilattice.

Semilattices and lattices are useful, as comparison can be characterised by algebraic operations, something that is not the case in general posets.

The set 2^X of subsets of X is a lattice under intersection and union.

Definition 1.1.15. A *closure system* in a lattice 2^X of subsets is a subfamily which contains X as a member and contains arbitrary intersections of members.

Proposition 1.1.16. *To each closure system $\mathcal{C} \subseteq 2^X$, there is an associated closure operator.*

In fact, all closure operators arise this way; the family

$$\mathcal{C}l = \{\mathrm{cl}(()Y) : Y \subseteq X\}$$

is a closure system. We call this operator *$\mathcal{C}$-generation* and can recover the operator as demonstrated below.

Proof. Let $\mathcal{C} \subseteq 2^X$ be a closure system. Fix $Y \subseteq X$ and denote by $\mathcal{C}_Y$ the family of closed subsets containing Y. The intersection $\bar{Y} = \bigcap \mathcal{C}_Y$ is in $\mathcal{C}$, and contains Y; indeed every $C \in \mathcal{C}_Y$ contains $\bar{Y}$ also. Fixing $Y \subseteq Z \subseteq X$, note that $\mathcal{C}_Z \subseteq \mathcal{C}_Y$ so $\bar{Y} \subseteq \bar{Z}$.

Therefore the map $Y \mapsto \bar{Y}$ is idempotent, nondecreasing and extensive. □

Corollary 1.1.17. *A closure system $C \subseteq 2^X$ naturally forms a complete lattice under set-theoretic intersection and taking closure of unions.*

Many substructural features we'll encounter naturally in what follows will arise as closure systems, and the above results allow us to treat them uniformly, results looking essentially as follows.

Proposition 1.1.18. *The set $\mathcal{E} \subseteq X \times X$ of equivalence relations on X forms a closure system.*

1.1.4 Enumerative Combinatorics

Definition 1.1.19. A *sequence* in X is a function $a : \mathbb{N} \to X$, usually written

$$\begin{aligned}(a_n)_n &= (a_1, a_2, a_3, \dots) \\ &= a_1, a_2, a_3, \dots\end{aligned}$$

as a tuple of images in order, often with brackets suppressed.

Occasionally, we use sequences indexed from zero rather than one.

Definition 1.1.20. Given a numerical sequence, that is a sequence of numbers, we can form its *ordinary (or power series) generating function* (OGF) as follows:

$$f(z) := \sum_{n=0}^{\infty} = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots.$$

Note that other kinds of generating functions exist, but we don't make any use of them in the sequel. Note also that a "generating function" may not be a function at all, and for this reason some authors prefer the term *generating series*, either to refer to OGFs or to the aforementioned. The series $\sum_{n=0}^{\infty}(n! \cdot x^n)$ is the generating function of the factorial sequence given by

$$0! = 1, \qquad n! = n \cdot (n-1)!$$

for integers $n \geq 1$. Standard techniques learned in an undergraduate course in mathematical analysis will show that this is undefined for any nonzero real number, and this is left to the reader as an exercise.

A sequence may be mutliply-indexed, in which case we write $(a_{n_1,n_2,\ldots,n_r})$ or similar. We may instead equivalently of sequences indexed by tuples; if $\mathbf{n} = (n_1, n_2, \ldots, n_r)$ then $a_{\mathbf{n}} := a_{n_1,n_2,\ldots,n_r}$ and the sequence $(a_{\mathbf{n}}) := (a_{n_1,n_2,\ldots,n_r})$.

For example, the doubly-indexed sequence

$$b_{n,k} = \binom{n}{k}$$

of binomial coefficients. The sequence may only be defined for certain values; it's fruitful to regard the binomial coefficients as being defined for all positive n and k, but for the purposes of drawing Pascal's triangle, or perhaps for establishing a general recurrence relation, one may require that $0 \leq k \leq n$.

Similarly, generating functions for multiply-indexed sequences can be used. We write

$$\begin{aligned}
f(z_1, z_2, \cdots, z_r) &= \sum_{\mathbf{n}=(n_1,\cdots,n_r)} a_{\mathbf{n}} z_1^{n_1} z_2^{n_2} \cdots z_r^{n_r} \\
&= a_{0,0,\ldots,0} + a_{1,0,0,\ldots,0} z_1 + a_{0,1,0,\ldots,0} z_2 + \cdots + a_{0,\ldots,0,1} z_r \\
&\quad + a_{2,0,\ldots,0} z_1^2 + a_{1,1,0,\ldots,0} z_1 z_2 + \cdots + a_{0,\ldots,0,1,1} z_{r-1} z_r + a_{0,\ldots,0,2} z_r^2 \\
&\quad \cdots + a_{n,0,\ldots,0} z_1^n + a_{n-1,1,0,\ldots,0} z_1^{n-1} z_2 + \cdots
\end{aligned}$$

where the sum runs over permissible tuples of values.

Definition 1.1.21. A *multiset* S indexed by $\mathcal{I}$ with elements in X is an equivalence class of tuples or sequences indexed by $\mathcal{I}$, where the equivalence classes comprise all those sequences $(a_i)_i$ so that (a_i) and (b_i) are in the same equivalence class (are *equivalent*), written $(a_i) \sim (b_i)$, whenever $b_i = a_{f(i)}$ for some permitation of $\mathcal{I}$ and for all i.

This technical definition obfuscates a simple concept. Namely, a multiset is a set with elements possibly appearing multiple times, and that cannot distinguish order or elements, much as a set cannot.

For example, the following object is a multiset:

$$\{1, 1, 2, 2, 2, 2, 3, 5, 6, 9, 9\} = \{1, 2, 3, 5, 6, 9, 1, 2, 6, 2, 2\}$$

The first representation emphasises *order*, and is the notation we will tend to favour. The second emphasises *multiplicity*.

Definition 1.1.22. The *multiplicity* $m_x(S)$ in a (finite) multiset S of elements from X, of x is the number of elements of S equal to x.

A set is therefore a multiset whose elements all have multiplicity one.

Definition 1.1.23. A *sequence of multiplicities* (for the multiset S with elements in X) is a sequence of numbers $(m_{x_i}(S))_i$ where (x_i) is some way of ordering X.

We conventionally choose X as the set of elements that appear, and choose as the indexing set $[\![n]\!]$ where n is the sum of the multiplicities, also known as the *cardinality*.

Definition 1.1.24. The *multinomial coefficient* is the number

$$\binom{n}{m_1, m_2, \ldots, m_k} := \frac{n!}{m_1! \cdot m_2! \cdots m_k!} \cdot \frac{1}{(n - \sum_i m_i)!}$$

where $0! = 1$ and $n! = n \cdot (n-1)!$ for each positive integer $n \in \mathbb{N}$.

When the sequence of denominators is one number we use the term *binomial*.

Proposition 1.1.25. *If S is a multiset of cardinality n with $m_1, m_2, \ldots, m_k$ a sequence of multiplicities then the number of tuples representing S is the multinomial coefficient* $\binom{n}{m_1,\ldots,m_k}$.

1.1.5 Asymptotics of sequences

Definition 1.1.26. We say that $a(n) \ll f(n)$ if there is a constant $M > 0$ and some $x_0 > 0$ such that

$$a(x) \leq M \cdot f(x) \quad \text{for all } x \geq x_0.$$

Say that $a(n) \gg f(n)$ if $f(n) \ll a(n)$, and that $a(n) \asymp f(n)$ if $a(n) \ll f(n)$ and $a(n) \gg f(n)$.

Definition 1.1.27. Denote by $O(f) = O(f(n))$ the set of all a whereby $a(n) \ll a(n)$.

Rather than writing $a \in O(f)$ it's conventional to write $a = O(f)$, as in there is a representative of $O(f)$ equal to a. This notation cleanly extends to function multiplication and composition, for example we take $a = f \cdot O(g)$ to mean $a = f \cdot h$ for some $h \ll g$.

1.1.6 Planar Combinatorics

Definition 1.1.28. The *closed upper (respectively lower) half-plane* is the set $\mathbb{R} \times \mathbb{R}_{\geq 0}$ of points in the plane $\mathbb{R}^2$ with nonnegative (resp. nonpositive) y-cooridinate. An *upper (lower) arch configutarion* on the pairs $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ (a_i and b_i pairwise distinct) is the family of semicircles in the upper (lower) half-plane bounded respectively by the pairs of points $(a_i, 0)$ and $(b_i, 0)$. We say that such a family is *noncrossing* if we write

$$m_i := \min(a_i, b_i), \qquad M_i := \max(a_i, b_i),$$

we do not have $m_i < m_j < M_i < M_j$ for $i \neq j$.

We ordinarily refer to arch configurations without reference to the upper or lower half-plane.

Proposition 1.1.29. *The noncrossing condition is equivalent to the semicircles being pairwise disjoint.*

Definition 1.1.30. A *meander sequence* of order n is a sequence $\mu_1, \mu_2, \ldots, \mu_{2n}$ containing each element of $[\![2n]\!]$ exactly once, and such that the respective upper and lower arch configurations

$$\{(\mu_1, \mu_2), (\mu_3, \mu_4), \ldots, (\mu_{2n-1}, \mu_{2n})\}, \qquad \{(\mu_2, \mu_3), (\mu_4, \mu_5), \ldots, (\mu_{2n-2}, \mu_{2n-1}), (\mu_{2n}, \mu_1)\}$$

are noncrossing. The *meander* determined by this sequence is simply the union of the above upper and lower arch configutarions.

There are several ways to permute this sequence while preserving the meander defined; clearly reflecting by swapping each μ_i with μ_{2n+1-i} preserve all the pairs above, reversing the ordering in each case. We could also cyclically

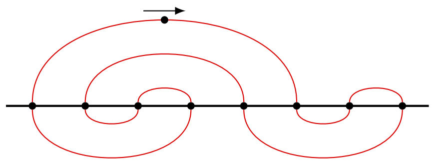


Figure 1.1: The meander of order 4 given by $1,6,7,8,5,2,3,4$ with basepoint and orientation shown.

shift by two, which entails replacing μ_{2n-1} and μ_{2n} with μ_1 and μ_2, and μ_i with μ_{i+2} for $i < 2n-1$. In fact this is essentially all one can do to the sequence while preserving the meander.

Proposition 1.1.31. *If $\mu_1, \mu_2, \ldots, \mu_{2n}$ and $\nu_1, \nu_2, \ldots, \nu_{2n}$ are two sequences representing the same meander then there is a sequence of length $k \geq 0$*

$$(\varrho_{0,1}, \varrho_{0,1}, \ldots, \varrho_{0,2n}),\ (\varrho_{1,1}, \varrho_{1,1}, \ldots, \varrho_{1,2n})\ \ldots,\ (\varrho_{k,1}, \varrho_{k,1}, \ldots, \varrho_{k,2n})$$

where $\varrho_{0,i} = \mu_i$ and $\varrho_{k,i} = \nu_i$ where the sequences $(\varrho_{r,i})_i$ and $(\varrho_{r+1,i})_i$ differ either by a reflection or a cyclic shift by two.

The n^{th} *meandric number* m_n is the number of distinct meanders of order n. Calculating these numbers is the *meander problem* [1], a difficult classical problem in combinatorics with roots in studies of Poincaré [64] in analytic and differential geometry. Different notions, mostly more-or-less equivalent, appear in these articles. Lacroix [11] dicusses several variations; we shall not.

Definition 1.1.32. A *circular matching* of order n is a partial matching whose n distinct labeller vertices lie on a circle, and whose edges are not loops and can be drawn as straght lines between the vertices they connect without any crossing. A circular matching is *in standard form* if the circle is the unit circle, the labels of the vertices are the numbers 1 to n, with n place at $(1,0)$ and the rest of the vertices spaced evenly with labels increasing anticlockwise except between n and 1.

Conventionally, we draw these using curves or circular arcs as in Figure 1.2. The number of distinct standard form matchings is the Motzkin number M_n,

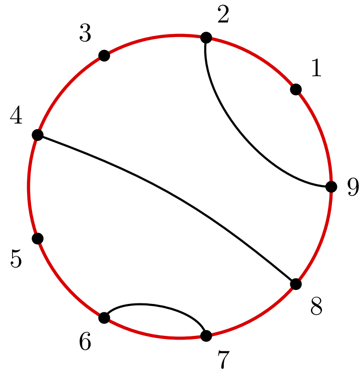


Figure 1.2: A circular matching of order 9. This is not a perfect matching nor a subgraph of any other circular matching.

which can be calculated recursively as

$$M_0 = M_1 = 1, \qquad M_n = M_{n-1} + \sum_{i=0}^{n-2} M_i M_{n-2-i}.$$

To see this, note that one can decompose a circular matching of order n in two ways. Either the point n is isolated, or n connects to $i+1 \in [\![n-1]\!]$. In the former case, one may remove it obtaining a matching of order one less. In the latter, the points between 1 and i don't connect to those from $i+2$ to $n-1$ by the noncrossing property. Therefore, we may regard these as two separate circular matchings, or order i and $n-2-i$.

The first few entries of this sequence are

$$1, 1, 2, 4, 9, 21, 51, 127, 323, \ldots$$

Definition 1.1.33. A circular matching of order $2n$ is *completable* if it is a subgraph of some perfect circular matching. A completable matching is *contiguous* if there exist a sequence

$$0 = u_0 < u_1 < \cdots < u_{l-1} < u_l = 2n$$

with each u_i even, and so that each u_i and $u_{i-1}+1$ are incident on a common edge. The number l is the *length* of the contiguous matching, and (u_i) its *defining sequence*.

A circular matching is completable in other words if one can draw in edges until no vertex is isolated, without needing to draw any edges crossing inside the circle. It's fairly clear that this is a hereditary property, and one which is invariant under adding a constant to the indices.

We define the *partial Catalan numbers* P_n as the number of nonisomorphic completable graphs of order $2n$. The *contiguous partial Catalan numbers* Q_n and R_{nl} are the number of nonisomorphic contiguous matchings there are of order $2n$, and respectively of length l.

Proposition 1.1.34. *The sequences P_n Q_n and R_{nl} satisfy*

$$P_0 = Q_0 = Q_1 = R_{nn} = 1, \qquad R_{n+1,0} = 0, \qquad R_{n+1,1} = P_n,$$
$$R_{n+1,l} = \sum_{i=0}^{n} P_{n-i} \cdot R_{2i,l-1}, \qquad Q_{n+1} = \sum_{l=1}^{n+1} R_{n+1,l},$$
$$P_{n+1} = \sum_{i=0}^{n+1} P_i \cdot P_{n-i-1}.$$

In the proof that follows, all matchings are compatible circular matchings.

Proof. The only circular matching of order 0 is empty, and is perfect and hence conriguous, so $P_0 = Q_0 = R_{0,0} = 1$ as needed. The only contiguous matching of order 2 is the prefect matching, hence $Q_1 = 1$. There are no empty contiguous matchings of nonzero order and zero length, so $R_{n+1,0} = 0$, and the only contiguous matching of length n and order $2n$ is the perfect matching

$$1 \sim 2, 3 \sim 4, \ldots, 2n-1 \sim 2n,$$

where $a \sim b$ denotes adjacency along an edge, giving the final boundary value $R_{n,n} = 1$.

Now let us consider a contiguous matching of length 1. Then $u_0 = 0$ and $u_1 = 2n$, meaning that $1 \sim 2n$. The remaining vertices 2 to $2n-1$ may be relabelled in order from 1 to $2n-2$. Every completable matching on the remaining vertices gives rise to a different contiguous matching of length 1, so $R_{n,1} = P_{n-1}$. Relabelling is a fundamental technique; for some interval $[i, j]$ of integers, we will refer simply to a matching on $[i,j]$ rather than considering these "up to relabelling in order."

Fix a contiguous matching Γ of length l and order $2n+2$. Then there is a contiguous matching of length $l-1$ on the elements 1 to i_{l-1}. Indeed, fixing i_{l-1}, there are $R_{n-i_{l-1},1} \cdot R_{i_{l-1},l-1} = P_{n-i_{l-1}-1} \cdot R_{i_{l-1},l-1}$ of these. Summing over possibly $i_{l-1} = 2i$, we get

$$R_{n+1,l} = \sum_{i=0}^{n} P_{n-i} \cdot R_{2i,l-1}.$$

Now let us consider a completable matching Γ of order $2n+2$. Our analysis splits into four mutually exclusive cases.

In the first case, neither $2n+2$ nor $2n+1$ is incident on an edge, in which case the points $1,\ldots,2n$ may form any of the P_n circular partitions.

In the second case, let's assume that $2n$ is incident on an edge with $2i+1$ for some $0 \le i \le n$. Then there are completable matchings on the points 1 to $2i$ and the $2(n-i)$ points from $2i+2$ to $2n+1$. There are $\sum_{i=0}^{n} P_i P_{n-i}$ completable matchings possible in this case.

The third case has $2n+1$ incident on an edge, but $2n+2$ isolated. Write $2i$ for the even integer edge-adjacent to $2n+1$ and $2m+1$ the smallest integer (possibly zero) such that the induced subgraph of Γ on the vertices in $[\![2m]\!]$ is contiguous. Then the induced subgraph on the $2i-2m-2$ vertices $2m+2$ to $2i-1$ forms a completable matching, as does that on those $2n-2i$ vertices from $2i+1$ to $2n$. Fixing i and m, we see there are $P_{n-i}P_{i-m-1}Q_m$ of these. Allowing i and m to vary, we obtain

$$P_{n+1} = \sum_{i=1}^{n} \sum_{m=0}^{i-1} P_{n-i}P_{i-m-1}Q_m. \qquad \square$$

a

Definition 1.1.35. A ballot sequence of length $2n$ is a sequence $(a_1 \cdots a_{2n})$ with each $a_i = \pm 1$ and each partial sum $\sum_{i=1}^{j} a_i$ nonnegative.

Ballot sequences are enumerated (see [72]) by the Catalan numbers C_n.

1.2 Semigroups

1.2.1 Basic Theory

Definition 1.2.1. A *magma* is a set S equipped with a *multiplication* map $(s,t) \mapsto s \cdot t$ associating to each pair of elements of S their *product*. A magma whose multiplication obeys the associative law, that is satisfies

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

for every choice of three elements $a, b, c \in S$, is called a *semigroup*.

A magma is *unital* if there is an element $e \in S$ with $e \cdot s = s \cdot e = s$ for every $s \in S$. The element e is called the *identity* for S. A unital semigroup is a *monoid*.

A *group* is a monoid S in which every element $g \in S$ has a *(global) inverse* $g^{-1} \in S$, that is to say an element for whom the products $g \cdot g^{-1} = g^{-1} = e$ equal the identity.

Given a magma S, it'll often be convenient to refer to the *opposite magma* S^{op}. Many concepts discussed will arise in dual pairs, one of which can be encoded with respect to S and the other respecting S^{op}. We also write $S^1 := S \cup \{1\}$ for a semigroup with adjoined element 1 acting as identity.

Definition 1.2.2. A subset $T \subseteq S$ is called a subsemigroup if every pair $s, t \in T$ of elements of T multiplies to an element $st \in T$. This is caled being *closed under multiplication*.

In practice we will omit the multiplication symbol, writing $st := s \cdot t$ for example.

We write $T \leq S$, and $S < T$ if the containment is proper, that is $S \neq T$. If 1 is the identity of the monoid M then S is a *submonoid* if $1 \in S$.

Proposition 1.2.3. *Let S be a semigroup. The intersection $T = \bigcap_{i \in \mathcal{I}} T_i$ of a family of subsemigroups of S is itself a subsemigroup of S.*

In other words, subsemigroups of S form a closure system.

Proof. If $s, t \in T$ then $s, t \in T_i$ for some i, so $st \in T_i$ and hence $st \in T$. $\square$

Corollary 1.2.4. *Let S be a semigroup and $X \subseteq S$ a subset, with $(T_i)_{i \in \mathcal{I}}$ the family of subsemigroups satisfying $X \subseteq T_i \subseteq S$. Then the intersection T is a semigroup containing X, and furthermore every subsemigroup of S which contains X must also contains T.*

This allows us to make the following definition.

Definition 1.2.5. Let $X \subseteq S$ be a subset. Denote by $\langle X \rangle$ the smallest subsemigroup of S and containing X. This is the subsemigroup *generated by* X; we say that X *generates* $\langle X \rangle$.

The above results and definitions can be ported to the domain of submonoids by assuming in each case that the subsemigroups contain the identity.

Definition 1.2.6. A map $f : S \to T$ between semigroups is a *homomorphism* if it respects multiplication. That is, $f(s) \cdot f(t) = f(s \cdot t)$ for all $s, t \in T$.

For monoids, we will require in addition that $f(1) = 1$; dropping this requirement we'll use the term *semigroup homomorphism*.

Definition 1.2.7. If $f : S \to T$ is a bijective homomorphism, then we say f is an *isomorphism*, and that the semigroups S and T are *isomorphic*.

Definition 1.2.8. An *action (on the right)* of a semigroup S on a set X is a map

$$\phi : X \times S \longrightarrow X : (x, s) \mapsto x^m$$

which is compatible with the semigroup structure, in the sense that $x^{(s \cdot t)} = (x^s)^t =: x^{st}$, and $x^1 = x$ if there is an identity element $1 \in S$.

The notation is suggestive of the exponentiation of numbers, and the usual laws of exponentiation are satisfied except that x^{st} is not guaranteed to equal x^{ts} unless we know that s and t commute.

In fact we can lift this defining map ϕ to a homomorphism

$$\hat{\phi} : S \longrightarrow X^X : s \mapsto \phi_s$$

where X^X is the monoid of maps acting on the right of X, and where $\phi_m : x \mapsto x^m$.

Proposition 1.2.9. *Every semigroup acts on itself via the right-multiplication mapping* $(s,t) \mapsto st$.

Corollary 1.2.10. *Let* $X \subseteq S$ *be a subset of a semigroup. Then* S *acts on the right ideal* XS *by right multiplication.*

Corollary 1.2.11. *Let* $T \leq S$ *be a subsemigroup. Then* T *acts on* S *by right multiplication.*

1.2.2 Special Elements

Definition 1.2.12. An element $s \in S$ of a semigroup is respectively:

- (an) *idempotent* if $s^2 = s$;
- a *left or right identity* if $st = t$, respectively $ts = t$, for all $t \in S$;
- a *left or right zero** if $st = t$, respectively $ts = s$, for all $t \in S$;
- a *zero or identity* if both left and right zero or identity;
- *regular* if there is f satisfying $efe = e$;
- a *(local) inverse* for t if $sts = s$ and $tst = t$;
- a *divisor* of $t \in S$ if $t = usv$ for some $u, v \in S^1$;
- a *zero divisor* if some nonzero $u, v \in S^1$ exist such that $usv = 0$.

We say that two elements $s, t \in S$ *commute* if $st = ts$.

The term *inverse* is used in two ways. In the context of a non-group semigroup, we will always mean local inverses — global inverses are not only rare in the wider world of semigroups, but don't exist except at the identity 1 in most examples we'll study here. In the context of groups, a local inverse for an element is the global inverse.

*We'll refer to *one-sided* identity and zero to mean right or left identity or zero.

Proposition 1.2.13. *Every regular element has an inverse.*

Proof. Let $a = axa$ be regular. Then xax is an inverse for a:

$$a(xax)a = ax(axa) = axa = a,$$
$$(xax)a(xax) = x(axaxa)x = xax. \qquad \square$$

We will use the characterisation of regularity as meaning "having an inverse."

Definition 1.2.14. A semigroup is *regular* if every element has an inverse, and *inverse* if inverses are unique. A semigroup is *commutative* if every pair of elements commute; a commutative group is said to be *abelian*.

Definition 1.2.15. The *natural partial order* on the idempotents of a semigroup is the relation given by

$$e \leq f \iff ef = fe = e.$$

That this is a partial order is not difficult, see [10], pp.23–24.

Definition 1.2.16. Fix S a semigroup and $x \in S$. If the *monogenic* semigroup $\langle x \rangle$ generated by the single element x is finite, then there is n such that $x^n = x^{2n}$. This is the *idempotent power* of x, often written x^ω.

If $x^\omega = x \cdot x^\omega$, we say that x is *aperiodic*, and we say that S is aperiodic if every element is aperiodic.

1.2.3 Ideals and Green's Pre-Orders

Given subsets $X, Y \subseteq S$ of a semigroup, write

$$XY := \{xy : x \in X, y \in Y\},$$

instead writing xY where $X = \{x\}$, or Xy if $Y = \{y\}$.

Definition 1.2.17. A subset $T \subseteq S$ is respectively a *left ideal, (two-sided) ideal* if it satisfies $S^1 T = T$, or $S^1 T S^1 = T$, respectively.

We usually omit "two-sided," including it only where necessary to describe ideals in contrast to, or alongside one-sided counterparts.

A *right ideal* in S is a left ideal in the dual S^{op}; an ideal is precisely a left ideal which is simultaneously a right ideal. The following results concerning left ideals each have a dual statement in terms of right-ideals, with multiplication reversed where applicable.

Proposition 1.2.18. *The intersection of a family of left ideals is a left ideal.*

Corollary 1.2.19. *The intersection of a family of ideals is an ideal.*

Proof. By duality, Proposition 1.2.18 implies that intersections of right ideals are right ideals. An intersection of ideals is simultaneously an intersection of left and right ideals, and is hence a left and right ideal, which is the same as being and ideal. □

Proposition 1.2.20. *For $s \in S$ and $X \subseteq S$, we have*

1. *The set S^1X is a left ideal in S;*
2. *The set S^1XS^1 is an ideal in S.*

These are the left ideal, respectively the ideal, generated by X. When $X = \{x\}$ is a singleton, we refer to the principal (left) ideal generated by x.

Proof. Similarly to Proposition 1.2.3, we'll rely on the observation that family of (left) ideals containing X has an intersection, itself a (left) ideal. Note that every left ideal (respectively ideal) containing X contains all $x \in X$ and each $sx \in SX$ (alongside each $xs \in XS$ and $sxt \in SXS$), so S^1X (S^1XS^1) is contained in the intersection, and hence equals it. □

We can compare elements with reference to the left, right and two-sided ideals by reference to a family of relations described by Sandy Green [30] in 1951.

Definition 1.2.21. Let S be a semigroup. The *Green's preorders* on S are the relations:

1. $s \leq_{\mathscr{L}} t$ precisely if $S^1 s \subseteq S^1 t$;

2. $s \leq_{\mathscr{R}} t$ precisely if $sS^1 \subseteq tS^1$;

3. $s \leq_{\mathscr{J}} t$ precisely if $S^1 s S^1 \subseteq S^1 t S^1$.

Then *Green's (equivalence) relations* are given by

1. $s \mathscr{L} t$ precisely if $S^1 s = S^1 t$;

2. $s \mathscr{R} t$ precisely if $sS^1 = tS^1$;

3. $s \mathscr{J} t$ precisely if $S^1 s S^1 = S^1 t S^1$;

4. $s \mathscr{H} t$ precisely if $s \mathscr{L} t$ and $s \mathscr{R} t$;

5. $s \mathscr{D} t$ precisely if there is $x \in S$ with $s \mathscr{L} x \mathscr{R} t$.

Duality will allow us to interchange $\mathscr{L}$ and $\mathscr{R}$ in the statement of theorems where convenient; this will be assumed herein. The following characterisations are well-known and useful.

Proposition 1.2.22. *As relations, we have:*

$$\mathscr{K} = \leq_{\mathscr{K}} \cap \geq_{\mathscr{K}}, \qquad \mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L} = \mathscr{L} \vee \mathscr{R},$$

for $\mathscr{K} = \mathscr{R}, \mathscr{L}, \mathscr{J}$, where the join is in the semilattice of equivalence relations on S.

Theorem 1.2.23. *Let H be an $\mathscr{H}$-class in the semigroup S. Then either the set $H \cdot H$ of products lies entirely outside H or is precisely H. In the latter case, H is a group.*

This result is well-known as *Green's Lemma*; it was proved in a 1951 paper of Sandy Green [30], as Theorem 7 and its immediate Corollary. The following result is Theorem 3 of the same paper.

Lemma 1.2.24. *For a finite semigroup, $\mathscr{D} = \mathscr{J}$.*

The following results give some insight into the structure of Green's classes in regular semigroups.

Proposition 1.2.25. *If $s \in S$ is regular and $s\mathscr{D}t$ then t is regular.*

Again Green [30], Theorem 6, observed the above as a corollary to Lemma 6 in [75]. We refer to these as regular $\mathscr{D}$-classes.

Proposition 1.2.26. *Every regular $s \in S$ is $\mathscr{L}$-related to an idempotent.*

Proof. Let s' be an inverse of s. Then $ss' = (ss's)s' = (ss')^2$. □

Proposition 1.2.27. *Let $e \in E(S)$ be an idempotent. Then $es = s$ for all $s \in \mathscr{R}_e$.*

Proof. If $s \in \mathscr{R}_e$ then by definition $s = ea$ for some $s \in S^1$, giving $es = e(ea) = ea = s$. □

Let S be a semigroup and T a subsemigroup. Writing $\mathscr{R}_T := \mathscr{R} \cap (T \times T)$ for the restriction of $\mathscr{R} = \mathscr{R}_S$ to T, we have the following result.

Proposition 1.2.28. *Let $T \leq S$ be a subsemigroup with T regular. Then the restrictions $\mathscr{R}_T, \mathscr{L}_T, \mathscr{H}_T$ of Green's relations from S to T coincide with Green's relations on T.*

Proof. By duality, a proof for $\mathscr{L}_T$ gives one for $\mathscr{R}_T$. Intersection gives $\mathscr{H}_T$, so we only prove that $\mathscr{L}$ restricts faithfully.

Let S be a semigroup and $T \subseteq S$ a regular subsemigroup, $s\mathscr{L}_S t$ be two elements in T. Then $as = t$ and $bt = s$ for some $a, b \in S$.

Pick $l \in S$ satisfying $l\mathscr{L}_S s$, and l' and s' respective inverses in T for l and s. We immediately have $l'l\mathscr{L}_T l$ and $s's\mathscr{L}_T s$, and therefore $l'l's$. Proposition 1.2.27 ensures $l'ls's = l'l$ and $s'sl'l = s's$ by duality. This implies, since these elements are in T, that $s's\mathscr{L}_T t't$, giving

$$s\mathscr{L}_T s's\mathscr{L}_T l'l\mathscr{L}_T l$$

□

1.2.4 Regular *-semigroups

Definition 1.2.29. Let S be a semigroup. A *semigroup involution* $a \mapsto a^*$ on S is a map satisfying $(s^*)^* = s$ and $(st)^* = t^*s^*$ for every pair of elements $s, t \in S$. The semigroup S is a *semigroup with involution* (or **-semigroup*) is a semigroup with a specified involution defined. A **-subsemigroup* of such S is a subsemigroup $T \leq S$ where $t \in T$ implies $t^* \in T$.

Some semigroups such as semigroups of size at least three comprising right zeroes, admit no involutions.

Definition 1.2.30. A *regular *-semigroup* is a regular semigroup S with a defined involution $s \mapsto s^*$ in which $s = s^*s$ for every $s \in S$.

Proposition 1.2.31. *A *-subsemigroup of a regular *-semigroup is itself a regular *-semigroup.*

Definition 1.2.32. A *projection* in a regular *-semigroup is an idempotent e which is a fixed point of the involution: $e^2 = e = e^*$.

Proposition 1.2.33. *In a regular *-semigroup S and $s, t \in S$, $e \in E(S)$ and $p \in E(S)$ is a projection, the following hold:*

1. *The elements ss^* and s^*s are projections;*
2. *Every projection $p \in S$ can be written as $p = t_1 t_1^* = t_2^* t_2$ for some $t_1, t_2 \in S$;*
3. *The element $e^* \in E(S)$ is idempotent; that's to say that $E(S)$ is closed under involution;*
4. *If $s \mathscr{L} t$ then $s^* \mathscr{R} t^*$;*
5. *For any $s \in S$ we have $s^* \mathscr{D} s$;*
6. *The idempotent e can be written as a product of two projections from its $\mathscr{D}$-class in a unique way.*

Proof. The first four items are obvious, since, respectively, $(ss^*)^* = ss^*$ and $ss * s = s$ ensuring idempotency of ss^* and s^*s; $pp^* = p = p^* = p^*p$ is an idempotent fixed under the involution if p is a projection; and we have $(e^*)2 = (e^2)^* = (e)^*$ for any idempotent e. The fourth follows from the fact that there exist a

To prove the later claims, fix $e \in S$. Then we have $e = (ee^*)e = e(e^*e)$, so $e \mathscr{L} e^*e$ and $e \mathscr{R} ee^*$. Dually, $e^* \mathscr{R} e^*e$ and $e^* \mathscr{L} ee^*$, meaning $e \mathscr{D} e$.

Now assume $e \in E(S)$ is idempotent. We can write $e = ee^*e = e(e^*)^2 e = (ee^*)(e^*e)$ as a product of two projections, respectively $\mathscr{R}$ and $\mathscr{L}$-related to e, and dually, respectively $\mathscr{L}$ and $\mathscr{R}$-related to e^*. Now assume $e = ss^*tt^*$ for some $s, t \in S$ so that $e^* = tt^*ss^*$. Clearly we have $e \leq_{\mathscr{R}} ss^*$ and $e^* \leq_{\mathscr{L}} ss^*$, but since $ss^* \mathscr{D} ss^*$, we have $e \mathscr{R} ss^* \mathscr{L} e^*$. That means that $ss^* \mathscr{H} ee^*$; Theorem 1.2.23 ensures that since this $\mathscr{H}$-class contains an idempotent, it is a group and hence contains exactly one, ensuring $ss^* = ee^*$. A dual argument quickly establishes $tt^* = e^*e$. □

Corollary 1.2.34. *If $e = pq$ is an idempotent and $p, q \mathscr{D} e$ are projections then $p = ee^*$ aand $t = e^*e$.*

1.3 Combinatorial Semigroup Theory

Definition 1.3.1. A *congruence* on a semigroup S is an equivalence relation which is *stable* under multiplication. That is, if $r, t \in S^1$ and $s \in [x]$, the equivalence class of $x \in S$, we have $rst \in [rxt]$.

We call such equivalence classes *congruence classes*. The mapping from element to congruence class is a homomorphism, and each homomorphism arises in this way. The congruence on the domain semigroup is recovered by observing that classes are exactly preimages of codomain elements.

Proposition 1.3.2. *A congruence is a subsemigroup of $S \times S$.*

Corollary 1.3.3. *The congruences form a closure system in the lattice of relations on S.*

There is therefore a well-defined mapping $\varrho \mapsto \varrho^\sharp$ associating to each relation $\varrho \subseteq S \times S$ the smallest congruence containing it. The congruences hence form a lattice whose meet is the intersection as a binary relation, and whose join is given by applying a closure operation returning the smallest congruence, when applied to the set-theoretic union.

Definition 1.3.4. The *quotient semigroup* S/ϱ of S by the congruence ϱ is the set of equivalence classes with the setwise multiplication induced by that of S.

This is guaranteed to be a well-defined semigroup by the properties characterising congruences, or equivalently, homomorphisms.

1.3.1 Free Semigroups and Formal Languages

Definition 1.3.5. Let X be a set. A *word* over X is a finite concatenation of letters from X. The *free semigroup* Σ^+ generated by Σ is the set

$$\bigcup_{i>0} \Sigma^i = \Sigma \cup (\Sigma \times \Sigma) \cup (\Sigma \times \Sigma \times \Sigma) \cup \cdots$$

of all words from Σ, with multiplication given by concatenation. If we adjoin an identity (represented by an empty word ε), we obtain the *free monoid* Σ^*.

Definition 1.3.6. Let $w \in \Sigma^*$ be a word. The *length* of w is the number of letters in w; equivalently the number l where $w \in \Sigma^l$. The *a-content* $\#_a(w)$, for some letter $a \in \Sigma$ is the number of occurrences of a in w. A *subword* of w is some word t such that $w = stu$ for some $s, u \in \Sigma^*$. The subword t is a *prefix* (respectively a *suffix*) of w if s (resp. u) can be chosen to be the empty word.

Definition 1.3.7. A *relative presentation* for a semigroup is a pair $\mathsf{P} = \langle S|R\rangle$ where S is a semigroup and $R \subseteq S \times S$ is a set of *relations*. The semigroup *presented by* P is $S/R^\sharp$ where $R^\sharp$ is the congruence generated by the relations in R. The term *presentation* is used when S is a free semigroup or monoid; in such a case, we usually write $\langle A|R\rangle$ where A is a set (usually in fact, just a list) of generators for A^+ or A^*.

Example 1.3.8. The presentation $\langle a, b \mid ab = b, a^2 = a, ba = a, b^2 = b\rangle$ defines a 2-element semigroup of right zeroes.

We regard a free monoid (semigroup) as a set of words. Throughout, we'll denote by Σ an alphabet, which is to say a finite set.

Definition 1.3.9. A *(formal) language* over Σ is a subset $L \subseteq \Sigma^*$. The *context* $\mathsf{C}_L(v)$ of a word v modulo the language L is the (possibly empty) set of pairs (u, w) such that $uvw \in L$. The syntactic equivalence $\sim_L$ of L is given by comparing contexts; if $\mathsf{C}_L(v) = \mathsf{C}_L(v')$ then $v \sim_L v'$ and otherwise $v \not\sim_L v'$.

Proposition 1.3.10. *For any fixed language $L \subseteq \Sigma^*$, this equivalence is a congruence.*

Proof. The relation is clearly an equivalence. To establish stability, fix $v \sim_L v'$ and fix $s, t \in \Sigma^*$. We need only establish that $svt \sim_L sv't$. Consider

$$\mathsf{C}_L(svt) = \{(u,w) : usvtw \in L\}.$$

Since $v \sim_L v'$ we have $usv'tw \in L$ for each $usvtw \in L$, meaning that $C_L(svt) = C_L(sv't)$ so $svt \sim_L sv't$. The equivalence is a congruence. □

There are many notions of complexity for languages, which in some sense determine how difficult it is to decide answers to questions such as membership, how many words there are of each length, as well as some sense of how complicated the semigroup $\mathsf{Syn}(L)$ is.

We frequently will rely on *regular expression* notation, which borrows from Stephen Kleene's notation for regular algebras [42].

Definition 1.3.11. Regular expression notation is defined recursively as follows.

- Every word w is a regular expression, and represents the language $\{w\}$;
- Every finite sum $\sum_{i=1}^n e_i$ of regular expressions e_i is a regular expression, representing the language $\bigcup_{i=1}^n L_i$ where e_i represents L_i;
- Every finite product $\sum_{i=1}^n e_i$ of regular expressions e_i is a regular expression, representing the language

  $$L_1 \cdot L_2 \cdots L_k = \{u_1 \cdots u_2 \cdots u_k : u_i \in L_i \text{ for } i = 1, \ldots, n\},$$

  where each language L_i is represented by e_i.

As a matter of practical concern, we'll abuse or extend this notation in some convenient ways. If L_i are languages (rather than expressions) we may write the product or sum to represent the concatenation or union.

Definition 1.3.12. A *grammar* $\mathcal{G}$ consists of a disjoint pair of finite alphabets $\mathfrak{N}$, $\mathfrak{T}$ of *nonterminals* and *terminals*, a *start symbol* $\mathbf{S} \in \mathfrak{N}$, and a finite set P of *production rules*, which are pairs in $\mathfrak{N} \times (\mathfrak{N} \cup \mathfrak{T})^*$, written $\mathbf{N} \longrightarrow w$.

The *language defined by* $\mathcal{G}$ is the set of terminal words $w \in \mathfrak{T}^*$ such that

$$\mathbf{S} = u_0 \longrightarrow u_1 \longrightarrow \cdots \longrightarrow u_k = w$$

is a sequence of applications of production rules.

We use certain conventions to encode grammars in a more obvious form to work with. Roman letters will be nonteminals while symbols in `typewriter` fonts will be terminals. More often than not, we use **S** for the start symbol, and write + for conjunction of production rules having the same left-hand side.

Definition 1.3.13. The *language* $L(\mathcal{G})$ *generated by a grammar* $\mathcal{G}$ comprises all those words with nonterminal letters, reachable from `S` by a sequence of replacements which consist of productions in $\mathcal{G}$.

In other words, if we have $n \to e$ a production in $\mathcal{G}$, then we write $unw \vdash_{\mathcal{G}} uvw$. If we have a sequence

$$u = u_0 \vdash_{\mathcal{G}} u_1 \vdash_{\mathcal{G}} \cdots \vdash_{\mathcal{G}} u_k = v$$

of these replacements, then we'll write $u \vdash_{\mathcal{G}}^* v$. We can then see that

$$L(\mathcal{G}) = \{w \in \mathcal{N}^* : \mathbf{S} \vdash_{\mathcal{G}} w\}.$$

Example 1.3.14. The *Dyck language* $\mathfrak{D}$ can be generated by a grammar with a nonterminal alphabet $\mathfrak{N} = \{\mathbf{S}\}$ consisting of only the start symbol, an alphabet $\mathfrak{T} = \{\,\texttt{[}, \texttt{]}\,\}$ of terminals and two production rules:

$$\mathbf{S} \longrightarrow \varepsilon, \qquad \mathbf{S} \longrightarrow \mathbf{S} \cdot \texttt{[} \cdot \mathbf{S} \cdot \texttt{]}\,.$$

We can write this with a single conjunction $\mathbf{S} \longrightarrow \varepsilon + \mathbf{S}\,\texttt{[}\,\mathbf{S}\,\texttt{]}$ in a natural, obvious extension of the regular expression notation.

The language $\mathscr{D}$ can be characterised as the smallest submonoid of $\mathfrak{T}^*$ closed under the map $w \mapsto \texttt{[} \cdot w \cdot \texttt{]}$.

This choice of grammar has some nice properties as we'll see in the next section.

1.3.2 The DSV Method

Definition 1.3.15. The *growth* of a language is the sequence $\gamma_0, \gamma_1, \ldots, \gamma_l, \ldots$ where

$$\gamma_l := \#\{w \in L : |w| = l\}.$$

The *growth series* is then the ordinary generating function for the $(\gamma_i)_i$:

$$f(z) = \sum_{i=0}^{\infty} \gamma_i z^i.$$

We can calculate the growth series fairly easily using certain nice grammars.

Example 1.3.16. There is an alternative grammar for the Dyck language, with productions

$$\mathbf{S} \longrightarrow \varepsilon + \mathbf{TT}, \qquad \mathbf{T} \longrightarrow S + \mathtt{[}\mathbf{S}\mathtt{]}.$$

We can take each nonterminal appearing and substitute for the formal symbol z, and for each nonterminal, substitute a formal power series, adding across conjunctions and equating across productions:

$$S(z) = 1 + T(z)^2, \qquad T(z) = S(z) + z \cdot S(z) \cdot z.$$

We make everything commute, giving

$$T(z) = S(z) + z^2 S(z) = (1 + z^2) S(z),$$

so

$$S(z) = 1 + (1 + z^2)^2 S(z)^2$$

The DSV method, named for Delaun, Schützenberger and Vali.

1.4 Diagrams

1.4.1 Diagrams

Definition 1.4.1. A picture in degree n is a graph whose vertices lie in the set

$$\pm[\![n]\!] := [\![n]\!] \cup -[\![n]\!] = \{\pm 1, \pm 2, \ldots, \pm n\}.$$

Two such graphs are equivalent if the connected components coincide. A *diagram* is an equivalence class of these graphs. The set of equivalence classes is the *partition monoid* $\mathcal{P}_n$.

When we draw diagrams, we consider two different representatives to be *equal* rather than merely equivalent in some sense, since we are only concerned with the connected components.

For a diagram $\delta \in \mathcal{P}_n$, we write $a \sim_\delta b$ if $a, b \in \pm[\![n]\!]$ lie in the same connected component in some representative of δ.

Proposition 1.4.2. *The representatives in δ, ordered by inclusion, form a join-semilattice.*

Definition 1.4.3. Given two diagrams $\delta, \nu \in \mathcal{P}_n$, we define the *(vertical) product* as the diagram $\delta \cdot \nu$ characterised by $a \sim_{\delta \cdot \nu} b$ for $a, b \in \pm[\![n]\!]$ if one of the following holds:

1. We have $a, b > 0$ with $a \sim_\delta b$;
2. Dually, $a, b < 0$ with $a \sim_\nu b$;
3. We have $b < 0 < a$ and there is a sequence $u_0, u_1, \ldots, u_{2k}$ so that each $u_i > 0$ with $a \sim_\delta -u_0$, $u_{2n} \sim_\nu b$ and

$$u_0 \sim_\nu u_1, -u_1 \sim_\delta -u_2, u_2 \sim_\nu u_3, \cdots, -u_{2n-1} \sim_\delta -u_{2n-1}, u_{2n-1} \sim_\nu u_{2n}.$$

It's essentially impossible to draw an equivalence class of pictures, so we pick a picture and use that to represent the whole class.

Definition 1.4.4. Given $\delta \in \mathcal{P}_n$ and $\nu \in \mathcal{P}_m$, and representatives $\Gamma \in \delta$ and $\Xi \in \nu$, we can form the *horizontal (tensor) sum* $\Gamma \otimes \Xi$, which is the picture of degree $n + m$ with

- $a \in \Gamma \otimes \Xi$ for $a, b \in \pm[\![n]\!]$ with $(a, b) \in \Gamma$;
- $a + m \in \Gamma \otimes \Xi$ for $a, b \in [\![m]\!]$ with $(a, b) \in \Xi$;
- $-a - m \in \Gamma \otimes \Xi$ for $a, b \in [\![m]\!]$ with $(-a, b) \in \Xi$;

- $a + m \in \Gamma \otimes \Xi$ for $a, b \in [\![m]\!]$ with $(a, -b) \in \Xi$.

Definition 1.4.5. Let Γ be a picture. There is a picture Γ^* which we call the *opposite* of Γ, given by swapping the signs of vertices in every edge. In other words, Γ^* is characterised by $(a, b) \in \Gamma$ precisely whenever $(-a, -b) \in \Gamma^*$

This map is an involution in the sense of Definition 1.2.29.

Proposition 1.4.6. *Involution of pictures induces an involution of diagrams. In other words, $\Gamma, \Xi \in \delta$ where $\delta \in \mathcal{P}_n$ for some n. If $\Gamma^* \in \nu$, then $\Xi^* \in \nu$.*

We write δ^* for the involution of a diagram.

