Operator algebras associated to semigroup actions

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Thesis submitted for the degree of
Doctor of Philosophy

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May 2019

Abstract

Reflexivity offers a way of reconstructing an algebra from a set of invariant subspaces. It is considered as Noncommutative Spectral Synthesis in association with synthesis problems in commutative Harmonic Analysis. Large classes of algebras are reflexive, the prototypical example being von Neumann algebras. The first example in the nonselfadjoint setting was the algebra of the unilateral shift which was shown by Sarason in the 1960s. Some further examples include the influential work of Arveson on CSL algebras, the $\mathbb{H}^p$ Hardy algebras examined by Peligrad, tensor products with the Hardy algebras and nest algebras. The concept of reflexivity was extended by Arveson who introduced the notion of hyperreflexivity. This is a measure of the distance to an algebra in terms of the invariant subspaces. It is a stronger property than reflexivity and examples include nest algebras, the free semigroup algebra and the algebra of analytic Toeplitz operators.

Here we consider these questions for the class of w*-semicrossed products, in particular, those arising from actions of the free semigroup and the free abelian semigroup. We show that they are hyperreflexive when the action is implemented by uniformly bounded row operators. Combining our results with those of Helmer, we derive that w*-semicrossed products of factors of any type are reflexive. Furthermore, we show that w*-semicrossed products of automorphic actions on maximal abelian selfadjoint algebras are reflexive. In each case it is also proved that the w*-semicrossed products have the bicommutant property if and only if the initial algebra of the dynamics does also. In addition we are interested in classifying the commuting endomorphisms of $\mathcal{B}(\mathcal{H})$ as an important example of dynamics implemented by a Cuntz family. Recall that $\mathcal{O}_n$ does not have a nice representation space in the sense that there is no countable collection of Borel functions that distinguish the unitary invariants. Therefore we focus our attention on the free atomic representations, which Davidson and Pitts classified up to unitary equivalence. Specifically we give a necessary and sufficient condition for an automorphism of $\mathcal{B}(\mathcal{H})$ to commute with a cyclic endomorphism.

Acknowledgements

There are many people I would like to thank for their help throughout the duration of my PhD. First and foremost I would like to thank my supervisor Dr Evgenios Kakariadis for his continued guidance and support. Throughout the last four years he has been a source of ideas and patience and he has shown me the way to become a better mathematician. I will be forever grateful for the care he has shown for both me and my project. I would also like to thank my second supervisor, Dr Michael Dritchel for both helping me with my PhD application and for advice given to me at the end of several talks. I am also grateful to my examiners Dr David Kimsey and Dr Christian Voigt.

I am indebted to EPSRC for granting my funding to complete my PhD, I am very grateful to have been given this opportunity to study an interesting and exciting topic. I would also like offer my sincerest thanks to all the staff at the department of Mathematics, Statistics and Physics at Newcastle University. In particular I would like to thank all the members of the analysis research group for several useful discussions and comments. I would also like to thank my friends and fellow PhD students for their interest and support. It has been a pleasure to work alongside everyone for the past three and a half years. In particular I would like to thank Dimitrios Chiotis for some interesting and helpful discussions on Hardy spaces and Toeplitz operators.

I would also like to thank my family, in particular, my parents Alison Bickerton and Robert Bickerton for their continued encouragement to pursue mathematics. Finally I would like to thank my partner Emma Jones for her love and support.

Contents

Chapter 1

Introduction and historic remarks

1.1 Introduction

The study of the reflexivity of operator algebras has its roots in the work of Radjavi-Rosenthal in [45] and is closely linked to the bicommutant property and invariant subspace problems. The term reflexivity is attributed to Halmos and is used to describe an algebra that is characterised by its invariant subspaces. That is, a unital algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is reflexive if it is equal to the algebra of bounded operators which leave invariant each subspace left invariant by every operator in $\mathcal{A}$. More specifically we can investigate the lattice of invariant subspaces of $\mathcal{A}$. If $A \subseteq \mathcal{B}(\mathcal{H})$ then define the lattice of $\mathcal{A}$ and the Alg as follows

$$\operatorname{Lat}\mathcal{A} = \{p \ : \ (1-p)ap = 0 \text{ for all } a \in \mathcal{A}\},$$

and,

$$\operatorname{Alg}\mathcal{A} = \{x \ : \ (1-p)xp = 0 \text{ for all } x \in \mathcal{A}\}.$$

We can also define the AlgLat as

$$\operatorname{AlgLat}(\mathcal{A}) = \{T \in \mathcal{B}(\mathcal{H}) \ : \ \operatorname{Lat}\mathcal{A} \subseteq \operatorname{Lat} T\}.$$

Then $\mathcal{A}$ is called reflexive if $\mathcal{A} = \operatorname{AlgLat}\mathcal{A}$. Reflexivity is considered as Noncommutative Spectral Synthesis in conjunction with synthesis problems in commutative Harmonic Analysis and it offers a systematic way of reconstructing an algebra from a set of invariant subspaces [6].

An operator algebra $\mathcal{A}$ is said to have the *bicommutant property* if it coincides with its bicommutant $\mathcal{A}''$. The prototypical examples are von Neumann algebras which are reflexive and have the bicommutant property. However results are less straightforward for nonselfadjoint algebras. In [4], Arveson introduced a function β to measure reflexivity. Define

$$\beta(T, \mathcal{A}) = \sup\{\|(1-p)Tp\| : p \in \operatorname{Lat} \mathcal{A}\},$$

then an algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is *hyperreflexive* if there is a constant C such that for each $T \in \mathcal{B}(\mathcal{H})$,

$$\operatorname{dist}(T, \mathcal{A}) \leq C\beta(T, \mathcal{A}).$$

A single operator A is called hyperreflexive if the wot-closed algebra generated by A and the identity is reflexive. This is a stronger property than reflexivity and so all hyperreflexive algebras are reflexive. Examples of hyperreflexive algebras include nest algebras and abelian von Neumann algebras. In fact, Rosenoer shows that abelian von neumann algebras are hyperreflexive with constant at most 2 [47]. However it is an open question if all von Neumann algebras are hyperreflexive. Kraus-Larson [39] and Davidson [15] showed that hyperreflexivity is a hereditary property. Bercovici [9] proved that a wot-closed algebra is hyperreflexive with distance constant at most 3 when its commutant contains two isometries with orthogonal ranges.

Our purpose is to examine the hyperreflexivity of operator algebras arising from dynamical systems which encode the action of the free semigroup or the free abelian semigroup. A dynamical system consists of an operator algebra $\mathcal{A}$ as well as its (uniformly bounded) endomorphisms. From this we can construct the w*-semicrossed product of the system. As in [33] and [18] we interpret a w*-semicrossed product as an algebra densely spanned by generalised analytic polynomials subject to a set of covariance relations. Examples of algebras related to dynamical systems were examined by Kastis-Power [34] and Katavolos-Power [35]. Then in [30], Kakariadis examined the reflexivity of one-variable systems. This work was extended by Helmer [27] to the examination of Hardy algebras of w*-correspondences. Semicrossed products and their norm-closed variants have been subject to a methodical programme of research since the latter half of the twentieth century. Here we further the arguments in [30] to semicrossed products over $\mathbb{F}_+^d$ and $\mathbb{Z}_+^d$. Algebras arising from the free semigroup $\mathbb{F}_+^d$ have previously been studied by numerous authors. Some examples include; Arias and Popescu [1], Davidson, Katsoulis and Pitts [20] and Fuller and Kennedy in [24].

Further motivation arises from the results of Helmer [27]. An application of Helmer's results demonstrate the reflexivity of semicrossed products of Type II or II factors over $\mathbb{F}_+^d$. Therefore, we wish to complete this programme by studying endomorphisms of $\mathcal{B}(\mathcal{H})$. Since every endomorphism of $\mathcal{B}(\mathcal{H})$ is spatial, we focus on actions where each generator is implemented by a Cuntz family. Dynamical systems implemented by Cuntz families have been examined in the works of Kakariadis and Peters [32], Laca [40] and Courtney-Muhly-Schmidt [13] amongst others.

In Chapter 3 we introduce the notion of dynamical systems over $\mathbb{F}_+^d$ and $\mathbb{Z}_+^d$ and use suitable covariance relations to define the algebras that play the role of the w*-semicrossed products. The key feature when working over $\mathbb{F}_+^d$ is the separation into left and right-lower triangular operators (clearly this distinction is redundant in the $\mathbb{Z}_+^d$ case).

In Chapter 4 we further examine dynamics of Cuntz families. Our setting accommodates $\mathbb{Z}_+^d$-actions where the generators are implemented by unitaries but where the unitaries implementing the actions may not commute. For example any two commuting automorphisms over $\mathcal{B}(\mathcal{H})$ are implemented by two unitaries that satisfy Weyl's relation and may not commute. In fact, the same holds if we consider a maximal abelian selfajoint subalgebra (m.a.s.a.) rather than $\mathcal{B}(\mathcal{H})$. We consider free atomic representations. These were classified by Davidson and Pitts [22]. We examine the case where the representation forms a cycle and we give a necessary and sufficient condition for this to commute with an automorphism of $\mathcal{B}(\mathcal{H})$. By appealing to the results of Laca [40] we determine when an automorphism of $\mathcal{B}(\mathcal{H})$ commutes with specific endomorphisms induced by Cuntz isometries.

In Chapter 5 we state the bicommutants of several w*-semicrossed products. In each case we identify the commutant with a twisted w*-semicrossed product over the commutant. Similar algebras (in the normed case) were examined in [32]. The twisting for w*-closed algebras was explored for automorphic $\mathbb{Z}_+$-actions in [30]. We apply similiar results for $\mathbb{Z}_+^d$-actions here by noting that twisting twice gives the w*-semicrossed product over the bicommutant.

In Chapter 6 we state our reflexivity results regarding the semicrossed products whose action is implemented by a Cuntz family. We then proceed in combination with [27] to tackle systems over any factor and automorphic systems over maximal abelian selfadjoint algebras. Alongside this we translate Helmer's reflexivity proof in our context for the right-sided version. We note that the methods described here appear to be generic and may be applicable to semicrossed products created from other semigroups. For example one may be able to examine semicrossed products arising from the discrete Heisenberg semigroup.

1.2 Known Results

The first result regarding reflexivity was developed by Sarason [49] in which it was proved that normal operators and the algebra of analytic Toeplitz operators, $\mathbb{H}^\infty$ are reflexive. This result was the inspiration for much of the research regarding reflexivity, for example Radjavi-Rosenthal [46]. Sarason's results have since been extended by Peligrad to the noncommutative Hardy Spaces [43] and to algebras of commuting isometries or tensor products with the Hardy Algebras obtained by Ptak [44]. Specifically a pair of isometries $\{V_1, V_2\}$ are called doubly commuting if V_1, V_2 commute and V_1, V_2^* commute. Ptak [44] showed that every such pair is reflexive by using a decomposition due to Slociński [51].

Further positive examples are given in the work of Arveson regarding commutative subspace lattice (CSL) algebras [3] and in the consideration of the class of the nest algebras [16], in which the hypereflexivity of nest algebras is established (with constant at most 1). An algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is called a nest algebra if $\mathcal{A} = \operatorname{Alg} \mathcal{N}$, where $\mathcal{N}$ is a nest, that is $\mathcal{N}$ is a totally ordered lattice of projections. A commutative subspace lattice is a lattice of mutually commuting projections which are sot-closed. Arveson shows that such lattices are always reflexive and then gives some examples of reflexive algebras related to CSLs.

In [15] Davidson examined the distance to the algebra of analytic Toeplitz operators $\mathbb{H}^\infty$. If we let M_f be the multiplication operator given by $(M_f h)(x) = f(x)h(x)$ for $f \in L^\infty$. Then recall that the Toeplitz operator with symbol f is given by $T_f = P_{H^2} M_f|_{H^2}$, where H^2 is the Hardy space on the disk. The algebra of analytic

Toeplitz operators is given by $\mathbb{H}^\infty = \{T_\phi : \phi \in \mathbb{H}^\infty\}$. Davidson [15] showed that the analytic Toeplitz algebra is hyperreflexive with distance constant 19. Key to his arugment is the existence of a linear projection π into the space of all Toeplitz operators $\{T_f : f \in L^\infty\}$. Exsistence of this projection was established by Arveson in [4]. Davidson and Pitts [22] showed that for $n \geq 2$, the left free semigroup algebra $\mathcal{L}_d := \overline{\text{alg}}^{wot}\{\mathbf{l}_\mu \ : \ \mu \in \mathbb{F}_+^d\}$ is hyperreflexive with constant at most 51. This was later improved by Bercovici [9] who reduced the distance constant to be at most 3.

An operator A is called quasinormal if A commutes with A^*A. The arguments of Davidson in [16] were further extended by Klis and Ptak [37] in order to tackle quasinormal operators. In [37] it is shown that quasinormal operators are hyperreflexive with distance constant at most 259 by using a result of Brown [11] who shows that every quasinormal operator is unitarily equivalent to $(A \otimes S) \oplus N$ where A is positive with $\ker A = \{0\}$, N is normal and S is the unilateral shift. Rosenoer [48] adapted Davidsons results to show that $\mathcal{B}(\mathcal{H}) \otimes \mathbb{H}^\infty$ is hyperreflexive with distance constant at most 19.

Fuller and Kennedy [24] examine reflexivity for isometric n-tuples. An n-tuple of operators $(V_1, \ldots, V_n)$ acting on a Hilbert space $\mathcal{H}$ is called isometric if the row operator $(V_1, \ldots, V_n) : \mathcal{H}^n \to \mathcal{H}$ is an isometry. By using a Lebesgue-von Neumann-Wold type decomposition for an isometry Fuller and Kennedy show that isometric tuples are hyperreflexive with constant 95 if $n = 1$ (that is isometries are hyperreflexive with constant 95) and constant 6 for $n \geq 2$.

In [30] Kakariadis showed that reflexivity holds for a number of w*-semicrossed products over $\mathbb{Z}_+$. For example, it is shown that the semicrossed product $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+$ is reflexive when $\mathcal{A}$ is reflexive and α is implemented by a unitary. It is further established that $\mathcal{A}$ has the bicommutant property if and only if the resulting semicrossed product does also.

1.3 Main Results

We are primarily interested in investigating the hyperreflexivity and bicommutantant property of various semicrossed products. Here we summarise the main results developed. These results appear in [8]. We examine actions implemented by invertible row operators that satisfy a uniform bound hypothesis. Specifically we say that $\{\alpha_i\}_{i\in[d]}$ is a *uniformly bounded spatial action* on a w*-closed algebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ if every α_i is implemented by an invertible row operator u_i and $\{u_i\}_{i\in[d]}$ is uniformly bounded.

Our main results regarding the bicommutant property are encapsulated in the following corollaries.

Corollary 1.3.1 (Corollary 5.1.2)**.** *Let $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ be a w*-dynamical system of a uniformly bounded spatial action. Then the following are equivalent*

(i) $\mathcal{A}$ has the bicommutant property;

(ii) $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ has the bicommutant property;

(iii) $\mathcal{A} \overline{\times}_\alpha \mathcal{R}_d$ has the bicommutant property;

(iv) $\mathcal{A} \otimes \mathcal{L}_d$ has the bicommutant property;

(v) $\mathcal{A} \otimes \mathcal{R}_d$ has the bicommutant property.

If any of the items above hold then all algebras are inverse closed.

For dynamical systems over $\mathbb{Z}_+^d$ we have a similar result.

Corollary 1.3.2 (Corollary 5.2.2)**.** *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ be a unital w*-dynamical system. Suppose that each $\alpha_{\mathbf{i}}$ is implemented by a uniformly bounded row operator $u_{\mathbf{i}}$. Then the following are equivalent*

(i) $\mathcal{A}$ has the bicommutant property;

(ii) $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ has the bicommutant property;

(iii) $\mathcal{A} \otimes \mathbb{H}^\infty(\mathbb{Z}_+^d)$ has the bicommutant property.

In short we show that if α is an action of $\mathbb{F}_+^d$ or $\mathbb{Z}_+^d$ on a w*-closed algebra $\mathcal{A}$ where each generator of α is implemented by a Cuntz family. Then $\mathcal{A}$ has the bicommutant property if and only if any of the resulting w*-semicrossed products does also.

Our main results regarding reflexivity are given as follows. Writing n_i for the multiplicity of the Cuntz family implementing the i-th generator of the action then we define

$$N := \sum_{i=1}^{d} n_i \text{ for } \mathbb{F}_+^d\text{-systems} \quad \text{and} \quad M := \prod_{i=1}^{d} n_i \text{ for } \mathbb{Z}_+^d\text{-systems}$$

for the *capacity* of the systems. We then show the following.

Theorem 1.3.3 (Theorem 6.1.2). *Let $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ be a w*-dynamical system of a uniformly bounded spatial action. Suppose that every α_i is given by an invertible row operator $u_i = [u_{i,j_i}]_{j_i\in[n_i]}$ and set $N = \sum_{i\in[d]} n_i$.*

(i) If $N \geq 2$ then every w-closed subspace of $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ or $\mathcal{A} \overline{\times}_\alpha \mathcal{R}_d$ is hyperreflexive. If K is the uniform bound related to $\{u_i\}$ then the hyperreflexivity constant is at most $3 \cdot K^4$.*

(ii) If $N = 1$ and $\mathcal{A}$ is reflexive then $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d = \mathcal{A} \overline{\times}_\alpha \mathcal{R}_d = \mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+$ is reflexive.

From this we obtain the following corollaries.

Corollary 1.3.4 (Corollary 6.1.3). *Let $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ be a w*-dynamical system so that every α_i is given by a Cuntz family $[s_{i,j_i}]_{j_i\in[n_i]}$. If $N = \sum_{i\in[d]} n_i \geq 2$ then every w*-closed subspace of $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ or $\mathcal{A} \overline{\times}_\alpha \mathcal{R}_d$ is hyperreflexive with distance constant at most 3.*

Corollary 1.3.5 (Corollary 6.1.4). *Let $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ be a system of w*-continuous automorphisms on a maximal abelian selfadjoint algebra $\mathcal{A}$. Then $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ and $\mathcal{A} \overline{\times}_\alpha \mathcal{R}_d$ are reflexive.*

In fact, we have similar results for the case of dynamics over $\mathbb{Z}_+^d$

Theorem 1.3.6 (Theorem 6.2.1). *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ be a unital w*-dynamical system. Suppose that every $\alpha_{\mathbf{i}}$ is uniformly bounded spatial, given by an invertible row operator $u_{\mathbf{i}} = [u_{i,j_i}]_{j_i\in[n_i]}$, and set $M = \prod_{i\in[d]} n_i$.*

(i) If $M \geq 2$ then every w-closed subspace of $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ is hyperreflexive. If K_i is the uniform bound associated to $u_{\mathbf{i}}$ (and its inverse) then the hyperreflexivity constant is at most $3 \cdot K^4$ for $K = \min\{K_i \mid n_i \geq 2\}$.*

(ii) If $M = 1$ and $\mathcal{A}$ is reflexive then $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ is reflexive.

