Interpolation problems, the symmetrized bidisc and the tetrablock

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Abstract

The spectral Nevanlinna-Pick interpolation problem is to find, if it is possible, an analytic function $f : \mathbb{D} \to \mathbb{C}^{k\times k}$ from the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to the space $\mathbb{C}^{k\times k}$ of $k \times k$ complex matrices, which interpolates a finite number of distinct points in $\mathbb{D}$ to the target matrices in $\mathbb{C}^{k\times k}$ subject to the spectral radius $r(f(\lambda)) \leq 1$, for every $\lambda \in \mathbb{D}$. For $k = 2$, this problem is connected to interpolation problem in $\mathrm{Hol}(\mathbb{D}, \Gamma)$, where $\mathrm{Hol}(\mathbb{D}, \Gamma)$ denotes the space of analytic functions from $\mathbb{D}$ to the closed symmetrized bidisc

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : z_1, z_2 \in \overline{\mathbb{D}}\} \subset \mathbb{C}^2.$$

In this thesis, we consider a special case of the three-point spectral Nevanlinna-Pick problem and give necessary and sufficient conditions for its solvability.

We also study interpolation problems from $\mathbb{D}$ to the tetrablock. The closed tetrablock is defined to be

$$\overline{\mathbb{E}} = \{x \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 zw \neq 0 \text{ for all } z, w \in \mathbb{D}\}.$$

Given n distinct points $\lambda_1, \cdots, \lambda_n$ in $\mathbb{D}$ and n points $x^1, \cdots, x^n$ in $\overline{\mathbb{E}}$, find, if is possible, an analytic function

$$\varphi : \mathbb{D} \to \overline{\mathbb{E}} \text{ such that } \varphi(\lambda_j) = x^j \text{ for } j = 1, \cdots, n.$$

This problem is closely connected to the μ_{Diag}-synthesis interpolation problem. For given data $\lambda_j \to W_j,\ 1 \leq j \leq n$, where λ_j are distinct points in $\mathbb{D}$ and W_j are complex 2×2 matrices, find, if it is possible, an analytic matrix function

$$F : \mathbb{D} \to \mathbb{C}^{2\times 2}$$

such that $F(\lambda_j) = W_j,\ 1 \leq j \leq n$, and $\mu_{\mathrm{Diag}}(F(\lambda)) \leq 1$ for all $\lambda \in \mathbb{D}$. We give criteria for the solvability of such interpolation problems. Here Diag is the space of 2×2 diagonal matrices, and for $A \in \mathbb{C}^{2\times 2}$,

$$\mu_{\mathrm{Diag}}(A) := \frac{1}{\inf\{\|X\| : X \in \mathrm{Diag},\ 1 - AX \text{ is singular}\}}.$$

If $1 - AX$ is non-singular for all $X \in \mathrm{Diag}$, then $\mu_{\mathrm{Diag}}(A) = 0$.

In addition, we give a realization theorem for analytic functions from the disc to the tetrablock.

Declaration on collaborative work

My thesis contains collaborative work with my supervisors Dr. Z. A. Lykova and Prof. N. J. Young. We have one joint paper [24]. The main problems and ideas how to solve these problems were provided to me by Lykova and Young. We have had weekly meetings to discuss mathematics, methods, new ideas and research papers related to my thesis. We have done research together.

The rest of each week I have worked independently on my thesis. I did calculations which were required in each step of proofs, searched for research material related to our research project, organised all research material in thesis. I have given several talks on topics of my thesis to Young Functional Analysts Workshops in Belfast, Glasgow and Newcastle and to Pure PhD workshops in Newcastle.

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Contents

Chapter 1

Introduction

The original Pick interpolation problem (1916) is to determine whether there exists an analytic function ϕ from the open unit disc $\mathbb{D}$ to the closed unit disc $\overline{\mathbb{D}}$ which satisfies some given interpolation conditions. The *spectral* Nevanlinna-Pick problem is the following. Given distinct points $\lambda_1, \cdots, \lambda_n$ in $\mathbb{D}$ and $k \times k$ complex matrices $W_1, \cdots, W_n$, find if possible an analytic $k \times k$ matrix-valued function $F : \mathbb{D} \to \mathbb{C}^{k \times k}$ such that

$$F(\lambda_j) = W_j \text{ for } j = 1, \cdots, n$$

and

$$r(F(\lambda)) \leq 1 \quad \text{for all } \lambda \in \mathbb{D},$$

where

$$r(W) := \sup\{|\lambda| : \lambda \quad \text{is an eigenvalue of } W\}$$

denotes the spectral radius of the matrix W.
In the case $k = 2$, J. Agler and N. J. Young [9] showed that the spectral interpolation problem is equivalent to the interpolation problem from $\mathbb{D}$ to the closed symmetrized bidisc

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : z_1, z_2 \in \overline{\mathbb{D}}\} :$$

for given n distinct points $\lambda_1, \cdots, \lambda_n$ in $\mathbb{D}$ and n points $z_1, \cdots, z_n$ in Γ, find, if it is possible, an analytic function

$$h : \mathbb{D} \to \Gamma \text{ such that } h(\lambda_j) = z_j \text{ for } j = 1, \cdots, n.$$

We give a criterion for solvability of a special three-point Γ-interpolation problem (Theorem 2.2.10).

The set Γ is a special μ-synthesis domain. In [17] John Doyle introduced the μ-synthesis problem involving the structured singular value $\mu(A)$ of a matrix A. The μ-synthesis problem is an interpolation problem for analytic matrix functions subject to structured uncertainty. The motivation came from the robust stabilization theory. The following definition of $\mu(\cdot)$ is given in [34, 28]. For $F \in \mathbb{C}^{m\times n}$ and any subspace Δ of $\mathbb{C}^{n\times m}$

$$\mu_\Delta(F) := \frac{1}{\inf\{\|X\| : X \in \Delta,\ 1 - FX \text{ is singular}\}}. \tag{1.0.1}$$

If $1 - FX$ is nonsingular for all $X \in \Delta$, then $\mu_\Delta(F) = 0$. Here $\|X\|$ is the operator norm of the matrix X. Two special cases of μ are the matrix norm $\|\cdot\|$ and the spectral radius r of a matrix F. Mathematically, the μ-synthesis interpolation problem is to find an analytic matrix function F on $\mathbb{D}$ which satisfies a finite number of interpolation conditions subject to $\mu(F(\lambda)) \leq 1$, for all $\lambda \in \mathbb{D}$.

Another case of μ-synthesis problem we consider here is $\mu = \mu_{\text{Diag}}$. Diag denotes the space of 2×2 diagonal matrices

$$\text{Diag} \stackrel{\text{def}}{=} \{\text{diag}(z, w) : z, w \in \mathbb{C}\}. \tag{1.0.2}$$

For $A \in \mathbb{C}^{2\times 2}$,

$$\mu_{\text{Diag}}(A) := \frac{1}{\inf\{\|X\| : X \in \text{Diag},\ 1 - AX \text{ is singular}\}}. \tag{1.0.3}$$

If $1 - AX$ is non-singular for all $X \in \text{Diag}$ then $\mu_{\text{Diag}}(A) = 0$.

The μ_{Diag}-synthesis interpolation problem was introduced by Abouhajar, White and Young in [1]. For given data $\lambda_j \to W_j,\ \ 1 \leq j \leq n$, where λ_j are distinct points in $\mathbb{D}$ and W_j are complex 2×2 matrices, find if possible, an analytic 2×2 matrix function $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ such that $F(\lambda_j) = W_j,\ \ 1 \leq j \leq n$, and $\mu_{\text{Diag}}(F(\lambda)) \leq 1$ for all $\lambda \in \mathbb{D}$. They constructed the domain

$$\mathbb{E} = \{x \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 zw \neq 0 \text{ for all } z, w \in \overline{\mathbb{D}}\}$$

called the tetrablock which has proven to have rich geometry and function theory. It was proved in [1] that an interpolation problem in $\text{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ (or an $\mathbb{E}$-interpolation problem) is equivalent to the μ_{Diag}-synthesis problem for 2×2 matrix functions. The symbol $\text{Hol}(\mathbb{D}, \Omega)$ is used throughout the text to denote the space of analytic functions $\psi : \mathbb{D} \to \Omega$. We denote by $\mathcal{S}^{2\times 2}$ the space of analytic 2×2 matrix functions $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ such that $\|F(\lambda)\| \leq 1$ for all $\lambda \in \mathbb{D}$. We study an $\mathbb{E}$-interpolation problem in this thesis. We use relations between $\mathcal{S}^{2\times 2}$ and $\mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$, given in [16, Theorem 7.1] to prove the following result (Theorem 3.3.2). For the given $\mathbb{E}$-interpolation data

$$\lambda_j \to x^j, \quad 1 \le j \le n,$$

where λ_j are distinct points in $\mathbb{D}$ and $x^j = (x_1^j,\ x_2^j,\ x_3^j)$ are points in $\mathbb{E}$, the existence of a solution of the $\mathbb{E}$-interpolation problem is equivalent to the existence of a solution of the Nevanlinna-Pick interpolation problem with data

$$\lambda_j \mapsto \begin{bmatrix} x_1^j & b_j \\ c_j & x_2^j \end{bmatrix}, \qquad 1 \le j \le n,$$

for some constants $b_j, c_j \in \mathbb{C}$ satisfying

$$b_j c_j = x_1^j x_2^j - x_3^j, \ \ 1 \le j \le n.$$

We show connections between the solution of a μ_{Diag}-synthesis problem and the Pick condition for the *solvability of* a family of matricial Nevanlinna-Pick interpolation problems.

1.1 Main results

We consider the following special case of the three-point spectral Nevanlinna-Pick Problem: Given the data

$$\begin{cases} \lambda_1 \to W_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \lambda_2 \to W_2 = \begin{bmatrix} -\alpha & 0 \\ 0 & -\alpha \end{bmatrix} \\ \lambda_3 \to W_3 \end{cases} \tag{1.1.1}$$

where distinct points $\lambda_1 = 0, \lambda_2, \lambda_3 \in \mathbb{D}$, $\alpha \in \mathbb{D} \setminus \{0\}$ and $W_3 \in \mathbb{C}^{2\times 2}$ has distinct eigenvalues and spectral radius $r(W_3) \le 1$, $\operatorname{tr} W_3 = s$ and $\det W_3 = p$; find if possible an analytic 2×2 matrix function F such that

$$F(\lambda_j) = W_j, \qquad j = 1, 2, 3,$$

and

$$r(F(\lambda)) \le 1 \ \text{ for all } \lambda \in \mathbb{D}.$$

The pseudo-hyperbolic distance between two points $\alpha, \lambda \in \mathbb{D}$ is defined by

$$\rho(\alpha,\ \lambda) = \left| \frac{\lambda - \alpha}{1 - \overline{\lambda}\alpha} \right|.$$

Theorem 2.2.10. *The spectral interpolation Problem* (1.1.1) *is solvable if and only if there exist $b_3, c_3 \in \mathbb{C}$ such that the quantities k_1, k_2, k_3, k_4 defined by*

$$\begin{aligned}
k_1 &= \rho(\lambda_2, \lambda_3)^2 \left| 1 + \frac{\alpha \overline{s}}{2\lambda_2 \overline{\lambda_3}} \right|^2 - \left| \frac{s}{2\lambda_3} + \frac{\alpha}{\lambda_2} \right|^2, \\
k_2 &= \rho(\lambda_2, \lambda_3)^2 \left| \frac{\alpha}{\lambda_2 \overline{\lambda_3}} \right|^2 - \left| \frac{1}{\lambda_3} \right|^2, \\
k_3 &= \rho(\lambda_2, \lambda_3)^2 \frac{\overline{\alpha}}{\overline{\lambda_2} \lambda_3} - \frac{\overline{\alpha}}{\overline{\lambda_2} \lambda_3} + \left(\frac{1}{2} \rho(\lambda_2, \lambda_3)^2 \left| \frac{\alpha}{\lambda_2 \overline{\lambda_3}} \right|^2 - \frac{1}{2} \left| \frac{1}{\lambda_3} \right|^2 \right) \overline{s}, \\
k_4 &= \frac{1}{4} s^2 - p,
\end{aligned}$$

satisfy

$$\begin{cases}
-\frac{k_2}{k_1} |k_4|^2 \leq |b_3|^2 \leq -\frac{k_1}{k_2} \\
-\frac{k_2}{k_1} |k_4|^2 \leq |c_3|^2 \leq -\frac{k_1}{k_2} \\
(k_1 k_2 - |k_3|^2)(|b_3|^2 + |c_3|^2) + k_1^2 + k_2^2 |k_4|^2 - 2Re(k_3^2 k_4) \geq 0 \\
b_3\ c_3 = k_4 \\
k_1 > 0 \\
k_2 < 0.
\end{cases}$$

Theorem 3.3.2 *Let $\lambda_1, \cdots, \lambda_n$ be distinct points in $\mathbb{D}$ and let $x^j = (x_1^j, x_2^j, x_3^j) \in \mathbb{E}$ for $j = 1, \cdots, n$. The following statements are equivalent.*

(1) There exists an analytic function $\varphi : \mathbb{D} \to \overline{\mathbb{E}}$ such that

$$\varphi(\lambda_j) = (x_1^j, x_2^j, x_3^j), \quad 1 \leq j \leq n;$$

(2) *There exist* $b_j, c_j \in \mathbb{C}$ *such that*

$$b_j c_j = x_1^j x_2^j - x_3^j, \qquad 1 \le j \le n,$$

and the Nevanlinna-Pick interpolation problem with data

$$\lambda_j \mapsto \begin{bmatrix} x_1^j & b_j \\ c_j & x_2^j \end{bmatrix}, \quad 1 \le j \le n,$$

is solvable.

Theorem 3.3.4 *Let* $\lambda_1, \cdots, \lambda_n$ *be distinct points in* $\mathbb{D}$ *and let* $W_j = \left(w_{ik}^j\right)_{i,k=1}^2$, $1 \le j \le n$, *be* 2×2 *matrices, such that* $w_{11}^j w_{22}^j \ne \det W_j$, $1 \le j \le n$. *The following two statements are equivalent:*

(1) *there exists an analytic* 2×2 *matrix function* F *on* $\mathbb{D}$ *such that* $F(\lambda_j) = W_j$, $1 \le j \le n$, *and* $\mu_{Diag}(F(\lambda)) \le 1$ *for all* $\lambda \in \mathbb{D}$;

(2) *there exist* $b_1, \cdots, b_n,\ c_1, \cdots, c_n \in \mathbb{C}$ *such that*

$$\left[\frac{I - \begin{bmatrix} w_{11}^i & b_i \\ c_i & w_{22}^i \end{bmatrix}^* \begin{bmatrix} w_{11}^j & b_j \\ c_j & w_{22}^j \end{bmatrix}}{1 - \overline{\lambda_i}\lambda_j} \right]_{i,j=1}^n \ge 0 \qquad (1.1.2)$$

where

$$b_j c_j = w_{11}^j w_{22}^j - \det W_j, \ \ 1 \le j \le n.$$

See Appendix B.2.1 for more details on inequality (1.1.2).

To state the realization formula for tetrablock, we use standard engineering notations. Let H, U and Y be Hilbert spaces and let

$$A : H \to H, \qquad B : U \to H,$$

$$C : H \to Y, \qquad D : U \to Y$$

be bounded linear operators. Then for any $z \in \mathbb{D}$, we define the operator-valued function

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](z) = D + Cz(1 - zA)^{-1}B : H \oplus U \to H \oplus Y$$

whenever $1 - Az$ is invertible.

Theorem 3.4.2 *A function*

$$x = (x_1,\ x_2,\ x_3) : \mathbb{D} \to \mathbb{C}^3$$

maps $\mathbb{D}$ *analytically into* $\overline{\mathbb{E}}$ *if and only if there exist a Hilbert space* H *and a unitary operator*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : H \oplus \mathbb{C}^2 \to H \oplus \mathbb{C}^2$$

such that

$$x_1 = \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right], \qquad x_2 = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right] \text{ and } x_3 = \det \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

where

$$B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} : \mathbb{C}^2 \to H, \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} : H \to \mathbb{C}^2 \text{ and } D = \left[D_{ij}\right]_{i,j=1}^{2}.$$

1.2 Description of results by chapter

This thesis is organised as follows.
In Chapter 1 we give a literature review of the subject. We introduce definitions of terms and notations used throughout the thesis.
In Chapter 2 we apply the Schur algorithm, presented in Appendix B.2, to obtain a necessary condition for solvability of a given Γ-interpolation problem known as the $\mathcal{C}_1$ condition.
$\mathcal{C}_1$ *condition:* Let λ_j be a finite number of distinct points in $\mathbb{D}$ and let $(s_j,\ p_j) \in \Gamma$ for $j = 1, \cdots, n$, we say that the data

$$\lambda_j \mapsto (s_j,\ p_j),\ \ j = 1, \cdots, n,$$

satisfy $\mathcal{C}_1$ if, for every Möbius function v, the Nevanlinna-Pick problem

$$\lambda_j \mapsto \frac{2p_j v(\lambda_j) - s_j}{2 - s_j v(\lambda_j)},\ \ j = 1, \cdots, n,$$

is solvable.
We give a criterion (Theorem 2.2.10) for the *solvability* of a special three-point spectral Nevanlinna-Pick problem of type (1.1.1).
In Chapter 3 we study the $\mathbb{E}$-interpolation problem. We reduce the problem of analytic interpolation $\mathbb{D} \to \overline{\mathbb{E}}$ to a family of classical Nevanlinna-Pick problems. We prove criteria for solvability of the μ_{Diag}-interpolation problem (Theorem 3.3.4). We give a realization theorem for analytic functions from the disc to the tetrablock.
In Apendix A we give some examples of solvable and unsolvable 3-point spectral Nevanlinna-Pick problems. We write matlab code that checks 3-point Γ-interpolation data that satisfy $\mathcal{C}_1$ condition. Appendix B contains basic definitions and general background materials. We present the Schur reduction and augmentation algorithms. In Appendix C we give examples of aligned and caddywhompus Γ-inner functions from Agler, Lykova and Young paper [5].

1.3 History and recent work

The interpolation problems for functions that are analytic on the unit disc was solved by George Pick in 1916 and independently by Rolf Nevanlinna in 1919. In [29] Pick carried out his research for interpolating functions $\mathbb{D} \to \{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0\}$, while in [25] Nevanlinna studied interpolating functions $\mathbb{D} \to \overline{\mathbb{D}}$. The classical Nevanlinna-Pick interpolation problem [31, 32] is the following. Given n-distinct points $\lambda_1, \cdots, \lambda_n$ in the unit disc $\mathbb{D}$ and n-points $\omega_1, \cdots, \omega_n$ in $\overline{\mathbb{D}}$, find if possible an analytic function $h : \mathbb{D} \to \overline{\mathbb{D}}$ such that

$$h(\lambda_j) = \omega_j, \quad j = 1, \cdots, n. \tag{1.3.1}$$

Pick determined that a solution of Nevanlinna-Pick problem exists if and only if the Pick matrix

$$\left[\frac{1-\overline{\omega_j}\omega_i}{1-\overline{\lambda_j}\lambda_i}\right]_{i,j=1}^{n}$$

is positive semi-definite.

Theorem 1.3.1. *[Pick's Theorem]*
The Nevanlinna-Pick interpolation problem (1.3.1) *has a solution ϕ in* $\operatorname{Hol}(\mathbb{D}, \overline{\mathbb{D}})$ *if and only if*

$$\left[\frac{1-\overline{\omega_j}\omega_i}{1-\overline{\lambda_j}\lambda_i}\right]_{i,j=1}^{n} \geq 0.$$

Moreover, the function ϕ is unique if and only if the Pick matrix has rank m strictly less than n. In this case, ϕ is a Blaschke product of degree m.

A necessary and sufficient condition for the existence a solution of Nevanlinna-Pick interpolation problem in a matrix version was stated in [11, Chapter 18]. In [12] Hari Bercovici, Ciprian Foias and Allen Tannenbaum gave necessary and sufficient conditions for the existence of interpolating function $F : \mathbb{D} \to \mathbb{C}^{k\times k}$ whose spectral radius, $r(F(\lambda)) \leq 1$, for all $\lambda \in \mathbb{D}$. The most intensively studied version of spectral interpolation problem is the 2×2 spectral Nevanlinna-Pick problem. An instance of 2×2 spectral Nevanlinna-Pick problem was studied by J. Agler and N. J. Young in [8, 9]. They constructed two dimensional complex domains $\mathbb{G}, \Gamma$, called the open and closed symmetrized bidiscs, and formulated a new interpolation problem called the Γ-interpolation problem which connects with 2×2 spectral Nevanlinna-Pick interpolation problem. This direction of research was used by Hari Bercovi [13] to give a different criteria for solvability of the spectral interpolation problem.

The Schur class of operator-valued or matricial functions is the set of analytic operator- or matrix-valued functions F on $\mathbb{D}$ such that the operator norm

$$\|F(\lambda)\| \leq 1 \quad \text{for all } \ \lambda \in \mathbb{D}.$$

The realization formula for functions of the Schur class is given in [7, Theorem 6.5]. In [2] Agler extended this representation to functions in the space $H^\infty(\mathbb{D}^2)$ of bounded analytic functions on $\mathbb{D}^2$. See also [7, Theorem 11.13]. He proved that there is a function f in the closed unit ball of $H^\infty(\mathbb{D}^2)$ if and only if there is a Hilbert space $H = H_1 \oplus H_2$ and a unitary operator

$$V = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathbb{C} \oplus H \to \mathbb{C} \oplus H$$

such that for P_1 the projection of H onto H_1 and P_2 the projection of H onto H_2 we have

$$f(z) = A + B(z_1P_1 + z_2P_2)(1 - D(z_1P_1 + z_2P_2))^{-1}C.$$

The operator theory approach generally helps us to describe the existence of a solution of Nevanlinna-Pick interpolation problem in terms of kernels of Hilbert space functions involving the Hardy space $H^2(\mathbb{D})$ of the disc. From this viewpoint, the sufficiency condition of the Pick's Theorem 1.3.1 can be interpreted as the property of the reproducing kernel for $H^2(\mathbb{D})$. Agler, Z. A. Lykova and Young used this method in [3] to show that the solvability of the n-point spectral Nevanlinna-Pick problem is equivalent to the existence of positive analytic kernels on the bidisc which satisfy a certain matrix inequality.

Another development in the study of the μ-synthesis problem is in connection to interpolation functions from $\mathbb{D}$ to the tetrablock, a region in $\mathbb{C}^3$, [1, 16]. The tetrablock was introduced by Abouhajar, White and Young in [1] due to its relationship to the μ_{Diag}-synthesis interpolation problem. They proved a Schwarz lemma for the tetrablock and used the lemma to obtain solvability criterion for a special case of two-point μ_{Diag}-synthesis problem. An infinitesimal version of the Schwarz lemma for the tetrablock was given in [35]. In [16] Brown, Lykova and Young described connections between the set of analytic functions $\mathbb{D} \to \mathbb{E}$ and the 2×2 matricial Schur class.

Due to many properties of the tetrablock, specialists in several complex variables and operator theory have showed interest in the study of $\mathbb{E}$. Several geometric properties of the tetrablock have been studied in [33]. In [14] Bhattacharyya and Sau studied the dilation theory of $\mathbb{E}$-contraction, involving a triple $(A,\ B,\ P)$ of commuting bounded operators having the closed tetrablock $\overline{\mathbb{E}}$ as spectral set. They showed that if $(R_1,\ R_2, U)$ and $(\tilde{R_1},\ \tilde{R_2},\ \tilde{U})$ are two unitary dilations of $(A,\ B,\ P)$ with the property that $\tilde{U}$ is the minimal unitary dilation of P, then the dilation $(\tilde{R_1},\ \tilde{R_2},\ \tilde{U})$ is unitarily equivalent to $(R_1,\ R_2,\ U)$.

In [6] Agler, Lykova and Young introduced another domain, which is connected to a μ-synthesis problem. The domain is called the pentablock. The pentablock is defined to be

$$\mathcal{P} = \{(a_{21},\ \operatorname{tr} A,\ \det A) : A = \left[a_{ij}\right]_{i,j=1}^{2} \in \mathbb{B}\}$$

where $\mathbb{B}$ denotes the open unit ball in the space $\mathbb{C}^{2\times 2}$ with the usual operator norm. They showed that $\mathcal{P}$ intersects $\mathbb{R}^3$ at a convex open domain with five faces and four vertices $(0,\ -2,\ 1)$, $(0,\ 2,\ 1)$, $(1,\ 0,\ -1)$ and $(-1,\ 0,\ -1)$. To establish a connection between $\mathcal{P}$ and the μ_E-synthesis problem, where

$$E = \{ \begin{bmatrix} z & w \\ 0 & z \end{bmatrix} : |w| \le 1 - |z|^2,\ z, w \in \mathbb{C}\},$$

they showed that a matrix $A = \left[a_{ij}\right] \in \mathbb{C}^{2\times 2}$ satisfies $\mu_E(A) < 1$ if and only if

$$(s,\ p) \in \mathbb{G} \text{ and } |a_{21}| \sup_{z\in\mathbb{D}} \frac{1 - |z|^2}{|1 - sz + pz^2|} < 1,$$

where $\mathbb{G}$ is the open symmetrized bidisc, $s = \operatorname{tr} A$ and $p = \det A$. Several geometric properties of $\mathcal{P}$ are proved in [6].