Proposition 1.4.7. *The semigroup $\mathcal{P}_n$ is a regular *-semigroup with respect to the induced involution *.*

Herein we won't use the term picture, instead identifying a diagram with a representative.

Definition 1.4.8. A semigroup of diagrams is a *diagram semigroup*. A diagram semigroup is *planar* if each diagram can be represented by a noncrossing partition of the ordered set $\pm[\![n]\!]$.

1.5 Algebra

Definition 1.5.1. A *semiring* is a set K endoed with two associative binary operators $+, \cdot$, the former being commutative, with respective units 0 and 1, which together satisfy the *distributive laws*:

$$a(b + c) = ab + ac, \qquad (a + b)c = ac + bc$$

for all $a, b, c \in K$ such that the additive identity 0 is a multiplicative zero. A semiring is *commutative* if $\cdot$ commutes.

A (commutative) *ring* is a semiring whose addition induces the structure of a(n abelian) group. An *integral domain* is a commutative ring if it has no zero divisors except 0. A *field* is a commutative ring whose nonzero elements form a group under mutliplication.

A **-ring* is a ring whose multiplicative monoid is a *-semigroup whose involution distributes over addition.

Definition 1.5.2. An *ideal* I in a ring R is a subset which inherits a subgroup structure with respect to addition and an ideal structure with respect to multiplication, written $I \trianglelefteq R$. We say that $a \equiv b \pmod{I}$ if $a - b \in I$. This equivalence relation is stable with respect to addition and multiplication; so the classes form a ring with respect to the setwise operations, and we write R/I for the induced *quotient ring*.

Given a quotient map, one can detect the factored ideal by checking what elements map to zero.

Proposition 1.5.3. *Subrings and ideals form closure systems in the set of subsets of a fixed ring R. We can talk about (sub)ring and ideal generation the same way as we do for semigroups.*

Example 1.5.4. The usual number systems provide good prototypical algebraic structures. The natural numbers $\mathbb{N}_0$ provide an example of a commutative semiring with no zero divisors. Extending to $\mathbb{Z}$ gives us an integral domain, and the rationals $\mathbb{Q}$, reals $\mathbb{R}$ and complex numbers $\mathbb{C}$ are fields.

Integral domains embed in fields (and are the only rings which do so). The *fraction field* is the smallest field in which an integral domain embeds, itself being embeddable in every such field.

If $\mathbb{K}$ is a semiring then we define the sets

$$K[X] = \left\{k_0 + k_1 X + \cdots + k_n X^n \;\middle|\; n \in \mathbb{N}_0, k_i \in K \text{ for each } 0 \leq i \leq n\right\}$$

$$K[\![X]\!] = \left\{\sum_{i=0}^{\infty} k_i X^i = k_0 + k_1 X + \cdots + k_n X^n + \cdots \;\middle|\; k_i \in K \text{ for each } i \in \mathbb{N}_0\right\}$$

of respectively *polynomials* and *formal series*.

These constructions usually preserve most of the structure. Commutativity (or lack thereof), possession of a unit, existence of additive inverses and lack of zero divisors translate forward in both cases. Checking this is an easy exercise for the reader.

Definition 1.5.5. For a ring R, a *(right) module over* R is an abelian group A (whose operation is written as addition) on which the multiplicative semigroup R acts compatibly with addition in A and R. In other words, for $a, b \in A$ and $r, s \in R$, we have

$$(a+b)r = ar + br, \qquad a(r+s) = ar + as,$$
$$a \cdot 1_R = a, \qquad 0_A \cdot r = a \cdot 0_R = 0_A$$

where 0_A is the identity in A and 1_r and 0_R are the identity and zero of R. A *vector space* is a module V over a field.

A *left module* is defined similarly, and an R*-bimodule* M is simultaneouly a left and right R-module such that if $r, s \in R$ and $m \in M$ then $(rm)s = r(ms)$.

We often use the term R*-module* instead of right module.

Definition 1.5.6. Let A be an R-module. A *generating set* is a subset B with $BR = \{br : b \in B, r \in R\}$ generates A as a group. A *basis* of an R-module A is a generating set whose elements are *linearly independent*, that is if $r_1, \ldots, r_n \in R$ are distinct with $b_1, \ldots b_n \in B$ then the *linear combination* $b_1r_1 + b_2r_2 + \cdots + b_nr_n = 0_A$ only when every r_i is zero. A module is *free* if it has a basis.

Definition 1.5.7. A *submodule* is a a subgroup $N \leq A$ such that for $n \in N$ and $r \in R$, $nr \in N$. As with ideals in rings, we can separate A into classes modulo N and the quotient inherits a natural R-module structure.

Submodules again form a closure system and we can talk meaningfully about generation.

Definition 1.5.8. Given a commutative ring R, an *algebra over* R (alternatively, R*-algebra*) is a ring A whose additive group is an R-bimodule such that for $a \in A$ and $r \in R$, $ar = ra$.

If R is a *-ring with involution written $'$, then a **-algebra* over R is a *-ring with an involution * which is an R-algebra such that, for $r \in R$ and $a \in A$, we have $(ra)^* = r'a^*$.

Algebras again comprise a closure system. Ideals being submodules, quotients are characterised by factoring out ideals just like with rings.

Example 1.5.9. Let $X = \{x_1, x_2, \ldots\}$ be a set of distinct formal letters. Then the *free R-algebra* $F_R(X)$ is the algebra with a basis comprising words over the letters whose multiplication is the linear extension of word concatenation. This algebra inherits a natural involution reversing the order of multiplication. Every algebra is a quotient of the free algebra over its generators.

The polynomial algebra $R[x_1, x_2, \ldots]$ is already known to us, and consists of polynomials, which are sums of commuting words. The ideal giving rise to the quotient from the free algebra is generated by the terms $x_i - x_j$.

We can write presentations for algebras similarly to semigroups.

Definition 1.5.10. Let $x_1, x_2, \ldots$ be a sequence of formal symbols, and $l_i = r_i$ be a sequence of equalities between elements of $R\langle X_1, X_2, \ldots\rangle$. Then the *algebra presented by* the above data is the quotient of the free algebra by the ideal generated by the $r_i - l_i$, written

$$\left\langle X_1, X_2, \ldots \,\middle|\, l_1 = r_1,\, l_2 = r_2,\, \ldots \right\rangle := \frac{R\langle X_1, X_2, \ldots\rangle}{I}$$

where

$$I = \{a_1(r_{i_1} - l_{i_1})b_1 + \cdots + a_n(r_{i_n} - l_{i_n})b_n : a_k, b_k \in R, k \leq n < \infty\}$$

is the ideal generated by the $(r_i - l_i)_i$.

Now let $X \subseteq A$ generate A as an R-algebra, and F_X denote the free R-algebra generated by X, with $\phi : F_X \to A$ the obvious quotient map. Given $s, t \in F_X$, say that the equality $s = t$ *holds in* A if $\phi(s) = \phi(t)$.

If $Q = \{l_i = r_i : l_i, r_i \in F_X, i \in \mathcal{I}\}$ is a set of equations that all hold in A, we say that Q *completely characterises* A if the set $\{l_i - r_i : i \in \mathcal{I}\}$ generates the ideal defining the quotient.

Definition 1.5.11. Given a semigroup S, and ring R the *semigroup ring* [of S over R] is the ring

$$RS = \{f : S \to R \mid f \text{ is a map}\}$$

whose elements are maps with only finitely many points mapping to nonzero elements in R. We add these in the image ($(f+g)(s) = f(s)+g(s)$ for each $s \in S$) and multiply by convolution:

$$(f * g)(s) = \sum_{x \in S} \sum_{y : xy = s} f(x) \cdot g(y).$$

When R is a field, we use the term *semigroup algebra*.

We often write these maps as sums, as follows

$$f = \sum_{s \in S} f(s) \cdot s,$$

adding using the distributive law. Nothing in particular restricts us to using rings rather than semirings, but we'll only use semirings in a couple of specific circumstances.

Definition 1.5.12. Let S be a semigroup and R a ring. A *twisting* is a map $\alpha : S \times S \to R$ satisfying

$$\alpha(x,y) \cdot \alpha(xy,z) = \alpha(x,yz) \cdot \alpha(y,z)$$

for all $x,y,z \in S$. The *twisted semigroup algebra* of S over R with respect to α is the R-algebra $R^{\alpha}[S]$ with basis S and multiplication defined by

$$x \cdot y := \alpha(x,y)(xy)$$

where (xy) is the product in S, and extended linearly.

1.6 Some Topology

Definition 1.6.1. A *topology* τ for a set X is a family of *open* subsets of X closed under arbitrary unions and finite intersection, and containing both X and the empty set. A *(topological) space* (X,τ) is a set equipped with a topology; we take τ as known and usually refer to X.

A map $f : X \to Y$ is *continuous* if the preimage of each open set is open. A *homeomorphism* is a continuous bijection whose inverse is bijective. Two spaces are *homeomorphic* if there is a homeomorphism mapping between.

Homeomorphism reflect the topological notion of "sameness" much as isomorphism reflects that notion in algebra. If two spaces X and Y are homeomorphic, we write $X \simeq Y$.

Proposition 1.6.2. *The union of a family of open subsets of a topological space X is open.*

Dually, intersections of closed sets are closed, meaning the closed sets form a closure system. There is then a well-defined maximal open set contained in some $Y \subseteq X$ and a minimal closed set containing Y. These are respectively the *interior* and *closure*, and the (possibly empty) difference is the *boundary*, which will be denoted by ∂Y.

Definition 1.6.3. A space is *connected* if it is not the union of two disjoint nonempty open subsets. A *connected component* is a connected subspace not properly containing any other connected subspace.

Definition 1.6.4. A *base* $\mathfrak{B}$ of open sets in a topology τ is a subset such that every open $U \in \tau$ is a union of sets in $\mathfrak{B}$.

Every topology on a set X can be completely described by a base of open subsets; we may fruitfully think of these as performing a similar function to generating sets in algebra.

Example 1.6.5. Let X be a set. The *discrete topology* is the set 2^X of all subsets, and has a base comprising the singletons $\{x\}$. Discrete spaces are not connected unless $|X| < 2$.

Given a family $\mathcal{X} = (X_i)_{i \in \mathcal{I}}$ of topological spaces, the *disjoint union* is the set

$$\coprod \mathcal{X} = \coprod_{i \in \mathcal{I}} X_i := \{(i, x) : i \in \mathcal{I},\, x \in X_i\}$$

with a topology given by the base of open sets $\{i\} \times O$, where $i \in \mathcal{I}$ and $O \subseteq X_i$ is open.

Let X be a topological space with a base $\mathfrak{B}$ of open sets and let $Y \subseteq X$ be a subset. The *subspace topology* on Y has a base $\mathfrak{B}_Y$ given by

$$\mathfrak{B}_Y = \{B \cap Y : B \in \mathfrak{B}\}.$$

Given a family (X_i) of topological spaces with topologies τ_i, the *box topology* on $\prod_i A_i$ has a base of open sets consisting of the set

$$\mathfrak{B} = \{\prod_{i \in \mathcal{I}} B_i : B_i \in \tau_i \text{ for each } i\}.$$

The standard base for the *product topology* is similar, except that only finitely many of the B_i in the product may be unequal to X_i. Clearly, for finite products, these bases, and hence topologies, coincide.

Denoting by $\mathbb{R}^n$ the vector space of n-tuples of real numbers under componentwise addition as usual, the sets I^n and S^{n-1} denote repectively the *unit cube* comprising vectors with all components in the unit interval$[0,1]$, and the $(n-1)$-sphere, comprising vectors whose components sum to 1 when squared. We have $\partial I^n \cong S^{n-1}$. Define I^0 to be a point.

Proposition 1.6.6. *Let $A = A_1 \sqcup A_2$ be a disjoint union of two topological spaces and B be a space. Then*

$$A \times B = (A_1 \times B) \sqcup (A_2 \times B).$$

The proof is left as an exercise to the reader.

Corollary 1.6.7. *Products distribute over disjoint unions.*

Definition 1.6.8. An mapping of I (S^1) into $\mathbb{R}^n$ is called a (closed) curve. If this mapping is an embedding then the (closed) curve is said to be *self-avoiding*.

Theorem 1.6.9 (Jordan Curve Theorem)**.** *Let $c : S^1 \to \mathbb{R}^2$ be a self-avoiding closed curve. Then the complement $\mathbb{R}^2 \smallsetminus c(S^1)$ of the image has two connected components.*

This theorem is folklore in topology, and highly nontrivial to prove. The reader is directed to [40,53,73,74]

Definition 1.6.10. A *neighbourhood* of a point $x \in X$ in a space is an open set containing x. Two points x, y are *separated by neighbourhoods* if there are neighbourhoods N_x, N_y of each which do not intersect. A space is *Hausdorff* if every pair of points is separated by neighbourhoods.

Definition 1.6.11. A space (X, τ) is *compact* if, given a family $C \subseteq \tau$ of open sets whose union is X, there is a finite subset $C' \subseteq C$ whose union is X.

Definition 1.6.12. A *metric space* X is a set with a well-defined notion of distance, which is to say having a map $d : X \times X \to \mathbb{R}_{\geq 0}$ so that $d(x, y) = 0$ precisely when $x = y$ and satisfying the triangle inequality for all $x, y, z \in X$:

$$d(x, z) \leq d(x, y) + d(y, z).$$

When dealing with several metric spaces, we will often denote by d_X the metric attached to X.

Definition 1.6.13. An *isometry* is a bijective map $f : X \to Y$ between metric spaces such that

$$d_X(x, x') = d_Y(f(x), f(x')).$$

We say that two spaces are *isometric* if there is an isometry between them.

Isometries serve the same function in the theory of metric spaces as homeomorphisms do in general topology and isomorphisms in algebra.

Definition 1.6.14. An *open ball* (around x of radius r) in a metric space is a set

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

A closed ball is similar, but we constrain the distance not to exceed r rather.

The open balls in a metric space form a base for its topology.

Definition 1.6.15. A *limit* of a sequence $x_1, x_2, \ldots$ of points in a metric space X is a point x such that for each positive real $\varepsilon > 0$ there is a nonnegative integer $N \geq 0$ so that $d(x_n, x) < \varepsilon$ for each $n > N$. A sequence of points in the metric space X is *Cauchy* if for every positive real $\varepsilon > 0$ there is a nonnegative integer $N \geq 0$ so that $d(x_n, x_{n+1}) < \varepsilon$ for all $n > N$.

A metric space is *complete* if every Cauchy sequence has a limit.

If x is the limit of a sequence $(x_i)_i$, we also say that the sequence converges to x (or that the x_i converge to x), written $x_i \underset{i \to \infty}{\to} x$.

Example 1.6.16. Any nonempty subset of a metric space is a metric space, whose metric is simply inherited from the ambient space.

Example 1.6.17. Let (X,d) be a metric space. Then the family $\mathcal{C}(X)$ of nonempty compact subsets inherits a metric, the *Haussdorff metric*. Writing Y_ε for the union of open ε-balls with centres in Y, we have

$$d_H(Y,Z) = \max\{\inf_{z\in Z}\sup_{y\in Y} d(z,y), \sup_{z\in Z}\inf_{y\in Y} d(z,y)\} \quad = \inf\{\varepsilon > 0 : Y \subseteq Z_\varepsilon \text{ and } Z \subseteq Y_\varepsilon\}.$$

Example 1.6.18. The real numbers form a metric space where $d(x,y) = |x-y|$, and any set-theoretic product $\mathbb{R}^d$ of is a metric space with the quadratic-mean distance

$$d\left((x_1, x_2, \dots x_d), (y_1, y_2, \dots, y_d)\right) = \sqrt{\sum_{i=1}^{d}(x_i - y_i)^2}.$$

This is called the *Euclidean space* $\mathbb{E}^n$.

Definition 1.6.19. Let (X,d) be a metric space. Then a map $f : X \to X$ is called a *contraction mapping* on X if there is $0 \leq k < 1$ such that

$$d(f(x), f(y)) \leq k \cdot d(x,y)$$

for each $x, y \in X$.

Theorem 1.6.20 (Contraction Mapping Theorem)**.** *A contraction mapping $f : X \to X$ where X is complete has a unique fixed point x_*. Furthermore, for every $x \in X$, the limit of the Cauchy sequence $x_0 = x$, $x_{i+1} = f(x)$ is x_*.*

This classical result can be found in any standard text on real analysis, see [68] Theorem 9.2.3.

Definition 1.6.21 ([9], p111–2)**.** The *faces* of I are $\{0\}$, $\{1\}$ and $[0,1]$ and the faces of I^n are products of faces of I.

A *cubical complex* $\mathscr{K}$ is a quotient of a finite disjoint union of unit cubes $X = \coprod_{\lambda\in\Lambda} I^{n_\lambda}$ by an equivalence relation $\sim$. The restrictions $p_\lambda : I^{n_\lambda} \to \mathscr{K}$ are of the quotient mapping $p : X \to \mathscr{K}$ are required to satisfy:

- for each λ, the map p_λ is injective;
- if $p_\lambda(I^{n_\lambda}) \cap p_{\lambda'}(I^{n_{\lambda'}}) \neq \varnothing$ then there is a metric isometry $h_{\lambda,\lambda'}$ from a face $T_\lambda \subseteq I^{n_\lambda}$ to a face $T_{\lambda'} \subseteq I^{n_{\lambda'}}$ such that $p_\lambda(x) = p_{\lambda'}(x')$ precisely when $x' = h_{\lambda,\lambda'}(x)$.

The *dimension* of the cube complex is the largest n_λ.

Proposition 1.6.22. *The product of two cubical complexes is a cubical complex.*

By Corollary 1.6.7, it suffices to check that the composition of the quotient maps in each component satisfies the conditions of Definition 1.6.21. This is a straightforward exercise for the reader.

Chapter 2

Introduction

Idempotents are of prominent importance in semigroup theory. They show us where groups occur inside the structure of semigroups and let us know where we can find "local" inverses inside the semigroup. The semigroups we'll study in this thesis are so-called diagram semigroups, which for our purposes are concrete combinatorially-defined structured whose elements are partitions of a certain type. The ones we're interested in have some convenient extra structure carried by involutions.

Related semigroup algebras and deformations have been extensively studied in several contexts. In particular, these are known in certain cases to be [76] *cellular*, and hence semisimple, emphasising the importance of understanding the idempotents in the algebra (and hence, the semigroup) in understanding certain aspects of the representation theory of the algebras.

There's also a rich variety theory surrounding many of the monoids of interest. It's known [3] that the pseudovarieties generated by the Jones monoids and Brauer monoids respectively comprise all finite aperiodic monoids and all finite monoids.

The work exposed in this thesis is closely related to that of the arXiv preprint [13] concerning the structure of idempotents in the planar diagram monoids, and which follows on from [12] which concerned itself with the non-planar cases. At this time, the former does not contain the method exposed herein for counting idempotents using generating functions, and this thesis is the first place that the algorithms used to generate the data shown in that paper are analysed.

There is significant overlap with [13], and as such I've elected to opt for a novel exposure of the material which will hopefully offer a deeper insight into the theory, and certainly more closely parallel the development of the algorithms which will echo the main results of the paper.

2.1 Diagram algebras and Semigroups

This section is based on the introduction from [12], whose content has been adapted and included at the suggestion of the external examiner.

There are many compelling reasons to study diagram algebras and semigroups. Besides their intrinsic appeal, they appear as key objects in several diverse areas of mathematics, from statistical mechanics to the representation theory of algebraic groups, often touching upon major combinatorial themes. In this introduction we seek to show the value of this study, though we can give only a superficial impression of all the connections that exist, with a particular emphasis on the types of problems we investigate, and we make no attempt to give an exhaustive description of an area that is exquisitely vast.

In 1927, Issai Schur [69] provided a vital link between permutation and matrix representations. This connection, now known as *Schur-Weyl duality*, shows that the *general linear group* $\mathrm{GL}_n(\mathbb{C})$ (consisting of all invertible $n \times n$ matrices over the complex field $\mathbb{C}$) and the complex group algebra $\mathbb{C}[\mathcal{S}_k]$ of the *symmetric group* $\mathcal{S}_k$ (consisting of all permutations of a k-element set) have commuting actions on k-fold tensor space $(\mathbb{C}^n)^{\otimes k}$, and that the irreducible components of these actions are intricately intertwined. In 1937, Richard Brauer [8] showed that an analogous duality holds between the *orthogonal group* $\mathrm{O}_n(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{C})$ and the so-called *Brauer algebra* $\mathbb{C}^\xi[\mathcal{B}r_k] \supseteq \mathbb{C}[\mathcal{S}_k]$. At the end of the 20th century, the *partition algebras* $\mathbb{C}^\xi[\mathcal{P}_k] \supseteq \mathbb{C}^\xi[\mathcal{B}r_k]$ were introduced by Paul Martin [55] in the context of Potts models in statistical mechanics. Martin later showed [58] that the partition algebras are in a kind of Schur-Weyl duality with the symmetric group $\mathcal{S}_n \subseteq \mathrm{O}_n(\mathbb{C})$ (in its disguise as the group of all $n \times n$ permutation matrices). Figure 2.1 shows the relationships between the various algebraic structures; vertical arrows indicate containment of algebras or groups and horizontal arrows indicate relationships between dual algebras and groups. It should be noted that there are several other Schur-Weyl dualities; for example, between the *partial Brauer algebra* $\mathbb{C}^{\xi}[\mathcal{PB}r_k]$ and $\mathrm{O}_n(\mathbb{C})$ [31, 35, 57, 59], and between (the semigroup algebras of) the symmetric and dual symmetric inverse semigroups [47].

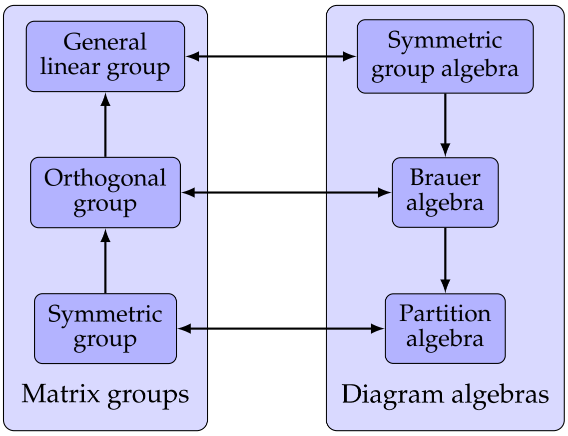


Figure 2.1: Schur-Weyl duality between matrix groups and diagram algebras.

The traditional approach to studying the above algebras, collectively referred to as *diagram algebras* since they have bases indexed by certain diagrams, has been via representation theory [34, 36, 56]. But recent investigations [2–5, 16–21, 44–46, 48, 49, 54, 59, 60] have taken a more direct approach, probing the so-called *partition monoids* (and other *diagram semigroups*) with the tools of semigroup theory, asking and answering the same kinds of questions of partition monoids as one would of any other kind of interesting semigroup, and thereby shedding new light on the internal structure of the diagram algebras.

There are two main reasons this approach has been so successful. The first is that the partition monoids naturally embed many important transformation semigroups on the same base set; these include the full (but not partial) transformation semigroups and the symmetric and dual symmetric inverse semigroups, allowing knowledge of these semigroups to lead to new information about the partition monoids. (See also [57] where a larger semigroup is defined that contains all of the above semigroups and more.) The second reason is that the partition algebras have natural bases consisting of diagrams (see below for the precise definitions), with the product of two basis elements always being a scalar multiple of another. Using this observation, Wilcox [76] realised the partition algebras as *twisted semigroup algebras* of the partition monoids, allowing the *cellularity* of the algebras to be deduced from structural information about the associated monoid. Cellular algebras were introduced by Graham and Lehrer [28] and provide a unified framework for studying several important classes of algebras, allowing one to obtain a great deal of information about the representation theory of the algebra; see [15] for the original study of cellular semigroup algebras and also [33] for some recent developments.

The elements C^λ_{st} of the cellular bases of the diagram algebras studied in [76] are all sums over elements from certain "$\mathscr{H}$-classes" in a corresponding diagram semigroup. Of importance to the cellular structure of the algebra is whether a product $C^\lambda_{st} \cdot C^\lambda_{uv}$ "moves down" in the algebra, and this is governed by the location of idempotents within the "$\mathscr{D}$-class" containing the elements involved in the sums defining C^λ_{st} and C^λ_{uv}. The twisted semigroup algebra structure has also been useful in the derivation of presentations by generators and relations [16,17]. But the benefits of the relationship do not only flow from semigroup theory to diagram algebras. Indeed, the partition monoids and other kinds of diagram semigroups have played vital roles in solving outstanding problems in semigroup theory itself, especially, so far, in the context of pseudovarieties of finite semigroups [2–5] and embeddings in regular $*$-semigroups [19,20].

It has long been recognised that the *biordered set* of idempotents $E(S) = \{x \in S : x^2 = x\}$ of a semigroup S often provides a great deal of useful information about the structure of the semigroup itself. In some cases, $E(S)$ is a subsemigroup of S (as in inverse semigroups, for example), but this is not generally the case. However, the subsemigroup generated by the idempotents of a semigroup is typically a very interesting object with a rich combinatorial structure. In many examples of finite semigroups, this subsemigroup coincides with the *singular ideal*, the set of non-invertible elements [25, 39, 66]; this is also the case with the partition and Brauer monoids [17, 20, 54]. Several studies have considered (minimal idempotent) generating sets of these singular ideals as well as more general ideals; see [21] and references therein. Another reason idempotent generated semigroups have received considerable attention in the literature is that they possess a universal property: every semigroup embeds into an idempotent generated semigroup [39] (indeed, in an idempotent generated regular $*$-semigroup [20]). There has also been a recent resurgence of interest in the so-called *free idempotent generated semigroups* (see [14, 29] and references therein) although, to the authors' knowledge, very little is currently known about the free idempotent generated semigroups arising from of the diagram semigroups we consider; we hope the current work will help with the pursuit of this knowledge.

Interestingly, although much is known [17, 20, 21, 54] about the semigroups generated by the idempotents of certain diagram semigroups, the idempotents themselves have so far evaded classification and enumeration, apart from the case of the Brauer monoid $\mathcal{B}r_n$ (see [48], where a different approach to ours leads to sums over set partitions). This stands in stark contrast to many other natural families of semigroup; for example, the idempotents of the symmetric inverse monoid $\mathcal{I}_X$ are the restrictions of the identity map, while the idempotents of the full transformation semigroup $\mathcal{T}_X$ are the transformations that map their image identically. These descriptions allow for easy enumeration; for example, $|E(\mathcal{I}_n)| = 2^n$, and $|E(\mathcal{T}_n)| = \sum_{k=1}^{n} \binom{n}{k} k^{n-k}$. It is the goal of this thesis to rectify the situation for several classes of diagram semigroups; specifically, the partition, Brauer and partial Brauer monoids $\mathcal{P}_n, \mathcal{B}r_n, \mathcal{PB}r_n$ (though much of what we say will also apply to various transformation semigroups such as $\mathcal{I}_n$ and $\mathcal{T}_n$).

For each semigroup, we completely describe the idempotents, and we give several formulae and recursions for the number of idempotents in the semigroup as well as in various Green's classes and ideals. We also give formulae for the number of idempotent basis elements in the corresponding diagram algebras; these depend on whether the constant that determines the twisting is a root of unity. Our approach is combinatorial in nature, and our results depend on certain equivalence relations and graphs associated to a partition.

Because Sloane's Online Encyclopedia of Integer Sequences (OEIS) [71] is an important resource in many areas of discrete mathematics, we record the sequences that result from our study. We remark that the approach outlined in Chapter 3 does not work for the so-called *Jones monoid* $\mathcal{J}_n \subseteq \mathcal{B}r_n$ (also sometimes called the *Temperley-Lieb monoid* and denoted $\mathcal{TL}_n$), which consists of all *planar* Brauer diagrams; values of $|E(\mathcal{J}_n)|$ up to $n = 19$ have been calculated by James Mitchell, using the Semigroups package in GAP [62], and may be found in Sequence A225798 on the OEIS [71].

Part II

Diagram Monoids

Chapter 3

Diagram Semigroups

The chapter is organised as follows. In Section 3.1, we define the diagram semigroups we will be studying, and we state and prove some of the basic properties we will need. The characterisation of the idempotents is given in Section 3.2, with the main result being Theorem 3.2.2. In Section 3.3, we enumerate the idempotents, first giving general results (Theorems 3.3.2, 3.3.3, 3.3.4) and then applying these to the partition, Brauer and partial Brauer monoids in Sections 3.3.1, 3.3.2 and 3.3.3. We describe an alternative approach to the enumeration of the idempotents in the Brauer and partial Brauer monoids in Section 3.4 (see Theorems 3.4.5 and 3.4.7). In Section 3.5, we classify and enumerate the idempotent basis elements in the partition, Brauer and partial Brauer algebras (see especially Theorems 3.5.1, 3.5.2, 3.5.3). Finally, in Section 3.6, we give several tables of calculated values.

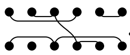
3.1 Preliminaries

Let X be a set, and X' a disjoint set in one-one correspondence with X via a mapping $X \to X' : x \mapsto x'$. If $A \subseteq X$ we will write $A' = \{a' : a \in A\}$. A *partition on* X is a collection of pairwise disjoint non-empty subsets of $X \cup X'$ whose union is $X \cup X'$; these subsets are called the *blocks* of the partition. The *partition monoid* on X is the set $\mathcal{P}_X$ of all such partitions, with a natural binary operation defined below.

When $n \in \mathbb{N} = \{0,1,2,\dots\}$ is a natural number and $X = [\![n]\!] = \{1,\dots,n\}$, we will write $\mathcal{P}_X = \mathcal{P}_n$. Note that $\mathcal{P}_0 = \mathcal{P}_\varnothing = \{\varnothing\}$ has a single element; namely, the empty partition, which we denote by $\varnothing$.

A partition may be represented as a graph on the vertex set $X \cup X'$; edges are included so that the connected components of the graph correspond to the blocks of the partition. Of course such a graphical representation is not unique, but we regard two such graphs as equivalent if they have the same connected components, and we typically identify a partition with any graph representing it. We think of the vertices from X (resp. X') as being the *upper vertices* (resp. *lower vertices*). For example, the partition

$$\alpha = \big\{\{1,4\},\{2,3,4',5'\},\{5,6\},\{1',3',6'\},\{2'\}\big\} \in \mathcal{P}_6$$

is represented by the graph $\alpha =$ .

In order to describe the product alluded to above, let $\alpha, \beta \in \mathcal{P}_X$. Consider now a third set X'', disjoint from both X and X', and in bijection with X via $x \mapsto x''$. Let $\alpha_\vee$ be the graph obtained from (a graph representing) α simply by changing the label of each lower vertex x' to x''. Similarly, let $\beta^\wedge$ be the graph obtained from β by changing the label of each upper vertex x to x''.

Consider now the graph $\Gamma(\alpha,\beta)$ on the vertex set $X \cup X' \cup X''$ obtained by joining $\alpha_\vee$ and $\beta^\wedge$ together so that each lower vertex x'' of $\alpha_\vee$ is identified with the corresponding upper vertex x'' of $\beta^\wedge$. Note that $\Gamma(\alpha,\beta)$, which we call the *product graph* of α and β, may contain multiple edges. We define $\alpha\beta \in \mathcal{P}_X$ to be the partition that satisfies the property that $x, y \in X \cup X'$ belong to the same block of $\alpha\beta$ if and only if there is a path from x to y in $\Gamma(\alpha,\beta)$. An example calculation (with X finite) is given in Figure 3.1.

We now define subsets

$$\mathcal{PB}r_X = \{\alpha \in \mathcal{P}_X : \text{each block of } \alpha \text{ has size at most } 2\}$$
$$\mathcal{B}r_X = \{\alpha \in \mathcal{P}_X : \text{each block of } \alpha \text{ has size } 2\}.$$

We note that $\mathcal{PB}r_X$ is a submonoid of $\mathcal{P}_X$ for any set X, while $\mathcal{B}r_X$ is a submonoid if and only if X is finite. For example, taking $X = \mathbb{N} = \{0,1,2,3,\dots\}$, the parti-

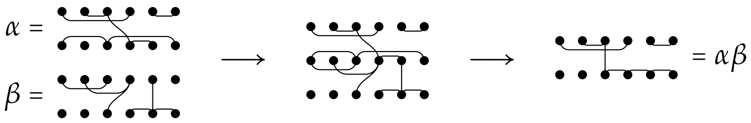


Figure 3.1: Two partitions $\alpha, \beta \in \mathcal{P}_6$ (left), their product $\alpha\beta \in \mathcal{P}_6$ (right), and the product graph $\Gamma(\alpha, \beta)$ (centre).

tions α, β pictured in Figure 3.2 both belong to $\mathcal{B}r_{\mathbb{N}}$, while the product $\alpha\beta$ (also pictured in Figure 3.2) does not. We call $\mathcal{PB}r_X$ the *partial Brauer monoid* and (in the case that X is finite) $\mathcal{B}r_X$ the *Brauer monoid* on X. Again, if $n \in \mathbb{N}$ and $X = [\![n]\!]$, we write $\mathcal{PB}r_n$ and $\mathcal{B}r_n$ for $\mathcal{PB}r_X$ and $\mathcal{B}r_X$, noting that $\mathcal{B}r_0 = \mathcal{PB}r_0 = \mathcal{P}_0 = \{\varnothing\}$.

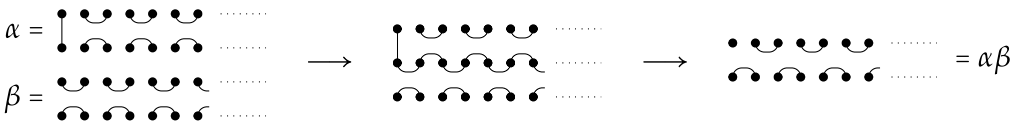


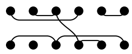
Figure 3.2: Two partitions $\alpha, \beta \in \mathcal{B}r_{\mathbb{N}}$ (left), their product $\alpha\beta \notin \mathcal{B}r_{\mathbb{N}}$ (right), and the product graph $\Gamma(\alpha, \beta)$ (centree).

We now introduce some notation and terminology that we will use throughout our study. Let $\alpha \in \mathcal{P}_X$. A block A of α is called a *transversal block* if $A \cap X \neq \varnothing \neq A \cap X'$, or otherwise an *upper* (resp. *lower*) *non-transversal block* if $A \cap X' = \varnothing$ (resp. $A \cap X = \varnothing$). The *rank* of α, denoted $\operatorname{rank}(\alpha)$, is equal to the number of transversal blocks of α. For $x \in X \cup X'$, let $[x]_\alpha$ denote the block of α containing x. We define the *upper* and *lower domains* of α to be the sets

$$\operatorname{dom}^\wedge(\alpha) = \{x \in X : [x]_\alpha \cap X' \neq \varnothing\} \qquad \text{and} \qquad \operatorname{dom}_\vee(\alpha) = \{x \in X : [x']_\alpha \cap X \neq \varnothing\}.$$

We also define the *upper* and *lower kernels* of α to be the equivalences

$$\ker^\wedge(\alpha) = \{(x,y) \in X \times X : [x]_\alpha = [y]_\alpha\} \qquad \text{and} \qquad \ker_\vee(\alpha) = \{(x,y) \in X \times X : [x']_\alpha = [y']_\alpha\}.$$

(The upper and lower domains and the upper and lower kernels have been called the *domain*, *codomain*, *kernel* and *cokernel* (respectively) in other works [16, 17, 20, 21], but there should be no confusion.) To illustrate these definitions, consider the partition $\alpha =$  from $\mathcal{P}_6$. Then $\operatorname{rank}(\alpha) = 1$, $\operatorname{dom}^\wedge(\alpha) = \{2,3\}$, $\operatorname{dom}_\vee(\alpha) = \{4,5\}$, and α has upper kernel-classes $\{1,4\}$, $\{2,3\}$, $\{5,6\}$, and lower kernel-classes $\{1,3,6\}$, $\{2\}$, $\{4,5\}$.

The next result was first proved for finite X in [48,76], and then in full generality in [22], though the language used in those papers was different from that used here; see also [59] on finite (partial and full) Brauer monoids.

Theorem 3.1.1 ([76, Theorem 17]). *For each $\alpha, \beta \in \mathcal{P}_X$, we have*

(i) $\alpha \mathscr{R} \beta$ *if and only if* $\operatorname{dom}^\wedge(\alpha) = \operatorname{dom}^\wedge(\beta)$ *and* $\ker^\wedge(\alpha) = \ker^\wedge(\beta)$;

(ii) $\alpha \mathscr{L} \beta$ *if and only if* $\operatorname{dom}_\vee(\alpha) = \operatorname{dom}_\vee(\beta)$ *and* $\ker_\vee(\alpha) = \ker_\vee(\beta)$;

(iii) $\alpha \mathscr{D} \beta$ *if and only if* $\operatorname{rank}(\alpha) = \operatorname{rank}(\beta)$. □

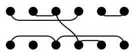
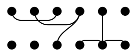
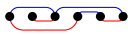
Finally, we define the *kernel* of α to be the join

$$\ker(\alpha) = \ker^\wedge(\alpha) \vee \ker_\vee(\alpha).$$

(The *join* $\varepsilon \vee \eta$ of two equivalence relations ε, η on X is the smallest equivalence relation containing the union $\varepsilon \cup \eta$; that is, $\varepsilon \vee \eta$ is the transitive closure of $\varepsilon \cup \eta$.) The equivalence classes of X with respect to $\ker(\alpha)$ are called the *kernel-classes* of α. We call a partition $\alpha \in \mathcal{P}_X$ *irreducible* if it has only one kernel-class; that is, α is irreducible if and only if $\ker(\alpha) = X \times X$. Some (but not all) partitions from $\mathcal{P}_X$ may be built up from irreducible partitions in a way we make precise below.

The equivalences $\ker^\wedge(\alpha), \ker_\vee(\alpha), \ker(\alpha)$ may be visualised graphically as follows. We define a graph $\Gamma^\wedge(\alpha)$ with vertex set X, and red edges drawn so that the connected components are precisely the $\ker^\wedge(\alpha)$-classes of X, and we define $\Gamma_\vee(\alpha)$ analogously but with blue edges. (Again, there are several possible choices for $\Gamma^\wedge(\alpha)$ and $\Gamma_\vee(\alpha)$, but we regard them all as equivalent.) We also define $\Gamma(\alpha)$ to be the graph on vertex set X with all the edges from both $\Gamma^\wedge(\alpha)$

and $\Gamma_\vee(\alpha)$. Then the connected components of $\Gamma(\alpha)$ are precisely the kernel-classes of α.

To illustrate these ideas, consider the partitions $\alpha =$  and $\beta =$  from $\mathcal{P}_6$. Then $\Gamma(\alpha) =$  and $\Gamma(\beta) =$ . So α is irreducible but β is not.

It will be convenient to conclude this section with two technical results that will help simplify subsequent proofs.

Lemma 3.1.2. *Let $\alpha, \beta \in \mathcal{P}_X$ and suppose $x, y \in X$. Then $(x, y) \in \ker_\vee(\alpha) \vee \ker^\wedge(\beta)$ if and only if x'' and y'' are joined by a path in the product graph $\Gamma(\alpha, \beta)$.*

Proof. If $(x, y) \in \ker_\vee(\alpha) \vee \ker^\wedge(\beta)$, then there is a sequence $x = x_0, x_1, \ldots, x_k = y$ such that $(x_0, x_1) \in \ker_\vee(\alpha)$, $(x_1, x_2) \in \ker^\wedge(\beta)$, $(x_2, x_3) \in \ker_\vee(\alpha)$, and so on. Such a sequence gives rise to a path $x'' = x_0'' \to x_1'' \to \cdots \to x_k'' = y''$ in the product graph $\Gamma(\alpha, \beta)$.

Conversely, suppose x'' and y'' are joined by a path in the product graph $\Gamma(\alpha, \beta)$. We prove that $(x, y) \in \ker_\vee(\alpha) \vee \ker^\wedge(\beta)$ by induction on the length of a path $x'' = z_0 \to z_1 \to \cdots \to z_t = y''$ in $\Gamma(\alpha, \beta)$. If $t = 0$, then $x = y$, and we are done, so suppose $t \geq 1$. If $z_r = w''$ for some $0 < r < t$, then an induction hypothesis applied to the shorter paths $x'' \to \cdots \to w''$ and $w'' \to \cdots \to y''$ tells us that (x, w) and (w, y), and hence also (x, y), belong to $\ker_\vee(\alpha) \vee \ker^\wedge(\beta)$. If none of $z_1, \ldots, z_{t-1}$ belong to X'', then they either all belong to X or all to X'. In the former case, it follows that $z_1, \ldots, z_{t-1}, y' \in [x']_\alpha$, so that $(x, y) \in \ker_\vee(\alpha) \subseteq \ker_\vee(\alpha) \vee \ker^\wedge(\beta)$. The other case is similar. □

Lemma 3.1.3. *Let $\alpha, \beta \in \mathcal{P}_X$ and suppose $A \cup B'$ is a transversal block of $\alpha\beta$. Then for any $a \in A$ and $b \in B$, and any $c, d \in X$ with $c' \in [a]_\alpha$, $d \in [b']_\beta$, we have $(c, d) \in \ker_\vee(\alpha) \vee \ker^\wedge(\beta)$.*

Proof. Consider a path $a = z_0 \to z_1 \to \cdots \to z_k = b'$ in the product graph $\Gamma(\alpha, \beta)$. Let $0 \leq r \leq k$ be the least index for which z_r does not belong to X, and let $0 \leq s \leq k$ be the greatest index for which z_s does not belong to X'. Then $z_r = x''$ and $z_s = y''$ for some $x, y \in X$ with $x' \in [a]_\alpha$ and $y \in [b']_\beta$. Since there is a path from

x'' to y'' in $\Gamma(\alpha,\beta)$, Lemma 3.1.2 tells us that $(x,y) \in \ker_\vee(\alpha) \vee \ker^\wedge(\beta)$. But we also have $(c,x) \in \ker_\vee(\alpha)$ and $(y,d) \in \ker^\wedge(\beta)$. Putting this all together gives $(c,d) \in \ker_\vee(\alpha) \vee \ker^\wedge(\beta)$, as required. □

3.2 Characterisation of idempotents

We now aim to give a characterisation of the idempotent partitions, and our first step in this direction is to describe the irreducible idempotents. (Recall that $\alpha \in \mathcal{P}_X$ is irreducible if $\ker(\alpha) = X \times X$.)

Lemma 3.2.1. *Suppose $\alpha \in \mathcal{P}_X$ is irreducible. Then α is an idempotent if and only if* $\mathrm{rank}(\alpha) \leq 1$.

Proof. It is clear that any partition of rank 0 is idempotent. Next, suppose $\mathrm{rank}(\alpha) = 1$, and let the unique transversal block of α be $A \cup B'$. Every non-transversal block of α is a block of α^2, so it suffices to show that $A \cup B'$ is a block of α^2. So suppose $a \in A$ and $b \in B$. Then there is a path from a to b' in (a graph representing) α, so it follows that there is a path from a to b'' and a path from a'' to b' in the product graph $\Gamma(\alpha,\alpha)$. Since α is irreducible, $\ker_\vee(\alpha) \vee \ker^\wedge(\alpha) = \ker(\alpha) = X \times X$, so Lemma 3.1.2 says that there is also a path from b'' to a''. Putting these together, we see that there is a path from a to b'. This completes the proof that α is idempotent.

Now suppose $\mathrm{rank}(\alpha) \geq 2$ and let $A \cup B'$ and $C \cup D'$ be distinct transversal blocks of α. Let a,b,c,d be arbitrary elements of A,B,C,D, respectively. Since α is irreducible, there is a path from b'' to d'' in the product graph $\Gamma(\alpha,\alpha)$. But since $a \in A$ and $b \in B$, there is a path from a to b'' in $\Gamma(\alpha,\alpha)$, and similarly there is a path from d'' to c. Putting these together, we see that there is a path from a to c, so that $A \cup C$ is contained in a block of α^2. But A and C are contained in different blocks of α, so it follows that α could not be an idempotent. □

We now show how idempotent partitions are built up out of irreducible ones. Suppose X_i $(i \in I)$ is a family of pairwise disjoint sets, and write $X =$

$\bigcup_{i\in I} X_i$. We define

$$\bigoplus_{i\in I} \mathcal{P}_{X_i} = \{\alpha \in \mathcal{P}_X : \text{each block of } \alpha \text{ is contained in } X_i \cup X_i' \text{ for some } i \in I\},$$

which is easily seen to be a submonoid of $\mathcal{P}_X$, and isomorphic to the direct product $\prod_{i\in I} \mathcal{P}_{X_i}$. Suppose $\alpha \in \bigoplus_{i\in I} \mathcal{P}_{X_i}$. For each $i \in I$, let $\alpha_i = \{A \in \alpha : A \subseteq X_i \cup X_i'\} \in \mathcal{P}_{X_i}$. We call α_i the *restriction of α to X_i*, and we write $\alpha_i = \alpha|_{X_i}$ and $\alpha = \bigoplus_{i\in I} \alpha_i$. We are now ready to prove the main result of this section, which gives a characterisation of the idempotent partitions. A precursor also appears in [48, 59] for finite (partial and full) Brauer monoids.

Theorem 3.2.2. *Let $\alpha \in \mathcal{P}_X$, and suppose the kernel-classes of α are X_i $(i \in I)$. Then α is an idempotent if and only if the following two conditions are satisfied:*

(i) *$\alpha \in \bigoplus_{i\in I} \mathcal{P}_{X_i}$, and*

(ii) *the restrictions $\alpha|_{X_i}$ all have rank at most 1.*

Proof. Suppose first that α is an idempotent, but that condition (i) fails. Then there is a block $A \cup B'$ of α such that $A \subseteq X_i$ and $B \subseteq X_j$ for distinct $i, j \in I$. Let $a \in A$ and $b \in B$. Since α is an idempotent, $A \cup B'$ is a block of α^2, and we also have $b' \in [a]_\alpha$ and $a \in [b']_\alpha$. So Lemma 3.1.3 tells us that $(a, b) \in \ker_\vee(\alpha) \vee \ker^\wedge(\alpha) = \ker(\alpha)$. But this contradicts the fact that $a \in X_i$ and $b \in X_j$, with X_i and X_j distinct kernel-classes. Thus, (i) must hold. It follows that $\alpha = \bigoplus_{i\in I} \alpha_i$ where $\alpha_i = \alpha|_{X_i}$ for each i. Then $\bigoplus_{i\in I} \alpha_i = \alpha = \alpha^2 = \bigoplus_{i\in I} \alpha_i^2$, so that each α_i is an irreducible idempotent, and (ii) now follows from Lemma 3.2.1.