Corollary 1.3.7 (Corollary 6.2.2). *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ be a unital w*-dynamical system. Suppose that at least one $\alpha_{\mathbf{i}}$ is implemented by a Cuntz family $[s_{i,j_i}]_{j_i \in [n_i]}$ with $n_i \geq 2$. Then every w*-closed subspace of $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ is hyperreflexive with distance constant 3.*

Corollary 1.3.8 (Corollary 6.2.3). *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ be a unital automorphic system over a maximal abelian selfadjoint algebra $\mathcal{A}$. Then $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ is reflexive.*

That is, we show that if α is an action of $\mathbb{F}_+^d$ or $\mathbb{Z}_+^d$ on $\mathcal{A}$ such that each generator of α is implemented by a Cuntz family. If the capacity of the system is greater than 1 then the resulting w*-semicrossed products are (hereditarily) hyperreflexive. If the capacity of the system is 1 and $\mathcal{A}$ is reflexive then the resulting w*-semicrossed products are reflexive. By applying the results of Bercovici [9] we get that the hyperreflexivity constant in Theorems 6.1.2 and 6.2.1 is at most $3 \cdot K^4$ when $N, M \geq 2$ (where K is the uniform bound for the invertible row operators). This follows since their commutant contains two isometries with orthogonal ranges.

For the free semigroup case the key strategy we rely on is to realise the semicrossed product as a subspace of $\mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{L}_N$, which is encapsulated in the following theorem.

Theorem 1.3.9 (Theorem 6.1.1). *Let $(\mathcal{B}(\mathcal{H}), \{\alpha_i\}_{i\in[d]})$ be a w*-dynamical system of a uniformly bounded spatial action. Suppose that every α_i is given by an invertible row operator $u_i = [u_{i,j_i}]_{j_i \in [n_i]}$ and set $N = \sum_{i\in[d]} n_i$. Then the w*-semicrossed product $\mathcal{B}(\mathcal{H}) \overline{\times}_\alpha \mathcal{L}_d$ is similar to $\mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_N$.*

This relies on the fact that every system on $\mathcal{B}(\mathcal{H})$ given by a Cuntz family of multiplicity n_i is equivalent to the trivial action of $\mathbb{F}_+^{n_i}$ on $\mathcal{B}(\mathcal{H})$. This was noted by Kakariadis and Katsoulis [31] and Kakariadis and Peters [33]. In the $\mathbb{Z}_+^d$ case we decompose the semicrossed product in each direction.

Proposition 1.3.10 (Proposition 3.3.15). *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ be a unital w*-dynamical system. Then $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ is unitarily equivalent to*

$$(\cdots((\mathcal{A} \overline{\times}_{\alpha_{\mathbf{1}}} \mathbb{Z}_+) \overline{\times}_{\widehat{\alpha}_{\mathbf{2}}} \mathbb{Z}_+) \cdots) \overline{\times}_{\widehat{\alpha}_{\mathbf{d}}} \mathbb{Z}_+$$

where $\widehat{\alpha}_{\mathbf{i}} = \alpha_{\mathbf{i}} \otimes^{(i-1)} I$ for $i = 2, \ldots, d$.

Following the work of Helmer [27], we obtain reflexivity for injectively reflexive systems. We call an algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ *injectively reducible* if there is a non-trivial reducing subspace M of $\mathcal{A}$ such that the representations

$$a \mapsto a|_M \quad \text{and} \quad a \mapsto a|_{M^\perp}$$

are both injective. Then, we can make the following definiton.

Definition 1.3.11. A w*-dynamical system $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ is *injectively reflexive* if

(i) $\mathcal{A}$ is reflexive.

(ii) $\mathcal{A}$ is injectively reducible by some M.

(iii) $\beta_\nu(\mathcal{A})$ is reflexive for all $\nu \in \mathbb{F}_+^d$ with

$$\beta_\nu(a) = \begin{bmatrix} a|_M & 0 \\ 0 & \alpha_\nu(a)|_{M^\perp} \end{bmatrix}.$$

Therefore we have the following results.

Theorem 1.3.12 (Theorem 6.1.7). *Let $(\mathcal{A}, \alpha, \mathbb{F}_+^d)$ be a unital w*-dynamical system. If $\mathcal{A}$ is injectively reflexive then the semicrossed products $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ and $\mathcal{A} \overline{\times}_\alpha \mathcal{R}_d$ are reflexive.*

Since dynamical systems over Type II or Type III factors are injectively reflexive we have the next corollary.

Corollary 1.3.13 (Corollary 6.1.8). *Let $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ be a unital w*-dynamical system on a factor $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$. Then $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ and $\mathcal{A} \overline{\times}_\alpha \mathcal{R}_d$ are reflexive.*

In a similar manner we can define injectively reflexive systems in the $\mathbb{Z}_+^d$ case.

Definition 1.3.14. A w*-dynamical system $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ is *injectively reflexive* if

(i) $\mathcal{A}$ is reflexive.

(ii) $\mathcal{A}$ is injectively reducible by M.

(iii) $\beta_{\underline{n}}(\mathcal{A})$ is reflexive for all $\underline{n} \in \mathbb{Z}_+^d$ with

$$\beta_{\underline{n}}(a) = \begin{bmatrix} a|_M & 0 \\ 0 & \alpha_{\underline{n}}(a)|_{M^\perp} \end{bmatrix}.$$

We have corresponding results for this case also.

Theorem 1.3.15 (Theorem 6.2.5)**.** *Let* $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ *be a unital* w^**-dynamical system. If the system is injectively reflexive then* $\mathcal{A} \,\overline{\times}_\alpha\, \mathbb{Z}_+^d$ *is reflexive.*

Corollary 1.3.16 (Corollary 6.2.7)**.** *Let* $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ *be a unital* w^**-dynamical system on a factor* $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ *for a separable Hilbert space* $\mathcal{H}$*. Then* $\mathcal{A} \,\overline{\times}_\alpha\, \mathbb{Z}_+^d$ *is reflexive.*

Chapter 2

Preliminaries

We shall begin with a brief survey of topologies on operator algebras before introducing von Neumann algebras. We then progress to a discussion of some general results regarding reflexivity and hyperreflexivity [42].

2.1 Topologies

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the set of bounded operators on $\mathcal{H}$. An operator algebra is a subalgebra of $\mathcal{B}(\mathcal{H})$ closed under multiplication of operators. A $*$-algebra is an algebra equipped with an involution. The study of operator algebras is primarily concerned with the study of subalgebras of $\mathcal{B}(\mathcal{H})$ closed under different topologies. For example a C*-algebra is a selfadjoint subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed with respect to the topology induced by the norm on $\mathcal{B}(\mathcal{H})$ [42].

2.1.1 Weak Topologies

Let $(X, \|\cdot\|)$ be a normed space, then its dual is

$$X^* = \mathcal{B}(X, \mathbb{C}) = \{f : X \to \mathbb{C} \ : \ f \text{ is continuous}\}.$$

By the Hahn-Banach theorem it can be seen that X embeds into X^{**} injectively. We can therefore define convergence in the following weak topologies.

Definition 2.1.1 (Weak Topologies)**.** We say that x_i converges to x in the *weak topology* on X (or $x_i \xrightarrow{\text{w}} x$) if $\phi(x_i) \to \phi(x)$ for all $\phi \in X^*$.

We say that ϕ_i converges to ϕ in the *w*-topology* on X^* (or $\phi_i \stackrel{\text{w*}}{\to} \phi(x)$) if $\phi_i(x) \to \phi(x)$ for all $x \in X$.

Note that there is also the weak topology on X^* induced by X^{**}. In the above definition, the w*-topology refers to X^*, however it may happen that $(X_1)^* = X = (X_2)^*$ then X admits two w*-topologies, one from X_1, the other from X_2.

2.1.2 Operator Topologies

We now move on to discuss the various operator topologies on $\mathcal{B}(\mathcal{H})$.

Definition 2.1.2 (Strong Operator Topology)**.** Let $\mathcal{H}$ be a Hilbert space, and $x \in \mathcal{H}$. Then the function

$$p_x : \mathcal{B}(\mathcal{H}) \to \mathbb{R} : \, T \mapsto \|Tx\|$$

is a seminorm on $\mathcal{B}(\mathcal{H})$. The locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the separating family $(p_x)_{x\in\mathcal{H}}$ is called the *strong operator topology* on $\mathcal{B}(\mathcal{H})$.

The strong operator topology is the topology of pointwise convergence for an operator. We say that T_i converges to T in the strong operator topology (or $T_i \stackrel{\text{sot}}{\to} T$) if $T_i x \to Tx$ for all $x \in \mathcal{H}$. A base for the sot is given by

$$B(T, \varepsilon, x_1 \cdots x_n) = \{W \, : \, \|(T - W)x_i\| < \varepsilon, \; i = 1, \cdots, n\}.$$

This topology is weaker than the norm topology on $\mathcal{B}(\mathcal{H})$. With respect to the sot, $\mathcal{B}(\mathcal{H})$ is a topological vector space, i.e. the operations of vector addition and scalar multiplication are strongly continuous. This is not the case in general for the operations of multiplication and involution. For example if v is the unilateral shift then $(v^n)^* \stackrel{\text{sot}}{\to} 0$ in the strong operator topology but $v^n \stackrel{\text{sot}}{\not\to} 0$.

Definition 2.1.3 (Weak Operator Topology)**.** Let $\mathcal{H}$ be a Hilbert space. Then the Hausdorff locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the separating family of seminorms

$$\mathcal{B}(\mathcal{H}) \to \mathbb{R}^+ : \, T \mapsto |\langle Tx, y\rangle|, \quad (x, y \in \mathcal{H})$$

is called the *weak operator topology* on $\mathcal{B}(\mathcal{H})$.

Say that T_i converges to T in the weak operator topology (or $T_i \stackrel{\text{wot}}{\to} T$) in the weak operator topology if and only if $\langle T_i x, y\rangle \to \langle Tx, y\rangle$ for all $x, y \in \mathcal{H}$. Linear functionals on $\mathcal{B}(\mathcal{H})$ are sot-continuous if and only if they are wot-continuous. Specifically, we have the following.

Proposition 2.1.4. *[12, Proposition 8.1] If $\Phi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is a linear functional then the following are equivalent.*

(i) Φ *is sot-continuous.*

(ii) Φ *is wot-continuous.*

(iii) There are vectors $f_1, \cdots, f_n, g_1, \cdots, g_n \in \mathcal{H}$ such that

$$\Phi(T) = \sum_{k=1}^{n} \langle T f_k, g_k \rangle \text{ for all } T \in \mathcal{B}(\mathcal{H}).$$

It follows that if E is a convex set then E is sot-closed if and only if E is wot-closed. It is immediate that the adjoint is continuous in the wot, whilst multiplication is separately continuous in the wot. The wot is strictly weaker than the sot. This is clear since if v is the unilateral shift and $\mathcal{H} = \ell^2$ then $v^n \overset{\text{wot}}{\to} 0$ but clearly $v^n \overset{\text{sot}}{\not\to} 0$. Continuity of multiplication in the weak topology does not hold in general. For example, again for the unilateral shift v we have that the sequences (v^{*n}) and (v^n) both converge weakly to zero, but the product sequence $((v^*)^n v^n)$ is $I_{\mathcal{H}}$.
Now suppose $u \in \mathcal{B}(\mathcal{H})$. Then u is called *trace-class* if $\|u\|_1 = \sum_{x \in E} \langle |u|x, x \rangle < \infty$ for an orthonormal basis E of the Hilbert space $\mathcal{H}$. For a trace class operator u define the *trace* of u by

$$\operatorname{tr}(u) = \sum_{x \in E} \langle ux, x \rangle .$$

In fact this definition of $\operatorname{tr}(u)$ is independent of the choice of basis E. Write $\mathcal{L}_1(\mathcal{H})$ for the collection of trace-class operators on $\mathcal{H}$. It can be shown that $\mathcal{L}_1(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$ and $\mathcal{L}_1(\mathcal{H})$ is closed with respect to $\|\cdot\|_1 = \operatorname{tr}(|u|)$. Then $u \in \mathcal{L}_1(\mathcal{H})$ if and only if

$$u = \sum_n \lambda_n x_n \otimes y_n^* \text{ for } \sum_n |\lambda_n| < \infty \quad \text{and} \quad \|x_n\|, \|y_n\| < 1.$$

Then $\operatorname{tr}(ub) : \mathcal{K}(\mathcal{H}) \to \mathbb{C}$ is continuous, hence the map

$$\mathcal{L}_1(\mathcal{H}) \to \mathcal{K}(\mathcal{H})^* : u \mapsto \operatorname{tr}(ub)$$

is an isometric linear isomorphism. Similarly for $v \in \mathcal{B}(\mathcal{H})$, $\mathrm{tr}(av) : \mathcal{L}_1(\mathcal{H}) \to \mathbb{C}$ is continuous, so the map

$$\mathcal{B}(\mathcal{H}) \to (\mathcal{L}_1(\mathcal{H}))^* : v \mapsto \mathrm{tr}(av)$$

is an isometric linear isomorphism [42, Chapter 4]. Thus $\mathcal{B}(\mathcal{H}) = (\mathcal{L}_1(\mathcal{H}))^* = (\mathcal{K}(\mathcal{H})^*)^*$.

Definition 2.1.5 (w*-topology)**.** If $\mathcal{H}$ is a Hilbert space, then the *w*-topology* on $\mathcal{B}(\mathcal{H})$ is the Hausdorff, locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the seminorms:

$$\mathcal{B}(\mathcal{H}) \to \mathbb{R}^+ : \ u \mapsto |\mathrm{tr}(uv)|, \quad (v \in \mathcal{L}_1(\mathcal{H})).$$

We say that T_i converges to T in the w*-topology (or $T_i \overset{\mathrm{w}^*}{\to} T$) if and only if $\mathrm{tr}(uT_i) \to \mathrm{tr}(uT)$ for all $u \in \mathcal{L}_1(\mathcal{H})$.

Thus the w*-topology is the w*-topology induced by $\mathcal{L}_1(\mathcal{H})$ on its dual, $\mathcal{B}(\mathcal{H})$. Hence the closed unit ball of $\mathcal{B}(\mathcal{H})$ is w*-compact by the Banach-Alaoglu theorem. The relative weak and w*-topologies on the closed unit ball of $\mathcal{B}(\mathcal{H})$ coincide, therefore the unit ball of $\mathcal{B}(\mathcal{H})$ is weakly compact.

Proposition 2.1.6. *[12, Proposition 20.1]*

(a) If $\mathcal{H}$ is separable then the closed unit ball of $\mathcal{B}(\mathcal{H})$ with the w-topology is a compact metric space.*

(b) The w-topology and the wot agree on bounded subsets of $\mathcal{B}(\mathcal{H})$.*

(c) A sequence in $\mathcal{B}(\mathcal{H})$ converges w if and only if it converges in the wot.*

Addition and scalar multiplication are w*-continuous, as is the involution. The weak operator topology is properly weaker than the w*-topology. This can be seen by way of the following example.

Example 2.1.7. Let $u_1, \ldots, u_n, \cdots \in \mathbb{N}$ be non-zero such that

$$u = \begin{bmatrix} u_1 & & \\ & u_2 & \\ & & \ddots \end{bmatrix} \in \mathcal{L}_1(\ell^2(\mathbb{Z}_+)).$$

Then let $\phi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be such that $\phi(v) = \mathrm{tr}(uv) = \sum_{k=1}^{\infty} \langle uve_k, e_k \rangle$. Then ϕ is w*-continuous. If ϕ is wot-continuous, then by Proposition 2.1.4 there are vectors $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ in $\mathcal{H}$ such that

$$\phi(v) = \sum_{k=1}^{\infty} \langle vx_k, y_k \rangle \text{ for all } v \in \mathcal{B}(\mathcal{H}).$$

We need to find a v such that $\langle vx_k, y_k \rangle = 0$ for all k. Let $W = \mathrm{span}\{x_1, \cdots x_n\}$ and $P = P_{W^\perp} = 1 - P_W$. Then there exists an M such that $e_M \notin W$, otherwise we would have $\ell^2(\mathbb{Z}_+) = \mathbb{C}^n$. Then $\sum_{k=1}^{\infty} \langle P_{W^\perp} x_k, y_k \rangle = 0$ and,

$$\begin{aligned}
\sum_{k=1}^{\infty} \langle P_{W^\perp} ue_k, e_k \rangle &= \sum_{k=1}^{\infty} \langle ue_k, P_{W^\perp} e_k \rangle \\
&= \sum_{k=1}^{\infty} u_k \langle P_{W^\perp} e_k, P_{W^\perp} e_k \rangle \\
&= \sum_{k=1}^{\infty} u_k \left\| P_{W^\perp} e_k \right\|^2 .
\end{aligned}$$

If this is equal to zero then $u_k \left\| P_{W^\perp} e_k \right\|^2 = 0$ for all k and since $u_k \neq 0$ we have $P_{W^\perp} e_k = 0$ for all k. Thus $e_k \in W$ and then $\ell^2(\mathbb{Z}_+) = \mathbb{C}^n$ which is a contradiction and so ϕ is not wot continuous.

2.2 von Neumann Algebras

A *von Neumann algebra* $\mathcal{R}$ is a w*-closed, unital, $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. Note that every von Neumann algebra is a C*-algebra. If C is a subset of an algebra $\mathcal{A}$, then we define its *commutant* C' to be the set of all elements of $\mathcal{B}(\mathcal{H})$ that commute with the elements of C. The *bicommutant* C'' is the set of all elements of $\mathcal{B}(\mathcal{H})$ which commute with the elements of C' [12].

Theorem 2.2.1 (Bicommutant Theorem)**.** *Let $\mathcal{R}$ be a $*$-algebra on a Hilbert space $\mathcal{H}$ and suppose that $\mathcal{I}_{\mathcal{H}} \in \mathcal{R}$. Then $\mathcal{R}$ is a von Neumann algebra on $\mathcal{H}$ if and only if $\mathcal{R} = \mathcal{R}''$.*

Definition 2.2.2 (Representation)**.** A *representation* π of a C*-algebra $\mathcal{A}$ is a pair $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is a Hilbert space and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism. A representation is called *faithful* if $\ker \pi = \{0\}$, that is, if π is injective.

If $\mathcal{A}$ is a C*-algebra acting on a Hilbert space $\mathcal{H}$ and $S \subseteq \mathcal{H}$, set

$$\mathcal{A}S = \operatorname{span}\{a\xi \mid a \in \mathcal{A}, \xi \in S\}$$

and set $[\mathcal{A}S]$ to be the closure of $\mathcal{A}S$. $\mathcal{A}$ acts *non-degenerately* on $\mathcal{H}$ if $[\mathcal{A}\mathcal{H}] = \mathcal{H}$. Equivalently, for each non-zero $\xi \in \mathcal{H}$ there is an $a \in \mathcal{A}$ such that $a\xi \neq 0$. It can be demonstrated that a von Neumann algebra is the dual space of a Banach space, however this is not true in general for C*-algebras. Due to the Spectral Theorem [50, Theorem 5.1] von Neumann algebras are useful as they are generated by their projections. It is possible to develop a comparison theory for these projections. This leads to the notion of 'type decomposition' for von Neumann algebras.