Chapter 2

The Γ-interpolation problem

We consider a special three-point Γ-interpolation problem. We study a necessary condition $\mathcal{C}_1$ for the solvability of this three-point problem. We apply Bercovici's theorem to find necessary and sufficient conditions for the solvability of the three point spectral Nevanlinna-Pick problem.

2.1 The symmetrized bidisc

The following sets were introduced by Agler and Young 2000.

Definition 2.1.1. [8] *The open and closed symmetrized bidiscs* $\mathbb{G}$ *and* Γ *are the subsets of* $\mathbb{C}^2$ *defined by*

$$\mathbb{G} = \{(s,p) = (z+w, zw) : z, w \in \mathbb{D}\}$$

and

$$\Gamma = \{(s,p) = (z+w, zw) : z, w \in \overline{\mathbb{D}}\}.$$

That is, the bidisc $\mathbb{D}^2 = \{(z,w) : z, w \in \mathbb{D}\}$ is mapped onto $\mathbb{G}$ by the symmetric function $\pi(z,w) = (z+w, zw)$.
By [10, Theorem 2.3], the symmetrized bidisc Γ is compact, starlike about the origin and polynomially convex.

Definition 2.1.2. *Let* Ω *be a domain in* $\mathbb{C}^n$ *with closure* $\overline{\Omega}$ *and let* $\mathcal{A}(\Omega)$ *be the algebra of continuous scalar functions on* $\overline{\Omega}$ *that are analytic on* Ω. *A boundary for* Ω *is a subset* K *of* $\overline{\Omega}$ *such that every function in* $\mathcal{A}(\Omega)$ *attains its maximum modulus on* K.

By [15, Corollary 2.2.10], at least when $\overline{\Omega}$ is polynomially convex, there is a smallest closed boundary of Ω, contained in all the closed boundaries of Ω and is called the distinguished boundary of Ω.

Theorem 2.1.3. *[10, Theorem 2.4] The distinguished boundary of* Γ *is the set*

$$b\Gamma = \{(s,p) : |s| \le 2,\ |p| = 1,\ s = \bar{s}p\}.$$

Topologically $b\Gamma$ *is a Mobius band.*

Definition 2.1.4. *A rational map* $\Phi : \mathbb{C}^3 \setminus \{(z,s,p) \in \mathbb{C}^3 : sz = 2\} \to \mathbb{C}$ *is defined by*

$$\Phi(z,s,p) = \frac{2pz - s}{2 - sz}, \tag{2.1.1}$$

for all $z \in \mathbb{C}$ *and* $(s,p) \in \mathbb{C}^2$ *such that* $sz \neq 2$.

Alternatively, the symbol $\Phi_z(s,p)$ will be used for $\Phi(z,s,p)$. The function Φ satisfies the following properties.

Proposition 2.1.5. [34, Proposition 2.3] *For every* $\omega \in \mathbb{T}$, Φ_ω *maps* $\mathbb{G}$ *analytically into* $\mathbb{D}$. *Conversely, if* $(s,p) \in \mathbb{C}^2$ *is such that* $|\Phi_\omega(s,p)| < 1$ *for all* $\omega \in \mathbb{T}$, *then* $(s,p) \in \mathbb{G}$.

The proposition below gives a complete characterization of points of $\mathbb{C}^2$ which belong to Γ, its distinguished boundary $b\Gamma$ or its topological boundary $\partial\Gamma$.

Proposition 2.1.6. [4, Proposition 3.2][10, Corollary 2.2] *Let* $(s,p) \in \mathbb{C}^2$. *Then*

(1) (s,p)$\in \mathbb{G}$ *if and only if* $|s - \bar{s}p| < 1 - |p|^2$;

(2) $(s,p) \in \mathbb{G}$ *if and only if* $|s| < 2$ *and, for all* $\omega \in \mathbb{T}$, $|\Phi_\omega(s,p)| < 1$;

(3) $(s,p) \in \Gamma$

if and only if $|s| \le 2$ *and* $|s - \bar{s}p| \le 1 - |p|^2$
if and only if $|s| \le 2$ *and, for all* ω *in a dense subset of* $\mathbb{T}$, $|\Phi(\omega,s,p)| \le 1$;

(4) $(s,p) \in b\Gamma$ *if and only if* $|s| \le 2$, $|p| = 1$ *and* $s = \bar{s}p$;

(5) $(s,p) \in \partial\Gamma$

if and only if $|s| \le 2$ *and* $|s - \bar{s}p| = 1 - |p|^2$
if and only if there exist $z \in \mathbb{T}$ *and* $w \in \overline{\mathbb{D}}$ *such that* $s = z + w$, $p = zw$.

Furthermore, for $\omega \in \mathbb{T}$ and $(s, p) \in \Gamma$,

$$|\Phi_\omega(s, p)| = 1 \text{ if and only if } \omega(s - \bar{s}p) = 1 - |p|^2 .$$

Definition 2.1.7. *A function $h \in Hol(\mathbb{D}, \Gamma)$ is Γ-inner if*

$$\lim_{r \to 1^-} h(r\lambda) \in b\Gamma \tag{2.1.2}$$

for almost all $\lambda \in \mathbb{T}$ with respect to Lebesgue measure.

By Fatou's Theorem (B.1.1), the radial limit (2.1.2) exists for almost all $\lambda \in \mathbb{T}$ with respect to Lebesgue measure.

Definition 2.1.8. *The royal variety $\mathfrak{R}$ is defined by*

$$\mathfrak{R} = \{(-2\lambda, \lambda^2) : \lambda \in \mathbb{C}\} = \{(s, p) \in \mathbb{C}^2 : s^2 = 4p\}.$$

Definition 2.1.9. *A point $\lambda \in \overline{\mathbb{D}}$ is called a royal node of a rational Γ-inner function $h = (s, p)$ if*

$$s^2(\lambda) - 4p(\lambda) = 0.$$

2.1.1 Interpolation in Hol($\mathbb{D}, \Gamma$)

The interpolation problems in Hol($\mathbb{D}, \Gamma$) was introduced by Agler and Young in [8] mainly because of its connection with a problem in control engineering.
A Γ-interpolation problem: Given n distinct points $\lambda_1, \cdots, \lambda_n$ in the open unit disc $\mathbb{D}$ and n points $z_1, \cdots, z_n$ in Γ, find if possible an analytic function

$$h : \mathbb{D} \to \Gamma \text{ such that } h(\lambda_j) = z_j \text{ for } j = 1, \cdots, n. \tag{2.1.3}$$

The data

$$\lambda_j \mapsto z_j, \qquad 1 \le j \le n, \tag{2.1.4}$$

are called *Γ-interpolation data.* The problem is said to be *solvable* if there exists an analytic function $h : \mathbb{D} \to \Gamma$ such that $h(\lambda_j) = z_j$ for $j = 1, \cdots, n$. Any such function h is called a *solution* of the Γ-interpolation problem with data (2.1.4).
The conditions $\mathcal{C}_\nu$ associated with the Γ-interpolation data (2.1.4) were introduced in [4].

Definition 2.1.10. *For Γ-interpolation data*

$$\lambda_j \mapsto (s_j,\ p_j),\ \ 1 \le j \le n, \tag{2.1.5}$$

we say that the data satisfy

Condition $\mathcal{C}_\nu(\lambda,\ s,\ p)$

if, for every Blaschke product υ of degree at most ν, the Nevanlinna-Pick problem

$$\lambda_j \mapsto \Phi(\upsilon(\lambda_j), s_j, p_j)$$

is solvable.

By the theorem below, the conditions $\mathcal{C}_\nu$ are all necessary for the solution of Γ-interpolation problem to exist.

Theorem 2.1.11. [4, Theorem 4.3] *Let $\lambda_1, \cdots, \lambda_n$ be distinct points in $\mathbb{D}$ and let $(s_j,\ p_j \in \mathbb{G}$, $j = 1, \cdots, n$. If there exists an analytic function $h : \mathbb{D} \to \Gamma$ such that*

$$h(\lambda_j) = (s_j,\ p_j), \quad j = 1, \cdots, n,$$

then for any function υ in the Schur class, the Nevanlinna-Pick problem with data

$$\lambda_j \mapsto \Phi(\upsilon(\lambda_j), s_j, p_j), \qquad 1 \le j \le n, \tag{2.1.6}$$

is solvable. In particular, the condition $\mathcal{C}_\nu(\lambda,\ s,\ p)$ holds for every non-negative integer ν.

Conjecture: It was conjectured by Agler, Lykova and Young in [4] that Condition $\mathcal{C}_{n-2}$ is necessary and sufficient for the solvability of an n-point Γ-interpolation problem.

For $n = 2$, $\mathcal{C}_0$ is sufficient for the solvability of the Nevanlinna-Pick problem. See [4, Theorem 4.4].
The following materials are taken from [4] and [5].

Definition 2.1.12. *[4, Definition 2.1] Let Ω be a domain, let $E \subset \mathbb{C}^N$, let $n \ge 1$, let $\lambda_1, \cdots, \lambda_n$ be distinct points in Ω and let $z_1, \cdots, z_n \in E$. The interpolation data*

$$\lambda_j \mapsto z_j : \Omega \to E,\ j = 1, \cdots, n$$

are said to be extremally solvable *if there exists a map $h \in Hol(\Omega, E)$ such that $h(\lambda_j) = z_j$ for $j = 1, \cdots, n$, but, for any open neighbourhood $\mathcal{U}$ of the closure of Ω, there is no $f \in Hol(\mathcal{U}, E)$ such that $f(\lambda_j) = z_j$ for $j = 1, \cdots, n$.*

Definition 2.1.13. *[4, Definition 2.1] The map $h \in Hol(\Omega, E)$ is called n-*extremal *(for $Hol(\Omega, E)$) if, for all choices of n distinct points $\lambda_1, \cdots, \lambda_n$ in Ω the interpolation data*

$$\lambda_j \mapsto h(\lambda_j) : \Omega \to E,\ j = 1, \cdots, n$$

are extremally solvable.

Definition 2.1.14. *[5, Definition 4.2] We say that $\mathcal{C}_\nu$ holds actively and extremally for the Γ-interpolation data $\lambda_j \mapsto (s_j,\ p_j),\quad 1 \le j \le n$, if $\mathcal{C}_\nu$ holds extremally and there is a Blaschke product m of degree ν such that the data*

$$\lambda_j \mapsto \Phi(m(\lambda_j),\ s_j,\ p_j),\ j = 1, \cdots, n, \tag{2.1.7}$$

are extremally solvable.

Denote by $\mathcal{B}l_n$ the collection of Blaschke products of degree at most n.

Definition 2.1.15. *[5, Definition 4.2] We say that $m \in \mathcal{S}$ or $\mathcal{B}l_\nu$ is an auxiliary extremal for the data* (2.1.5) *if the data* (2.1.7) *are extremally solvable.*

Definition 2.1.16. *Let $h = (s, p)$ be a rational $\mathbb{G}$-inner function. We say that h is* aligned *if $h(\mathbb{D}) \subset \mathbb{G}$, the degree of h is at most 4 and there exist at least $d(p) - 1$ distinct royal nodes of h in $\mathbb{T}$ and, if $d(p) = 4$, there are distinct royal nodes $\omega_1, \omega_2, \omega_3$ of h in $\mathbb{T}$ such that the points $\frac{1}{2}s(\omega_1), \frac{1}{2}s(\omega_2), \frac{1}{2}s(\omega_3) \in \mathbb{T}$ are distinct and in the opposite cyclic order to $\omega_1, \omega_2, \omega_3$.*

Definition 2.1.17. *A rational Γ-inner function $h = (s, p)$ is caddywhompus if $h(\mathbb{D}) \subset \Gamma$, the degree of h is equal to 4, h has at least 3 distinct royal nodes in $\mathbb{T}$ and for every choice of 3 distinct royal nodes w_1, w_2, w_3 in $\mathbb{T}$, the points $\frac{1}{2}\overline{s(w_1)},\ \frac{1}{2}\overline{s(w_2)},\ \frac{1}{2}\overline{s(w_3)} \in \mathbb{T}$ are not in the same cyclic order as $w_1,\ w_2,\ w_3$.*

One can find examples from [5, Example 13.2] of aligned and caddywhompus Γ-inner functions in Appendix C. To state [5, Theorem 1.1], we need to describe the associated problem.
The associate problem to the Γ-interpolation problem (2.1.3):
Given data $\lambda_j \to (s_j, p_j),\ j = 1, 2, 3$, that satisfy condition $\mathcal{C}_1$ extremally with auxiliary extremal $m \in \text{Aut}\ \mathbb{D}$ find a Blaschke product p of degree at most 4 such that

$$p(\lambda_j) = p_j, \qquad j = 1, 2, 3, \tag{2.1.8}$$

and

$$p(\tau_l) = \overline{m}(\tau_l)^2, \qquad l = 1, \cdots, d(mq), \tag{2.1.9}$$

where the τ_l are the roots of the equation $mq(\tau) = 1$ and q is the unique function in the Schur class such that

$$q(\lambda_j) = \Phi(m(\lambda_j), s_j, p_j), \qquad j = 1, 2, 3.$$

Theorem 2.1.18. *[5, Theorem 1.1] Let $\lambda_1, \lambda_2, \lambda_3$ be distinct points in $\mathbb{D}$ and let $z_1, z_2, z_3 \in \mathbb{G}$. The following statement are equivalent.*

(1) There exists an aligned $\mathbb{G}$*-inner function h of degree at most 4 such that* $h(\lambda_j) = z_j$ *for* $j = 1, 2, 3$;

(2) condition $\mathcal{C}_1(\lambda, z)$ *holds extremally and actively, and the associated problem is solvable.*

However, in [23, Example 2.2], A.S. Kamara gave a counter-example with three-node Γ-interpolation data which satisfy $\mathcal{C}_1$ and showed that the corresponding spectral Nevanlinna-Pick problem is not solvable. We consider $\mathcal{C}_1$ condition and a specific three-point spectral Nevanlinna-Pick problem and give a criterion for its *solvability.* The following is a well known result.

Lemma 2.1.19. *Let S be the linear transformation*

$$S(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d, \in \mathbb{C}$ *are such that* $ad - bc \neq 0,\ c \neq 0$ *and* $cz + d \neq 0$ *for all* $z \in \overline{\mathbb{D}}$*, and so S does not have a pole in* $\overline{\mathbb{D}}$*. Then*

$$S(\mathbb{D}) = \{z \in \mathbb{C} : |z - C| < R\}$$

where

$$C = \frac{b\overline{d} - a\overline{c}}{|d|^2 - |c|^2} \quad \text{and} \quad R = \left| \frac{ad - bc}{|d|^2 - |c|^2} \right|.$$

denote the centre and radius of the disc $|z - C| < R$.

Proof. Let $S(z) = \dfrac{az + b}{cz + d}$. In matrix notation the linear transformation is given by

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and its inverse

$$S^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{with } ad - bc \neq 0.$$

If $w = S(z)$, then $z = S^{-1}(w) = \dfrac{dw - b}{-cw + a}$. The value

$$S^{-1}(\infty) = \lim_{w \to \infty} \frac{dw - b}{-cw + a} = -\frac{d}{c}.$$

Note that $S^{-1}(C)$ and $S^{-1}(\infty)$ are conjugates with respect to $\mathbb{T}$. That is,

$$\overline{S^{-1}(C)} \cdot S^{-1}(\infty) = 1,$$

and so

$$\overline{S^{-1}(C)}(-\frac{d}{c}) = 1.$$

Therefore

$$S^{-1}(C) = -\frac{\overline{c}}{\overline{d}}.$$

Thus

$$\begin{aligned} C &= S(-\frac{\overline{c}}{\overline{d}}) \\ &= \frac{a(-\frac{\overline{c}}{\overline{d}}) + b}{c(-\frac{\overline{c}}{\overline{d}}) + d} \\ &= \frac{b\overline{d} - a\overline{c}}{|d|^2 - |c|^2}. \end{aligned}$$

The radius R is

$$\begin{aligned} R &= |S(1) - C| \\ &= \left| \frac{a+b}{c+d} - \frac{b\overline{d} - a\overline{c}}{|d|^2 - |c|^2} \right| \\ &= \left| \frac{(a+b)(|d|^2 - |c|^2) - (c+d)(b\overline{d} - a\overline{c})}{(c+d)(|d|^2 - |c|^2)} \right| \\ &= \left| \frac{a\,|d|^2 - a\,|c|^2 + b\,|d|^2 - b\,|c|^2 - b\overline{d}c + a\,|c|^2 - b\,|d|^2 + ad\overline{c}}{(c+d)(|d|^2 - |c|^2)} \right| \\ &= \left| \frac{a\,|d|^2 - b\overline{d}c + ad\overline{c} - b\,|c|^2}{(c+d)(|d|^2 - |c|^2)} \right| \\ &= \left| \frac{(\overline{c} + \overline{d})(ad - bc)}{(c+d)(|d|^2 - |c|^2)} \right| \\ &= \left| \frac{ad - bc}{|d|^2 - |c|^2} \right|. \end{aligned}$$

□

2.1.2 A special Γ-interpolation problem

Consider the Γ-interpolation problem: given $\lambda_1 = 0,\ \lambda_2,\ \lambda_3 \in \mathbb{D}$ where $\lambda_2 \neq 0,\ \ \lambda_3 \neq 0$ and $\lambda_2 \neq \lambda_3$, and $(s_1, p_1) = (0, 0),\ (s_2, p_2) = (-2\alpha, \alpha^2),\ \alpha \in \mathbb{D} \setminus \{0\}$ and $(s_3, p_3) =$

(s,p) in $\mathbb{G}$, find if possible a function $f:\mathbb{D}\to\mathbb{G}$ such that $f(\lambda_j)=(s_j,p_j),\ j=1,2,3$.

The Γ-interpolation data for this problem are the following

$$\begin{cases} \lambda_1 \mapsto (0,0) \\ \lambda_2 \mapsto (-2\alpha,\alpha^2), \text{ where } \alpha\in\mathbb{D}\setminus\{0\} \\ \lambda_3 \mapsto (s,p)\in\Gamma. \end{cases} \tag{2.1.10}$$

Let us describe the case where $(s,p)\in\mathfrak{R}$.

Proposition 2.1.20. *Let $\lambda_j\mapsto z_j,\ j=1,2,3$, be the $\Gamma-$interpolation data* (2.1.10) *where $(s,p)\in\mathfrak{R}$. Let $(s,p)=(-2\eta,\eta^2),\quad \eta\in\mathbb{D}$. Suppose that there exists $m\in\mathcal{S}$ such that $m(0)=0,\ m(\lambda_2)=\alpha$ and $m(\lambda_3)=\eta$. Then, for $k(\lambda)=(-2\lambda,\lambda^2)$, the function $h(\lambda)=(k\circ m)(\lambda)$ is a solution of the $\Gamma-$interpolation problem* (2.1.10).

Let us consider the case when $(s,p)\in\mathbb{G}$ and $s\neq 0$.

Proposition 2.1.21. *Let $\lambda_j\mapsto z_j,\ j=1,2,3$, be the $\Gamma-$interpolation data* (2.1.10) *such that $(s,p)\in\mathbb{G},\ (s,p)\notin\mathfrak{R}$ and $s\neq 0$. Suppose these data satisfy Condition $\mathcal{C}_1$. Then the following inequalities hold*

$$|\alpha|\le|\lambda_2|, \tag{2.1.11}$$

$$\frac{2|s-\bar{s}p|+|s^2-4p|}{4-|s|^2}\le|\lambda_3|. \tag{2.1.12}$$

If

(1.1) $$|\alpha|<|\lambda_2| \quad \textit{and} \quad \frac{2\left|s-\bar{s}p\right|+\left|s^2-4p\right|}{4-|s|^2}<|\lambda_3|,$$

then we have the following

$$\left|\overline{\lambda_2}\lambda_3 s+2\overline{\alpha}p\right|<\left|2\overline{\lambda_2}\lambda_3+\overline{\alpha}s\right| \tag{2.1.13}$$

and

$$\frac{\left|b\overline{d}-a\overline{c}\right|+|ad-bc|}{|d|^2-|c|^2}\le\rho(\lambda_2,\lambda_3) \tag{2.1.14}$$

where

$$\rho(\lambda_2,\lambda_3)=\left|\frac{\lambda_3-\lambda_2}{1-\overline{\lambda_3}\lambda_2}\right|,$$
$$a=2\lambda_2 p+\alpha\lambda_3 s,$$
$$b=-(2\alpha\lambda_3+\lambda_2 s),$$
$$c=-(\overline{\lambda}_2\lambda_3 s+2\overline{\alpha}p),$$
$$d=2\overline{\lambda}_2\lambda_3+\overline{\alpha}s.$$

If $|\alpha| = |\lambda_2|$, *then we have*

$$(1.2a) \qquad s = -\frac{2\alpha\lambda_3}{\lambda_2}, \quad p = \frac{\alpha^2\lambda_3^2}{\lambda_2^2} \ \text{and} \ \frac{2|s - \bar{s}p| + |s^2 - 4p|}{4 - |s|^2} = |\lambda_3|.$$

If $\dfrac{2|s - \bar{s}p| + |s^2 - 4p|}{4 - |s|^2} = |\lambda_3|$ *then we have*

$$(1.2b) \qquad |\alpha| = |\lambda_2|; \ \text{and as in (1.2a)}, \ s = -\frac{2\alpha\lambda_3}{\lambda_2}, \quad p = \frac{\alpha^2\lambda_3^2}{\lambda_2^2}.$$

Proof. By Definition 2.1.4, for all $z \in \mathbb{D}$,

$$\begin{aligned}
\Phi_z(0,0) &= 0, \\
\Phi_z(-2\alpha, \alpha^2) &= \frac{2\alpha^2 z - (-2\alpha)}{2 - (-2\alpha)z} \\
&= \frac{2\alpha^2 z + 2\alpha}{2 + 2\alpha z} \\
&= \frac{2\alpha(\alpha z + 1)}{2(1 + \alpha z)} \\
&= \alpha,
\end{aligned}$$

and

$$\Phi_z(s,p) = \frac{2pz - s}{2 - sz}.$$

Condition $\mathcal{C}_1$ for the data (2.1.10) is that, for every $\nu \in Bl_1$,

$$\begin{cases} \lambda_1 \mapsto 0, \\ \lambda_2 \mapsto \alpha, \\ \lambda_3 \mapsto \frac{2p\nu(\lambda_3) - s}{2 - s\nu(\lambda_3)} \end{cases} \tag{2.1.15}$$

are solvable Nevanlinna-Pick data. Hence, since $\nu(\lambda_3)$ takes on all values in $\mathbb{D}$ as ν varies over Bl_1, the Blaschke products of degree at most one, $\mathcal{C}_1$ condition for the data (2.1.10) is satisfied if the Nevanlinna-Pick problem with data

$$\begin{cases} \lambda_1 \mapsto 0 \\ \\ \lambda_2 \mapsto \alpha \\ \\ \lambda_3 \mapsto \frac{2pz - s}{2 - sz}, \end{cases} \tag{2.1.16}$$

is solvable for every $z \in \overline{\mathbb{D}}$.
Suppose $\mathcal{C}_1$ *holds for the data* (2.1.10). Consider any $z \in \overline{\mathbb{D}}$ and let $\omega_1 = 0,\ \omega_2 = \alpha,\ \omega_3 = \dfrac{2pz - s}{2 - sz}$. Then (2.1.16) becomes a standard Nevanlinna-Pick interpolation problem. By assumption, for each $z \in \mathbb{D}$, there is an *analytic function* $h \in \mathcal{S}$ satisfying

$$h(\lambda_1) = 0$$

$$h(\lambda_2) = \alpha$$

$$h(\lambda_3) = \frac{2pz - s}{2 - sz}.$$

The function h is depended on z.
Step 1. Reduction at λ_1: Fix $z \in \mathbb{D}$. By Proposition B.2.7, the Schur reduction h_1 of h at λ_1 is analytic. That is,

$$h_1 = \frac{B_{\omega_1} \circ h}{B_{\lambda_1}} \in \mathcal{S}$$

implying

$$h_1(\lambda) = \frac{B_{\omega_1} \circ h}{B_{\lambda_1}}(\lambda).$$

Substituting $\lambda_1 = \omega_1 = 0$ we have

$$h_1(\lambda) = \frac{h(\lambda)}{\lambda}, \quad \lambda \neq \lambda_1. \tag{2.1.17}$$

Then,

$$h_1(\lambda_j) = \frac{h(\lambda_j)}{\lambda_j}, \ \ j = 2, 3. \tag{2.1.18}$$

Since

$$h(\lambda_2) = \alpha \ \text{ and } \ h(\lambda_3) = \frac{2pz - s}{2 - sz},$$

it follows that

$$h_1(\lambda_2) = \frac{\alpha}{\lambda_2} \ \text{ and } \ h_1(\lambda_3) = \frac{2pz - s}{(2 - sz)\lambda_3}.$$

Since $h_1 \in \mathcal{S}$, that is, $|h_1(\lambda_j)| \leq 1, \ \ j = 2, 3,$ the new interpolation data

$$\begin{cases} \lambda_2 \mapsto \frac{\alpha}{\lambda_2} \\ \\ \lambda_3 \mapsto \frac{2pz - s}{(2 - sz)\lambda_3} \end{cases} \tag{2.1.19}$$

satisfy

$$\left|\frac{\alpha}{\lambda_2}\right| \leq 1 \ \text{ and } \ \left|\frac{2pz - s}{(2 - sz)\lambda_3}\right| \leq 1.$$

Therefore, the inequalities

$$\begin{cases} |\alpha| \leq |\lambda_2|, \\ \\ \left|\frac{2pz-s}{2-sz}\right| \leq |\lambda_3|. \end{cases} \tag{2.1.20}$$

hold for all $z \in \overline{\mathbb{D}}$. By assumption, $(s, p) \in \mathbb{G}$, $s \neq 0$, and $(s, p) \notin \mathfrak{R}$. Therefore for all $z \in \overline{\mathbb{D}}$, $|sz| \leq |s| < 2$ and $4p - s^2 \neq 0$. Hence $2 - sz \neq 0$ for all $z \in \overline{\mathbb{D}}$. Thus, by Lemma 2.1.19, the map $S : z \to \dfrac{2pz - s}{2 - sz}$, maps $\mathbb{D}$ to the open disc with radius $R = \dfrac{|s^2 - 4p|}{4 - |s|^2}$ and centre $C = \dfrac{2\bar{s}p - 2s}{4 - |s|^2}$. Since $|S(z)| \leq |\lambda_3|$ for all $z \in \mathbb{D}$, we have $|C| + R \leq |\lambda_3|$. Therefore, if the interpolation problem with the data (2.1.10) satisfies Condition $\mathcal{C}_1$, then

$$\begin{cases} |\alpha| \leq |\lambda_2|, \\ \\ \dfrac{2|s - \bar{s}p| + |s^2 - 4p|}{4 - |s|^2} \leq |\lambda_3|. \end{cases} \tag{2.1.21}$$

Case (1.1): If $|\alpha| < |\lambda_2|$ and $\dfrac{2\,|s - \bar{s}p| + \left|s^2 - 4p\right|}{4 - |s|^2} < |\lambda_3|$, then we carry out a second reduction to obtain a parametrization of the solutions of Problem (2.1.16).