Conversely, suppose (i) and (ii) both hold, and write $\alpha = \bigoplus_{i\in I} \alpha_i$ where $\alpha_i \in \mathcal{P}_{X_i}$ for each i. Since $\mathrm{rank}(\alpha_i) \leq 1$, Lemma 3.2.1 says that each α_i is an idempotent. It follows that $\alpha^2 = \bigoplus_{i\in I} \alpha_i^2 = \bigoplus_{i\in I} \alpha_i = \alpha$. □

3.3 Enumeration of idempotents

For a subset Σ of the partition monoid $\mathcal{P}_X$, we write $E(\Sigma) = \{\alpha \in \Sigma : \alpha^2 = \alpha\}$ for the set of all idempotents from Σ, and we write $e(\Sigma) = |E(\Sigma)|$. In this section, we aim to derive formulae for $e(\mathcal{K}_X)$ where $\mathcal{K}_X$ is one of $\mathcal{P}_X, \mathcal{B}r_X, \mathcal{PB}r_X$. The infinite case is essentially trivial, but we include it for completeness.

Proposition 3.3.1. *If X is infinite, then $e(\mathcal{B}r_X) = e(\mathcal{PB}r_X) = e(\mathcal{P}_X) = 2^{|X|}$.*

Proof. Since $\mathcal{B}r_X \subseteq \mathcal{PB}r_X \subseteq \mathcal{P}_X$ and $|\mathcal{P}_X| = 2^{|X|}$, it suffices to show that $e(\mathcal{B}r_X) = 2^{|X|}$. Let $\mathscr{A} = \{A \subseteq X : |X \setminus A| \geq \aleph_0\}$, and let $A \in \mathscr{A}$. Let β_A be any element of $\mathcal{B}r_X$ with $\text{dom}^\wedge(\beta_A) = \text{dom}_\vee(\beta_A) = A$ and such that $\{a, a'\}$ is a block of β_A for all $a \in A$. Then β_A is clearly an idempotent. The map $\mathscr{A} \to E(\mathcal{B}r_X) : A \mapsto \beta_A$ is clearly injective, so the result follows since $|\mathscr{A}| = 2^{|X|}$. □

The rest of the paper concerns the finite case so, unless stated otherwise, X will denote a finite set from here on.

For a subset Σ of $\mathcal{P}_X$, we write $C(\Sigma)$ for the set of all irreducible idempotents of Σ. So, by Theorem 3.2.2, $C(\Sigma)$ consists of all partitions $\alpha \in \Sigma$ such that $\ker(\alpha) = X \times X$ and $\text{rank}(\alpha) \leq 1$. We will also write $c(\Sigma) = |C(\Sigma)|$. Our next goal is to show that we may deduce the value of $e(\mathcal{K}_n)$ from the values of $c(\mathcal{K}_n)$ when $\mathcal{K}_n$ is one of $\mathcal{P}_n, \mathcal{B}r_n, \mathcal{PB}r_n$.

Recall that an *integer partition of n* is a k-tuple $\mu = (m_1, \ldots, m_k)$ of integers, for some k, satisfying $m_1 \geq \cdots \geq m_k \geq 1$ and $m_1 + \cdots + m_k = n$. We write $\mu \vdash n$ to indicate that μ is an integer partition of n. With $\mu \vdash n$ as above, we will also write $\mu = (1^{\mu_1}, \ldots, n^{\mu_n})$ to indicate that, for each i, exactly μ_i of the m_j are equal to i. By convention, we consider $\mu = \varnothing$ to be the unique integer partition of 0.

Recall that a *set partition of X* is a collection $\mathbf{X} = \{X_i : i \in I\}$ of pairwise disjoint non-empty subsets of X whose union is X. We will write $\mathbf{X} \vDash X$ to indicate that $\mathbf{X}$ is a set partition of X. Suppose $\mathbf{X} = \{X_1, \ldots, X_k\} \vDash [\![n]\!]$. For $i \in [\![n]\!]$, write $\mu_i(\mathbf{X})$ for the cardinality of the set $\{j \in \mathbf{k} : |X_j| = i\}$, and put $\mu(\mathbf{X}) = (1^{\mu_1(\mathbf{X})}, \ldots, n^{\mu_n(\mathbf{X})})$, so $\mu(\mathbf{X}) \vdash n$. For $\mu = (1^{\mu_1}, \ldots, n^{\mu_n}) \vdash n$, we write $\pi(\mu)$ for the number of set partitions $\mathbf{X} \vDash [\![n]\!]$ such that $\mu(\mathbf{X}) = \mu$. It is easily seen (and well-known) that

$$\pi(\mu) = \frac{n!}{\prod_{i=1}^n \mu_i!(i!)^{\mu_i}}.$$

If $\alpha \in \mathcal{P}_n$ has kernel classes $X_1, \ldots, X_k$, we write $\mu(\alpha) = \mu(\mathbf{X})$ where $\mathbf{X} = \{X_1, \ldots, X_k\}$.

Note that if $|X_1| \geq \cdots \geq |X_k|$, then $\mu(\alpha) = (|X_1|, \ldots, |X_k|)$ in the alternative notation for integer partitions.

Theorem 3.3.2. *If $\mathcal{K}_n$ is one of $\mathcal{P}_n, \mathcal{B}r_n, \mathcal{PB}r_n$, then*

$$e(\mathcal{K}_n) = n! \cdot \sum_{\mu \vdash n} \prod_{i=1}^{n} \frac{c(\mathcal{K}_i)^{\mu_i}}{\mu_i!(i!)^{\mu_i}}.$$

The numbers $e(\mathcal{K}_n)$ satisfy the recurrence:

$$e(\mathcal{K}_0) = 1, \quad e(\mathcal{K}_n) = \sum_{m=1}^{n} \binom{n-1}{m-1} c(\mathcal{K}_m)\, e(\mathcal{K}_{n-m}) \qquad \text{for } n \geq 1.$$

The values of $c(\mathcal{K}_n)$ are given in Propositions 3.3.6, 3.3.9 and 3.3.13.

Proof. Fix an integer partition $\mu = (m_1, \ldots, m_k) = (1^{\mu_1}, \ldots, n^{\mu_n}) \vdash n$. We count the number of idempotents α from $\mathcal{K}_n$ with $\mu(\alpha) = \mu$. We first choose the kernel-classes $X_1, \ldots, X_k$ of α, with $|X_i| = m_i$ for each i, which we may do in $\pi(\mu)$ ways. For each i, the restriction of α to X_i is an irreducible idempotent of $\mathcal{K}_{X_i}$, and there are precisely $c(\mathcal{K}_{X_i}) = c(\mathcal{K}_{m_i})$ of these. So there are $c(\mathcal{K}_{m_1}) \cdots c(\mathcal{K}_{m_k}) = c(\mathcal{K}_1)^{\mu_1} \cdots c(\mathcal{K}_n)^{\mu_n}$ idempotents with kernel classes $X_1, \ldots, X_k$. Multiplying by $\pi(\mu)$ and summing over all μ gives the first equality.

For the recurrence, note first that $E(\mathcal{K}_0) = \mathcal{K}_0 = \{\varnothing\}$, where $\varnothing$ denotes the empty partition. Now suppose $n \geq 1$ and let $m \in [\![n]\!]$. We will count the number of idempotents α from $\mathcal{K}_n$ such that the kernel-class A of α containing 1 has size m. We first choose the remaining $m-1$ elements of A, which we may do in $\binom{n-1}{m-1}$ ways. The restriction $\alpha|_A$ is an irreducible idempotent from $\mathcal{K}_A$, and may be chosen in $c(\mathcal{K}_A) = c(\mathcal{K}_m)$ ways, while the restriction $\alpha|_{[\![n]\!] \setminus A}$ is an idempotent from $\mathcal{K}_{[\![n]\!] \setminus A}$, and may be chosen in $e(\mathcal{K}_{[\![n]\!] \setminus A}) = e(\mathcal{K}_{n-m})$ ways. Summing over all $m \in [\![n]\!]$ gives the recurrence, and completes the proof. □

It will also be convenient to record a result concerning the number of idempotents of a fixed rank. If $\mathcal{K}_n$ is one of $\mathcal{P}_n, \mathcal{B}r_n, \mathcal{PB}r_n$ and $0 \leq r \leq n$, we write

$$D_r(\mathcal{K}_n) = \{\alpha \in \mathcal{K}_n : \operatorname{rank}(\alpha) = r\}.$$

So, by Theorem 3.1.1 and the fact that $\mathcal{B}r_n$ and $\mathcal{PB}r_n$ are regular subsemigroups of $\mathcal{P}_n$, the sets $D_r(\mathcal{K}_n)$ are precisely the $\mathscr{D}$-classes of $\mathcal{K}_n$. Note that $D_r(\mathcal{B}r_n)$ is non-empty if and only if $n \equiv r \pmod 2$, as each non-transversal block of an element of $\mathcal{B}r_n$ has size 2. For a subset $\Sigma \subseteq \mathcal{P}_n$ and an integer partition $\mu \vdash n$, we write

$$E_\mu(\Sigma) = \{\alpha \in E(\Sigma) : \mu(\alpha) = \mu\} \qquad \text{and} \qquad e_\mu(\Sigma) = |E_\mu(\Sigma)|.$$

For a subset Σ of $\mathcal{P}_n$, and for $r \in \{0,1\}$, let

$$C_r(\Sigma) = \{\alpha \in C(\Sigma) : \operatorname{rank}(\alpha) = r\} \qquad \text{and} \qquad c_r(\Sigma) = |C_r(\Sigma)|.$$

So by Lemma 3.2.1, $c(\Sigma) = c_0(\Sigma) + c_1(\Sigma)$.

Theorem 3.3.3. *Suppose $\mathcal{K}_n$ is one of $\mathcal{P}_n, \mathcal{B}r_n, \mathcal{PB}r_n$ and $0 \le r \le n$. Then*

$$e(D_r(\mathcal{K}_n)) = \sum_{\mu \vdash n} e_\mu(D_r(\mathcal{K}_n)).$$

If $\mu = (m_1, \dots, m_k) = (1^{\mu_1}, \dots, n^{\mu_n}) \vdash n$, then

$$e_\mu(D_r(\mathcal{K}_n)) = \frac{n!}{\prod_{i=1}^n \mu_i!(i!)^{\mu_i}} \sum_{\substack{A \subseteq \mathbf{k} \\ |A|=r}} \left(\prod_{i \in A} c_1(\mathcal{K}_{m_i}) \cdot \prod_{j \in \mathbf{k} \setminus A} c_0(\mathcal{K}_{m_j}) \right).$$

The values of $c_r(\mathcal{K}_n)$ are given in Propositions 3.3.6, 3.3.9 and 3.3.13.

Proof. Fix $\mu = (1^{\mu_1}, \dots, n^{\mu_n}) = (m_1, \dots, m_k) \vdash n$, and suppose $\alpha \in E(D_r(\mathcal{K}_n))$ is such that $\mu(\alpha) = \mu$. We choose the kernel-classes $X_1, \dots, X_k$ (where $|X_i| = m_i$) of α in $\pi(\mu)$ ways. Now, $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_k$, with $\alpha_i \in C(\mathcal{K}_{X_i})$ for each i, and $r = \operatorname{rank}(\alpha) = \operatorname{rank}(\alpha_1) + \cdots + \operatorname{rank}(\alpha_k)$. So we require precisely r of the α_i to have rank 1. For each subset $A \subseteq \mathbf{k}$ with $|A| = r$, there are $\prod_{i \in A} c_1(\mathcal{K}_{m_i}) \cdot \prod_{j \in \mathbf{k} \setminus A} c_0(\mathcal{K}_{m_j})$ ways to choose the α_i so that $\operatorname{rank}(\alpha_i) = 1$ if and only if $i \in A$. Summing over all such A gives the result. □

Remark 1. If $\mathcal{K}_n$ is one of $\mathcal{P}_n, \mathcal{B}r_n, \mathcal{PB}r_n$, then the ideals of $\mathcal{K}_n$ are precisely the sets

$$I_r(\mathcal{K}_n) = \bigcup_{s \le r} D_s(\mathcal{K}_n).$$

(See [21,49].) It follows immediately that the number of idempotents in such an ideal is given by $e(I_r(\mathcal{K}_n)) = \sum_{s\le r} e(D_s(\mathcal{K}_n))$, so these values may be deduced from the values of $e(D_s(\mathcal{K}_n))$ given above.

We also give a recurrence for the numbers $e(D_r(\mathcal{K}_n))$.

Theorem 3.3.4. *The numbers $e(D_r(\mathcal{K}_n))$ satisfy the recurrence*

$$e(D_n(\mathcal{K}_n)) = 1 \qquad e(D_0(\mathcal{K}_n)) = \rho(\mathcal{K}_n)^2$$

and

$$e(D_r(\mathcal{K}_n)) = \sum_{m=1}^{n} \binom{n-1}{m-1}\Big(c_0(\mathcal{K}_m)e(D_r(\mathcal{K}_{n-m})) + c_1(\mathcal{K}_m)e(D_{r-1}(\mathcal{K}_{n-m}))\Big)$$

if $1 \le r \le n-1$, where $\rho(\mathcal{K}_n)$ is the number of $\mathscr{R}$-classes in $D_0(\mathcal{K}_n)$; these values are given in Lemma 3.3.5. The values of $c_r(\mathcal{K}_n)$ are given in Propositions 3.3.6, 3.3.9 and 3.3.13.

Proof. Note that $D_n(\mathcal{K}_n)$ is the group of units of $\mathcal{K}_n$ (which is the symmetric group $\mathcal{S}_n$), so $e(D_n(\mathcal{K}_n)) = 1$ for all n. Also, since every element of $D_0(\mathcal{K}_n)$ is an idempotent, it follows that $e(D_0(\mathcal{K}_n)) = |D_0(\mathcal{K}_n)| = \rho(\mathcal{K}_n)^2$. Now consider an element $\alpha \in D_r(\mathcal{K}_n)$ where $1 \le r \le n-1$, and suppose the kernel-class A of α containing 1 has size $m \in [\![n]\!]$. Then, as in the proof of Theorem 3.3.2, the restriction $\alpha|_A$ belongs to $C(\mathcal{K}_A)$ and the restriction $\alpha|_{[\![n]\!]\setminus A}$ belongs to $E(\mathcal{K}_{[\![n]\!]\setminus A})$. But, since $\operatorname{rank}(\alpha) = r$, it follows that either

(i) $\alpha|_A \in C_0(\mathcal{K}_A)$ and $\alpha|_{[\![n]\!]\setminus A} \in E(D_r(\mathcal{K}_{[\![n]\!]\setminus A}))$, or

(ii) $\alpha|_A \in C_1(\mathcal{K}_A)$ and $\alpha|_{[\![n]\!]\setminus A} \in E(D_{r-1}(\mathcal{K}_{[\![n]\!]\setminus A}))$.

The proof concludes in a similar fashion to the proof of Theorem 3.3.2. □

As usual, for an odd integer k, we write $k!! = k(k-2)\cdots 3\cdot 1$, and we interpret $(-1)!! = 1$.

Lemma 3.3.5. *If $\rho(\mathcal{K}_n)$ denotes the number of $\mathscr{R}$-classes in $D_0(\mathcal{K}_n)$ where $\mathcal{K}_n$ is one of $\mathcal{P}_n, \mathcal{B}r_n, \mathcal{PB}r_n$, then*

$$\rho(\mathcal{P}_n) = B(n), \qquad \rho(\mathcal{B}r_n) = \begin{cases} (n-1)!! & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$
$$\rho(\mathcal{PB}r_n) = a_n,$$

where $B(n)$ is the nth Bell number, and a_n satisfies the recurrence

$$a_0 = a_1 = 1, \quad a_n = a_{n-1} + (n-1)a_{n-2} \text{ for } n \geq 2.$$

Proof. The results concerning $\mathcal{P}_n$ and $\mathcal{B}r_n$ are well-known; see for example the proof of [21, Theorems 7.5 and 8.4]. For the $\mathcal{PB}r_n$ statement, note that, since $\mathrm{dom}^\wedge(\alpha) = \varnothing$ for all $\alpha \in D_0(\mathcal{PB}r_n)$, Theorem 3.1.1 says that the $\mathscr{R}$- classes of $\mathcal{PB}r_n$ are indexed by the equivalence relations ε on $[\![n]\!]$ that satisfy the condition that each equivalence class has size 1 or 2. Such equivalences are in one-one correspondence with the involutions (i.e., self-inverse permutations) of $[\![n]\!]$, of which there are a_n (see A000085 on the OEIS [71]). □

Remark 2. The *completely regular* elements of a semigroup are those that are $\mathscr{H}$-related to an idempotent. Because the $\mathscr{H}$-class of any idempotent from a (non-empty) $\mathscr{D}$-class $D_r(\mathcal{K}_n)$ is isomorphic to the symmetric group $\mathcal{S}_r$, it follows that the number of completely regular elements in $\mathcal{K}_n$ is equal to $\sum_{r=0}^{n} r! e(D_r(\mathcal{K}_n))$.

3.3.1 The partition monoid

In this section, we obtain formulae for $c_0(\mathcal{P}_n), c_1(\mathcal{P}_n), c(\mathcal{P}_n)$. Together with Theorems 3.3.2, 3.3.3 and 3.3.4, this yields formulae and recurrences for $e(\mathcal{P}_n)$ and $e(D_r(\mathcal{P}_n))$. The key step is to enumerate the pairs of equivalence relations on $[\![n]\!]$ with specified numbers of equivalence classes and whose join is equal to the universal relation $[\![n]\!] \times [\![n]\!]$.

Let $E(n)$ denote the set of all equivalence relations on $[\![n]\!]$. If $\varepsilon \in E(n)$, we denote by $[\![n]\!]/\varepsilon$ the quotient of $[\![n]\!]$ by ε, which consists of all ε-classes of $[\![n]\!]$.

For $r,s \in [\![n]\!]$, we define sets

$$E(n,r) = \{\varepsilon \in E(n) : |[\![n]\!]/\varepsilon| = r\},$$
$$E(n,r,s) = \{(\varepsilon,\eta) \in E(n,r) \times E(n,s) : \varepsilon \vee \eta = [\![n]\!] \times [\![n]\!]\},$$

and we write $e(n,r,s) = |E(n,r,s)|$.

Proposition 3.3.6. *If $n \geq 1$, and $i = 0,1$ then*

$$c_i(\mathcal{P}_n) = \sum_{r,s \in [\![n]\!]} (rs)^i \cdot e(n,r,s), \quad c(\mathcal{P}_n) = \sum_{r,s \in [\![n]\!]} (1+rs)e(n,r,s).$$

A recurrence for the numbers $e(n,r,s)$ is given in Proposition 3.3.7.

Proof. Let $r,s \in [\![n]\!]$ and consider a pair $(\varepsilon,\eta) \in E(n,r,s)$. We count the number of idempotent partitions $\alpha \in C(\mathcal{P}_n)$ such that $\ker^{\wedge}(\alpha) = \varepsilon$ and $\ker_{\vee}(\alpha) = \eta$. Clearly there is a unique such α satisfying $\operatorname{rank}(\alpha) = 0$. To specify such an α with $\operatorname{rank}(\alpha) = 1$, we must also specify one of the ε-classes and one of the η-classes to form the unique transversal block of α, so there are rs of these. Since there are $e(n,r,s)$ choices for (ε,η), the statements follow after summing over all r,s. □

Remark 3. The numbers $c_0(\mathcal{P}_n)$ count the number of pairs of equivalences on $[\![n]\!]$ whose join is $[\![n]\!] \times [\![n]\!]$. These numbers may be found in Sequence A060639 on the OEIS [71].

For the proof of the following result, we denote by $\varepsilon_{ij} \in E(n)$ the equivalence relation whose only non-trivial equivalence class is $\{i,j\}$. On a few occasions in the proof, we will make use of the (trivial) fact that if $\varepsilon \in E(n,r)$, then $\varepsilon \vee \varepsilon_{ij}$ has at least $r-1$ equivalence classes. As usual, we write $S(n,r) = |E(n,r)|$; these are the (unsigned) Stirling numbers of the second kind.

Proposition 3.3.7. *The numbers $e(n,r,s) = |E(n,r,s)|$ satisfy the recurrence:*

$$e(n,r,1) = S(n,r)$$
$$e(n,1,s) = S(n,s)$$
$$e(n,r,s) = s \cdot e(n-1,r-1,s) + r \cdot e(n-1,r,s-1) + rs \cdot e(n-1,r,s)$$
$$+ \sum_{m=1}^{n-2} \binom{n-2}{m} \sum_{a=1}^{r-1} \sum_{b=1}^{s-1} \big(a(s-b) + b(r-a)\big) e(m,a,b) e(n-m-1,r-a,s-b).$$

where $r,s \geq 2$.

Proof. The $r = 1$ and $s = 1$ cases are clear, so suppose $r,s \geq 2$. Consider a pair $(\varepsilon,\eta) \in E(n,r,s)$. We consider several cases. Throughout the proof, we will write $[\![n]\!]^\flat = \{1,\ldots,n-1\}$.

Case 1. Suppose first that $\{n\}$ is an ε-class. Let $\varepsilon' = \varepsilon \cap ([\![n]\!]^\flat \times [\![n]\!]^\flat)$ and $\eta' = \eta \cap ([\![n]\!]^\flat \times [\![n]\!]^\flat)$ denote the induced equivalence relations on $[\![n]\!]^\flat$. Then we clearly have $\varepsilon' \in E(n-1,r-1)$. Also, $\{n\}$ cannot be an η-class, or else then $\{n\}$ would be an $\varepsilon \vee \eta$-class, contradicting the fact that $\varepsilon \vee \eta = [\![n]\!] \times [\![n]\!]$. It follows that $\eta' \in E(n-1,s)$.

Next we claim that $\varepsilon' \vee \eta' = [\![n]\!]^\flat \times [\![n]\!]^\flat$. Indeed, suppose to the contrary that $\varepsilon' \vee \eta'$ has $k \geq 2$ equivalence classes. Let $\eta'' \in E(n)$ be the equivalence on $[\![n]\!]$ obtained from η' by declaring $\{n\}$ to be an η''-class. Then $\varepsilon \vee \eta''$ has $k+1$ equivalence classes. But $\eta = \eta'' \vee \varepsilon_{in}$ for some $i \in [\![n]\!]^\flat$. It follows that $\varepsilon \vee \eta = (\varepsilon \vee \eta'') \vee \varepsilon_{in}$ has $(k+1)-1 = k \geq 2$ equivalence classes, contradicting the fact that $\varepsilon \vee \eta = [\![n]\!] \times [\![n]\!]$. So this establishes the claim.

It follows that $(\varepsilon',\eta') \in E(n-1,r-1,s)$. So there are $e(n-1,r-1,s)$ such pairs. We then have to choose which block of η' to put n into when creating η, and this can be done in s ways. So it follows that there are $s \cdot e(n-1,r-1,s)$ pairs (ε,η) in Case 1.

Case 2. By symmetry, there are $r \cdot e(n-1,r,s-1)$ pairs (ε,η) in the case that $\{n\}$ is an η-class.

Case 3. Now suppose that $\{n\}$ is neither an ε-class nor an η-class. Again, let ε',η' be the induced equivalences on $[\![n]\!]^\flat$. This time, $\varepsilon' \in E(n-1,r)$ and $\eta' \in E(n-1,s)$. We now consider two subcases.

Case 3.1. If $\varepsilon' \vee \eta' = [\![n]\!]^\flat \times [\![n]\!]^\flat$, then $(\varepsilon',\eta') \in E(n-1,r,s)$. By similar reasoning to that above, there are $rs \cdot e(n-1,r,s)$ pairs (ε,η) in this case.

Case 3.2. Finally, suppose $\varepsilon' \vee \eta' \neq [\![n]\!]^\flat \times [\![n]\!]^\flat$, and denote by k the number of $\varepsilon' \vee \eta'$-classes. We claim that $k = 2$. Indeed, suppose this is not the case. By assumption, $k \neq 1$, so it follows that $k \geq 3$. Let ε'' and η'' be the equivalence relations on $[\![n]\!]$ obtained from ε' and η' by declaring $\{n\}$ to be an ε''- and η''-class. Then $\varepsilon'' \vee \eta''$ has $k+1$ equivalence classes, and $\varepsilon = \varepsilon'' \vee \varepsilon_{in}$ and $\eta = \eta'' \vee \varepsilon_{jn}$

for some $i, j \in [\![n]\!]^\flat$. So $\varepsilon \vee \eta = (\varepsilon'' \vee \eta'') \vee \varepsilon_{in} \vee \varepsilon_{jn}$ has at least $(k+1) - 2 = k - 1 \geq 2$ equivalence classes, a contradiction. So we have proved the claim.

Denote by B_1 the $\varepsilon' \vee \eta'$-class of $[\![n]\!]^\flat$ containing 1, and let the other $\varepsilon' \vee \eta'$-class be B_2, noting that $1 \leq |B_1| \leq n-2$. If $|B_1| = m$, then there are $\binom{n-2}{m-1}$ ways to choose B_1 (and $B_2 = [\![n]\!]^\flat \setminus B_1$ is then fixed).

For $i = 1, 2$, let $\varepsilon_i = \varepsilon \cap (B_i \times B_i)$ and $\eta_i = \eta \cap (B_i \times B_i)$. Note that $\varepsilon_i \vee \eta_i = B_i \times B_i$ for each i. Let $a = |B_1/\varepsilon_1|$ and $b = |B_1/\eta_1|$. So $1 \leq a \leq r-1$ and $1 \leq b \leq s-1$, and also $|B_2/\varepsilon_2| = r - a$ and $|B_2/\eta_2| = s - b$. So, allowing ourselves to abuse notation slightly, we have $(\varepsilon_1, \eta_1) \in E(m, a, b)$ and $(\varepsilon_2, \eta_2) \in E(n-m-1, r-a, s-b)$. So there are $e(m, a, b)e(n-m-1, r-a, s-b)$ ways to choose $\varepsilon_1, \varepsilon_2, \eta_1, \eta_2$.

We must also choose which blocks of ε' and η' to add n to, when creating ε, η from ε', η'. But, in order to ensure that $\varepsilon \vee \eta = [\![n]\!] \times [\![n]\!]$, if we add n to one of the ε'-classes in B_1, we must add n to one of the η'-classes in B_2, and vice versa. So there are $a(s-b) + b(r-a)$ choices for the blocks to add n to.

Multiplying the quantities obtained in the previous three paragraphs, and summing over the appropriate values of m, a, b, we get a total of

$$\sum_{m=1}^{n-2} \binom{n-2}{m} \sum_{a=1}^{r-1} \sum_{b=1}^{s-1} \big(a(s-b) + b(r-a)\big) e(m, a, b) e(n-m-1, r-a, s-b)$$

pairs (ε, η) in Case 3.2.

Adding the values from all the above cases gives the desired result. □

3.3.2 The Brauer monoid

We now apply the general results above to derive a formula for $e(\mathcal{B}r_n)$. As in the previous section, the key step is to obtain formulae for $c_0(\mathcal{B}r_n), c_1(\mathcal{B}r_n), c(\mathcal{B}r_n)$, but the simple form of these values (see Proposition 3.3.9) allows us to obtain neat expressions for $e(\mathcal{B}r_n)$ and $e(D_r(\mathcal{B}r_n))$ (see Theorem 3.3.10). But first it will be convenient to prove a result concerning the graphs $\Gamma(\alpha)$, where α belongs to the larger partial Brauer monoid $\mathcal{PB}r_n$, as it will be useful on several occasions (these graphs were defined after Theorem 3.1.1).

Lemma 3.3.8. *Let $\alpha \in C(\mathcal{PB}r_X)$ where X is finite. Then $\Gamma(\alpha)$ is either a cycle or a path.*

Proof. The result is trivial if $|X| = 1$, so suppose $|X| \geq 2$. In the graph $\Gamma(\alpha)$, no vertex can have two red or two blue edges coming out of it, so it follows that the degree of each vertex is at most 2. It follows that $\Gamma(\alpha)$ is a union of paths and cycles. Since $\Gamma(\alpha)$ is connected, we are done. □

Proposition 3.3.9. *If $n \geq 1$, then*

$$c_0(\mathcal{B}r_n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (n-1)! & \text{if } n \text{ is even,} \end{cases}$$

$$c_1(\mathcal{B}r_n) = \begin{cases} n! & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

$$c(\mathcal{B}r_n) = \begin{cases} n! & \text{if } n \text{ is odd} \\ (n-1)! & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let α be an irreducible idempotent from $\mathcal{B}r_n$. By Lemma 3.2.1, and the fact that $\operatorname{rank}(\beta) \in \{n, n-2, n-4, \ldots\}$ for all $\beta \in \mathcal{B}r_n$, we see that $\operatorname{rank}(\alpha) = 0$ if n is even, while $\operatorname{rank}(\alpha) = 1$ if n is odd. If n is even, then by Lemma 3.3.8 and the fact that $\Gamma(\alpha)$ has the same number of red and blue edges, $\Gamma(\alpha)$ is a cycle $1 - i_2 - i_3 - \cdots - i_n - 1$, where $\{i_2, \ldots, i_n\} = \{2, \ldots, n\}$, and there are precisely $(n-1)!$ such cycles. Similarly, if n is odd, then $\Gamma(\alpha)$ is a path $i_1 - i_2 - i_3 - \cdots - i_n$, where $\{i_1, \ldots, i_n\} = [\![n]\!]$, and there are $n!$ such paths. □

Theorem 3.3.10. *Let $n \in \mathbb{N}$ and put $k = \lfloor \frac{n}{2} \rfloor$. Then*

$$e(\mathcal{B}r_n) = \sum_{\mu \vdash n} \frac{n!}{\prod_{i=1}^{n} \mu_i! \cdot \prod_{j=1}^{k} (2j)^{\mu_{2j}}}.$$

If $0 \leq r \leq n$, then

$$e(D_r(\mathcal{B}r_n)) = \sum_{\substack{\mu \vdash n \\ \mu_1 + \mu_3 + \cdots = r}} \frac{n!}{\prod_{i=1}^{n} \mu_i! \cdot \prod_{j=1}^{k} (2j)^{\mu_{2j}}}.$$

Proof. By Theorem 3.3.2 and Proposition 3.3.9,

$$e(\mathcal{B}r_n) = n! \cdot \sum_{\mu \vdash n} \frac{(1!)^{\mu_1}(1!)^{\mu_2}(3!)^{\mu_3}(3!)^{\mu_4}\cdots}{\mu_1!\cdots\mu_n! \cdot (1!)^{\mu_1}(2!)^{\mu_2}(3!)^{\mu_3}(4!)^{\mu_4}\cdots} = n! \cdot \sum_{\mu \vdash n} \frac{1}{\mu_1!\cdots\mu_n! \cdot 2^{\mu_2} \cdot 4^{\mu_4}\cdots},$$

establishing the first statement. For the second, suppose $\mu = (m_1, \ldots, m_k) = (1^{\mu_1}, \ldots, n^{\mu_n}) \vdash n$. Theorem 3.3.3 gives

$$e_\mu(D_r(\mathcal{B}r_n)) = \frac{n!}{\prod_{i=1}^n \mu_i!(i!)^{\mu_i}} \sum_{\substack{A \subseteq \mathbf{k} \\ |A|=r}} \left(\prod_{i \in A} c_1(\mathcal{B}r_{m_i}) \cdot \prod_{j \in \mathbf{k} \setminus A} c_0(\mathcal{B}r_{m_j}) \right).$$

By Proposition 3.3.9, we see that for $A \subseteq \mathbf{k}$ with $|A| = r$,

$$\begin{aligned} \prod_{i \in A} c_1(\mathcal{B}r_{m_i}) \cdot \prod_{j \in \mathbf{k} \setminus A} c_0(\mathcal{B}r_{m_j}) \neq 0 &\iff m_i \text{ is odd for all } i \in A \text{ and } m_j \text{ is even for all } j \in \mathbf{k} \setminus A \\ &\iff A = \{i \in \mathbf{k} : m_i \text{ is odd}\}. \end{aligned}$$

So

$$\begin{aligned} e_\mu(D_r(\mathcal{B}r_n)) \neq 0 &\iff \{i \in \mathbf{k} : m_i \text{ is odd}\} \text{ has size } r \\ &\iff \mu_1 + \mu_3 + \cdots = r, \end{aligned}$$

in which case,

$$e_\mu(D_r(\mathcal{B}r_n)) = n! \cdot \frac{(1!)^{\mu_1}(1!)^{\mu_2}(3!)^{\mu_3}(3!)^{\mu_4}\cdots}{\mu_1!\cdots\mu_n! \cdot (1!)^{\mu_1}(2!)^{\mu_2}(3!)^{\mu_3}(4!)^{\mu_4}\cdots} = n! \cdot \frac{1}{\mu_1!\cdots\mu_n! \cdot 2^{\mu_2} \cdot 4^{\mu_4}\cdots}.$$

Summing over all $\mu \vdash n$ with $\mu_1 + \mu_3 + \cdots = r$ gives the desired expression for $e(D_r(\mathcal{B}r_n))$. □

Remark 4. The formula for $e(\mathcal{B}r_n)$ may be deduced from [48, Proposition 4.10]. Note that $e(D_r(\mathcal{B}r_n)) \neq 0$ if and only if $n \equiv r \pmod 2$.

Proposition 3.3.9 also leads to a simple form of the recurrences from Theorems 3.3.2 and 3.3.4 for the numbers $e(\mathcal{B}r_n)$ and $e(D_r(\mathcal{B}r_n))$.

Theorem 3.3.11. *The numbers $e(\mathcal{B}r_n)$ satisfy the recurrence:*

$$e(\mathcal{B}r_0) = 1, \quad e(\mathcal{B}r_n) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1}{2i-1}(2i-1)!\, e(\mathcal{B}r_{n-2i}) + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2i}(2i+1)!\, e(\mathcal{B}r_{n-2i-1}) \qquad \text{for } n \geq 1.$$

□

Theorem 3.3.12. *The numbers $e(D_r(\mathcal{B}r_n))$ satisfy the following recurrence for $n \geq 1$:*

$$e(D_n(\mathcal{B}r_n)) = 1$$

$$e(D_0(\mathcal{B}r_n)) = \begin{cases} (n-1)!!^2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$$e(D_r(\mathcal{B}r_n)) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1}{2i-1}(2i-1)!\, e(D_r(\mathcal{B}r_{n-2i})) + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2i}(2i+1)!\, e(D_{r-1}(\mathcal{B}r_{n-2i-1})).$$

□

3.3.3 The partial Brauer monoid

As usual, the key step in calculating $e(\mathcal{PB}r_n)$ is to obtain formulae for $c(\mathcal{PB}r_n)$.

Proposition 3.3.13. *If $n \geq 1$, then*

$$c_0(\mathcal{PB}r_n) = \begin{cases} n! & \text{if } n \text{ is odd} \\ (n+1)\cdot(n-1)! & \text{if } n \text{ is even,} \end{cases}$$

$$c_1(\mathcal{PB}r_n) = \begin{cases} n! & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

$$c(\mathcal{PB}r_n) = \begin{cases} 2\cdot n! & \text{if } n \text{ is odd} \\ (n+1)\cdot(n-1)! & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let α be an irreducible idempotent from $\mathcal{PB}r_n$. Suppose first that n is odd. By Lemma 3.3.8, whether $\operatorname{rank}(\alpha)$ is equal to 0 or 1, $\Gamma(\alpha)$ is a path $i_1-i_2-i_3-\cdots-i_n$, and there are $n!$ such paths. Now suppose n is even. Then $\Gamma(\alpha)$ is either a cycle $1-i_2-i_3-\cdots-i_n-1$, of which there are $(n-1)!$, or else a path $i_1-i_2-i_3-\cdots-i_n$ or $i_1-i_2-i_3-\cdots-i_n$, of which there are $n!/2$ of both kinds. All of these have $\operatorname{rank}(\alpha) = 0$, and adding them gives $n! + (n-1)! = (n+1)\cdot(n-1)!$. □

Theorem 3.3.14. *Let $n \in \mathbb{N}$ and put $k = \lfloor \frac{n}{2} \rfloor$. Then*

$$e(\mathcal{PB}r_n) = n! \cdot \sum_{\mu \vdash n} \frac{\prod_{j=1}^{k} (1 + \frac{1}{2j})^{\mu_{2j}}}{\prod_{i=1}^{n} \mu_i!} 2^{\mu_1+\mu_3+\cdots}.$$

If $0 \le r \le n$, then

$$e(D_r(\mathcal{PB}r_n)) = n! \cdot \sum_{\substack{\mu \vdash n \\ \mu_1+\mu_3+\cdots \ge r}} \frac{\prod_{j=1}^{k} (1 + \frac{1}{2j})^{\mu_{2j}}}{\prod_{i=1}^{n} \mu_i!} \binom{\mu_1+\mu_3+\cdots}{r}.$$

Proof. By Theorem 3.3.2 and Proposition 3.3.13,

$$\begin{aligned} e(\mathcal{PB}r_n) &= n! \cdot \sum_{\mu \vdash n} \frac{(2\cdot 1!)^{\mu_1}(2\cdot 3!)^{\mu_3}\cdots(3\cdot 1!)^{\mu_2}(5\cdot 3!)^{\mu_4}\cdots}{\mu_1!\cdots\mu_n!\cdot(1!)^{\mu_1}(3!)^{\mu_3}\cdots(2!)^{\mu_2}(4!)^{\mu_4}\cdots} \\ &= n! \cdot \sum_{\mu \vdash n} \frac{2^{\mu_1+\mu_3+\cdots}}{\mu_1!\cdots\mu_n!} \left(\frac{3}{2}\right)^{\mu_2} \left(\frac{5}{4}\right)^{\mu_4} \cdots, \end{aligned}$$

giving the first statement. For the second, suppose $\mu = (m_1, \ldots, m_k) = (1^{\mu_1}, \ldots, n^{\mu_n}) \vdash n$. Theorem 3.3.3 gives

$$e_\mu(D_r(\mathcal{PB}r_n)) = \frac{n!}{\prod_{i=1}^{n} \mu_i!(i!)^{\mu_i}} \sum_{\substack{A \subseteq \mathbf{k} \\ |A|=r}} \left(\prod_{i \in A} c_1(\mathcal{PB}r_{m_i}) \cdot \prod_{j \in \mathbf{k} \setminus A} c_0(\mathcal{PB}r_{m_j}) \right).$$

Let $B_\mu = \{i \in \mathbf{k} : m_i \text{ is odd}\}$. By Proposition 3.3.13, we see that for $A \subseteq \mathbf{k}$ with $|A| = r$,

$$\prod_{i \in A} c_1(\mathcal{PB}r_{m_i}) \cdot \prod_{j \in \mathbf{k} \setminus A} c_0(\mathcal{PB}r_{m_j}) \ne 0 \iff m_i \text{ is odd for all } i \in A \iff A \subseteq B_\mu.$$

In particular, $e_\mu(D_r(\mathcal{PB}r_n)) \neq 0$ if and only if $\mu_1 + \mu_3 + \cdots = |B_\mu| \geq r$. For such a $\mu \vdash n$ and for $A \subseteq B_\mu$ with $|A| = r$,

$$\begin{aligned}\prod_{i\in A} c_1(\mathcal{PB}r_{m_i}) \cdot \prod_{j\in \mathbf{k}\setminus A} c_0(\mathcal{PB}r_{m_j}) &= \prod_{i\in A} c_1(\mathcal{PB}r_{m_i}) \cdot \prod_{i\in B_\mu\setminus A} c_0(\mathcal{PB}r_{m_i}) \cdot \prod_{j\in \mathbf{k}\setminus B_\mu} c_0(\mathcal{PB}r_{m_j}) \\ &= \prod_{i\in B_\mu} m_i! \cdot \prod_{j\in \mathbf{k}\setminus B_\mu} (m_j+1)\cdot(m_j-1)! \\ &= (1!)^{\mu_1}(3!)^{\mu_3}\cdots(3\cdot 1!)^{\mu_2}(5\cdot 3!)^{\mu_4}\cdots.\end{aligned}$$

Since there are $\binom{\mu_1+\mu_3+\cdots}{r}$ subsets $A \subseteq B_\mu$ with $|A| = r$, it follows that

$$\begin{aligned}e_\mu(D_r(\mathcal{PB}r_n)) &= \binom{\mu_1+\mu_3+\cdots}{r} \cdot n! \cdot \frac{(1!)^{\mu_1}(3!)^{\mu_3}\cdots(3\cdot 1!)^{\mu_2}(5\cdot 3!)^{\mu_4}\cdots}{\mu_1!\cdots\mu_n!\cdot(1!)^{\mu_1}(3!)^{\mu_3}\cdots(2!)^{\mu_2}(4!)^{\mu_4}\cdots} \\ &= \binom{\mu_1+\mu_3+\cdots}{r}\frac{n!}{\mu_1!\cdots\mu_n!}\left(\frac{3}{2}\right)^{\mu_2}\left(\frac{5}{4}\right)^{\mu_4}\cdots.\end{aligned}$$

Summing over all $\mu \vdash n$ with $\mu_1 + \mu_3 + \cdots \geq r$ gives the required expression for $e(D_r(\mathcal{PB}r_n))$. □

Again, the recurrences for the numbers $e(\mathcal{PB}r_n)$ and $e(D_r(\mathcal{PB}r_n))$ given by Theorems 3.3.2 and 3.3.4 take on a neat form.

Theorem 3.3.15. *The numbers $e(\mathcal{PB}r_n)$ satisfy the recurrence:*

$$\begin{aligned}e(\mathcal{PB}r_0) = 1, \quad e(\mathcal{PB}r_n) &= \sum_{i=1}^{\lfloor \frac{n}{2}\rfloor}\binom{n-1}{2i-1}(2i+1)\cdot(2i-1)!\,e(\mathcal{PB}r_{n-2i}) \\ &\quad + 2\cdot\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1}{2i}(2i+1)!\,e(\mathcal{PB}r_{n-2i-1})\end{aligned}$$ □

for $n \geq 1$.

Theorem 3.3.16. *The numbers $e(D_r(\mathcal{PB}r_n))$ satisfy the recurrence:*

$$e(D_n(\mathcal{PB}r_n)) = 1$$

$$e(D_0(\mathcal{PB}r_n)) = \begin{cases} a_n^2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$$e(D_r(\mathcal{PB}r_n)) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1}{2i-1}(2i+1)\cdot(2i-1)!\, e(D_r(\mathcal{PB}r_{n-2i})) + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2i}(2i+1)!\Big(e(D_r(\mathcal{PB}r_{n-2i-1})) + e(D_{r-1}(\mathcal{PB}r_{n-2i-1}))\Big)$$

for $n \geq 1$, where the numbers a_n are defined in Lemma 3.3.5. □

3.3.4 Other subsemigroups

We conclude this section with the observation that the general results above (Theorems 3.3.2, 3.3.3, 3.3.4) apply to many other subsemigroups of $\mathcal{P}_n$ (though the initial conditions need to be slightly modified in Theorem 3.3.4). As observed in [16,17,20], the full transformation semigroup and the symmetric and dual symmetric inverse semigroups $\mathcal{T}_n, \mathcal{I}_n, \mathcal{I}_n^*$ are all (isomorphic to) subsemigroups of $\mathcal{P}_n$:

- $\mathcal{T}_n \cong \{\alpha \in \mathcal{P}_n : \text{dom}^\wedge(\alpha) = [\![n]\!] \text{ and } \ker_\vee(\alpha) = \Delta\}$,
- $\mathcal{I}_n \cong \{\alpha \in \mathcal{P}_n : \ker^\wedge(\alpha) = \ker_\vee(\alpha) = \Delta\}$,
- $\mathcal{I}_n^* \cong \{\alpha \in \mathcal{P}_n : \text{dom}^\wedge(\alpha) = \text{dom}_\vee(\alpha) = [\![n]\!]\}$,

where $\Delta = \{(i,i) : i \in [\![n]\!]\}$ denotes the trivial equivalence (that is, the equality relation), and the above mentioned theorems apply to these subsemigroups. For example, one may easily check that $c(\mathcal{T}_n) = c_1(\mathcal{T}_n) = n$, so that Theorem 3.3.2 gives rise to the formula

$$e(\mathcal{T}_n) = n! \cdot \sum_{\mu \vdash n} \prod_{i=1}^{n} \frac{1}{\mu_i!((i-1)!)^{\mu_i}},$$

and the recurrence

$$e(\mathcal{T}_0) = 1, \quad e(\mathcal{T}_n) = \sum_{m=1}^{n} \binom{n-1}{m-1} \cdot m \cdot e(\mathcal{T}_{n-m}) \quad \text{for } n \geq 1.$$

As noted in the Introduction, the usual formula is $e(\mathcal{T}_n) = \sum_{k=1}^{n} \binom{n}{k} k^{n-k}$. The recurrence for $e(\mathcal{I}_n^*)$, combined with the fact that $e(\mathcal{I}_n^*) = B(n)$ is the nth Bell number [23], leads to

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k),$$

a well-known identity. We leave it to the reader to explore further if they wish.

3.4 A different approach for $\mathcal{B}r_n$ and $\mathcal{PB}r_n$

We now outline an alternative method for determining $e(\mathcal{B}r_n)$ and $e(\mathcal{PB}r_n)$. This approach will also allow us to determine the number of idempotents in an arbitrary $\mathscr{R}$-, $\mathscr{L}$- and $\mathscr{D}$-class of $\mathcal{B}r_n$ and $\mathcal{PB}r_n$. One advantage of this method is that we do not need to take sums over integer partitions; rather, everything depends on sequences defined by some fairly simple recurrence relations (see Theorems 3.4.5 and 3.4.7). The key idea is to define a variant of the graph $\Gamma(\alpha)$ in the case of $\alpha \in \mathcal{PB}r_n$.