Definition 2.2.3 (Factors)**.** Let $\mathcal{R}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. Then $\mathcal{R}$ is said to be a *factor* if $\mathcal{R} \cap \mathcal{R}' = \mathbb{C}I$, where $I = \mathrm{id}_{\mathcal{H}}$.

Definition 2.2.4 (Central Carrier)**.** The *central carrier* of an element a in a von Neumann algebra $\mathcal{R}$, denoted by C_a, is the smallest projection p in $C = \mathcal{R} \cap \mathcal{R}'$ for which $pa = a$.

Two projections p, q in $\mathcal{R}$ are said to be *(Murray-von Neumann) equivalent* when $v^*v = p$ and $vv^* = q$ for some $v \in \mathcal{R}$. If $p, q \in \mathcal{R}$ then p is *weaker* than q (written $p \precsim q$) when p is equivalent to a subprojection of q. Similarly, we say that q is *stronger* than p. It can be demonstrated that '$\precsim$' is a partial order on the (equivalence classes of) projections in a von Neumann algebra and that the equivalence classes of projections in a factor are totally ordered [29, Chapter 6].

Definition 2.2.5 (Projections)**.** If $p \in \mathcal{R}$ is a projection in a von Neumann algebra then p is said to be:

(i) *Abelian*: if $p\mathcal{R}p$ is abelian.

(ii) *Infinite*: (relative to $\mathcal{R}$) when $p \sim p_0 < p$ for some projection $p_0 \in \mathcal{R}$.

(iii) *Finite*: (relative to $\mathcal{R}$) if it is not infinite.

(iv) *Properly Infinite*: if p is infinite and cp is either 0 or infinite, for each central projection c.

(v) *Countably decomposable* if any collection of mutually orthogonal non-zero subprojections of p is countable.

$\mathcal{R}$ is *finite*, *properly infinite* or *countably decomposable* if I is respectively finite, properly infinite or countably decomposable.

We now have the terminology to define the different Types of von Neumann algebras.

Definition 2.2.6 (Types of von Neumann Algebra)**.** A von Neumann algebra $\mathcal{R}$ is said to be of:

(i) *Type I* if it has an abelian projection with central carrier I. $\mathcal{R}$ is of Type I_n if I is the sum of n equivalent abelian projections.

(ii) *Type II* if $\mathcal{R}$ has no non-zero abelian projections but has a finite projection with central carrier I.

- $\mathcal{R}$ is *Type* II_1 if I is finite.
- $\mathcal{R}$ is *Type* II_∞ if I is infinite.

(iii) *Type III* if $\mathcal{R}$ has no non-zero finite projections.

Every von Neumann algebra can be decomposed into a direct sum of von Neumann algebras of Types I, II_1, II_∞ and III. A factor is one and only one of the Type I, Type II_1, Type II_∞ or Type III. A factor of Type I is isomorphic to $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Type I von Neumann algebras are often referred to as discrete von Neumann algebras to indicate that fact that the identity can be decomposed into a sum of central projections, each of which is the 'discrete' sum of projections, minimal with the given central projection as central carrier. Type II von Neumann algebras are often called continuous. Factors of Type I_n and II_1 are finite von Neumann algebras, while factors of the other Types are properly infinite von Neumann algebras. Each factor of Type II_1 has infinite linear dimension.

It transpires that the comparison of projections in von Neumann algebra $\mathcal{R}$ relates to the nature of the dimension functions on $\mathcal{R}$. It is therefore possible to reformulate the type decomposition of $\mathcal{R}$ in terms of properties of the corresponding dimension function [29].

Definition 2.2.7 (Dimension Function)**.** Let $\mathcal{R}_p$ denote the set of projections in a von Neumann algebra $\mathcal{R}$ and let $p, q \in \mathcal{R}_p$. A *dimension function* on $\mathcal{R}$ is a function $d : \mathcal{R}_+ \to [0, \infty]$ such that

(i) $d(p) = d(q)$ when $p \sim q$.

(ii) $d(p+q) = d(p) + d(q)$ if $p \perp q$

(iii) d extends to a function on $M_n(\mathcal{R})_+$ with the same properties.

We say that d is normalised if $\sup\{d(p) \,:\, p \in \mathcal{R}_+\} = 1$.

Let $\mathcal{R}$ be a factor and let $\mathcal{R}_p$ denote the set of projections in $\mathcal{R}$. Then it is possible to construct a dimension function $d : \mathcal{R}_p \to [0, \infty]$ such that $p \precsim q$ if and only if $d(p) \leq d(q)$ [29]. The range of d determines the type of $\mathcal{R}$ as illustrated in the following theorem.

Theorem 2.2.8. *Let $\mathcal{R}$ be a countably decomposable factor. Then there is a dimension function d, which is unique up to normalization. Then the range of d has the following possibilities:*

(i) $\{0, 1, \ldots, n\}$ *if $\mathcal{R}$ is Type I_n.*

(ii) $\{0, 1, 2, \ldots, \infty\}$ *if $\mathcal{R}$ is Type I_∞.*

(iii) $[0, 1]$ *if $\mathcal{R}$ is Type II_1.*

(iv) $[0, \infty)$ *if $\mathcal{R}$ is Type II_∞.*

(v) $\{0, \infty\}$ *if $\mathcal{R}$ is Type III.*

We can use the projections in a von Neumann algebra to describe the structure of their weakly closed ideals. If $\mathcal{I}$ is a left ideal in a von Neumann algebra $\mathcal{R}$, $p \in \mathcal{I}$ a projection and v a partial isometry in $\mathcal{R}$ with initial projection p then $v \in \mathcal{I}$ for $v = vp$. Also, if $\mathcal{I}$ is two sided and $q \precsim p$ then $q \in \mathcal{I}$, for $q \sim p_0 \leq p$. The set of operators with finite range projection in $\mathcal{R}$ forms a two sided ideal. It can be shown that each non-zero two sided ideal in a factor $\mathcal{A}$ contains this ideal [29, Section 6.8]. This implies that if I is finite relative to the factor $\mathcal{A}$ then I lies in each non-zero, two sided ideal in $\mathcal{A}$ and so $\mathcal{A}$ has no proper two sided ideals. Therefore we have the following lemmas.

Lemma 2.2.9. *Let $\mathcal{R}$ be a factor, then:*

(i) If $\mathcal{R}$ is finite then it is simple with respect to norm-closed ideals.

(ii) If $\mathcal{R}$ is a countably decomposable Type III factor then it is simple with respect to norm-closed ideals.

(iii) No proper two-sided ideal in a countably decomposable factor contains an infinite projection.

Lemma 2.2.10. *If $\mathcal{I}$ is a weakly closed left (or right) ideal in the von Neumann algebra $\mathcal{R}$ then $\mathcal{I} = \mathcal{R}p$ (or $\mathcal{I} = p\mathcal{R}$) for some projection $p \in \mathcal{R}$. If $\mathcal{I}$ is a two sided ideal then p is a central projection in $\mathcal{R}$.*

Thus the weakly closed ideals in $\mathcal{R}$ are the principal ideals generated by the (central) projections in $\mathcal{R}$. It follows that each weakly-closed two sided ideal $\mathcal{I}$ in a von Neumann algebra $\mathcal{R}$ is a self adjoint. Note that if $\mathcal{R}$ is a factor then by Lemma 2.2.10 it must have no weakly-closed ideals.

2.3 Reflexivity

A unital subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is said to be reflexive if it can be determined by its invariant subspaces. An operator is called reflexive if the algebra it generates is reflexive. The concept of reflexivity has been examined since the latter half of the twentieth century, beginning with Sarason's proof that the unilateral shift is reflexive [49]. von Neumann algebras are reflexive due to the Bicommutant Theorem. For now we require the following definitions.

Definition 2.3.1 (Invariant Subspace)**.** Let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$ where $\mathcal{H}$ is a Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator. Then $\mathcal{M}$ is said to be an *invariant subspace* for A if $h \in \mathcal{M}$ implies $Ah \in \mathcal{M}$. That is, if $A\mathcal{M} \subseteq \mathcal{M}$. If $\mathcal{M}^\perp$ is the orthogonal complement of $\mathcal{M}$ then $\mathcal{M}$ is called a *reducing subspace* for A if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant subspaces for A.

Definition 2.3.2 (Reflexive Cover)**.** For a subspace $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ the reflexive cover of $\mathcal{S}$ is given by

$$\mathrm{Ref}(\mathcal{S}) = \{T \in \mathcal{B}(\mathcal{H}) \;:\; T\xi \in \overline{\mathcal{S}\xi}, \text{ for all } \xi \in \mathcal{H}\}.$$

It is clear that $\operatorname{Ref}(\mathcal{A})$ is a weakly-closed subspace of $\mathcal{B}(\mathcal{H})$ which contains $\mathcal{A}$. The reflexive cover proves to be useful when $\mathcal{A}$ is a subspace of operators which need not be an algebra or contain the identity.

Let $\mathcal{H}$ be a Hilbert space, $S(\mathcal{H})$ be the collection of all closed subspaces of H and $P(\mathcal{H})$ be the collection of all orthogonal projections on $\mathcal{H}$. Then for $s_\lambda \in S(\mathcal{H})$ let $\vee s_\lambda$ denote the projection on the closed linear span of $\cup s_\lambda \mathcal{H}$, and let $\wedge s_\lambda$ denote the projection on the intersection $\cap s_\lambda \mathcal{H}$. Then this makes $S(\mathcal{H})$ into a lattice. Recall that there is a bijective correspondence between closed subspaces and orthogonal projections. This allows us to transfer the lattice structure of $S(\mathcal{H})$ to $P(\mathcal{H})$. So if $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ then the set

$$\operatorname{Lat}\mathcal{S} = \{P \in P(\mathcal{H}) \,:\, SP = PSP \text{ for all } S \in \mathcal{S}\}$$

is a complete sublattice of $P(\mathcal{H})$ called the invariant subspace lattice of $\mathcal{S}$. It can be seen that the ranges of the projections in $\operatorname{Lat}\mathcal{S}$ are precisely the closed $\mathcal{S}$-invariant subspaces.

If $\mathcal{L}$ is a subspace lattice, we can define

$$\operatorname{Alg}\mathcal{L} = \{S \in B(H) \,:\, SP = PSP \text{ for all projections } P \in \mathcal{L}\}$$

which is the algebra of all operators leaving invariant the projections of $\mathcal{L}$. A unital algebra $\mathcal{A}$ of operators on a Hilbert space is reflexive if any $T \in \mathcal{B}(\mathcal{H})$ which leaves invariant all $\mathcal{A}$-invariant subspaces (that is, all elements of the lattice $\operatorname{Lat}(\mathcal{A})$) is in $\mathcal{A}$.

Definition 2.3.3 (AlgLat)**.** For an algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ the AlgLat of $\mathcal{A}$ is given by

$$\operatorname{AlgLat}(\mathcal{A}) = \{T \in \mathcal{B}(\mathcal{H}) \,:\, \operatorname{Lat}(\mathcal{A}) \subseteq \operatorname{Lat} T\}.$$

We say that $\mathcal{A}$ is reflexive if $\operatorname{AlgLat}(\mathcal{A}) = \mathcal{A}$. A unital algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is called *hereditarily reflexive* if every w*-closed subalgebra of $\mathcal{A}$ is reflexive.

Note that $\operatorname{AlgLat}(\mathcal{A})$ is the unital algebra of all operators leaving invariant all $\mathcal{A}$-invariant subspaces.

Theorem 2.3.4. *[12, Proposition 22.3(e)] If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a unital subalgebra for a Hilbert space $\mathcal{H}$ then* $\operatorname{Ref}(\mathcal{A}) = \operatorname{AlgLat}(\mathcal{A})$.

In general, $\mathrm{Ref}(\mathcal{A}) \subseteq \mathrm{AlgLat}(\mathcal{A})$, and the inclusion may be strict. In fact, it can be demonstrated that for a subspace $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$, the smallest reflexive algebra containing $\mathcal{S}$ is $\mathrm{AlgLat}(\mathcal{A})$. It is clear that any reflexive algebra must be unital. By the Bicommutant Theorem if $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a unital $*$-algebra, then $\mathrm{Ref}(\mathcal{A}) = \overline{\mathcal{A}}^{w^*}$. Therefore von Neumann algebras are reflexive (note that $p \in \mathrm{Lat}\,\mathcal{A}$ for $\mathcal{A}$ selfadjoint if and only if p is reducing if and only if $p \in \mathcal{A}'$).
Reflexivity is preserved under operations such as taking adjoints and similarity. Recall that two (unital) algebras $\mathcal{A}$ and $\mathcal{B}$ are said to be *similar* if $\mathcal{A} = W\mathcal{B}W^{-1}$ for some invertible operator $W \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.

Lemma 2.3.5. *Let $\mathcal{L}$ be a lattice and let $\mathcal{S}$ be a w*-closed subspace of $\mathcal{B}(\mathcal{H})$. If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a unital subalgebra for a Hilbert space $\mathcal{H}$ and $W \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is invertible then*

(i) $W(\mathrm{Alg}\,\mathcal{L})W^{-1} = \mathrm{Alg}(W\mathcal{L})$.

(ii) $W\,\mathrm{Lat}\,\mathcal{A} = \mathrm{Lat}(W\mathcal{A}W^{-1})$.

(iii) $W\,\mathrm{Ref}(\mathcal{S})W^{-1} = \mathrm{Ref}(W\mathcal{S}W^{-1})$.

We call a w*-closed subspace $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ hereditarily reflexive if each of its w*-closed subspaces is reflexive. It is immediate that hereditary reflexivity is also preserved under similarities.

Lemma 2.3.6. *(a) If $\{\mathcal{S}_n\}$ is a sequence of reflexive subspaces then $\oplus_n \mathcal{S}_n$ is reflexive.*

(b) If $\{\mathcal{S}_i\}$ is any collection of reflexive subspaces then $\cap_i \mathcal{S}_i$ is reflexive.

2.4 Hyperreflexivity and the $\mathbb{A}_1$-Property

Hyperreflexivity is a stronger property than reflexivity, introduced by Arveson, which provides a measurement for reflexivity. For an algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and an operator T, the distance from T to $\mathcal{A}$ is given by

$$\mathrm{dist}(T, \mathcal{A}) = \inf\{\|T - A\| \,:\, A \in \mathcal{A}\}.$$

We can also define the distance quantity

$$\beta(T, \mathcal{A}) = \sup\{\|(1 - P)TP\| \,:\, P \in \mathrm{Lat}\,\mathcal{A}\}.$$

In general we have that $\beta(T,\mathcal{A}) \leq \operatorname{dist}(T,\mathcal{A})$ for all $T \in \mathcal{B}(\mathcal{H})$, since if $A \in \mathcal{A}$ and $P \in \operatorname{Lat}\mathcal{A}$ then

$$\|(1-P)TP\| = \|(1-P)(T-A)P\| \leq \|1-P\| \, \|T-A\| \, \|P\| = \|T-A\| \, .$$

Definition 2.4.1 (Hyperreflexivity)**.** A w*-closed algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is *hyperreflexive* (with distance constant at most C) if there exists a constant C such that

$$\operatorname{dist}(T,\mathcal{A}) \leq C\beta(T,\mathcal{A}) \text{ for all } T \in \mathcal{B}(\mathcal{H}).$$

Note that if $T \in \mathcal{B}(\mathcal{H})$ then $\beta(T,\mathcal{A}) = 0$ if and only if $T \in \operatorname{Ref}\mathcal{A}$. It follows that a hyperreflexive algebra is reflexive. When $\mathcal{H}$ is finite dimensional a subspace $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is reflexive if and only if it is hyperreflexive. A subspace $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is called *hereditarily hyperreflexive* is each of its w*-closed subspaces is hyperreflexive.

There is a connection between hyperreflexivity and the preannihilator. Recall the following definition.

Definition 2.4.2 (Preannihilator)**.** If $\mathcal{X}$ is a Banach space and $Y \subseteq \mathcal{X}^*$ then the *preannihilator* of Y is the set

$$Y_\perp = \{x \in \mathcal{X} \; : \; y^*(x) = 0 \text{ for all } y^* \in Y\}.$$

The following lemma demonstrates that the reflexive cover of a subspace can be recovered via rank one operators.

Lemma 2.4.3. *[12, Theorem 56.9] If $\mathcal{S}$ is any linear subspace of $\mathcal{B}(\mathcal{H})$ then $(\operatorname{Ref}\mathcal{S})_\perp$ is the closed linear span of the rank one operators it contains.*

Consequently, we have that a w*-closed subspace of $\mathcal{B}(\mathcal{H})$ is reflexive if and only if its preannihilator is the closed linear span of the rank one operators it contains, that is $\mathcal{S}$ is reflexive if and only if $\mathcal{S}_\perp = (\operatorname{Ref}\mathcal{S})_\perp$.

Similarly it is possible to characterise hyperreflexivity through $\mathcal{A}_\perp$. Arveson showed that a w*-closed unital algebra $\mathcal{A}$ is hyperreflexive if and only if for every $\phi \in \mathcal{A}_\perp$ there are rank one functionals $\phi_n \in \mathcal{A}_\perp$ such that $\phi = \sum_n \phi_n$ and $\sum_n \|\phi_n\| < \infty$. The hyperreflexivity constant is at most C when $\sum_n \|\phi_n\| \leq C \cdot \|\phi\|$ for $\phi = \sum_n \phi_n \in \mathcal{A}_\perp$ as above. Specifically, Arveson demonstrated the following theorem.

Theorem 2.4.4. *[6, Theorem 7.4]. Let $\mathcal{A}$ be a w^*-closed subalgebra of $\mathcal{B}(\mathcal{H})$. Then $\mathcal{A}$ is hyperreflexive if and only if every $\phi \in \mathcal{A}_\perp$ has a representation $\phi = \sum_{n=1}^{\infty} \phi_n$ where each ϕ_n is a elementary functional (that is, a functional of rank at most one) in $\mathcal{A}_\perp$ and $\sum_{n=1}^{\infty} \|\phi_n\| < \infty$.*

We give an outline of the proof here for the purposes of self-containment. For one direction assume that every $\phi \in \mathcal{A}_\perp$ can be written as $\phi = \sum_{n=1}^{\infty} \phi_n$ for elementary functionals ϕ_n and $\sum_{n=1}^{\infty} \|\phi_n\| \leq \infty$. Hyperreflexivity shall follow by demonstrating that there is a constant C such that

$$\sum_{n=1}^{\infty} \|\phi_n\| \leq C \cdot \|\phi\| . \tag{2.1}$$

First we define a new norm on $\mathcal{A}_\perp$ as follows. Let $|\phi|$ be the infimum of all numbers such that $\sum_{n=1}^{\infty} \|\phi_n\| < \infty$ where $\phi_1, \phi_2, \ldots$ is a sequence of elementary functionals in $\mathcal{A}_\perp$ such that

$$\sum_{n=1}^{\infty} \|\phi_n\| < \infty \quad \text{and} \quad \phi = \sum_{n=1}^{\infty} \phi_n.$$

Then by hypothesis $\|\phi\| \leq |\phi| < \infty$ for all $\phi \in \mathcal{A}_\perp$. It follows by an application of the Open Mapping Theorem that there is a constant C such that $|\phi| \leq C \cdot \|\phi\|$ (take the identity map from $(\mathcal{A}_\perp, |\cdot|) \to (\mathcal{A}_\perp, \|\cdot\|)$). We have to show that $\mathcal{A}$ is hyperreflexive, that is, that

$$\operatorname{dist}(B, \mathcal{A}) = \sup\{|\phi(B) : \phi \in \mathcal{A}_\perp, \|\phi\| \leq 1\} \leq C \cdot \sup_{p \in \operatorname{Lat} \mathcal{A}} \|(1-p)Bp\| .$$

To show this it suffices to show that for each $\phi \in \mathcal{A}_\perp$ such that $\|\phi\| \leq 1$ we have

$$|\phi(B)| \leq C \cdot \sup_{p \in \operatorname{Lat} \mathcal{A}} \|(1-p)Bp\| .$$

This is established via direct computations.