Step 2. Reduction at λ_2: Let $z \in \mathbb{D}$ and let h_2 be the Schur reduction of h_1 at λ_2. Then

$$h_2(\lambda) = \frac{B_{\frac{\alpha}{\lambda_2}}(h_1(\lambda))}{B_{\lambda_2}(\lambda)}, \qquad \lambda \neq \lambda_2. \tag{2.1.22}$$

Therefore

$$h_2(\lambda_3) = \frac{B_{\frac{\alpha}{\lambda_2}}(h_1(\lambda_3))}{B_{\lambda_2}(\lambda_3)}.$$

By substituting $h_1(\lambda_3) = \dfrac{h(\lambda_3)}{\lambda_3} = \dfrac{1}{\lambda_3}\Phi_z(s, p)$ to equation (2.1.22) we obtain

$$h_2(\lambda_3) = B_{\frac{\alpha}{\lambda_2}}\left(\frac{1}{\lambda_3}\Phi_z(s,p)\right) \cdot \frac{1}{B_{\lambda_2}(\lambda_3)}$$

$$= \frac{\frac{1}{\lambda_3}\frac{2pz-s}{2-sz} - \frac{\alpha}{\lambda_2}}{1 - \frac{\overline{\alpha}}{\overline{\lambda}_2}\frac{1}{\lambda_3}\frac{2pz-s}{2-sz}} \cdot \frac{1-\overline{\lambda}_2\lambda_3}{\lambda_3 - \lambda_2}$$

$$= \frac{\overline{\lambda}_2}{\lambda_2} \cdot \frac{\lambda_2\frac{2pz-s}{2-sz} - \alpha\lambda_3}{\overline{\lambda}_2\lambda_3 - \overline{\alpha}\frac{2pz-s}{2-sz}} \cdot \frac{1-\overline{\lambda}_2\lambda_3}{\lambda_3 - \lambda_2}$$

$$= \frac{\overline{\lambda}_2}{\lambda_2} \cdot \frac{\lambda_2(2pz-s) - \alpha\lambda_3(2-sz)}{\overline{\lambda}_2\lambda_3(2-sz) - \overline{\alpha}(2pz-s)} \cdot \frac{1-\overline{\lambda}_2\lambda_3}{\lambda_3 - \lambda_2}$$

$$= \frac{\overline{\lambda_2}}{\lambda_2} \cdot \frac{(2\lambda_2 p + \alpha\lambda_3 s)z - (2\alpha\lambda_3 + \lambda_2 s)}{-(\overline{\lambda}_2\lambda_3 s + 2\overline{\alpha}p)z + 2\overline{\lambda}_2\lambda_3 + \overline{\alpha}s} \cdot \frac{1}{\rho(\lambda_2,\ \lambda_3)}.$$

Since $|h_2(\lambda_3)| \leq 1$ for all $z \in \mathbb{D}$, we have

$$\sup_{z\in\mathbb{T}} \left| \frac{(2\lambda_2 p + \alpha\lambda_3 s)z - (2\alpha\lambda_3 + \lambda_2 s)}{-(\overline{\lambda}_2\lambda_3 s + 2\overline{\alpha}p)z + 2\overline{\lambda}_2\lambda_3 + \overline{\alpha}s} \right| \leq \rho(\lambda_2, \lambda_3).$$

That is,

$$\sup_{z\in\overline{\mathbb{D}}} \left| \frac{(2\lambda_2 p + \alpha\lambda_3 s)z - (2\alpha\lambda_3 + \lambda_2 s)}{-(\overline{\lambda}_2\lambda_3 s + 2\overline{\alpha}p)z + 2\overline{\lambda}_2\lambda_3 + \overline{\alpha}s} \right| \leq \rho(\lambda_2, \lambda_3). \tag{2.1.23}$$

Consider the linear fraction transformation

$$S_1 : z \mapsto \frac{(2\lambda_2 p + \alpha\lambda_3 s)z - (2\alpha\lambda_3 + \lambda_2 s)}{-(\overline{\lambda}_2\lambda_3 s + 2\overline{\alpha}p)z + 2\overline{\lambda}_2\lambda_3 + \overline{\alpha}s}.$$

Let $C',\ R'$ be the centre and radius of the disc $S_1(\mathbb{D})$. Note that

$$S_1^{-1} : w \mapsto \frac{(2\overline{\lambda}_2\lambda_3 + \overline{\alpha}s)w + 2\alpha\lambda_3 + \lambda_2 s}{(\overline{\lambda}_2\lambda_3 s + 2\overline{\alpha}p)w + 2\lambda_2 p + \alpha\lambda_3 s}.$$

Then

$$S_1^{-1}(\infty) = \frac{2\overline{\lambda}_2\lambda_3 + \overline{\alpha}s}{\overline{\lambda}_2\lambda_3 s + 2\overline{\alpha}p}$$

and

$$S_1^{-1}(\infty)\overline{S_1^{-1}(C')} = 1.$$

Hence

$$\overline{S_1^{-1}(C')} = \frac{\overline{\lambda}_2\lambda_3 s + 2\overline{\alpha}p}{2\overline{\lambda}_2\lambda_3 + \overline{\alpha}s} \in \mathbb{D}.$$

We obtain

$$\left|\overline{\lambda}_2\lambda_3 s + 2\overline{\alpha}p\right| < \left|2\overline{\lambda}_2\lambda_3 + \overline{\alpha}s\right|. \tag{2.1.24}$$

By Lemma 2.1.19, since the inequality (2.1.23) holds,

$$C' = \frac{b\overline{d} - a\overline{c}}{|d|^2 - |c|^2} \text{ and } R' = \frac{|ad - bc|}{|d|^2 - |c|^2}.$$

where

$$\begin{aligned} a &= 2\lambda_2 p + \alpha\lambda_3 s \\ b &= -(2\alpha\lambda_3 + \lambda_2 s) \\ c &= -(\overline{\lambda}_2\lambda_3 s + 2\overline{\alpha}p) \\ d &= 2\overline{\lambda}_2\lambda_3 + \overline{\alpha}s. \end{aligned}$$

Because the inequality (2.1.23) holds, the inequality $|C'| + R' \leq \rho(\lambda_2, \lambda_3)$ is satisfied. Therefore

$$\frac{\left|b\overline{d} - a\overline{c}\right| + |ad - bc|}{|d|^2 - |c|^2} \leq \rho(\lambda_2, \lambda_3). \tag{2.1.25}$$

Case $(1.2a)$: Suppose $|\alpha| = |\lambda_2|$, that is, $\left|\frac{\alpha}{\lambda_2}\right| = 1$. Then for h_1 from equation (2.1.18),

$$h_1(\lambda_2) = \frac{\alpha}{\lambda_2} \in \mathbb{T}.$$

Therefore, by Schwarz lemma,

$$h_1(\lambda) = \frac{\alpha}{\lambda_2} \text{ for all } \lambda \in \mathbb{D}.$$

Hence, by equation (2.1.17),

$$h_1(\lambda) = \frac{h(\lambda)}{\lambda}.$$

Thus

$$h(\lambda) = \frac{\alpha\lambda}{\lambda_2} \text{ for all } \lambda \in \mathbb{D}. \tag{2.1.26}$$

It is clear that $h(\lambda_1) = 0$, and $h(\lambda_2) = \alpha$. Note that h solves data (2.1.16) if

$$h(\lambda_3) = \frac{2pz - s}{2 - sz} \text{ for all } z \in \mathbb{D}.$$

One can see that

$$\frac{2pz-s}{2-sz} = -\frac{s}{2} + \frac{(4p-s^2)z}{2(2-sz)} \quad \text{for all} \quad z \in \mathbb{D}.$$

Thus

$$h(\lambda_3) = -\frac{s}{2} + \frac{(4p-s^2)z}{2(2-sz)} = \frac{\alpha\lambda_3}{\lambda_2} \quad \text{for all} \quad z \in \mathbb{D}. \tag{2.1.27}$$

In particular, for $z = 0$,

$$-\frac{s}{2} = \frac{\alpha\lambda_3}{\lambda_2}.$$

That is,

$$s = -\frac{2\alpha\lambda_3}{\lambda_2},$$

and hence

$$\frac{(4p-s^2)z}{2(2-sz)} = 0 \ \text{ for all } \ z \in \mathbb{D}.$$

Therefore

$$4p - s^2 = 0.$$

Thus since

$$4p = s^2, \ \text{ we have } \quad p = \frac{\alpha^2\lambda_3^2}{\lambda_2^2}.$$

Substituting these values of s, p in (2.1.12) we obtain

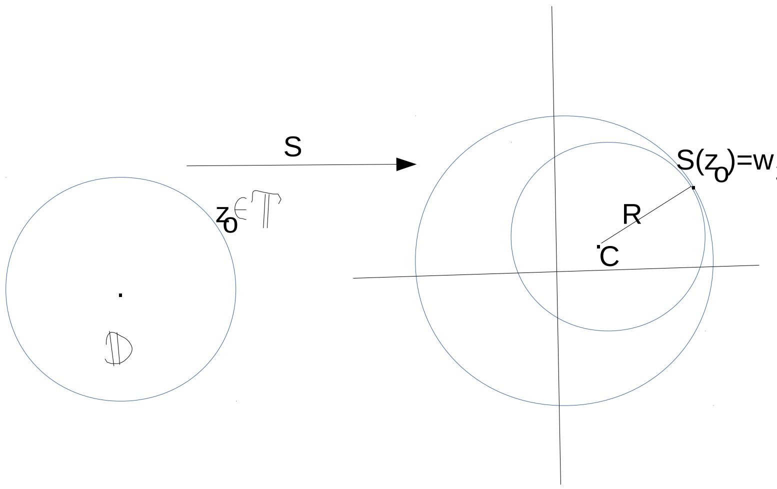
$$\begin{aligned}
\frac{2|s-\bar{s}p| + |s^2-4p|}{4-|s|^2} &= \frac{2\left|-\frac{2\alpha\lambda_3}{\lambda_2} - \left(-\frac{2\overline{\alpha}\overline{\lambda_3}}{\overline{\lambda_2}}\right) \cdot \frac{\alpha^2\lambda_3^2}{\lambda_2^2}\right| + \left|\frac{4\alpha^2\lambda_3^2}{\lambda_2^2} - \frac{4\alpha^2\lambda_3^2}{\lambda_2^2}\right|}{4 - \left|-\frac{2\alpha\lambda_3}{\lambda_2}\right|^2} \\
&= \frac{4\left|\frac{\alpha\lambda_3}{\lambda_2}\right|\left(1 - \left|\frac{\alpha\lambda_3}{\lambda_2}\right|^2\right)}{4 - 4\left|\frac{\alpha\lambda_3}{\lambda_2}\right|} \\
&= \frac{4\left|\lambda_3\right|\left[1 - \left|\lambda_3\right|^2\right]}{4\left[1 - \left|\lambda_3\right|^2\right]} \\
&= |\lambda_3|
\end{aligned}$$

Case (1.2b). Let

$$\frac{2\left|s - \bar{s}p\right| + \left|s^2 - 4p\right|}{4 - \left|s\right|^2} = \left|\lambda_3\right|.$$

As in Step 1, this equation gives us

$$\left|\frac{2pz - s}{2 - sz}\right| \leq |\lambda_3|\,, \text{ for all } z \in \overline{\mathbb{D}}.$$



That is, the unit disc $\mathbb{D}$ is mapped onto the disc $S(\mathbb{D}) = \{w \in \mathbb{C} : |w - C| < R\}$, and there exists $z_0 \in \mathbb{T}$ such that $S(z_0) = \dfrac{spz_0 - s}{2 - sz_0} = \omega_3$ and $|\omega_3| = |\lambda_3|$. It follows from (2.1.18) that h_1 attains modulus 1 at $\lambda_3 \in \mathbb{D}$. Using (2.1.17) with

$$h(\lambda_3) = \frac{2pz_0 - s}{2 - sz_0} = \omega_3$$

and

$$h_1(\lambda) = \frac{h(\lambda)}{\lambda}, \quad \lambda \neq \lambda_1,$$

we have

$$|h_1(\lambda_3)| = \left|\frac{\omega_3}{\lambda_3}\right| = 1, \quad \lambda_3 \in \mathbb{D}.$$

Therefore by maximum modulus, h_1 is constant, and

$$h_1(z) = \frac{\omega_3}{\lambda_3} \text{ for all } z \in \mathbb{D}.$$

By (2.1.18),

$$h_1(\lambda_2) = \frac{\alpha}{\lambda_2} = \frac{\omega_3}{\lambda_3},$$

and

$$\left|\frac{\omega_3}{\lambda_3}\right| = 1 \text{ implies } \frac{|\alpha|}{|\lambda_2|} = 1.$$

Therefore $|\alpha| = |\lambda_2|$. It has been shown that for $|\alpha| = |\lambda_2|$, we have

$$s = -\frac{2\alpha\lambda_3}{\lambda_2}, \quad p = \frac{\alpha^2\lambda_3^2}{\lambda_2^2}.$$

Notice also that for such (s, p), (2.1.12) holds with equality. □

Remark 2.1.22. *If the case (1.2a) or (1.2b) holds, then the solution of Problem* (2.1.16) *is given by* $h(\lambda) = \dfrac{\alpha\lambda}{\lambda_2}$ *for all* $\lambda \in \mathbb{D}$.

Sufficient conditions for the data (2.1.10) to satisfy $\mathcal{C}_1$ condition are the following.

Proposition 2.1.23. *Given* $\lambda_1 = 0,\ \lambda_2 \neq \lambda_3$ *in* $\mathbb{D},\ \ \alpha \in \mathbb{D} \setminus \{0\},\ \ (s, p) \in \mathbb{G}$. *Suppose*

$$|\alpha| < |\lambda_2|, \tag{2.1.28}$$

$$\frac{2|s - \bar{s}p| + |s^2 - 4p|}{4 - |s|^2} < |\lambda_3|, \tag{2.1.29}$$

$$\left|\bar{\lambda}_2\lambda_3 s + 2\bar{\alpha}p\right| < \left|2\overline{\lambda_2}\lambda_3 + \bar{\alpha}s\right|, \tag{2.1.30}$$

and

$$\frac{\left|b\bar{d} - a\bar{c}\right| + |ad - bc|}{|d|^2 - |c|^2} < \rho(\lambda_2, \lambda_3), \tag{2.1.31}$$

where

$$\begin{aligned}
a &= 2\lambda_2 p + \alpha\lambda_3 s \\
b &= -(2\alpha\lambda_3 + \lambda_2 s) \\
c &= -(\bar{\lambda}_2\lambda_3 s + 2\bar{\alpha}p) \\
d &= 2\bar{\lambda}_2\lambda_3 + \bar{\alpha}s.
\end{aligned}$$

Then the data $0 \mapsto (0, 0)$, $\lambda_2 \mapsto (-2\alpha, \alpha^2)$, $\lambda_3 \mapsto (s, p)$ *satisfy* $\mathcal{C}_\nu$ *for all* $\nu \geq 1$.

Proof. Let $z \in \mathbb{D}$. Conditions (2.1.28) and (2.1.29) imply

$$\left|\frac{\alpha}{\lambda_2}\right| < 1 \text{ and } \left|\frac{1}{\lambda_3}\Phi_z(s, p)\right| < 1,$$

hence

$$\frac{\alpha}{\lambda_2}, \frac{1}{\lambda_3}\Phi_z(s,p) \in \mathbb{D}.$$

Therefore

$$\sup_{z\in\mathbb{D}} |\Phi_z(s,p)| < |\lambda_3|$$

and

$$\sup_{z\in\mathbb{D}} \left| B_{\frac{\alpha}{\lambda_2}} \left(\frac{1}{\lambda_3}\Phi_z(s,p) \right) \right| < \rho(\lambda_2,\lambda_3).$$

Consider the constant function

$$h_2(\lambda) = \frac{B_{\frac{\alpha}{\lambda_2}}\left(\frac{1}{\lambda_3}\cdot\Phi_z(s,p)\right)}{B_{\lambda_2}(\lambda_3)} = \beta, \text{ for all } \lambda \in \mathbb{D}. \tag{2.1.32}$$

We apply the Schur augmentation technique, see Section B.2.
Let $h_1 : \mathbb{D} \to \mathbb{D}$ be the Schur augmentation of h_2 at λ by λ_2, $\frac{\alpha}{\lambda_2}$. Then

$$h_1(\lambda) = B_{-\frac{\alpha}{\lambda_2}} \circ \left(B_{\lambda_2}(\lambda)h_2(\lambda)\right). \tag{2.1.33}$$

We have

$$\begin{aligned}
h_1(\lambda_2) &= B_{-\frac{\alpha}{\lambda_2}} \circ \left(B_{\lambda_2}(\lambda_2)h_2(\lambda_2)\right)\\
&= B_{-\frac{\alpha}{\lambda_2}}(0)\\
&= \frac{\alpha}{\lambda_2}, \text{ and}\\
h_1(\lambda_3) &= B_{-\frac{\alpha}{\lambda_2}} \circ \left(B_{\lambda_2}(\lambda_3)\cdot\frac{B_{\frac{\alpha}{\lambda_2}}\left(\frac{1}{\lambda_3}\cdot\Phi_z(s,p)\right)}{B_{\lambda_2}(\lambda_3)} \right)\\
&= B_{-\frac{\alpha}{\lambda_2}} \circ \left(B_{\frac{\alpha}{\lambda_2}}\left(\frac{1}{\lambda_3}\cdot\Phi_z(s,p)\right)\right)\\
&= \frac{1}{\lambda_3}\cdot\Phi_z(s,p).
\end{aligned}$$

Define $h : \mathbb{D} \to \mathbb{D}$ by $h(\lambda) = \lambda h_1(\lambda)$ for all $\lambda \in \mathbb{D}$. Then we have

$$\begin{aligned}
&h(\lambda_1) = 0,\\
&h(\lambda_2) = \lambda_2 h_1(\lambda_2) = \alpha, \text{ and}\\
&h(\lambda_3) = \lambda_3 h_1(\lambda_3) = \Phi_z(s,p).
\end{aligned}$$

Since $0 = \Phi_z(0,0)$, $\alpha = \Phi_z(-2\alpha,\alpha^2)$,and $\Phi_z(s,p) \in \mathbb{D}$ for all $z \in \mathbb{D}$, it follows that the data $0 \mapsto (0,0)$, $\lambda_2 \mapsto (-2\alpha,\alpha^2)$, and $\lambda_3 \mapsto (s,p)$ satisfy $\mathcal{C}_\nu$ for all $\nu \geq 1$. □

2.2 The spectral Nevanlinna-Pick interpolation problem

We consider μ-synthesis problem for the special case of μ, the spectral radius of a square matrix A, $r(A)$.
The spectral Nevanlinna-Pick problem $\mu = r$ is stated as follows: given distinct points $\lambda_1, \cdots, \lambda_n \in \mathbb{D}$ and $k \times k$ matrices $W_1, \cdots, W_n$, construct an analytic $k \times k$ matrix function F on $\mathbb{D}$ such that

$$F(\lambda_j) = W_j \qquad \text{for } j = 1, \cdots, n \tag{2.2.1}$$

and

$$r(F(\lambda)) \leq 1 \qquad \text{for all } \lambda \in \mathbb{D}. \tag{2.2.2}$$

We describe several approaches to the solution of this problem. We use Hari Bercovici's result [13] to prove a solvability criterion for a special case of the three point spectral interpolation problem.
The Nevanlinna-Pick problem for $k \geq 2$: given distinct points $\lambda_j \in \mathbb{D}$, $1 \leq j \leq n$, and $k \times k$ complex matrices $W_1, \cdots, W_n$, find necessary and sufficient conditions for the existence of an analytic $k \times k$ matrix valued function

$$F : \mathbb{D} \to \mathbb{C}^{k\times k} \text{ with } F(\lambda_j) = W_j,\ 1 \leq j \leq n, \text{ and such that } \|F\| \leq 1. \tag{2.2.3}$$

For $W \in \mathbb{C}^{k\times k}$, its conjugate transpose is denoted W^*.

Theorem 2.2.1. [11, Pick's criteria, Chapter 18] Let $\lambda_1, \cdots \lambda_n$ be distinct points in $\mathbb{D}$ and let $W_1, \cdots, W_1$ be $k \times k$ matrices with entries in $\mathbb{C}$. The following statements are equivalent.

(i) There exists an analytic $k \times k$ matrix valued function $F : \mathbb{D} \to \mathbb{C}^{k\times k}$ such that

$$F(\lambda_j) = W_j,\ 1 \leq j \leq n,$$

and

$$\|F\| \leq 1.$$

(ii) The matrix

$$\left[(I - W_j^* W_i)/(1 - \overline{\lambda_j}\lambda_i)\right]_{i,j=1}^n,$$

is positive semi-definite.

When we consider $k \times k$ matrices with $k = 1$, this problem is the classical Nevanlinna-Pick problem, for which there is a criteria by Pick's theorem. There is an analytic theory for spectral Nevanlinna-Pick problem with $k = 2$, obtained by Agler and Young. It states as follows:

Theorem 2.2.2. [9, Main Theorem 0.1] *Let $\lambda_1, \lambda_2 \in \mathbb{D}$ be distinct points, let W_1, W_2 be non-scalar 2×2 matrices of spectral radius less than 1 and let $s_j = \operatorname{tr} W_j,\ p_j = \det W_j$ for $j = 1, 2$. The following three statements are equivalent:*

(1) there exists an analytic function $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ such that

$$F(\lambda_1) = W_1, \qquad F(\lambda_2) = W_2$$

and

$$r(F(\lambda)) \leq 1, \qquad \text{for all } \lambda \in \mathbb{D};$$

(2)

$$\max_{\omega \in \mathbb{T}} \left| \frac{(s_2 p_1 - s_1 p_2)\omega^2 + 2(p_2 - p_1)\omega + s_1 - s_2}{(s_1 - \overline{s_2} p_1)\omega^2 - 2(1 - p_1 \overline{p_2})\omega + \overline{s_2} - s_1 \overline{p_2}} \right| \leq \left| \frac{\lambda_1 - \lambda_2}{1 - \overline{\lambda_2}\lambda_1} \right|;$$

(3)

$$\left[\frac{\overline{(2 - \omega s_i)}(2 - \omega s_j) - \overline{(2\omega p_i - s_i)}(2\omega p_j - s_j)}{1 - \overline{\lambda}_i \lambda_j} \right]_{i,j=1}^{2} \geq 0$$

for all $\omega \in \mathbb{T}$.

In fact, for target 2×2 matrices, the solvability of the spectral Nevanlinna-Pick problem is equivalent to the existence of a map $f : \mathbb{D} \to \Gamma$ satisfying the property stated below.

Theorem 2.2.3. *[9, Theorem 1.1] Let $\lambda_1, \cdots, \lambda_n$ be distinct in $\mathbb{D}$ and let $W_1, \cdots, W_n$ be 2×2 matrices. Suppose that either all or none of $W_1, \cdots, W_n$ are scalar matrices. The following statements are equivalent.*

(1) there exists an analytic 2×2 matrix function F in $\mathbb{D}$ such that $F(\lambda_j) = W_j, \quad j = 1, \cdots, n$ and $r(F(\lambda)) \leq 1$ for all $\lambda \in \mathbb{D}$;

(2) there exists an analytic function $f : \mathbb{D} \to \Gamma$ such that $f(\lambda_j) = (\operatorname{tr} W_j, \det W_j), \quad j = 1, \cdots, n$. □

Here $\operatorname{tr} W$ and $\det W$ denote the trace and the determinant of a matrix W.