Let $\alpha \in \mathcal{PB}r_X$. We define $\Lambda^\wedge(\alpha)$ (resp. $\Lambda_\vee(\alpha)$) to be the graph obtained from $\Gamma^\wedge(\alpha)$ (resp. $\Gamma_\vee(\alpha)$) by adding a red (resp. blue) loop at each vertex $i \in X$ if $\{i\}$ (resp. $\{i'\}$) is a block of α. And we define $\Lambda(\alpha)$ to be the graph with vertex set X and all the edges from both $\Lambda^\wedge(\alpha)$ and $\Lambda_\vee(\alpha)$. Some examples are given in Figure 3.3 with X finite. Note that $\Lambda(\alpha) = \Gamma(\alpha)$ if and only if $\alpha \in \mathcal{B}r_X$.

Since the graph $\Lambda^\wedge(\alpha)$ (resp. $\Lambda_\vee(\alpha)$) determines (and is determined by) $\text{dom}^\wedge(\alpha)$ and $\ker^\wedge(\alpha)$ (resp. $\text{dom}_\vee(\alpha)$ and $\ker_\vee(\alpha)$), we immediately obtain the following from Theorem 3.1.1.

Corollary 3.4.1. *Let X be any set (finite or infinite). For each $\alpha, \beta \in \mathcal{PB}r_X$, we have*

(i) *$\alpha \mathscr{R} \beta$ if and only if $\Lambda^\wedge(\alpha) = \Lambda^\wedge(\beta)$,*

(ii) *$\alpha \mathscr{L} \beta$ if and only if $\Lambda_\vee(\alpha) = \Lambda_\vee(\beta)$,*

(iii) *$\alpha \mathscr{H} \beta$ if and only if $\Lambda(\alpha) = \Lambda(\beta)$.* □

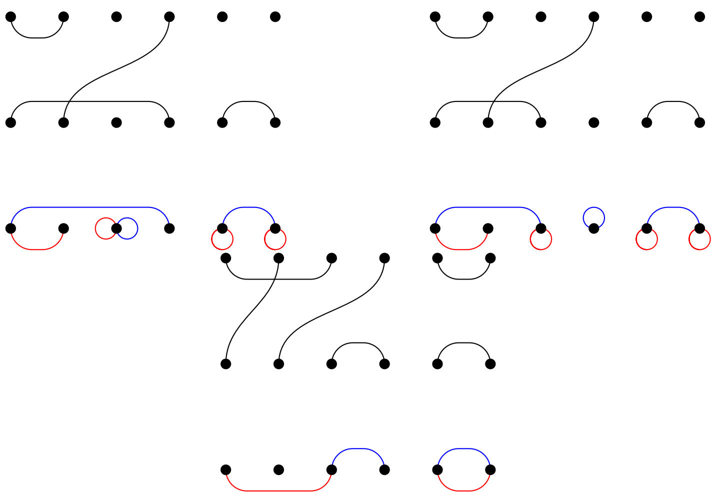
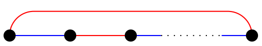


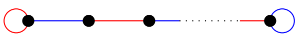
Figure 3.3: Elements α, β, γ (left to right) of the partial Brauer monoid $\mathcal{PB}r_6$ and their graphs $\Lambda(\alpha), \Lambda(\beta), \Lambda(\gamma)$ (below).

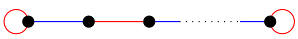
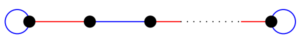
We now aim to classify the graphs on vertex set X that are of the form $\Lambda(\alpha)$ for some $\alpha \in \mathcal{PB}r_X$, and we will begin with the irreducible idempotents.

Lemma 3.4.2. *Let $\alpha \in C(\mathcal{PB}r_X)$ where X is finite. Then $\Lambda(\alpha)$ is of one of the following four forms:*

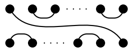
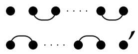
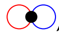
(1) : *an alternating-colour path of even length,*

(2)  : *an alternating-colour circuit of even length,*

(3)  : *an alternating-colour path of even length with loops,*

(4)  *or*  : *an alternating-colour path of odd length with loops.*

If $\alpha \in C(\mathcal{B}r_X)$, then $\Lambda(\alpha)$ is of the form (1) *or* (2).

Proof. By Lemma 3.3.8, we know that $\Gamma(\alpha)$ is either a cycle or a path. If $\Gamma(\alpha)$ is a cycle, then $\Lambda(\alpha) = \Gamma(\alpha)$ is of type (2). Next suppose $\Gamma(\alpha)$ is a path, and write $n = |X|$. We consider the case in which n is odd (so the path is of even length). Re-labelling the elements of X if necessary, we may assume that $\Gamma(\alpha)$ is the path $1-2-3-\cdots-n$. Since $\Gamma(\alpha)$ completely determines $\ker^{\wedge}(\alpha)$ and $\ker_{\vee}(\alpha)$, it follows that α must be one of  or , in which case $\Lambda(\alpha)$ is of type (1) or (3), respectively. (Note that the preceeding discussion include the case $n = 1$, where we have $\Gamma(\alpha) = \bullet$, so the two possibilities for $\Lambda(\alpha)$ are $\bullet$ or , which are of type (1) and (3), respectively.) The case in which n is even is similar, and leads to $\Lambda(\alpha)$ being of type (4). The statement concerning $\mathcal{B}r_X$ is clear, seeing as elements of $\mathcal{B}r_X$ have no singleton blocks. □

Now consider a graph Λ with edges coloured red or blue. We say that Λ is *balanced* if it is a disjoint union of finitely many subgraphs of types (1–4) from Lemma 3.4.2. We call a balanced graph Λ *reduced* if it is a disjoint union of finitely many subgraphs of types (1–2) from Lemma 3.4.2. If X is a finite set, we write $\mathrm{Bal}(X)$ (resp. $\mathrm{Red}(X)$) for the set of all balanced (resp. reduced balanced) graphs with vertex set X.

Proposition 3.4.3. *If X is a finite set, then the maps*

$$\Phi : E(\mathcal{PB}r_X) \to \mathrm{Bal}(X) : \alpha \mapsto \Lambda(\alpha)$$
$$\Psi = \Phi|_{E(\mathcal{B}r_X)} : E(\mathcal{B}r_X) \to \mathrm{Red}(X) : \alpha \mapsto \Lambda(\alpha) = \Gamma(\alpha)$$

are bijections. If $\alpha \in E(\mathcal{PB}r_X)$, then $\mathrm{rank}(\alpha)$ is equal to the number of connected components of $\Lambda(\alpha)$ of type (1) as listed in Lemma 3.4.2.

Proof. Let $\alpha \in E(\mathcal{PB}r_X)$, and write $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_k$ where $\alpha_1, \ldots, \alpha_k$ are the irreducible components of α. Then $\Lambda(\alpha)$ is the disjoint union of the subgraphs $\Lambda(\alpha_1), \ldots, \Lambda(\alpha_k)$, and is therefore reduced, by Lemma 3.4.2. If, in fact, $\alpha \in E(\mathcal{B}r_X)$, then each of $\Lambda(\alpha_1), \ldots, \Lambda(\alpha_k)$ must be of the form (1) or (2), since α has no blocks of size 1. This shows that Φ and Ψ do indeed map $E(\mathcal{PB}r_X)$ and $E(\mathcal{B}r_X)$ to $\mathrm{Bal}(X)$ and $\mathrm{Red}(X)$, respectively. Note also that $\mathrm{rank}(\alpha) = \mathrm{rank}(\alpha_1) + \cdots + \mathrm{rank}(\alpha_k)$ is equal to the number of rank 1 partitions among $\alpha_1, \ldots, \alpha_k$, and that the rank of some $\beta \in C(\mathcal{PB}r_Y)$ is equal to 1 if and only if $\Lambda(\beta)$ is of type (1).

Let $\Lambda \in \mathrm{Bal}(X)$, and suppose $\Lambda_1, \ldots, \Lambda_k$ are the connected components of Λ, with vertex sets $X_1, \ldots, X_k$, respectively. Then there exist irreducible idempotents $\alpha_i \in C(\mathcal{P}_{X_i})$ with $\Lambda(\alpha_i) = \Lambda_i$ for each i, and it follows that $\Lambda = \Lambda(\alpha_1 \oplus \cdots \oplus$

α_k), showing that Φ is surjective. If $\Lambda \in \text{Red}(X)$, then $\alpha_1 \oplus \cdots \oplus \alpha_k \in \mathcal{B}r_X$. Finally, if $\alpha, \beta \in E(\mathcal{PB}r_X)$ are such that $\Lambda(\alpha) = \Lambda(\beta)$, then $\alpha \mathscr{H} \beta$ by Corollary 3.4.1, so that $\alpha = \beta$ (as $\mathscr{H}$ is idempotent-separating), whence Φ (and hence also Ψ) is injective. □

3.4.1 The Brauer monoid

For $\alpha \in \mathcal{B}r_n$, we write

$$R_\alpha(\mathcal{B}r_n) = \{\beta \in \mathcal{B}r_n : \Lambda^\wedge(\beta) = \Lambda^\wedge(\alpha)\} \qquad \text{and} \qquad L_\alpha(\mathcal{B}r_n) = \{\beta \in \mathcal{B}r_n : \Lambda_\vee(\beta) = \Lambda_\vee(\alpha)\}.$$

By Corollary 3.4.1, these are precisely the $\mathscr{R}$- and $\mathscr{L}$-classes of α in $\mathcal{B}r_n$. At this point, it will be convenient to introduce an indexing set. Put $[\![n]\!]^0 = [\![n]\!] \cup \{0\}$, and let $I(n) = \{r \in [\![n]\!]^0 : n - r \in 2\mathbb{Z}\}$. For $r \in I(n)$, let

$$D_r(\mathcal{B}r_n) = \{\alpha \in \mathcal{B}r_n : \text{rank}(\alpha) = r\}.$$

By Theorem 3.1.1, we see that these are precisely the $\mathscr{D}$-classes of $\mathcal{B}r_n$. We will need to know the number of $\mathscr{R}$-classes (which is equal to the number of $\mathscr{L}$-classes) in a given $\mathscr{D}$-class of $\mathcal{B}r_n$.

Lemma 3.4.4 (See the proof of [21, Theorem 8.4]). *For $n \in \mathbb{N}$ and $r = n - 2k \in I(n)$, the number of $\mathscr{R}$-classes (and $\mathscr{L}$-classes) in the $\mathscr{D}$-class $D_r(\mathcal{B}r_n)$ is equal to*

$$\rho_{nr} = \binom{n}{r}(2k-1)!! = \frac{n!}{2^k k! r!}. \qquad \square$$

Theorem 3.4.5. *Define a sequence a_{nr}, for $n \in \mathbb{N}$ and $r \in I(n)$, by*

$$\begin{aligned} a_{nn} &= 1 && \textit{for all } n \\ a_{n0} &= (n-1)!! && \textit{if } n \textit{ is even} \\ a_{nr} &= a_{n-1,r-1} + (n-r)a_{n-2,r} && \textit{if } 1 \le r \le n-2. \end{aligned}$$

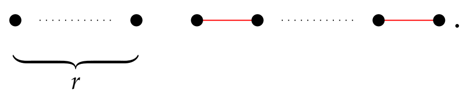
Then for any $n \in \mathbb{N}$ and $r \in I(n)$, and with ρ_{nr} as in Lemma 3.4.4:

(i) *$e(R_\alpha(\mathcal{B}r_n)) = e(L_\alpha(\mathcal{B}r_n)) = a_{nr}$ for any $\alpha \in D_r(\mathcal{B}r_n)$,*

(ii) $e(D_r(\mathcal{B}r_n)) = \rho_{nr} a_{nr}$,

(iii) $e(\mathcal{B}r_n) = \sum_{r \in I(n)} \rho_{nr} a_{nr}$.

Proof. Note that (iii) follows from (ii), which follows from (i) and Lemma 3.4.4, so it suffices to prove (i). Let $\alpha \in D_r(\mathcal{B}r_n)$. Re-labelling the points from $[\![n]\!]$, if necessary, we may assume that

$$\Gamma^\wedge(\alpha) = \Lambda^\wedge(\alpha) =$$ 

Let A_{nr} be the set of reduced balanced graphs on vertex set $[\![n]\!]$ with the same red edges as $\Gamma^\wedge(\alpha)$. By Corollary 3.4.1 and Proposition 3.4.3, $|A_{nr}| = e(R_\alpha(\mathcal{B}r_n))$. Put $a_{nr} = |A_{nr}|$. We show that a_{nr} satisfies the stated recurrence. By symmetry, $e(L_\alpha(\mathcal{B}r_n)) = a_{nr}$.

Clearly $a_{nn} = 1$ for all n. If n is even, then a_{n0} is the number of ways to match the vertices from $[\![n]\!]$ with $n/2$ non-intersecting (blue) arcs, which is equal to $(n-1)!!$. Suppose now that $1 \le r \le n-2$. Elements of A_{nr} come in two kinds:

1. those for which 1 is a connected component of its own, and
2. those for which 1 is an endpoint of an even length alternating path.

There are clearly $a_{n-1,r-1}$ elements of A_{nr} of type 1. Suppose now that $\Gamma \in A_{nr}$ is a graph of type 2. There are $n-r$ possible vertices for vertex 1 to be joined to by a blue edge. Suppose the vertex adjacent to 1 is x. Removing these two vertices, as well as the blue edge $1-x$ and the red edge adjacent to x (and relabelling the remaining vertices), yields an element of $A_{n-2,r}$. Since this process is reversible, there are $(n-r)a_{n-2,r}$ elements of A_{nr} of type 2. Adding these gives $a_{nr} = a_{n-1,r-1} + (n-r)a_{n-2,r}$. □

3.4.2 The partial Brauer monoid

For $\alpha \in \mathcal{PB}r_n$, we write

$$R_\alpha(\mathcal{PB}r_n) = \{\beta \in \mathcal{PB}r_n : \Lambda^\wedge(\beta) = \Lambda^\wedge(\alpha)\}$$
$$L_\alpha(\mathcal{PB}r_n) = \{\beta \in \mathcal{PB}r_n : \Lambda_\vee(\beta) = \Lambda_\vee(\alpha)\}.$$

By Corollary 3.4.1, these are precisely the $\mathscr{R}$- and $\mathscr{L}$-classes of α in $\mathcal{PB}r_n$. For $r \in [\![n]\!]^0$, let

$$D_r(\mathcal{PB}r_n) = \{\alpha \in \mathcal{PB}r_n : \text{rank}(\alpha) = r\}.$$

Again, these are precisely the $\mathscr{D}$-classes of $\mathcal{PB}r_n$. But unlike the case of $\mathcal{B}r_n$, it is not true that any two $\mathscr{D}$-related elements of $\mathcal{PB}r_n$ are $\mathscr{R}$-related to the same number of idempotents. So we will obtain a formula for $e(R_\alpha(\mathcal{PB}r_n))$ that will depend on the paramaters r, t, where $r = \text{rank}(\alpha)$ and t is the number of singleton non-transversal upper-kernel classes. Note that n, r, t are constrained by the requirement that $n - r - t$ is even. With this in mind, we define an indexing set

$$J(n) = \{(r,t) \in [\![n]\!]^0 \times [\![n]\!]^0 : t \in I(n-r)\} = \{(r,t) \in [\![n]\!]^0 \times [\![n]\!]^0 : n - r - t \in 2\mathbb{Z}\}.$$

There is a dual statement of the following lemma, but we will not state it.

Lemma 3.4.6. *For $n \in \mathbb{N}$ and $(r,t) \in J(n)$, with $n - r - t = 2k$, the number of $\mathscr{R}$-classes in $D_r(\mathcal{PB}r_n)$ in which each element has t singleton non-transversal upper-kernel classes is equal to*

$$\begin{aligned} \rho_{nrt} &= \binom{n}{r}\binom{n-r}{t}(2k-1)!! \\ &= \frac{n!}{2^k k! r! t!}. \end{aligned}$$

Proof. By Corollary 3.4.1, the number of such $\mathscr{R}$-classes is equal to the number of graphs on vertex set $[\![n]\!]$ with r vertices of degree 0, t vertices with a single loop, and the remaining $n - r - t$ vertices of degree 1. To specify such a graph, we first choose the vertices of degree 0 in $\binom{n}{r}$ ways. We then choose the vertices with loops in $\binom{n-r}{t}$ ways. And finally, we choose the remaining edges in $(n - r - t - 1)!! = (2k-1)!!$ ways. □

Theorem 3.4.7. *Define a sequence a_{nrt}, for $n \in \mathbb{N}$ and $(r,t) \in J(n)$, by*

$$\begin{aligned} a_{nn0} &= 1 && \text{for all } n \\ a_{n0t} &= a_n && \text{if } n - t \text{ is even} \\ a_{nrt} &= a_{n-1,r-1,t} + (n-r-t)a_{n-2,r,t} && \text{if } n \geq 2 \text{ and } 1 \leq r \leq n-1, \end{aligned}$$

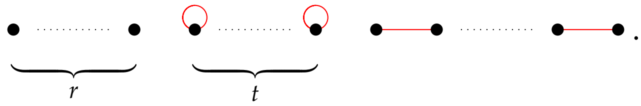
where the sequence a_n is defined in Lemma 3.3.5. Then for any $n \in \mathbb{N}$ and $(r,t) \in J(n)$, and with ρ_{nrt} as in Lemma 3.4.6:

(i) $e(R_\alpha(\mathcal{PB}r_n)) = a_{nrt}$ *for any* $\alpha \in D_r(\mathcal{PB}r_n)$ *with* t *singleton non-transversal upper-kernel classes,*

(ii) $e(D_r(\mathcal{PB}r_n)) = \sum_{t \in I(n-r)} \rho_{nrt} a_{nrt}$,

(iii) $e(\mathcal{PB}r_n) = \sum_{(r,t) \in J(n)} \rho_{nrt} a_{nrt}$.

Proof. Again, it suffices to prove (i). Let $\alpha \in D_r(\mathcal{PB}r_n)$. Re-labelling the points from $[\![n]\!]$, if necessary, we may assume that

$\Lambda^\wedge(\alpha) =$ 

Let A_{nrt} be the set of all balanced graphs on vertex set $[\![n]\!]$ with the same red edges as $\Lambda^\wedge(\alpha)$. Again, by Corollary 3.4.1 and Proposition 3.4.3, it suffices to show that the numbers $a_{nrt} = |A_{nrt}|$ satisfy the stated recurrence.

It is clear that $a_{nn0} = 1$ for all n. If $r = 0$ (and $n - t$ is even), then we may complete $\Lambda^\wedge(\alpha)$ to a graph from A_{n0t} by adding as many (non-adjacent) blue edges as we like, and adding blue loops to the remaining vertices. Again, such assignments of blue edges are in one-one correspondence with the involutions of $[\![n]\!]$, of which there are a_n. Now suppose $n \geq 2$ and $1 \leq r \leq n-1$. By inspection of (1–4) in Lemma 3.4.2, we see that elements of A_{nrt} come in two kinds:

1. those for which 1 is a connected component of its own, and

2. those for which 1 is an endpoint of an even length alternating path (with no loops).

The proof concludes in similar fashion to that of Theorem 3.4.5. □

3.5 Idempotents in diagram algebras

Let $\alpha, \beta \in \mathcal{P}_n$. Recall that the product $\alpha\beta \in \mathcal{P}_n$ is defined in terms of the product graph $\Gamma(\alpha,\beta)$. Specifically, A is a block of $\alpha\beta$ if and only if $A = B \cap ([\![n]\!] \cup [\![n]\!]') \neq$

$\varnothing$ for some connected component B of $\Gamma(\alpha,\beta)$. In general, however, the graph $\Gamma(\alpha,\beta)$ may contain some connected components strictly contained in the middle row $[\![n]\!]''$, and the *partition algebra* $\mathcal{P}_n^\xi$ is designed to take these components into account. We write $m(\alpha,\beta)$ for the number of connected components of the product graph $\Gamma(\alpha,\beta)$ that are entirely contained in the middle row. It is important to note (and trivially true) that $m(\alpha,\beta) \leq n$ for all $\alpha,\beta \in \mathcal{P}_n$. Now let F be a field and fix some $\xi \in F$.

We denote by $\mathcal{P}_n^\xi$ the F-algebra with basis $\mathcal{P}_n$ and product $\circ$ defined on basis elements $\alpha,\beta \in \mathcal{P}_n$ (and then extended linearly) by

$$\alpha \circ \beta = \xi^{m(\alpha,\beta)}(\alpha\beta).$$

If $\alpha,\beta,\gamma \in \mathcal{P}_n$, then $m(\alpha,\beta)+m(\alpha\beta,\gamma) = m(\alpha,\beta\gamma)+m(\beta,\gamma)$, and it follows that $\mathcal{P}_n^\xi$ is an associative algebra. We may also speak of the subalgebras of $\mathcal{P}_n^\xi$ spanned by $\mathcal{B}r_n$ and $\mathcal{PB}r_n$; these are the *Brauer* and *partial Brauer algebras* $\mathcal{B}r_n^\xi$ and $\mathcal{PB}r_n^\xi$, respectively. See [36] for a survey-style treatment of the partition algebras.

In this section, we determine the number of partitions $\alpha \in \mathcal{P}_n$ such that α is an idempotent basis element of $\mathcal{P}_n^\xi$; that is, $\alpha \circ \alpha = \alpha$. These numbers depend on whether ξ is a root of unity. As such, we define

$$M = \begin{cases} m & \text{if } \xi \text{ is an } m\text{th root of unity where } m \leq n \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{K}_n$ is one of $\mathcal{P}_n, \mathcal{B}r_n, \mathcal{PB}r_n$, we will write

$$E^\xi(\mathcal{K}_n) = \{\alpha \in \mathcal{K}_n : \alpha = \alpha \circ \alpha \text{ in } \mathcal{K}_n^\xi\} \qquad \text{and} \qquad e^\xi(\mathcal{K}_n) = |E^\xi(\mathcal{K}_n)|.$$

Theorem 3.5.1. *Let $\alpha \in \mathcal{P}_n$, and suppose the kernel-classes of α are $X_1,\ldots,X_k$.*

Then the following are equivalent:

(1) $\alpha \in E^\xi(\mathcal{P}_n)$,

(2) $\alpha \in E(\mathcal{P}_n)$ *and* $\operatorname{rank}(\alpha) \equiv k \pmod{M}$,

(3) *the following three conditions are satisfied:*

(i) $\alpha \in \mathcal{P}_{X_1} \oplus \cdots \oplus \mathcal{P}_{X_k}$,

(ii) *the restrictions* $\alpha|_{X_i}$ *all have rank at most* 1,

(iii) *the number of restrictions* $\alpha|_{X_i}$ *of rank* 0 *is a multiple of* M.

Proof. First, note that if $\alpha \in E(\mathcal{P}_n)$, then Theorem 3.2.2 gives $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_k$ where $\alpha_i = \alpha|_{X_i} \in C(\mathcal{P}_{X_i})$ for each i, and $r = \operatorname{rank}(\alpha) = \operatorname{rank}(\alpha_1) + \cdots + \operatorname{rank}(\alpha_k)$ with $\operatorname{rank}(\alpha_i) \in \{0, 1\}$ for each i. Re-labelling the X_i if necessary, we may suppose that $\operatorname{rank}(\alpha_1) = \cdots = \operatorname{rank}(\alpha_r) = 1$. Then the connected components contained entirely in X'' in the product graph $\Gamma(\alpha, \alpha)$ are precisely the sets $X''_{r+1}, \ldots, X''_k$. So $m(\alpha, \alpha) = k - r$.

Now suppose (1) holds. Then $\alpha = \alpha \circ \alpha = \xi^{m(\alpha,\alpha)}(\alpha^2)$, so $\alpha = \alpha^2$ and $m(\alpha, \alpha) \in M\mathbb{Z}$. Since $\alpha \in E(\mathcal{P}_n)$, it follows from the first paragraph that $k - r = m(\alpha, \alpha) \in M\mathbb{Z}$, and so (2) holds.

Next, suppose (2) holds. Since $\alpha \in E(\mathcal{P}_n)$, Theorem 3.2.2 tells us that (i) and (ii) hold. Write $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_k$ where $\alpha_i = \alpha|_{X_i} \in C(\mathcal{P}_{X_i})$ for each i. The set $\{i \in \mathbf{k} : \operatorname{rank}(\alpha_i) = 0\}$ has cardinality $m(\alpha, \alpha)$, which is equal to $k - \operatorname{rank}(\alpha)$ by the first paragraph. By assumption, $k - \operatorname{rank}(\alpha) \in M\mathbb{Z}$, so (iii) holds.

Finally, suppose (3) holds and write $\alpha_i = \alpha|_{X_i}$ for each i. Since $\operatorname{rank}(\alpha_i) \leq 1$ and X_i is a kernel-class of α, it follows that $\alpha_i \in \mathcal{P}_{X_i}$ is irreducible and so $\alpha_i \in C(\mathcal{P}_{X_i})$ by Lemma 3.2.1. For each $i \in \mathbf{k}$, let

$$l_i = \begin{cases} 0 & \text{if } \operatorname{rank}(\alpha_i) = 1 \\ 1 & \text{if } \operatorname{rank}(\alpha_i) = 0. \end{cases}$$

Then $l_1 + \cdots + l_k$ is a multiple of M by assumption, and $\alpha_i \circ \alpha_i = \xi^{l_i}(\alpha_i^2) = \xi^{l_i}\alpha_i$ in $\mathcal{P}^\xi_{X_i}$ for each i. But then $\alpha \circ \alpha = \xi^{l_1 + \cdots + l_k}\alpha = \alpha$ so that (1) holds. □

Remark 5. If $M = 0$, then part (2) of the previous theorem says that $\operatorname{rank}(\alpha) = k$. Also, conditions (ii) and (iii) in part (3) may be replaced with the simpler statement that the restrictions $\alpha|_{X_i}$ all have rank 1. If $M = 1$, then $\xi = 1$ so $\mathcal{P}^\xi_n$ is the (non-twisted) semigroup algebra of $\mathcal{P}_n$ and $E^\xi(\mathcal{P}_n) = E(\mathcal{P}_n)$; in this case, Theorem 3.5.1 reduces to Theorem 3.2.2.

We are now ready to give formulae for $e^\xi(\mathcal{K}_n)$ where $\mathcal{K}_n$ is one of $\mathcal{P}_n, \mathcal{B}r_n, \mathcal{PB}r_n$.

It will be convenient to give separate statements depending on whether $M = 0$ or $M > 0$. The next result is proved in an almost identical fashion to Theorem 3.3.2, relying on Theorem 3.5.1 rather than Theorem 3.2.2.

Theorem 3.5.2. *If $M = 0$ and $\mathcal{K}_n$ is one of $\mathcal{P}_n, \mathcal{B}r_n, \mathcal{PB}r_n$, then*

$$e^\xi(\mathcal{K}_n) = n! \cdot \sum_{\mu \vdash n} \prod_{i=1}^{n} \frac{c_1(\mathcal{K}_i)^{\mu_i}}{\mu_i!(i!)^{\mu_i}}.$$

The numbers $e^\xi(\mathcal{K}_n)$ satisfy the recurrence:

$$e^\xi(\mathcal{K}_0) = 1, \quad e^\xi(\mathcal{K}_n) = \sum_{m=1}^{n} \binom{n-1}{m-1} c_1(\mathcal{K}_m) e^\xi(\mathcal{K}_{n-m}) \quad \textit{for } n \geq 1.$$

The values of $c_1(\mathcal{K}_n)$ are given in Propositions 3.3.6, 3.3.9 and 3.3.13. □

Recall that if $\alpha \in \mathcal{P}_n$ has kernel-classes $X_1, \ldots, X_k$ with $|X_1| \geq \cdots \geq |X_k|$, then the integer partition $\mu(\alpha)$ is defined to be $(|X_1|, \ldots, |X_k|)$. For a subset $\Sigma \subseteq \mathcal{P}_n$ and an integer partition $\mu \vdash n$, we write

$$E^\xi_\mu(\Sigma) = \{\alpha \in E^\xi(\Sigma) : \mu(\alpha) = \mu\} \qquad \text{and} \qquad e^\xi_\mu(\Sigma) = |E^\xi_\mu(\alpha)|.$$

If $\mu = (m_1, \ldots, m_k) \vdash n$, we call k the *height of* μ, and we write $k = h(\mu)$. The next result follows quickly from Theorem 3.5.1.

Theorem 3.5.3. *Suppose $M > 0$, and let $\mathcal{K}_n$ be one of $\mathcal{P}_n, \mathcal{B}r_n, \mathcal{PB}r_n$. Then*

$$e^\xi(\mathcal{K}_n) = \sum_{\mu \vdash n} e^\xi_\mu(\mathcal{K}_n).$$

If $\mu \vdash n$ and $k = h(\mu)$, then

$$e^\xi_\mu(\mathcal{K}_n) = \sum_{\substack{0 \leq r \leq n \\ r \equiv k \pmod{M}}} e_\mu(D_r(\mathcal{K}_n)).$$

The values of $e_\mu(D_r(\mathcal{K}_n))$ are given in Theorem 3.3.3. □

We may also derive recurrences for the values of $e^\xi(D_r(\mathcal{K}_n))$ in the case $M = 0$. Things get more complicated when $M > 0$ since the question of whether or not an element of $E(\mathcal{K}_n)$ belongs additionally to $E^\xi(\mathcal{K}_n)$ depends not just on its rank but also on the number of kernel classes. We will omit the $M > 0$ case.

Theorem 3.5.4. *If $M = 0$ and $\mathcal{K}_n$ is one of $\mathcal{P}_n, \mathcal{B}r_n, \mathcal{PB}r_n$, then the numbers $e^\xi(D_r(\mathcal{K}_n))$ satisfy the recurrence:*

$$\begin{aligned} e^\xi(D_n(\mathcal{K}_n)) &= 1 \\ e^\xi(D_0(\mathcal{K}_n)) &= 0 \\ e^\xi(D_r(\mathcal{K}_n)) &= \sum_{m=1}^{n} \binom{n-1}{m-1} c_1(\mathcal{K}_m) e^\xi(D_{r-1}(\mathcal{K}_{n-m})) \end{aligned}$$

for $n \geq 1$ and $1 \leq r \leq n-1$. The values of $c_1(\mathcal{K}_n)$ are given in Propositions 3.3.6, 3.3.9 and 3.3.13. □

We now use Theorems 3.5.2 and 3.5.3 to derive explicit values for $e^\xi(\mathcal{B}r_n)$ and $e^\xi(\mathcal{PB}r_n)$ in the case $M = 0$. In fact, since $c_1(\mathcal{B}r_n) = c_1(\mathcal{PB}r_n)$ by Propositions 3.3.9 and 3.3.13, it follows that $e^\xi(\mathcal{B}r_n) = e^\xi(\mathcal{PB}r_n)$ in this case. These numbers seem to be Sequence A088009 on the OEIS [71], although it is difficult to understand why.

Theorem 3.5.5. *If $n \in \mathbb{N}$ and $M = 0$, then*

$$e^\xi(\mathcal{B}r_n) = e^\xi(\mathcal{PB}r_n) = \sum_\mu \frac{n!}{\mu_1! \mu_3! \cdots \mu_{2k+1}!},$$

where $k = \lfloor \frac{n-1}{2} \rfloor$, and the sum is over all integer partitions $\mu = (1^{\mu_1}, \ldots, n^{\mu_n}) \vdash n$ with $\mu_{2i} = 0$ for $i = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$.

The numbers $e^\xi(\mathcal{B}r_n) = e^\xi(\mathcal{PB}r_n)$ satisfy the recurrence:

$$e^\xi(\mathcal{B}r_0) = 1, \quad e^\xi(\mathcal{B}r_n) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2i} (2i+1)!\, e^\xi(\mathcal{B}r_{n-2i-1}) \qquad \text{for } n \geq 1.$$ □

Theorem 3.5.4 yields a neat recurrence for the numbers $e^\xi(D_r(\mathcal{B}r_n)) = e^\xi(D_r(\mathcal{PB}r_n))$ in the case $M = 0$.

Theorem 3.5.6. *If $M = 0$, then the numbers $e^{\xi}(D_r(\mathcal{B}r_n)) = e^{\xi}(D_r(\mathcal{PB}r_n))$ satisfy the recurrence:*

$$\begin{aligned}
&e^{\xi}(D_n(\mathcal{B}r_n)) = 1 \\
&e^{\xi}(D_0(\mathcal{B}r_n)) = 0 && \text{if } n \geq 1 \\
&e^{\xi}(D_r(\mathcal{B}r_n)) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2i} (2i+1)!\, e^{\xi}(D_{r-1}(\mathcal{B}r_{n-2i-1})) && \text{if } 1 \leq r \leq n-1. \qquad \square
\end{aligned}$$

We may also use the methods of Section 3.4 to derive recurrences for the number of elements from an $\mathscr{R}$-, $\mathscr{L}$- or $\mathscr{D}$-class of $\mathcal{B}r_n$ or $\mathcal{PB}r_n$ that satisfy $\alpha = \alpha \circ \alpha$ in $\mathcal{P}_n^{\xi}$ with $M = 0$. Recall that $I(n) = \{r \in [\![n]\!]^0 : n - r \in 2\mathbb{Z}\}$.

Theorem 3.5.7. *Define a sequence b_{nr}, for $n \in \mathbb{N}$ and $r \in I(n)$, by*

$$\begin{aligned}
&b_{nn} = 1 && \text{for all } n \\
&b_{n0} = 0 && \text{if } n \geq 2 \text{ is even} \\
&b_{nr} = b_{n-1,r-1} + (n-r) b_{n-2,r} && \text{if } 1 \leq r \leq n-2.
\end{aligned}$$

Then for any $n \in \mathbb{N}$ and $r \in I(n)$, and with $M = 0$ and ρ_{nr} as in Lemma 3.4.4:

(i) $e^{\xi}(R_\alpha(\mathcal{B}r_n)) = e^{\xi}(L_\alpha(\mathcal{B}r_n)) = b_{nr}$ *for any $\alpha \in D_r(\mathcal{B}r_n)$,*

(ii) $e^{\xi}(D_r(\mathcal{B}r_n)) = \rho_{nr} b_{nr}$,

(iii) $e^{\xi}(\mathcal{B}r_n) = \sum_{r \in I(n)} \rho_{nr} b_{nr}$.

Proof. The proof is virtually identical to the proof of Theorem 3.4.5, except we require $b_{n0} = 0$ if $n \geq 2$ is even since reduced balanced graphs with n vertices and $n/2$ red (and blue) edges correspond to elements of $D_0(\mathcal{B}r_n)$, and these do not belong to $E^{\xi}(\mathcal{B}r_n)$ by Theorem 3.5.1. $\square$

Remark 6. If $\alpha \in \mathcal{PB}r_n$, then $E(R_\alpha(\mathcal{PB}r_n))$ is non-empty if and only if $R_\alpha(\mathcal{PB}r_n)$ has non-trivial intersection with $\mathcal{B}r_n$, in which case $e(R_\alpha(\mathcal{PB}r_n)) = b_{nr}$ where $r = \operatorname{rank}(\alpha)$. (This is because $\beta \in E^{\xi}(R_\alpha(\mathcal{PB}r_n))$ if and only if the connected components of $\Lambda(\beta)$ are all of type (1) as stated in Lemma 3.4.2, in which case $\beta \in E^{\xi}(\mathcal{B}r_n)$.) A dual statement can be made concerning $\mathscr{L}$-classes. It follows that $e^{\xi}(D_r(\mathcal{PB}r_n)) = e^{\xi}(D_r(\mathcal{B}r_n)) = \rho_{nr} b_{nr}$ for all $r \in I(n)$, and $e^{\xi}(\mathcal{PB}r_n) = e^{\xi}(\mathcal{B}r_n) = \sum_{r \in I(n)} \rho_{nr} b_{nr}$.

3.6 Calculated values

In this section, we list calculated values of $c_0(\mathcal{K}_n), c_1(\mathcal{K}_n), c(\mathcal{K}_n), e(\mathcal{K}_n), e^{\xi}(\mathcal{K}_n)$ where $\mathcal{K}_n$ is one of $\mathcal{P}_n, \mathcal{B}r_n, \mathcal{PB}r_n$ and where $M = 0$. We also give values of $e(D_r(\mathcal{K}_n))$ and $e^{\xi}(D_r(\mathcal{K}_n))$ where $M = 0$, and $e(R_\alpha(\mathcal{B}r_n))$ and $e^{\xi}(R_\alpha(\mathcal{B}r_n))$ for $\alpha \in D_r(\mathcal{B}r_n)$.

n	0	1	2	3	4	5	6	7	8	9	10
$c_0(\mathcal{B}r_n)$		0	1	0	6	0	120	0	5040	0	362880
$c_1(\mathcal{B}r_n)$		1	0	6	0	120	0	5040	0	362880	0
$c(\mathcal{B}r_n)$		1	1	6	6	120	120	5040	5040	362880	362880
$e(\mathcal{B}r_n)$	1	1	2	10	40	296	1936	17872	164480	1820800	21442816
$e^{\xi}(\mathcal{B}r_n)$	1	1	1	7	25	181	1201	10291	97777	1013545	12202561

Table 3.1: Calculated values of $c_0(\mathcal{B}r_n), c_1(\mathcal{B}r_n), c(\mathcal{B}r_n), e(\mathcal{B}r_n), e^{\xi}(\mathcal{B}r_n)$ with $M = 0$.

n	0	1	2	3	4	5	6	7	8	9	10
$c_0(\mathcal{PB}r_n)$		1	3	6	30	120	840	5040	45360	362880	3991680
$c_1(\mathcal{PB}r_n)$		1	0	6	0	120	0	5040	0	362880	0
$c(\mathcal{PB}r_n)$		2	3	12	30	240	840	10080	45360	725760	3991680
$e(\mathcal{PB}r_n)$	1	2	7	38	241	1922	17359	180854	2092801	26851202	376371799
$e^{\xi}(\mathcal{PB}r_n)$	1	1	1	7	25	181	1201	10291	97777	1013545	12202561

Table 3.2: Calculated values of $c_0(\mathcal{PB}r_n), c_1(\mathcal{PB}r_n), c(\mathcal{PB}r_n), e(\mathcal{PB}r_n), e^{\xi}(\mathcal{PB}r_n)$ with $M = 0$.

n	0	1	2	3	4	5	6	7	8	9	10
$c_0(\mathcal{P}_n)$		1	3	15	119	1343	19905	369113	8285261	219627683	6746244739
$c_1(\mathcal{P}_n)$		1	5	43	529	8451	167397	3984807	111319257	3583777723	131082199809
$c(\mathcal{P}_n)$		2	8	58	648	9794	187302	4353920	119604518	3803405406	137828444548
$e(\mathcal{P}_n)$	1	2	12	114	1512	25826	541254	13479500	389855014	12870896154	478623817564
$e^{\xi}(\mathcal{P}_n)$	1	1	6	59	807	14102	301039	7618613	223586932	7482796089	281882090283

Table 3.3: Calculated values of $c_0(\mathcal{P}_n), c_1(\mathcal{P}_n), c(\mathcal{P}_n), e(\mathcal{P}_n), e^{\xi}(\mathcal{P}_n)$ with $M = 0$.

$n \setminus r$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		1									
2	1		1								
3		9		1							
4	9		30		1						
5		225		70		1					
6	225		1575		135		1				
7		11025		6615		231		1			
8	11025		132300		20790		364		1		
9		893025		873180		54054		540		1	
10	893025		16372125		4054050		122850		765		1

Table 3.4: Calculated values of $e(D_r(\mathcal{B}r_n))$.

$n \setminus r$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		1									
2	1		1								
3		3		1							
4	3		5		1						
5		15		7		1					
6	15		35		9		1				
7		105		63		11		1			
8	105		315		99		13		1		
9		945		693		143		15		1	
10	945		3465		1287		195		17		1

Table 3.5: Calculated values of $e(R_\alpha(\mathcal{B}r_n)) = e(L_\alpha(\mathcal{B}r_n))$ where $\alpha \in D_r(\mathcal{B}r_n)$.

$n \setminus r$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		1									
2	0		1								
3		2		1							
4	0		4		1						
5		8		6		1					
6	0		24		8		1				
7		48		48		10		1			
8	0		192		80		12		1		
9		384		480		120		14		1	
10	0		1920		168		195		16		1

Table 3.6: Calculated values of $e^{\xi}(R_\alpha(\mathcal{B}r_n)) = e^{\xi}(L_\alpha(\mathcal{B}r_n)) = e^{\xi}(R_\alpha(\mathcal{PB}r_n)) = e^{\xi}(L_\alpha(\mathcal{PB}r_n))$ where $\alpha \in D_r(\mathcal{B}r_n)$ and $M = 0$.

$n \setminus r$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	4	2	1								
3	16	18	3	1							
4	100	88	48	4	1						
5	676	860	280	100	5	1					
6	5776	6696	4020	680	180	6	1				
7	53824	76552	35196	13580	1400	294	7	1			
8	583696	805568	531328	131936	37240	2576	448	8	1		
9	6864400	10765008	6159168	2571744	397656	88200	4368	648	9	1	
10	90174016	141145120	101644560	32404800	9780960	1027152	187320	6960	900	10	1

Table 3.7: Calculated values of $e(D_r(\mathcal{PB}r_n))$.

$n \setminus r$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	4	7	1								
3	25	70	18	1							
4	225	921	331	34	1						
5	2704	15191	6880	995	55	1					
6	41209	304442	163336	29840	2345	81	1				
7	769129	7240353	4411190	958216	95760	4739	112	1			
8	17139600	200542851	134522725	33395418	3992891	252770	8610	148	1		
9	447195609	6372361738	4595689200	1267427533	174351471	13274751	581196	14466	189	1	
10	13450200625	229454931097	174564980701	52345187560	8059989925	709765413	37533657	1205460	2289	235	1

Table 3.8: Calculated values of $e(D_r(\mathcal{P}_n))$.

$n \setminus r$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		1									
2	0		1								
3		6		1							
4	0		24		1						
5		120		60		1					
6	0		1080		120		1				
7		5040		5040		210		1			
8	0		80640		16800		336		1		
9		362880		604800		45360		504		1	
10	0		9072000		3024000		105840		720		1

Table 3.9: Calculated values of $e^{\xi}(D_r(\mathcal{B}r_n)) = e^{\xi}(D_r(\mathcal{PB}r_n))$ with $M = 0$.

$n \setminus r$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	0	5	1								
3	0	43	15	1							
4	0	529	247	30	1						
5	0	8451	4795	805	50	1					
6	0	167397	108871	22710	1985	75	1				
7	0	3984807	2855279	697501	76790	4130	105	1			
8	0	111319257	85458479	23520966	3070501	209930	7658	140	1		
9	0	3583777723	2887069491	871103269	129732498	10604811	495054	13062	180	1	
10	0	131082199809	109041191431	35334384870	5843089225	549314745	30842427	1046640	20910	225	1

Table 3.10: Calculated values of $e^{\xi}(D_r(\mathcal{P}_n))$ with $M = 0$.

Chapter 4

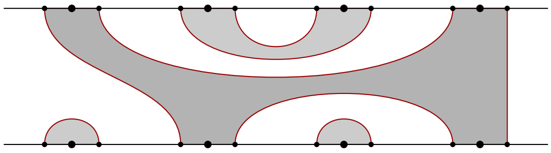
Planar Diagram Semigroups and Their Idempotents

Write $\mathfrak{c}_n$ for the chain $n' < (n-1)' < \cdots 1' < 1 < 2 \cdots n$. The *planar part* πS of a diagram semigroup $S \leq \mathcal{P}_n$ is the set of diagrams in S which may be represented by a noncrossing partition P in the sense that is $a < b < c < d$ are in $\mathfrak{c}_n$, then if a and c are in the same component of P then either b and d are in different components, or all four lie in the same component. A *planar diagram semigroup* is one which is equal to its planar part.

We define the monoids $\pi\mathcal{P}_n$, $\mathcal{M}_n$ and $\mathcal{J}_n$ as respectively the planar parts of $\mathcal{P}_n$, $\mathcal{PB}r_n$ and $\mathcal{B}r_n$. These are respectively the *planar partition monoid*, the *Motzkin monoid* and *Jones monoid*. The set of all noncrossing partitions of degree n is $\pi\mathcal{P}_n$, so if $S \subseteq \mathcal{P}_n$ then $\pi S = S \cap \pi\mathcal{P}_n$. The following result was brought to my attention by James East, and establishes the importance of the Jones monoid in the study of planar diagram semigroups.

Proposition 4.0.1. *The semigroups $\mathcal{J}_n$ and $\pi\mathcal{P}_{2n}$ are isomorphic.*

This result is explored in the algebra setting in [43]. The following picture is suggestive of how the bijection is constructed in terms of the pictures.



To understand how to interpret this picture, observe that the lines connecting the smaller dots are in fact the edges of an element of $\mathcal{J}_{2n}$. The grey areas bounded within these lines connect the larger dots together, the groupings observed giving rise to a noncrossing partition in $\pi\mathcal{P}_n$.

In this chapter, we define a structure on the idempotents of $\mathcal{K}_n = \mathcal{M}_n, \mathcal{J}_n$ which simultaneously refines the natural order, and partitions them into easy-to-count sets. This structure is one of a cubical set, and our ability to decompose it into very simple consitituent parts will later provide leverage to design a fast algorithm for iterating through, or enumerating, the idempotents in the semigroups of interest here.

The process of iterating in this way naturally splits naturally into three parts:

1. Define a function $\hat{\cdot} : E(\mathcal{K}_n) \to D$ mapping idempotents into some subset $D \subseteq E(\mathcal{K}_n)$ which fixes D pointwise;

2. Find a way to quickly iterate over D computationally;

3. Find a way to quickly iterate over the preimage of some $\delta \in D$.

This chapter concerns itself primarily with providing a decomposition of $E = E(\mathcal{K}_n)$ that will be useful to solve the first and third tasks. The second is solved in the Appendix A, using results from [63].

This chapter loosely follows Section 3 of [13], albeit in more detail.

4.1 Idempotents in $\mathcal{M}_n$

Recall that an element e of a semigroup is idempotent if $e^2 = e$. It's conventional to denote the set of idempotents in the semigroup S by $E(S)$, or E if S is understood. In the case of semigroups, this set has a well-known ordered structure;

the following, classical definition from the theory of semigroups will suffice for our purposes.

Recall that $e \leq f$ in the natural order on idempotents if $e \leq f \iff e = ef = fe$.

For an element of the Motzkin monoid $\mathcal{M}_n$, there is a simple test for idempotency which can be executed quickly — in $O(n)$-time. We first need to define some terminology.