For the converse we shall require the use of the following facts. The first lemma is a standard fact from Banach space theory.

Lemma 2.4.5. *Let $\mathcal{B}$ be a bounded set in a Banach space E such that $r \cdot \mathcal{B} \subseteq \mathcal{B}$ for each $0 \leq r \leq 1$ and such that the closure of $\mathcal{B}$ contains the unit ball of E. Then* *every element $x \in E$ has a representation $x = \sum_{n=1}^{\infty} \theta_n x_n$ where $x_n \in \mathcal{B}$, $0 \leq \theta_n < \infty$.*

Lemma 2.4.6. *[6, Lemma 1] Let $\phi \in \mathcal{A}_\perp$ and assume there is a projection $p \in \operatorname{Lat}\mathcal{A}$ for which $\phi(B) = \phi((1-p)Bp)$, for all $B \in \mathcal{B}(\mathcal{H})$. Then ϕ has a representation $\phi = \sum_{n=1}^{\infty} \phi_n$ where each ϕ_n is elementary in $\mathcal{A}$ and $\sum_{n=1}^{\infty} \|\phi_n\| \leq \|\phi\|$.*

Returning to the proof of Theorem 2.4.4, assume that $\mathcal{A}$ is hyperreflexive with distance constant C. For every $p \in \operatorname{Lat}\mathcal{A}$, define $\mathcal{B}_p$ to be the space of all functionals ϕ of $\mathcal{B}(\mathcal{H})$ such that $\phi(B) = \phi((1-p)Bp)$, for all $B \in \mathcal{B}(\mathcal{H})$. Also, let $\mathcal{B}$ denote the set of all finite sums of the form $\phi = \phi_1 + \cdots + \phi_n$, where $\phi_i \in \mathcal{B}_{p_i}$ for some $p_i \in \operatorname{Lat}\mathcal{A}$ and $\|\phi_1\| + \cdots + \|\phi_n\| \leq C$. It can be shown that $\operatorname{ball}\mathcal{A}_\perp \subseteq \overline{\mathcal{B}}$.

We claim that we may write each ϕ_n as a finite sum

$$\phi_n = g_1^n + \cdots + g_m^n$$

where $g_j^n(B) = g_j^n((1-p_j^n)Bp_j^n)$ for some $p_j^n \in \operatorname{Lat}\mathcal{A}$ and $\sum_k g_k^n \leq C$. This follows by using Lemma 2.4.5 to write $\phi = \sum_{n=1}^{\infty} \theta_n \phi_n$, where $\phi_n \in \mathcal{B}$, $\theta_n \geq 0$ and $\sum_n \theta_n < \infty$. We may then use the definition of $\mathcal{B}$ to write each ϕ_n as the required finite sum. Applying Lemma 2.4.6 allows us to write each g_k^n as

$$g_k^n = h_{nk_1} + h_{nk_2} + \cdots$$

(i.e. as a possibly infinite sum of elementary functionals in $\mathcal{A}_\perp$) where

$$\sum_{r=1}^{\infty} \|h_{nkr}\| \leq \|g_k^n\|,$$

where each h_{nkr} is an elementary functional in $\mathcal{A}_\perp$. So it follows that f can be expressed as a series of elementary functionals in $\mathcal{A}_\perp$ that can be shown (directly) to be absolutely convergent. This completes the proof of the theorem.
Note that from this proof we get that the hyperreflexivity constant arises from the absolute convergence of the series $\sum_n \|\phi_n\|$ in the sense of (2.1).

Corollary 2.4.7. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ be w*-closed. If $\mathcal{A}$ is similar to $\mathcal{B}$ via an invertible operator U and $\mathcal{B}$ is hyperreflexive with constant at most C then $\mathcal{A}$ is hyperreflexive with constant at most $(\max\{\|U\|, \|U^{-1}\|\})^4 \cdot C$.*

Proof. Let $\psi : \mathcal{A} \to \mathcal{B}$ be given by $\psi(A) = UAU^{-1}$ and let $\phi \in \mathcal{A}_\perp$. Set $g = \phi \circ \psi^{-1}$. Then

$$g(\psi(A)) = \phi(A) = 0 \text{ for all } A \in \mathcal{A},$$

therefore $g \in \mathcal{B}_\perp$. Therefore by [6, Theorem 7.4] there are elementary functionals (g_n) such that

$$g = \sum_{n=1}^{\infty} g_n \quad \text{and} \quad \sum_{n=1}^{\infty} \|g_n\| \leq C \cdot \|g\|.$$

Now set $\phi_n = g_n \circ \psi$, then $g_n = \phi_n \circ \psi^{-1}$ and we have that

$$\phi = g \circ \psi = \sum_{n=1}^{\infty} g_n \circ \psi = \sum_{n=1}^{\infty} \phi_n.$$

Also we see that if $g_n = 0$ then $\phi_n = g_n \circ \psi = 0$. Note that each ϕ_n is elementary since if $g_n(A) = \langle A\xi_n, \eta_n \rangle$ then

$$\phi_n(A) = g_n \circ \psi(A) = \langle \psi(A)\xi_n, \eta_n \rangle = \left\langle UAU^{-1}\xi_n, \eta_n \right\rangle = \left\langle AU^{-1}\xi_n, U^{-1}\eta_n \right\rangle$$

In addition,

$$\begin{aligned}
\sum_{n=1}^{\infty} \|\phi_n\| \leq \sum_{n=1}^{\infty} \|g_n\| \cdot \|\psi\| = \|\psi\| \sum_{n=1}^{\infty} \|g_n\| &= \|\psi\| \cdot C \cdot \|g\| \\
= \|\psi\| \cdot C \cdot \|\phi\| \left\|\psi^{-1}\right\| &= \|U\|^2 \left\|U^{-1}\right\|^2 \cdot C \cdot \|\phi\| \\
\leq (\max\{\|U\|, \left\|U^{-1}\right\|\})^4 \cdot C \cdot \|\phi\|,
\end{aligned}$$

showing that the hyperreflexivity constant is at most $(\max\{\|U\|, \|U^{-1}\|\})^4 \cdot C$. ∎

There is a corresponding notion of hyperreflexivity for linear subspaces, reflecting the connection between reflexive subspaces and reflexive algebras. If $\mathcal{S}$ is a linear subspace of $\mathcal{B}(\mathcal{H})$ we can define an algebra associated to this subspace as

$$\mathcal{A}_\mathcal{S} = \left\{ \begin{bmatrix} \lambda I & T \\ 0 & \mu I \end{bmatrix} \,:\, T \in \mathcal{S} \text{ and } \lambda, \mu \in \mathbb{C} \right\}.$$

We can see that $\mathcal{A}_\mathcal{S}$ is a subalgebra of $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ and we have the following lemma.

Lemma 2.4.8. *[12, Proposition 56.4] Let $\mathcal{S}$ be a linear subspace of $\mathcal{B}(\mathcal{H})$ and let $\mathcal{A}_\mathcal{S}$ be the algebra defined above. Then a closed subspace, $\mathcal{M}$ of $\mathcal{H} \oplus \mathcal{H}$ is in* Lat $\mathcal{A}_\mathcal{S}$

if and only if $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, where $\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathcal{H}$ and $\mathcal{S}\mathcal{M}_2 \subseteq \mathcal{M}_1$. In addition $\mathcal{S}$ is reflexive if and only if $\mathcal{A}_{\mathcal{S}}$ is reflexive. Also if $\mathcal{A}_{\mathcal{S}}$ is hyperreflexive with distance constant C then $\mathcal{S}$ is hyperreflexive with distance constant at most C.

A hyperreflexive algebra is reflexive, however not all reflexive algebras are hyperreflexive as illustrated in the following example due to Kraus and Larson [38].

Example 2.4.9. [38, Section 2] Let $\{e_1, e_2\}$ be an orthonormal basis for a two dimensional Hilbert space $\mathcal{H}_2$. Fix ε with $0 < \varepsilon < 1/3$. Put

$$u_1 = e_1, \quad u_2 = \frac{e_1 + \varepsilon e_2}{\sqrt{1+\varepsilon^2}}.$$

Also, let g_1, g_2 be the rank one projections $g_j = u_j \otimes u_j$ and let $\mathcal{E}$ be the linear span of g_1 and g_2. Set $\mathcal{S} = \mathcal{E}^\perp = \{A \in \mathcal{B}(\mathcal{H}_2) \ : \ \mathrm{tr}(Af) = 0, \ f \in \mathcal{E}\}$. We can see that this is a linear span of its rank one projections therefore $\mathcal{S}$ is reflexive. In fact it is hyperreflexive as it is acting on a finite dimensional space. Kraus and Larson [38] show that the constant of hyperreflexivity is at least $\frac{1}{3\varepsilon}$.

Now, for each $n \geq 4$ let $\mathcal{S}_n$ be the linear subspace just constructed with $\varepsilon = n^{-1}$. Therefore, by the arguments above we know that $\mathcal{S}_n$ is hyperreflexive with constant at least $n/3$. Then let $\mathcal{A}_{\mathcal{S}_n}$ be the algebra constructed prior to Lemma 2.4.8 corresponding to the subspace $\mathcal{S}_n$. Then $\mathcal{A}_{\mathcal{S}_n}$ is hyperreflexive with constant at least $n/3$. Let $\mathcal{H}_n$ denote the Hilbert space upon which $\mathcal{A}_{\mathcal{S}_n}$ acts and let $\mathcal{A} = \oplus_{n=4}^{\infty} \mathcal{A}_{\mathcal{S}_n}$. We have just shown that every $\mathcal{A}_{\mathcal{S}_n}$ is reflexive, therefore by Lemma 2.3.6 $\mathcal{A}$ is reflexive. If $T_n \in \mathcal{B}(\mathcal{H}_n)$ with $\mathrm{dist}(T_n, \mathcal{A}_{\mathcal{S}_n}) \geq \frac{n}{3}\beta(T_n, \mathcal{A}_{\mathcal{S}_n})$ let $\hat{T}_n = \bigoplus\limits_{i=4}^{\infty} X_i$ where $X_n = T_n$ and $S_i = 0$ for $i \neq n$. Therefore (by definition of $\hat{T}$ we have that $\mathrm{dist}(\hat{T}_n, \mathcal{A}_{\mathcal{S}_n}) = \mathrm{dist}(T_n, \mathcal{A}_{\mathcal{S}_n})$. However, $\mathrm{Lat}\,\mathcal{A} = \oplus_{n=4}^{\infty} \mathrm{Lat}\,\mathcal{A}_n$. Therefore $\mathrm{dist}(\hat{T}_n, \mathcal{A}_n) = \mathrm{dist}(T_n, \mathcal{A}_n) > \frac{n}{3}\alpha(T_n, \mathcal{A}_n) = \alpha(\hat{T}_n, \mathcal{A}_n)$ and thus $\mathcal{A}$ is not hyperreflexive.

We now proceed to define the $\mathbb{A}_1$-property. Note that for any w*-closed subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ there is a predual given by $\mathcal{L}_1(\mathcal{H})/\mathcal{S}_\perp$.

Definition 2.4.10 ($\mathbb{A}_1$-property)**.** A linear subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is said to have the $\mathbb{A}_1$-property if $\mathcal{S}$ is w*-closed and for every w*-continuous linear functional ϕ on $\mathcal{S}$ there are vectors $h, g \in \mathcal{H}$ such that $\phi(S) = \langle Sh, g\rangle$ for all $S \in \mathcal{S}$.
In particular, if $r \geq 1$ say that $\mathcal{S}$ has property $\mathbb{A}_1(r)$ if $\mathcal{S}$ is w*-closed and for every

$\varepsilon > 0$ and every w*-continuous linear functional ϕ on $\mathcal{S}$ there are vectors $h, g \in \mathcal{H}$ such that $\phi(S) = \langle Sh, g\rangle$ for all $S \in \mathcal{S}$ and

$$\|h\|\,\|g\| < (r+\varepsilon)\,\|\phi\|\,.$$

The usefulness of this property becomes apparent with the following results.

Lemma 2.4.11. *If $\mathcal{S}$ is a w*-closed subspace of $\mathcal{B}(\mathcal{H})$ that has property $\mathbb{A}_1(r)$ for some $r \geq 1$ then $\mathcal{S}$ is wot-closed.*

Theorem 2.4.12. *A subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is hereditarily reflexive if and only if $\mathcal{S}$ is reflexive and has property $\mathbb{A}_1$.*

We have a similar result if we are considering hyperreflexive spaces.

Theorem 2.4.13. *If $\mathcal{S}$ is a hyperreflexive subspace of $\mathcal{B}(\mathcal{H})$ then then every w*-closed subspace of $\mathcal{S}$ is hyperreflexive if and only if $\mathcal{S}$ has property $\mathbb{A}_1(1)$.*
If $\mathcal{S}$ is hyperreflexive and has property $\mathbb{A}_1(r)$ for some $r \geq 1$, then for every w-closed subspace of $\mathcal{T}$ of $\mathcal{S}$,*

$$\kappa(\mathcal{T}) \leq r + (1+r)\kappa(\mathcal{S}),$$

where $\kappa(\mathcal{S})$ is the smallest possible .

Kraus-Larson [38] and Davidson [15] have shown that the above result holds if we consider a w*-closed algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ instead of a linear subspace. If $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is a w*-closed subspace then we write $\mathcal{S}^{(\infty)}$ for the inflation of $\mathcal{S}$, that is

$$\mathcal{S}^{(\infty)} = \left\{ \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & \ddots \end{bmatrix} : S \in \mathcal{S} \right\}$$

Proposition 2.4.14. *If $\mathcal{S}$ is a w*-closed subspace of $\mathcal{B}(\mathcal{H})$ then $\mathcal{S}^{(\infty)}$ is reflexive.*

This follows since $\mathcal{B}(\mathcal{H})^{(\infty)}$ has the $\mathbb{A}_1$-property. Therefore an application of Theorem 2.4.12 implies that it suffices to show that $\mathcal{B}(\mathcal{H})^{(\infty)}$ is reflexive. However it is a von Neumann algebra and thus is reflexive. We end this chapter by considering the following examples in order to demonstrate the difference between reflexivity and the bicommutant property.

Examples 2.4.15. (1) Reflexive and has Bicommutant Property:
If we take any w*-closed algebra $\mathcal{A} = \mathcal{A}''$ in $\mathcal{B}(\mathcal{H})$ then consider its inflation $\mathcal{A}^{(\infty)} \in \mathcal{B}(\mathcal{H}^{(\infty)})$. However $\mathcal{B}(\mathcal{H}^{(\infty)})$ has the $\mathbb{A}_1(1)$-property therefore it is hereditarily hyperreflexive and thus since $\mathcal{A}$ is w*-closed $\mathcal{A}^{(\infty)}$ is reflexive. We will now show that $\mathcal{A}^{(\infty)}$ has the bicommutant property. Suppose that $T \in (\mathcal{A}^{(\infty)})'$ so that

$$\begin{aligned} AT &= \begin{bmatrix} a & 0 & \cdots \\ 0 & a & \cdots \\ \vdots & \cdots & \ddots \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} & \cdots \\ t_{21} & t_{22} & \cdots \\ \vdots & \cdots & \ddots \end{bmatrix} \\ &= \begin{bmatrix} t_{11} & t_{12} & \cdots \\ t_{21} & t_{22} & \cdots \\ \vdots & \cdots & \ddots \end{bmatrix} \begin{bmatrix} a & 0 & \cdots \\ 0 & a & \cdots \\ \vdots & \cdots & \ddots \end{bmatrix} = TA, \end{aligned}$$

for $a \in \mathcal{A}$. Therefore,

$$\begin{bmatrix} at_{11} & at_{12} & \cdots \\ at_{21} & at_{22} & \cdots \\ \vdots & \cdots & \ddots \end{bmatrix} = \begin{bmatrix} t_{11}a & t_{12}a & \cdots \\ t_{21}a & t_{22}a & \cdots \\ \vdots & \cdots & \ddots \end{bmatrix}.$$

Thus if $T \in (\mathcal{A}^{(\infty)})'$ then every entry of T is in $\mathcal{A}'$. Therefore $T \in \mathcal{M}_\infty(\mathcal{A}')$. The reverse containment follows by completing the inverse computation. Hence the commutant of the inflation consists of matrices whose entries are in $\mathcal{A}'$.

We can now perform a similar computation to obtain that $(\mathcal{A}^{(\infty)})'' = (\mathcal{A}'')^{(\infty)}$. In this case we require, for $B \in \mathcal{B}(\mathcal{H}^{(\infty)})$ that

$$TB = BT \text{ for all } T \in \mathcal{M}_\infty(\mathcal{A}'). \tag{2.2}$$

Applying (2.2) for $t_{11} = I$ and every other entry equal to zero we have

$$\begin{bmatrix} b_{11} & b_{12} & \cdots \\ 0 & 0 & \cdots \\ \vdots & \cdots & \ddots \end{bmatrix} = \begin{bmatrix} b_{11} & 0 & \cdots \\ b_{21} & 0 & \cdots \\ \vdots & \cdots & \ddots \end{bmatrix}.$$

Similarly, applying for $t_{21} = I$ and every other entry equal to zero we have

$$\begin{bmatrix} 0 & 0 & \cdots \\ b_{11} & b_{12} & \cdots \\ 0 & \cdots & \ddots \end{bmatrix} = \begin{bmatrix} b_{12} & 0 & \cdots \\ b_{22} & 0 & \cdots \\ \vdots & \cdots & \ddots \end{bmatrix}$$

Repeating these calculations for each entry of the first column of T equal to the identity we see that all non-diagonal entries must be equal to zero. Similarly using the second set of equalities forces $b_{11} = b_{nn}$ for all n. Hence $(\mathcal{A}'')^{(\infty)}$ consists of diagonal matrices with constant diagonal whose entries are in $\mathcal{A}''$.
The reverse containment follows by a similar computation and so $(\mathcal{M}_\infty(\mathcal{A}'))' = (\mathcal{A}'')^{(\infty)}$. Therefore $(\mathcal{A}'')^\infty = (\mathcal{A}^\infty)''$.