In [3], Agler, Lykova and Young studied the spectral Nevanlinna-Pick interpolation problem as a quadratic semidefinite program subject to certain matrix inequalities. They proved the following.

Theorem 2.2.4. [3, Theorem 8.1] *Let $n \geq 1$, let $\lambda_1, \cdots, \lambda_n$ be distinct point in $\mathbb{D}$, and let $(s_j, p_j) \in \Gamma$ for $j = 1, \cdots, n$. Let z_1, z_2, z_3 be distinct points in $\mathbb{D}$. The following three conditions are equivalent.*

(1) *There exists an analytic function* $h : \mathbb{D} \to \Gamma$ *satisfying*

$$h(\lambda_j) = (s_j, p_j) \text{ for } j = 1, \cdots, n; \tag{2.1}$$

(2) *there exists a rational* Γ*-inner function h satisfying* (2.1);

(3) *there exists positive* $3n$*-square matrices* $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *of rank at most* 1 *and* $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *such that, for* $1 \le i, j \le n$ *and* $1 \le l, k \le 3$,

$$1 - \overline{\left(\frac{2z_l p_i - s_i}{2 - z_l s_i}\right)} \frac{2z_k p_j - s_j}{2 - z_k s_j} = (1 - \overline{z_l} z_k) N_{il,jk} + (1 - \overline{\lambda_i}\lambda_j) M_{il,jk}; \tag{2.2}$$

(4) *there exist* $3n$*-square matrices* $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *of rank at most* 1 *and* $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *such that*

$$\left[1 - \overline{\left(\frac{2z_l p_i - s_i}{2 - z_l s_i}\right)} \frac{2z_k p_j - s_j}{2 - z_k s_j}\right] \ge [(1 - \overline{z_l} z_k) N_{il,jk} + (1 - \overline{\lambda_i}\lambda_j) M_{il,jk}]; \tag{2.2}$$

Note: In Theorem 2.2.4 (4), we have a condition that rank $N \le 1$, and so the problem is not convex.

A close relationship between Theorem 2.2.4 and a criterion for μ-synthesis problem was stated in [3, Theorem 8.4]. A similar result for the existence of solutions for n-point spectral Nevanlinna-Pick problem for the generic case that none of the W_j, $j = 1, \cdots, n$, is a scalar multiple of the identity was earlier obtained by Agler and Young:

Theorem 2.2.5. [8, Main Theorem 0.1] *Let* $\lambda_1, \cdots, \lambda_n$ *be distinct points in* $\mathbb{D}$ *for some* $n \in \mathbb{N}$ *and let* $W_1 \cdots, W_n$ *be* 2×2 *matrices, none of them a scalar multiple of the identity. The following two statements are equivalent:*

(1) *there exists an analytic* 2×2 *matrix function F on* $\mathbb{D}$ *such that* $F(\lambda_j) = W_j$, $1 \le j \le n$, *and* $r(F(\lambda)) \le 1$ *for all* $\lambda \in \mathbb{D}$;

(2) *there exists* $b_1 \cdots, b_n, c_1 \cdots, c_n \in \mathbb{C}$ *such that*

$$\left[\frac{I - \begin{bmatrix} \frac{1}{2}s_i & b_i \\ c_i & -\frac{1}{2}s_i \end{bmatrix}^* \begin{bmatrix} \frac{1}{2}s_j & b_j \\ c_j & -\frac{1}{2}s_j \end{bmatrix}}{1 - \overline{\lambda_i}\lambda_j}\right]_{i,j=1}^{n} \ge 0$$

where

$$s_j = \operatorname{tr} W_j, \qquad p_j = \det W_j$$

and

$$b_j c_j = p_j - \frac{s_j^2}{4}, \qquad 1 \le j \le n.$$

A refinement of the result of Agler and Young was obtained by Hari Bercovici [13]. Bercovici's result admits some target data W_j that are scalar multiples of the identity matrix. The result shows a close relationship between bounding the operator F with norm and bounding F by its spectral radius. It is stated below.

Theorem 2.2.6. [13, Theorem 2.2] *Fix a natural number n, distinct points $\lambda_1, \cdots, \lambda_n \in \mathbb{D}$, and matrices $W_1, \cdots, W_n \in \mathbb{C}^{2\times 2}$ such that at least one of W_j has distinct eigenvalues. The following are equivalent.*

(1) *There exists an analytic function $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ such that $F(\lambda_j) = W_j$, $1 \le j \le n$, and $r(F(\lambda)) \le 1$ for $\lambda \in \mathbb{D}$.*

(2) *There exists a bounded analytic function satisfying the conditions in (1).*

(3) *There exists an analytic function $G : \mathbb{D} \to \mathbb{C}^{2\times 2}$ such that $G(\lambda_j)$ is similar to W_j, $j = 1, \cdots, n$, and $\|G(\lambda)\| \le 1$ for $\lambda \in \mathbb{D}$.*

(4) *There exists an analytic function G satisfying the conditions in (3) such that*

$$G(\lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & a(\lambda) \end{bmatrix} \text{ for some analytic functions } a, b, c \text{ on } \mathbb{D} \text{ and for all } \lambda \in \mathbb{D}.$$

(5) *There exist matrices W_j' similar to W_j, $j = 1, \cdots, n$, such that*

$$\left[\frac{I - W_i'^* W_j'}{1 - \overline{\lambda}_i \lambda_j}\right]_{i.j=1}^{n} \ge 0.$$

(6) *There exist complex numbers $b_1, \cdots, b_n$, $c_1, \cdots, c_n \in \mathbb{C}$ with the following properties:*

  (a) $b_j c_j = \frac{1}{4} \operatorname{tr}^2 W_j - \det W_j$;

  (b) *if W_j is a scalar multiple of the identity, then $b_j = c_j = 0$;*

  (c) *if $\frac{1}{4} \operatorname{tr}^2 W_j - \det W_j = 0$ but W_j is not a scalar multiple of the identity then $b_j = 0 \neq c_j$; and* *(d) we have*

$$\left[\frac{I - W_i'^* W_j'}{1-\overline{\lambda_i}\lambda_j}\right]_{i,j=1}^n \geq 0,$$

$$\text{where } W_j' = \begin{bmatrix} a_j & b_j \\ c_j & a_j \end{bmatrix}, \text{ with } a_j = \frac{1}{2}\operatorname{tr} W_j.$$

□

2.2.1 Connection between interpolation into $\mathrm{Hol}(\mathbb{D}, \Gamma)$ and interpolation into $\mathrm{Hol}(\mathbb{D}, \Sigma)$

The sets Γ and $\mathbb{G}$ are connected with the spectral unit balls

$$\Sigma = \{A \in M_2(\mathbb{C}) : r(A) \leq 1\},$$

and

$$\Sigma^0 = \{A \in M_2(\mathbb{C}) : r(A) < 1\},$$

by the facts that $A \in \Sigma$ if and only if $(\operatorname{tr} A, \det A) \in \Gamma$ and $A \in \Sigma^0$ if and only if $(\operatorname{tr} A, \det A) \in \mathbb{G}$. The introduction of these sets gave one approach to the study of the 2×2 spectral Nevanlinna-Pick interpolation problem. *Theorem 2.2.3* states a connection between interpolation into Γ and interpolation into Σ which holds when either all target matrices are non-derogatory or scalar. When some target matrices are scalar, there is additional connection involving derivatives, see [8, Theorem 2.9]. The next theorem follows from [8, Theorem 2.9].

Theorem 2.2.7. *Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{D}$ and let $W_1, W_2, W_3 \in \Sigma$, where $W_1 = 0$, $W_2 = -\alpha I$, $\operatorname{tr} W_3 = s_0$, $\det W_3 = p_0$. The following statements are equivalent.*

(1) There exists an analytic 2×2 matrix function F such that

$$F(\lambda_j) = W_j \qquad 1 \leq j \leq 3$$

and

$$r(F(\lambda)) \leq 1 \quad \textit{for all} \quad \lambda \in \mathbb{D}$$

(2) There exists an analytic function

$$h : \mathbb{D} \to \Gamma : \lambda \mapsto (s, p)$$

such that

$$h(0) = (0,0),\ h(\lambda_2) = (-2\alpha, \alpha^2),\ h(\lambda_3) = (s_0, p_0)$$

and

$$p'(0) = 0, \qquad \alpha s'(\lambda_2) + p'(\lambda_2) = 0.$$

Proof. Suppose (1) holds. Define F in $S^{2\times 2}$ by

$$F = \begin{bmatrix} \frac{1}{2}s & b \\ c & \frac{1}{2}s \end{bmatrix},$$

where b, c are analytic functions on $\mathbb{D}$ and

$$p = \frac{1}{4}s^2 - bc.$$

Then

$$F(0) = \begin{bmatrix} 0 & b(0) \\ c(0) & 0 \end{bmatrix}, \quad \text{where } b(0)c(0) = 0,$$

$$F(\lambda_2) = \begin{bmatrix} -\alpha & b(\lambda_2) \\ c(\lambda_2) & -\alpha \end{bmatrix}, \quad \text{where } b(\lambda_2)c(\lambda_2) = 0,$$

and

$$F(\lambda_3) = \begin{bmatrix} \frac{1}{2}s_0 & b(\lambda_3) \\ c(\lambda_3) & \frac{1}{2}s_0 \end{bmatrix} \quad \text{where } b(\lambda_3)c(\lambda_3) = \frac{1}{4}s_0^2 - p_0.$$

Hence $\frac{1}{4}s^2 - p$ has double zero at 0 and λ_2. Consequently, the mapping

$$h = (\operatorname{tr} F, \det F)$$

is analytic from $\mathbb{D} \to \Gamma$ and satisfy the interpolation conditions

$$h(\lambda_j) = (\operatorname{tr} W_j, \det W_j), \ \ j \leq 3.$$

Secondly the mapping h satisfies the differential equation

$$(\frac{1}{4}s^2 - p)'(\lambda_j) = 0, \ j = 1, 2.$$

That is,

$$(\frac{1}{2}ss' - p')(\lambda_j) = 0, j = 1, 2.$$

We have

$$\begin{cases} p'(0) = 0 \\ \\ \alpha s'(\lambda_2) + p'(\lambda_2) = 0. \end{cases}$$

Conversely, suppose (2) holds. Then, by Riesz factorization theorem, (Theorem B.1.4), every function $f \in \mathcal{S}$ has a unique inner-outer factorization, expressible in the form $f = \varphi\psi$, where φ is inner and $\psi = e^c$ is outer and $e^c(0) \geq 0$. Thus $f = (\varphi\psi^{\frac{1}{2}})\psi^{\frac{1}{2}}$, here $\psi^{\frac{1}{2}} = e^{\frac{1}{2}c}$.

Consider $\frac{1}{4}s^2 - p = bc$. Let $\frac{1}{4}s^2 - p = \psi\varphi_1\varphi_2$ where ψ is outer, φ_1, φ_2 are inner and $\varphi_1(\lambda_j) = 0 = \varphi_2(\lambda_j),\ j = 1,2$. Then we can take $b = \psi^{\frac{1}{2}}\varphi_1,\quad c = \psi^{\frac{1}{2}}\varphi_2$ if and only if $\frac{1}{4}s^2 - p$ has double zero at λ_1, λ_2. We define the analytic matrix function F on $\mathbb{D}$ by

$$F = \begin{bmatrix} \frac{1}{2}s & b \\ c & \frac{1}{2}s \end{bmatrix}$$

such that

$$F(\lambda_j) = W_j,\ 1 \leq j \leq 3,\ r(F(\lambda)) \leq 1 \text{ for all } \lambda \in \mathbb{D}$$

where

$$W_1 = 0,\ W_2 = -\alpha I,\ \operatorname{tr} W_3 = s_0,\ \det W_3 = p_0.$$

Therefore there exists an analytic function $h : \mathbb{D} \to \Gamma$ such that Γ-interpolation problem

$$h(0) = (0,0),\ h(\lambda_2) = (-2\alpha, \alpha^2),\ h(\lambda_3) = (s_0, p_0),\ \alpha s'(\lambda_2) + p'(\lambda_2) = 0$$

is *solvable.* □

The following examples from [8] shows that non-derogatory structure of the target matrices are indispensable.

Example 2.2.8. [8, Example 2.3] Let $\lambda_1 = 0,\ \lambda_2 = \beta \in (0,\ 1),\ W_1 = 0$ and

$$W_2 = \begin{bmatrix} 0 & 1 \\ 0 & \frac{2\beta}{1+\beta} \end{bmatrix}.$$

Here W_1 is derogatory and W_2 is non-derogatory, the analytic function $\mathbb{D} \to \Gamma$ defined by

$$f(\lambda) = \left(\frac{2\lambda(1-\beta)}{1-\beta\lambda},\ \frac{\lambda(\lambda-\beta)}{1-\beta\lambda} \right) \tag{2.2.4}$$

satisfy $f(\lambda) = (\operatorname{tr} W_j,\ \det W_j),\ j = 1,2,$ but there is no analytic function $F : \mathbb{D} \to \Sigma$ such that $F(\lambda_j) = W_j,\ j = 1,2.$ Suppose in contradiction such an F exists. Since each entry of

F vanishes at 0, we can write $F(\lambda) = \lambda G(\lambda)$ for some analytic function $G : \mathbb{D} \to \mathbb{C}^{2\times 2}$. By Vesentini's Theory [26, Theorem 2.3.12] the function

$$\mathbb{D} \to \mathbb{R}^+ : \lambda \mapsto r(G(\lambda))$$

is subharmonic, and so attains its maximum over the disc $\{z : |z| \le t\}$ at a point of the circle $\{z : |z| = t\}$, for any $t \in (0, 1)$. Hence

$$\sup_{|\lambda|\le t} r(G(\lambda)) = \sup_{|\lambda|=t} r\left(\frac{1}{\lambda}F(\lambda)\right) = \sup_{|\lambda|=t} \frac{1}{t} r(F(\lambda)) \le \frac{1}{t}, \quad 0 < t < 1.$$

This implies that $G(\lambda) \in \Sigma$ for all $\lambda \in \mathbb{D}$. But for

$$G(\beta) = \beta^{-1} W_2 = \begin{bmatrix} 0 & \frac{1}{\beta} \\ 0 & \frac{2}{1+\beta} \end{bmatrix},$$

the eigenvalues of $G(\beta)$ are 0 and $\dfrac{2}{1+\beta}$. Since $\dfrac{2}{1+\beta} > 1$ this contradicts $G(\beta) \in \Sigma$. The postulated $F : \mathbb{D} \to \Sigma$ cannot therefore exist.

Example 2.2.9. [8, Example 2.4] Let $\lambda_1 = 0$, $\lambda_2 = \beta \in (0, 1)$, and for $\alpha \in \mathbb{C}$,

$$W_1(\alpha) = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}, \; W_2 = \begin{bmatrix} 0 & 1 \\ 0 & \frac{2\beta}{1+\beta} \end{bmatrix}.$$

In the present example, for $\alpha \neq 0$, there is an interpolation function. For then both $W_1(\alpha)$ and W_2 are non-derogatory, and by [8, Theorem 2.1], the desired interpolating function exists if and only if there is an analytic function from $\mathbb{D} \to \Gamma$ satisfying

$$f(0) = (0, 0), \; f(\beta) = \left(\frac{2\beta}{1+\beta}, 0\right).$$

The function $f : \mathbb{D} \to \Gamma$ is given by equation (2.2.4).

When W_1 is a scalar matrix as in Example 2.2.8 we may use the Schur reduction technique to eliminate the interpolation condition. See [8, Theorem 2.4].

2.2.2 3-point spectral Nevanlinna-Pick interpolation problem

Given the spectral interpolation data

$$\begin{cases} \lambda_1 \to W_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \\ \lambda_2 \to W_2 = \begin{bmatrix} -\alpha & 0 \\ 0 & -\alpha \end{bmatrix} \\ \lambda_3 \to W_3 \end{cases} \tag{2.2.5}$$

where distinct points $\lambda_1 = 0, \lambda_2, \lambda_3 \in \mathbb{D}$, $\alpha \in \mathbb{D} \setminus \{0\}$, and $W_3 \in \mathbb{C}^{2\times 2}$ has distinct eigenvalues and spectral radius, $r(W_3) \leq 1$, $\operatorname{tr} W_3 = s$ and $\det W_3 = p$. Find an analytic 2×2 matrix function F such that

$$F(\lambda_j) = W_j, \qquad j = 1, 2, 3$$

and

$$r(F(\lambda)) \leq 1 \text{ for all } \lambda \in \mathbb{D}.$$

Notice that the target data comprise both scalar and nonscalar matrices. We shall derive solvability conditions for this 3-point Nevanlinna-Pick data using the result of Hari Bercovici [13]. It will help us to generate examples of solvable and unsolvable 3-point spectral Nevanlinna-Pick problems.

Theorem 2.2.10. *The spectral interpolation Problem* (2.2.5) *is solvable if and only if there exist* $b_3, c_3 \in \mathbb{C}$ *such that the quantities* k_1, k_2, k_3, k_4 *defined by*

$$\begin{aligned}
k_1 &= \rho(\lambda_2, \lambda_3)^2 \left|1 + \frac{\alpha \bar{s}}{2\lambda_2 \overline{\lambda_3}}\right|^2 - \left|\frac{s}{2\lambda_3} + \frac{\alpha}{\lambda_2}\right|^2, \\
k_2 &= \rho(\lambda_2, \lambda_3)^2 \left|\frac{\alpha}{\lambda_2 \overline{\lambda_3}}\right|^2 - \left|\frac{1}{\lambda_3}\right|^2, \\
k_3 &= \rho(\lambda_2, \lambda_3)^2 \frac{\bar{\alpha}}{\overline{\lambda_2}\lambda_3} - \frac{\bar{\alpha}}{\overline{\lambda_2}\lambda_3} + \left(\frac{1}{2}\rho(\lambda_2, \lambda_3)^2 \left|\frac{\alpha}{\lambda_2 \overline{\lambda_3}}\right|^2 - \frac{1}{2}\left|\frac{1}{\lambda_3}\right|^2\right) \bar{s}, \\
k_4 &= \frac{1}{4}s^2 - p
\end{aligned}$$

satisfy

$$\begin{cases}
-\frac{k_2}{k_1}\left|k_4\right|^2 \leq \left|b_3\right|^2 \leq -\frac{k_1}{k_2} \\
-\frac{k_2}{k_1}\left|k_4\right|^2 \leq \left|c_3\right|^2 \leq -\frac{k_1}{k_2} \\
(k_1 k_2 - |k_3|^2)(|b_3|^2 + |c_3|^2) + k_1^2 + k_2^2 \left|k_4\right|^2 - 2Re(k_3^2 k_4) \geq 0 \\
b_3\, c_3 = k_4 \\
k_1 > 0 \\
k_2 < 0.
\end{cases} \tag{2.2.6}$$

Proof. By Bercovici's Theorem 2.2.6 [(1) is equivalent to (6)], the spectral interpolation problem (2.2.5) is solvable if and only if there are $b_3, c_3 \in \mathbb{C}$ such that the following Nevanlinna-Pick interpolation problem is solvable

$$\begin{cases} \lambda_1 \mapsto W_1' = 0 \\ \lambda_2 \mapsto W_2' = -\alpha I, \\ \lambda_3 \mapsto W_3' = \begin{bmatrix} \frac{1}{2}s & b_3 \\ c_3 & \frac{1}{2}s \end{bmatrix}, \end{cases} \quad W_j' \in \mathbb{C}^{2\times 2} \tag{2.2.7}$$

where $s = \operatorname{tr} W_3,\ p = \det W_3$. To solve the Nevanlinna-Pick problem (2.2.7) for some $b_3, c_3 \in \mathbb{C}$ satisfying

$$b_3 c_3 = \frac{1}{4}s^2 - p,$$

we need to find an analytic function $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ such that

$$F(\lambda_j) = W_j', \ \ j = 1, 2, 3,$$

and

$$\|F(\lambda)\| \leq 1 \text{ for all } \ \lambda \in \mathbb{D}.$$

We shall apply Schur reduction to solve the above problem. Let F be any analytic function which interpolates the data of problem (2.2.7) and let $G(\lambda)$ be the Schur reduction of F at λ_1. Then

$$G(\lambda) = \frac{F(\lambda)}{\lambda},\ \lambda \neq \lambda_1.$$

Therefore

$$\begin{cases} G(\lambda_2) = -\frac{\alpha}{\lambda_2} I \\ \\ G(\lambda_3) = \frac{1}{\lambda_3} W_3'. \end{cases}$$

Similarly, if H is the Schur reduction of G at λ_2, then

$$H(\lambda) = \frac{1}{B_{\lambda_2}(\lambda)}(G(\lambda) + \frac{\alpha}{\lambda_2} I)(I + \frac{\overline{\alpha}}{\overline{\lambda_2}} G(\lambda))^{-1}.$$

At $\lambda = \lambda_3$

$$H(\lambda_3) = \frac{1 - \overline{\lambda_2}\lambda_3}{\lambda_3 - \lambda_2}(\frac{1}{\lambda_3} W_3' + \frac{\alpha}{\lambda_2} I)(I + \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3} W_3')^{-1}$$

The problem (2.2.7) is solvable if and only if

$$\|H(\lambda_3)\| = \|\frac{1 - \overline{\lambda_2}\lambda_3}{\lambda_3 - \lambda_2}\left(\frac{1}{\lambda_3} W_3' + \frac{\alpha}{\lambda_2} I\right)\left(I + \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3} W_3'\right)^{-1}\| \leq 1.$$

Therefore,

$$\|(\frac{1}{\lambda_3}W_3' + \frac{\alpha}{\lambda_2}I)(I + \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3}W_3')^{-1}\| \leq \left|\frac{\lambda_3 - \lambda_2}{1 - \overline{\lambda_2}\lambda_3}\right| = \rho(\lambda_2, \lambda_3)$$

In view of Proposition B.1.6, $\rho(\lambda_2, \lambda_3)^2 I - T^*T \geq 0$ where

$$T = (\frac{1}{\lambda_3}W_3' + \frac{\alpha}{\lambda_2}I)(I + \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3}W_3')^{-1}.$$

Therefore,

$$\rho(\lambda_2, \lambda_3)^2 I - \left[(\frac{1}{\lambda_3}W_3' + \frac{\alpha}{\lambda_2}I)(I + \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3}W_3')^{-1}\right]^* \left[(\frac{1}{\lambda_3}W_3' + \frac{\alpha}{\lambda_2}I)(I + \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3})^{-1}\right] \geq 0$$

That is,

$$\rho(\lambda_2, \lambda_3)^2 I - (I + \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3}W_3')^{-1*}(\frac{1}{\lambda_3}W_3' + \frac{\alpha}{\lambda_2}I)^*(\frac{1}{\lambda_3}W_3' + \frac{\alpha}{\lambda_2}I)(I + \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3}W_3')^{-1} \geq 0$$

Left multiplication by $(I + \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3}W_3')^*$ and right multiplication by $(I + \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3}W_3')$ give

$$\rho(\lambda_2, \lambda_3)^2(I + \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3}W_3')^*(I + \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3}W_3') - (\frac{1}{\lambda_3}W_3' + \frac{\alpha}{\lambda_2}I)^*(\frac{1}{\lambda_3}W_3' + \frac{\alpha}{\lambda_2}I) \geq 0$$

Let

$$D = \rho(\lambda_2, \lambda_3)^2 \underbrace{(I + \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3}W_3')^*}_{A} \underbrace{(I + \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3}W_3')}_{B} - \underbrace{(\frac{1}{\lambda_3}W_3' + \frac{\alpha}{\lambda_2}I)^*}_{C} \underbrace{(\frac{1}{\lambda_3}W_3' + \frac{\alpha}{\lambda_2}I)}_{F}$$

where

$$A = \begin{bmatrix} 1 + \frac{\alpha\overline{s}}{2\lambda_2\overline{\lambda_3}} & \frac{\alpha\overline{c_3}}{\lambda_2\overline{\lambda_3}} \\ \frac{\alpha\overline{b_3}}{\lambda_2\overline{\lambda_3}} & 1 + \frac{\alpha\overline{s}}{2\lambda_2\overline{\lambda_3}} \end{bmatrix}, \quad B = \begin{bmatrix} 1 + \frac{\overline{\alpha}s}{2\overline{\lambda_2}\lambda_3} & \frac{\overline{\alpha}b_3}{\overline{\lambda_2}\lambda_3} \\ \frac{\overline{\alpha}c_3}{\overline{\lambda_2}\lambda_3} & 1 + \frac{\overline{\alpha}s}{2\overline{\lambda_2}\lambda_3} \end{bmatrix},$$

$$C = \begin{bmatrix} \frac{\overline{s}}{2\overline{\lambda_3}} + \frac{\overline{\alpha}}{\overline{\lambda_2}} & \frac{\overline{c_3}}{\overline{\lambda_3}} \\ \frac{\overline{b_3}}{\overline{\lambda_3}} & \frac{\overline{s}}{2\overline{\lambda_3}} + \frac{\overline{\alpha}}{\overline{\lambda_2}} \end{bmatrix}, \; F = \begin{bmatrix} \frac{s}{2\lambda_3} + \frac{\alpha}{\lambda_2} & \frac{b_3}{\lambda_3} \\ \frac{c_3}{\lambda_3} & \frac{s}{2\lambda_3} + \frac{\alpha}{\lambda_2} \end{bmatrix};$$