The *rank*[*] of $\alpha \in \mathcal{P}_n$ is simply the number of transverse edges in a diagram, which are those meeting points on both the top and bottom of the picture. In the case of Motzkin elements this is the number of edges from the top to the bottom of the diagram. Elements of rank which is below 2 will be important in our study here.

4.1.1 The Interface Graph of a Motzkin Element

Recall that given $\alpha \in \mathcal{PB}r_n$, there is an associated graph $\Lambda(\alpha)$, edges coloured and possibly with mutliple appearing. We define a modification here, to be more consistent with the notation of the latter paper.

Given an element $\alpha \in \mathcal{M}_n$, the *interface graph* $\Gamma(\alpha)$ is obtained by taking $\Lambda(\alpha)$, removing the loops and marking certain vertices with up- or down-facing half-edges, which we'll refer to as *stubs*. We add an upward-facing stub if the vertex is incident in $\Lambda(\alpha)$ on no blue edge, and a downward-facing stub if the vertex is incident on no red edges.

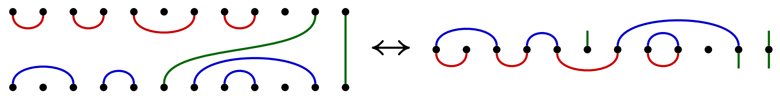
There are two reasons for this slight change in notation. First, logistically, the chapter on the Brauer monoids was added after the rest of the thesis was completed at the suggestion of the external examiner, and has been adapted from the source material in [12].

The second reason is that, when performing calculations with diagrams, one draws them one below the other to denote multiplication. There is a certain *generic neighbourhood* of the line of interface between these diagrams, which is

[*]Certain authors [37] prefer the term *propagating number*.

any open neighbourhood in the bounding rectangle of the concatenated diagrams which contains all non-transversal edges meeting the interface line between the diagrams.

Any such neighbourhood, and specifically the edges contained in it, will contain enough information to reconstruct an element α from the multiplication diagram $\Gamma(\alpha, \alpha) = \Gamma(\alpha)$. Below we can see the correspondence between an element α and the interface graph $\Gamma(\alpha)$



(4.1)

The upward and downward edges are coloured blue and red as before, but now the upper and lower stubs are emphasised in green.

Formally, this object consists of a pair of graphs. The upper edges lie in $\Gamma^{\vee}_{\alpha}$ of non-loops and $S^{\vee}_{\alpha}$ if they are loops, so that the resulting graph is a partial matching, possibly with some loops on unmatched vertices. The lower edges are similar, with $\Gamma^{\wedge}_{\alpha}$ consisting of non-loops and $S^{\wedge}_{\alpha}$ comprising loops. Such a graph only represents some $\alpha \in \mathcal{M}_n$ precisely if there are equally many upper and lower loops (the number of these is the rank), so we assume this hereafter without further comment.

We conventionally write the elements of $S^{\vee}$ and $S^{\wedge}$ in order:

$$S^{\vee}_{\alpha} = \{s^{\vee}_{\alpha;1} < s^{\vee}_{\alpha;2} < \cdots < s^{\vee}_{\alpha;k}\},$$

where $k = |S^{\vee}_{\alpha}|$. A handy mnemonic is that in the $\vee$ version of graphs, the edges point down and in $\wedge$ they point upwards. Where α is understood, we write Γ, $\Gamma^{\vee}$, $S^{\vee}$ and so on for brevity.

Proposition 4.1.1. *Let $n > 0$ be a positive integer. A graph of the above form is an interface graph of an element of $\mathcal{M}_n$ precisely if the following hold:*

1. *the reflexive closure of the adjacency relation in $\Gamma^{\vee}$ defines a noncrossing partition;*

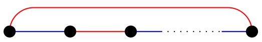
2. *for any $0 < i < s < j \leq n$, we have $(i, j) \notin \Gamma^{\vee}$ or $s \notin S^{\vee}$;*

3. *likewise for* $\Gamma^\wedge$ *and* $S^\wedge$.

Akin to Lemma 3.4.2, we have four kinds of connected components. We'll recycle the notation $C(\mathcal{K}_X)$ to refer to a connected component in a diagram in $\mathcal{K}_n$ where $X \subseteq [\![n]\!]$. We can now recast our prior result as follows.

Lemma 4.1.2. *Let* $\alpha \in C(\mathcal{M}_X)$ *where* $X \subseteq [\![n]\!]$ *is finite. Then* $\Gamma(\alpha)$ *is of one of the following four forms:*

(1) : *an alternating-colour path of even length bounded by two stubs,*

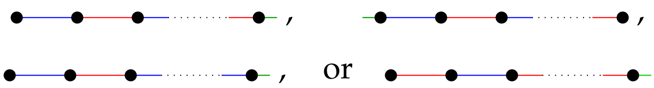
(2)  : *an alternating-colour circuit of even length,*

(3) : *an alternating-colour path of even length,*

(4) *or* : *an alternating-colour path of odd length.*

If $\alpha \in C(\mathcal{J}_X)$, *then* $\Lambda(\alpha)$ *is of the form* (1) *or* (2).

Note that non-irreducible components may also have one of two other possible forms, namely an alternating-colour path of arbitrary length, ending on one stub:



(4.2)

There are also paths of odd length bounded by two stubs:

 or (4.3)

Definition 4.1.3. The components defined in Lemma 4.1.2 we refer to respectively as *trans-active paths*, *cycles*, and the latter two types we refer to as *inert paths*. The components described in (4.2) we refer to as *half-rays* and those in (4.3) are referred to as *cis-active paths*.

The prefixes cis- and trans- come from Latin "this side of" and "across,on the far side;" their application in this context is from Des FitzGerald.

Observation 4.1.4. *The elements of $\mathcal{M}_n$ are (up to homeomorphism) planar graphs. If one were to delete the transverse edges and specify which upper and lower vertices were incident on such edges one would only have one way (up to homeomorphism) of adding transverse edges to the graph in the correct places without any crossing. In other words, the interface graph $\Gamma(\alpha)$ determines α.*

Proposition 4.1.5. *An element $\alpha \in \mathcal{M}_n$ is idempotent precisely if its interface graph contains no cis-active paths and no half-rays.*

Such features will be referred to as *obstacles (to idempotency)*.

Proof. Let $\alpha \in \mathcal{M}_n$ for $n \geq 0$. We note firstly that the interface graph $\Gamma := \Gamma_\alpha$ is a certain generic neighbourhood of the interface between the two copies of α in the multiplication diagram for α^2.

Now let's assume to derive a contradiction that Γ possessed a cis-active path. By *-regularity, we may assume this is bounded by $s_i^\wedge$ and $s_j^\wedge$ in $S_\alpha^\wedge$. Let us now denote by a prime anything in the bottom half of the multiplication diagram; there is a path

$$s_i^\vee \xrightarrow[\alpha]{} s_i^\wedge = a' \xrightarrow[\alpha']{} b' \xrightarrow[\alpha]{} \cdots \xrightarrow[\alpha']{} r = s_j^\wedge \xrightarrow[\alpha]{} s_j^\vee,$$

giving us an edge $(s_i^\vee, s_j^\vee)$ in α^2 but not α, a clear contradiction.

Now, let's assume that none of these obstacles occur, i.e. that Γ has no connected components that are either half-rays or cis-active paths. Cycles and inert paths can't obstruct idempotency, so we need only consider the active paths to prove idempotency.

At this point, planarity allows us to number the paths in the order they appear left-to-right; the stubs at either end of a trans-active path therefore must agree on index. Pick such a path. Since the stubs at the boundary of the path agree on index, the interface propagates this path through the multiplication diagram as required. □

Corollary 4.1.6. *Any element of rank zero in $\mathcal{M}_n$ is idempotent.*

Cis-active paths and half-rays contain transversal edges, the number of which is the rank.

4.2 A Reduction Process on Idempotents

In this section, we will develop a rewriting process on the elements of the Motzkin monoid of degree n. This will turn out to be rank-reducing, have unique irreducible elements (i.e. be a *complete* rewriting system), and idempotency, rank-parity and membership in the Jones monoid $\mathcal{J}_n$ will all three be left invariant under such a process of rewriting. The irreducible elements in components containing an idempotent will turn out to be precisely the idempotents whose ranks are 0 or 1.

The structure underlying such a process is a directed graph Δ. The main tools are classical from the perspective of rewriting theory.

Definition 4.2.1. A *rewriting system* on a set X is a graph Γ whose vertex set is X and whose *reductions* consist of directed edges.

We concern ourselves with some properties of the directed paths in the graph, of which there may be infinitely many; all paths in this chapter will be assumed directed without further comment.

Are there arbitrarily long finite paths? Are there infinite sequences of reductions $x_0 \to x_1 \to \cdots \to x_n \to \cdots$? If two paths start at u, does there exist paths to v from their endpoints?

We refer to a familiar terminology where convenient, so if $x \to y$, then y is the *child* of x and x is the *parent*. Similarly if $x = a_0 \to a_1 \to \cdots \to a_n = y$ then x and y are respectively an *ancestor* and *descendant* of one another, in which case we write $x \xrightarrow{*} y$. An element x in a rewriting system is *reducible* it has a child and *irreducible* otherwise.

Definition 4.2.2. A rewriting system is

- *terminating* if there are no infinite sequences of reductions;
- *locally confluent* if any two children of any element have a common descendent;
- *semi-confluent* if any child and descendent of any element have a common descendent;

- *confluent* if any two descendents of any node have a common descendent;
- *Church-Rosser* if any two weakly-connected nodes share a common descendent.

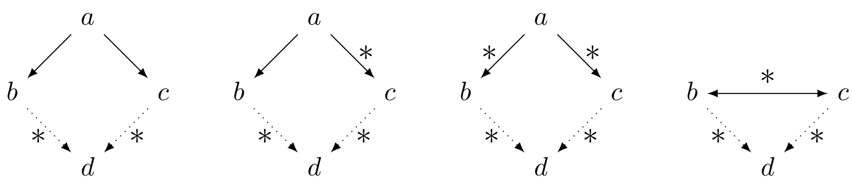


Figure 4.1: Diagrams describing the notions of local and semi-confluence, confluence and the Church-Rosser property for rewriting systems. The solidly-drawn lines are assumptions, and if a, b and c satisfy these assumptions then in each case, there exists a common descendent d.

It's fairly clear that

$$\text{Church-Rosser} \implies \text{confluent} \implies \text{semi-confluent} \implies \text{locally confluent}$$

The following classical theorem, due to Newman, gives a partial converse.

Lemma 4.2.3 (Diamond Lemma)**.** *A terminating rewriting system is locally-confluent precisely if it is Church-Rosser.*

Proposition 4.2.4. *Let $\mathscr{X} = (X, \to)$ be a rewriting system with $x \in X$. If $\mathscr{X}$ is terminating then there is an irreducible y with $x \overset{*}{\to} y$. If $\mathscr{X}$ is Church-Rosser and y, y' are irreducibles with $x \overset{*}{\to} y$ and $x \overset{*}{\to} y$ then $y = y'$.*

Proof. Assume that $\mathscr{X}$ is terminating. Then there are no directed ends, so any sequence of edges with neighbouring edges adjacent on a vertex is finite, and hence a path. Pick a path from x which is maximal in the sense that no edge exists from the endpoint. This must exist, otherwise every such path could be extended to a directed end. The endpoint is an irreducible.

The other claim is immediate as the paths from x to y and y' together comprise an undirected path from y to y', and the Church-Rosser property guarantees these have a common descendant. As the two are assumed irreducible in the statement of the proposition, they must be equal. □

We call a rewriting system *complete* if terminating and Church-Rosser. These are precisely the rewriting systems under which rewrites are guaranteed to finish with unique irreducibles; those are elements which do not rewrite further, and they are unique in the sense that, given an irreducible, one may not rewrite any ancestor to a different irreducible.

One may associate to a complete rewriting system $\mathscr{X} = (X, \to)$ a map $x \mapsto \hat{x}$ which associates to each $x \in X$ the unique irreducible $\hat{x}$ satisfying $x \overset{*}{\leftrightarrow} \hat{x}$.

4.2.1 A Complete Rewriting System

We can define a rewriting system on the elements of $\mathcal{M}_n$ as follows:

1. Let α be an element of $\mathcal{M}_n$ and $\Gamma := \Gamma(\alpha)$ its interface graph.
2. Pick some odd $i < |S^\vee|$.
3. Remove the i-th and $(i+1)$-th entries from $S^\vee$, adding the edge $(s_i^\vee, s_{i+1}^\vee)$ to $\Gamma^\vee$.
4. Do the same to $S^\wedge$ and $\Gamma^\wedge$.

Pictorally, this amounts to drawing an edge from an odd-indexed stub to the next [even-indexed] stub on the interface graph, and doing this above and below the interface line. If α' can be obtained from α by such an above operation write $\alpha \rightsquigarrow \alpha'$.

This process has some interesting features.

Proposition 4.2.5. *Let $\alpha \rightsquigarrow \alpha'$. Then:*

1. *The rank of α' is two less than that of α;*
2. *We have $\alpha \in \mathcal{J}_n$ precisely if $\alpha' \in \mathcal{J}_n$.*

Proof. The first implication needs no proof; this follows straight from the definition.

For the second, observe there are n edges in total on a diagram in $\mathcal{J}_n$, as it may be represented by a perfect matching on $2n$ points. These pairings (i, j) are either realisable as edges belonging to $\Gamma^\vee$ or to $\Gamma^\wedge$ in the interface graph, or as a pairing of a vertex $i \in [\![n]\!]$ with one from $j' \in [\![n]\!]'$ a part in $S^\vee$ and $S^\wedge$. The stubs in the S index the transverse edges, matching with a partner across the diagram; we have

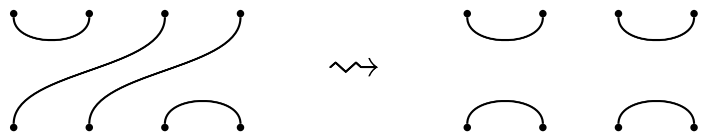
$$|\Gamma_\alpha^\vee| + |\Gamma_\alpha^\wedge| + |S_\alpha^\vee| = n$$

precisely if $\alpha \in \mathcal{J}_n$.

Rewriting does not affect the above parameter; we decrease $|S_\alpha^\vee|$ by two when we exchange two pairs of stubs representing a pair of transverse edges, for two non-transverse edges, in turn increasing $|\Gamma_\alpha^\vee|$ and $|\Gamma_\alpha^\wedge|$ each by one. □

Corollary 4.2.6. *If D denotes those elements of rank at most one, then every $\delta \in D$ is irreducible.*

Example 4.2.7. The following example shows a rewrite $\alpha \rightsquigarrow \alpha'$.



We have

$$S_\alpha^\vee = \{3,4\}, \qquad S_\alpha^\wedge = \{1,2\}, \qquad S_{\alpha'}^\vee = S_{\alpha'}^\wedge = \varnothing,$$
$$\Gamma_\alpha^\vee = \{(1,2)\}, \qquad \Gamma_\alpha^\wedge = \{(3,4)\}, \qquad \Gamma_{\alpha'}^\vee = \Gamma_{\alpha'}^\wedge = \{(1,2),(3,4)\}.$$

The only odd $i < |S_\alpha^\vee|$ to pick was 1; we were bound to rewrite replacing the only two transverse edges.

The rank-reducing nature implies that, as a rewriting system, this is cycle-free. In particular, by the finiteness of $\mathcal{M}_n$, this is a terminating system.

Proposition 4.2.8. *The rewriting system on $\mathcal{M}_n$ given by $\rightsquigarrow$ is complete.*

Proof. Termination is automatic, since rewriting reduces rank, which starts non-negative and finite. By the Diamond Lemma (Lemma 4.2.3), it hence suffices to prove local confluence. Let $\alpha \rightsquigarrow \beta$ and $\alpha \rightsquigarrow \gamma$. Then we seek to prove that there is η such that $\beta \overset{*}{\rightsquigarrow} \eta \overset{*}{\leftsquigarrow} \gamma$.

If $\gamma = \beta$ then $\eta = \beta$ suffices, so assume without loss of generality that $\gamma \neq \beta$. Write $\Gamma^\vee_\xi$ and $S^\vee_\xi$, for example, for the relevant features of the interface graph of ξ.

By definition, if $\alpha \to \xi$ there is odd i_ξ such that

$$S^\vee_\alpha = S^\vee_\xi \cup \{s^\vee_{\alpha;i_\xi}, s^\vee_{\alpha;i_\xi+1}\} \quad \text{and} \quad \Gamma^\vee_\xi = \Gamma^\vee_\alpha \cup \{(s^\vee_{\alpha;i_\xi}, s^\vee_{\alpha;i_\xi+1})\}. \tag{4.4}$$

In particular, this is true for both $\xi = \beta, \gamma$. Note that $|i_\beta - i_\gamma| \geq 2$ since $\beta \neq \gamma$, and also $S^\vee_\beta \cup S^\vee_\gamma = S^\vee_\alpha$.

The above observations all hold for the other side of the interface line (i.e. the $\wedge$-counterparts of each of the above) because of the involution *.

In particular, if we define η by its interface graph, writing

$$S^\vee_\eta := S^\vee_\beta \cap S^\vee_\gamma \quad \text{and} \quad \Gamma^\vee_\eta := \Gamma^\vee_\beta \cup \Gamma^\vee_\gamma,$$

then we see that $\beta \rightsquigarrow \eta \leftsquigarrow \gamma$ as required. □

Corollary 4.2.9. *There is a mapping $\alpha \mapsto \hat{\alpha}$ associated with the rewrite system $(\mathcal{M}_n, \rightsquigarrow)$ asssociating to each $\alpha \in \mathcal{M}_n$ the unique irreducible in its weakly connected component.*

We will call this mapping the *hat map*.

Corollary 4.2.10. *The fibres of this map are exactly the weakly connected components of the rewriting system $(\mathcal{M}_n, \rightsquigarrow)$.*

Corollary 4.2.11. *If $\alpha \in \mathcal{M}_n$, then $\alpha = \hat{\alpha}$ precisely if the rank of α is at most one.*

This rests on the fact that the rank is the number $r = |S^\vee| = |S^\wedge|$. If there are at least two edges, then one can rewrite.

Corollary 4.2.12. *Let $r, \hat{r}$ denote the ranks of $\alpha, \hat{\alpha} \in \mathcal{M}_n$. Then $r - \hat{r} = 2k$ for some nonnegative integer k.*

Here k is the number of rewrites, as each reduces rank by two.

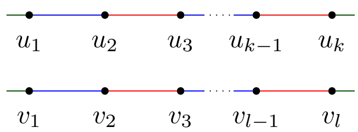
Corollary 4.2.13. *The map associated to the complete rewriting system $(\mathcal{M}_n, \rightsquigarrow)$ maps onto the set D comprising all elements of rank at most 1, fixing D pointwise and preserving the parity of the rank under mapping.*

The study of the fibers of this map, were we able to calculate detailed statistics for them, would reduce the study of the entire set of elements (respectively idempotents) to those of rank at most 1. This somewhat-fuzzy heuristic will turn out to be of importance later on; we will make it precise via construction of a combinatorial cell complex whose 0-cells are precisely the idempotents, whose 1-cells are rewrites $\alpha \rightsquigarrow \alpha'$, and whose connected components are precisely these fibres.

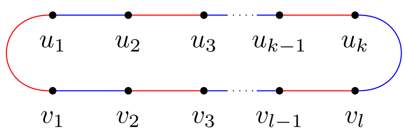
Proposition 4.2.14. *Let $\alpha \rightsquigarrow \beta$, with $\alpha \in E(\mathcal{M}_n)$ an idempotent. Then β is idempotent.*

Proof. Given that α is idempotent, there are no obstacles to idempotency in Γ_α. We need only check that there are no obstacles to idempotency then in Γ_β, by 4.1.5.

The interface diagram of α therefore contains only active trans-paths, cycles and inert paths. Since we remove two transversal edges from the diagram and replace them with two nontransversal edges, we are replacing a pair of active trans paths



with the cycle



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No new obstacles are introduced, and β is idempotent. □

Corollary 4.2.15. *For any idempotent $\alpha \in \mathcal{M}_n$, $\hat{\alpha}$ is idempotent.*

4.2.2 Reversing Rewriting

We have seen that rewriting with the $\rightsquigarrow$-relation preserves idempotency, but does it *reflect* idempotency? That is, is it the case that if $\alpha \to \beta$ with β idempotent, is it guaranteed that α is idempotent? Unfortunately, the answer is no; Example 4.2.7 shows us that a non-idempotent α of rank 2 rewrites to an element $\hat{\alpha}$ of rank zero, which must be idempotent by Corollary 4.1.6.

The aim of this subsection is to reverse the process of rewriting in such a way as to preserve idempotency. To do this, we define another rewriting system, which coincides on $E(\mathcal{M}_n)$ with the reverse of that induced by the $\rightsquigarrow$ relation. We do this using the combinatorics of the interface diagram.

Definition 4.2.16. Let $\alpha, \beta \in \mathcal{M}_n$ with $\alpha \rightsquigarrow \beta$. Say hat α *rewrites up* to β (written $\beta \twoheadleftarrow \alpha$) if there is a cycle in the interface graph of α and two connected edges, (i,j) in $\Gamma_\alpha^\vee$ and (k,l) in $\Gamma_\alpha^\wedge$, so that

$$\Gamma_\beta^\vee = \Gamma_\alpha^\vee \smallsetminus \{(i,j)\}, \qquad \Gamma_\beta^\wedge = \Gamma_\alpha^\wedge \smallsetminus \{(k,l)\},$$
$$S_\alpha^\vee = S_\beta^\vee \smallsetminus \{i,j\}, \qquad S_\beta^\wedge = S_\alpha^\wedge \smallsetminus \{k,l\}.$$

Write $\overset{*}{\twoheadleftarrow}$ for the transitive closure of $\twoheadleftarrow$, so that $\alpha \overset{*}{\twoheadleftarrow} \beta$ precisely if there is a sequence $\alpha = s_0, s_1, \ldots, s_k = \beta$ where $s_{i-1} \twoheadleftarrow s_i$ for each $i \in [\![k]\!]$.

Proposition 4.2.17. *Let $\alpha, \beta \in \mathcal{M}_n$ satisfy $\alpha \twoheadleftarrow \beta$. Then $\hat{\alpha} = \hat{\beta}$.*

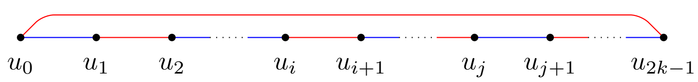
This is a direct consequence of $(\mathcal{M}_n, \rightsquigarrow)$ being complete.

Proposition 4.2.18. *Let $\alpha \in E(\mathcal{M}_n)$ be an idempotent and $\beta \in \mathcal{M}_n$ satisfy $\alpha \twoheadleftarrow \beta$. Then $\beta \in E(\mathcal{M}_n)$ is also an idempotent and $\hat{\alpha} = \hat{\beta}$.*

Proof. First observe that the interface graph of β coincides with α except in the one connected component in that of α that is replaced with two in that of β. It's clear that, since this component in $\Gamma(\alpha)$

By Proposition 4.1.5, an element is idempotent precisely if its interface graph lacks cis-active paths and half-rays.

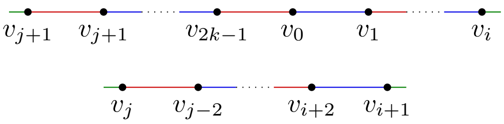
We observe that Γ_α has no such obstacles, and we need only establish that rewriting up produces none. Writing $\wedge$ for an upward edge and $\vee$ for a lower edge, the interface graph of α contains a cycle



of length $2k$, such that

$$S_\beta^\vee = S_\alpha^\vee \sqcup \{v_i, v_{i+1}\}, \qquad S_\beta^\wedge = S_\alpha^\wedge \sqcup \{v_j, v_{j+1}\},$$
$$\Gamma_\alpha^\vee = \Gamma_\beta^\vee \sqcup \{(v_i, v_{i+1})\}, \qquad \Gamma_\alpha^\wedge = \Gamma_\beta^\wedge \sqcup \{(v_j, v_{j+1})\},$$

Hence we produce two trans-active paths



where v_i and v_{i+1} have downward stubs, and v_j and v_{j+1} have upward stubs. Hence, β has no newly-introduced obstacles to idempotency not present in α, and therfore none. It is hence idempotent by Proposition 4.1.5. □

Theorem 4.2.19. *Let $\alpha \overset{*}{\twoheadleftarrow} \beta$ in $\mathcal{M}_n$. Then α is idempotent precisely if β is idempotent.*

By Proposition 4.2.18, if $\alpha \in E$ then $\beta \in E$, and by Proposition 4.2.14, the reverse holds.

4.3 A CW-Complex Structure on the set of Motzkin monoid idempotents E($\mathcal{M}_n$)

We've already seen that for $\alpha \in E(\mathcal{M}_n)$, there is a corresponding idempotent $\hat{\alpha}$ of low rank which can be obtained from it by some sequence of $\rightsquigarrow$-rewrites.

Assume we have $\alpha \rightsquigarrow \beta$. Since $S_\alpha^\vee$ and company impose very strict constraints on what $S_\beta^\vee$ are, we'll drop the αs in the subscript as follows. We write

$$S_\alpha^\vee = \{s_1^\vee(\alpha) < s_2^\vee(\alpha) < \cdots < s_r^\vee(\alpha)\}$$

where r is the rank of α, and similar for $S_\alpha^\wedge$ and the $s_i^\wedge(\alpha)$, omitting α where convenient. Whenever we have

$$S_\alpha^\wedge = S_\beta^\wedge \cup \{s_i^\wedge, s_{i+1}^\wedge\}, \qquad \Gamma_\beta^\wedge = \Gamma_\alpha^\wedge \cup \{(s_i^\wedge, s_{i+1}^\wedge)\},$$
$$S_\alpha^\vee = S_\beta^\vee \cup \{s_i^\vee, s_{i+1}^\vee\} \quad \text{and} \quad \Gamma_\beta^\vee = \Gamma_\alpha^\vee \cup \{(s_i^\vee, s_{i+1}^\vee)\},$$

then we write $\alpha \rightsquigarrow_i \beta$.

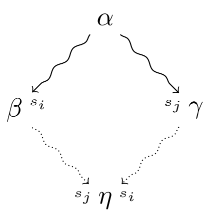
Let $\alpha \in \mathcal{M}_n$ be fixed herein and r be its rank. The following observations follow directly from the definition.

Proposition 4.3.1. *If $\beta \twoheadleftarrow \alpha$ then there is $i < r$ whereby $\alpha \underset{i}{\rightsquigarrow} \beta$.*

Proposition 4.3.2. *Let $i < r$. There is at most one β such that $\alpha \rightsquigarrow_{s_i} \beta$.*

Proposition 4.3.3. *There is at most one α such that $\alpha \rightsquigarrow_{s_i} \beta$.*

Proposition 4.3.4. *If α, β and γ satisfy $\alpha \underset{s_i}{\rightsquigarrow} \beta$ and $\alpha \underset{s_j}{\rightsquigarrow} \beta$ as in (4.5), then there is a (unique) η which satisfies either $\beta \underset{s_i}{\rightsquigarrow} \eta$ or $\gamma \underset{s_i}{\rightsquigarrow} \eta$. In this case, the relations are satisfied simultaneously.*



(4.5)

This follows immediately from the definition. We can induct upon these observations to great effect; if $\mathbf{s} = (s_1, \ldots, s_n)$ is a sequence of numbers such that

$$\alpha = \zeta_0 \rightsquigarrow_{s_1} \zeta_1 \rightsquigarrow_{s_2} \cdots \rightsquigarrow_{s_{(n-1)}} \zeta_{(n-1)} \rightsquigarrow_{s_n} \zeta_n = \beta.$$

then we can extend the notation $\rightsquigarrow$ notation to write $\alpha \rightsquigarrow_{\mathbf{s}} \beta$ in such a case. Defining the *content*

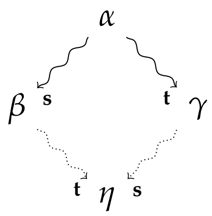
$$\coprod \mathbf{s} = \{s_1, s_2, \ldots, s_n\}$$

of $\mathbf{s}$ as the set of entries, the following follows from an iterative application of Proposition 4.3.2.

Corollary 4.3.5. *Given $\mathbf{s}$, the relation $\rightsquigarrow_{\mathbf{s}}$ is an injective partial function. That is, given α (respectively β) there is at most one β (resp. α) such that $\alpha \rightsquigarrow_{\mathbf{s}} \beta$.*

In fact, by inducting on Proposition 4.3.4 we can attain the following

Corollary 4.3.6. *If α, β and γ satisfy the solidly-drawn relationship in (4.6), and* $\mathbf{s}$ *and* $\mathbf{t}$ *have disjoint contents, then there is a (unique) η which satisfies either relation drawn in dots. In this case, the relations are satisfied simultaneously.*



(4.6)

We then have the following by induction on the above.

Corollary 4.3.7. *If $S = \coprod \mathbf{s} = \coprod \mathbf{t}$, then we have $\alpha \rightsquigarrow_{\mathbf{s}} \beta$ precisely if $\alpha \rightsquigarrow_{\mathbf{t}} \beta$.*

In this case, we can write $\alpha \rightsquigarrow_S \beta$. Aided by these observations, we'll inductively build a complex whose cells are cubes which are in some sense "indexed" by sets of these relationships fixed at basepoints.

Definition 4.3.8. The *mutation complex* $\mathscr{E}(\mathcal{M}_n)$ is a cubical complex defined as follows:

- The vertices (0-cubes) are the idempotents in $E(\mathcal{M}_n)$;
- The k-cubes ($1 \leq k \leq \frac{n}{2}$) are indexed by the triples (α, S, β) where $\alpha \rightsquigarrow_S \beta$ such that $|S| = k$;
- The boundary of an k-cube (α, S, β) is the union of all the $(k-1)$-cubes (γ, T, η) where $T \subset S$ and either $\alpha = \gamma$ or $\beta = \eta$.

The purpose of the rest of this section is the proof of the following.

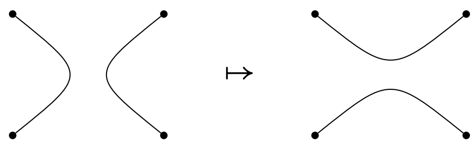
Theorem 4.3.9. *Each connected component in $\mathscr{E}(\mathcal{M}_n)$ is a product of rooted trees of height at most 1, as pointed CW-complexes.*

The above result has some profound implications for understanding the set of idempotents in $\mathcal{M}_n$ and any submonoids whose membership is preserved by the hat map, such as the Jones monoid. We've taken pains by this point to lay the ground so that the result now follows quickly from previous observations.

Proof. We fix $\alpha \in E(\mathcal{M}_n)$ of rank at most one, to be the basepoint of this complex. The only connected components in α are trans-active paths (at most one by assumption), cycles or inert paths. The inert paths aren't the possible result or rewriting, nor is the (potentially nonexitent) trans-active path, so we need only study the cycles.

We seek to build a coherent "coordinate structure" on the connected component $[\alpha]$ whose entries take values in the rooted trees of height 1 discussed in the statement of the theorem. To do this, we need to examine the reverse of the $\rightsquigarrow$ relation.

The relation $\rightsquigarrow$ produces a cycle connected component left of the rightmost transversal (if present) in the interface graph where previously there were two trans-active paths, and does so by *mutating* the transverse edges to two non-transverse return edges on the same four points:



(4.7)

In general we use this term to mean any such action on four points in a diagram where two edges are swapped for two other edges. Since the purpose of our study is semigroups of planar diagrams, we'll require in addition that the new edges do not cross.

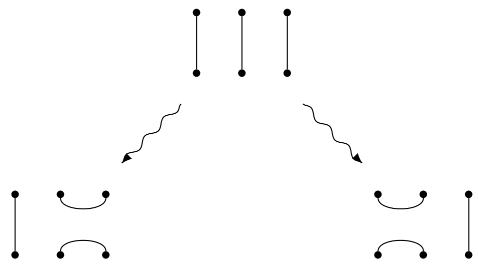
The reverse relation therefore consists of all rewrites of pairs of return edges in the same connected component. Fix a connected component θ of α, and denote by $R^{\vee}(\theta)$ the set of upper return-edges in θ and $R^{\wedge}(\theta)$. Let T_θ denote the set

$$T_\theta := \{*\} \cup \Big(R^{\vee}(\theta) \times R^{\wedge}(\theta)\Big).$$

Identify each element t of $T_\theta - \{*\}$ with the element of $[\theta]$ that has that pair of return edges opened to a transversal. Then we have $t \rightsquigarrow \theta$, and $\alpha_t \rightsquigarrow \alpha$, where α_t is α with θ replaced by t, giving us the edges we desire.

This process can be carries out independently across the connected components left of the rightmost tansversal, giving rise to the desired product structure. The basepoint is, as expected, the element α, which appears with all its connected components mutated. □

The attention paid to the rightmost transversal is to prevent double-counting. We only mutate on *odd* indexed stubs for the same reason; we could have something like the following situation in which one element rewrites to separate irreducible elements.



(4.8)

Observation 4.3.10. *Fix $\alpha \in E(\mathcal{M}_n)$ an idempotent of rank at most 1, and θ a connected component. The size of T_θ is $1 + t_\theta \cdot b_\theta$ where t and b are respectively the numbers of top and bottom return edges.*

Corollary 4.3.11. *The number of idempotents in $[\alpha]$ is*

$$\big|[\alpha]\big| = \prod_\theta (1 + b_\theta \cdot t_\theta),$$

with the product over cycles left of the rightmost transversal.

Part III

Enumeration: Theory and Implementation

Chapter 5

Convexity and Enumerating Idempotents in Planar Diagram Semigroups

This chapter builds a theoretical framework for extending the results from the last several sections into a family of methods for counting families of elements by parameters other than degree, such as by rank or number of blocks providing an obstacle to idempotency, for example.

The framework in question provides a partial answer to the question of what sorts of information can be gleaned from detailed study of the cube complex $\mathscr{E}(\mathcal{M}_n)$ defined in Chapter 4.

We also detail some results concerning elements of particularly high rank, and provide a set of polynomial bounds for the sequence $e_{n,n-d}$ of numbers idempotents of degree n for each fixed d. Formulae for the sequences are calculated in some small examples, but as with many enumeration problems in these monoids, it doesn't seem an easy task to find a general formula.

5.1 Convexity and Enumeration by Parameter Profile

Looking at Table E.1 and Table E.3 in the appendices, a few patterns become apparent.

The numbers $e_{n,n}$ of idempotents of rank n are equal to 1 for each $n \in \mathbb{N}_0$. This is very easy to see, since the only element with n transversal blocks in $\mathcal{M}_n$ or $\mathcal{J}_n$ is the identity, which is always idempotent.

The numbers $e_{n,n-1}$ for $\mathcal{M}_n$ seem to satisfy $e_{n,n-1} = n$ for the values of n we can see there. This is as one might expect, as there are n partial identities with $n - 1$ transversal blocks, one for each pair i, i' that aren't connected.

Definition 5.1.1. The *sequence of differences* $\Delta(a_i)$ of a sequence $(a_i)_i$ is the sequence $(a_{i+1} - a_i)_i$ with the same index. The *sequence of n^{th} differences* $\Delta^n(a_i)$ of $(a_i)_i$ is the sequence of differences of the sequence $\Delta^{n-1}(a_i)$, where $\Delta^0(a_i) = a_i$

More examination suggests that after a small number of terms, the terms $e_{n,n-d}$ seem to settle into a pattern where the sequence of d^{th} differences for $\mathcal{M}_n$ is constant. In order for this to be the case, the sequence must be equal to some degree-d polynomial, after the behaviour stabilises. Examination of the table for $\mathcal{J}_n$ suggests this is also the case there, but the polynomial is degree d for terms of the form $e_{n,n-2d}$.

It is not immediately obvious why this may be the case. We explore this phenomenon using a tame generalisation of integer partitions.

5.1.1 Ordered Partitions and Words

Recall a *composition* of n is a sequence of positive integers whose sum is n. The following observation is classical enumerative combinatorics, see [52].

Observation 5.1.2. *There are 2^{n-1} such sequences. To see this, consider n dots drawn on a piece of paper in a row. There are $n - 1$ gaps between, and hence 2^{n-1} ways to add vertical lines to the interior of the sequence so that no two are adjacent. The (necessarily* *positive) numbers of dots appearing consecutively gives rise to a composition, as seen below:*

$$(\bullet\bullet\bullet\bullet \mid \bullet\bullet\bullet\bullet\bullet \mid \bullet\bullet \mid \bullet\bullet) \longleftrightarrow 4+5+3+2+2.$$

Indeed, given a composition of n, it's easy to see how to insert lines to ensure the dots will be grouped into sets of the correct size. This correspondence is a bijection.

A tangentially-related set of problems to counting compositions is that of counting words. If I have an alphabet $\{a_1, \dots a_n\}$ of letters, and I give each a respective *weight* of r_i, one might ask how many ways there are to make words whose letters' weights sum to r. This generalises length, which may be interpreted as a weight given by an alphabet of unit-weight letters, and depends on the (multi)set of weights rather than the letters themselves, as we can see by permuting the alphabet.

The following example is an instructive, if brief, departure in this direction.

Example 5.1.3 (A Change Counting Problem [65])**.** Since decimalisation in the UK, there are coins of value (in pence) 1, 2, 5, 10, 20, 50, 100, and 200 in recent years. Some coins circulated recently (those denominations up to a £1) may have part of the shield of the UK on (and may also not). We'll assume that all two pound coins are indistinguishable to make the combinatorics more interesting, and perhaps because the author does not know better at the time of writing.

We may assign to each distinguishable coin a letter whose weight is its value in pence. For our purposes, the actual letters don't matter, so we will instead define just a multiset of weights as follows:

$$S = \{1, 1, 2, 2, 5, 5, 10, 10, 20, 20, 50, 50, 100, 100, 200\}.$$

This multiset will be encoded with the generating function

$$s(z) = 2(z + z^2 + z^5 + z^{10} + z^{20} + z^{50} + z^{100}) + z^{200} = \sum_{n \in S} z^n$$

where the coefficient of z^i is the multiplicity in S of the weight i. Note that, formally, we have

$$(1-z)(1+z+z^2+z^3+\cdots) = 1, \qquad \frac{1}{1-z} = 1+z+z^2+z^3+\cdots$$

Two piles of coins differ if the sequence of coins appearing from top to bottom is not the same. The number of ways of making a pile worth k pence with exactly n coins is given by the coefficient of z^k in $(s(z))^n$, and the total number of ways of giving change is given by the coefficient of z^k in the series

$$s^*(z) = \sum_{n=0}^{\infty} \big(s(z)\big)^n = \frac{1}{1 - s(z)}.$$

For our purposes, an approximate answer may suffice. To enumerate ways of making 10 pence, it suffices to consider only piles of size up to 10, since the smallest weight is 1. So we need only consider the coefficient of z^{10} in $s(z)^{10}$. In general, for fixed k, the polynomial $s(z)^k$ will suffice to exactly enumerate sequences needing at most k weights summed from S.

It will also happen that if k' is close to k in size, that $s(z)^{k'}$ will provide a suitable approximation for $s^*(z)$. For example, if $k' = 9$ in the above example, we only lose the ability to count piles of size 10. We already know there are only $2^{10} = 1024$ of these, so this approximation is fairly good. Indeed, the approximation will be better when k is larger, and in particular when the number of letters having small weight is small.

Construction of compositions can be seen as a specific instance of this coin counting-type problem, with respect to the set of weights $\mathbb{N}$ of all positive integers. The generating function of the multiset of positive integers is then

$$p(z) = \sum_{n=1}^{\infty} z^n = \frac{z}{1-z} = z + z^2 + z^3 + \cdots$$

Writing $p := p(z)$, we then have

$$\begin{aligned} p^*(z) &= \sum_{n=0}^{\infty} p = 1 + p + p^2 + p^3 + \cdots \\ &= \frac{1}{1-p} = \frac{1}{1 - \frac{z}{1-z}} = \frac{1-z}{1-2z} = 1 + \frac{z}{1-2z}. \end{aligned}$$

We see that $p^*(z) = 1 + 2z + 4z^2 + 8z^3 + \cdots$ is the generating function of the set of integer partitions, following Observation 5.1.2.

5.1.2 Filtering Powers of Generating Functions using Modular Arithmetic

When dealing with the twisted variants of the monoids we're interested in, it's convenient to be able to "ignore" terms in a power series whereby the powers of certain indeterminates don't have the right modular arithmetic properties. For example, if $f(z) = \sum_n a_n z^n$, one may wish to examine the function $g(z) = \sum_n a_{5n} z^{5n}$, omitting terms whose powers don't divide by 5.

We have the following tool, whose proof will be undertaken in the remainder of the subsection.

Theorem 5.1.4 (Root-of-unity filter)**.** *Let $f(z) = \sum_n a_n z^n$. Then for any primitive k-th root of unity ζ we have*

$$a_{kn} z^{kn} = \frac{1}{k} \sum_{i=0}^{k-1} f(\zeta^i \cdot z).$$

The term is due to Q. Yuan [77], who proves it in a somewhat more involved fashion than we do here.

Proof. Observe that the sum of all k-th roots of unity is zero. We have $z^k - 1 = (z+1) \cdot (1 + z + z^2 + \cdots z^{k-1})$, and hence

$$\sum_{i=0}^{k-1} z^i = 1 + z + z^2 + \cdots z^{k-1} = \frac{z^k - 1}{z+1}.$$

Now letting ζ be a k-th root of unity; we immediately infer that

$$\sum_{i=0}^{k-1} \zeta^i = \frac{\zeta^k - 1}{\zeta + 1} = 0.$$

Assume that ζ is a primitive k-th root of unity, and apply this observation to the formal sums $f(z\zeta^i)$; we obtain

$$f_{\mathrm{mod}\ k}(z) = \sum_{i=0}^{k-1} f(z\zeta^i) = \sum_{i=0}^{k-1} \sum_{n=0}^{\infty} a_n \zeta^{in} z^n = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{k-1} \zeta^{in} \right) a_n z^n. \tag{5.1}$$

Now, note that if $k|n$ then each term in the bracketed summation is 1, so this is k itself.

Otherwise, let d be the greatest common divisor of $k = pd$ and $n = qd$. Then the bracketed sum in (5.1) is equal to

$$\sum_{i=0}^{k-1} \zeta^{in} = \sum_{i=0}^{k-1} \zeta^{iqd} = d \sum_{i=0}^{p-1} \zeta^{iq} = 0$$

since $\zeta^{qpd} = \zeta^{kq} = 1$. Therefore, writing $f_{\text{mod } k}(z) = \sum_n b_n z^n$, we have $b_n = 0$ where $k \nmid n$, and otherwise $b_n = k a_n$. Therefore we see that

$$f_{\text{mod } k}(z) = \sum_{n=0}^{\infty} k a_{kn} z^{kn} = k \sum_{n=0}^{\infty} a_{kn} z^{kn}$$

as required. □

5.1.3 Tensor Product, Convexity and Generating Idempotents

Definition 5.1.5. Let $n > 0$ and $\alpha \in \mathcal{M}_n$. Write $a \sim_\alpha b$ whenever $a \leq b \leq c$ and one of a and a' is in the same block as one of c and c'. Now define $\sim_\alpha^*$ as the smallest equivalence relation finer than $\sim_\alpha$.

We say that α is *convex* if $\sim_\alpha^*$ is universal, and call the classes of this relation the *convex components* of α.

Example 5.1.6. Both elements of $\mathcal{M}_1$ are convex. The non-convex elements of $\mathcal{M}_2$ are the partial identities. The non-projections in $\mathcal{J}_3$ are all convex, but the projections are not, and indeed projections in $\mathcal{J}_{2n+1}$ are always non-convex for $n \geq 1$.

We don't define this relation or the notion of irreducibility for degree-zero objects for technical reasons that will soon become apparent.

Recall that for each $n, m \geq 0$ there is a *horizontal tensor* map

$$\otimes : \mathcal{M}_n \times \mathcal{M}_m \to \mathcal{M}_{n+m}.$$

This is in fact an injective homomorphism, and the image of $\mathcal{J}_n \times \mathcal{J}_m$ is contained within $\mathcal{J}_{n+m}$ in each case. We will conventionally refer to the tensor as acting on the union of the $\mathcal{M}_n$, rather than specifying degree.

Observation 5.1.7. *Every element α in $\mathcal{M}_n$ ($n > 0$) has a decomposition as a tensor product of some elements of degree at most n. The decomposition is* proper *if $\alpha = \alpha_1 \otimes \alpha_2$ is non-convex, and may not be unique in this case.*

When α is convex, there is no proper decomposition into a tensor of two Motzkin elements. In general, is a unique decomposition as a tensor product of convex Motzkin elements each having positive degree.

Observation 5.1.8. *The element $\alpha' = (\alpha_1, \ldots, \alpha_k)$ in $\prod_i \mathcal{M}_{n_i}$ is idempotent precisely if the α_i are all idempotent, and hence, so is*

$$\alpha = \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_k.$$

The following parameters capture some desirable information about elements of $\mathcal{M}_n$, many of which reflect some important Boolean/non-parametric properties in a combinatorial fashion. The properties listed below are all *additive* in the tensor, in the sense that if $f(\alpha)$ denotes any of the following, we have $f(\alpha \otimes \beta) = f(\alpha) + f(\beta)$:

- The degree n of α;
- The *block number*, *singleton block number* and *convex component number* of α;
- The *nonidempotency*, the number of blocks in α which comprise obstacles to idempotency;
- The *cycle, half-ray and (trans-, cis-) active and inert path numbers* in Γ_α;
- The *rank* r of α, or the *(rank) deficiency* $n - r$;
- The *cup and cap numbers* $|\Gamma_\alpha^\vee|$ and $|\Gamma_\alpha^\wedge|$;
- The *crossing number* of $\gamma \in \mathcal{P}_n$, that being the smallest number of crossings in a graph in the usual bounding rectangle whose connected components are the blocks of γ;
- The *discrepancy*, or the number of non-singleton blocks in α which are not a block in any $\beta \in \mathcal{J}_n$.

Many of these parameters allow us to distinguish interesting subsets of $\mathcal{M}_n$, either unilaterally or in concert with others. Most are additive over not only over the direct sum construction discussed in Section 3.2, and many of the results that follow could be fruitfully recast in terms of irreducibles rather than convex elements.