(2) Reflexive but does not have the Bicommutant Property:
Suppose that $\mathcal{A}$ is a w*-closed subalgebra of $\mathcal{B}(\mathcal{H})$ which doesn't have the bicommutant property and consider its inflation $\mathcal{A}^{(\infty)} \in \mathcal{B}(\mathcal{H}^{(\infty)})$. Then we have seen that $\mathcal{A}^{(\infty)}$ is reflexive. Also, from Example (1) we have that $(\mathcal{A}'')^{(\infty)} = (\mathcal{A}^{(\infty)})''$. Taking compressions of $(\mathcal{A}^{(\infty)})''$ to the $(0,0)$-entry we have that $P_0\mathcal{A}''P_0 = \mathcal{A}$ thus implying that $\mathcal{A}'' = \mathcal{A}$, which contradicts the assumption that $\mathcal{A}$ did not have the bicommutant property.

(3) Not reflexive but has the Bicommutant Property:
An example of a non-reflexive algebra is the 2×2 lower triangular matrices over $\mathbb{C}$. Set

$$\mathcal{A} = \left\{ \begin{bmatrix} \lambda & 0 \\ \mu & \lambda \end{bmatrix} \,:\, \lambda, \mu \in \mathbb{C} \right\}$$

Then

$$\mathcal{L} = \mathrm{Lat}(\mathcal{A}) = \{M \subseteq \mathbb{C}^2 \,:\, AM \subseteq M \text{ for all } A \in \mathcal{A}\} = \{\{0\}, \mathbb{C}e_2, \mathbb{C}^2\}$$

Thus

$$\mathrm{AlgLat}(\mathcal{A}) = \left\{ \begin{bmatrix} \lambda & 0 \\ \mu & \nu \end{bmatrix} \,:\, \lambda, \mu, \nu \in \mathbb{C} \right\}.$$

Therefore $\mathrm{AlgLat}(\mathcal{A}) \supsetneq \mathcal{A}$ and therefore $\mathcal{A}$ is not reflexive.

However $\mathcal{A} = \mathcal{A}' = \mathcal{A}''$ in this case. Indeed, for $T \in \mathcal{A}'$ and $A \in \mathcal{A}$ we require

$$TA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ \mu & \lambda \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ \mu & \lambda \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = AT.$$

That is,

$$\begin{bmatrix} a\lambda + b\mu & b\lambda \\ c\lambda + d\mu & d\lambda \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \mu a + \lambda c & \mu b + \lambda d \end{bmatrix}.$$

Applying for $\lambda = 0, \mu = 1$ we have that

$$\begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}.$$

Comparing entry-wise, we see that $b = 0$ and $a = d$. Therefore $\mathcal{A} = \mathcal{A}'$ and so $\mathcal{A} = \mathcal{A}' = \mathcal{A}''$.

Chapter 3

Semicrossed Products

We now discuss a generalised version of Fejér's Theorem and its application to operators. We examine lower triangular operators and use Fejér's theorem to realise a gauge action of $\mathbb{T}$ on $\mathcal{B}(\mathcal{H}) \otimes \ell^2(\mathbb{F}_+^d)$.

3.1 Fejér's Theorem

We shall begin by presenting Féjer's theorem for functions [17, Section 14.6]. We include a proof for self-containment. Recall the definition of a positive kernel.

Definition 3.1.1 (Positive Kernel)**.** We call a family of 2π-periodic continuous functions k_n $n \in \mathbb{N}$ a *positive kernel* if

(i) $k_n(t) \geq 0$ for all $t \in \mathbb{R}$.

(ii) $\int_{-\pi}^{\pi} k_n(t)\ \mathrm{dt} = 1$.

(iii) If $\delta \in (0, \pi)$ then k_n converges uniformly to zero on $[-\pi, -\delta] \cup [\delta, \pi]$.

Given $f : [-\pi, \pi] \to \mathbb{R}$ we write $\sigma_n(f)(x) = \int_{-\pi}^{\pi} f(x+t) k_n(t)\ \mathrm{dt}$.

Lemma 3.1.2. *[17, Theorem 14.6.4] If $f : [-\pi, \pi] \to \mathbb{R}$ is continuous and 2π-periodic and if $\{k_n\}_n$ is a positive kernel then:*

$$\sigma_n(f) \to f \quad \textit{uniformly.}$$

Proof. The idea is to appropriately cut the integral of $\sigma_n(f)$ into two parts: $[-\delta, \delta]$ and $[-\pi - \delta, \delta + \pi]$. So, fix $\varepsilon > 0$ and let $M = \sup\{|f(x)| : x \in [-\pi, \pi]\}$. Since

f is uniformly continuous, for $\varepsilon > 0$ we find $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/2$ wherever $|x - y| < \delta$ and find $n_0 > 0$ such that $k_n(x) < \frac{\varepsilon}{8\pi M}$ for all $n \geq n_0$, for all $x \in [-\pi, -\delta] \cup [\delta, \pi]$. Combining these,

$$\begin{aligned}
|\sigma_n(f)(x) - f(x)| &= |\sigma_n(f)(x) - f(x) \int_{-\pi}^{\pi} k_n(t) \text{ dt}| \\
&\leq \int_{-\pi}^{\pi} |f(x+t) - f(x)| k_n(t) \text{ dt} \\
&= \int_{-\delta}^{\delta} |f(x+t) - f(x)| k_n(t) \text{ dt} + \\
&+ \int_{[-\pi,-\delta]\cup[\pi,\delta]} |f(x+t) - f(x)| k_n(t) \text{ dt}.
\end{aligned}$$

Then, since $|f(x) - f(y)| < \frac{\varepsilon}{2}$ we have that

$$\begin{aligned}
\int_{-\delta}^{\delta} |f(x+t) - f(x)| k_n(t) \text{ dt} + \int_{[-\pi,-\delta]\cup[\pi,\delta]} &|f(x+t) - f(x)| k_n(t) \text{ dt} \leq \\
&\leq \frac{\varepsilon}{2} \int_{-\pi}^{\pi} k_n(t) \text{ dt} + 2M \int_{[-\pi,-\delta]\cup[\pi,\delta]} k_n(t) \text{ dt} \\
&\leq \frac{\varepsilon}{2} \int_{-\pi}^{\pi} k_n(t) \text{ dt} + 2M \frac{\varepsilon}{8\pi M} \int_{[-\pi,-\delta]\cup[\pi,\delta]} \text{ dt} \\
&\leq \frac{\varepsilon}{2} \int_{-\pi}^{\pi} k_n(t) \text{ dt} + \frac{\varepsilon}{4\pi} \int_{[-\pi,\pi]} \text{ dt} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

■

Write $K_{n+1}(x) = \frac{1}{2\pi} \sum_{k=-n}^{n} (1 - \frac{|k|}{n+1}) e^{ikx}$ for the Fejér kernel. It can be verified that this is a positive kernel. Therefore we have that $\sigma_n(f) \to f$ uniformly for f being 2π-periodic as in Lemma 3.1.2. It is convenient to have a formula for $\sigma_n(f)$ when computed with respect to the Fejér kernel. To this end we obtain

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x-t) K_{n+1}(t) \text{ dt} &= \int_{x-\pi}^{x+\pi} f(x-t) K_{n+1}(t) \text{ dt} \\
&= \int_{-\pi}^{\pi} f(t) K_{n+1}(x-t) \text{ dt}.
\end{aligned}$$

Inputting the Fejér kernel $K_{n+1}(x)$ then gives

$$\begin{aligned}\int_{-\pi}^{\pi} f(x-t)K_{n+1}(t)\ \mathrm{dt} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{k=-n}^{n}(1-\frac{|k|}{n+1})e^{ik(x-t)}\ \mathrm{dt}\\ &= \frac{1}{2\pi}\sum_{k=-n}^{n}(1-\frac{|k|}{n+1})(\int_{-\pi}^{\pi} f(t)e^{-ikt}\ \mathrm{dt})e^{ikx}\\ &= \sum_{k=-n}^{n}(1-\frac{|k|}{n+1})C_k\cdot e^{ikx},\end{aligned}$$

where $C_k = \left\langle f, e^{ikt}\right\rangle_{L^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\ \mathrm{dt}$. This is convenient as it shows that $\{e^{ikt}\}_{k\in\mathbb{Z}}$ forms a basis in $L^2([-\pi,\pi])$.

3.2 Lower Triangular Operators

We now proceed to consider lower triangular operators. We begin by examining an application of Fejér's theorem in operator theory for operators on the free semigroup.

3.2.1 Free Semigroup Operators

For $d \in \mathbb{Z}_+ \cup \{\infty\}$ let $\mathbb{F}_+^d$ be the free semigroup on d generators. Also let $\mathcal{K} = \mathcal{H} \otimes \ell^2(\mathbb{F}_+^d)$ for a Hilbert space $\mathcal{H}$. Write $|\mu|$ for the length of a word $\mu = \mu_m \dots \mu_1 \in \mathbb{F}_+^d$. For $z \in \mathbb{T}$ define $u_z : \ell^2(\mathbb{F}_+^d) \to \ell^2(\mathbb{F}_+^d)$ such that $u_z(e_\mu) = z^{|\mu|}e_\mu$ and set $U_z = I \otimes u_z$. Note that every u_z is a unitary. For $m \in \mathbb{N}$ we define the *m-th Fourier coefficient* to be the expression

$$G_m(T) := \frac{1}{2\pi}\int_{-\pi}^{\pi} U_{e^{it}}TU_{e^{-it}}e^{-imt}\ \mathrm{dt},$$

where the integral is the w*-limit of Riemann sums.

Theorem 3.2.1. *Let $T \in \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{F}_+^d))$ and write*

$$\sigma_{n+1}(T) := \sum_{k=-n}^{n}(1-\frac{|k|}{n+1})G_k(t).$$

Then $\sigma_{n+1}(T) \xrightarrow{w^*} T$.

Proof. Let $\phi \in L^1(\mathcal{H})$ then we have to show $\phi(\sigma_{n+1}(T)) \xrightarrow{|\cdot|} \phi(T)$. Define $f : [\pi, \pi] \to \mathbb{R}$, given by $f(t) = \phi(U_{e^{it}} T U_{e^{-it}})$. Then $\sigma_n(f) \to f$ uniformly by Lemma 3.1.2. Hence $\sigma_n(f)(0) \to \sigma(f)(0) = \phi(T)$. Therefore we have to show that $\sigma_n(f)(0) = \phi(\sigma_n(T))$. We compute:

$$\begin{aligned}\sigma_{n+1}(f)(0) &= \frac{1}{2\pi}\sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)\int_{-\pi}^{\pi}\phi(U_{e^{it}}TU_{e^{-it}})e^{-kt}\ \mathrm{dt}\\ &= \sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)\phi\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(U_{e^{it}}TU_{e^{-it}})e^{-kt}\ \mathrm{dt}\right)\\ &= \sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)\phi(G_k(T))\\ &= \phi(\sigma_{n+1}(T)),\end{aligned}$$

and the proof is complete. ∎

Now proceed to fix a Hilbert space $\mathcal{H}$ and consider the space $\mathcal{K} = \mathcal{H} \otimes \ell^2(\mathbb{F}_+^d)$. Then let $\mu, \nu \in \mathbb{F}_+^d$ then we can endow $\mathbb{F}_+^d$ with a (right) partial order given by

$$\nu \leq_r \mu \text{ if there exists } z \in \mathbb{F}_+^d \text{ such that } \mu = \nu z.$$

We can similarly define a left partial ordering by

$$\nu \leq_l \mu \text{ if there exists } z \in \mathbb{F}_+^d \text{ such that } \mu = z\nu.$$

For a word $\mu = \mu_k \dots \mu_1$ we write $\overline{\mu} := \mu_1 \dots \mu_k$ for the reversed word of μ. We define the following creation operators on the Hilbert space $\ell^2(\mathbb{F}_+^d)$ by

$$\mathbf{l}_\mu e_w = e_{\mu w} \quad \text{and} \quad \mathbf{r}_\nu e_w = e_{w\overline{\nu}}.$$

For $\mu, \nu \in \mathbb{F}_+^d$ we write

$$L_\mu := I_{\mathcal{H}} \otimes \mathbf{l}_\mu \quad \text{and} \quad R_\nu := I_{\mathcal{H}} \otimes \mathbf{r}_\nu.$$

Then we can define the free semigroup algebras

$$\mathcal{L}_d := \overline{\mathrm{alg}}^{wot}\{\mathbf{l}_\mu \,:\, \mu \in \mathbb{F}_+^d\} \quad \text{and} \quad \mathcal{R}_d := \overline{\mathrm{alg}}^{wot}\{\mathbf{r}_\nu \,:\, \nu \in \mathbb{F}_+^d\}.$$

Example 3.2.2. For the purposes of illustration we shall now calculate the values of $\sigma_{n+1}(T)$ for a specific operator T. For $x \in \mathcal{B}(\mathcal{H})$, let $T = 3x \otimes I + \sqrt{2}x \otimes l_{12} + \pi x \otimes l_{31} - 4x \otimes l_{1321}$ in $\mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{F}_+^d))$. By definition we have

$$\begin{aligned} G_0(T) &= G_0(3x \otimes I) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} U_{e^{it}}(3x \otimes I) U_{e^{-it}} \ \mathrm{dt} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \ \mathrm{dt}(3x \otimes I) \\ &= 3x \otimes I. \end{aligned}$$

Note that all the other terms of T will be equal to zero. We can proceed similarly to see that the values of $G_m(T)$ are

$$\begin{aligned} G_0(T) &= 3x \otimes 1 \\ G_1(T) &= 0 \\ G_2(T) &= \sqrt{2}x \otimes l_{12} + \pi x_{31} \otimes l_{31} \\ G_3(T) &= 0 \\ G_4(T) &= 4x \otimes l_{1321}, \end{aligned}$$

and $G_m(T) = 0$ for $m \geq 4$. Hence

$$\begin{aligned} \sigma_1(T) &= G_0(T) \\ \sigma_2(T) &= \sum_{k=-1}^{1} (1 - \frac{|k|}{2}) G_k(T) = G_0(T) + \frac{1}{2} G_1(T) \\ \sigma_3(T) &= \sum_{k=-2}^{2} (1 - \frac{|k|}{3}) G_k(T) = G_0(T) + \frac{2}{3} G_1(T) + \frac{1}{3} G_2(T) \\ \sigma_4(T) &= \sum_{k=-3}^{3} 1 - \frac{|k|}{3}) G_k(T) = 0 + G_0(T) + \frac{3}{4} G_1(T) + \frac{1}{2} G_2(T) + \frac{1}{4} G_3(T) \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma_{n+1}(T) &= G_0(T) + \frac{n}{n+1} G_1(T) + \cdots + \frac{1}{n+1} G_n(T) \\ &= G_0(T) + (1 - \frac{2}{n+1}) G_2(T) + (1 - \frac{4}{n+1}) G_4(T). \end{aligned}$$

Hence $\sigma_{n+1}(T) \to T$.

The following application of Fejér's Theorem, shows that we can consider the algebra $\mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_d$ to be closed in the w*-topology instead.

Proposition 3.2.3. *Let* $\mathcal{A}_1 = \overline{\mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_d}^{w^*}$ *and* $\mathcal{A}_2 = \overline{\mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_d}^{wot}$ *then* $\mathcal{A}_1 = \mathcal{A}_2$.

Proof. The fact that $\mathcal{A}_1 \subseteq \mathcal{A}_2$ is clear. If $x \in \mathcal{A}_1$ and $x = w^* - \lim_i x_i$, $x_i \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_d$ then $x = \mathrm{wot} - \lim_i x_i$ and so $x \in \mathcal{A}_2$.

Conversely let $x \in \mathcal{A}_2$ and consider the m-th Fourier coefficient

$$G_m(x) = \mathrm{wot} - \sum_{|\mu|=m} L_\mu \left(\mathrm{wot} - \sum_{w \in \mathbb{F}_+^d} T_{\mu w, w} \otimes p_w \right) = \mathrm{wot} - \sum_{|\mu|=m} L_\mu a_\mu \otimes I.$$

However, $\|G_m(T)\| \leq \|x\|$. Suppose that $\{\mu \in \mathbb{F}_+^d : |\mu| = m\}$ is infinite. We need to show that the wot sum is the same as the w* sum. Since the sum is countably infinite then if $\mathcal{F} \subseteq \{\mu \in \mathbb{F}_+^d : |\mu| = m\}$ is a finite subset then

$$\sum_{\mu \in \mathcal{F}} a_\mu \otimes l_\mu = (\sum_{\mu \in \mathcal{F}} L_\mu L_\mu^*) \cdot G_m(x).$$

Then taking norms we have that

$$\left\| \sum_{\mu \in \mathcal{F}} a_\mu \otimes l_\mu \right\| = \left\| \sum_{\mu \in \mathcal{F}} L_\mu L_\mu^* \cdot G_m(x). \right\| \leq \|G_m(x)\|.$$

Thus the wot-sum is bounded and therefore also converges in the w* sense. ∎

Therefore the Fourier coefficients coincide in the wot and w*-topologies. Then from the end of Section 3.1 we have that $\mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{F}_+^d))$ admits a w*-continuous action induced by the unitaries

$$U_s \xi \otimes e_w = e^{i|w|s} \xi \otimes e_w \text{ for all } \xi \otimes e_w,$$

with $s \in [-\pi, \pi]$. Also the m-th Fourier coefficient is given by

$$G_m(T) := \frac{1}{2\pi} \int_{-\pi}^{\pi} U_s T U_s^* e^{-ims} \, \mathrm{ds}.$$

For $T \in \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{F}_+^d))$ we write $T_{\mu,\nu} \in \mathcal{B}(\mathcal{H})$ for the (μ, ν)-entry given by

$$\langle T_{\mu,\nu}\xi, \eta\rangle := \langle T\xi \otimes e_\nu, \eta \otimes e_\mu\rangle \text{ for all } \xi, \eta \in \mathcal{H}.$$

We can now define lower triangular operators in this setting.

Definition 3.2.4. An operator $T \in \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{F}_+^d))$ is a *right lower triangular operator* if $T_{\mu,\nu} = 0$ whenever $\nu \not\leq_r \mu$. Similarly an operator T is a *left lower triangular operator* if $T_{\mu,\nu} = 0$ whenever $\nu \not\leq_l \mu$.

Writing p_w for the projection onto e_w we then have the following result.