$AB = \left((AB)_{ij}\right)_{i,j=1}^2$ where

$$(AB)_{11} = \left|1 + \frac{\alpha\overline{s}}{2\lambda_2\overline{\lambda_3}}\right|^2 + \left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 |c_3|^2$$

$$(AB)_{12} = \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3} b_3 + \frac{1}{2}\left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 \bar{s} b_3 + \frac{\alpha}{\lambda_2\overline{\lambda_3}}\overline{c_3} + \frac{1}{2}\left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 s\overline{c_3}$$

$$(AB)_{21} = \frac{\alpha}{\lambda_2\overline{\lambda_3}}\overline{b_3} + \frac{1}{2}\left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 s\overline{b_3} + \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3} c_3 + \frac{1}{2}\left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 \bar{s} c_3$$

$$(AB)_{22} = \left|1 + \frac{\alpha\bar{s}}{2\lambda_2\overline{\lambda_3}}\right|^2 + \left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 |b_3|^2$$

and $CF = \left((CF)_{ij}\right)_{i,j=1}^2$ where

$$(CF)_{11} = \left|\frac{s}{2\lambda_3} + \frac{\alpha}{\lambda_2}\right|^2 + \left|\frac{1}{\lambda_3}\right|^2 |c_3|^2$$

$$(CF)_{12} = \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3} b_3 + \frac{1}{2}\left|\frac{1}{\lambda_3}\right|^2 \bar{s} b_3 + \frac{\alpha}{\lambda_2\overline{\lambda_3}}\overline{c_3} + \frac{1}{2}\left|\frac{1}{\lambda_3}\right|^2 s\overline{c_3}$$

$$(CF)_{21} = \frac{\alpha}{\lambda_2\overline{\lambda_3}}\overline{b_3} + \frac{1}{2}\left|\frac{1}{\lambda_3}\right|^2 s\overline{b_3} + \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3} c_3 + \frac{1}{2}\left|\frac{1}{\lambda_3}\right|^2 \bar{s} c_3$$

$$(CF)_{22} = \left|\frac{s}{2\lambda_3} + \frac{\alpha}{\lambda_2}\right|^2 + \left|\frac{1}{\lambda_3}\right|^2 |b_3|^2.$$

Then

$$D = \rho(\lambda_2, \lambda_3)^2 AB - CF = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

where

$$d_{11} = \underbrace{\rho(\lambda_2,\lambda_3)^2\left|1 + \frac{\alpha\bar{s}}{2\lambda_2\overline{\lambda_3}}\right|^2 - \left|\frac{s}{2\lambda_3} + \frac{\alpha}{\lambda_2}\right|^2}_{k_1} + \underbrace{\left[\rho(\lambda_2,\lambda_3)^2\left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 - \left|\frac{1}{\lambda_3}\right|^2\right]}_{k_2} |c_3|^2$$

$$d_{12} = \underbrace{\left[\rho(\lambda_2,\lambda_3)^2\frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3} - \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3} + \left(\frac{1}{2}\rho(\lambda_2,\lambda_3)^2\left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 - \frac{1}{2}\left|\frac{1}{\lambda_3}\right|^2\right)\bar{s}\right]}_{k_3} b_3$$

$$+ \underbrace{\left[\rho(\lambda_2,\lambda_3)^2\frac{\alpha}{\lambda_2\overline{\lambda_3}} - \frac{\alpha}{\lambda_2\overline{\lambda_3}} + \left(\frac{1}{2}\rho(\lambda_2,\lambda_3)^2\left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 - \frac{1}{2}\left|\frac{1}{\lambda_3}\right|^2\right) s\right]}_{\overline{k_3}} \overline{c_3}$$

$$d_{21} = \underbrace{\left[\rho(\lambda_2,\lambda_3)^2 \frac{\alpha}{\lambda_2\overline{\lambda_3}} - \frac{\alpha}{\lambda_2\overline{\lambda_3}} + \left(\frac{1}{2}\rho(\lambda_2,\lambda_3)^2\left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 - \frac{1}{2}\left|\frac{1}{\lambda_3}\right|^2\right) s\right]}_{\overline{k_3}} \overline{b_3}$$

$$+ \underbrace{\left[\rho(\lambda_2,\lambda_3)^2 \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3} - \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3} + \left(\frac{1}{2}\rho(\lambda_2,\lambda_3)^2\left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 - \frac{1}{2}\left|\frac{1}{\lambda_3}\right|^2\right) \overline{s}\right]}_{k_3} c_3$$

$$d_{22} = \underbrace{\rho(\lambda_2,\lambda_3)^2\left|1 + \frac{\alpha\overline{s}}{2\lambda_2\overline{\lambda_3}}\right|^2 - \left|\frac{s}{2\lambda_3} + \frac{\alpha}{\lambda_2}\right|^2}_{k_1} + \underbrace{\left[\rho(\lambda_2,\lambda_3)^2\left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 - \left|\frac{1}{\lambda_3}\right|^2\right]}_{k_2} |b_3|^2.$$

Let

$$k_1 = \rho(\lambda_2,\lambda_3)^2\left|1 + \frac{\alpha\overline{s}}{2\lambda_2\overline{\lambda_3}}\right|^2 - \left|\frac{s}{2\lambda_3} + \frac{\alpha}{\lambda_2}\right|^2,$$

$$k_2 = \rho(\lambda_2,\lambda_3)^2\left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 - \left|\frac{1}{\lambda_3}\right|^2 \text{ and}$$

$$k_3 = \rho(\lambda_2,\lambda_3)^2 \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3} - \frac{\overline{\alpha}}{\overline{\lambda_2}\lambda_3} + \left(\frac{1}{2}\rho(\lambda_2,\lambda_3)^2\left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 - \frac{1}{2}\left|\frac{1}{\lambda_3}\right|^2\right) \overline{s};$$

and consider the system

$$\begin{cases} d_{11} \geq 0 \\ d_{22} \geq 0 \\ \det D \geq 0 \\ b_3\, c_3 = \frac{1}{4}s^2 - p. \end{cases} \tag{2.2.8}$$

The problem (2.2.7) is *solvable* if and only if (2.2.8) holds. Whenever problem (2.2.7) is *solvable*, the problem (2.1.10) is also solvable [8, Theorem 2.9]. By Theorem 2.1.11, the *solvability* of problem (2.1.10) implies that condition C_1 is satisfied whereby $|\alpha| \leq |\lambda_2|$. It follows that

$$\begin{aligned} k_2 &= \rho(\lambda_2,\lambda_3)^2\left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 - \left|\frac{1}{\lambda_3}\right|^2 \\ &= \left|\frac{1}{\lambda_3}\right|^2 \left[\rho(\lambda_2,\lambda_3)^2\left|\frac{\alpha}{\lambda_2}\right|^2 - 1\right] \\ &= \left|\frac{1}{\lambda_3}\right|^2 \left(\rho(\lambda_2,\lambda_3)\left|\frac{\alpha}{\lambda_2}\right| + 1\right)\left(\rho(\lambda_2,\lambda_3)\left|\frac{\alpha}{\lambda_2}\right| - 1\right). \end{aligned}$$

Clearly,

$$\left|\frac{1}{\lambda_3}\right|^2 \left(\rho(\lambda_2,\lambda_3)\left|\frac{\alpha}{\lambda_2}\right| + 1\right) > 0.$$

However , since $\lambda_2 \neq \lambda_3$ we have

$$\left(\rho(\lambda_2,\lambda_3)\left|\frac{\alpha}{\lambda_2}\right| - 1\right) < 0.$$

Therefore $k_2 < 0$.

Note that if $k_1 \leq 0$ then since $k_2 < 0$, this will imply that $d_{11} < 0$, and $d_{22} < 0$. In this case the matrix function D cannot be positive semi-definite and hence there is no solution. Therefore for some $b_3, c_3 \in \mathbb{C}$,

$$d_{11} = k_1 + k_2 |c_3|^2 \geq 0 \text{ and } d_{22} = k_1 + k_2 |b_3| \geq 0$$

if and only if $k_1 > 0$.

For this case, write $k_4 = \frac{1}{4}s^2 - p$, implying $|b_3|\,|c_3| = |k_4|$. Since $k_1 > 0$, then for some $b_3, c_3 \in \mathbb{C}$, the system

$$\begin{cases} k_1 + k_2 |c_3|^2 \geq 0 \\ k_1 + k_2 |b_3|^2 \geq 0 \\ k_1^2 + k_1 k_2(|b_3|^2 + |c_3|^2) + k_2^2 |k_4|^2 - [|k_3|^2 (|b_3|^2 + |c_3|^2) + 2Re(k_3^2 k_4)] \geq 0 \\ b_3 c_3 = k_4 \\ k_1 > 0 \\ k_2 < 0 \end{cases}$$

is equivalent to

$$\begin{cases} |c_3|^2 \le -\frac{k_1}{k_2} \\ |b_3|^2 \le -\frac{k_1}{k_2} \\ (k_1k_2 - |k_3|^2)(|b_3|^2 + |c_3|^2) + k_1^2 + k_2^2\,|k_4|^2 - 2Re(k_3^2k_4) \ge 0 \\ b_3c_3 = k_4 \\ k_1 > 0 \\ k_2 < 0. \end{cases}$$

Substituting

$$|c_3| = \frac{|k_4|}{|b_3|}, \text{ implying, } |b_3|^2 \ge -\frac{k_2\,|k_4|^2}{k_1}$$

in the first argument and

$$|b_3| = \frac{|k_4|}{|c_3|}, \text{ implying, } |c_3|^2 \ge -\frac{k_2\,|k_4|^2}{k_1}$$

in the second argument, lead to the condition

$$\begin{cases} -\frac{k_2}{k_1}\,|k_4|^2 \le |b_3|^2 \le -\frac{k_1}{k_2} \\ -\frac{k_2}{k_1}\,|k_4|^2 \le |c_3|^2 \le -\frac{k_1}{k_2} \\ (k_1k_2 - |k_3|^2)(|b_3|^2 + |c_3|^2) + k_1^2 + k_2^2\,|k_4|^2 - 2Re(k_3^2k_4) \ge 0 \\ b_3c_3 = k_4 \\ k_1 > 0 \\ k_2 < 0. \end{cases} \tag{2.2.9}$$

□

Chapter 3

An interpolation problem for the tetrablock

3.1 The tetrablock

The set $\mathbb{E} \subset \mathbb{C}^3$ called the *tetrablock* was introduced in [1] in connection with a μ-synthesis problem.

Definition 3.1.1. *The* tetrablock *is the domain defined by*

$$\mathbb{E} = \{x \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 zw \neq 0 \textit{ for all } z, w \in \overline{\mathbb{D}}\}.$$

The closure of $\mathbb{E}$ is denoted by $\overline{\mathbb{E}}$. The rational functions

$$\Psi : \mathbb{C}^4 \setminus \{(z,\ x_1,\ x_2,\ x_3) \in \mathbb{C}^4 : x_2 z = 1\} \to \mathbb{C}$$

and

$$\Upsilon : \mathbb{C}^4 \setminus \{(z,\ x_1,\ x_2,\ x_3) \in \mathbb{C}^4 : x_1 z = 1\} \to \mathbb{C}$$

which are associated with $\mathbb{E}$, are defined for all $z \in \mathbb{C}$ and $x \in \mathbb{C}^3$ such that $x_2 z \neq 1$ and $x_1 z \neq 1$ respectively by

$$\Psi(z, x) = \frac{x_3 z - x_1}{x_2 z - 1}$$

and

$$\Upsilon(z, x) = \frac{x_3 z - x_2}{x_1 z - 1}.$$

For $x \in \mathbb{E}$, the linear fractional map $\Psi(., x)$ maps $\mathbb{D}$ to the open disc with centre and radius

$$\frac{x_1 - \overline{x_2} x_3}{1 - |x_2|^2} \text{ and } \frac{|x_1 x_2 - x_3|}{1 - |x_2|^2},$$

respectively. Similarly if $x \in \mathbb{E}$, $\Upsilon(.,x)$ maps $\mathbb{D}$ to the open disc with centre and radius

$$\frac{x_2 - \overline{x_1}x_3}{1-|x_1|^2}, \quad \frac{|x_1x_2 - x_3|}{1-|x_1|^2},$$

respectively. For $x \in \mathbb{E}$ such that $x_1x_2 = x_3$, the functions $\Psi(.,x)$ and $\Upsilon(.,x)$ are constant equal to x_1 and x_2 respectively. Hence we have

$$\begin{aligned} \|\Psi(.,x)\|_{H^\infty} &= \sup_{z\in\mathbb{D}} |\Psi(z,x)| \\ &= \begin{cases} \frac{|x_1-\overline{x_2}x_3|+|x_1x_2-x_3|}{1-|x_2|^2} & \text{if} \quad |x_2|<1,\ x_1x_2 \neq x_3 \\ |x_1| & \text{if} \quad x_1x_2 = x_3 \\ \infty & \textit{otherwise} \end{cases} \end{aligned} \tag{3.1.1}$$

and

$$\begin{aligned} \|\Upsilon\|_{H^\infty} &= \sup_{z\in\mathbb{D}} |\Upsilon(z,x)| \\ &= \begin{cases} \frac{|x_2-\overline{x_1}x_3|+|x_1x_2-x_3|}{1-|x_1|^2} & \text{if} \quad |x_1|<1,\ x_1x_2 \neq x_3 \\ |x_2| & \text{if} \quad x_1x_2 = x_3 \\ \infty & \text{otherwise.} \end{cases} \end{aligned} \tag{3.1.2}$$

For a 2×2 matrix A, to determine whether $\mu(A) \leq 1$ in $\mathbb{C}^3$, we need to know the number $(a_{11}, a_{22}, \det A) \in \mathbb{C}^3$. From [1], the closed tetrablock is described as the set

$$\overline{\mathbb{E}} = \left\{ (a_{11},\ a_{22},\ \det A) : A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ with } \|A\| \leq 1 \right\} \subset \mathbb{C}^3.$$

In other words, $\mathbb{E}$ is the image of the Cartan domain of the open unit ball in the space of 2×2 matrices under the map

$$\mathbb{C}^{2\times 2} \ni [a_{ij}] \to (a_{11},\ a_{22},\ \det[a_{ij}]) \in \mathbb{C}^3.$$

By [16, Proposition 3.3] the following statements hold.

Proposition 3.1.2. *Let $x = (x_1, x_2, x_3) \in \mathbb{C}^3$. The following statements are equivalent.*

(1) $x \in \overline{\mathbb{E}}$;

(2) $|\Upsilon(z,x)| \leq 1$ *for all* $z \in \mathbb{D}$ *and if* $x_1x_2 = x_3$ *then, in addition,* $|x_1| \leq 1$;

(3) $|\Psi(z,x)| \leq 1$ *for all* $z \in \mathbb{D}$ *and if* $x_1x_2 = x_3$ *then, in addition* $|x_2| \leq 1$;

(4) $|x_2 - \overline{x_1}x_3| + |x_1x_2 - x_3| \le 1 - |x_1|^2$ *and if* $x_1x_2 = x_3$ *then in addition* $|x_2| \le 1$;

(5) $|x_1 - \overline{x_2}x_3| + |x_1x_2 - x_3| \le 1 - |x_2|^2$ *and if* $x_1x_2 = x_3$ *then in addition* $|x_1| \le 1$;

(6) $|x_1|^2 + |x_2|^2 - |x_3|^2 + 2\,|x_1x_2 - x_3| \le 1$ *and* $|x_3| \le 1$;

(7) *there is a* 2×2 *matrix* $A = [a_{ij}]_{i,j=1}^2$ *such that* $\|A\| \le 1$ *and* $x = (a_{11}, a_{22}, \det A)$;

(8) *there is a symmetric* 2×2 *matrix* $A = [a_{ij}]_{i,j=1}^2$ *such that* $\|A\| \le 1$ *and* $x = (a_{11}, a_{22}, \det A)$.

3.2 Interpolation in Hol$(\mathbb{D}, \mathbb{E})$

Denote by π the mapping

$$\pi : \mathbb{C}^{2\times 2} \to \mathbb{C}^3 : A = [a_{ij}] \mapsto (a_{11}, a_{22}, \det A). \tag{3.2.1}$$

The mapping π is used to prove a connection between $\mathbb{E}$ and the set of matrices for which $\mu_{\mathrm{Diag}}(\cdot) < 1$ (see the definition (1.0.3)).

Theorem 3.2.1. *[1, Theorem 9.1] An element x of $\mathbb{C}^3$ belongs to $\mathbb{E}$ if and only if there exists $A \in \mathbb{C}^{2\times 2}$ such that $\mu_{Diag}(A) < 1$ and $x = \pi(A)$. Similarly, $x \in \overline{\mathbb{E}}$ if and only if there exists $A \in \mathbb{C}^{2\times 2}$ such that $\mu_{Diag}(A) \le 1$ and $x = \pi(A)$.*

Consequently, the interpolation problems for the set $\{A \in \mathbb{C}^{2\times 2} : \mu_{\mathrm{Diag}}(A) < 1\}$ and the tetrablock are equivalent according to the theorem below.

Theorem 3.2.2. *[1, Theorem 9.2] Let $\lambda_1, \cdots, \lambda_n$ be distinct points in $\mathbb{D}$ and let*

$$W_j = \begin{bmatrix} w_{11}^j & w_{12}^j \\ w_{21}^j & w_{22}^j \end{bmatrix}, \; j = 1, \cdots, n,$$

be 2×2 matrices such that

$$w_{11}^j w_{22}^j \ne \det W_j \text{ and } \mu_{Diag}(W_j) < 1, \; j = 1, \cdots, n.$$

The following conditions are equivalent.

(1) *There exists an analytic 2×2 matrix function F on $\mathbb{D}$, such that*

$$F(\lambda_j) = W_j, \qquad 1 \le j \le n,$$

and

$$\sup_{\lambda \in \mathbb{D}} \mu_{Diag}(F(\lambda)) < 1;$$

(2) there exist an analytic function $\varphi \in Hol(\mathbb{D}, \mathbb{E})$ *such that*

$$\varphi(\lambda_j) = (w_{11}^j,\ w_{22}^j,\ \det W_j) \text{ for } j = 1, \cdots, n.$$

We denote by $\mathcal{A}(\mathbb{E})$ the algebra of continuous functions on $\overline{\mathbb{E}}$ that are analytic on $\mathbb{E}$. A boundary for $\mathbb{E}$ is a subset B of $\overline{\mathbb{E}}$ such that every function in $\mathcal{A}(\mathbb{E})$ attains its maximum modulus on B. By [1, Theorem 2.9], $\mathbb{E}$ is polynomially convex, and so the maximal ideal space of $\mathcal{A}(\mathbb{E})$ is $\overline{\mathbb{E}}$. It follows from the theory of uniform algebras [15, Corollary 2.2.10] that there is a smallest closed boundary of $\mathbb{E}$, contained in all the closed boundaries of $\mathbb{E}$ and is called the distinguished boundary of $\mathbb{E}$ [or the Shilov boundary of $\mathcal{A}(\mathbb{E})$] denoted by $b\mathbb{E}$. The following alternative description of $b\mathbb{E}$ are given in [1, Theorem 7.1].

Proposition 3.2.3. *[1, Theorem 7.1] Let* $x = (x_1,\ x_2,\ x_3) \in \mathbb{C}^3$. *The following are equivalent.*

(1) $x \in b\mathbb{E}$;

(2) $x \in \mathbb{E}$ *and* $|x| = 1$;

(3) $x_1 = \overline{x_2} x_3,\ |x_3| = 1$ *and* $|x_2| \leq 1$.

Definition 3.2.4. *An* $\mathbb{E}$*-inner function is an analytic function* $\phi = (\phi_1,\ \phi_2,\ \phi_3) : \mathbb{D} \to \overline{\mathbb{E}}$ *such that the radial limit*

$$\lim_{r \to 1^-} \phi(r\lambda) \tag{3.2.2}$$

exists and belongs to $b\mathbb{E}$ *for almost all* $\lambda \in \mathbb{T}$ *with respect to Lebesgue measure.*

By Fatou's Theorem, Theorem B.1.1, the radial limit (3.2.2) exists for almost all $\lambda \in \mathbb{T}$ with respect to the Lebesgue measure.

3.3 An $\overline{\mathbb{E}}$-interpolation problem and a matricial Nevanlinna-Pick problem

An $\overline{\mathbb{E}}$-interpolation problem: Given n distinct points $\lambda_1, \cdots, \lambda_n$ in the open unit disc $\mathbb{D}$ and n points $x^1, \cdots, x^n$ in $\overline{\mathbb{E}}$, find if possible an analytic function

$$\varphi : \mathbb{D} \to \overline{\mathbb{E}} \text{ such that } \varphi(\lambda_j) = x^j \text{ for } j = 1, \cdots, n. \tag{3.3.1}$$

The data

$$\lambda_j \to x^j, \qquad 1 \leq j \leq n, \tag{3.3.2}$$

are called $\overline{\mathbb{E}}$-*interpolation data*. The problem is said to be solvable if there exists an analytic function $\varphi : \mathbb{D} \to \overline{\mathbb{E}}$ such that $\varphi(\lambda_j) = x^j$ for $j = 1, \cdots, n$. Any such function φ is called a solution of the $\overline{\mathbb{E}}$-interpolation problem with data (3.3.2).

One of our aims is to find criteria for the *solvability* of the $\overline{\mathbb{E}}$-interpolation problem. Brown, Lykova and Young proved the following result.

Theorem 3.3.1. *[16, Theorem 7.1] Let $x = (x_1, x_2, x_3) \in \mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$. There exists a unique function*

$$F = [F_{ij}]_1^2 \in \mathcal{S}^{2\times 2}$$

such that

$$x = (F_{11}, F_{22}, \det F),$$

and

$$|F_{12}| = |F_{21}| \text{ a.e. on } \mathbb{T},\ F_{21} \text{ is either 0 or outer, and } F_{21}(0) \geq 0.$$

Moreover, for all $\mu, \lambda \in \mathbb{D}$ and all $w, z \in \mathbb{C}$ such that

$$1 - F_{22}(\mu)w \neq 0 \text{ and } 1 - F_{22}(\lambda)z \neq 0,$$

$$\begin{aligned} 1 - \overline{\Psi(w, x(\mu))}\Psi(z, x(\lambda)) &= (1 - \overline{w}z)\overline{\gamma(\mu, w)}\gamma(\lambda, z) \\ &\quad + \eta(\mu, w)^*(I - F(\mu)^*F(\lambda))\eta(\lambda, z), \end{aligned} \tag{3.3.3}$$

where

$$\gamma(\lambda, z) := (1 - F_{22}(\lambda)z)^{-1}F_{21}(\lambda) \text{ and } \eta(\lambda, z) := \begin{bmatrix} 1 \\ z\gamma(\lambda, z) \end{bmatrix}. \tag{3.3.4}$$

The analytic matrix function F constructed in [16, Theorem 7.1] relates the property of mapping from Hol($\mathbb{D}$, $\overline{\mathbb{E}}$) and membership of the Schur class. Recall from Proposition 3.1.2 that for all $x = (x_1,\ x_2,\ x_3) \in \mathbb{E}$ such that $x_1x_2 = x_3$, we have $|x_1(\lambda)|,\ |x_2(\lambda)| \leq 1$ for all $\lambda \in \mathbb{D}$. Then by the method of construction of F in [16, Theorem 7.1], to every function $x \in$ Hol($\mathbb{D}$, $\mathbb{E}$) corresponds a unique function $F = [F_{ij}] \in \mathcal{S}^{2\times 2}$ such that $x = (F_{11},\ F_{22},\ \det F)$ and $|F_{12}| = |F_{21}|$ a.e. on $\mathbb{T}$ and F_{21} is outer or zero and $F_{21}(0) \geq 0$. Two cases arise. If $x \in$ Hol($\mathbb{D}$, $\mathbb{E}$) is such that $x_1x_2 = x_3$ then a function corresponding to it in $\mathcal{S}^{2\times 2}$ is given by

$$F = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix},$$

and satisfies the property that

$$x = (F_{11},\ F_{22},\ \det F) \text{ with } |F_{12}| = |F_{21}| = 0.$$

In the case that $x_1x_2 \neq x_3$, the $H^\infty(\mathbb{D})$ function $x_1x_2 - x_3$ is nonzero and so, by Theorem B.1.4, it has inner-outer factorization which can be written in the form

$$x_1x_2 - x_3 = \phi e^C,$$

where ϕ is inner, e^C is outer and $e^C(0) \geq 0$. Let F be defined by

$$F = \begin{bmatrix} x_1 & \phi e^{\frac{1}{2}C} \\ e^{\frac{1}{2}C} & x_2 \end{bmatrix}.$$

Then clearly,

$$\det F = x_1x_2 - \phi e^C = x_1x_2 - x_1x_2 + x_3 = x_3,$$

and

$$|F_{12}| = e^{\mathrm{Re}\left(\frac{1}{2}C\right)} = |F_{21}| \text{ a.e. on } \mathbb{T}, \ F_{21} \text{ is outer, and } F_{21}(0) \geq 0.$$

Note that, by Proposition 3.1.2 (1) $\Leftrightarrow$ (7), $x \in \overline{\mathbb{E}}$ if and only if there is a 2×2 matrix $A = [a_{i,j}]_{i,j=1}^2$ such that

$$\|A\| \leq 1 \text{ and } x = (a_{11},\ a_{22},\ \det A),$$

we have

$$(F_{11}(\lambda),\ F_{22}(\lambda),\ \det F(\lambda)) \in \overline{\mathbb{E}}$$

for all $\lambda \in \mathbb{D}$.