Theorem 5.1.9. *Let $\alpha \in \mathcal{M}_n$. The above tensor-stable parameters encode the following information:*

- *The element $\alpha \in \mathcal{J}_n$ precisely if the block number is n, or the singleton block number is zero;*
- *The nonidempotency of α is zero precisely if $\alpha^2 = \alpha \in E(\mathcal{M}_n)$;*
- *The active path number and rank coincide;*
- *For $\gamma \in \mathcal{PB}r_n$ (respectively in $\mathcal{B}r_n$), the crossing number is zero precisely if $\gamma \in \mathcal{M}_n$ ($\gamma \in \mathcal{J}_n$);*
- *If α is an idempotent in $\mathcal{J}_n$, then the element represented by α in the one-deformation parameter Temperley-Lieb algebra $TL_n(\delta)$ is idempotent for all δ precisely if the cycle number is zero;*
- *If α is idempotent in $\mathcal{J}_n$, then the element represented in $TL_n(\delta)$ by α is idempotent if δ is a c^{th} toot of unity, where c is the cycle number;*
- *The elements of discrepancy zero form a semigroup $\mathcal{PJ}_n$.*

The proof of the first several parts is basic combinatorics. Examination of relationships with one-deformation parameter Temperley-Lieb algebras is a straightforward corollary to the results given in the last section of [49].

5.1.4 A Parametric Study of $\mathscr{E}(\mathcal{M}_n)$

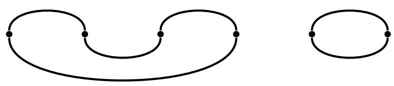
We will refer to such numbers as *tensor-additive parameters*, or simply *parameters*. The main thrust of this section is to develop some sense of when these parameters might be useful in the study of $\mathcal{M}_n$, or of interesting subsets and subsemigroups.

Of primary concern is understanding when parameters interact nicely, or in interesting ways, with the rewriting systems discussed in Chapter 4. For example, the nonidempotency of any idempotent is zero, which is not very interesting when considering mutations, since they preserve the property of being idempotent, and hence don't affect idempotency.

We haven't examined possible ways to generalise our notion of mutation to one which allows us to produce high-rank non-idempotents from idempotents, or even from arbitrary low-rank elements. This is certainly possible, perhaps allowing creating of half-rays by allowing more general forms of mutations on edges in one or two components, which may not be assumed cycles.

On the other hand, one has rank, which is a prototypical example of a well-behaved parameter. Proposition 4.2.5 describes how rewriting affects rank, and we can observe as a corollary that if one mutates on $\alpha \in \mathcal{M}_n$ once, the resulting α' has rank exactly two larger.

Example 5.1.10. Let

$\alpha =$ 

Then the fibre $[\alpha]$ in $\mathscr{E}(\mathcal{M}_n)$ is a product of an I-shaped tree (a 2-chain) and a V-shaped tree, and in particular has six elements.

To each element in this component of the complex, we'll attach a monomial, and sum these over the whole fibre. The eventual aim will be to quickly assemble a generating function that encodes a lot of information about the number of idempotents in $\mathcal{M}_n$, and also some more detailed information such as distribution by rank, cycle number etc.

So, to the degree parameter n, we associate the formal variable z. To rank r we associate the formal variable y.

The basepoint of the fibre has degree 6 and rank zero, so gets a value of z^6. There are three elements in the product at distance one from the basepoint, and two at distance 2, giving a factor of $1+3r+2r^2 = (1+r)(1+2r)$. The terms $1+r$ and $1+2r$ describe the ranks of elements in the trees, where in each case the tree is regarded as having at each vertex the irreducible summand of some $\delta \in [\alpha]$ on which the mutation takes place that contributes to the rank change.

The following definition recalls the direct-sum decomposition from Section 3.2.

Definition 5.1.11. A parameter *plays nicely* with the decomposition into irreducibles if for $\alpha \in \mathcal{P}_X$, we have $f(\alpha) = \sum_{i \in I} f(\alpha_i)$ where each α_i is an irreducible defined on some subset X_i of X.

We write

$$\alpha = \bigoplus_{i \in I} \alpha_i$$

in such a case.

Proposition 5.1.12. *Let f_j be a sequence of parameters that play nicely, and z_j be a corresponding sequence of distinct formal variables, both indexed by $j \in \mathcal{I}$. Then, for $\alpha \in E(\mathcal{M}_n)$ one associates to each irreducible summand $\alpha \in E(\mathcal{M}_{X_i})$ the monomial*

$$F(\alpha_i) = \prod_{j \in J} z_j^{f_j(\alpha_i)}.$$

Setting $F[\alpha] = \sum_{\beta \in \alpha} F(\beta)$, then we have

$$F[\alpha] = \prod_{i \in I} F[\alpha_i].$$

This is a familiar sort of utilitarian result from the theory of generating functions, and is a fairly direct consequence of these parameters playing nicely.

A *parameter profile* is a collection of parameters. Write $\mathbf{p}(\alpha) = (n,r,p,d,t,c)$ for the *standard parameter profile* where n is the degree of α, r is its rank, p is the discrepancy, d is deficiency, t is the number of convex components, and c is the number of connected components in Γ_α. Then for $\mathbf{v} \in \{n\} \times \mathbb{N}_0^5$, write

$$\hat{E}(\mathbf{v}) := \{\alpha \in \mathcal{M}_n : \mathbf{p}(\alpha) = \mathbf{v}\}.$$

Now define the *convex parameter profile* $\mathbf{c}(\alpha) = (n, r, p, d, c)$ where the entries are as above, and let $\mathbf{u} \in \{n\} \times \mathbb{N}_0^4$; write

$$\hat{C}(\mathbf{u}) = \left\{\alpha \in \mathcal{M}_n : \mathbf{c}(\alpha) = \mathbf{v}, \mathsf{v}(\alpha) = 1\right\}.$$

Then we observe the following regarding decompositions of elements of $\mathcal{M}_n$.

Proposition 5.1.13. *The set $E(\mathcal{M}_n)$ decomposes as a disjoint union*

$$E(\mathcal{M}_n) = \coprod_{\mathbf{v} \in \{n\} \times \mathbb{N}_0^5} \hat{E}(\mathbf{v})$$

where the union is over all $\mathbf{v}$ with first entry n. Furthermore, we have

$$\hat{E}(\mathbf{v}) = \bigcup_{k=1}^{n} \left\{ \bigotimes_{i=1}^{k} \hat{C}(\mathbf{u}_i) : \sum_{i=1}^{k} \left(\mathbf{u}_i\right) = \mathbf{v}, \forall_i \mathbf{u}_i \in \mathbb{N}_0^5 \right\}.$$

Define the following statistics

$$\hat{c}(\mathbf{u}) = \#\left\{ \alpha \in E(\mathcal{M}_n) \ : \ \alpha \text{ is convex with } \mathbf{c}(\alpha) = \mathbf{u} \right\},$$
$$\hat{e}(\mathbf{v}) = \#\left\{ \alpha \in E(\mathcal{M}_n) \ : \ \mathbf{p}(\alpha) = \mathbf{v} \right\}$$

Let $\mathbf{x} = (z, y, x, w, v, u)$ be a tuple of 6 indeterminates; for $\mathbf{v} = (n, r, p, d, t, c)$ as above, write

$$\mathbf{x}^{\mathbf{v}} := z^n \cdot y^r \cdot x^p \cdot w^d \cdot v^t \cdot u^c.$$

Then the *standard generating function* for the planar diagram monoids is

$$U(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{\alpha \in \mathcal{M}_n} \mathbf{x}^{\mathbf{p}(\alpha)} = \sum_{\mathbf{v} \in \mathbb{N}_0^6} \hat{e}(\mathbf{v}) \cdot \mathbf{x}^{\mathbf{v}}$$

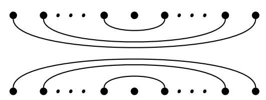
This function contains a lot of the data we are interested in for bot $\mathcal{J}_n$ and for $\mathcal{M}_n$. Recalling Theorem 5.1.9

Of independent interest, idempotents of rank-zero with only one connected component in the interface graph correspond precisely with meanders. So in particular, differentiating with respect to u and then sustituting $u = y = 0$ we obtain the generating function of the meanders; we cannot then do better than numerical asymptotic and/or low-degree approximation to this function without finding a solution to that of enumerating meanders.

Observation 5.1.14. *For any element $\alpha \in \mathcal{M}_n$ with $\mathsf{p}(\alpha) = p$, $\mathsf{d}(\alpha) = d$, $\mathsf{r}(\alpha) = r$, we have*

$$d,\ p,\ r,\ r+d,\ p+d \leq n.$$

Furthermore $d = 0$ implies $p = 0$, but we can have $d = 1$ and $p = n-1$ (the largest possible):



(5.2)

5.1.5 Convex Generating Function

Part of the reason we've developed this convex machinery in the first place is to reduce our study from the whole semigroup to the convex elements therein. Given the fibration techniques developed earlier, these methods will allow us to reduce our study to simply the convex or irreducible idempotents in the bottom $\mathscr{D}$-classes; for the sake of simplicity, we opt for the former.

If we can get a hold on some behaviour of the $\hat{c}$ then we can approximate U to low degree as follows. Firstly, let $\mathbf{q}$ be a parameter profile and $\mathbf{z}$ be an indeterminate of the same size. Then define

$$V_{\mathbf{q}}(\mathbf{z}) = \sum_{n=0}^{\infty} \sum_{\substack{\alpha \in \mathcal{M}_n \\ \alpha \text{ convex}}} \mathbf{z}^{\mathbf{q}(\alpha)} = \sum_{\mathbf{u}} \hat{c}(\mathbf{u}) \mathbf{z}^{\mathbf{u}}.$$

If we have some indeterminate $z \notin \mathbf{z}$, we can build a parameter profile-sensitive generating function for the non-convex guys. Note that each $\alpha = \alpha_1 \otimes \cdots \otimes a_k$ for some k and sequence of convex α_i. Then we have

$$\begin{aligned} U_{\mathbf{q}}(\mathbf{z}) &= \sum_{n} \sum_{\alpha \in \mathcal{M}_n} \mathbf{z}^{\mathbf{q}(\alpha)} \\ &= \sum_{k} \left(z^k \cdot \sum_{\otimes} \mathbf{z}^{\mathbf{q}(\alpha_1)} \cdot \mathbf{z}^{\mathbf{q}(\alpha_2)} \ldots \mathbf{z}^{\mathbf{q}(\alpha_k)} \right) \\ &= \sum_{k} \big(z V_{\mathbf{q}}(\mathbf{z}) \big) = \frac{1}{1 - zF(\mathbf{z})}. \end{aligned}$$

where the inner sum on the second line is being summed over all decompositions $\alpha_1 \otimes \cdots \otimes \alpha_k$.

This approach would work for any parameter profile that one would care to build.

One can generate huge amounts of data very quickly when one has a handle on the behaviour of these parameters around the fibres in $\mathscr{E}$.

Given the ease with which nonplanar diagrams were handled in [12], it may be tempting to conjecture that one could make some progress by starting with the Brauer monoid and filtering out nonplanarity by substituting zero into the indeterminate that counts nonplanarity. This approach may well prove fruitful for a judicious choice of parameter profile, but one needs only consider the fact that in the planar world, we need only fix two parameters (rank zero and one connected component) to recover the meander counting problem, whose nature has eluded much more sophisticated techniques than these.

The convex generating function $U_{\mathbf{p}}$ for the standard parameter profile is printed in the appendices; the terms of degree at most 9 take up almost 3 pages. To give one an idea of the amount of data that this method generates, the function $V_{\mathbf{p}}$ would take almost 20 pages, presented the same way. to give one an idea of the amount of data that this method generates. GAP has some useful tools for manipulating polynomials, and so it's possible to build a fairly comprehensive picture of the semigroup just by massaging the data through GAP's tools.

5.2 Counting Idempotents of High Rank

Our work with convex idempotents yields certain classifications of idempotents of particularly high rank, and the work with combinatorics on words yields some low-rank results. This section concerns itself mainly with results of the following type.

Proposition 5.2.1. *The number of idempotents of rank $n-4$ in $\mathcal{J}_n$ is equal to*

$$\frac{1}{2}\left(9(n-4)^2+19(n-4)-24\right)$$

for $n>4$, is 4 if $n=4$ and 0 otherwise.

We'll obtain the following asymptotic result.

Theorem 5.2.2. *Let $\mathcal{K}_n$ denote one of $\mathcal{J}_n$, $\mathcal{PJ}_n$ and $\mathcal{M}_n$; let $e_{n;r}$ be the number of idempotents of rank r in $\mathcal{K}_n$.*

Then for $\delta := \delta_{\mathcal{K}}$ some fixed positive integer and $p := p_{\mathcal{K}}$ some polynomial of degree δ, the sequence $(e_{n;n-\delta})_n$ coincides for $n \geq 3\delta$ with $p(n)$.

This theorem is "effective" in the sense that we know exactly what the polynomial is for rank-deficiency δ up to some coefficients that depend on the choice of $\mathcal{K}_\bullet$, although calculating the polynomial directly is computationally expensive. These coefficients are the numbers of convex idempotents of various ranks in certain degrees.

5.2.1 Idempotents of High Rank

One may define the *rank deficiency* δ of an element in $\mathcal{P}_n$ as the degree n minus the rank r. The prototype for the sort of low-rank deficiency result we're looking to prove is as follows.

Observation 5.2.3. *The number of idempotents in $\mathcal{J}_n$ of rank deficiency $\delta = 2$ is*

$$3n - 5$$

We will, where possible, express these sort of results in terms of the universal generating function if we have enough information about the $\mathscr{D}$-classes in question. In this case, all these idempotents are in $\mathcal{J}_n$ and hence $p = d = 0$, so we can only say the following.

We'll first note as an observation that there is exactly one idempotent of full rank in $\mathcal{M}_n$, who is also in $\mathcal{J}_n$, for each $n \geq 0$. That is, writing $R_{n,r}$ for the number of idempotents of degree n and rank r, we have $R_{n,n} = 1$.

There are n idempotents of rank-deficiency $\delta = n - r$ equal to 1; namely the partial identities on the subsets of $[\![n]\!]$ which comprise all-but-one point, so $R_{n,n-1} = n$.

In table 5.1, which stratifies the enumeration of Motzkin idempotents by rank and degree, we can see certain patterns when reading the columns. The two examples above are easy to understand with pure combinatorics, but what about those of rank deficiency 2?

$n \backslash n-r$	0	1	2	3	4	5	6	7	8	9	10	11
0	1											
1	1	1										
2	1	2	4									
3	1	3	11	16								
4	1	4	19	48	81							
5	1	5	28	93	266	441						
6	1	6	38	152	549	1 492	2 601					
7	1	7	49	226	947	3 211	9 042	16 129				
8	1	8	61	316	1 480	5 784	20 004	56 712	104 329			
9	1	9	74	423	2 169	9 432	37 048	127 676	369 689	697 225		
10	1	10	88	548	3 036	14 402	62 149	241 268	841 945	2 477,806	4 787,344	
11	1	11	103	692	4 104	20 968	97 697	413 629	1 612,936	5 682,635	17 026,951	33 616,804

Table 5.1: Numbers of idempotents in the Motzkin monoids listed by degree and rank-deficiency

After a little thought, one concludes that one is dealing with something that looks like the identity almost everywhere, with a Jones generator c_i, possibly multiplied with a partial identity, or missing strands $e_i e_j$. There are $n-1$ choices of c_i, and for each we can either rub one of the two edges out or leave both; there are $\binom{n}{2} = \frac{1}{2}n(n-1)$ ways to choose a pair $e_i e_j$ from $\mathcal{E}_n$.

Writing $R_{n,r}$ for the number of idempotents of degree n and rank r, we have, for $n > 3$,

$$R_{n,n-2} = \binom{n}{2} + 3n - 3 = \frac{n^2}{2} + 2n - 3.$$

After this, the waters become murky very quickly.

5.2.2 Coins, Partitions and Count-Summing

This subsection loosely follows example 5.1.3.

Recall that an ordered integer partition of n is a tuple $\mathbf{v}$ of positive integers summing to n. There are two notions of size associated with these objects, whereas we've only taken note of one up to this point. We have the *sum* $\sigma\mathbf{v}$, and the order $|\mathbf{v}|$ which counts the number of elements.

If we'll return briefly to change-counting, example 5.1.3, we note that we may be interested in the *number* of coins given in the change and not merely the amount. For instance, few people would be happy to pay for a 50p chocolate bar with a 5 note and be given the change in pennies.

We recall the multiset of coin values (we could distinguish in example 5.1.3 between the newer coat-of-arms coins and the others) and the generating function:

$$S = \{1,1,2,2,5,5,10,10,20,20,50,50,100,100,200\},$$
$$s(z) = 2(z + z^2 + z^5 + z^{10} + z^{20} + z^{50} + z^{100}) + z^{200} = \sum_{n \in S} z^n$$

We can deform the generating function to one which will track the number of coins as well as the total value by multiplying by v. Write $\hat{s}(z,v) = v \cdot s(z)$. Then we can use the same method as before to build a generating function to track the coin-number and value in change given:

$$\hat{s}^*(z,v) = \sum_{n=0}^{\infty} \left(\hat{s}(z)\right)^n = \frac{1}{1 - v \cdot s(z)}.$$

The coefficient of $v^k z^n$ here will tell us how many piles of k coins from the set S can be distinguished from one another, whose value in pence is n. The first few terms in the power-series expansion, up to v^2, are:

$$\begin{gathered}
z^{400}v^2 + 4z^{300}v^2 + 4z^{250}v^2 + 4z^{220}v^2 + 4z^{210}v^2 + 4z^{205}v^2 \\
+4z^{202}v^2 + 4z^{201}v^2 + 4z^{200}v^2 + 8z^{150}v^2 + 8z^{120}v^2 + 8z^{110}v^2 \\
+8z^{105}v^2 + 8z^{102}v^2 + 8z^{101}v^2 + 4z^{100}v^2 + 8z^{70}v^2 + 8z^{60}v^2 \\
+8z^{55}v^2 + 8z^{52}v^2 + 8z^{51}v^2 + 4z^{40}v^2 + 8z^{30}v^2 + 8z^{25}v^2 \\
+8z^{22}v^2 + 8z^{21}v^2 + 4z^{20}v^2 + 8z^{15}v^2 + 8z^{12}v^2 + 8z^{11}v^2 \\
+4z^{10}v^2 + 8z^7v^2 + 8z^6v^2 + 4z^4v^2 + 8z^3v^2 + 4z^2v^2 \\
+z^{200}v + 2z^{100}v + 2z^{50}v + 2z^{20}v + 2z^{10}v + 2z^5v + 2z^2v + 2zv + 1
\end{gathered}$$

Note that no z^9v^2 term appears, consistent with the fact that no pair of elements of S sum to 9. The term $4z^{20}v^2$ accounts for the four ways to make 20p; the two coins must be worth 10p, and we can independently allow the first and/or second to be a coat-of-arms coin. Counting in this sense is as easy as coefficient extraction.

A fairly easy generalised setting for this sort of technique to work is starting with some set X and some positive-integer-valued weight function

$$\hbar : X \longrightarrow \mathbb{N}_{\geq 0}$$

and extending this to the free monoid X^* by way of

$$\hbar(x_1 x_2 \cdots x_k) = \sum_{i=1}^{k} \hbar(x_i).$$

This set-up has a combinatorial significance when the set X_n of elements of a given weight n is finite for each n. We define the generating function

$$f_X(z) = \sum_{n=1}^{\infty} z^n \cdot |X_n|$$

to count how many elements weigh n, and apply the same transformation as before. The number of "strings" weighing n in the free monoid X^* is precisely

$$[z^n]\left(\frac{1}{1 - f_X(z)}\right). \tag{5.3}$$

We can count length in this by deforming by a factor of v at the first stage as before, giving the number of strings of weight n and length l as

$$[z^n v^l]\left(\frac{1}{1 - v \cdot f_X(z)}\right)$$

in essentially the same way as before.

Before we go any further, we're interested in certain families $(X_n \subseteq \mathcal{PB}_n)_n$ of subsets of the monoids $\mathcal{PB}_n$. Write

$$\mathcal{PB}_\infty := \bigcup_{n=0}^{\infty} \mathcal{PB}_n.$$

for the set-union of these. Given that the tensor operation is associative and unital (the unique element of degree zero), we observe that this set inherits a monoid structure that is compatible in some nice way with the monoid structure in each degree. Namely that for $s, t \in \mathcal{PB}_n$ and $x, y \in \mathcal{PB}_r$, we have the following *interchange law*

$$(s \cdot t) \otimes (x \cdot y) = (s \otimes x) \cdot (t \otimes y) \in \mathcal{PB}_{n+r}.$$

This monoid is readily verified to be free, and the tensor-irreducible elements of $\mathcal{PB}_n$ (those which don't deompose properly into a tensor of elements of lesser degree) will be denoted $\beta\mathcal{PB}_n$. Similarly, writing $\beta\mathcal{PB}_\infty$ for the union across all degrees, we see that $\mathcal{PB}_\infty = (\beta\mathcal{PB}_\infty)^*$ with respect to the tensor operation.

Definition 5.2.4. Let the sets $X_n \subseteq \mathcal{PB}_n$ be a family of subsets of the $\mathcal{PB}_n$. This family is called *(tensor)-stable* if for each $i, j \geq 0$, we have

$$X_i \otimes X_j := \{x \otimes y : x \in X_i, y \in X_j\} \subseteq X_{i+j}.$$

We will usually just refer to stable families.

Example 5.2.5. The following sets form stable families:

- The sequences $(\varnothing)_n$ of empty sets and full monoids $(\mathcal{PB}_n)_n$;
- Most of the families of monoids $(\mathcal{M}_n)_n$, $(\mathcal{J}_n)_n$, $(\mathcal{PB}r_n)$, $(S_n)_n, \ldots$ of interest to us;
- The family of sets of idempotent (regular, invertible, ...) elements in any stable family;

The (partial) annular Jones and Motzkin monoids and the cyclic groups do not form stable families*; the smallest stable families containing these are respectively the (partial) Brauer monoids and the symmetric or alternating groups depending on the parity of the degree.

Letting $\mathscr{X} = (X_n)_n$ denote a stable family, then $\beta\mathscr{X} = (\beta X_n)_n$ comprises, in each degree n, those elements which don't decompose, for any $0 < r < n$, into tensor products from $X_{n-r} \otimes X_r$. Recalling (5.3), we obtain the following.

*These are the only families of monoids featured in figure **??** that do not form stable families.

Lemma 5.2.6. *Let $\mathscr{X} = (X_n)_n$ be a stable family. Then*

$$X_n = \bigsqcup_{k=1}^{n} \beta X_k \otimes X_{n-k} = \beta X_n \sqcup \bigsqcup_{k=1}^{n-1} X_k \otimes X_{n-k}.$$

5.2.3 2-Compositions of (n,r) and Spartans

A *(2-)composition*[*] *of (the pair)* (n,r) *of dimension* k is a $2 \times k$ matrix

$$S = \begin{pmatrix} n_1 & n_2 & \cdots & n_k \\ r_1 & r_2 & \cdots & r_k \end{pmatrix} \tag{5.4}$$

where the row sums are respectively n and r; write $S \vdash (n,r)$. We call n the degree, r the rank and $d = n - r$ its deficiency. These will be used to represent elements of $\mathcal{M}_n$, so a 2-composition of (n,r) into, say (n_1, n_2) and (r_1, r_2) will correspond to an element of $\mathcal{M}_n$ which decomposes into two convex components of respective degree n_1 and n_2, and respective rank r_1 and r_2.

We say that two compositions are equivalent if they differ by a trailing sequence of columns comprising a pair of 1s, written $S \approx T$. These equivalence classes, which we often identify with their representatives of least dimension, we will call *spartans*; we call the (optionally) omitted columns of 1s *trivial columns*.

Note that the columns omitted when representing a Spartan by a 2-composition will contribute nothing to deficiency, contribute 1 each to rank, degree and dimension, so tracking the dimension of a composition allows us to recover it from its Spartan representative. We will assume, often implicitly throughout, that all spartans will comprise classes of matrices whose nontrivial columns are all top-heavy in the sense that $n_i \geq 3r_i$.

We can juxtapose a pair of 2-compositions and get another 2-composition in an obvious way; given $S_i \vdash (n_i, r_i)$ $(i = 1,2)$ as above, we have

$$S_1 \otimes S_2 = \begin{pmatrix} n_{1,1} & n_{1,2} & \cdots & n_{1,k_1} & n_{2,1} & \cdots & n_{2,k_2} \\ r_{1,1} & r_{1,2} & \cdots & r_{1,k_1} & r_{2,1} & \cdots & r_{2,k_2} \end{pmatrix} \vdash (n_1 + n_2, r_1 + r_2).$$

[*] The prefix "2-" will be dropped, and the suffixal descriptions "of (n,r)" and "of dimension k" are omitted unless necessary.

Write $\mathcal{K}_n$ for any of $\mathcal{M}_n$, $\mathcal{PJ}_n$ and $\mathcal{M}_n$. Label each idempotent in $E(\mathcal{K}_n)$ by the composition of degrees and ranks of its convex components; the resulting mapping will have as its image all compositions S, as in (5.4), satistfying $n_i \geq 3r_i$ for each $n_i > 1$:

Lemma 5.2.7. *Let $\alpha \in \mathcal{M}_n$ be a convex idempotent for some $n > 1$. Then the rank of α is at most $\frac{n}{3}$.*

Proof. Let $\alpha \in \mathcal{M}_n$ be a convex idempotent of rank r.

Every contribution to rank comes from an active or inert path component by definition. By idempotency, there are only active paths, each of which has odd length. If some path has length 1 then either $n = 1$ or α is nonconvex, so for $n > 1$ all paths have length at least 3 and the rank is at most $\frac{n}{3}$ as required. □

We can recover the original by reordering the columns (there are multinomially many ways to do this) and attaching to each column $\binom{n}{r}$ some element $e \in C_{n,r}$ among the set of convex idempotents of rank r in $\mathcal{K}_n$. We'll write $c_{n,r}$ for the size of this set. Counting all idempotents then is as simple as multiplying the product of the c_{n_i,r_i} across the columns in S by the multinomial coefficient counting the number of nonidentical rearrangements of S there are.

In practice, since idempotents of low-deficiency comprise (for all but the smallest degree) many components of rank 1 and deficiency 1, we may reduce our study to spartan representatives and count the smaller multinomial problem of how to arrange the nontrivial columns from the spartan representative among the remaining array of 1s.

Example 5.2.8. The number of idempotents of deficiency 2 in $\mathcal{J}_n$ is given by the formula $3n-5$. One can verify this by checking that for all diagrams, there exists i such that:

- the cup and cap appear opposite one another, i.e. we have non-transverse components $(i, i+1)$ and $(i', (i+1)')$ and every other component is of the form (j, j') for some $j \neq i, i+1$;

- the cup and cap appear offset by +1, i.e. there are nontransverse components $(i, i+1)$ and $((i+1)', (i+2)')$ with transverse components $(i+2, i')$ with every other component of the form (j, j') for some $j \neq i, i+1, i+2$;
- the cup and cap are offset by -1, i.e. there are nontransverse components $(i, i+1)$ and $((i-1)', i')$ with transverse components $(i-1, (i+1)')$ with every other component of the form (j, j') for some $j \neq i-1, i, i+1$;
- the cup and cap are offset by more than one in either direction, and hence the element isn't idempotent.

To see how this works in the algebraic context, note that there are three spartans of deficiency 2:

$$\begin{array}{c|c|cc} 3 & 2 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{array}.$$

The first corresponds to the convex idempotents in rank 1, degree 3. One can arrange to put one of these in any of the first $n-2$ positions, giving $\binom{n-2}{1} = n-2$ choices for its position; there are $c_{3,1} = 2$ of these, so this term contributes $2n-4$ to the count.

The second, corresponds to the non-identity element in $\mathcal{J}_2$; this may be placed freely on the first $\binom{n-1}{1} = n-1$ columns, and there is only $c_{2,0} = 1$ of these.

The third corresponds to a pair of partial identities, which don't exist in the Jones monoid. Examining like we would for the Motzkin or similar, we note that we can arrange these two into any of the n positions, giving $\binom{n}{2}$ ways to do so, having chosen the particular convex components in order from the $c_{1,0}^2$-many ways of doing so, of which there are non in $\mathcal{J}_2$.

Adding across the spartans gives us the formula, $3n-5$.

5.2.4 The Polynomial Recurrences

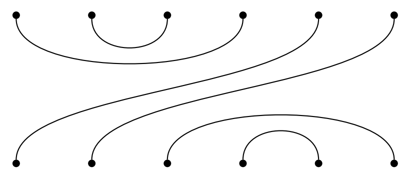
Theorem 5.2.9. *Let $0 \leq r$ be a nonnegative integer. Then, for $n > r$ the number of idempotents of degree n and rank $n-r$ is given by a polynomial of degree $n-r$.*

$n-r \backslash n$	0	1	2	3	4	5	6	7	8	9
0		1	3	9	44	23	1331	8089	51435	338193
1		1		2	4	3	14	864	5088	32326
2							2	8	76	440
3										2

Table 5.2: Numbers of convex idempotents in the Motzkin monoids listed by degree and rank-deficiency. Note that the rows and columns have been switched compared to table 5.1

To prove this, we first observe the following fact.

This bound is attained for degree dividing three by a pair of mutually-involutive Jones elements that "look like" parallel copies of the convex degree-three rank-one idempotents:



(5.5)

We'll now get around to proving the main result of this section.

Theorem 5.2.2. *Let $\mathcal{K}_n$ denote one of $\mathcal{J}_n$, $\mathcal{PJ}_n$ and $\mathcal{M}_n$; let $e_{n;r}$ be the number of idempotents of rank r in $\mathcal{K}_n$.*

Then for $\delta := \delta_{\mathcal{K}}$ some fixed positive integer and $p := p_{\mathcal{K}}$ some polynomial of degree δ, the sequence $(e_{n;n-\delta})_n$ coincides for $n \geq 3\delta$ with $p(n)$.

The proof lies on three observations.

Firstly, we can associate to each $\alpha \in \cup_n E(\mathcal{M}_n)$ a 2-composition whose columns reflect the degrees and ranks of the tensor-indecomposable factors α_i where

$$\alpha = \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_k.$$

Proposition 5.2.10. *Given $\delta > 0$, there are finitely-many spartans representing idempotents in $\cup_n \mathcal{M}_n$ of deficiency δ.*

Proof. The deficiency of an idempotent α is an invariant of its representing spartan from the definition; it's simply the sum of the deficiencies across the convex components α_i. The columns in the 2-composition representing α encode the contribution of each convex component to the total deficiency.

Given a spartan of degree n and rank r, we can choose some 2-composition of minimal dimension represented by it, unique up to column-reordering. The columns in this all contribute positively to the deficiency, and there are only finitely-many columns $\binom{a}{b}$ which could appear first since $1 \leq a \leq n$ and $0 \leq b \leq \min(r, a)$, certainly no more than n^2. Since each column contributes positively to degree, there are at most n such columns, hence at most n^{2n} 2-compositions of degree n and rank r, and hence no more than this many spartans. $\square$

We can improve upon this bound considerably, but we need only show that for fixed deficiency, the number of spartans is (eventually) constant with respect to varying degree.

Proposition 5.2.11. *Let ς be a spartan of degree k and dimension l. The number of idempotents of degree $n \geq k$ represented by ς is $p(n)$ where $p := p_\varsigma$ is a polynomial of degree l depending only on ς.*

Proof. Let ς be a spartan of degree k, rank $k - \delta$ and dimension l; choose a minimal-dimension 2-composition $\hat{\varsigma}$ represented by ς. The number of choices m_ς of $\hat{\varsigma}$ depends only on ς and is bounded above by $k!$.

Now fix $n \geq k$ and some $l + n - k$-dimensional 2-composition $\tilde{\varsigma}$ which comprises a $2 \times (n - k)$ matrix of 1s, with the l columns of $\hat{\varsigma}$ shuffled in, in order. There are $\binom{l+n-k}{l}$ ways to do this, independent of the choice of $\hat{\varsigma}$.

We write $c_{s,t}$ denote the number of convex idempotents in $\mathcal{M}_s$ rank t and note that $c_{1,1} = 1$. Then the number of idempotents represented by

$$\tilde{\varsigma} = \begin{pmatrix} s_1 & s_2 & \cdots & s_l \\ t_1 & t_2 & \cdots & t_l \end{pmatrix}$$

is exactly $\prod_{i=1}^{l} c_{s_i,t_i}$. The number of idempotents represented by $\hat{\varsigma}$ is m_ς times this number, giving the number of idempotents in degree n represented by ς as

$$\binom{l+n-k}{l} \cdot m_{\varsigma} \cdot \prod_{i=1}^{l} c_{s_i,t_i},$$

which is a constant (in n) multiple of

$$\binom{l+n-k}{l} = \frac{1}{l!}(l+n-k)(l-1+n-k)\cdots(1+n-k),$$

itself a degree-l polynomial in n. □

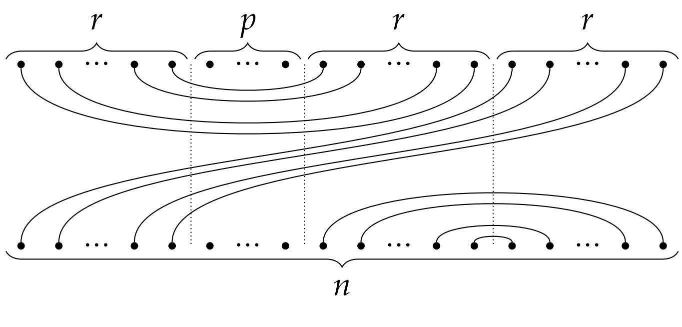
To apply this result to counting idempotents of high rank, we need to bound the dimension of the spartan ς in its deficiency δ.

Proposition 5.2.12. *Let $0 \leq r \leq n$, and write $\mathsf{c}_{n,r}$ for the smallest number of convex components in an idempotent $\alpha \in E(\mathcal{M}_n)$ of rank-r. Then*

1. *If $n = 1$ or $r \leq \frac{n}{3}$, then $\mathsf{c}_{n,r} = 1$;*

2. *Otherwise $\mathsf{c}_{n,r} =$.*

Proof. All elements of $\mathcal{M}_1$ are convex, so to prove the first claim, we need only establish that, for $2 \leq n = 3r + p$, there exist convex idempotents of rank r in $\mathcal{M}_n$. Fix n, p and r s; to witness the validity of the claim, note that the following idempotent of degree n and rank $r = r_1 + r_2$ is convex when r is small enough that r_2 can be assumed zero (i.e. when n holds):



Alongside Lemma 5.2.7, this observation says that an idempotent having rank and degree satisfy $n \geq 3r$ characterises the existence of convex idempotents. In other words, for any 2-composition M with the property that all columns are either pairs of 1s or have rank ϱ no larger than a third of the degree ν, there's an idempotent represented by it. We need only bound the dimension of this in the deficiency; $c_{n,r}$ has an obvious interpretation as the smallest possible dimension of such a 2-composition.

Each column $M_{-,i}$ having degree ν_i and rank ϱ_i contributes at least $\frac{2}{3}\nu_i$. There is no contribution to deficiency from the columns of 1s; if we write l for the number of columns which are not of this form (i.e. those which do contribute deficiency) then the number of these is $k-l$ where k is the dimension of the 2-composition M. If S is the set of indices of nontrivial columns, we have

$$\delta = n - r \geq \frac{2}{3}\sum_{i=1}^{k} \nu_i = \frac{2}{3}\sum_{s\in S} \nu_s.$$

In other words, the dimension of the representing spartan is bounded above by $\frac{3}{2}\delta$ where δ is its deficiency. □

This bound is sharp, up to discarding the fractional part, and can be attained by the parallel copies of the degree-three, rank-one idempotents as in (5.5).

Proof of Theorem 5.2.2. This proof is for $\mathcal{K}_n = \mathcal{M}_n$, but straightforward adjustments can be made to the statements and proofs of the above auxiliary results in order to get the result for other $\mathcal{K}_n$. Indeed, it works for arbitrary stable families in $\mathcal{M}_n$.

Firstly, we recall from Proposition 5.2.10 that only finitely-many spartans represent any given $\mathscr{D}$-class in $\mathcal{M}_n$, and that for fixed δ and sufficiently large n, the $\mathscr{D}$-classes of rank $n-\delta$ elements in $\mathcal{M}_n$ and those of $n+1-\delta$ in $\mathcal{M}_{n+1}$ coincide.

The set of spartans representing idempotents, in other words, is fixed for sufficiently large n (when $n \geq= 3\delta$). We the can obtain leverage from Proposition 5.2.11. □

The effective nature of the theorem relies on the constructibility of spartans, and calculating these mutlinomial counts; we do not address these issues here, rather in the appendix.

Chapter 6

Algorithm Design

One of the primary aims of this thesis was to describe the algorithm developed in the joint paper [13]. The results for that paper were tested against the `semigroups` package for `GAP`, which can do everything that we set out to do in terms of enumeration, but whose methods are much slower, being of a more general nature. Indeed, if given a finite semigroup in a format that it understands, and assuming that the elements can reasonably fit in memory, `semigroups` can compute the number of idempotents. This may take a huge amount of time in general, and is not particularly fast for Motzkin, Jones or partial Jones monoids.

The current "best version" of the code following is due to the external examiner, James Mitchell, who implemented a very fast multi-thread version of these algorithms in C [61]. The focus here will be on developing the algorithm and comparing its performance with that of the general methods used in `semigroups`.

Some comparisons will be provided later between the implementation from James Mitchell's Jones package and `semigroups`using the time utility for unix and `GAP`'s own time command.

6.1 Preliminaries on Computing

6.1.1 The GAP system

GAP is a computational algebra tool, comprising a standalone application and a suite of packages and libraries extending its functionality. Its original intended purpose was to provide a piece of software to do group-theoretic computations, and due to the close connection between many branches of semigroup theory and group theory, and possibly the proximity to a large cadre of semigroup theorists at St Andrews, one of the principal centres for its development, it seems a natural place for semigroup algorithms to be implemented and run.

There is a high-performance application, hpcGAP, being built by the GAP group, which implements parallel computing with shared memory on top of the core GAP functionality. The hpcGAP development branch is not stable, but the GAP group is working to merge this parallel processing functionality back into the main branch as of v4.8.

During a short visit to St Andrews, supported by the CoDiMa grant, some of the algorithms designed here were implemented in hpcGAP, but we realised at the time that a distributed implementation would serve our needs better; the code is structured into a massive number of very small tasks which don't require feedback from one another and don't have much data to pass around, so can be happily executed orthogonally on independent nodes.

This led us to consider Alex Konovalov's SCSCP protocol (implemented as the SCSCP package in GAP) to distribute the functionality. We managed to calculate the size of $E(\mathcal{J}_{27})$ in about 2 days, having almost a month to deal with $\mathcal{J}_{24}$ sequentially; with James Mitchell's implementation of the code, runtimes have reduced significantly.

A particular draw of GAP is the existence of some functionality to deal natively with the sort of objects we're interested in, some in the core GAP application and some provided by the semigroups package. For instance, the Jones monoid can be called with

```
JonesMonoid( n );
```

with the `semigroups` package, where n is the degree of the monoid. As of version 2.6, `semigroups` also has commands to construct the Motzkin and partial Jones monoids in the desired degree.

6.1.2 Parallel computing

A computational model can be thought of an abstract representation (or approximation) of the machine for which a programmer writes instructions. An important and growing field in computer science, and one of particular interest for our purposes, is that of parallel computing. A parallel computational model must, at least, model execution and data movement; that is to say it must contain information about how code may be executed, and how data may be moved around between processors and memory devices in the machine (or cluster of machines) on which code is being run. The increased freedom in terms of what code may be running, and when, can produce extra engineering concerns for the programmer. How can she ensure the code executes as intended, at least to the extent that subroutines of a task run in an order that preserves the integrity of data throughout task execution? How can she ensure that the computational resources are being utilised effectively?

For most of the history of computing, much of it has been conducted in serial, that is to say with tasks being executed by one processor node, never actively working on more than one instruction at a specific point in time. In particular, personal computing (computing for the use of the general public) was broadly single-core only until about a decade ago, although multicore personal and protable computing is widespread and increasing [27, 41].

Two models have emerged for computing in parallel:

- The *shared memory* model, where all CPUs have access to a shared cache of memory;
- The *distributed* model, where each CPU has its own cache of memory.

This is not an exhaustive description of all possible non-serial computational models, but the dichotomy provides a useful starting point. In practice, one usually works with hybrids of these models, whereby each processor has its own cache and the whole bank of processors (or pehaps a subset thereof) has a shared cache. The design of modern computers and contemporary portable devices tends to follow such a model for their design, with a hierarchy of storage from processor caches through (sometimes) shared caches, RAM and non-volatile storage such as solid-state drives and hard disk drives, with earlier storage closer being faster and smaller than latter devices.

There are advantages to both systems certainly, and the engineering of hybrid systems is often planned so as to take advantage of the strengths of each. The main strength of the distributed model is that it's inherently effective in situations where the cost of passing data between processors can be relatively expensive compared to processing, particularly in situations where one has access to many moderately-powerful networked machines but not to a single computer with many processor cores. In particular, it's very well-suited to deployment over networked computers; the SCSCP protocol [26], used in `GAP`, and the HTCondor framework [50,51] are built principally to take advantage of the strengths of this paradigm; they are primarily used for computation conducted over a network rather than on one powerful machine.

The shared-memory model is intended to take advantage of a low-latency shared cache of memory. It's appropriate in situations where data is easy to pass between nodes. The implementation of a shared-memory system is usually a hardware concern, rather than one of networking for example. This is because modern processors are fast enough that circuit distance between devices can quickly introduce such a penalty to computing that circuit distances are usually measured in tiny increments — below the micrometer scale. A result of the shared memory being available is that it's easy to check task execution and memory state during computing. This makes ensuring ongoing data consistency less expensive in this model, and makes task scheduling faster, potentially reducing processor idle time and improving performance. The HPCGAP*

*High-Performance Computing GAP, whose functionality is being merged into the core GAP distribution as of v4.8; see the change summary in [32].

system [6] and the Deepchem system for drug discovery [67] are designed to take advantage of the shared memory model.

The main thrust of parallelisation in this study will be distributed in nature. Distributed computing excels with tasks that decompose into subtasks that are:

- largely-independent from one another, so that a minimal amount of information needs to move around during subtask execution;
- slow to finish running, compared to the time taken to transport the data required to define a subtask or a batch of subtasks.

As we shall shortly see, the algorithms of interest here satisfy these criteria in a strong sense.

6.2 Bounds on computing

There are several methods discussed here. They more or less satisfy the following specification.

1. Take as input n and construct some computational representation* of $\mathcal{K}_n = \mathcal{PJ}_n, \mathcal{J}_n, \mathcal{M}_n$.
2. Define some subset X of $\mathcal{K}_n$ and map $f : \mathcal{K}_n \to X$ in which:
   - X is easy to iterate over;
   - $f(e)$ is idempotent for any idempotent e;
   - it is easy to enumerate the number of idempotents in such an $f(e)$.
3. Sum the counts over X.

Before we discuss exact bounds, first let's prove some bounds on the numbers of elements in special classes of the semigroups of interest.

*The term *representation* has a specific mathematical meaning. We don't use it in that sense, so will conventionally drop the adjective *computational*.

First we'll recall some notation. As usual $\mathcal{K}_n$ is $\mathcal{M}_n$, $\mathcal{PJ}_n$ or $\mathcal{J}_n$. We denote by $D_r = D_r(\mathcal{K}_n)$ the set of all rank-r elements in $\mathcal{K}_n$, and $D = D(\mathcal{K}_n)$ those elements of rank at most one. Write

$$\mathcal{O}_n^l := \{\mathbf{v} = (v_1, \ldots, v_l) : 1 \leq v_1 < v_2 < \cdots < v_l \leq n$$

for the ordered tuples of length l containing only entries from $[\![n]\!]$ and C_n, M_n, P_n for the Catalan, Motzkin and partial Catalan numbers.

Theorem 6.2.1. *With the above notation, we have*

$$|\mathcal{M}_n| = M_{2n}, \qquad |\mathcal{J}_n| = C_n$$

$$|D_r(\mathcal{M}_n)| = \sum \left\{ M_{u_1-1} \cdot M_{v_1-1} \cdot \left(\prod_{i=2}^{n} M_{u_i-u_{i-1}} M_{u_i-u_{i-1}}\right) \cdot M_{n-u_l} \cdot M_{n-v_l} : \mathbf{u}, \mathbf{v} \in \mathcal{O}_n^l \right\}$$

$$|D(\mathcal{M}_n)| = M_n^2 + \sum_{i,j=1}^{n} M_{i-1} M_{n-i} M_{j-1} M_{n-j},$$

$$|D(\mathcal{J}_n)| = C_{\lceil \frac{n}{2} \rceil}.$$

This theorem is a knitting together of several results from the Appendix A.

Table 6.1 details approximate runtimes. The $O(n)$ term in each accounts for the runtime taken to iterate forwards in X, and to count the number of idempotents in the `count`.

X	`count`	$\mathcal{K}_n$	Runtime
$\mathcal{K}_n$	`IsIdempotent`	$\mathcal{J}_n$	$C_n \cdot O(n)$
		$\mathcal{PJ}_n$	$\sum_{i=0}^{n} \left(C_n \cdot \binom{2n}{i} \cdot O(n)\right)$
		$\mathcal{M}_n$	$M_{2n} \cdot O(n)$
D	`Hatter`	$\mathcal{J}_n$	$C_{\lceil \frac{n}{2} \rceil} \cdot O(n)$
		$\mathcal{M}_n$	$O(n) \cdot \left(M_n^2 + \sum_{i,j=1}^{n} M_{i-1} M_{n-i} M_{j-1} M_{n-j}\right)$

Table 6.1: Comparison of runtimes of unoptimised idempotent enumeration methods

6.3 Algorithm design

A first attempt in GAP to enumerate the idempotents in these structures looks as follows:

```
List( [ 1 .. 20 ],
    n -> NrIdempotents( JonesMonoid( n ) )
  );
```

Whereas GAP is very adept at encoding certain very large structures (indeed, some infinite), it isn't possible to handle sets of `Idempotents` so well in a uniform manner. On a moderately-powerful desktop computer, the above code is guaranteed to execute while an element of $\mathcal{J}_n$ will fit in memory. The computations will become longer than the life of the universe so far long before memory is an issue, however, due to the exponential growth in size of $\mathcal{J}_n$ and $\mathcal{M}_n$.