Proposition 3.2.5. *If T is a left lower triangular operator in $\mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{F}_+^d))$ then*

$$G_m(T) = \begin{cases} \sum_{|\mu|=m} \sum_{w\in\mathbb{F}_+^d} L_\mu(T_{\mu w,w} \otimes p_w) & \text{if } m \geq 0, \\ 0 & \text{if } m < 0, \end{cases}$$

where the sum is taken in the w-topology. In a dual way if T is a right lower triangular operator in $\mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{F}_+^d))$ then*

$$G_m(T) = \begin{cases} \sum_{|\mu|=m} \sum_{w\in\mathbb{F}_+^d} R_\mu(T_{w\overline{\mu},w} \otimes p_w) & \text{if } m \geq 0, \\ 0 & \text{if } m < 0, \end{cases}$$

where the sum is taken in the w-topology.*

Proof. Here we give the proof for the right case, the left case is proven in [8]. Fix $\nu, \nu' \in \mathbb{F}_+^d$ and $\xi, \eta \in \mathcal{H}$. Then

$$\begin{aligned} \langle G_m(T)\xi \otimes e_\nu, \eta \otimes e_{\nu'}\rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle T\xi \otimes e_\nu, \eta \otimes e_{\nu'}\rangle\, e^{i(-m-|\nu|+|\nu'|)s} \,\mathrm{ds} \\ &= \langle T\xi \otimes e_\nu, \eta \otimes e_{\nu'}\rangle \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(-m-|\nu|+|\nu'|)s} \,\mathrm{ds} \\ &= \langle T_{\nu',\nu}\xi, \eta\rangle \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(-m-|\nu|+|\nu'|)s} \,\mathrm{ds} \\ &= \delta_{|\nu'|,m+|\nu|} \langle T_{\nu',\nu}\xi, \eta\rangle \end{aligned}$$

for all $T \in \mathcal{B}(\mathcal{K})$. Let T be a right lower triangular operator and consider the case where $m < 0$. We have that $\langle G_m(T)\xi \otimes e_\nu, \eta \otimes e_{\nu'}\rangle = 0$ when $|\nu'| \neq m + |\nu|$, so assume equality holds. If $|\nu'| = m + |\nu|$ then $|\nu'| < |\nu|$ and thus $\nu \not\leq_r \nu'$. Then we get that $\langle T_{\nu',\nu}\xi, \eta\rangle = 0$ since we have assumed that T is lower right triangular. Therefore $G_m(T) = 0$ when $m < 0$. Now consider the case when $m \geq 0$. We have that $\langle T_{\nu',\nu}\xi, \eta\rangle = 0$ whenever $\nu \not\leq_r \nu'$. Therefore we obtain that

$$\begin{aligned}\langle G_m(T)\xi \otimes e_\nu, \eta \otimes e_{\nu'}\rangle &= \begin{cases}\langle T_{\nu',\nu}\xi, \eta\rangle & \text{if } \nu \leq_r \nu' \text{ and } |\nu'| = m + |\nu|, \\ 0 & \text{otherwise,}\end{cases} \\ &= \begin{cases}\langle T_{\nu',\nu}\xi, \eta\rangle & \text{if } \nu' = \nu z \text{ with } |z| = m, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

On the other hand we compute

$$\begin{aligned}&\sum_{|\mu|=m}\sum_{w\in\mathbb{F}_+^d}\langle R_\mu T_{w\overline{\mu},w} \otimes p_w(\xi \otimes e_\nu), \eta \otimes e_{\nu'}\rangle = \\ &\qquad = \sum_{|\mu|=m}\sum_{w\in\mathbb{F}_+^d}\left\langle T_{w\overline{\mu},w} \otimes p_w(\xi \otimes e_\nu), \mathcal{R}_\mu^*(\eta \otimes e_{\nu'})\right\rangle \\ &\qquad = \sum_{|\mu|=m}\left\langle T_{\nu\overline{\mu},\nu}\xi \otimes e_\nu, \eta \otimes \mathbf{r}_\mu^* e_{\nu'}\right\rangle \\ &\qquad = \sum_{|\mu|=m}\left\langle T_{\nu\overline{\mu},\nu}\xi \otimes e_{\nu\overline{\mu}}, \eta \otimes e_{\nu'}\right\rangle.\end{aligned}$$

If $\nu' = \nu\overline{z}$ for some $z \in \mathbb{F}_+^d$ with length m then we have that

$$\begin{aligned}&\sum_{|\mu|=m}\sum_{w\in\mathbb{F}_+^d}\langle R_\mu(T_{w\overline{\mu},w} \otimes p_w)\xi \otimes e_\nu, \eta \otimes e_{\nu'}\rangle = \langle T_{\nu\overline{z},\nu}\xi \otimes e_\nu, \eta \otimes e_{\nu'}\rangle = \\ &\qquad = \begin{cases}\langle T_{\nu',\nu}\xi, \eta\rangle & \text{if } \nu' = \nu\overline{z} \text{ and } |\overline{z}| = m, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Now, notice that if we have $\nu' = \nu x$ then $\nu' = \nu\overline{z}$ for $z = \overline{x}$, and we are done. ∎

3.2.2 Operators on $\mathbb{Z}_+^d$

It is possible to develop results analogous to those in the previous section for $\mathbb{Z}_+^d$ which is done as follows. Consider the Hilbert space $\mathcal{H} \otimes \ell^2(\mathbb{Z}_+^d)$. There is a partial order on $\mathbb{Z}_+^d$ given by saying

$$n \leq m \text{ if there exists } \underline{z} \in \mathbb{Z}_+^d \text{ such that } \underline{m} = \underline{z} + \underline{n}.$$

Analogously to the previous section we can define creation operators in $\ell^2(\mathbb{Z}_+^d)$ given by $\mathbf{l}_{\underline{m}} e_{\underline{w}} = e_{\underline{m}+\underline{w}}$. We write $\mathbb{H}^\infty(\mathbb{Z}_+^d)$ for the wot-closed algebra generated by these creation operators. An appeal to Fejér's theorem again gives that implies that we may consider the w*-closure instead. We can define a w*-continuous action on $\mathcal{H} \otimes \ell^2(\mathbb{Z}_+^d)$ by considering the unitaries

$$U_{\underline{s}} \xi \otimes e_{\underline{w}} = e^{i \sum_{i=1}^d w_i s_i} \xi \otimes e_{\underline{w}} \text{ for all } \xi \otimes e_{\underline{w}}.$$

Then the Fourier coefficients on $T \in \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{Z}_+^d))$ are given by

$$G_{\underline{m}}(T) := \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} U_{\underline{s}} T U_{\underline{s}}^* e^{-i \sum_{i=1}^d m_i s_i} d\underline{s} \quad \text{for } \underline{m} \in \mathbb{Z}_+^d,$$

where the integral is the w*-limit of Riemann sums. For $T \in \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{Z}_+^d))$ we write $T_{\underline{m},\underline{n}} \in \mathcal{B}(\mathcal{H})$ for the operator given by

$$\langle T_{\underline{m},\underline{n}} \xi, \eta \rangle = \langle T \xi \otimes e_{\underline{n}}, \eta \otimes e_{\underline{m}} \rangle .$$

Again we can define lower triangular operators as follows.

Definition 3.2.6. An operator $T \in \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{Z}_+^d))$ is a *lower triangular operator* if $T_{\underline{m},\underline{n}} = 0$ whenever $\underline{n} \not\leq \underline{m}$.

Set $L_{\underline{m}} = I_{\mathcal{H}} \otimes \mathbf{l}_{\underline{m}}$ and write $p_{\underline{w}}$ for the projection of $\ell^2(\mathbb{Z}_+^d)$ to $e_{\underline{w}}$. We then have the following proposition in analogy to Proposition 3.2.5.

Proposition 3.2.7. *If T is a lower triangular operator in $\mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{Z}_+^d))$ then*

$$G_{\underline{m}}(T) = \begin{cases} \sum_{\underline{w} \in \mathbb{Z}_+^d} L_{\underline{m}} (T_{\underline{m}+\underline{w},\underline{w}} \otimes p_{\underline{w}}) & \textit{if } \underline{m} \in \mathbb{Z}_+^d, \\ 0 & \textit{otherwise.} \end{cases}$$

where the sum is taken in the w-topology.*

Proof. The proof follows similarly to Proposition 3.2.5. Let T be a lower triangular operator. Then for $\underline{n}, \underline{n'} \in \mathbb{Z}_+^d$ and $\xi, \eta \in \mathcal{H}$ we obtain

$$\begin{aligned}
&\langle G_{\underline{m}}(T)\xi \otimes e_{\underline{n}}, \eta \otimes e_{\underline{n'}} \rangle = \\
&\quad = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \langle T\xi \otimes e_{\underline{n}}, \eta \otimes e_{\underline{n'}} \rangle \, e^{-i\sum_{i=1}^d (m_i+n_i-n_i')s_i} d\underline{s} \\
&\quad = \frac{1}{(2\pi)^d} \langle T\xi \otimes e_{\underline{n}}, \eta \otimes e_{\underline{n'}} \rangle \int_{[-\pi,\pi]^d} e^{-i\sum_{i=1}^d (m_i+n_i-n_i')s_i} d\underline{s} \\
&\quad = \delta_{\underline{n'},\underline{m}+\underline{n}} \langle T_{\underline{n'},\underline{n}}\xi, \eta \rangle .
\end{aligned}$$

Therefore $\langle G_{\underline{m}}(T)\xi \otimes e_{\underline{n}}, \eta \otimes e_{\underline{n'}} \rangle = 0$ when $\underline{n'} \neq \underline{m} + \underline{n}$. If $\underline{n'} = \underline{m} + \underline{n}$ for $\underline{m} \notin \mathbb{Z}_+^d$ then there exists an $i = 1, \dots, d$ such that $n_i' < n_i$. In this case $\underline{n} \not< \underline{n'}$ hence $T_{\underline{n'},\underline{n}} = 0$ and thus $G_{\underline{m}}(T) = 0$, since we assumed that T is lower triangular. On the other hand if $\underline{m} \in \mathbb{Z}_+^d$ then

$$\begin{aligned}
&\sum_{\underline{w} \in \mathbb{Z}_+^d} \langle L_{\underline{m}}(T_{\underline{m}+\underline{w}} \otimes p_{\underline{w}})\xi \otimes e_{\underline{n}}, \eta \otimes e_{\underline{n'}} \rangle = \\
&\qquad = \sum_{\underline{w} \in \mathbb{Z}_+^d} \left\langle (T_{\underline{m}+\underline{w},\underline{w}} \otimes p_{\underline{w}})\xi \otimes e_{\underline{n}}, \eta \otimes \mathbf{l}_{\underline{m}}^* e_{\underline{n'}} \right\rangle \\
&\qquad = \sum_{\underline{w} \in \mathbb{Z}_+^d} \left\langle T_{\underline{m}+\underline{w},\underline{w}}\xi \otimes p_{\underline{w}} e_{\underline{n}}, \eta \otimes \mathbf{l}_{\underline{m}}^* e_{\underline{n'}} \right\rangle \\
&\qquad = \left\langle T_{\underline{m}+\underline{n},\underline{n}}\xi \otimes e_{\underline{n}}, \eta \otimes \mathbf{l}_{\underline{m}}^* e_{\underline{n'}} \right\rangle \\
&\qquad = \langle T_{\underline{m}+\underline{n},\underline{n}}\xi \otimes e_{\underline{m}+\underline{n}}, \eta \otimes e_{\underline{n'}} \rangle \\
&\qquad = \delta_{\underline{n'},\underline{m}+\underline{n}} \langle T_{\underline{m}+\underline{n},\underline{n}}\xi, \eta \rangle .
\end{aligned}$$

Hence

$$\langle G_{\underline{m}}(T)\xi \otimes e_{\underline{n}}, \eta \otimes e_{\underline{n'}} \rangle = \sum_{\underline{w} \in \mathbb{Z}_+^d} \langle L_{\underline{m}}(T_{\underline{m}+\underline{w}} \otimes p_{\underline{w}})\xi \otimes e_{\underline{n}}, \eta \otimes e_{\underline{n'}} \rangle$$

and we are done. ∎

3.3 Tensoring with $\mathcal{B}(\mathcal{H})$

The primary purpose of this section is to provide a proof of the reflexivity of $\mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_d$. Bercovici in [9] shows that a wot-closed algebra is hyperreflexive with distance constant at most 3 when its commutant contains two isometries with orthogonal ranges. Davidson and Pitts [22] show that the wot-closure of the algebraic tensor product of $\mathcal{B}(\mathcal{H})$ with $\mathcal{L}_d$ satisfies the $\mathbb{A}_1(1)$-property, when $d \geq 2$. Their arguments again depend on the existence of two isometries with orthogonal ranges in the commutant; thus they also apply for the tensor product of $\mathcal{B}(\mathcal{H})$ with $\mathcal{R}_d$. By following this idea we have that every w*-closed subalgebra of $\mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_d$ is hyperreflexive with distance constant at most 3, when $d \geq 2$, as its commutant contains $I_{\mathcal{H}} \otimes \mathcal{R}_d$. Thus the reflexivity of $\mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_d$ can be derived from the hyperreflexivity results of Bercovici [9]. However the method stated here gives an independent proof of reflexivity.

We shall require the following notation. Write $\mathbb{B}_d$ for the ℓ^2-unit ball in d dimensions. For $\lambda \in \mathbb{B}_d$ and $w = w_m \cdots w_1 \in \mathbb{F}_+^d$ write $w(\lambda) = \lambda_{w_m} \cdots \lambda_{w_1}$. Then by [22, Theorem 2.6] the eigenvectors of $\mathcal{L}_d^* = \overline{\operatorname{alg}}^{w^*}\{\mathbf{l}_\mu^* : \mu \in \mathbb{F}_+^d\}$ are the vectors

$$\nu_\lambda = (1 - \|\lambda\|^2)^{1/2} \sum_{w \in \mathbb{F}_+^d} \overline{w(\lambda)} e_w = (1 - \|\lambda\|^2)^{1/2} (I - \sum_{i=1}^d \overline{\lambda}_i L_i)^{-1} e_1 \text{ for } \lambda \in \mathbb{B}_d.$$

They are well defined because if $\lambda \in \mathbb{B}_d$ then ν_λ is defined for $\lambda \in \mathbb{B}_d$ and

$$\begin{aligned}
\sum_w |w(\lambda)|^2 &= \sum_{k \geq 0} \sum_{|w| = k} |w(\lambda)|^2 \\
&= \sum_{k \geq 0, m_i \geq 0} \sum_{\sum m_i = k} \frac{k!}{m_1! \cdots m_d!} |\lambda_1|^{2m_1} \cdots |\lambda_d|^{2m_d} \\
&= \sum_{k \geq 0} \left(\sum_{i=1}^d |\lambda_i|^2 \right)^k \\
&= (1 - \|\lambda\|^2)^{-1} < \infty.
\end{aligned}$$

Also, since

$$\left\| \sum_{i=1}^d \overline{\lambda_i} L_i \right\|^2 = \sum_{i=1}^d |\lambda_i|^2 = \|\lambda\|^2 < 1$$

we have that $(I - \sum_{i=1}^d \overline{\lambda}_i L_i)$ is invertible with inverse given by

$$\sum_{k \geq 0} \left(\sum_{i=1}^d \overline{\lambda_i} L_i \right)^k = \sum_{w \in \mathbb{F}_+^d} \overline{w(\lambda)} L_w,$$

by the above computation, thus rearranging gives the equality. Thus we can now establish the following.

Proposition 3.3.1. *[1], [22] The algebras $\mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_d$ and $\mathcal{B}(\mathcal{H}) \otimes \mathcal{R}_d$ are reflexive.*

Proof. We shall just show that $\mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_d$ is reflexive. Since the gauge action of $\mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{F}_+^d))$ restricts to a gauge action of $\mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_d$, it suffices to show that every $G_m(T)$ is in $\mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_d$ whenever T is in $\operatorname{Ref}(\mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_d)$.
For $\xi, \eta \in \mathcal{H}$ and $\nu, \mu \in \mathbb{F}_+^d$ there is a sequence $X_n \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_d$ such that

$$\langle T_{\mu,\nu}\xi, \eta\rangle = \langle T\xi \otimes e_\nu, \eta \otimes e_\mu\rangle = \lim_n \langle X_n\xi \otimes e_\nu, \eta \otimes e_\mu\rangle = \lim_n \langle [X_n]_{\mu,\nu}\xi, \eta\rangle .$$

Taking $\nu \not<_l \mu$ gives that T is left lower triangular because each X_n is. Therefore it suffices to show that $T_{\mu z,z} = T_{\mu,\emptyset}$ for all $z \in \mathbb{F}_+^d$. In fact when this holds, we can write

$$\begin{aligned} G_m(T) &= \begin{cases} \sum_{|\mu|=m} \sum_{w\in\mathbb{F}_+^d} L_\mu(T_{\mu w,w} \otimes p_w) & \text{if } m \geq 0, \\ 0 & \text{if } m < 0, \end{cases} \\ &= \begin{cases} \sum_{|\mu|=m} L_\mu(T_{\mu,\emptyset} \otimes I) & \text{if } m \geq 0, \\ 0 & \text{if } m < 0, \end{cases} \end{aligned}$$

and thus $G_m(T) \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_d$. An application of Fejér's Lemma will give that $T \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_d$ as required. For convenience we use the notation

$$T_{(\mu)} := L_\mu^* G_m(T) = \sum_{w\in\mathbb{F}_+^d} T_{\mu w,w} \otimes p_w.$$

We treat the cases $m = 0$ and $m \geq 1$ separately.

• **The case** $m = 0$. Let $z \in \mathbb{F}_+^d$ and assume that $\{z_1, \dots, z_{|z|}\} \subseteq [d']$ for some finite d'. If $d < \infty$ then take $d' = d$. Let $\lambda \in \mathbb{B}_{d'} \subseteq \mathbb{B}_d$ such that $\lambda_i \neq 0$ for every $i \in [d']$, and consider the vector

$$g = \sum_{w\in\mathbb{F}_+^{d'}} w(\lambda)e_w.$$

From [22] we have that g is an eigenvector for $\mathcal{L}_d^*$. Therefore the vector $(L_\mu(x \otimes I))^*\xi \otimes g$ is in the closure of $\{y\xi \otimes g \mid y \in \mathcal{B}(\mathcal{H})\}$. Thus for $\xi \in \mathcal{H}$ there exists a sequence (x_n) in $\mathcal{B}(\mathcal{H})$ such that

$$G_0(T)^*\xi \otimes g = \lim_n x_n^*\xi \otimes g. \tag{3.1}$$

Hence for $\eta \in \mathcal{H}$, taking inner products we get

$$\begin{aligned}
w(\lambda) \langle \xi, T_{w,w}\eta \rangle &= \langle \xi, T_{w,w}\eta \rangle \langle g, e_w \rangle \\
&= \langle \xi \otimes g, T_{w,w}\eta \otimes e_w \rangle \\
&= \langle G_0(T)^*\xi \otimes g, \eta \otimes e_w \rangle \\
&= \lim_n \langle x_n^*\xi \otimes g, \eta \otimes e_w \rangle \\
&= \lim_n \langle \xi, x_n\eta \rangle \langle g, e_w \rangle \\
&= w(\lambda) \lim_n \langle \xi, x_n\eta \rangle .
\end{aligned}$$

Then applying for $w = \emptyset$ and $w = z$ we have that $T_{z,z} = T_{\emptyset,\emptyset}$ as $z(\lambda) \neq 0$. Therefore since z was arbitrary we have that $G_0(T) = T_{\emptyset,\emptyset} \otimes I$.