We have the following result which reduces the $\overline{\mathbb{E}}$-interpolation problem to a standard matricial Nevanlinna-Pick problem.

Theorem 3.3.2. *Let $\lambda_1, \cdots, \lambda_n$ be distinct points in $\mathbb{D}$ and let $x^j = (x_1^j, x_2^j, x_3^j) \in \overline{\mathbb{E}}$ for $j = 1, \cdots, n$. The following statements are equivalent.*

(1) There exists an analytic function $x : \mathbb{D} \to \overline{\mathbb{E}}$ such that

$$x(\lambda_j) = (x_1^j, x_2^j, x_3^j), \quad 1 \leq j \leq n; \tag{3.3.5}$$

(2) There exist $b_j, c_j \in \mathbb{C}$ satisfying

$$b_jc_j = x_1^jx_2^j - x_3^j, \qquad 1 \leq j \leq n, \tag{3.3.6}$$

such that the Nevanlinna-Pick interpolation problem with data

$$\lambda_j \mapsto \begin{bmatrix} x_1^j & b_j \\ c_j & x_2^j \end{bmatrix} \qquad 1 \leq j \leq n. \tag{3.3.7}$$

is solvable.

Proof. (1) $\Rightarrow$ (2) Suppose there is an analytic function $x : \mathbb{D} \to \overline{\mathbb{E}}$ such that (3.3.5) holds. Then, by Theorem 3.3.1, there is an 2×2 matrix analytic function F on $\mathbb{D}$ such that $\|F\| \leq 1$,

$$x = (x_1,\ x_2,\ x_3) = (F_{11},\ F_{22},\ \det F), \tag{3.3.8}$$

and

$$|F_{12}| = |F_{21}| \text{ a.e. on } \mathbb{T},\ F_{21} \text{ is either 0 or outer, and } F_{21}(0) \geq 0. \tag{3.3.9}$$

Let $b_j = F_{12}(\lambda_j)$ and $c_j = F_{21}(\lambda_j),\ \ i \leq j \leq n$. Then

$$F(\lambda_j) = \begin{bmatrix} x_1(\lambda_j) & F_{12}(\lambda_j) \\ F_{21}(\lambda_j) & x_2(\lambda_j) \end{bmatrix} = \begin{bmatrix} x_1^j & b_j \\ c_j & x_2^j \end{bmatrix},$$

and so

$$x_3^j = x_3(\lambda_j) = x_1^j x_2^j - b_j c_j.$$

Thus

$$b_j c_j = x_1^j x_2^j - x_3^j.$$

Hence equations (3.3.6) are satisfied and for this choice of b_j and c_j the matricial Nevanlinna-Pick problem with the data (3.3.7) is solvable by F.
(2) $\Rightarrow$ (1) Let $b_j,\ c_j$ exist such that the equations (3.3.6) hold. Let the Nevanlinna-Pick problem with data (3.3.7) be solvable by an 2×2 matrix analytic function $F = \left[F_{ij}\right]_1^2 : \mathbb{D} \to \mathbb{C}^{2\times 2}$. That is F is a 2×2 Schur function such that for all $\lambda_j \in \mathbb{D}$,

$$F(\lambda_j) = \begin{bmatrix} x_1^j & b_j \\ c_j & x_2^j \end{bmatrix}, \quad 1 \leq j \leq n.$$

Define an analytic function $x : \mathbb{D} \to \overline{\mathbb{E}}$ by

$$\begin{aligned} x_1(\lambda) &= F_{11}(\lambda), \\ x_2(\lambda) &= F_{22}(\lambda), \\ x_3(\lambda) &= \det F(\lambda) = F_{11}(\lambda)F_{22}(\lambda) - F_{12}(\lambda)F_{21}(\lambda). \end{aligned}$$

Note that since conditions (3.3.6) are satisfied, for $j = 1, \cdots, n$,

$$\begin{aligned} x_1(\lambda_j) &= F_{11}(\lambda_j) = x_1^j, \\ x_2(\lambda_j) &= F_{22}(\lambda_j) = x_2^j, \\ x_3(\lambda_j) &= \det F(\lambda_j) = x_1(\lambda_j)x_2(\lambda_j) - b_j c_j = x_3^j. \end{aligned}$$

□

Corollary 3.3.3. *Let $\lambda_j \in \mathbb{D}$, $1 \le j \le n$, be distinct points in $\mathbb{D}$ and let $(x_1^j,\ x_2^j,\ x_3^j) \in \overline{\mathbb{E}}$ such that $x_1^i x_2^j \neq x_3^j$, $1 \le j \le n$. The following statement are equivalent.*

(1) There exists an analytic function $x : \mathbb{D} \to \overline{\mathbb{E}}$ such that

$$x(\lambda_j) = (x_1,\ x_2,\ x_3)(\lambda_j) = (x_1^j,\ x_2^j,\ x_3^j),\ \ 1 \le j \le n. \tag{3.3.10}$$

(2) There exists a rational $\overline{\mathbb{E}}$-inner function $x : \mathbb{D} \to \overline{\mathbb{E}}$ such that

$$x(\lambda_j) = (x_1,\ x_2,\ x_3)(\lambda_j) = (x_1^j,\ x_2^j,\ x_3^j),\ \ 1 \le j \le n. \tag{3.3.11}$$

(3) There exist $b_j, c_j \in \mathbb{C}$ satisfying

$$b_j c_j = x_1^j x_2^j - x_3^j, \qquad 1 \le j \le n, \tag{3.3.12}$$

such that the Nevanlinna-Pick interpolation problem with data

$$\lambda_j \mapsto \begin{bmatrix} x_1^j & b_j \\ c_j & x_2^j \end{bmatrix} \qquad 1 \le j \le n. \tag{3.3.13}$$

is solvable.

Proof. We have (1) is equivalent to (2) by [16, Theorem 8.1] and (3) is eqivalent to (1) by Theorem 3.3.2. Therefore result holds. □

Our next result shows that the *solvability* condition for an $\mathbb{E}$-interpolation problem can be represented in terms of a family of positive semi-definite matrices.

Theorem 3.3.4. *Let $\lambda_1, \cdots, \lambda_n$ be distinct points in $\mathbb{D}$ and let $W_j = \left(w_{ik}^j\right)_{i,k=1}^2$, $1 \le j \le n$, be 2×2 matrices, such that $w_{11}^j w_{22}^j \neq \det W_j$, $1 \le j \le n$. The following two statements are equivalent:*

(1) there exists an analytic function $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ such that $F(\lambda_j) = W_j$, $1 \le j \le n$, and $\mu_{Diag}(F(\lambda)) \le 1$ for all $\lambda \in \mathbb{D}$;

(2) there exist $b_1, \cdots, b_n,\ c_1, \cdots, c_n \in \mathbb{C}$ such that

$$\left[\frac{I - \begin{bmatrix} w_{11}^i & b_i \\ c_i & w_{22}^i \end{bmatrix}^* \begin{bmatrix} w_{11}^j & b_j \\ c_j & w_{22}^j \end{bmatrix}}{1 - \overline{\lambda_i}\lambda_j}\right]_{i,j=1}^n \ge 0 \tag{3.3.14}$$

and

$$b_j c_j = w_{11}^j w_{22}^j - \det W_j,\ \ 1 \le j \le n.$$

Proof. By Theorem 3.2.2, since $w^j_{11}w^j_{22} \neq \det W_j$, $1 \leq j \leq n$, the existence of the desired analytic function $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ is equivalent to the existence of an analytic function $x : \mathbb{D} \to \overline{\mathbb{E}}$ such that

$$x(\lambda_j) = (w^j_{11},\ w^j_{22},\ \det W_j),\ \ 1 \leq j \leq n.$$

In other words, the μ_{Diag}-synthesis interpolation problem with data

$$\lambda_j \to W_j,\ \ 1 \leq j \leq n,$$

is solvable if and only if the $\mathbb{E}$-interpolation problem with data

$$\lambda_j \to (w^j_{11},\ w^j_{22},\ \det W_j),\ 1 \leq j \leq n, \tag{3.3.15}$$

is solvable. By Theorem 3.3.2, the $\mathbb{E}$-interpolation problem (3.3.15) is solvable if and only if there exist some complex numbers $b_j,\ c_j$ satisfying

$$b_j c_j = w^j_{11}w^j_{22} - \det W_j, \qquad 1 \leq j \leq n.$$

such that the matricial Nevanlinna-Pick problem with data

$$\lambda_j \mapsto \begin{bmatrix} w^j_{11} & b_j \\ c_j & w^j_{22} \end{bmatrix}, \qquad 1 \leq j \leq n.$$

is solvable. By the matricial version of Pick's Theorem 2.2.1, the last problem is solvable if and only if the Pick type condition (3.3.14) is satisfied. □

Corollary 3.3.5. *Let $\lambda_1, \cdots, \lambda_n$ be distinct points in $\mathbb{D}$ and let $W_j = \left(w^j_{ik}\right)^2_{i,k=1}$, $1 \leq j \leq n$, be 2×2 matrices, such that $w^j_{11}w^j_{22} \neq \det W_j$, $1 \leq j \leq n$. The following statements are equivalent.*

(1) There exists an analytic function $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ such that $F(\lambda_j) = W_j$, $1 \leq j \leq n$, and $\mu_{Diag}(F(\lambda)) \leq 1$ for all $\lambda \in \mathbb{D}$.

(2) There exists a rational function $x : \mathbb{D} \to \overline{\mathbb{E}}$ such that

$$x(\lambda_j) = (x_1,\ x_2,\ x_3)(\lambda_j) = (w^j_{11},\ w^j_{22}\ \det W_j), \quad 1 \leq j \leq n. \tag{3.3.16}$$

(3) There exist $b_1, \cdots, b_n,\ c_1, \cdots, c_n \in \mathbb{C}$ such that

$$\left[\frac{I - \begin{bmatrix} w^i_{11} & b_i \\ c_i & w^i_{22} \end{bmatrix}^* \begin{bmatrix} w^j_{11} & b_j \\ c_j & w^j_{22} \end{bmatrix}}{1 - \overline{\lambda_i}\lambda_j} \right]^n_{i,j=1} \geq 0 \tag{3.3.17}$$

where

$$b_j c_j = w_{11}^j w_{22}^j - \det W_j, \;\; 1 \le j \le n.$$

Proof. (1) ⇔ (2). By [16, Theorem 1.1], an analytic function $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ interpolates a finite number of distinct points $\lambda_j \in \mathbb{D}$ to the target matrices $W_j = \left(w_{ik}^j\right)_{i,k=1}^2$ for each $j = 1, \cdots, n$, subject to $\mu_{\text{Diag}}(F) \le 1$ if and only if there exists a rational function $x : \mathbb{D} \to \overline{\mathbb{E}}$ which satisfies equation (3.3.16).
(1) ⇔ (3). Statements (1) and (3) are equivalent by Theorem 3.3.4. Thus we have (2) if and only if (1) if and only if (3). □

3.4 Realization theory for the tetrablock

A realization formula for a class of functions is an expression for a general function in the class in terms of operators on Hilbert space. In this section, we give a realization formula for the class $\text{Hol}(\mathbb{D}, \overline{\mathbb{E}})$. The classical realization theorem is for the Schur class [7, Theorem 6.5].
In a block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A is non-singular, the Schur complement of A is defined to be

$$D - CA^{-1}B.$$

By virture of identity

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ CA^{-1} & 1 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} 1 & A^{-1}B \\ 0 & 1 \end{bmatrix}.$$

It will be convinient to use some standard engineering notation.

Let H, U and Y be Hilbert spaces and let

$$A : H \to H, \qquad B : U \to H,$$

$$C : H \to Y, \qquad D : U \to Y$$

be bounded linear operators. Then for any $z \in \mathbb{D}$, we define the operator-valued function

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](z) = D + Cz(1 - zA)^{-1}B : H \oplus U \to H \oplus Y$$

whenever $1 - Az$ is invertible. By [7, Theorem 6.5], we have the following statement.

Proposition 3.4.1. *Let H, U and Y be Hilbert spaces and let*

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] : H \oplus U \to H \oplus Y$$

be a contractive operator; then for any $z \in \mathbb{D}$,

$$\left\| D + Cz(1 - zA)^{-1}B \right\| \leq 1.$$

Theorem 3.4.2. *A function*

$$x = (x_1,\ x_2,\ x_3) : \mathbb{D} \to \mathbb{C}^3$$

maps $\mathbb{D}$ analytically into $\overline{\mathbb{E}}$ if and only if there exist a Hilbert space H and a unitary operator

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : H \oplus \mathbb{C}^2 \to H \oplus \mathbb{C}^2$$

such that, for $\lambda \in \mathbb{D}$,

$$x_1(\lambda) = \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right](\lambda),\ x_2(\lambda) = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right](\lambda) \text{ and } x_3(\lambda) = \det \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](\lambda),$$

where

$$B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} : \mathbb{C}^2 \to H, \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} : H \to \mathbb{C}^2 \quad \text{and} \quad D = \left[D_{ij}\right]_{i,j=1}^{2}.$$

Proof. Given the analytic function $x = (x_1,\ x_2,\ x_3) \in \mathrm{Hol}(\mathbb{D},\ \overline{\mathbb{E}})$, by Theorem 3.3.1, there is a unique function F in the Schur class,

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix},$$

such that

$$\begin{aligned} x_1(\lambda) &= F_{11}(\lambda), \\ x_2(\lambda) &= F_{22}(\lambda), \\ x_3(\lambda) &= \det F(\lambda) = F_{11}(\lambda)F_{22}(\lambda) - F_{21}(\lambda)F_{12}(\lambda), \quad \lambda \in \mathbb{D}. \end{aligned}$$

By the Realization Theorem [7, Theorem 6.5], there exist a Hilbert space H and a unitary operator $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ on $H \oplus \mathbb{C}^2$ such that, for all $\lambda \in \mathbb{D}$,

$$\begin{aligned} F(\lambda) &= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](\lambda) \\ &= D + C\lambda(1-\lambda A)^{-1}B. \end{aligned}$$

Since F is a contraction, that is, the operator norm $\|F\| \le 1$, we have $\|D\| \le 1$ and $1-\lambda A$ is invertible for all $\lambda \in \mathbb{D}$. Let $K = \lambda\,(1-\lambda A)^{-1} : H \to \mathbb{C}^2$. Then for all $\lambda \in \mathbb{D}$ and $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{C}^2$, we have

$$\begin{aligned} \begin{bmatrix} F_{11}(\lambda) & F_{12}(\lambda) \\ F_{21}(\lambda) & F_{22}(\lambda) \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \left(\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} K(\lambda) \begin{bmatrix} B_1 & B_2 \end{bmatrix}\right)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ &= \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} K(\lambda) \begin{bmatrix} B_1 & B_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ &= \begin{bmatrix} D_{11}z_1 + D_{12}z_2 \\ D_{21}z_1 + D_{22}z_2 \end{bmatrix} + \begin{bmatrix} C_1K(\lambda)(B_1z_1 + B_2z_2) \\ C_2k(\lambda)(B_1z_1 + B_2z_2) \end{bmatrix} \\ &= \begin{bmatrix} (D_{11} + C_1K(\lambda)B_1)\,z_1 + (D_{12} + C_1K(\lambda)B_2)\,z_2 \\ (D_{21} + C_2K(\lambda)B_1)\,z_1 + (D_{22} + C_2K(\lambda)B_2)\,z_2 \end{bmatrix} \\ &= \begin{bmatrix} D_{11} + C_1K(\lambda)B_1 & D_{12} + C_1K(\lambda)B_2 \\ D_{21} + C_2K(\lambda)B)_1 & D_{22} + C_2K(\lambda)B_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{aligned}$$

Thus,

$$F_{11}(\lambda) = D_{11} + C_1\lambda(1-\lambda A)^{-1}B_1 = \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right](\lambda) = x_1(\lambda),$$

$$F_{22}(\lambda) = D_{22} + C_2\lambda(1-\lambda A)^{-1}B_2 = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right](\lambda) = x_2(\lambda)$$

and

$$\det F(\lambda) = \det\left(D + C\lambda(1-\lambda A)^{-1}B\right) = \det\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](\lambda) = x_3(\lambda).$$

Conversely, let H be a Hilbert space and let

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : H \oplus \mathbb{C}^2 \to H \oplus \mathbb{C}^2$$

be a unitary operator such that, for all $\lambda \in \mathbb{D}$,

$$x_1(\lambda) = \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right](\lambda), \quad x_2(\lambda) = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right](\lambda) \text{ and } x_3(\lambda) = \det \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](\lambda).$$

Then

$$D_{11} + C_1\lambda(1 - \lambda A)^{-1}B_1 = y_1 : H \oplus \mathbb{C}^2 \to H \oplus \mathbb{C}^2, \quad \lambda \in \mathbb{D}$$

and

$$D_{22} + C_2\lambda(1 - \lambda A)^{-1}B_2 = y_2 : H \oplus \mathbb{C}^2 \to H \oplus \mathbb{C}^2, \quad \lambda \in \mathbb{D}.$$

Let, for $\lambda \in \mathbb{D}$,

$$\chi(\lambda) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](\lambda) = \left[\chi_{ij}\right](\lambda),$$

so that $\|\chi\|_\infty \leq 1$ by the Realization Theorem for the Schur class, [7, Theorem 6.5]. That is,

$$\chi_{jj}(\lambda) = \left[\begin{array}{c|c} A & B_j \\ \hline C_j & D_{jj} \end{array}\right](\lambda), \qquad j = 1, 2, \quad \lambda \in \mathbb{D}$$

and so

$$x_1 = \chi_{11},\ x_2 = \chi_{22} \text{ and } x_3 = \det \chi,$$

on $\mathbb{D}$. Hence, for all $\lambda \in \mathbb{D}$,

$$x(\lambda) = (\chi_{11}(\lambda),\ \chi_{22}(\lambda),\ \det \chi(\lambda)).$$

Since χ is a contraction, it follows that for all $z \in \mathbb{D}$, the mapping

$$\chi(z) = D + Cz(1 - zA)^{-1}B = \begin{bmatrix} \chi_{11}(z) & \chi_{22}(z) \\ \chi_{21}(z) & \chi_{22}(z) \end{bmatrix}$$

belongs to $\mathcal{S}^{2\times 2}$. Hence by Proposition 3.1.2, $x = (x_1,\ x_2,\ x_3) \in \mathrm{Hol}(\mathbb{D},\ \mathbb{E})$. $\square$

Appendix A

Examples

A.1 Kamara's example

In a reaction to Agler, Lykova and Young's $\Gamma-$interpolation conjecture [4, Conjecture 4.1] and Agler and Young's result, [9, Theorem 1.1], A. S. Kamara gave the following example, [23, Example 2.2]:
Let

$$\lambda_1 = 0, \;\; \lambda_2 = -0.12 + 0.5i \;\text{ and }\; \lambda_3 = -0.874, \tag{A.1.1}$$

and let

$$\alpha = -0.32 + 0.15i, \quad \beta = 0.5 + 0.77i, \quad \gamma = -0.38; \tag{A.1.2}$$

set $s = \beta + \gamma$ and $p = \beta\gamma$. Then the Γ-interpolation data

$$\begin{cases} 0 = \lambda_1 \mapsto (0, 0), \\ \qquad \lambda_2 \mapsto (-2\alpha, \alpha^2), \\ \qquad \lambda_3 \mapsto (s, p) \end{cases} \tag{A.1.3}$$

satisfy $\mathcal{C}_1$. He showed in [23] that the following spectral Nenanlinna-Pick problem, to find an analytic function $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ such that

$$\begin{cases} \lambda_1 \mapsto W_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \\ \lambda_2 \mapsto W_2 = \begin{bmatrix} -\alpha & 1 \\ 0 & -\alpha \end{bmatrix} \\ \\ \lambda_3 \mapsto W_3 = \begin{bmatrix} \beta & 1 \\ 0 & \gamma \end{bmatrix} \end{cases} \tag{A.1.4}$$

and $r(F(\lambda)) \leq 1$ for $\lambda \in \mathbb{D}$, is not solvable. Note that if the interpolation problem with data

$$\begin{cases} \lambda_1 \mapsto 0, \\ \lambda_2 \mapsto -\alpha I, \\ \lambda_3 \mapsto \begin{bmatrix} \beta & 0 \\ 0 & \gamma \end{bmatrix}, \end{cases} \tag{A.1.5}$$

is solvable then problem (A.1.4) is solvable, see Theorems (2.2.5) and (2.2.6).
Here we use Theorem (2.2.10) to show that the spectral interpolation problem with data (A.1.5) is not solvable.

Lemma A.1.1. *The data* (A.1.3) *satisfy* $\mathcal{C}_1$.

Proof. By Proposition 2.1.21, sufficient conditions for $\mathcal{C}_1$ are (2.1.11), (2.1.12), (2.1.13) and (2.1.14).

The condition (2.1.11), $|\alpha| < |\lambda_2|$, holds clearly since

$$0.3534 = |\alpha| < |\lambda_2| = 0.5142.$$

The condition (2.1.12) is $\dfrac{2|s-\bar{s}p| + |s^2-4p|}{4-|s|^2} < |\lambda_3|$ and we have

$$0.8479 = \frac{2|s-\bar{s}p| + |s^2-4p|}{4-|s|^2} < |\lambda_3| = 0.8740.$$

The condition (2.1.13) is $\left|\overline{\lambda_2}\lambda_3 s + 2\bar{\alpha}p\right| < \left|2\overline{\lambda_2}\lambda_3 + \bar{\alpha}s\right|$.
Since

$$\left|\overline{\lambda_2}\lambda_3 s + 2\bar{\alpha}p\right| = |-0.2901 + 0.3775i| = 0.4761$$

and

$$\left|2\overline{\lambda_2}\lambda_3 + \bar{\alpha}s\right| = |0.2869 + 0.6096i| = 0.6737,$$

the condition (2.1.13) is satisfied.