We can start to use the multithread capability available on modern computers, and indeed on performance machines, by subdividing the problem. Some potential avenues to do this are as follows:

- Count by $\mathscr{D}$-class. There are $\frac{n}{2}$ of these in $\mathcal{J}_n$ and $n+1$ in $\mathcal{M}_n$. There are a Fibonacci number (exponentially-many) in $\mathcal{PJ}_n$.
- Count by $\mathscr{R}$-class. There are exponentially-many of these in each case.
- Find a many-to-one endofunction $f : \mathcal{K}_n \to \mathcal{K}_n$ which fixes its image, whose image is easy to iterate over and such that the number of idempotents mapping to a given element is easy to enumerate.

The last method can be achieved by considering the hat map. We've already proven a battery of results that tell us exactly how to do this step-by-step.

Our approach is exectued as follows:

- Take as input the number n.
- Construct an iterator for the bottom $\mathscr{D}$-class D of $\mathcal{J}_n$;

- For each element, evaluate the size of the hat map's fibre rooted at this point, and add to a counter **`count`**;

- After iterating over D, return **`count`**.

The iterator in the second step comprises, in our implementation, two identical iterators over the $\mathscr{L}$-classes (dual to the $\mathscr{R}$-classes), and the elements are their (nontrivial) intersections. Morally, we're summing the entries in a symmetric $n \times n$ matrix whose rows are $\mathscr{L}$-classes and whose columns are $\mathscr{R}$-classes. So, for a runtime speedup by a factor of almost 2, we can copy the state of the $\mathscr{L}$-class iterator into the $\mathscr{R}$-class iterator and count all entries above the diagonal twice.

These steps straightforwardly generalise to $\mathcal{M}_n$, although we now need to consider all elements of rank one also, and ignore nonidempotents. In $\mathcal{J}_n$ this is one $\mathscr{D}$-class, whereas in $\mathcal{M}_n$ it's two (rank-zero and rank-one) for $n > 1$. While every rank 1 element of $\mathcal{J}_n$ is idempotent (indeed they form a rectangular band for n odd), and the rank-zero partitions form a rectangular band, there exist non-idempotent elements of rank 1 in both $\mathcal{PJ}_n$ ($n > 2$) and $\mathcal{M}_n$ ($n > 1$).

In $\mathcal{PJ}_n$ things are more complicated. There are three separate classes of elements of rank at most 1 for $n > 1$: the minimal ideal of rank-zero elements, the rank-one elements with an odd-odd transversal and those with even-even transversal).

The fourth step leans on the combinatorics of the product decomposition for the mutation complex that we proved in Section 4.3, specialised to $\mathcal{J}_n$, but working essentially identically in the case of $\mathcal{M}_n$. This approach cannot work for $\mathcal{PJ}_n$, however, as the Motzkin hat map does not preserve membership there, but we have another rewriting system and hat map which works there.

We'll call these steps respectively the *iter*, *hatter* and *wrapper*. The above algorithms can all be specialised to the partial Jones monoid; we'll discuss this later.

6.3.1 Correctness

We'll restate the above wrapper algorithm as follows to find the size of $\mathcal{M}_n$:

```
  input: n
    count = 0
    for d in iter do
        count += size( hatfibre( d ) )
    end do
output: count
```

where **hatfibre** is a data structure that contains enough information to enumerate the weakly connected component of d in $\mathscr{E}(\mathcal{M}_n)$ and **iter** iterates over the ideal D of elements of rank at most 1.

Theorem 6.3.1. *The algorithm above is correct if the following conditions are all met:*

1. *The* **hatfibre** *function works correctly and iter iterates over computational representations for all elements of D;*
2. *The hat procedure is a function $s \mapsto \hat{s}$ defined for all idempotents;*
3. *The image of the hat procedure is the ideal D;*
4. *Every idempotent $e \in E$ is produced by some mutation on $\hat{e} \in D$;*
5. *The inverse hat process only produces idempotents in the fibre of the hat map.*

This is fairly clear from brief study of the algorithm.

The first part is computational in nature and clearly depends on the chosen representation of the elements used in the algorithm, so we'll leave that open until we discuss representation.

The second is Corollary 4.2.9 and the third is Corollary 4.2.6 and Corollary 4.2.13. The fourth is Theorem 4.2.19 and fifth follow from Theorem 4.2.19.

The hatter algorithm is implemented essentially as follows, executing on an idempotent $d \in D$:

```
input: d
  count = 1
  for comp in components(d) do
    if comp is active then
      continue
    else if comp is half-ray then
      output 0
    end if
    count *= 1 + #top_return_edges * #bot_return_edges
  end do
output: count
```

In practice, as we incrementally optimise the algorithm, this algorithm will be adapted to the new representation — the steps which iterate the components and enumerate return edges are hence omitted for now. It's not difficult to see that it's correct from Corollary 4.3.11.

We'll proceed assuming the correctness of the algorithms up to the unstated subroutines, and devise some results that will potentially allow us to optimise its progress in the following subsection.

6.3.2 Optimisation

This section contains some optimisations in implementation which reduce runtimes by some small amount, and is of practical, rather than theoretical, interest.

6.3.2.1 Staying left of the leftmost transversal

A connected component $\theta \subseteq [\![n]\!]$ of the interface graph of α is *left of the leftmost transversal* if there is $c \in \theta$ such that for any transversal component θ' and $x \in \theta'$, $c < x$.

Set c_θ the leftmost point in the component. If the leftmost transversal is θ' (the only one in rank at most 1) then the information about the elements of the hat map's fibre is contained left of $c_{\theta'}$. In particular, if we can quickly access

$c_{\theta'}$ and c_θ for each component, we can reduce runtime by ignoring components that don't contribute to the count.

Furthermore, if this is the case, and if in addition the components are ordered with $\phi < \theta$ when $c_\phi < c_\theta$ then we can iterate over these quickly.

Considering the map reversing the order of the indices in the interface graph should convince the reader that this process roughly halves runtime, doing better when the transversal component is large. This is representation-dependent, as access to the c_θ is a defining factor in the utility of this approach.

6.3.2.2 Involution

We recall here that all the semigroups of interest herein are regular $*$-semigroups, meaning in particular that there is an anti-isomorphic involution on the semigroups which necessarily maps $\mathscr{R}$-classes to $\mathscr{L}$-classes and vice-versa. This map specifically maps idempotents to idempotents.

We can effectively iterate over elements ($\mathscr{H}$-classes) H in D_0 if we do the following:

- order, then iterate over the $\mathscr{L}$-classes L in D_0;
- for each L, iterate over $\mathscr{L}$-classes L' dual to those no earlier than L, take $H = L^* \cap L'$.

If there are k such $\mathscr{L}$-classes, then we have effectively halved (since typically $k \gg 0$) calls to the hatter procedure from k^2 in number to only $\frac{1}{2}k^2 + \frac{1}{2}k$. Note that in each family of monoids, k is exponential in n. We apply a similar process to $\mathscr{L}$-classes of rank-1 elements in studying $\mathcal{M}_n$ for $n \geq 1$.

6.3.2.3 Encoding as Dyck words and pattern-avoiding permutations

Appendix A is devoted largely to developing some combinatorics surrounding the language of Dyck words, which is shown to index the elements of the Jones monoids $\mathcal{J}_n$. Another language is described which serves the same role for $\mathcal{M}_n$.

Calculating $\mathscr{L}$-, $\mathscr{R}$- and $\mathscr{D}$-classes for a bipartition semigroup* is not entirely

*By a bipartition semigroup, I mean the term as it's used in the GAP `semigroups` package.

trivial, whereas there is a reasonably efficient algorithm for producing the Dyck words of up to a given length. In practice, we encode as lists of nonnegative integers formatted in such a way as to aid moving around interface graphs:

$$\begin{aligned}\text{Dyck words} &\leftrightarrow \text{Noncrossing perfect matchings}\\ &\leftrightarrow \text{Noncrossing involutions}\\ &\leftrightarrow \text{Lists of images.}\end{aligned}$$

A slight modification is made for Motzkin words and partial transformations avoiding certain patterns, which will be clearer presented in context.

Definition 6.3.2. The *permutation* $\varsigma := \varsigma_w$ *associated to a Dyck word* $w = w_1 w_2 \cdots w_{2n}$ (of length $2n$) is the fixpoint-free involution of degree $2n$ given by mapping each index $1 \le k \le 2n$ to the index of the bracket matching w_k in w.

The orbits of these involutions satisfy the *noncrossing* property, stated in the following.

Proposition 6.3.3. *Given a Dyck word* $w = w_1 w_2 \cdots w_{2n}$ *and associated permutation* $\sigma = \sigma_w$, *if* $\{w_i, w_l\}$ *and* $\{w_j, w_k\}$ *are orbits with* $i < j < l$ *then* $i < k < l$.

Definition 6.3.4. Given a Dyck word $w = w_1 w_2 \cdots w_{2n}$, a pair w_i, w_j of letters ($i < j$) in w constitutes a *matched pair* if $w = u \cdot w_i \cdot v \cdot w_j \cdot x$ for some Dyck words u, v and x.

Example 6.3.5. The following example demonstrates this fairly clearly, writing $w \mapsto \varsigma_w$:

$$\varepsilon \mapsto (), \qquad \texttt{[][]} \mapsto (12)(34), \qquad \texttt{[[]]} \mapsto (14)(23),$$
$$\texttt{[][][[]]} \mapsto (12)(34)(58)(67), \qquad \texttt{[[][[][]]]} \mapsto (1,10)(23)(49)(56)(78).$$

The image list is then the list

$$[\![n]\!]^{\varsigma} = [1^{\varsigma}, 2^{\varsigma}, \ldots, (2n)^{\varsigma}]$$

of images, exactly as one would imagine. This is a well-known notation for permutations often described as *one-line notation*.

The Motzkin language $\mathfrak{M}$ can be identified with noncrossing involutions whose domain is a subset of the domains of the permutations above. That is, given $\alpha \in \mathcal{M}_n$, the elements of $\pm[\![n]\!]$ paired off by this partial involution ι are exactly the non-singleton blocks in the partition α. We employ a similar one-line notation for these objects, writing 0 in any entry which is undefined, and the image otherwise. If the image is defined, we will always have $\iota(x) = x$, or $\iota(x) = y$ and $\iota(y) = x$. This is sufficient to index $\mathscr{R}$-classes of $\mathcal{M}_n$, and is a convenient representation to move around in the interface graph, since all it takes is an array lookup.

Chapter 7

Conclusions, Next Steps

The main result of this thesis is the existence of a refinement of the natural partial order on idempotents for the semigroups $\mathcal{J}_n$ and $\mathcal{M}_n$. The structure of this ordering gives rise to a rewriting system with a particularly nice cube complex structure that allows us to locate all idempotents quickly in terms of those in the $\mathscr{D}$-classes of elements with rank at most 1.

From there, we can quickly calculate the sizes of connected components in this cell complex, in $O(n)$ time where n is the degree. Iterating this counting process over the aforementioned $\mathscr{D}$-classes gives us a fast algorithm for counting idempotents in such semigroups.

From there, these methods can be extended using the theory of stable families. This is an untested methodology that shows a lot of promise, both for increasing the amount of information available about the elements of these semigroups, but also as something of theoretical interest on its own. Through this methodology, we've managed to reduce part of the coarse problem of enumerating idempotents to that of counting meanders — a hard problem, and perhaps a sign that not much more progress is likely to be made withut bringing some new machinery to the task.

Given the deep relationships between the $\mathcal{J}_n$ and the one-deformation parameter algebras $TL_n(\delta)$, one may be tempted to ask whether semigroup theory could contribute to, or benefit from study of other related structures. To my dissatisfaction, I wasn't aware until fairly recently of most of the research I'm about to talk about, which is a shame as some of it looks amenable to the methods dicussed here. I have taken some time to examine the *affine* or *annular Jones monoid*, which is simply the regular Jones monoid, except that has a cyclic group of units whose order is the degree.

There are several families of semigroups closely related to those, including the Kauffman monoids discussed in [4], the partial Jones monoids $\mathcal{PJ}_n$, which do not appear in the literature, but consist of diagrams which look like elements of $\mathcal{J}_n$ but with some edges missing. The ideal structure is vastly more complicated in both cases — the former has infinitely-long chains of ideals for $n > 1$ and the ordering on the $\mathscr{D}$-classes of $\mathcal{PJ}_n$ has exponentially-wide anti-chains corresponding to many incomparable ideals. We have some partial results using spartans and FitzGerald's so-called exotic statistics, which are recurrences involving the numbers γ and ω discussed in [49]

Noncrossing of strands seems to be a critical factor in how easy the semigroups in question are to analyse in this way — Motzkin $\mathcal{M}_n$ and Jones $\mathcal{J}_n$ are much harder to analyse than the partition algebras and the Brauer families, but the affine case seems for all intents and purposes impenetbrable. Indeed, although we have a method for dealing with partial Jones $\mathcal{PJ}_n$, for example, it's much slower and its irreducible idempotents may be distributed more widely throughout the semigroup.

It's tempting to ask, given the fact that these families each contain all the finite aperiodic semigroups, what sort of implications this work has for finite apriodic semigroups in general. In particular, do they perhaps possess some of the structures that we've seen here, or are these semigroups just rather special? Do other families containing all the aperoidic monoids have such a structure pehaps?

7.1 Limitations and Possible Improvements

One possible "limitation" of the enriched generating function approach is that certain parameter profiles force one to do calculations that will reduce (in either $\mathcal{J}_n$ or $\mathcal{M}_n$) to counting meanders. The problem of enumerating meanders is a classical one that has defied a century's worth of combinatorial innovations and greatest minds; it has deep connections to theoretical computer science and constraint satisfaction problems, to permutation group theory, to representation theory of groups and algebras, to theoretical physics and many other areas of combinatorics and algebra.

In other words, an efficient solution to the problem of calculating fine-grained information for $\mathcal{M}_n$ would give one an efficient solution to the problem of meander enumeration.

Several ideas of have come together during the write-up phase that haven't had the time to coalesce into actual mathematical content. Des FitzGerald has started looking at that in [13]. A useful example, the cycle number enumeration discussed above, or the hatter process for $\mathcal{PJ}_n$, which has not been mentioned. It would also be nice to have a fuller description of the structure of $\mathcal{PJ}_n$.

Current methods have all been completely resisted by the annular monoids $\mathcal{PAJ}_n$ and $\mathcal{AJ}_n$. Their structure does not have enough symmetry to use regularity to much effect like in the case of $\mathcal{B}r_n$ and cousins, but appears to have enough to get in the way of using convexity or generating functions as a useful toolkit.

Part IV

Appendices

Appendix A

Combinatorics on Words and Indexing Motzkin Elements

The work found in this chapter has largely been subsumed by a better implementation of the main algorithm by the external examiner. At his suggestion, this chapter was left in for completeness, but has been moved to the appendices to reflect its relative lack of importance in the implementation.

We will not prove it here, but the ordering implied for Dyck words in [63] is identical to the one given here, albeit our previous methods for producing this ordering or iterating over it were significantly less efficient.

We will develop several indexing processes which make extensive use of concepts from formal language theory and combinatorics on words. By the end of this chapter we will be able to iterate over the $\mathscr{L}$-classes (hence $\mathscr{R}$- and $\mathscr{H}$-classes) in some fixed $\mathscr{D}$-class in $\mathcal{M}_n$ and $\mathcal{J}_n$.

A.1 Dyck Words: Ordering, the Jones monoid and Grammar

The language $\mathfrak{D}$ of Dyck words has a number of equivalent formulations, including

- The set of *balanced bracketings* over an alphabet consisting of a matched pair of open and closed brackets;
- Those words over such an alphabet for which any prefix contains at least as many opening brackets as closing brackets, and whose suffixes have at most as many.
- The set of words generated by the following productions

$$\mathbf{S} \longrightarrow \epsilon + \mathbf{S} \cdot \mathbf{S} + \mathtt{[S]}$$

  with the usual conventions about terminal and nonterminal symbols;
- The smallest submonoid of the monoid $\{\mathtt{[}, \mathtt{]}\}^*$ closed under the mapping $w \mapsto \mathtt{[} \cdot w \cdot \mathtt{]}$.

These are well-known to be equivalent, see [24, 38, 70], and we'll refer to them interchangeably herein.

It's well-known [72] that there are C_n-many words of length $2n$ in D, where

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

is the n-th Catalan number.

Example A.1.1. The Dyck words of various small lengths are listed and enumerated below:

Length	Words	Number
0	ε.	1
2	`[]`.	1
4	`[[]]`, `[][]`.	2
6	`[[[]]]`, `[[][]]`, `[[]][]`, `[][[]]`, `[][][]`.	5

The first few values for the Catalan numbers (see the online encyclopedia of integer sequences, A000108 [71]) are 1, 1, 2, 5, 14, 42, ...; the first values of this sequence coincide with those above.

This sequence is important because it enumerates the Jones monoid in several ways, and will inspire asymptotic results we'll derive later. We'll now construct an explicit bijection between $\mathcal{J}_n$ and the C_n-many Dyck words of length $2n$ of a given size n using the interface graph.

A.1.1 Dyck words and the Jones monoid

Let $\alpha \in \mathcal{J}_n$. We define a map Ψ_α associating to each $k \in \pm[\![n]\!]$ either **[** if k is connected in α to a larger element, or **]** otherwise.

Proposition A.1.2. *Let $\alpha \in \mathcal{J}_n$. Then the word*

$$w_\alpha = \Psi_\alpha(n') \cdot \Psi_\alpha((n-1)') \cdots \Psi_\alpha(1') \cdot \Psi_\alpha(1) \cdot \Psi_\alpha(2) \cdots \Psi_\alpha(n)$$

is a Dyck word. Furthermore, the map $\alpha \mapsto w_\alpha$ is a bijection from $\mathcal{J}_n$ onto the set of all Dyck words of length $2n$.

Proof. Ballot sequences are enumerated (see [72]) by the Catalan numbers C_n, so in order to prove the theorem, we need only establish bijections from $\mathcal{J}_n$ to the ballot sequences of length $2n$ and from there to the Dyck words of length $2n$, whose composition is $\alpha \mapsto w_\alpha$.

Set $\mathfrak{c}_n = \{n' < \cdots < 2' < 1' < 1 < \cdots n\}$ as before, and let i and j be connected in α. Then set $d_\alpha(i) = -1$ if $j < i$ and 1 otherwise.

This $(d_\alpha(i))_{i \in \mathfrak{c}_n}$ is a ballot sequence. To see this, assume otherwise and observe that each $d_\alpha(i)$ is ± 1, so by assumption some partial sum $\sum_{i=n'}^{j} d_\alpha(i)$ must be negative. Let j be the first number such that this is true and denote by $j-1$ its predecessor in $\mathfrak{c}_n$. Then write D_j for the j-th partial sum and note that $D_{j-1} = \sum_{i=n'}^{j-1} d_\alpha(i) = 0$, so $D_j = 1$. We know that each $i < j$ has $D_i \geq 0$, so these i are all paired off up to $j-1$ therefore j can't connect to any $i < j$, a contradiction. The map $\alpha \mapsto (d_\alpha(i))_i$ is clearly an injection, and since $\mathcal{J}_n$ has C_n elements, see [7], it's a bijection.

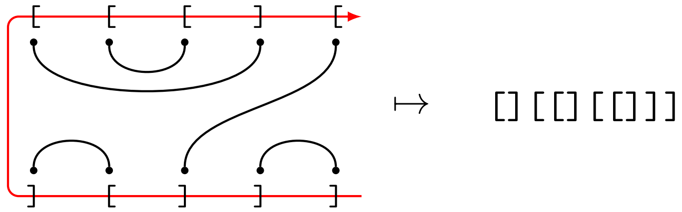


Figure A.1: The cyclic indexing of an element of $\mathcal{J}_5$ by a Dyck word.

We can construct a Dyck word $u = (u_i)_i$ by setting $u_i = [$ if $a_i = 1$ and $u_i =]$ otherwise. This is clearly a bijection, and it's clear that the composition is the desired mapping. □

The above indexing is important, because the induced subgraph of α supported on its vertices of one sign determines $\mathscr{L}$- and $\mathscr{R}$-classes, so those are indexed by the n-long prefixes of the w_α. Recall that Dyck words are characterised by having all prefixes (and suffixes, respectively) containing no more (resp. less) closed than open brackets in the word.

The condition given by dropping the requirements on suffixes characterises prefixes of Dyck words, and vice-versa. The map which switches the brackets over induces a word-reversing automorphism of the Dyck language, so without any loss of generality we'll forget about prefixes for now.

Definition A.1.3. A *Dyck suffix* v is a suffix of a Dyck word $w = l \cdot l$. The *quasirank* $\mathsf{q}(r)$ of a Dyck suffix r is the number of unpaired closing brackets in r.

Writing $\vec{\mathfrak{D}}$ for the set of Dyck suffixes, we have

$$\vec{\mathfrak{D}} = \coprod_{r=0}^{\infty} \left(\mathfrak{F} \cdot] \right)^r \cdot \mathfrak{D}$$

Given $\xi \in \mathcal{J}_n$, write r_ξ for the length-n suffix of the Dyck word w_ξ.

Proposition A.1.4. *Let $\alpha, \beta \in \mathcal{M}_n$, we have $r_\alpha = r_\beta$ precisely if $\alpha \mathscr{R} \beta$.*

We'll use the r_α notation hereafter for this suffix.

Proof. Let $\alpha \mathscr{R} \beta$ be elements of $\mathcal{J}_n$. Then $\alpha = \beta s$ and $\beta = \alpha t$ for some $s, t \in \mathcal{J}_n$. It's clear from the definition of multiplication that $\Gamma_\alpha^\vee = \Gamma_\beta^\vee$ since postmuliplication may only add edges to these sets, implying mutual containment.

Indeed, since $\alpha = \beta s = (\alpha t)s = \alpha(ts)$, neither t nor s may reduce rank. Post-multiplication can only affect $S^\vee$ by removing elements, which means that $S_\alpha^\vee = S_\beta^\vee$.

Assuming now that $S_\xi^\vee = S_\gamma^\vee$ and $\Gamma_\xi^\vee = \Gamma_\gamma^\vee$ for some $\xi, \gamma \in \mathcal{J}_n$. Then we have

$$S_\xi^\vee = S_{\xi\xi^*}^\vee = S_\xi^\vee = S_{\xi\xi^*}^\wedge = S_{\gamma\gamma^*}^\vee = S_{\gamma\gamma^*}^\wedge,$$
$$\Gamma_\xi^\vee = \Gamma_{\xi^*\xi}^\vee = \Gamma_{\xi^*\xi}^\wedge = \Gamma_{\gamma^*\gamma}^\vee = \Gamma_{\gamma^*\gamma}^\wedge.$$

Since the interface graph determines an element of $\mathcal{J}_n$, we see that $\xi\xi^* = \gamma\gamma^*$. By virtue of being closed under the involution on $\mathcal{P}_n$, $\mathcal{J}_n$ is a regular *-semigroup, which means that

$$\xi = \xi\xi^*\xi = \xi\gamma^*\gamma, \qquad \gamma = \gamma\gamma^*\gamma = \gamma\xi^*\xi,$$

so $\xi \mathscr{R} \gamma$.

Given that each Dyck word w is equal to w_ξ for some $\xi \in \mathcal{J}_n$, each length-n suffix must be the suffix of some w_ξ. Suffixes are determined by the connections on $1, 2, \ldots, n$, so are determined by the $\vee$-part of the interface graph. □

Given $w \in \{\mathtt{[}, \mathtt{]}\}^*$, we write w^{-1} for the word obtained from w by reversing the order of symbols and swapping each `[` for a `]` and vice-versa. Writing l_α for the n-long prefix of w_α, we obtain the following results.

Corollary A.1.5. *Let $\alpha \in \mathcal{J}_n$. Then $l_\alpha = r_{\alpha^*}^{-1}$.*

Corollary A.1.6. *For $\alpha, \beta \in \mathcal{J}_n$, we have $\alpha \mathscr{L} \beta$ precisely if $l_\alpha = l_\beta$.*

Proposition A.1.7. *Let $\alpha \in \mathcal{J}_n$. The quasirank $\mathsf{q}(r_\alpha)$ is the rank of α.*

Proof. The unmatched brackets in r_α correspond to transverse blocks in α, the number of which is exactly the rank. □

Corollary A.1.8. *Let $\alpha, \beta \in \mathcal{J}_n$. Then $\mathsf{q}(r_\alpha) = \mathsf{q}(r_\beta)$ precisely if $\alpha \mathscr{D} \beta$.*

Proof. Fix $\alpha, \beta \in \mathcal{J}_n$ of the same rank k. Then $l_\alpha \cdot r_\alpha$ and $l_\beta \cdot r_\beta$ are Dyck words with r_β and r_α having quasirank equal to k.

We know that $\alpha \mathscr{D} \beta$ precisely when there is γ such that $\alpha \mathscr{R} \gamma \mathscr{L} \beta$, and that $\gamma \mapsto l_\gamma$ and $\gamma \mapsto r_\gamma$ determine the $\mathscr{L}$ and $\mathscr{R}$-classes respectively.

So, let $w = l_\beta \cdot r_\alpha$. Proposition A.1.2 guarantees that there is γ with $w = w_\gamma$, so by Proposition A.1.4 and Corollary A.1.5, $\alpha \mathscr{R} \gamma \mathscr{L} \beta$ as was requried. □

A.1.2 An ordering on Dyck words

We will now develop a partial order (a well-ordering as it turns out, but we'll have to prove this later) on Dyck words that will later generalize in several directions, and will be used to build fast iterators with low memory overhead to apply to counting problems in the semigroups.

The *Dyck order* is constructed iteratively from the following:

(O1) The ordering is length-monotone: if $|u| < |v|$ then $u < v$;

(O2) The first word in $\mathfrak{D}^{(2n)}$ is $(\,\texttt{[]}\,)^n = \texttt{[][]}\cdots\texttt{[]}$;

(O3) The last word in $\mathfrak{D}^{(2n)}$ is $\texttt{[}^n\texttt{]}^n = \texttt{[[}\cdots\texttt{[]}\cdots\texttt{]]}$;

(O4) Let $u \in \mathfrak{D}^{(2l)}$, and $v \in \mathfrak{D}^{(2k)}$, not the last of its length, and put $w = \texttt{[}u\texttt{]}v$. Then the successor w' of w is $\texttt{[}u\texttt{]}v'$ where v' is the successor to v;

(O5) For $u \in \mathfrak{D}^{(2k)}$, $v \in \mathfrak{D}^{(2l)}$ and $w = \texttt{[}u\texttt{]}v$, with v the last word of its length, the following hold:

  (a) If u is the last word in $\mathfrak{D}^{(2k)}$ and $l > 0$ then w's successor is $\texttt{[}u_0\texttt{]}v_0$ where u_0 is the first word in $\mathfrak{D}^{(2k+2)}$ and v_0 is the first word in $\mathfrak{D}^{(2l-2)}$;

  (b) If u is not the last of its length then the next word is $u'\texttt{[}v_0\texttt{]}$ where u' is the successor of u, and v_0 is the first word in $\mathfrak{D}^{(2l)}$.

Example A.1.9. We can read off the smallest few words

$$\varepsilon < \texttt{[]} < \texttt{[][]} < \texttt{[[]]} < \texttt{[][][]}$$

directly. We can then get

$$\texttt{[][][]} < \texttt{[][[]]} < \texttt{[[]][]} << \texttt{[[][]]} < \texttt{[[[]]]}$$

from (respectively) (O4) applied with $u = \varepsilon$ and $v = \texttt{[][]}$, then (O5a) with $u = \varepsilon$ and $v = \texttt{[[]]}$, (O5a) again with $u = v = \texttt{[]}$, then (O5b) with $u = \texttt{[][]}$ and $v = \varepsilon$ to get the last word of length 6.

The Dyck order clearly has the property that each word has finitely-many predecessors by (O1). We still must prove that it's a total order, which is not completely obvious.

Observation A.1.10. *For given u, the restriction of the above ordering to the set* $\texttt{[}u\texttt{]}\mathfrak{D}^{(2n)}$ *is a total order precisely if the restriction to* $\mathfrak{D}^{(2n)}$ *is a total order.*

Lemma A.1.11. *Let s and t be Dyck words. Then* $s < t$ *precisely when* $\texttt{[}s\texttt{]} < \texttt{[}t\texttt{]}$.

That is to say that conjugation $w \mapsto \texttt{[}w\texttt{]}$ by brackets preserves and reflects the order.

Proof. If $|s| < |t|$ both directions are obvious, therefore we let $|s| = |t|$.

Let $s < t$. Clearly, then, s is not the last word of its length. Therefore there is a chain

$$s =: s_0 < s_0' = s_1 < s_1' = s_2 < \cdots < s_r := t$$

since $s < t$. Then, writing $u_i = s_i$, $v_i = \varepsilon$ and $w_i := \texttt{[}u_i\texttt{]}v_i = \texttt{[}s_i\texttt{]}$, we can repeatedly apply (O5a) to get

$$\texttt{[}s\texttt{]} = w_0 < w_0' = w_1 < \cdots w_r = \texttt{[}t\texttt{]}.$$

Now, let $\texttt{[}s\texttt{]} < \texttt{[}t\texttt{]}$, we see that such a chain of w_i exists precisely when the chain $(s_i)_i$ exists. □

Corollary A.1.12. *Let* $u, s, t \in \mathfrak{D}$ *be Dyck words. If* $s < t$ *then* $\texttt{[}u\texttt{]}s < \texttt{[}u\texttt{]}t$.

The proof follows that of Lemma A.1.11, letting $w_i = \texttt{[}u\texttt{]}s_i$ and repeatedly comparing using (O4).

Corollary A.1.13. *The Dyck order is a well-ordering of* $\mathfrak{D}$.

Proposition A.1.14. *Let $u_1, u_2, \ldots, u_k \in \mathfrak{D}$ be Dyck words. Then the successor to $[u_1][u_2]\cdots[u_k]$ is equal to $[u_1]w$ for some w unless $k = 2$ and $u_2 = [^n]^n$ for some n.*

This follows from the definition.

Corollary A.1.15. *The successor to $[u_1][u_2]\cdots[u_k]$ is $[u_1][u_2]\cdots[u_{k-2}]w$ for some $w \in \mathfrak{D}$.*

Corollary A.1.16. *If $|u| = |v|$ with $u < v$, then $[u]w < [v]w$.*

Proof. Set $|w| = 2l$ and write $U = ([\,])^l$ and $W = [^l]^l$ for the first and last words of the same length. By well-ordering, we can define a finite sequence $u = u_0 < u_1 < u_2 < \cdots < u_r = v$ of successive terms in the Dyck order.

By repeated application of Corollary and (O5b), we have

$$[u]w \leq [u]W \prec [u_1]U \leq [u_1]W \prec \cdots \prec [v]U \leq [v]w.$$

□

Proposition A.1.17. *The Dyck order is compatible with concatenation on Dyck words, in the sense that if u, v, w, x are Dyck words with $u \leq v$ and $w \leq x$, then*

$$uw \leq vw \leq vx. \tag{A.1}$$

Proof. If $|u| < |v|$ then the first inequation of (A.1) holds, and if $|w| < |x|$ then the second holds; if the reverse length-inequalities hold then neither of (A.1) never hold.

We need only prove the assertion in the case that $|u| = |v|$ and $|w| = |x|$. Assume that $u \leq v$ are in $D^{(2l)}$ and $w \leq x$ are in $D^{(2k)}$. Then, by well-ordering, we have sequences $u = u_0, u_1, \ldots u_k = v$ and $w = w_0, w_1, \ldots w_r = x$, where in both cases, each term is followed by its successor in the Dyck order.

Then by Corollary A.1.15, Corollary A.1.2 and Corollary A.1.16, we have the sequence

$$\begin{aligned} u_0 w_0 < u_0 w_1 < \ldots < u_0 w_r &\leq u_0 W \\ < u_1 U \leq u_1 w_0 < u_1 w_1 < \ldots < u_1 w_r &\leq u_1 W \\ < \cdots \\ < u_k U \leq u_k w_0 < u_k w_1 < \ldots < u_k w_r. \end{aligned}$$

where $U = (\texttt{[]})^k$ is the first word of length $|w| = 2k$ and $W = \texttt{[}^k\texttt{]}^k$ is the last. □

A.1.3 Prefix Codes and Free Monoids

Definition A.1.18. A *prefix code* is a language C such that for any two distinct words in C, neither is a prefix of the other.

Prefix codes turn out to generate submonoids of free monoids which are themselves free, and whose rank is the size of the code. The Dyck language is a free submonoid of $\{\texttt{[},\texttt{]}\}^*$ generated by the countable prefix code consisting of *Dyck primes*, which are those words of the form $\texttt{[}w\texttt{]}$ with $w \in \mathfrak{D}$. Clearly, the Dyck primes are all Dyck words.

We can characterise the language $\mathfrak{D}'$ of Dyck primes as the words $\texttt{[}u\texttt{]}$ where $u \in \mathfrak{D}$. The prime prefix of a nonempty Dyck word w is therefore the unique Dyck prime $\texttt{[}u\texttt{]}$ where $w = \texttt{[}u\texttt{]}v$ where u and v are Dyck words.

Proposition A.1.19. *The language $\vec{\mathfrak{D}}$ of Dyck suffixes is a free monoid.*

Proof. The set of Dyck primes augmented with the word $\texttt{]}$ comprises a prefix code. This set is contained in $\vec{\mathfrak{D}}$, and so is any concatenation of its elements. We need only establish then that each non-Dyck word $w \in \vec{\mathfrak{D}} - \mathfrak{D}$ decomposes as

$$w = w' \cdot \texttt{]} \cdot \vec{w} \tag{A.2}$$

for some Dyck word $w' \in \mathfrak{D}$ and Dyck suffix $\vec{w} \in \vec{\mathfrak{D}}$.

So let $w \in \vec{\mathfrak{D}}$ be a Dyck suffix, and assume $w \notin \mathfrak{D}$, and in particular that w is nonempty.

Let $\tilde{w}$ be a Dyck prefix such that $\tilde{w} \cdot w \in \mathfrak{D}$ is a Dyck word. Then $\tilde{w} \cdot w$ decomposes as a concatenation of some $l > 0$ Dyck primes $\mathtt{[}u_1\mathtt{]} \cdot \mathtt{[}u_2\mathtt{]} \cdots \mathtt{[}u_l\mathtt{]}$, where each u_i is a Dyck word.

Choose r so that $\mathtt{[}u_r\mathtt{]}$ is the first word not completely contained in $\tilde{w}$. Then the trailing bracket $\mathtt{]}$ is part of the suffix $\vec{w}$ and we have

$$\tilde{w} = \mathtt{[}u_1\mathtt{]}\,\mathtt{[}u_2\mathtt{]} \cdots \mathtt{[}u_r', \qquad w = u_r''\mathtt{]} \cdots \mathtt{[}u_l\mathtt{]},$$

where $u_r = u_r' \cdot u_r''$ denotes the decomposition of u_r across the factorisation $\tilde{w} \cdot w$. If u_r'' is in $\mathfrak{D}$ then we have a decomposition of the sort prescibed by (A.2).

If this is not the case, then u_r'' is a Dyck suffix and is shorter than w. In this situation, we may repeat this process; each time we do the word at the border is shorter, so the process must terminate after a finite number of steps, giving a (possibly empty) Dyck word at the last iteration, as required to establish the result. □

Corollary A.1.20. *The language of Dyck prefixes is a free monoid.*

A.2 Motzkin Words and the Motzkin Monoid

The language $\mathfrak{M}$ of Motzkin words comprises a somewhat richer language than that of the Dyck words. It decomposes into a union of languages $\mathfrak{M}_n$ for

We obtain another hierarchical decomposition of the language as a union of languages, and the various parts in this decomposition correspond to the $\mathscr{D}$-classes of the monoids $\mathcal{M}_n$. This extra structure, and the fact that this family of languages is not so ubiquitous in its appearance throughout combinatorics as the Dyck language, will require a slightly more comprehensive treatment.

Definition A.2.1. The language $\mathfrak{M}$ of Motzkin words is that generated by the context-free grammar with start symbol S, an additional nonterminal T, terminals $\Sigma = \{\mathtt{[},\mathtt{]},\mathtt{|},\mathtt{O}\}$ and productions

$$\mathbf{S} \longrightarrow \mathbf{S} \cdot \mathbf{S} + \mathbf{T} + \mathtt{|}; \tag{A.3}$$

$$\mathbf{T} \longrightarrow \mathbf{T} \cdot \mathbf{T} + \mathtt{[}\mathbf{T}\mathtt{]} + \mathtt{O} + \varepsilon. \tag{A.4}$$

Given a word $w \in \mathfrak{M}$, its rank $\mathsf{r}(w)$ is simply the number of instances of the letter $\mathtt{|}$. Denote by $\mathfrak{M}_k$ the set of words of rank k.

Observation A.2.2. *The language $\mathfrak{M}_0$ is the same as the language generated from* $\mathtt{T}$ *using only the productions in* (A.4).

Proposition A.2.3. *The language $\mathfrak{M}_0$ is a free submonoid of Σ^*. In fact,*

$$\mathfrak{M}_0 = \left(\mathtt{[}\mathfrak{M}_0\mathtt{]} + \mathtt{O} \right)^*.$$

Proof. The set $\mathtt{[}\mathfrak{M}_0\mathtt{]} + \mathtt{O}$ has the prefix property, so it generates a free submonoid of Σ^*, whose elements are all of rank zero.

We need only observe from the production (A.4) that every word in $\mathfrak{M}_0$ is either empty or decomposes into a concatenation of either $\mathtt{O}$ or words of the form $\mathtt{[}w\mathtt{]}$ where $w \in \mathfrak{M}_0$. □

Similar examination of (A.3) will confirm the following result.

Corollary A.2.4. *The language $\mathfrak{M} \subseteq \Sigma^*$ is a free submonoid, generated by the prefix code* $\left(\mathtt{[}\mathfrak{M}_0\mathtt{]} + \mathtt{O} + \mathtt{|} \right)$.

Definition A.2.5. The *shuffle (product)** $u \sqcup\!\sqcup v$ of two words is the set of all possible alternating concatenations

$$u_0 \cdot v_1 \cdot u_1 \cdots u_{k-1} \cdot v_k \cdot u_k,$$

where k is allowed to vary between 1 and the shorter of the two lengths, with $u = u_0 \cdots u_k$ and $v = v_1 \cdots v_k$ varying over all possible decompositions into subwords with u_0 and u_k possibly empty and every other v_i and u_i nonempty.

The shuffle product $K \sqcup\!\sqcup L$ of two languages is the language given by

$$K \sqcup\!\sqcup L = \bigcup \left\{ k \sqcup\!\sqcup l : k \in K, l \in L \right\}.$$

Proposition A.2.6. *The language $\mathfrak{M}_0$ decomposes as a shuffle $D \sqcup\!\sqcup \mathtt{O}^*$.*

---

*The shuffle product means something slightly more general in algebra, applying to formal sums or polynomials in general. The language-theoretic shuffle is a particular instance in which the underlying algebra of formal sums takes coefficients in the Boolean semiring.

We can therefore use the following result to describe $\mathfrak{M}_0$.

Proposition A.2.7. *Let L and K be two languages on disjoint alphabets which have well-orders that are monotone with respect to length. Then there is a well-order on $L \sqcup\!\sqcup K$ which is monotone respecting length.*

The above result is established constructively in a way that is also *effective* in the sense that if comparisons in both of L and K are respectively cheap or fast to compute, then so are those in $L \sqcup\!\sqcup K$. Before attempting the proof, we need the term *scattered subword* (of w) which refers to a word whose letters appear in w in order, but not necessarily successively, as it the case with the more familiar notion of a (sequential) subword.

Proof. Let $L \subseteq \Phi^*$ and $K \subseteq \Xi^*$ be languages on disjoint alphabets Φ and Ξ, and having well-orderings defined. By disjointness, if $w \in L \sqcup\!\sqcup K$ decomposes into letters $w_1 \cdots w_r$, then each $w_i \in \Phi$ or $w_i \in \Xi$, so we can find two scattered subwords $w \upharpoonright_\Phi := w_{i_1} w_{i_2} \cdots w_{i_l} \in L$ and $w \upharpoonright_\Xi := w_{j_1} \cdots w_{j_l} \in K$ containing, between them, every letter in w.

We call l and k the Φ- and Ξ-length of w, written $|w|_\Phi$ and $|w|_\Xi$. Write $\mathrm{i}_\Phi(w)$ for the sequence of indices $i_1, \ldots, i_l$.

Define an order as follows.

- If u is a shorter word than v then $u < v$;
- If $|u| = |v|$ and $u \upharpoonright_\Phi < v \upharpoonright_\Phi$ then $u < v$;
- If $u \upharpoonright_\Phi = v \upharpoonright_\Phi$ and $u \upharpoonright_\Xi < v \upharpoonright_\Xi$ then $u < v$;
- If $u \upharpoonright_\Phi = v \upharpoonright_\Phi$ and $u \upharpoonright_\Xi = v \upharpoonright_\Xi$ and m is the first index where the vectors $\mathrm{i}_\Phi(u)$ and $\mathrm{i}_\Phi(v)$ have differing entries, say i_m and i'_m, then $u < v$ if $i_m < i'_m$.

If none of the above are the case then the words are equal.

First we observe that this partial order is monotone respecting the length. Next, we note that the cases are mutually exclusive, and exhaustive, meaning that the ordering is total. Finally, we must establish well-ordering, which follows from the fact this is an ordering on a (subset of a) finitely-generated free monoid which respects length. □

We recall that $\mathfrak{M}_k$ denotes the set of Motzkin words of rank k, and observe that

$$\mathfrak{M}_k = (\mathfrak{M}_0 \cdot \mathbf{|})^k \cdot \mathfrak{M}_0.$$

In particular, we can induce a *relative shortlex* order on $\mathfrak{M}_k$ with respect to $\mathfrak{M}_0$ as follows. If $|w| < |w'|$ then we require that $w < w'$. Otherwise, note that every $w \in \mathfrak{M}_k$ admits a decomposition into

$$w = u_0 \cdot \mathbf{|} \cdot u_1 \cdot \mathbf{|} \cdot u_2 \cdots u_{k-1} \cdot \mathbf{|} \cdot u_k,$$

where each $u_i \in \mathfrak{M}_0$. Clearly, then, if $w \neq w'$ are in $\mathfrak{M}_k$, the respective sequences $(u_i)_i$ and $(u'_i)_i$ of rank-zero subwords from $\mathfrak{M}_0$ do not agree. If r is the first place at which they disagree then $w < w'$ if $u_r < u'_r$.

We can then, if required, extend this in a natural way to a well-ordering on the whole of the language $\mathfrak{M}$ by imposing length-monotonicity, then rank-monotonicity for words of equal length, then comparing equally-long words of equal rank using the ordering on the $\mathfrak{M}_k$.

A.3 Noncrossing partitions, Correspondences and Computation

The main purpose of this section is to establish an explicit bijection between the set of $\mathscr{R}$-classes in $\mathcal{J}_n$ and a set of lists of integers. From there, we carefully extend this correspondence to $\mathscr{R}$-classes in $\mathcal{PJ}_n$ and $\mathcal{M}_n$ using results from earlier in the chapter. We proceed by introducing intermediary structures, many of which have known bijections.

A partition $P = P_1 \sqcup \cdots \sqcup P_k$ of $[\![n]\!]$ is *noncrossing* if, for $a < b < c < d \in [\![n]\!]$, if $a, c \in P_i$ and $b, d \in P_j$ implies $i = j$. This is equivalent to being able to arrange the points 1 to n around a circle and draw some graph whose connected components are exactly the parts P_i, such that the graph is planar, i.e. the edges are noncrossing.

The *weak orbit* relation of a partial function $f : X \to X$ is the coarsest equivalence relation on X containing each pair $(x, f(x))$. A weak orbit of f is a then class in this relation.

A permutation (respectively, a partial bijection) of $[\![n]\!]$ is then said to be noncrossing if its weak orbits form a noncrossing partition of $[\![n]\!]$. A noncrossing permutation (partial bijection) is *visibly noncrossing* if for every fixed point b, if $a < b < c \in [\![n]\!]$ then a and c lie in different (weak) orbits. A permutation (respectively, a partial bijection) π is called a *(partial) involution* if $\pi^2(k) = k$ for all $k \in [\![n]\!]$ (on which π is defined). The *one-line notation* for a permutation π is simply the n-tuple

$$\pi[\![n]\!] = (\pi(1), \pi(2), \ldots \pi(n)).$$

This notation can be extended to encompass partial bijections by writing $\pi k = 0$ for any undefined values.

Proposition A.3.1. *The correspondence from partial bijections in degree n to their one-line notation is an injection, and its image is the set of tuples without repeated nonzero entries.*

Proof. Clearly $\pi \neq \varsigma$ precisely if, for some i we have $\pi(i) \neq \varsigma(i)$, either in the sense that preciselye one is defined, or that both are defined but unequal. Therefore we have $\pi[\![n]\!] \neq \varsigma[\![n]\!]$, meaning one-line notation faithfully represents partial bijections.

To see the converse, note that the sets are equinumerous. A partial bijection from k points in $[\![n]\!]$ is uniquely determined by the sets of points mapped from and to, and the order of the points in the image, of which there are $\binom{n}{k}^2 k!$. Similarly, an n-tuple with k distinct nonzero entries is uniquely determined by the locations, values and order of appearance of those entries, giving the same number of lists. □

We call a tuple

$$\mathbf{v} = (v_1, v_2, \cdots, v_n) \in [\![n]\!]^n$$

a *Dyck list (of size n)* if it is one-line notation for a noncrossing involution of $[\![n]\!]$. Such a tuple is a *Motzkin list* if it is one-line notation for a noncrossing partial involution.

Definition A.3.2. Let S be a regular $\star$-semigroup and $e \in S$. Then e is a *projection* if $e^* = e$.