• **The case** $m \geq 1$**.** We have to show that $T_{\mu z,z} = T_{\mu,\emptyset}$ for all $z \in \mathbb{F}_+^d$ and $|\mu| = m$. Note that every μ of length m can be written as $\mu = qi^\omega$ for some $i \in [d]$ and $\omega \geq 1$ and $q = q_{|q|} \cdots q_1$ with $q_1 \neq i$. We shall consider the case when $i = 1$. Then substituting this for $i \in \{2, \dots, d\}$ shall complete the proof.
Hence fix a word $\mu = q1^\omega$ of length $m = |q| + \omega$ with

$$\omega \geq 1 \quad \text{and} \quad q = q_{|q|} \dots q_1 \text{ with } q_1 \neq 1 \text{ or } q = \emptyset.$$

We will use induction on $|z|$. To this end fix an $r \in (0, 1)$. For $w = w_{|w|} \dots w_1 \in \mathbb{F}_+^d$ we write

$$w(t) = w_t \dots w_1 \quad \text{for } t = 1, \dots, |w|.$$

- For $|z| = 1$: First suppose that $q \neq \emptyset$. Let the vectors

$$v := e_\emptyset + \sum_{k=1}^\infty r^k e_{1^k} \quad \text{and} \quad \mathbf{l}_{q(t)} v = e_{q(t)} + \sum_{k=1}^\infty r^k e_{q(t)1^k} \text{ for } t = 1, \dots, |q|$$

and fix $\xi \in \mathcal{H}$. Again, an application of [22, Theorem 2.6] yields that v is an eigenvector for $\mathcal{L}_d^*$. Therefore we get that $X^*\xi \otimes \mathbf{l}_q v$ is in the closure of

$$\{x\xi \otimes v + \sum_{t=1}^{|q|} x_t \xi \otimes \mathbf{l}_{q(t)} v \mid x, x_t \in \mathcal{B}(\mathcal{H}), t = 1, \ldots, |q|\}$$

for all $X \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_d$. Hence there are sequences (x_n) and $(x_{t,n})$ in $\mathcal{B}(\mathcal{H})$ such that

$$G_m(T)^*\xi \otimes \mathbf{l}_q v = \lim_n x_n^* \xi \otimes v + \sum_{t=1}^{|q|} x_{t,n}^* \xi \otimes \mathbf{l}_{q(t)} v. \tag{3.2}$$

Furthermore for $|\mu'| = m$ note that we have $(\mathbf{l}_{\mu'})^* \mathbf{l}_q v = \delta_{\mu',\mu} r^\omega v$. Now for all $\eta \in \mathcal{H}$ and $z \in \mathbb{F}_+^d$ we get that

$$\langle G_m(T)^*\xi \otimes \mathbf{l}_q v, \eta \otimes e_z \rangle = r^\omega \langle \xi, T_{q1^\omega z, z} \eta \rangle \langle v, e_z \rangle .$$

Every $\mathbf{l}_{q(t)} v$ is supported on $q(t)1^k$ with $|q(t)1^k| \geq t \geq 1$ and so $\left\langle \mathbf{l}_{q(t)} v, e_\emptyset \right\rangle = 0$ for all t. By taking the inner product with $\eta \otimes e_\emptyset$ in (3.2) we get

$$\begin{aligned} \langle G_m(T)^*\xi \otimes \mathbf{l}_q v, \eta \otimes e_\emptyset \rangle &= r^\omega \langle \xi, T_{q1^\omega,\emptyset} \eta \rangle \langle v, e_\emptyset \rangle \\ &= r^\omega \langle \xi, T_{q1^\omega,\emptyset} \eta \rangle \\ &= \lim_n \langle \xi, x_n \eta \rangle . \end{aligned}$$

On the other hand the only vector of length 1 in the support of $\mathbf{l}_{q(t)} v$ is achieved when $t = 1$ and $k = 0$, in which case it is $q(1) \neq 1$ by assumption. Therefore by taking inner product with $\eta \otimes e_1$ in (3.2) we obtain

$$\begin{aligned} \langle G_m(T)^*\xi \otimes \mathbf{l}_q v, \eta \otimes e_1 \rangle &= r^{\omega+1} \langle \xi, T_{q1^\omega 1,1} \eta \rangle \\ &= \lim_n r \langle \xi, x_n \eta \rangle . \end{aligned}$$

Therefore by rearranging, $\langle \xi, T_{q1^\omega 1,1} \eta \rangle = \lim_n r^{-\omega} \langle \xi, x_n \eta \rangle = \langle \xi, T_{q1^\omega,\emptyset} \eta \rangle$ which implies that $T_{q1^\omega 1,1} = T_{q1^\omega,\emptyset}$ when $q \neq \emptyset$.

On the other hand if $q = \emptyset$ then we can repeat the above argument by substituting $\mathbf{l}_{q(t)} v$ with zeros to get again that $T_{1^\omega 1,1} = T_{1^\omega,\emptyset}$. Therefore in every case we have that $T_{\mu 1,1} = T_{\mu,\emptyset}$.

In a similar manner we next show that $T_{\mu 2,2} = T_{\mu,\emptyset}$. Similarly to above, let the vectors

$$w = e_\emptyset + \sum_{k=1}^{\infty} r^k e_{2^k} \quad \text{and} \quad \mathbf{l}_{\mu(s)} w = e_{\mu(s)} + \sum_{k=1}^{\infty} r^k e_{\mu(s) 2^k} \text{ for } s = 1, \dots, m.$$

Again, w is an eigenvector for $\mathcal{L}_d^*$ and thus following similar reasoning as above we have that for $\xi \in \mathcal{H}$ there are sequences (y_n) and $(y_{s,n})$ in $\mathcal{B}(\mathcal{H})$ such that

$$G_m(T)^* \xi \otimes \mathbf{l}_\mu w = \lim_n y_n^* \xi \otimes w + \sum_{s=1}^{m} y_{s,n}^* \xi \otimes \mathbf{l}_{\mu(s)} w \tag{3.3}$$

since w is an eigenvector of $\mathcal{L}_d^*$. Once again we have that $(\mathbf{l}_{\mu'})^* \mathbf{l}_\mu w = \delta_{\mu',\mu} w$ when $|\mu'| = m$. Now for $\eta \in \mathcal{H}$ and $z \in \mathbb{F}_+^d$ we get

$$\langle G_m(T)^* \xi \otimes \mathbf{l}_\mu w, \eta \otimes e_z \rangle = \langle \xi, T_{\mu z, z} \eta \rangle \langle w, e_z \rangle .$$

For $z = \emptyset$ we have that

$$\left\langle \mathbf{l}_{\mu(s)} w, e_\emptyset \right\rangle = \left\langle e_{\mu(s)} + \sum_{k=1}^{\infty} r^k e_{\mu(s) 2^k}, e_\emptyset \right\rangle = 0$$

for all $s \in [m]$. Therefore taking inner products with $\eta \otimes e_\emptyset$ in (3.3) gives

$$\langle \xi, T_{\mu,\emptyset} \eta \rangle = \lim_n \langle \xi, y_n \eta \rangle .$$

For $z = 2$ we have that $\left\langle \mathbf{l}_{\mu(1)} w, e_2 \right\rangle = \langle \mathbf{l}_1 w, e_2 \rangle = 0$. Moreover we have that $\left\langle \mathbf{l}_{\mu(s)} w, e_2 \right\rangle = 0$ when $s \geq 2$. Therefore (3.3) gives

$$r \langle \xi, T_{q 1^\omega 2, 2} e_2 \rangle = \lim_n r \langle \xi, y_n \eta \rangle .$$

Therefore we have that $\langle \xi, T_{\mu 2, 2} e_2 \rangle = \langle \xi, T_{\mu,\emptyset} \eta \rangle$ and thus $T_{\mu 2, 2} = T_{\mu,\emptyset}$. Applying for $i \in \{3, \dots, d\}$ yields $T_{\mu i, i} = T_{\mu,\emptyset}$ for all $i \in [d]$.

- Inductive hypothesis: Assume that $T_{q 1^\omega z, z} = T_{q 1^\omega, \emptyset}$ when $|z| \leq N$. We will show that the same is true for words of length $N + 1$.

First consider the word $1z$ with $|z| = N$. Suppose that $q \neq \emptyset$ so that $q(1) \neq 1$. We apply the same arguments as above for the vectors $\mathbf{r}_z v$ and $\mathbf{r}_z \mathbf{l}_{q(t)} v$ with $t = 1, \dots, |q|$.

Since $\mathbf{r}_z$ commutes with every $\mathbf{l}_\nu$ we get that

$$\mathbf{r}_z(\mathbf{r}_z)^*(\mathbf{l}_\nu)^*\mathbf{r}_z v = \mathbf{r}_z(\mathbf{l}_\nu)^* v \quad \text{and} \quad \mathbf{r}_z(\mathbf{r}_z)^*(\mathbf{l}_\nu)^*\mathbf{r}_z\mathbf{l}_{q(t)} v = \mathbf{r}_z(\mathbf{l}_\nu)^*\mathbf{l}_{q(t)} v.$$

As every $R_z(R_z)^*$ commutes with every $x \otimes I$ for $x \in \mathcal{B}(\mathcal{H})$, we have that for a fixed $\xi \in \mathcal{H}$ there are sequences (x_n) and $(x_{t,n})$ in $\mathcal{B}(\mathcal{H})$ such that

$$R_z(R_z)^* G_m(T)^* \xi \otimes \mathbf{r}_z\mathbf{l}_q v = \lim_n x_n^*\xi \otimes \mathbf{r}_z v + \sum_{t=1}^{|q|} x_{t,n}^*\xi \otimes \mathbf{r}_z\mathbf{l}_{q(t)} v. \tag{3.4}$$

Arguing as above for $\eta \otimes e_z$ and $\eta \otimes e_{1z}$ (i.e. taking inner products now in (3.4)) yields that

$$\langle \xi, T_{q1^\omega 1z,1z}\eta\rangle = \langle \xi, T_{q1^\omega z,z}\eta\rangle .$$

Consequently $T_{q1^\omega 1z,1z} = T_{q1^\omega z,z}$ which is $T_{q1^\omega,\emptyset}$ by the inductive hypothesis.

On the other hand if $q = \emptyset$ then we repeat the above arguments by substituting the $\mathbf{l}_{q(t)}v$ with zeros. Therefore in either case we have that $T_{\mu 1z,1z} = T_{\mu,\emptyset}$.

For $2z$ with $|z| = N$ we take the vectors $\mathbf{r}_z w$ and $\mathbf{r}_z\mathbf{l}_{\mu(s)}w$ for $s \in [m]$. Then for a fixed $\xi \in \mathcal{H}$ there are sequences (y_n) and $(y_{s,n})$ in $\mathcal{B}(\mathcal{H})$ such that

$$R_z(R_z)^* G_m(T)^* \xi \otimes \mathbf{r}_z\mathbf{l}_\mu w = \lim_n y_n^*\xi \otimes \mathbf{r}_z w + \sum_{s=1}^{m} y_{s,n}^*\xi \otimes \mathbf{r}_z\mathbf{l}_{\mu(s)} w. \tag{3.5}$$

Taking inner product with $\eta \otimes e_z$ and $\eta \otimes e_{2z}$ gives that $\langle \xi, T_{\mu 2z,2z}\eta\rangle = \langle \xi, T_{\mu z,z}\eta\rangle$. As η and ξ are arbitrary we then derive that $T_{\mu 2z,2z} = T_{\mu z,z}$ which is $T_{\mu,\emptyset}$ by the inductive hypothesis. Applying for $i \in \{3, \dots, d\}$ in place of 2 gives the same conclusion, thus $T_{\mu iz,iz} = T_{\mu,\emptyset}$ for all $i \in [d]$ and $|z| = N$. The induction then shows that $T_{\mu z,z} = T_{\mu,\emptyset}$ for all $z \in \mathbb{F}_+^d$.
Substituting the letter 1 in the word μ by any letter $i \in \{2, \dots, d\}$ completes the proof. ∎

We can use similar methods to show the reflexivity of $\mathcal{B}(\mathcal{H}) \otimes \mathbb{H}^\infty(\mathbb{Z}_+^d)$.

Theorem 3.3.2. *[44, Section 3.] The algebra $\mathcal{B}(\mathcal{H}) \otimes \mathbb{H}^\infty(\mathbb{Z}_+^d)$ is reflexive.*

Proof. By definition we have that $\mathcal{B}(\mathcal{H}) \otimes \mathbb{H}^\infty(\mathbb{Z}_+^d)$ is reflexive if $\mathcal{B}(\mathcal{H}) \otimes \mathbb{H}^\infty(\mathbb{Z}_+^d) \supseteq \operatorname{Ref}(\mathcal{B}(\mathcal{H}) \otimes \mathbb{H}^\infty(\mathbb{Z}_+^d))$. Since the gauge action of $\mathcal{B}(\mathcal{H}) \otimes \ell^2(\mathbb{Z}_+^d)$ restricts to an action

of $\mathcal{B}(\mathcal{H})\otimes\mathbb{H}^\infty(\mathbb{Z}^d_+)$ it suffices to show that every $G_{\underline{m}}(T)\in\mathcal{B}(\mathcal{H})\otimes\mathbb{H}^\infty(\mathbb{Z}^d_+)$ whenever T is in the reflexive cover.
Now let $\xi,\eta\in\mathcal{H}$ and let $\underline{m},\underline{n}\in\mathbb{Z}^d_+$, then there is a sequence $A_n\in\mathcal{B}(\mathcal{H})\otimes\mathbb{H}^\infty(\mathbb{Z}^d_+)$ such that

$$\langle T_{\underline{m},\underline{n}}\xi,\eta\rangle=\langle T\xi\otimes e_{\underline{n}},\eta\otimes e_{\underline{m}}\rangle=\lim_n\langle A_n\xi\otimes e_{\underline{n}},\eta\otimes e_{\underline{m}}\rangle=\lim_n\langle (A_n)_{\underline{m},\underline{n}}\xi,\eta\rangle\,.$$

If we take $\underline{n}<\underline{m}$ then we have that T is lower triangular since each A_n is. Thus it suffices to show that $T_{\underline{m}+\underline{w},\underline{w}}=T_{\underline{m},0}$ for all $\underline{m}\in\mathbb{Z}^d_+$. This follows because if $T_{\underline{m}+\underline{w},\underline{w}}=T_{\underline{m},0}$ then,

$$G_0(T)=\sum_{\underline{w}\in\mathbb{Z}^d_+}T_{\underline{w},\underline{w}}\otimes p_{\underline{w}}=\sum_{\underline{w}\in\mathbb{Z}^d_+}T_{0,0}\otimes p_{\underline{w}}=T_{0,0}\otimes I\in\mathcal{B}(\mathcal{H})\otimes\mathbb{H}^\infty(\mathbb{Z}^d_+),$$

and,

$$G_{\underline{m}}(T)=L_{\underline{m}}\sum_{\underline{w}\in\mathbb{Z}^d_+}T_{\underline{m}+\underline{w},\underline{w}}\otimes p_{\underline{w}}=L_{\underline{m}}(T_{\underline{m},0}\otimes I)=T_{\underline{m},0}\otimes\mathbf{l}_{\underline{m}}\in\mathcal{B}(\mathcal{H})\otimes\mathbb{H}^\infty(\mathbb{Z}^d_+).$$

Let $r\in(0,1)$ and consider the vector $v=\sum_{\underline{\ell}\in\mathbb{Z}^d_+}r^{\underline{\ell}}\xi\otimes e_{\underline{\ell}}$. Then

$$\|v\|^2=\left\|\sum_{\underline{\ell}\in\mathbb{Z}^d_+}r^{\underline{\ell}}\xi\otimes e_{\underline{\ell}}\right\|^2=\sum_{\ell_1}\cdots\sum_{\ell_d}r^{2\ell_1}\cdots r^{2\ell_d}\|\xi\|^2=\sum_{\ell_1}r^{2\ell_1}\cdots\sum_{\ell_d}r^{2\ell_d}\|\xi\|^2\,.$$

Hence, $\|v\|^2=(1-r^2)^{-d}\|\xi\|^2$ that is, $\|v\|=(1-r^2)^{-d/2}\|\xi\|$. Note that the space $\mathcal{K}:=\overline{\operatorname{span}}\{(x\otimes I)v:x\in\mathcal{B}(\mathcal{H})\}$ is $(\mathcal{B}(\mathcal{H})\otimes\mathbb{H}^\infty(\mathbb{Z}^d_+))^*$-invariant since

$$\begin{aligned}(x\otimes\mathbf{l}^*_{\underline{m}})v&=(x\otimes\mathbf{l}^*_{\underline{m}})(\sum_{\underline{\ell}\in\mathbb{Z}^d_+}r^{\underline{\ell}}\xi\otimes e_{\underline{\ell}})=\sum_{\underline{\ell}\in\mathbb{Z}^d_+}r^{\underline{\ell}}x\xi\otimes\mathbf{l}^*_{\underline{m}}e_{\underline{\ell}}=\sum_{\underline{\ell}\geq\underline{m}}r^{\underline{\ell}}x\xi\otimes e_{\underline{\ell}-\underline{m}}\\&=\sum_{\underline{w}\in\mathbb{Z}^d_+}r^{\underline{w}+\underline{m}}x\xi\otimes e_{\underline{w}}=r^{\underline{m}}(x\otimes I)v.\end{aligned}$$

Hence

$$(I\otimes\mathbf{l}^*_{\underline{m}})(x\otimes I)v=(x\otimes I)(I\otimes\mathbf{l}^*_{\underline{m}})v=r^{\underline{m}}(x\otimes I)v\in\mathcal{K}.$$

Therefore $A^*\mathcal{K}\subseteq\mathcal{K}$ for all $A\in\mathcal{B}(\mathcal{H})\otimes\mathbb{H}^\infty(\mathbb{Z}^d_+)$ and so $G_{\underline{m}}(T)^*\mathcal{K}\subseteq\mathcal{K}$.