The condition (2.1.14) is $\dfrac{\left|b\bar{d} - a\bar{c}\right| + |ad-bc|}{|d|^2 - |c|^2} < \rho(\lambda_2, \lambda_3)$,

where

$$\begin{cases} a = 2\lambda_2 p + \alpha\lambda_3 s = 0.4727 + 0.0798i \\ b = -(2\alpha\lambda_3 + \lambda_2 s) = -0.16 + 0.2946i \\ c = -(\overline{\lambda_2}\lambda_3 s + 2\bar{\alpha}p) = 0.2901 - 0.3775i \\ d = 2\overline{\lambda_2}\lambda_3 + \bar{\alpha}s = 0.2869 + 0.6096i. \end{cases}$$

Calculations show that

$$0.8792 = \frac{\left|b\bar{d} - a\bar{c}\right| + |ad - bc|}{|d|^2 - |c|^2} < \rho(\lambda_2, \lambda_3) = 0.9083,$$

hence the inequality (2.1.14) is satisfied. Therefore, by Proposition 2.1.23, $\mathcal{C}_1$ holds for the data (A.1.3). □

Lemma A.1.2. *The spectral interpolation problem*

$$\begin{cases} \lambda_1 \mapsto 0, \\ \lambda_2 \mapsto -\alpha I, \\ \lambda_3 \mapsto \begin{bmatrix} \beta & 0 \\ 0 & \gamma \end{bmatrix}, \end{cases} \tag{A.1.6}$$

is solvable if and only if there exist $b_3, c_3 \in \mathbb{C}$ *satisfying the system*

$$\begin{cases} 0.2878 \leq |b_3|^2 \leq 0.4060, \\ 0.2878 \leq |c_3|^2 \leq 0.4060, \\ |b_3|^2 + |c_3|^2 \geq 0.6440, \\ b_3c_3 = 0.0454 + 0.3388i. \end{cases} \tag{A.1.7}$$

Proof. We apply **Theorem** 2.2.10 to obtain the complex numbers b_3, c_3. We have

$$\begin{aligned} \rho(\lambda_2, \lambda_3) &= 0.9083, \\ k_1 &= \rho(\lambda_2, \lambda_3)^2 \left|1 + \frac{\alpha\bar{s}}{2\lambda_2\overline{\lambda_3}}\right|^2 - \left|\frac{s}{2\lambda_3} + \frac{\alpha}{\lambda_2}\right|^2 \\ &= 0.3244, \\ k_2 &= \rho(\lambda_2, \lambda_3)^2 \left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 - \left|\frac{1}{\lambda_3}\right|^2 \\ &= -0.799, \\ k_3 &= \rho(\lambda_2, \lambda_3)^2 \frac{\bar{\alpha}}{\overline{\lambda_2}\lambda_3} - \frac{\bar{\alpha}}{\overline{\lambda_2}\lambda_3} + \left(\frac{1}{2}\rho(\lambda_2, \lambda_3)^2 \left|\frac{\alpha}{\lambda_2\overline{\lambda_3}}\right|^2 - \frac{1}{2}\left|\frac{1}{\lambda_3}\right|^2\right)\bar{s} \\ &= 0.038 + 0.2i, \\ &\textit{and} \\ k_4 &= \frac{1}{4}s^2 - p \\ &= 0.0454 + 0.3388i. \end{aligned}$$

Thus

$$-\frac{k_2}{k_1}|k_4|^2 = 0.2878, \quad -\frac{k_1}{k_2} = 0.406.$$

The inequalities

$$0.2878 \leq |b_3|^2 \leq 0.406 \quad \text{and} \quad 0.2878 \leq |c_3|^2 \leq 0.406 \tag{A.1.8}$$

clearly hold for some $b_3, c_3 \in \mathbb{C}$. Similarly, there are infinitely many $b_3, c_3 \in \mathbb{C}$ such that

$$b_3c_3 = 0.0454 + 0.3388i. \tag{A.1.9}$$

It remains to show that the inequality

$$(k_1k_2 - |k_3|^2)(|b_3|^2 + |c_3|^2) + k_1^2 + k_2^2|k_4|^2 - 2Re(k_3^2k_4) \geq 0$$

is not satisfied for any complex numbers with the properties in (A.1.8) and (A.1.9). Now

$$k_1k_2 - |k_3|^2 = -0.3006, \text{ and } k_1^2 + k_2^2|k_4|^2 - 2Re(k_3^2k_4) = 0.1936.$$

We have

$$-0.3006(|b_3|^2 + |c_3|^2) + 0.1936 \geq 0, \text{ implying } |b_3|^2 + |c_3|^2 \leq 0.6440.$$

The solution set of the required complex numbers $b_3, c_3 \in \mathbb{C}$ is given by the system

$$\begin{cases} 0.2878 \leq |b_3|^2 \leq 0.4060, \\ 0.2878 \leq |c_3|^2 \leq 0.4060, \\ |b_3|^2 + |c_3|^2 \leq 0.6440, \\ b_3c_3 = 0.0454 + 0.3388i. \end{cases} \tag{A.1.10}$$

Let $|b_3|^2 = x$ and $|c_3|^2 = y$, Then $xy = |k_4|^2 = 0.1168$. We transform (A.1.10) to the equivalent system

$$\begin{cases} 0.2878 \leq x \leq 0.4060, \\ 0.2878 \leq y \leq 0.4060, \\ x + y \leq 0.6440, \\ xy = 0.1168. \end{cases} \tag{A.1.11}$$

The hyperbola $xy = 0.1168$ is not in the region $x + y \leq 0.6440$ as shown in the graph. Therefore the spectral interpolation problem (A.1.6) is not solvable. □

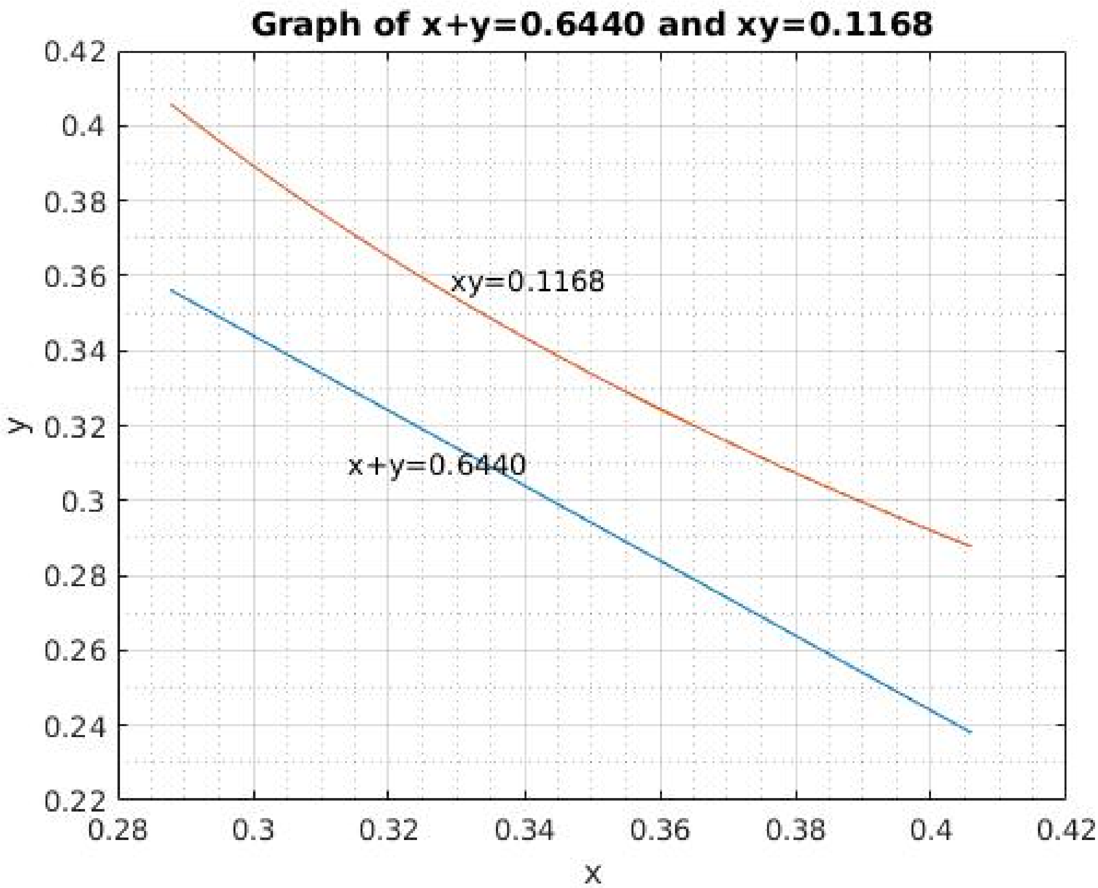

Graph of x+y=0.6440 and xy=0.1168
0.42
0.4
0.38
0.36
0.34
0.32
0.3
0.28
0.26
0.24
0.22
y
0.28
0.3
0.32
0.34
0.36
0.38
0.4
0.42
x
xy=0.1168
x+y=0.6440

Figure A.1

A.1.1 Matlab code 1

This code is used to determine a Γ-interpolation data that satisfy $\mathcal{C}_1$ condition. It is also used to find criteria for solvability of a special case of three-point spectral interpolation problem. We have used it here to cross check that the spectral interpolation problem with the data (A.1.4) where λ_1, λ_2, λ_3, α, β, γ given by equations (A.1.1) and (A.1.2) is not solvable.

```
function [s, p, rho, k_1, k_2, k_3, k_4] =
    GammaInterFunction(lambda1,lambda2,alpha, beta, gamma);
lambda1=0
lambda2=complex(-0.12,0.5)
lambda3=complex(-0.874,0)
alpha=complex(-0.32,0.15)
beta=complex(0.5,0.77)
gamma=complex(-0.38,0)
s=beta+gamma
p=beta*gamma
fprintf('We proceed to verify that the data satisfy c_1.\n')
fprintf('By proposition 2.2.6, necessary conditions for c_1 are the \n')
fprintf('conditions (2.2.5), (2.2.6), (2.2.7), (2.2.8).\n')
fprintf('The condition (2.2.5) is abs(alpha)<=abs(lambda2).\n')
fprintf('We have \n')
modalpha=abs(alpha)
modlambda2=abs(lambda2)
modlambda3=abs(lambda3)
if (modalpha<=modlambda2)
    fprintf('Clearly condition (2.2.5) holds.\n')
else
    fprintf('Condition (2.2.5) does not hold.\n')
end
fprintf('We turn to condition (2.2.6).\n')
fprintf('Let lhs226 denote the left hand side of inequality (2.2.6).\n')
fprintf('Then\n')
lhs226=(2*abs(s-conj(s)*p)+abs(s^2-4*p))/(4-abs(s)^2)
if(lhs226<=modlambda3)
    fprintf('Here we go! Condition (2.2.6) is satisfied.\n')
else
    fprintf('Condition(2.2.6) does not hold\n')
```

```
end
fprintf('We further check that condition (2.2.7) holds.\n')
fprintf('We denote the left hand side and right hand side of the \n')
fprintf('inequality (2.2.7) by lhs227 and rhs227 respectively.\n')
fprintf('Then\n')
lhs227=abs(conj(lambda2)*lambda3*s+2*conj(alpha)*p)
rhs227=abs(2*conj(lambda2)*lambda3+conj(alpha)*s)
if(lhs227<rhs227)
    fprintf('In other words, condition (2.2.7) is true.\n')
else
    fprintf('Condition (2.2.7) does not work.')
end
fprintf('Finally, we verify that condition (2.2.8) also holds.\n')
fprintf('Let \n')
a=2*lambda2*p+alpha*lambda3*s
b=-(2*alpha*lambda3+lambda2*s)
c=-(conj(lambda2)*lambda3*s+2*conj(alpha)*p)
d=2*conj(lambda2)*lambda3+conj(alpha)*s
rho=abs((lambda3-lambda2)/(1-conj(lambda2)*lambda3))
fprintf('Denote by lhs228 the left hand side of inequality (2.2.8).\n')
fprintf('We have \n')
lhs228=(abs(b*conj(d)-a*conj(c))+abs(a*d-b*c))/(abs(d)^2-abs(c)^2)
if(le(lhs228,rho))
    fprintf('Yes lhs228 is less than rho.\n')
    fprintf('Therefore condition (2.2.8) holds.\n')
else
    fprintf('No! condition (2.2.8) does not hold\n')
end

fprintf('One may want to enquire if there are complex numbers b_3, c_3 \n')
fprintf('such that the data form 3-point spectral interpolation data.\n')
fprintf('To this end, \n')
fprintf('we check that 3-point spectral interpolation conditions,\n')
fprintf('Theorem 2.3.9, hold.\n')
fprintf('Recall\n')
q_1=rho^2*abs(1+(alpha*conj(s))/(2*lambda2*conj(lambda3)))^2
q_2=abs(s/(2*lambda3)+alpha/lambda2)^2
k_1=q_1-q_2
k_2=rho^2*abs(alpha/(lambda2*conj(lambda3)))^2-abs(1/lambda3)^2
```

```
m1=conj(alpha)/(conj(lambda2)*lambda3)*(rho^2-1)
m2=0.5*(rho^2*[abs(alpha/(lambda2*conj(lambda3)))]^2-abs(1/lambda3)^2)*conj(s)
k_3=m1+m2
k_4=1/4*s^2-p
fprintf('Define\n')
coeff=k_1*k_2-abs(k_3)^2
const=k_1^2+k_2^2*abs(k_4)^2-2*real(k_3^2*k_4)
g=-const/coeff
fprintf('Let lb denote the greatest lower bound for b_3, c_3, and \n')
fprintf('let rb denote the least upper bound for b_3, c_3.\n')
fprintf('Then\n')
lb=-(k_2/k_1)*abs(k_4)^2
rb=-(k_1/k_2)
fprintf('Clearly, there are complex numbers b_3, c_3 whose moduli\n')
fprintf('lie between lb and by rb.\n')
fprintf('Note that |b_3|^2+|c_3|^2 is less than %f.\n', g)
fprintf('The spectral interpolation problem (2.3.8) is not solvable.\n')
fprintf('See graph.\n')
fprintf('Let x=|b_3|^2 and y=|c_3|^2.\n')
fprintf('Then\n')
fprintf('xy=|k_4|^2=%f\n', abs(k_4)^2)
fprintf('x+y>=%f\n',g)
c1graph3=figure(3)
x=[0.2878:0.0001:0.4060];
y1=0.6440-x;
y2=0.1168./x;
plot(x,y1);
hold on;
plot(x,y2);
hold off;
xlabel('x');
ylabel('y');
title('Graph of x+y=0.6440 and xy=0.1168');
grid on;
grid minor;
end
```

A.2 Solvable example of spectral Nevanlinna-Pick problem

Example A.2.1. *Let*

$$\lambda_1 = 0, \ \lambda_2 = -0.05 + 0.5i, \ \text{and } \lambda_3 = -0.91$$

and let

$$\alpha = -0.01 + 0.15i, \ \beta = 0.45 + 0.25i, \ \gamma = 0.05 + 0.1i;$$

set $s = \beta + \gamma$ and $p = \beta\gamma$. Then the spectral Nevanlinna-Pick problem, find an analytic function $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ such that

$$\begin{cases} \lambda_1 \mapsto W_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \\ \lambda_2 \mapsto W_2 = \begin{bmatrix} -\alpha & 0 \\ 0 & -\alpha \end{bmatrix} \\ \\ \lambda_3 \mapsto W_3 = \begin{bmatrix} \beta & 0 \\ 0 & \gamma \end{bmatrix} \end{cases} \tag{A.2.1}$$

and $r(F(\lambda)) \leq 1$ for $\lambda \in \mathbb{D}$, is solvable.

Proof. Calculations using Theorem 2.2.10 give the constants

$$\begin{aligned} k_1 &= 0.07105, \\ k_2 &= -1.119, \\ k_3 &= -0.2402 + 0.1958i, \\ k_4 &= 0.0344 + 0.0300i; \end{aligned}$$

and the solution set of all complex numbers b_3, c_3 satisfying (A.2.1):

$$\begin{cases} 0.0033 \leq |b_3|^2 \leq 0.6390, \\ 0.0033 \leq |c_3|^2 \leq 0.6390, \\ |b_3|^2 + |c_3|^2 \leq 0.5647, \\ b_3 c_3 = 0.0344 + 0.03i. \end{cases} \tag{A.2.2}$$

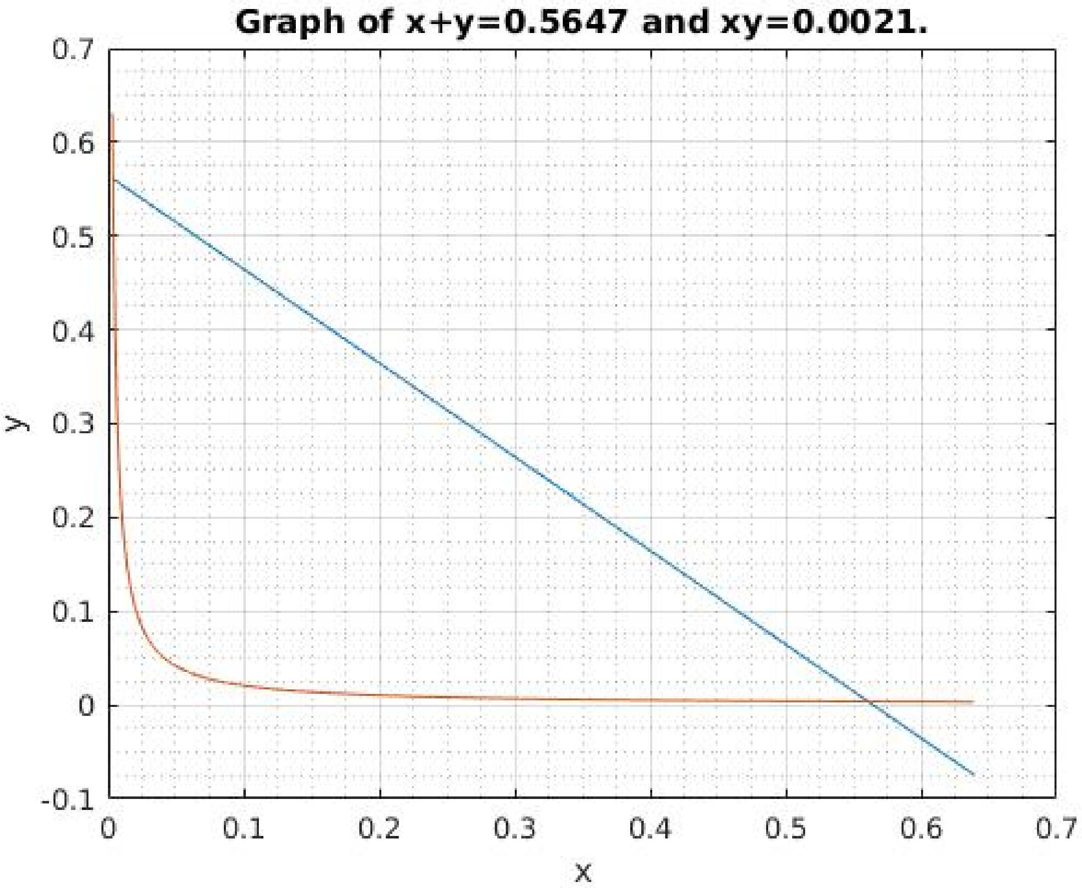


Figure A.2

Letting $|b_3|^2 = x$ and $|c_3|^2 = y$, so that $xy = |k_4|^2 = 0.0021$. We obtain the equivalent system

$$\begin{cases} 0.0033 \leq x \leq 0.6390, \\ 0.0033 \leq y \leq 0.6390, \\ x + y \leq 0.5647, \\ xy = 0.0021 \end{cases} \tag{A.2.3}$$

The hyperbola $xy = 0.0021$ lies in the region $x + y \leq 0.5647,\ 0.0033 \leq x, y < 0.561$, as shown in the graph. Therefore the spectral interpolation problem (A.2.1) is solvable. □

A.2.1 Matlab code 2

This code is used to check a Γ-interpolation data that satisfy $\mathcal{C}_1$ condition. It is also used to find criteria for solvability of a special case of three-point spectral interpolation problem. We have used it here to show that Example A.2.1 is a solvable example of 3-point spectral Nevanlinna-Pick interpolation problem.

```
function [s, p, rho, k_1, k_2, k_3, k_4] =
    GammaInterFunction2(lambda1,lambda2,alpha, beta,gamma);
lambda1=0
lambda2=complex(-0.05,0.5)
lambda3=complex(-0.91,0)
alpha=complex(-0.01,0.15)
beta=complex(0.45,0.25)
gamma=complex(0.05,0.10)
s=beta+gamma
p=beta*gamma
fprintf('We proceed to verify that the data satisfy c_1.\n')
fprintf('By proposition 2.2.6, necessary conditions for c_1 are \n')
fprintf('(2.2.5), (2.2.6), (2.2.7), and (2.2.8).\n')
fprintf('The condition (2.2.5) is abs(alpha)<=abs(lambda2).\n')
fprintf('We have \n')
modalpha=abs(alpha)
modlambda2=abs(lambda2)
modlambda3=abs(lambda3)
if (modalpha<=modlambda2)
    fprintf('Clearly condition (2.2.5) holds.\n')
else
    fprintf('Condition (2.2.5) does not hold.\n')
end
fprintf('We turn to condition (2.2.6).\n')
fprintf('Let lhs226 denote the left hand side of inequality (2.2.6).\n')
fprintf('Then\n')
lhs226=(2*abs(s-conj(s)*p)+abs(s^2-4*p))/(4-abs(s)^2)
if(lhs226<=modlambda3)
    fprintf('Here we go! Condition (2.2.6) is satisfied.\n')
else
    fprintf('Condition(2.2.6) does not hold\n')

end
```

```
fprintf('We further check that condition (2.2.7) holds.\n')
fprintf('We denote the left hand side and right hand side of the\n')
fprintf('inequality (2.2.7) by lhs227 and rhs227 respectively.\n')
fprintf('Then\n');
lhs227=abs(conj(lambda2)*lambda3*s+2*conj(alpha)*p)
rhs227=abs(2*conj(lambda2)*lambda3+conj(alpha)*s)
if(lhs227<rhs227)
    fprintf('In other words, condition (2.2.7) is true.\n')
else
    fprintf('Condition (2.2.7) does not work.')
end
fprintf('Finally, we verify that condition (2.2.8) also holds.\n')
fprintf('Let \n')
a=2*lambda2*p+alpha*lambda3*s
b=-(2*alpha*lambda3+lambda2*s)
c=-(conj(lambda2)*lambda3*s+2*conj(alpha)*p)
d=2*conj(lambda2)*lambda3+conj(alpha)*s
rho=abs((lambda3-lambda2)/(1-conj(lambda2)*lambda3))
fprintf('Denote by lhs228 the left hand side of inequality (2.2.8).\n')
fprintf('We have \n')
lhs228=(abs(b*conj(d)-a*conj(c))+abs(a*d-b*c))/(abs(d)^2-abs(c)^2)
if(le(lhs228,rho))
    fprintf('Yes lhs228 is less than rho.\n')
    fprintf('Therefore condition (2.2.8) holds.\n')
else
    fprintf('No! condition (2.2.8) does not hold\n')
end

fprintf('One may want to enquire if there are complex numbers b_3, c_3 \n')
fprintf('such that the data form 3-point spectral interpolation data.\n')
fprintf('To this end,\n')
fprintf('we check that 3-point spectral interpolation conditions,\n')
fprintf('Theorem 2.3.9, hold.\n')
fprintf('Recall\n')
q_1=rho^2*abs(1+(alpha*conj(s))/(2*lambda2*conj(lambda3)))^2
q_2=abs(s/(2*lambda3)+alpha/lambda2)^2
k_1=q_1-q_2
k_2=rho^2*abs(alpha/(lambda2*conj(lambda3)))^2-abs(1/lambda3)^2
m1=conj(alpha)/(conj(lambda2)*lambda3)*(rho^2-1)
```

```
m2=0.5*(rho^2*[abs(alpha/(lambda2*conj(lambda3)))]^2-abs(1/lambda3)^2)*conj(s)
k_3=m1+m2
k_4=1/4*s^2-p
fprintf('Define\n')
coeff=k_1*k_2-abs(k_3)^2
const=k_1^2+k_2^2*abs(k_4)^2-2*real(k_3^2*k_4)
g=-const/coeff
fprintf('Denote left boundary, lb, and right boundary, rb, for b_3, c_3.\n')
fprintf('Then')
lb=-(k_2/k_1)*abs(k_4)^2
rb=-(k_1/k_2)
fprintf('Clearly, there are complex numbers b_3, c_3 \n')
fprintf('whose moduli lie between lb and by rb.\n')
fprintf('Moreover |b_3|^2+|c_3|^2 is less than %f.\n', g)
fprintf('The spectral interpolation problem (2.3.8) is solvable.\n')
fprintf('See graph.\n')
fprintf('Let x=|b_3|^2 and y=|c_3|^2.\n')
fprintf('Then\n')
fprintf('xy=|k_4|^2=%f\n', abs(k_4)^2)
fprintf('x+y>=%f\n',g)
c1graph4=figure(4)
x=[0.0033:0.00001:0.6390];
y1=0.5647-x;
y2=0.002082./x;
plot(x,y1);
hold on;
plot(x,y2);
hold off;
xlabel('x');
ylabel('y');
title('Graph of x+y=0.5647 and xy=0.0021.');
grid on;
grid minor;
end
```

Appendix B

Background material

B.1 Basic definitions and general background materials

Let $\mathbb{D}$ be the open unit disc of the complex plane $\mathbb{C}$. For $1 \leq p < \infty$, the Hardy space $H^p(\mathbb{D})$ is the space of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\sup_{0 \leq r \leq 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{it}) \right|^p dt \right)^{\frac{1}{p}} < \infty.$$

The norm of $f \in H^p(\mathbb{D})$ is

$$\|f\|_p = \sup_{0 \leq r \leq 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{it}) \right|^p dt \right)^{\frac{1}{p}}.$$

The space $H^\infty(\mathbb{D})$ consists of all bounded analytic functions $f : \mathbb{D} \to \mathbb{C}$ with norm given by

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

An $H^\infty(\mathbb{D})$ function $f : \mathbb{D} \to \mathbb{C}$ such that $|f(\lambda)| = 1$ almost everywhere for $\lambda \in \mathbb{T}$ is called an *inner function*. An *outer function* is an analytic function f in the unit disc of the form

$$f(z) = \lambda \exp \left[\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} k(\theta) d\theta \right] \tag{B.1.1}$$

where $k \in L^1(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{R} : f \text{ is integrable on } \mathbb{T}\}$ and $\lambda \in \mathbb{T}$. The outer function f lies in $H^1(\mathbb{D})$ if and only if the exponential function e^C is integrable where

$$C(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} k(\theta) d\theta, \quad z \in \mathbb{D}.$$

We denote by $L^\infty(\mathbb{T})$ the space of all (equivalent classes of) essentially bounded functions on $\mathbb{T}$ with essential supremum norm relative to the Lebesgue measure. By Fatou's Theorem, a bounded analytic function on the disc has radial limits at every point of the unit circle.

Theorem B.1.1. *[30, Fatou] To every $f \in H^p(\mathbb{D})$ corresponds a function $g \in L^p(\mathbb{T})$, $1 \leq p \leq \infty$, defined almost everywhere on $\mathbb{T}$ by*

$$g(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}).$$

The equality $\|f\|_{H^p} = \|g\|_{L^p}$ holds.

Proposition B.1.2. *[27, pg 62] Let $f : \mathbb{D} \to \mathbb{C}$ be an outer function*

$$f(z) = \lambda \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{e}^{i\theta}+z}{\mathrm{e}^{i\theta}-z} k(\theta)\mathrm{d}\theta\right], \quad z \in \mathbb{D},$$

where $\lambda \in \mathbb{T}$ and $k \in L^1(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{R} : f \text{ is integrable on } \mathbb{T}\}$. Then

$$k(\theta) = \log\left|f(e^{i\theta})\right|$$

almost everywhere on $\mathbb{T}$.

The following are characterizations of outer functions.