We first aim to prove the following correspondences:

$$\begin{aligned} \mathscr{R}\text{-classes in } \mathcal{J}_n &\leftrightarrow \text{Projections} \\ &\leftrightarrow \text{Visibly noncrossing involutions of } [\![n]\!] \\ &\leftrightarrow \text{Dyck lists of size } n \end{aligned}$$

There are similar correspondences between the $\mathscr{R}$-classes in the Motzkin monoids $\mathcal{M}_n$ and Motzkin lists of size n, and between the $\mathscr{R}$-classes in partial Jones monoids $\mathcal{PJ}_n$ and XXX lists of size n.

Proposition A.3.3. *Let S be a regular $\star$-semigroup and $s \in S$. The elements ss^* and s^*s are projections. Furthermore, if $e = st$ is a projection satisfying $es = s$ then $e = ss^*$.*

Proof. The involution on a $\star$-semigroup is antihomomorphic, meaning $(ab)^* = b^*a^*$ and $(a^*)^*$ for all elements a and b. Therefore, write $(ss^*)^* = (s^*)^* \cdot s^* = ss^*$; s^*s follows similarly easily.

Now let $es = sts = s$. □

This gets us from $\mathscr{R}$

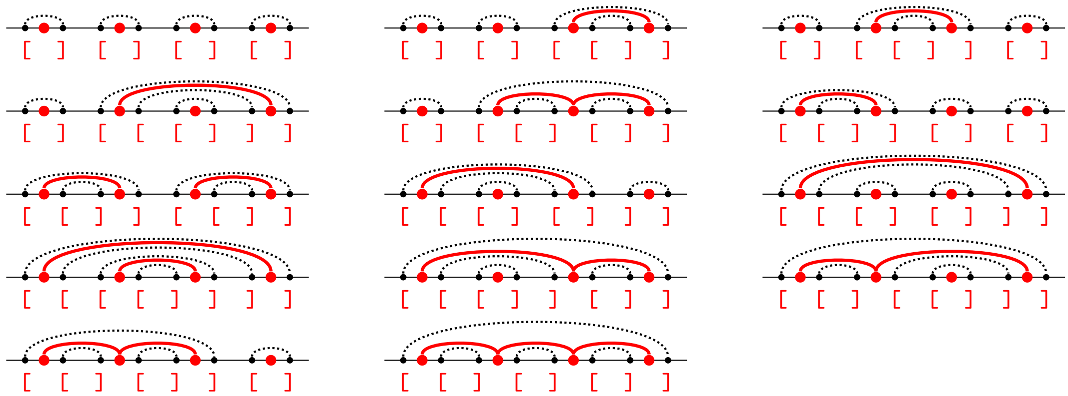


Figure A.2: Explicit correspondence between Dyck words of semi-length n and non-crossing partitions of $[\![n]\!]$ with noncrossing partitions of $[\![2n]\!]$ shown as intermediary.

Appendix B

The Ordered Integer Partition Dictionary

This chapter lists and enumerates the pairs $(\mathbf{n}, \mathbf{r})$ of all pairs of ordered integer partitions which satisfy the following constraints:

$$\mathbf{n} = (n_1, \ldots, n_k) \vdash n, \qquad \mathbf{r} = (r_1, \ldots, r_k) \vdash r,$$
$$r_i \leq \max\left(\frac{n_i}{3}, 1\right), \qquad r_i = n_i \implies r_i > 0$$
$$i > j \implies n_i \leq n_j, \qquad i < j, n_i = n_j \implies r_i \geq r_j$$

The first two define the notation and ensure that they have identical order, while the fourth is a "properness" condition, ensuring that the order is bounded by n and that there are not infinitely many possibilities. The third constraint is necessary for the number $c_{\mathbf{n},\mathbf{r}}$ as defined in Section 5.1 to be nonzero. The last two simply state that we've ordered the pairs by degree, highest-to-lowest, and thereafter by rank lowest-to-highest; for our purposes we only care about number up to reordering.

For the above, we'll write

$$\begin{Bmatrix} \mathbf{n} \\ \mathbf{r} \end{Bmatrix} = \begin{Bmatrix} n_1 & n_2 & \cdots & n_k \\ r_1 & r_2 & \cdots & r_k \end{Bmatrix} \vdash (n, r)$$

For $r = n$ we have only the following pair of partitions

$$\left\{\begin{matrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{matrix}\right\}.$$

In this notation, we'll list partitions up to column-reordering, with the convention adopted that there are precisely enough trailing columns $\binom{1}{1}$ to make the rows add to n and r respectively. We can pick a canonical representative for some equivalence class up to column-operations by stipulating that the top entries in columns are nonincreasing, and that the bottom are nonincreasing when the top entries are the same.

This can be achieved independent of n. For $r = n - 1$:

$$\left\{\begin{matrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \end{matrix}\right\}.$$

There are $\binom{n}{1}$ reorderings. For succinctness, we will drop the trailing columns of $\{1;1\}$:

$$\left\{\begin{matrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \end{matrix}\right\}, \quad \left\{\begin{matrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{matrix}\right\}.$$

For $r = n - 2$:

$$\left\{\begin{matrix} 3 \\ 1 \end{matrix}\right\}, \quad \left\{\begin{matrix} 2 \\ 0 \end{matrix}\right\}, \quad \left\{\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}\right\}.$$

There are respectively $\binom{n-2}{1}$, $\binom{n-1}{1}$ and $\binom{n}{2}$ reorderings.

For $r = n - 3$:

$$\left\{\begin{matrix} 4 \\ 1 \end{matrix}\right\}, \quad \left\{\begin{matrix} 3 \\ 0 \end{matrix}\right\}, \quad \left\{\begin{matrix} 3 & 1 \\ 1 & 0 \end{matrix}\right\}, \quad \left\{\begin{matrix} 2 & 1 \\ 0 & 0 \end{matrix}\right\}, \quad \left\{\begin{matrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{matrix}\right\}.$$

There are respectively $\binom{n-3}{1}$, $\binom{n-2}{1}$, $\binom{n-3}{2}$, $\binom{n-1}{2}$ and $\binom{n}{3}$ ways to place these.

For $r = n - 4$:

$$\left\{\begin{matrix} 6 \\ 2 \end{matrix}\right\}, \quad \left\{\begin{matrix} 5 \\ 1 \end{matrix}\right\}, \quad \left\{\begin{matrix} 4 \\ 0 \end{matrix}\right\}, \quad \left\{\begin{matrix} 4 & 1 \\ 1 & 0 \end{matrix}\right\}, \quad \left\{\begin{matrix} 3 & 1 \\ 0 & 0 \end{matrix}\right\}, \quad \left\{\begin{matrix} 3 & 2 \\ 1 & 0 \end{matrix}\right\},$$

$$\left\{\begin{matrix} 3 & 1 & 1 \\ 1 & 0 & 0 \end{matrix}\right\} \quad \left\{\begin{matrix} 2 & 2 \\ 0 & 0 \end{matrix}\right\}, \quad \left\{\begin{matrix} 2 & 1 & 1 \\ 0 & 0 & 0 \end{matrix}\right\}, \quad \left\{\begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}\right\}.$$

There are respectively $\binom{n-5}{1}$, $\binom{n-4}{1}$, $\binom{n-3}{1}$, $\binom{n-3}{2}$, $\binom{n-2}{2}$, $\binom{n-3}{2}$, $\binom{n-2}{3}$, $\binom{n-2}{2}$, $\binom{n-1}{3}$ and $\binom{n}{4}$ ways to place these.

For greater rank deficiency, the collections and insertion possibilities quickly become unwieldy. For example the spartan

$$\left\{\begin{matrix} 4 & 4 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{matrix}\right\}$$

can be inserted $\binom{n-9}{5}$ ways and rearranged $\binom{5}{2,3}$ ways, where

$$\binom{n}{r_1, r_2, \dots, r_l} = \frac{n!}{r_1! \cdot r_2! \cdots r_l!}$$

is the multinomial coefficient, with r some partition of n written as a list.

We need some automatic method of generating these. So, define

$$Sp_{n,r,p,q} := \left\{ \left\{\begin{matrix} \mathbf{n} \\ \mathbf{r} \end{matrix}\right\} \vdash (n,r) \;\middle|\; \max \mathbf{n} \leq p, \max \mathbf{r} \leq q \right\}.$$

Then $Sp_{n,r,p,q}$ contains all those Spartans which can be juxtaposed to the right of

$$\left\{\begin{matrix} p \\ q \end{matrix}\right\}$$

to form other spartans $s \vdash (n+p, r+q)$ without needing to reorder columns. Write

$$\mathbf{n} \otimes \mathbf{m} = (n_1, \dots, n_r, m_1, \dots, m_k)$$

where $\mathbf{n} = (n_1, \dots, n_r)$ and $\mathbf{m} = (m_1, \dots, m_k)$, and

$$\left\{\begin{matrix} \mathbf{n}_1 \\ \mathbf{r}_1 \end{matrix}\right\} \otimes \left\{\begin{matrix} \mathbf{n}_2 \\ \mathbf{r}_2 \end{matrix}\right\} = \left\{\begin{matrix} \mathbf{n}_1 \otimes \mathbf{n}_2 \\ \mathbf{r}_1 \otimes \mathbf{r}_2 \end{matrix}\right\}$$

Proposition B.0.1. *Let ς be a spartan of weight $(n-d, r-d)$ and length l whose columns appear with multiplicity $m_1, m_2, \dots m_k$. Then there are*

$$\binom{l+d}{d, m_1, \dots, m_k}$$

simultaneous partitions of weight (n, r) and length l which have no zero columns and which are equivalent to ς.

Note that

$$\binom{l+d}{d, m_1, \ldots, m_k} = \binom{l}{m_1, \ldots, m_k} \cdot \binom{l+d}{l}.$$

Write c_{nr} for the number of convex idempotents of degree n and rank r.

Corollary B.0.2. *Given $\varsigma = (\lambda_{i,j})_{i \le l-d; j=1,2}$ as above, there are*

$$c_{\lambda_{11},\lambda_{21}} c_{\lambda_{12},\lambda_{22}} \cdots c_{\lambda_{l1},\lambda_{l2}} \cdot \binom{l}{d, m_1, m_2, \ldots, m_k}$$

idempotents in $\mathcal{M}_n$ of rank d whose associated spartan is ς.

This Corollary tells us that to understand idempotents of high rank, we need only understand:

1. The spartans representing elements of degree n and rank r;
2. The simultaneous partitions representing these elements;
3. The number of convex elements of given rank and degree.

We've already determined how to determine how many simultaneous partitions are represented by some given spartan, solving 2 modulo a solution to 1. To address 3 requires some more work.

First, we state the main result. Let $S\binom{n}{r}$ be the set of spartans of weight (n, r). Let $|\varsigma|$ denote the length of the spartan ς and $m_i(\varsigma)$ the multiplicities of the columns in ordered form.

Theorem B.0.3. *The number of idempotents of degree n and rank r is*

$$\sum_{d=0}^{r} \sum_{\varsigma} c_{\lambda_{11},\lambda_{12}} \cdots c_{\lambda_{|\varsigma|,1} \lambda_{|\varsigma|,2}} \binom{l+d}{d, m_1(\varsigma), \ldots, m_{|\varsigma|}(\varsigma)}, \tag{B.1}$$

where the second sum is over ς in $S\binom{n-d}{r-d}$.

Fixing δ and letting n with $r = n - \delta$ constrained, we have only a finite set of spartans representing idempotents in $E_r(\mathcal{M}_n)$ for each n. Indeed, the sequence of sets $S\binom{n}{n-\delta}$ of possible spartans stabilises at around $n = \frac{3}{2}\delta$, as we'll see shortly. The multinomials in the sum in (B.1) are all polynomial in n of degree uniformly bounded by δ, and hence so is the sum.

Theorem B.0.4. *Let $\delta \ge 0$ be fixed, and $n > \frac{3}{2}\delta$. The number of idempotents of degree n and rank $r = n - \delta$ in $\mathcal{M}_n$ is a polynomial in n of degree at most δ.*

B.0.1 There are Finitely Many Spartans of given Rank-deficiency

Proposition B.0.5. *Let $\alpha \in \mathcal{M}_n$ be a convex idempotent with $n > 1$. Then the rank of α is at most $r \leq \frac{n}{3}$.*

Proof. Let $\alpha \in \mathcal{M}_n$ be a convex idempotent.

Every contribution to rank comes from an active or inert path by definition, and specifically an active path by idempotency, each of which has odd length (including a contribution of length one for bounding stubs). If some path has length 1 then either $n = 1$ or α is nonconvex, so all paths in α have length at least 3, witnessing the bounds. □

Proposition B.0.6. *Let $\varepsilon > 0$ and $a_n = \frac{n-r}{n}$ for $n \in \mathbb{N}_0$. Then there is $N \in \mathbb{N}$ such that for $n > N$, $a_n > 1 - \varepsilon$.*

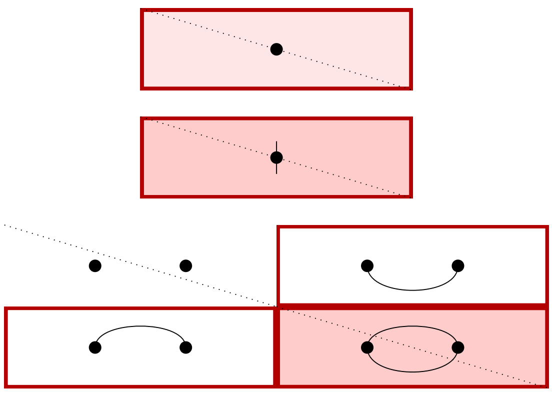
Corollary B.0.7. *The sequence a_n eventually exceeds $\frac{1}{3}$ forever.*

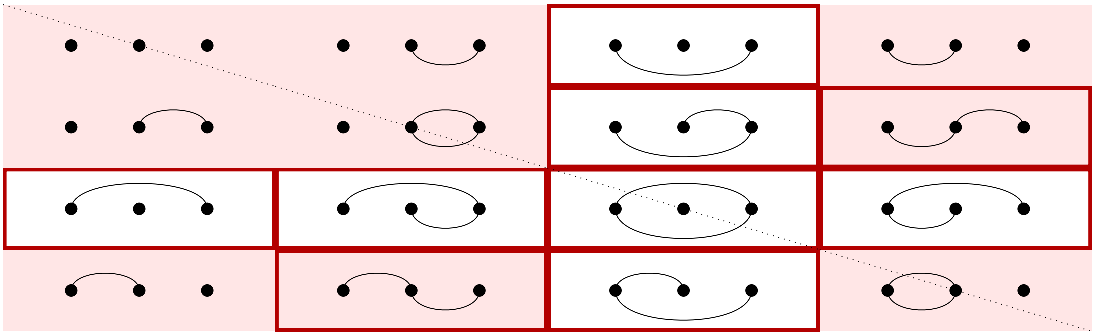
Appendix C

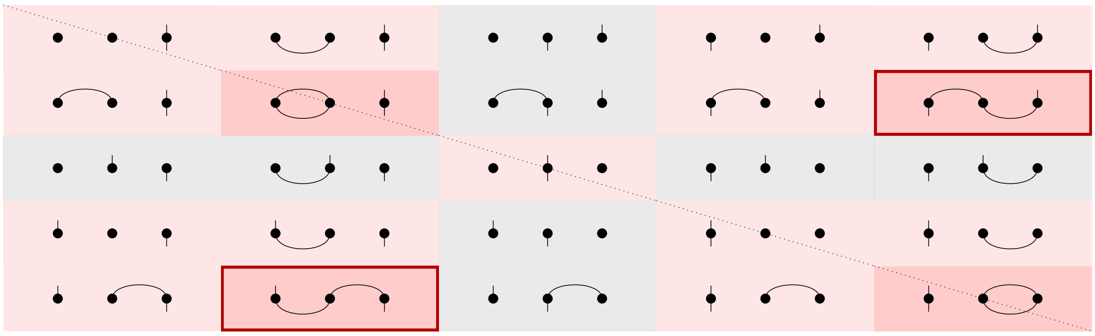
The Convex Idempotent Dictionary

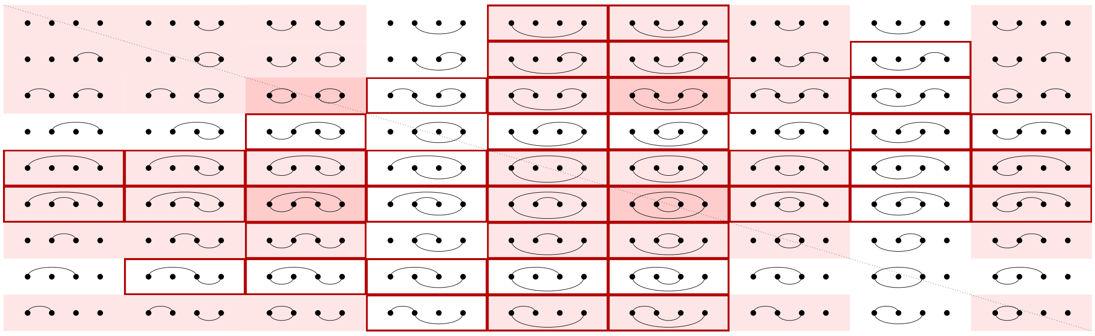
This section shows all the idempotents of degree $n \leq 5$ and rank at most one. Those in the partial Jones and Jones monoids $\mathcal{PJ}_n$ and $\mathcal{J}_n$ are shaded light and medium pink, convex idempotents are highlighted in a box and non-idempotents are greyed out. There is a diagonal line drawn across each $\mathscr{D}$-class; in each case those elements it meets are the projections.

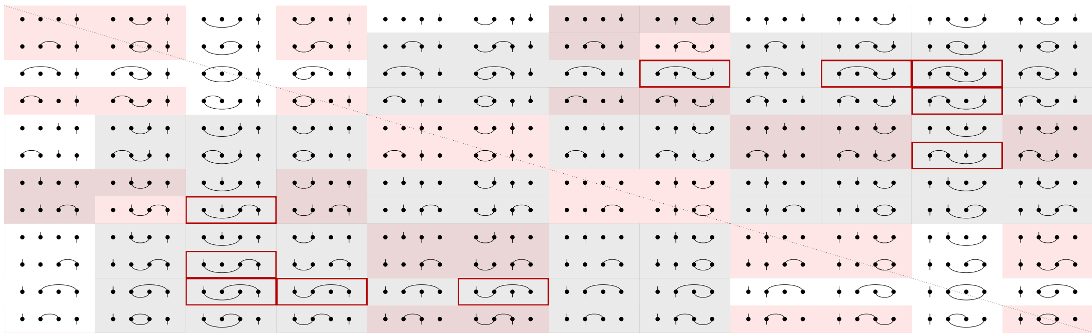
The classes are presented in ascending degree and rank. The $\mathscr{D}$-classes of low-rank elements in degree 5 are on a separate landscape page, as they are too wide to display easily in portrait.

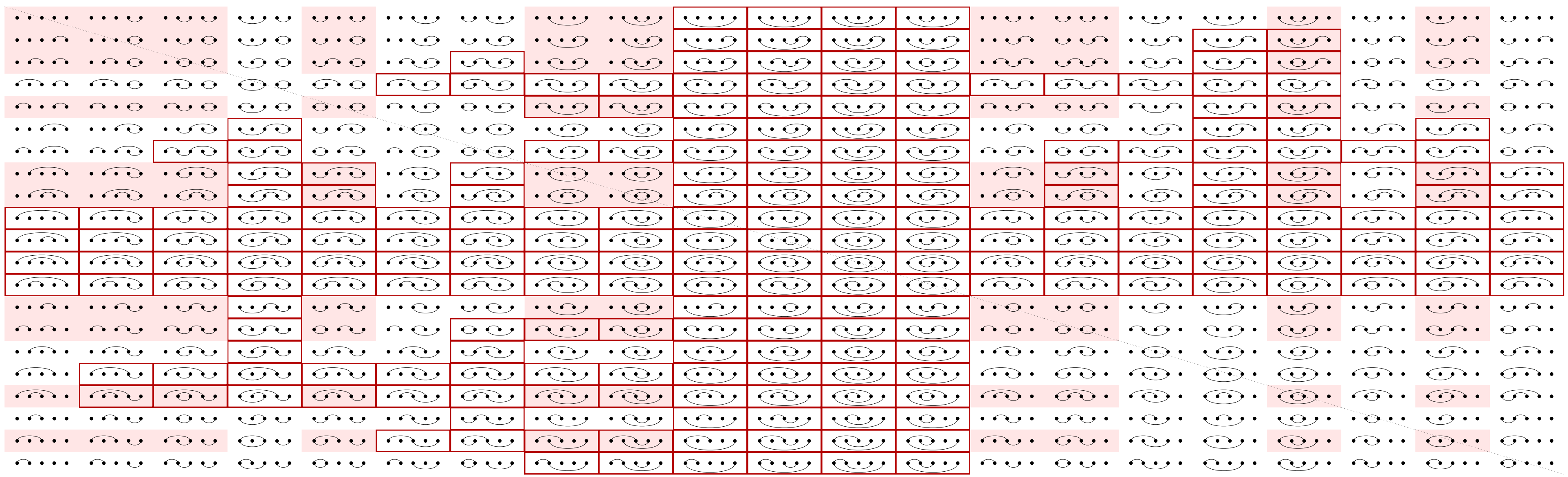


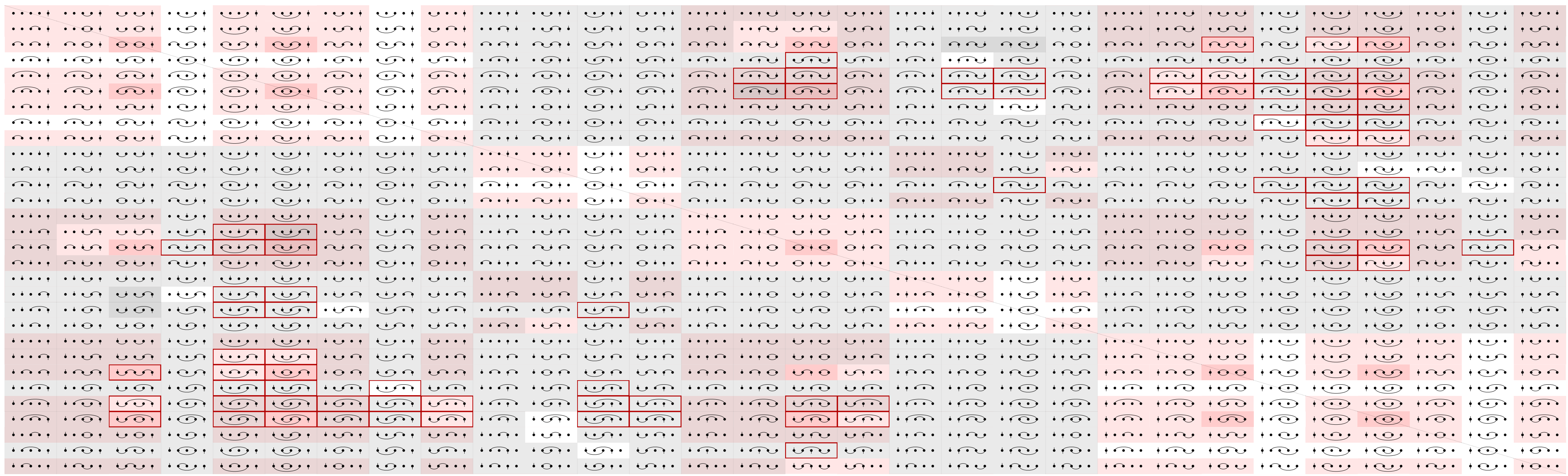












Appendix D

The Convex Generating Function for $\mathcal{M}_n$ is Low Degree

Included below is a printout of a variant of the universal generating function $U(\mathbf{x})$ described in Chapter 5. We use this as a reference to check some observations above.

We isolated the contibution from the convex part by evaluating $C(\mathbf{x})|_{u=0}$ where

$$C(\mathbf{x}) = \frac{U(\mathbf{x}) - 1}{u}.$$

The full generating function U would take some 20 pages to print out in a similar format, and can be recovered in degree up-to 9 by taking

$$U(\mathbf{x}) = \frac{1}{1 - uC(\mathbf{x})}.$$

The Taylor expansion of the right-hand side up to the ninth degree will agree with U for all terms with the power of z not exceeding 9. The convex part of the function is given below.

$$z^9x^2w^7v^8u + 2z^9xw^8v^8u + 6z^9yx^2w^6v^7u + 9z^9x^4w^5v^7u + 36z^9x^3w^6v^7u + 54z^9x^2w^7v^7u + 20z^9yx^4w^4v^6u + 52z^9yx^3w^5v^6u$$
$$+10z^9x^6w^3v^6u + 60z^9x^5w^4v^6u + 238z^9x^4w^5v^6u + 402z^9x^3w^6v^6u + 24z^9x^2w^6v^7u + 70z^9xw^7v^7u + 30z^9y^2x^4w^3v^5u + 6z^9yx^6w^2v^5u$$
$$+32z^9yx^5w^3v^5u + 164z^9yx^4w^4v^5u + 106z^9yx^2w^5v^6u + 8z^9yw^6v^7u + z^9x^8wv^5u + 8z^9x^7w^2v^5u + 76z^9x^6w^3v^5u + 398z^9x^5w^4v^5u$$
$$+980z^9x^4w^5v^5u + 64z^9x^4w^4v^6u + 486z^9x^3w^5v^6u + 1198z^9x^2w^6v^6u + 12z^9x^2w^5v^7u + 24z^9xw^6v^7u + 14z^9w^7v^7u + 6z^9y^2x^6wv^4u$$
$$+26z^9y^2x^5w^2v^4u + 16z^9yx^6w^2v^4u + 138z^9yx^5w^3v^4u + 84z^9yx^4w^3v^5u + 554z^9yx^3w^4v^5u + 64z^9yx^2w^4v^6u + 64z^9yxw^5v^6u + 8z^9x^7w^2v^4u$$
$$+140z^9x^6w^3v^4u + 12z^9x^6w^2v^5u + 772z^9x^5w^4v^4u + 226z^9x^5w^3v^5u + 1720z^9x^4w^4v^5u + 32z^9x^4w^3v^6u + 5154z^9x^3w^5v^5u + 128z^9x^3w^4v^6u$$
$$+494z^9x^2w^5v^6u + 1022z^9xw^6v^6u + 12z^8yx^2w^5v^6u + 6z^8x^2w^6v^6u + 2z^8w^7v^7u + 8z^9y^2x^6wv^3u + 62z^9y^2x^4w^2v^4u + 10z^9y^2x^2w^3v^5u$$

$$+44z^9yx^6w^2v^3u + 60z^9yx^5w^2v^4u + 698z^9yx^4w^3v^4u + 36z^9yx^4w^2v^5u + 116z^9yx^3w^3v^5u + 808z^9yx^2w^4v^5u + 118z^9yw^5v^6u + 12z^9x^7w^2v^3u$$
$$+4z^9x^7wv^4u + 200z^9x^6w^3v^3u + 100z^9x^6w^2v^4u + 6z^9x^6wv^5u + 1314z^9x^5w^3v^4u + 36z^9x^5w^2v^5u + 6696z^9x^4w^4v^4u + 462z^9x^4w^3v^5u$$
$$+2888z^9x^3w^4v^5u + 9064z^9x^2w^5v^5u + 112z^9x^2w^4v^6u + 362z^9xw^5v^6u + 234z^9w^6v^6u + 28z^8yx^4w^3v^5u + 76z^8yx^3w^4v^5u + 8z^8x^4w^4v^5u$$
$$+56z^8x^3w^5v^5u + 12z^8x^2w^5v^6u + 52z^8xw^6v^6u + z^8w^6v^7u + 32z^9y^2x^5wv^3u + 32z^9y^2x^4wv^4u + 12z^9y^2x^3w^2v^4u + 250z^9yx^5w^2v^3u$$
$$+224z^9yx^4w^2v^4u + 1726z^9yx^3w^3v^4u + 330z^9yx^2w^3v^5u + 680z^9yxw^4v^5u + 56z^9yw^4v^6u + 12z^9x^7w^2v^2u + 4z^9x^7wv^3u + 164z^9x^6w^2v^3u$$
$$+20z^9x^6wv^4u + 2570z^9x^5w^3v^3u + 290z^9x^5w^2v^4u + 3972z^9x^4w^3v^4u + 56z^9x^4w^2v^5u + 20202z^9x^3w^4v^4u + 634z^9x^3w^3v^5u + 3254z^9x^2w^4v^5u$$
$$+28z^9x^2w^3v^6u + 6358z^9xw^5v^5u + 56z^9xw^4v^6u + 82z^9w^5v^6u + 4z^8y^2x^4w^2v^4u + 4z^8yx^6wv^4u + 22z^8yx^5w^2v^4u + 140z^8yx^4w^3v^4u$$
$$+106z^8yx^2w^4v^5u + 14z^8x^5w^3v^4u + 128z^8x^4w^4v^4u + 6z^8x^4w^3v^5u + 84z^8x^3w^4v^5u + 378z^8x^2w^5v^5u + 6z^8x^2w^4v^6u + 12z^8xw^5v^6u$$
$$+48z^8w^6v^6u + z^7x^2w^5v^6u + 2z^7xw^6v^6u + 108z^9y^2x^4wv^3u + 30z^9y^2x^2w^2v^4u + 886z^9yx^4w^2v^3u + 338z^9yx^3w^2v^4u + 2440z^9yx^2w^3v^4u$$
$$+84z^9yx^2w^2v^5u + 112z^9yxw^3v^5u + 682z^9yw^4v^5u + 4z^9x^7wv^2u + 240z^9x^6w^2v^2u + 30z^9x^6wv^3u + 990z^9x^5w^2v^3u + 24z^9x^5wv^4u$$
$$+12128z^9x^4w^3v^3u + 650z^9x^4w^2v^4u + 14z^9x^4wv^5u + 6986z^9x^3w^3v^4u + 56z^9x^3w^2v^5u + 27790z^9x^2w^4v^4u + 670z^9x^2w^3v^5u + 2296z^9xw^4v^5u$$
$$+1444z^9w^5v^5u + 4z^8yx^6wv^3u + 54z^8yx^5w^2v^3u + 40z^8yx^4w^2v^4u + 416z^8yx^3w^3v^4u + 52z^8yx^2w^3v^5u + 12z^8yxw^4v^5u + 4z^8x^6w^2v^3u$$
$$+68z^8x^5w^3v^3u + 8z^8x^5w^2v^4u + 92z^8x^4w^3v^4u + 3z^8x^4w^2v^5u + 932z^8x^3w^4v^4u + 12z^8x^3w^3v^5u + 150z^8x^2w^4v^5u + 744z^8xw^5v^5u$$
$$+36z^8w^5v^6u + 4z^7yx^2w^4v^5u + 4z^7x^4w^3v^5u + 16z^7x^3w^4v^5u + 28z^7x^2w^5v^5u + 12z^9y^2x^3wv^3u + 10z^9y^2x^2wv^4u + 4z^9y^2xw^2v^4u$$
$$+12z^9yx^4wv^3u + 1850z^9yx^3w^2v^3u + 792z^9yx^2w^2v^4u + 2088z^9yxw^3v^4u + 346z^9yw^3v^5u + 4z^9x^7wvu + 30z^9x^6wv^2u + 1616z^9x^5w^2v^2u$$
$$+108z^9x^5wv^3u + 3008z^9x^4w^2v^3u + 92z^9x^4wv^4u + 28342z^9x^3w^3v^3u + 962z^9x^3w^2v^4u + 8400z^9x^2w^3v^4u + 84z^9x^2w^2v^5u + 17476z^9xw^4v^4u$$
$$+354z^9xw^3v^5u + 606z^9w^4v^5u + 4z^8y^2x^4wv^3u + 8z^8y^2x^2w^2v^4u + 4z^8yx^6wv^2u + 16z^8yx^5wv^3u + 288z^8yx^4w^2v^3u + 20z^8yx^4wv^4u$$
$$+68z^8yx^3w^2v^4u + 452z^8yx^2w^3v^4u + 6z^8yw^4v^5u + 8z^8x^6w^2v^2u + 16z^8x^5w^2v^3u + 672z^8x^4w^3v^3u + 16z^8x^4w^2v^4u + 366z^8x^3w^3v^4u$$
$$+2702z^8x^2w^4v^4u + 48z^8x^2w^3v^5u + 292z^8xw^4v^5u + 410z^8w^5v^5u + 9z^8w^4v^6u + 4z^7yx^4w^2v^4u + 12z^7yx^3w^3v^4u + z^7x^6wv^4u$$
$$+6z^7x^5w^2v^4u + 36z^7x^4w^3v^4u + 86z^7x^3w^4v^4u + 12z^7x^2w^4v^5u + 40z^7xw^5v^5u + 44z^9y^2x^2wv^3u + 4z^9y^2w^2v^4u + 64z^9yx^4wv^2u$$
$$+72z^9yx^3wv^3u + 2676z^9yx^2w^2v^3u + 36z^9yx^2wv^4u + 484z^9yxw^2v^4u + 1770z^9yw^3v^4u + 84z^9yw^2v^5u + 30z^9x^6wvu + 204z^9x^5wv^2u$$
$$+5840z^9x^4w^2v^2u + 288z^9x^4wv^3u + 5776z^9x^3w^2v^3u + 56z^9x^3wv^4u + 34476z^9x^2w^3v^3u + 1204z^9x^2w^2v^4u + 14z^9x^2wv^5u + 5872z^9xw^3v^4u$$
$$+28z^9xw^2v^5u + 3860z^9w^4v^4u + 102z^9w^3v^5u + 44z^8yx^5wv^2u + 92z^8yx^4wv^3u + 648z^8yx^3w^2v^3u + 146z^8yx^2w^2v^4u + 68z^8yxw^3v^4u$$
$$+120z^8x^5w^2v^2u + 146z^8x^4w^2v^3u + 2640z^8x^3w^3v^3u + 64z^8x^3w^2v^4u + 768z^8x^2w^3v^4u + 12z^8x^2w^2v^5u + 3376z^8xw^4v^4u + 24z^8xw^3v^5u$$
$$+286z^8w^4v^5u + 6z^7y^2x^4wv^3u + 14z^7yx^4w^2v^3u + 34z^7yx^2w^3v^4u + 6z^7yw^4v^5u + 6z^7x^5w^2v^3u + 54z^7x^4w^3v^3u + 8z^7x^4w^2v^4u$$
$$+90z^7x^3w^3v^4u + 314z^7x^2w^4v^4u + 6z^7x^2w^3v^5u + 12z^7xw^4v^5u + 10z^7w^5v^5u + 4z^9y^2xwv^3u + 312z^9yx^3wv^2u + 312z^9yx^2wv^3u$$
$$+2388z^9yxw^2v^3u + 912z^9yw^2v^4u + 168z^9x^5wvu + 558z^9x^4wv^2u + 12408z^9x^3w^2v^2u + 444z^9x^3wv^3u + 7272z^9x^2w^2v^3u + 140z^9x^2wv^4u$$
$$+20896z^9xw^3v^3u + 810z^9xw^2v^4u + 1598z^9w^3v^4u + 4z^8y^2x^2wv^3u + 8z^8y^2w^2v^4u + 152z^8yx^4wv^2u + 94z^8yx^3wv^3u + 654z^8yx^2w^2v^3u$$
$$+36z^8yx^2wv^4u + 16z^8yxw^2v^4u + 40z^8yw^3v^4u + 588z^8x^4w^2v^2u + 508z^8x^3w^2v^3u + 5536z^8x^2w^3v^3u + 114z^8x^2w^2v^4u + 1264z^8xw^3v^4u$$
$$+1464z^8w^4v^4u + 90z^8w^3v^5u + 44z^7yx^3w^2v^3u + 16z^7yx^2w^2v^4u + 16z^7yxw^3v^4u + 8z^7x^5w^2v^2u + 4z^7x^5wv^3u + 68z^7x^4w^2v^3u$$
$$+4z^7x^4wv^4u + 448z^7x^3w^3v^3u + 16z^7x^3w^2v^4u + 108z^7x^2w^3v^4u + 308z^7xw^4v^4u + 8z^6yx^2w^3v^4u + 4z^6x^2w^4v^4u + 2z^6w^5v^5u$$
$$+2z^9y^3v^3u + 6z^9y^2wv^3u + 700z^9yx^2wv^2u + 260z^9yxwv^3u + 2206z^9yw^2v^3u + 168z^9ywv^4u + 496z^9x^4wvu + 1220z^9x^3wv^2u$$
$$+14972z^9x^2w^2v^2u + 640z^9x^2wv^3u + 5306z^9xw^2v^3u + 56z^9xwv^4u + 4598z^9w^3v^3u + 270z^9w^2v^4u + 308z^8yx^3wv^2u + 224z^8yx^2wv^3u$$
$$+146z^8yxw^2v^3u + 8z^8yw^2v^4u + 1712z^8x^3w^2v^2u + 1084z^8x^2w^2v^3u + 5868z^8xw^3v^3u + 168z^8xw^2v^4u + 894z^8w^3v^4u + 15z^8w^2v^5u$$
$$+2z^7y^2x^2wv^3u + 96z^7yx^2w^2v^3u + 42z^7yw^3v^4u + 4z^7x^5wv^2u + 104z^7x^4w^2v^2u + 14z^7x^4wv^3u + 154z^7x^3w^2v^3u + 998z^7x^2w^3v^3u$$
$$+20z^7x^2w^2v^4u + 86z^7xw^3v^4u + 82z^7w^4v^4u + 4z^6yx^4wv^3u + 14z^6yx^3w^2v^3u + 12z^6x^3w^3v^3u + 4z^6x^2w^3v^4u + 26z^6xw^4v^4u$$
$$+z^6w^4v^5u + 776z^9yxwv^2u + 892z^9ywv^3u + 28z^9yv^4u + 1104z^9x^3wvu + 1510z^9x^2wv^2u + 9656z^9xw^2v^2u + 466z^9xwv^3u$$
$$+1500z^9w^2v^3u + 28z^9wv^4u + 16z^8y^2wv^3u + 364z^8yx^2wv^2u + 16z^8yxwv^3u + 76z^8yw^2v^3u + 32z^8x^3wv^2u + 3356z^8x^2w^2v^2u$$
$$+30z^8x^2wv^3u + 1656z^8xw^2v^3u + 2336z^8w^3v^3u + 252z^8w^2v^4u + 16z^7yx^3wv^2u + 12z^7yx^2wv^3u + 72z^7yxw^2v^3u + 20z^7yw^2v^4u$$
$$+4z^7x^5wvu + 22z^7x^4wv^2u + 484z^7x^3w^2v^2u + 16z^7x^3wv^3u + 236z^7x^2w^2v^3u + 5z^7x^2wv^4u + 856z^7xw^3v^3u + 10z^7xw^2v^4u$$
$$+26z^7w^3v^4u + 4z^6yx^4wv^2u + 24z^6yx^2w^2v^3u + 6z^6x^4w^2v^2u + 8z^6x^3w^2v^3u + 68z^6x^2w^3v^3u + 2z^6x^2w^2v^4u + 4z^6xw^3v^4u$$
$$+24z^6w^4v^4u + z^5x^2w^3v^4u + 2z^5xw^4v^4u + 1240z^9ywv^2u + 186z^9yv^3u + 1368z^9x^2wvu + 1296z^9xwv^2u + 2232z^9w^2v^2u$$
$$+186z^9wv^3u + 8z^8y^2v^3u + 72z^8yxwv^2u + 16z^8ywv^3u + 80z^8x^3wvu + 224z^8x^2wv^2u + 3720z^8xw^2v^2u + 120z^8xwv^3u$$
$$+1118z^8w^2v^3u + 40z^8wv^4u + 52z^7yx^2wv^2u + 116z^7yw^2v^3u + 22z^7x^4wvu + 76z^7x^3wv^2u + 912z^7x^2w^2v^2u + 36z^7x^2wv^3u$$

$$+210z^7xw^2v^3u+228z^7w^3v^3u+20z^6yx^3wv^2u+12z^6yx^2wv^3u+4z^6yxw^2v^3u+40z^6x^3w^2v^2u+16z^6x^2w^2v^3u+156z^6xw^3v^3u$$

$$+16z^6w^3v^4u+2z^5yx^2w^2v^3u+z^5x^4wv^3u+4z^5x^3w^2v^3u+10z^5x^2w^3v^3u+408z^9yv^2u+1104z^9xwvu+408z^9wv^2u$$

$$+24z^8y^2v^2u+48z^8ywv^2u+352z^8x^2wvu+544z^8xwv^2u+1580z^8w^2v^2u+240z^8wv^3u+5z^8v^4u+40z^7yxwv^2u$$

$$+40z^7ywv^3u+88z^7x^3wvu+116z^7x^2wv^2u+768z^7xw^2v^2u+20z^7xwv^3u+70z^7w^2v^3u+40z^6yx^2wv^2u+2z^6yw^2v^3u$$

$$+136z^6x^2w^2v^2u+40z^6xw^2v^3u+96z^6w^3v^3u+4z^6w^2v^4u+4z^5x^3w^2v^2u+4z^5x^2w^2v^3u+18z^5xw^3v^3u+262z^9yvu$$

$$+262z^9wvu+624z^8xwvu+512z^8wv^2u+30z^8v^3u+112z^7ywv^2u+10z^7yv^3u+150z^7x^2wvu+116z^7xwv^2u$$

$$+208z^7w^2v^2u+10z^7wv^3u+4z^6yxwv^2u+6z^6x^2wv^2u+228z^6xw^2v^2u+50z^6w^2v^3u+4z^5yx^2wv^2u+4z^5yw^2v^3u$$

$$+4z^5x^3wv^2u+36z^5x^2w^2v^2u+2z^5x^2wv^3u+4z^5xw^2v^3u+6z^5w^3v^3u+336z^8wvu+64z^8v^2u+44z^7yv^2u$$

$$+156z^7xwvu+44z^7wv^2u+2z^6y^2v^2u+4z^6ywv^2u+24z^6x^2wvu+32z^6xwv^2u+128z^6w^2v^2u+12z^6wv^3u$$

$$+4z^5x^3wvu+8z^5x^2wv^2u+48z^5xw^2v^2u+4z^4yx^2wv^2u+2z^4x^2w^2v^2u+2z^4w^3v^3u+42z^8vu+42z^7yvu$$

$$+42z^7wvu+72z^6xwvu+48z^6wv^2u+2z^6v^3u+8z^5ywv^2u+14z^5x^2wvu+8z^5xwv^2u+16z^5w^2v^2u$$

$$+8z^4xw^2v^2u+z^4w^2v^3u+48z^6wvu+8z^6v^2u+4z^5yv^2u+24z^5xwvu+4z^5wv^2u+8z^4w^2v^2u$$

$$+z^3x^2wv^2u+2z^3xw^2v^2u+8z^6vu+8z^5yvu+8z^5wvu+8z^4xwvu+4z^4wv^2u+8z^4wvu$$

$$+z^4v^2u+4z^3xwvu+2z^4vu+2z^3yvu+2z^3wvu+2z^2wvu+z^2vu+zyvu$$

$$+zwvu$$

Appendix E

Tables of results

Jones Monoid $\mathcal{J}_n$

$d\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	1	1	1	1	1	1	1	1	1	1	1
2			1	4	7	10	13	16	19	22	25	28	31
4					4	25	57	98	148	207	275	352	438
6							25	196	522	1 006	1 673	2 550	3 664
8									196	1 764	5 206	10 837	19 261
10											1 764	17 424	55 319
12													17 424
C_n	1	1	2	5	14	42	132	429	1 430	4 862	16 796	58 786	208 012
e_n	1	1	2	5	12	36	96	300	886	3 000	8 944	31 192	96 138
e_n/C_n	1	1	1	1	.857	.727	.725	.620	.617	.533	.531	.462	.461
d_n^*	1	1	1	4	7	25	57	196	522	1 764	5 206	17 424	55 319
d_n^*/e_n	1	1	.5	.8	.583	.694	.594	.630	.589	.588	.582	.559	.575

Table E.1: Numbers of idempotents in $\mathcal{J}_n$ stratified by rank, for degree $n < 13$; relative contributions of idempotents in the whole monoid and largest $\mathscr{D}$-class among the idempotents.

n	13	14	15	16	17
C_n	742 900	2674 440	35 357 670	129 644 790	477 638 700
e_n	342 562	1083 028	3923 351	12 656 024	46 455 770
e_n/C_n	.461	.405	.405	.358	358
d_n^*	184 041		2044 490		23 639 044
d_n^*/e_n	.537		.521		.508
n	18	19	20	21	22
C_n	1767 263 190	1767 263 190	6564 120 420	24 466 267 020	91 482 563 640
e_n	152 325 850	565 212 506	1878 551 444	7033 866 580	23 645 970 022
e_n/C_n	.319	.320	.286	.287	.258
d_n^*		282 105 616		3455 793 796	
d_n^*/e_n		.499		.491	
n	23	24	25	26	
C_n	343 059 613 650	1289 904 147 324	4861 946 401 452	18 367 353 072 152	
e_n	89 222 991 344	302 879 546 290	1150 480 017 950		
e_n/C_n	.260	.235	.237		
d_n^*	43 268 992 144		551 900 410 000		
d_n^*/e_n	.485		.480		

Table E.2: Table E.1 for degrees 13–26 with available data shown; no stratification by rank.

Motzkin Monoids $\mathcal{M}_n$

$r\backslash n$	0	1	2	3	4	5	6	7	8	9
0	1	1	4	16	81	441	2601	16 129	104 329	697 225
1		1	2	11	48	266	1492	9 042	56 712	369 689
2			1	3	19	93	549	3 211	20 004	127 676
3				1	4	28	152	947	5 784	37 048
4					1	5	38	226	1480	9 432
5						1	6	49	316	2 169
6							1	7	61	423
7								1	8	74
8									1	9
9										1
m_{2n}	1	2	9	51	323	2 188	15 511	113 634	853 467	6536 329
e_n	1	2	7	31	153	834	4 839	29 612	188 695	1243 746
e_n/m_{2n}	1	1	.778	.608	.474	.381	.312	.261	.221	.190
d_n^*	1	1	4	16	81	441	2601	16 129	104 329	697 225
e_n/m_{2n}	1	.5	.571	.516	.529	.529	.538	.545	.552	.561

Table E.3: Table E.1 for Motzkin monoids $\mathcal{M}_n$ in degree and rank up to 9.

List of Notation

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