Proposition B.1.3. *[27, pg 62] Let f be a nonzero function in $H^1(\mathbb{D})$. The following are equivalent.*

(i) *f is an outer function.*

(ii) *If g is any function in $H^1(\mathbb{D})$ such that $|f| = |g|$ almost everywhere on $\mathbb{T}$, then $|g(z)| \geq |f(z)|$ at each point of $z \in \mathbb{D}$.*

(iii) $\log|f(0)| = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \log\left|f(e^{i\theta})\right| d\theta.$

Theorem B.1.4. *[27, pg 63] Let f be a nonzero function in $H^1(\mathbb{D})$. Then f can be written in the form $f = \phi\psi$ where ϕ is an inner function and ψ is an outer function. This factorization is unique up to a constant of modulus one and the outer function ψ is in $H^1(\mathbb{D})$.*

Proof. Define ψ by

$$\psi(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\left|f(e^{i\theta})\right| d\theta\right),$$

then ψ is an outer function in $H^1(\mathbb{D})$. Also $\phi = f/\psi$ is an inner function. This factorization is unique for if f has another factorization $f = \phi_1\psi_1$ with ϕ_1 inner and ψ_1 is outer then $|\psi| = |\psi_1|$ a.e. on $\mathbb{T}$. One can see then that $\psi = \lambda\psi_1$ for some $\lambda \in \mathbb{T}$. So we have $\lambda\phi_1\psi_1 = \phi_1\psi_1$ and $\phi_1 = \lambda\phi$. $\square$

Let $\mathbb{C}^n$ be the set of complex n-tuples. For $v = (v_1, \cdots, v_n)$, $w = (w_1, \cdots, w_n) \in \mathbb{C}^n$, and let $\langle v, w\rangle = \sum_{j=1}^n v_j\overline{w_j}$ denote the usual inner product. The inner product $\langle \cdot, \cdot \rangle$ generates a norm on $\mathbb{C}^n$ given by

$$\|x\|_{\mathbb{C}^n} = \langle x, x\rangle^{\frac{1}{2}} = \left(\sum_{j=1}^n |x_j|^2\right)^{\frac{1}{2}}.$$

An operator $x \mapsto Tx : \mathbb{C}^n \to \mathbb{C}^m$ is a complex $m \times n$ matrix $Tx = Ax,\ x \in \mathbb{C}^n$. The operator norm of a matrix

$$A = [a_{i,j}]_{i=1,j=1}^{m,n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is given by

$$\begin{aligned} \|A\| &= \sup_{\|x\|_{\mathbb{C}^n} \le 1} \|Ax\|_{\mathbb{C}^m} \\ &= \sup_{\|x\|_{\mathbb{C}^n} \le 1} \left\| \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right\|_{\mathbb{C}^m}. \end{aligned}$$

Let X be a Banach space. A bounded linear operator $F : X \to X$ is *invertible* if there exists a bounded linear operator $F^{-1} : X \to X$ such that

$$F \circ F^{-1} = I_X \text{ and } F^{-1} \circ F = I_X.$$

Here I_X is the identity operator. The spectrum of a bounded linear operator $T : X \to X$ is the set

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}.$$

It is known that the spectrum $\sigma(T)$ is included in the closed ball of radius $\|T\|$.

Definition B.1.5. *A matrix $A \in \mathbb{C}^{n\times n}$ is called positive semi-definite if for all non-zero column vectors $z \in \mathbb{C}^n$, we have*

$$z^* A z \geq 0 \tag{B.1.2}$$

where z^* denotes the conjugate transpose of z.
If the inquality (B.1.2) *holds strictly for all $z \in \mathbb{C}^n \setminus \{0\}$, we simply say that the matrix A is positive definite.*

The following is well known.

Proposition B.1.6. *The following hold for any $T, A, B, C \in \mathbb{C}^{n\times n}$.*

(1) $I - T^*T \geq 0$ if and only if $\|T\| \leq 1$.

(2) $A \geq B$ if and only if $A - B \geq 0$.

(3) If $A \geq B$ then $C^*AC \geq C^*BC$.

Definition B.1.7. *Let $z_0 \in \mathbb{D}$. A function $x = (x_1,\ x_2,\ x_3) : \mathbb{D} \to \mathbb{E} \subset \mathbb{C}^3$ is said to be complex differentiable at z_0 if the limit,*

$$\lim_{z\to z_0} \frac{x(z) - x(z_0)}{z - z_0}$$

exists in $(\mathbb{C}^3,\ \|\cdot\|_{\mathbb{C}^3})$. We denote this limit by $x'(z_0)$ and call it the derivative of x at z_0. A function x is said to be analytic in $\mathbb{D}$ if it is complex differentiable at every point $z_0 \in \mathbb{D}$, that is, for every point $z_0 \in \mathbb{D}$, there exists $x'(z_0) \in \mathbb{C}^3$ such that

$$\lim_{z\to z_0} \left\| \frac{x(z) - x(z_0)}{z - z_0} - x'(z_0) \right\|_{\mathbb{C}^3} = 0.$$

Proposition B.1.8. *A function $x = (x_1,\ x_2,\ x_3) : \mathbb{D} \to \mathbb{E}$ is analytic on $\mathbb{D}$ if and only if each $x_i : \mathbb{D} \to \mathbb{C}$ is analytic on $\mathbb{D}$.*

Theorem B.1.9. *[18, Theorem 8.21] Let $Q : \mathbb{D} \to \mathbb{C}^{p\times m}$ be a rational $H^\infty(\mathbb{D})$ function, and let Δ be a subspace of $\mathbb{C}^{m\times p}$. Then $\mu_\Delta(Q(.))$ attains its maximum over Δ at a point on $\mathbb{T}$.*

Definition B.1.10. *A compact subset X of $\mathbb{C}^n$ is said to be polynomially convex if for every point $z \in \mathbb{C}^n \setminus X$ there is a polynomial p such that*

$$|p(z)| > \sup\{|p(x)| : x \in X\}.$$

Definition B.1.11. *A unitary operator is a bijective linear map $U : H \to H$ on a Hilbert space H such that for all $x, y \in H$, we have*

$$\langle Ux, Uy\rangle_H = \langle x, y\rangle_H.$$

Definition B.1.12. *A bounded linear mapping $T : H_1 \to H_2$ between Hilbert spaces H_1 and H_2 with $\|T\| \leq 1$, is called a contraction.*

B.2 Schur reduction and augmentation

For $\alpha, \lambda \in \mathbb{D}$, we define

$$B_\alpha(\lambda) := \frac{\lambda - \alpha}{1 - \overline{\alpha}\lambda}.$$

When $\alpha \in \mathbb{D}$, the rational function B_α is called a Blaschke factor. A Mӧbius function is a function of the form cB_α for some $\alpha \in \mathbb{D}$ and $|c| = 1$. The set of all Mӧbius functions forms the group of automorphisms of $\mathbb{D}$. A finite Blaschke product is a function which is expressible as

$$B(z) = c\prod_{j=1}^{n} \frac{z - \alpha_j}{1 - \overline{\alpha_j} z}$$

where $|\alpha_j| < 1$ and $|c| = 1$.

The following results are basic. Here $\mathcal{S}$ denotes the Schur class the analytic functions $f : \mathbb{D} \to \overline{\mathbb{D}}$.

Lemma B.2.1. *[21, Schwarz's Lemma] Suppose $f \in \mathcal{S}$ and$f(0) = 0$. Then*

$$\begin{cases} |f(z)| \leq |z|, & \text{for all } z \in \mathbb{D} \setminus \{0\}, \\ |f'(0)| \leq 1. \end{cases} \tag{B.2.1}$$

If either $|f(z)| = |z|$ for some $z \neq 0$ or $|f'(0)| = 1$ then $f(z) = e^{i\varphi}z$, for some real constant φ.

Definition B.2.2. *For a function $f : U \to \mathbb{C}$, we say that $|f|$ has a local maximum at $z_o \in U$ if there exists $\epsilon > 0$ such that $\{z \in U : |z - z_o| < \epsilon\} = N_\epsilon(z_0) \subset U$, and $|f(z)| \leq |f(z_0)|$ for all $z \in N_\epsilon(z_0)$. It is called a strict local maximum if for all $z \neq z_0$ with $|z - z_0| < \epsilon$ we have $|f(z)| < |f(z_0)|$.*

Proposition B.2.3. *[30, Theorem 10.24] Let U be a bounded domain. An analytic function $f : U \to \mathbb{C}$ has no strict local maximum of its modulus in U. If it has a local maximum, then it is constant.*

Corollary B.2.4 (Maximum modulus theorem)**.** *Let $U \subseteq \mathbb{C}$ be a bounded domain. Let f be a continuous function on $\overline{U}$ that is analytic in U. Then the maximum value of $|f|$ on $\overline{U}$ (which must occur since $\overline{U}$ is closed and bounded) must occur on ∂U.*

Definition B.2.5. *Let $f : \mathbb{D} \to \overline{\mathbb{D}}$ be analytic at $z_0 \in \mathbb{D}$ and let $f(z_0) = w_0$. The* Schur reduction *of f at z_0 is a function which is defined by*

$$g = \frac{B_{w_0} \circ f}{B_{z_0}},$$

where B_{z_0} is a Blaschke factor vanishing at z_0 :

$$B_{z_0}(z) = \frac{z - z_0}{1 - \overline{z_0} z}, \quad z \in \mathbb{D}.$$

Definition B.2.6. *Let $g : \mathbb{D} \to \overline{\mathbb{D}}$ be the Schur reduction of an analytic function $f : \mathbb{D} \to \overline{\mathbb{D}}$ satisfying $f(z_0) = w_0$. Then $f : \mathbb{D} \to \overline{\mathbb{D}}$ is called the augmentation of g at z by z_0, w_0 and is given by*

$$f(z) = B_{-w_0} \circ (B_{z_0}(z) g(z)), \quad z \in \mathbb{D}.$$

Proposition B.2.7. [20] *If a function $g : \mathbb{D} \to \overline{\mathbb{D}}$ is analytic in a neighbourhood of a closed disc $\overline{\mathbb{D}}$ and vanishes at $\alpha \in \mathbb{D}$ then either the function $\dfrac{g}{B_\alpha}$ is analytic in $\mathbb{D}$, with a removable singularity at α and maps $\mathbb{D} \to \mathbb{D}$, or $g = cB_\alpha$ for some $c \in \mathbb{T}$.*

Proof. By assumption, g is analytic in a neighbourhood of a closed disc $\overline{\mathbb{D}}$. Then its modulus $|g(z)| \leq 1$ for every $z \in \mathbb{T}$. Note that $\frac{g}{B_\alpha}$ has a removable singularity at α. Therefore, since $|B_\alpha(z)| = 1$ for every $z \in \mathbb{T}$, it follows that $|\frac{g}{B_\alpha}(z)| \leq 1$ for every $z \in \mathbb{T}$. By the maximum principle, $|\frac{g}{B_\alpha}(z)| \leq 1$ for all $z \in \mathbb{D}$. In fact $|\frac{g}{B_\alpha}(z)| < 1$ for all $z \in \mathbb{D}$ (since $\frac{g}{B_\alpha}$ is analytic and has no strict local maxima in $\mathbb{D}$) unless $g = cB_\alpha$ for some $c \in \mathbb{T}$. Therefore $\frac{g}{B_\alpha} \in \text{Hol}(\mathbb{D}, \mathbb{D})$ or $g = cB_\alpha$ for some $c \in \mathbb{T}$. □

The Schur reduction technique

The technique is well known and we will use to demonstrate the proof of Pick's Theorem.

Suppose for n distinct points $\lambda_1, \cdots, \lambda_n$ in the unit disc $\mathbb{D}$ and n points $\omega_1, \cdots, \omega_n$ in $\mathbb{D}$, an analytic function $h : \mathbb{D} \to \mathbb{D}$ satisfies

$$h(\lambda_j) = \omega_j, \quad j = 1, \cdots, n. \tag{B.2.2}$$

Then

$$B_{\omega_1} \circ h(\lambda_1) = 0. \tag{B.2.3}$$

We will now parametrize all solutions $h \in \text{Hol}(\mathbb{D}, \mathbb{D})$ of equation (B.2.2) using Proposition B.2.7 and equation (B.2.3). Let h be a solution of (B.2.2). Two cases arise.

Case 1: $h_1 = \frac{B_{\omega_1} \circ h}{B_{\lambda_1}} : \mathbb{D} \to \mathbb{D}$ is analytic.

Case 2: $h_1 = \frac{B_{\omega_1} \circ h}{B_{\lambda_1}} = c_1 \quad for \quad some \quad c_1 \in \mathbb{T}$.

The mapping h_1 is the Schur reduction of h at λ_1. Let us consider the two cases.

In **Case 2**

$$B_{\omega_1} \circ h = c_1 B_{\lambda_1}.$$

Therefore, if the problem (1.3.1) is solvable, there is a unique solution

$$h(\lambda) = B_{-\omega_1} \circ (c_1 B_{\lambda_1}(\lambda)) = \frac{c_1 B_{\lambda_1}(\lambda) + \omega_1}{1 + \overline{\omega}_1 c_1 B_{\lambda_1}(\lambda)}, \;\; \lambda \in \mathbb{D}. \tag{B.2.4}$$

Then h is the Schur augmentation of h_1 at λ_1. In this case the interpolation data (1.3.1) satisfy

$$\omega_j = \frac{c_1 B_{\lambda_1}(\lambda_j) + \omega_1}{1 + \overline{\omega_1} c_1 B_{\lambda_1}(\lambda_j)}, \qquad j = 2, \cdots, n. \tag{B.2.5}$$

This situation is non-generic.

On the other hand, if **Case 1** holds, then

$$h_1 = \frac{B_{\omega_1} \circ h}{B_{\lambda_1}},$$

and so

$$h_1(\lambda) = \frac{1 - \overline{\lambda_1}\lambda}{\lambda - \lambda_1} \cdot \frac{h(\lambda) - \omega_1}{1 - \overline{\omega}_1 h(\lambda)}, \;\; \lambda \in \mathbb{D}. \tag{B.2.6}$$

This is the generic case. Therefore the problem (1.3.1) is reduced to finding an analytic function $h_1 : \mathbb{D} \to \mathbb{D}$ such that

$$h_1(\lambda_j) = \omega_j^{(1)} \quad j = 2, \cdots, n. \tag{B.2.7}$$

where

$$\omega_j^{(1)} := \frac{1 - \overline{\lambda_1}\lambda_j}{\lambda_j - \lambda_1} \cdot \frac{\omega_j - \omega_1}{1 - \overline{\omega}_1 \omega_j}, \; j = 2, \cdots, n.$$

If any of $\omega_j^{(1)}$, $j = 2, \cdots, n$, does not lie in $\mathbb{D}$, then the problem (1.3.1) is not solvable. Otherwise, if $w_j^{(1)} \in \mathbb{D}$ for all $j = 2, \cdots, n$, then we have the following interpolation problem: for $\lambda_2, \cdots, \lambda_n \in \mathbb{D}$ and $w_j^{(1)}$, $j = 2, \cdots, n$, in $\mathbb{D}$, find an analytic function $h_1 : \mathbb{D} \to \mathbb{D}$ such that $h_1(\lambda_j) = \omega_j^{(1)}, \quad 2 \le j \le n$.
We then repeat the procedure to determine the Schur reduction of h_1 at λ_2. If the interpolation data are solvable at each λ_j then the process continues until we reduce the original problem to one-point interpolation problem which can be solved by the Schwarz-Pick lemma.

Lemma B.2.8. [7, Lemma 0.3][Schwarz-Pick] *For any analytic function $h : \mathbb{D} \to \mathbb{D}$, and $\lambda_1 \neq \lambda_2$ in $\mathbb{D}$,*

$$\rho\left(h(\lambda_1), h(\lambda_2)\right) \le \rho\left(\lambda_1, \lambda_2\right).$$

For completion, let us demonstrate the Schur augmentation process. We begin with the two point interpolation data

$$\begin{cases} \lambda_{n-1} \mapsto w_1^{(n-1)} \\ \lambda_n \mapsto w_2^{(n)}. \end{cases} \tag{B.2.8}$$

and write the constant function

$$h_{n-1}(\lambda) = \frac{B_{w_1^{(n-1)}}(h_{n-2}(\lambda_n))}{B_{\lambda_{n-1}}(\lambda_n)} = c, \qquad \lambda \in \mathbb{D}.$$

Define $h_{n-2} : \mathbb{D} \to \mathbb{D}$ by

$$h_{n-2}(\lambda) = B_{-w_1^{(n-1)}} \circ \left(B_{\lambda_{n-1}}(\lambda) h_{n-1}(\lambda)\right), \qquad \lambda \in \mathbb{D}. \tag{B.2.9}$$

Calculate for each λ_j, $1 \leq j \leq n$, the value of $h_{n-2}(\lambda_j)$, from (B.2.9). Again define h_{n-3} in terms of h_{n-2} and repeat the same principle. The procedure will continue until we obtain h, the solution of the original interpolation problem.

B.2.1 Pick condition from Theorem 3.3.4

Recall the matricial Pick condition from Theorem 3.3.4 (2) is

$$\left[\frac{I - \begin{bmatrix} w_{11}^i & b_i \\ c_i & w_{22}^i \end{bmatrix}^* \begin{bmatrix} w_{11}^j & b_j \\ c_j & w_{22}^j \end{bmatrix}}{1 - \overline{\lambda_i}\lambda_j} \right]_{i,j=1}^n \geq 0, \tag{B.2.10}$$

for some $b_1, \cdots, b_n$, $c_1, \cdots, c_n$ in $\mathbb{C}$.
Equivalently, one can write (B.2.10) as

$$\begin{bmatrix} \frac{I - W_1^* W_1}{1 - \overline{\lambda_1}\lambda_1} & \frac{I - W_1^* W_2}{1 - \overline{\lambda_1}\lambda_2} & \cdots & \frac{I - W_1^* W_n}{1 - \overline{\lambda_1}\lambda_n} \\ \frac{I - W_2^* W_1}{1 - \overline{\lambda_2}\lambda_1} & \frac{I - W_2^* W_2}{1 - \overline{\lambda_2}\lambda_2} & \cdots & \frac{I - W_2^* W_n}{1 - \overline{\lambda_2}\lambda_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{I - W_n^* W_1}{1 - \overline{\lambda_n}\lambda_1} & \frac{I - W_n^* W_2}{1 - \overline{\lambda_n}\lambda_2} & \cdots & \frac{I - W_n^* W_n}{1 - \overline{\lambda_n}\lambda_n} \end{bmatrix} \geq 0,$$

where

$$W_j = \begin{bmatrix} w^j_{11} & b_j \\ c_j & w^j_{22} \end{bmatrix},\ 1 \le j \le n$$

and

$$\frac{I - W_i^* W_j}{1 - \overline{\lambda_i}\lambda_j} = \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} w^i_{11} & b_i \\ c_i & w^i_{22} \end{bmatrix}^* \begin{bmatrix} w^i_{11} & b_j \\ c_j & w^j_{22} \end{bmatrix}}{1 - \overline{\lambda_i}\lambda_j}$$

$$= \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \overline{w^i_{11}} & \overline{c_i} \\ \overline{b_i} & \overline{w^i_{22}} \end{bmatrix} \begin{bmatrix} w^j_{11} & b_j \\ c_j & w^j_{22} \end{bmatrix}}{1 - \overline{\lambda_i}\lambda_j}$$

$$= \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \overline{w^i_{11}} w^j_{11} + \overline{c_i} c_j & \overline{w^i_{11}} b_j + \overline{c_i} w^j_{22} \\ \overline{b_i} w^j_{11} + \overline{w^i_{22}} c_j & \overline{b_i} b_j + \overline{w^i_{22}} w^j_{22} \end{bmatrix}}{1 - \overline{\lambda_i}\lambda_j}$$

$$= \begin{bmatrix} \dfrac{1 - \overline{w^i_{11}} w^j_{11} - \overline{c_i} c_j}{1 - \overline{\lambda_i}\lambda_j} & \dfrac{-\overline{w^i_{11}} b_j - \overline{c_i} w^j_{22}}{1 - \overline{\lambda_i}\lambda_j} \\ \dfrac{-\overline{b_i} w^j_{11} - \overline{w^1_{22}} c_j}{1 - \overline{\lambda_i}\lambda_j} & \dfrac{1 - \overline{b_i} b_j - \overline{w^i_{22}} w^j_{22}}{1 - \overline{\lambda_i}\lambda_j} \end{bmatrix}.$$

Appendix C

Examples of aligned and caddywhompus Γ-inner functions

Here we give examples of aligned and caddywhompus Γ-inner functions which were constructed by Agler, Lykova and Young in [5].

Example C.0.1. *[5, Example 13.2]*
(1) Consider the Γ-inner function

$$h(\lambda) = \left(2(1-r)\frac{\lambda^2}{1+r\lambda^3}, \frac{\lambda(\lambda^3+r)}{1+r\lambda^3} \right), \qquad \lambda \in \mathbb{D}. \tag{C.0.1}$$

The royal nodes of h in $\mathbb{T}$ are the three cube roots w_j of -1 and $\frac{1}{2}\overline{s(w_j)} = -w_j$ for each j. Hence h is aligned.

(2) Let $0 < \alpha < 1$ and let h be the symmetrization of the two Blaschke products λ^2 and $B_\alpha B_{-\alpha}$, that is,

$$h(\lambda) = (\lambda^2 + B_\alpha B_{-\alpha}(\lambda),\ \lambda^2 B_\alpha B_{-\alpha}(\lambda))$$

where

$$B_\alpha(\lambda) = \frac{\lambda - \alpha}{1 - \overline{\alpha}\lambda}.$$

The royal nodes of h are the points λ for which $\lambda^2 = B_\alpha B_{-\alpha}(\lambda) = B_{\alpha^2}(\lambda^2)$, which are the points $\lambda = 1,\ i,\ -1,\ -i$. The table of the royal nodes w_j and the target values $\frac{1}{2}\overline{s(w_j)}$ is given below. Clearly, for any choice of 3 royal nodes w_j, there are two corresponding target values $\frac{1}{2}\overline{s(w_j)}$, and hence the target values are not in the same cyclic order as the nodes. Hence, the degree 4 Γ-inner function h is caddywhompus.

j	1	2	3	4
Royal nodes w_j	1	i	-1	-i
$\frac{1}{2}\overline{s(w_j)}$	1	-1	1	-1.

(3) Let $-1 < \alpha < 1$ and h be a symmetrization of the Blaschke products λ^3 and B_α, so that

$$h(\lambda) = (\lambda^3 + B_\alpha(\lambda),\ \lambda^3 B_\alpha(\lambda)). \tag{C.0.2}$$

Here

$$(s^2 - 4p)(\lambda) = \frac{(\lambda^2-1)^2(\alpha\lambda^2 - \lambda + \alpha)^2}{(1-\alpha\lambda)^2}$$

and so the royal nodes of h are the points $1,\ -1$ and

$$\frac{1 \pm \sqrt{1-4\alpha^2}}{2\alpha}. \tag{C.0.3}$$

Thus if $|\alpha| < \frac{1}{2}$ then h has 4 royal nodes in $\mathbb{R}$, to wit 1, -1, and the two points (C.0.3) of which one is in $\mathbb{D}$ and one lies outside $\overline{\mathbb{D}}$. When $\alpha = \pm\frac{1}{2}$ the only royal nodes of h are 1 and -1. Thus for $|\alpha| \leq \frac{1}{2}$, h is neither aligned or caddywhompus. When $\frac{1}{2} < |\alpha| < 1$, though, the nodes (C.0.3) lie in $\mathbb{T}$, and so h has four royal nodes in $\mathbb{T}$. For example when $\alpha = \frac{-1}{\sqrt{3}}$ one has a royal node $w = e^{i5\pi/6}$ and $\frac{1}{2}\overline{s(w)} = -i$. The images of the nodes under $\frac{1}{2}\bar{s}$ are in opposite cyclic order to the nodes themselves. I follows that $\frac{1}{2}\bar{s}$ maps every triple of royal nodes to a triple of distinct points in $\mathbb{T}$ in the opposite cyclic order. Thus h is caddywhompus.

(4) Let $h(\lambda) = (\lambda^2 + B_\alpha(\lambda),\ \lambda^2 B_\alpha(\lambda))$ where $-1 < \alpha < 1$. The function h is a Γ-inner function of degree 3 having 1 as a royal node in $\mathbb{T}$. There are 3 cases. If $\frac{1}{3} < \alpha < 1$ then h has 3 distinct royal nodes in $\mathbb{T}$, to wit $1,\ w,\ \overline{w}$ where

$$w = \frac{1}{2\alpha}(1 - \alpha + i\sqrt{(3\alpha-1)(1+\alpha)}).$$

Since h has degree 3 and has 2 royal nodes h is aligned.
For $\alpha \leq \frac{1}{3}$ there is only one royal node of h in $\mathbb{T}$ (to wit, the point 1), and so h is not aligned. When $-1 < \alpha < \frac{1}{3}$ there are two other royal nodes, of which one is in $\mathbb{D}$ and the other is in $\mathbb{C} \setminus \overline{\mathbb{D}}$. When $\alpha = \frac{1}{3}$,

$$(s^2 - 4p)(\lambda) = \frac{(\lambda-1)^6}{(3-\lambda)^2}$$

and all the royal nodes coalesce at 1.

We state here the associated problem of [5, Theorem 1.1].
Given data $\lambda_j, s_j, p_j,\ \ j = 1,2,3,$ that satisfy condition $\mathcal{C}_1$ extremally with auxiliary extremal $m \in \text{Aut}\,\mathbb{D}$ find a Blaschke product p of degree at most 4 such that

$$p(\lambda_j) = p_j, \qquad j = 1,2,3, \tag{C.0.4}$$

and

$$p(\tau_l) = \overline{m}(\tau_l)^2, \qquad l = 1, \cdots, d(mq), \tag{C.0.5}$$

where the τ_l are the roots of the equation $mq(\tau) = 1$ and q is the unique function in the Schur class such that

$$q(\lambda_j) = \Phi(m(\lambda_j), s_j, p_j), \qquad j = 1,2,3.$$

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