Since $v \in \mathcal{K}$ there exists $x_n \in \mathcal{B}(\mathcal{H})$ such that $G_{\underline{m}}(T)^* v = \lim_n (x_n^* \otimes I)v$. Then,

$$\begin{aligned} G_{\underline{m}}(T)^* v &= (\sum_{\underline{w} \in \mathbb{Z}_+^d} T^*_{\underline{m}+\underline{w},\underline{w}} \otimes p_{\underline{w}})(I \otimes \mathbf{l}_{\underline{m}})(\sum_{\underline{\ell} \in \mathbb{Z}_+^d} r^{\underline{\ell}} \xi \otimes e_{\underline{\ell}}) \\ &= (\sum_{\underline{w} \in \mathbb{Z}_+^d} T^*_{\underline{m}+\underline{w},\underline{w}} \otimes p_{\underline{w}}) r^{\underline{m}} \cdot v \\ &= r^{\underline{m}} (\sum_{\underline{\ell} \in \mathbb{Z}_+^d} \sum_{\underline{w} \in \mathbb{Z}_+^d} r^{\underline{\ell}} T^*_{\underline{m}+\underline{w},\underline{w}} \xi \otimes p_{\underline{w}} e_{\underline{\ell}}) \\ &= r^{\underline{m}} (\sum_{\underline{w} \in \mathbb{Z}_+^d} r^{\underline{w}} T^*_{\underline{m}+\underline{w},\underline{w}} \xi \otimes e_{\underline{w}}). \end{aligned}$$

Also,

$$\lim_n (x_n^* \otimes I)v = \lim_n (x_n^* \otimes I)(\sum_{\underline{\ell} \in \mathbb{Z}_+^d} r^{\underline{\ell}} \xi \otimes e_{\underline{\ell}}) = \lim_n \sum_{\underline{\ell} \in \mathbb{Z}_+^d} r^{\underline{\ell}} x_n^* \xi \otimes e_{\underline{\ell}}.$$

Then taking inner products with $\eta \otimes e_{\underline{s}}$ gives,

$$\langle G_{\underline{m}}(T)^* v, \eta \otimes e_{\underline{s}} \rangle = \left\langle r^{\underline{m}} \sum_{\underline{w} \in \mathbb{Z}_+^d} r^{\underline{w}} T^*_{\underline{m}+\underline{w},\underline{w}} \xi \otimes e_{\underline{w}}, \eta \otimes e_{\underline{s}} \right\rangle = r^{\underline{m}} r^{\underline{s}} \left\langle T^*_{\underline{m}+\underline{s},\underline{s}} \xi, \eta \right\rangle.$$

On the other hand

$$\left\langle \lim_n (x_n^* \otimes I)v, \eta \otimes e_{\underline{s}} \right\rangle = \left\langle \lim_n \sum_{\underline{\ell} \in \mathbb{Z}_+^d} r^{\underline{\ell}} x_n^* \xi \otimes e_{\underline{\ell}}, \eta \otimes e_{\underline{s}} \right\rangle = r^{\underline{s}} \lim_n \langle x_n^* \xi, \eta \rangle$$

for all $\underline{s}$. Thus

$$r^{\underline{m}} T^*_{\underline{m},0} = \lim_n x_n^* \xi = r^{\underline{m}} T^*_{\underline{m}+\underline{s},\underline{s}} \xi$$

for all $\underline{s}$. That is, $r^{\underline{m}} T_{\underline{m},0} = \lim_n x_n \xi = r^{\underline{m}} T_{\underline{m}+\underline{s},\underline{s}} \xi$. Since ξ is arbitrary we have that $T^*_{\underline{m},0} = T^*_{\underline{m}+\underline{s},\underline{s}}$ for all $\underline{s} \in \mathbb{Z}_+^d$. That is, $T_{\underline{m},0} = T_{\underline{m}+\underline{s},\underline{s}}$ for all $\underline{s} \in \mathbb{Z}_+^d$. Hence $G_{\underline{m}}(T) \in \mathcal{B}(\mathcal{H}) \otimes \mathbb{H}^\infty(\mathbb{Z}_+^d)$ as required. ∎

3.3.1 Semicrossed Products over $\mathbb{F}_+^d$

A w*-semicrossed product is is a nonselfadjoint analogue of the crossed product and our aim is to study the reflexivity of the w*-semicrossed product in various cases. From now on we fix a w*-closed unital subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and we write $\mathrm{End}(\mathcal{A})$ for the continuous completely bounded endomorphisms of $\mathcal{A}$.

Definition 3.3.3 (Dynamical System)**.** A *dynamical system* denoted $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$, consists of an operator algebra $\mathcal{A}$ and d unital $\alpha_i \in \mathrm{End}(\mathcal{A})$ such that each α_i is uniformly bounded, that is

$$\sup\{\|\alpha_\mu\| \,:\, \mu \in \mathbb{F}_+^d\} < \infty,$$

where $\alpha_\mu = \alpha_{\mu_n} \cdots \alpha_{\mu_1}$ for $\mu = \mu_n \cdots \mu_1 \in \mathbb{F}_+^d$.

Now, given such a dynamical system we define the following two representations acting on $\mathcal{K} = \mathcal{H} \otimes \ell^2(\mathbb{F}_+^d)$

$$\pi(a)\xi \otimes e_\mu = \alpha_\mu(a)\xi \otimes e_\mu \quad \text{and} \quad \overline{\pi}(a)\xi \otimes e_\mu = \alpha_{\overline{\mu}}(a)\xi \otimes e_\mu.$$

Recall that we previously defined

$$L_\mu := I_{\mathcal{H}} \otimes \mathbf{l}_\mu \quad \text{and} \quad R_\nu := I_{\mathcal{H}} \otimes \mathbf{r}_\nu.$$

This leads us to define the following.

Definition 3.3.4. Let $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ be a w*-dynamical system. We define the *w*-semicrossed products*

$$\mathcal{A} \,\overline{\times}_\alpha\, \mathcal{L}_d := \overline{\mathrm{span}}^{\mathrm{w}^*}\{L_\mu \overline{\pi}(a) \mid a \in \mathcal{A}, \mu \in \mathbb{F}_+^d\}$$

and

$$\mathcal{A} \,\overline{\times}_\alpha\, \mathcal{R}_d := \overline{\mathrm{span}}^{\mathrm{w}^*}\{R_\mu \pi(a) \mid a \in \mathcal{A}, \mu \in \mathbb{F}_+^d\}.$$

It transpires that $(\overline{\pi}, \{L_i\}_{i=1}^d)$ and $(\pi, \{R_i\}_{i=1}^d)$ satisfy the following *covariance relations*

$$\overline{\pi}(a)L_i = L_i\overline{\pi}\alpha_i(a) \quad \text{and} \quad \pi(a)R_i = R_i\pi\alpha_i(a)$$

for all $a \in \mathcal{A}$ and $i \in [d]$. We shall show the right version. For every $w \in \mathbb{F}_+^d$ we have that

$$\pi(a)R_i\xi \otimes e_w = \alpha_{wi}(a)\xi \otimes e_{wi} = \alpha_w\alpha_i(a)\xi \otimes e_{wi} = R_i\pi\alpha_i(a)\xi \otimes e_w$$

and similarly for the left version. Therefore $\mathcal{A} \,\overline{\times}_\alpha\, \mathcal{L}_d$ and $\mathcal{A} \,\overline{\times}_\alpha\, \mathcal{R}_d$ are algebras. Note that for the unitaries $U_s \in \mathcal{B}(\mathcal{K})$ for $s \in [-\pi, \pi]$ defined previously we have

$$U_s\pi(a)U_s^* = \pi(a) \quad \text{and} \quad U_sR_\nu U_s^* = e^{i|\nu|s}R_\nu,$$

and similarly for the left version we have

$$U_s\overline{\pi}(a)U_s^* = \overline{\pi}(a) \quad \text{and} \quad U_sL_\nu U_s^* = e^{i|\nu|s}L_\nu.$$

Therefore applying Proposition 3.2.3 we have that $T \in \mathcal{A}\overline{\times}_\alpha\mathcal{R}_d$ if and only if $G_m(T) \in \mathcal{A}\overline{\times}_\alpha\mathcal{R}_d$ for all $m \in \mathbb{Z}$. and similarly for the left version. This leads to the following proposition.

Proposition 3.3.5. *Let $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ be a unital w*-dynamical system. Then an operator $T \in \mathcal{B}(\mathcal{K})$ is in $\mathcal{A}\overline{\times}_\alpha\mathcal{L}_d$ if and only if it is left lower triangular and*

$$G_m(T) = \sum_{|\mu|=m} L_\mu\overline{\pi}(a_\mu) \quad \textit{for } a_\mu \in \mathcal{A}$$

for all $m \in \mathbb{Z}_+$. Similarly an operator $T \in \mathcal{B}(\mathcal{K})$ is in $\mathcal{A}\overline{\times}_\alpha\mathcal{R}_d$ if and only if it is right lower triangular and

$$G_m(T) = \sum_{|\mu|=m} R_\mu\pi(a_\mu) \quad \textit{for } a_\mu \in \mathcal{A}$$

for all $m \in \mathbb{Z}_+$.

Proof. We shall show the right version, the left follows similarly and is provided in [8]. Note that if $T = R_z\pi(a)$ with $|z| = m$ then it is right lower triangular since

$$\begin{aligned}
\langle R_z\pi(a)\xi \otimes e_{\nu'}, \eta \otimes e_\nu\rangle &= \langle R_z\alpha_{\nu'}(a)\xi \otimes e_{\nu'}, \eta \otimes e_\nu\rangle \\
&= \langle \alpha_{\nu'}(a)\xi \otimes e_{\nu'\overline{z}}, \xi \otimes \eta\rangle \\
&= \delta_{\nu'\overline{z},\nu}\,\langle \alpha_{\nu'}(a)\xi, \eta\rangle\,.
\end{aligned}$$

This is zero whenever $\nu'\overline{z} \neq \nu$.
If $\nu'\overline{z} = \nu$ then $|\nu| > |\nu'|$ and thus $\nu \not<_r \nu'$ and thus T is right lower triangular. Furthermore, for $T = R_z\pi(a)$ with $|z| = m$ then

$$\begin{aligned}
\langle T_{w\overline{z},w}\xi, \eta\rangle &= \langle T\xi \otimes e_w, \eta \otimes e_{w\overline{z}}\rangle \\
&= \langle R_z\pi(a)\xi \otimes e_w, \eta \otimes e_{w\overline{z}}\rangle \\
&= \langle R_z\pi(a)\xi \otimes e_w, R_z\eta \otimes e_w\rangle \\
&= \langle \pi(a)\xi \otimes e_w, \eta \otimes e_w\rangle\,.
\end{aligned}$$

Hence $\sum_{w \in \mathbb{F}_+^d} T_{w\bar{z},w} \otimes p_w = \pi(a)$. Therefore $G_m(T) = \sum_{|\mu|=m} R_\mu \pi(a_\mu)$ where $a_z = a$ and $a_\mu = 0$ for $\mu \neq z$.

Conversely if T satisfies the conditions above then we shall show that every $G_m(T)$ is in $\mathcal{A} \overline{\times}_\alpha \mathcal{R}_d$. This follows since for every finite subset of words F, of length m, we have that

$$\left\| \sum_{\mu \in F} R_\mu \pi(a_\mu) \right\| = \left\| \sum_{\mu \in F} R_\mu (R_\mu)^* G_m(T) \right\| \leq \| G_m(T) \| .$$

Thus $(\sum_{\mu \in F} R_\mu \pi(a_\mu))_{F_m}$ is bounded and therefore every $G_m(T)$ is in $\mathcal{A} \overline{\times}_\alpha \mathcal{R}_d$. Then applying Fejér's Lemma shows that for every T we have $T \in \mathcal{A} \overline{\times}_\alpha \mathcal{R}_d$ and completes the proof. ∎

We can also form dynamical systems when the action is induced by an invertible row operator. We begin with the following definitions.

Definition 3.3.6 (Invertible Row Operator)**.** For $n \in \{1, \dots, \infty\}$ a row operator $u = [u_1 \dots u_n \dots] \in \mathcal{B}(\mathcal{H} \otimes \ell^2(n), \mathcal{H})$ is *invertible* if there exists a column operator $v = [v_1 \dots v_n \dots]^t \in \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \ell^2(n))$ such that

$$vu = I_{\mathcal{H} \otimes \ell^2(n)} \quad \text{and} \quad \sum_{i \in [n]} u_i v_i = I_{\mathcal{H}},$$

where the sum is considered in the sot.

Definition 3.3.7. Let $\{u_i\}_{i \in [d]}$ be a family of invertible row operators such that $u_i = [u_{i,j_i}]_{j_i \in [n_i]}$. We say that $\{u_i\}_{i \in [d]}$ is *uniformly bounded* if the operators

$$\widehat{u}_{\mu_m \cdots \mu_1} = u_{\mu_m} \cdot (u_{\mu_{m-1}} \otimes I_{[n_{\mu_m}]}) \cdots (u_{\mu_1} \otimes I_{[n_{\mu_m} \cdots n_{\mu_2}]})$$

and their inverses

$$\widehat{v}_{\mu_1 \cdots \mu_m} = (v_{\mu_1} \otimes I_{[n_{\mu_m} \cdots n_{\mu_2}]}) \cdots (v_{\mu_{m-1}} \otimes I_{[n_{\mu_m}]}) \cdot v_{\mu_m}$$

are uniformly bounded with respect to $\mu_m \dots \mu_1 \in \mathbb{F}_+^d$.

Note that when every $n_i = 1$ we have that $\widehat{u}_{\mu_m \cdots \mu_1} = u_{\mu_m} \cdots u_{\mu_1} = u_\mu$. More generally $\widehat{u}_{\mu_m \cdots \mu_1}$ is the row operator of all possible products of the u_{μ_i, j_i}. We illustrate this in the following example for finite multiplicities.

Example 3.3.8. Let the row operators u_1, u_2, u_3 with $n_1 = 2$, $n_2 = 3$, $n_3 = 2$. Then the operators $\widehat{u}_{312}$ is given by

$$\begin{aligned}\widehat{u}_{312} &= u_3 \cdot (u_1 \otimes I_{[n_3]}) \cdot (u_2 \otimes I_{[n_3 \cdot n_1]}) \\ &= u_3 \cdot (u_1 \otimes I_2) \cdot (u_2 \otimes I_4) \\ &= [u_{3,1}u_{1,1}u_{2,1} \;\; u_{3,1}u_{1,1}u_{2,2} \;\; u_{3,1}u_{1,1}u_{2,3} \;\; u_{3,1}u_{1,2}u_{2,1} \;\; u_{3,1}u_{1,2}u_{2,2} \;\; u_{3,1}u_{1,2}u_{2,3} \\ &\qquad u_{3,2}u_{1,1}u_{2,1} \;\; u_{3,2}u_{1,1}u_{2,2} \;\; u_{3,2}u_{1,1}u_{2,3} \;\; u_{3,2}u_{1,2}u_{2,1} \;\; u_{3,2}u_{1,2}u_{2,2} \;\; u_{3,2}u_{1,2}u_{2,3}].\end{aligned}$$

Thus we see that the $\widehat{u}$ is the row opertator of all the possible products of the u_1, u_2, u_3.

So, now suppose that we have $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ where each α_i is given by

$$\alpha_i(a) = \sum_{j_i \in [n_i]} u_{i,j_i}\, a\, v_{i,j_i} \text{ for all } a \in \mathcal{A}.$$

Applying u_{i,j_i} and v_{i,j_i} on each side we have that

$$\alpha_i(a)u_{i,j_i} = u_{i,j_i}a \quad \text{and} \quad v_{i,j_i}\alpha_i(a) = av_{i,j_i}.$$

We call $\{\alpha_i\}_{i\in[d]}$ a *uniformly bounded spatial action* on $\mathcal{A}$ if every α_i is implemented by an invertible row operator u_i and $\{u_i\}_{i\in[d]}$ is uniformly bounded.

Proposition 3.3.9. *Let α be an endomorphism of $\mathcal{B}(\mathcal{H})$ induced by an invertible row operator $u = [u_i]_{i\in[n]}$ for some $n \in \mathbb{Z}_+ \cup \{\infty\}$. Then for any $x, y \in \mathcal{B}(\mathcal{H})$ we have that*

$$\alpha(x)y = y\alpha(x) \quad \textit{if and only if} \quad x \cdot v_j y u_k = v_j y u_k \cdot x \textit{ for all } j, k \in [n]$$

where $v = [v_i]_{i\in[n]}$ is the inverse of u.

Proof. Suppose first that $\alpha(x)y = y\alpha(x)$. Then it follows that

$$xv_jyu_k = v_j\alpha(x)yu_k = v_jy\alpha(x)u_k = v_jyu_kx$$

for all $j, k \in [n]$. Conversely if $xv_jyu_k = v_jyu_kx$ for all $j, k \in [n]$ then we get

$$v_j\alpha(x)yu_k = xv_jyu_k = v_jyu_kx = v_jy\alpha(x)u_k.$$

Therefore we obtain

$$\alpha(x)y = \sum_{j\in[n]}\sum_{k\in[n]} u_j(v_j\alpha(x)yu_k)v_k = \sum_{j\in[n]}\sum_{k\in[n]} u_j(v_jy\alpha(x)u_k)v_k = y\alpha(x),$$

and the proof is complete. ∎

Uniformly bounded spatial actions can be extended to all of $\mathcal{B}(\mathcal{H})$. In fact, if $\alpha \in \mathrm{End}(\mathcal{A})$ is implemented by an invertible row operator u then α extends to an endomorphism of $\mathcal{A}''$. This is because if we apply the above proposition we have that $v_jyu_k \in \mathcal{A}'$ for all $y \in \mathcal{A}'$ since $\mathcal{A}' \subseteq \alpha(\mathcal{A})'$. Therefore for $z \in \mathcal{A}''$ we have that

$$zv_jyu_k = v_jyu_kz.$$

Therefore $\alpha(z) \in \mathcal{A}''$ again by the above proposition. Thus given a w*-dynamical system $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ where each α_i is implemented by an invertible row operator u_i then we also have the systems $(\mathcal{B}(\mathcal{H}), \{\alpha_i\}_{i\in[d]})$ and $(\mathcal{A}'', \{\alpha_i\}_{i\in[d]})$. Hence the w*-semicrossed products over these systems are all well defined.

We end this section by defining two other semicrossed products. Let $\{\alpha_i\}_{i\in[d]}$ be endomorphisms of $\mathcal{B}(\mathcal{H})$ where each α_i is induced by an invertible row operator u_i. Then form the free semigroup

$$\mathbb{F}_+^N = \langle (i,j) \mid i \in [d], j \in [n_i] \rangle = *_{i\in[d]}\mathbb{F}_+^{n_i}.$$

for $N = n_1 + \cdots + n_d$. Similarly to above, define the following operators

$$V_{i,j} = u_{i,j} \otimes \mathbf{l}_i \quad \text{and} \quad W_{i,j} = u_{i,j} \otimes \mathbf{r}_i \ \text{ for all } \ (i,j) \in ([d],[n_i]).$$

We also define the representation $\rho(x) = x \otimes I$. This allows us to make the following definition.

Definition 3.3.10. We define the w*-semicrossed products

$$\mathcal{A}' \,\overline{\times}_u\, \mathcal{L}_d := \overline{\mathrm{alg}}^{\mathrm{w}^*}\{V_{i,j}, \rho(b) \mid (i,j) \in ([d],[n_i]), b \in \mathcal{A}'\}$$

and

$$\mathcal{A}' \,\overline{\times}_u\, \mathcal{R}_d := \overline{\mathrm{alg}}^{\mathrm{w}^*}\{W_{i,j}, \rho(b) \mid (i,j) \in ([d],[n_i]), b \in \mathcal{A}'\}.$$

We can use the following proposition to show that the algebras in Definition 3.3.10 are spaces of generalised polynomials.

Proposition 3.3.11. *Let $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ be a w*-dynamical system such that each α_i is implemented by a uniformly bounded invertible row operator u_i. Then*

$$\mathcal{A}' \overline{\times}_u \mathcal{L}_d = \overline{\operatorname{span}}^{\mathrm{w}^*}\{V_{\mathbf{w}}\rho(b) \mid \mathbf{w} \in \mathbb{F}_+^N, b \in \mathcal{A}'\}$$

and

$$\mathcal{A}' \overline{\times}_u \mathcal{R}_d = \overline{\operatorname{span}}^{\mathrm{w}^*}\{W_{\mathbf{w}}\rho(b) \mid \mathbf{w} \in \mathbb{F}_+^N, b \in \mathcal{A}'\}$$

where $\mathbf{w} = (w_k, j_{w_k}) \dots (w_1, j_{w_1}) \in \mathbb{F}_+^N$.

Proof. We prove the left version. The right version follows by similar arguments. By the above comments the linear span on the right hand side is an algebra. It suffices to show that $\rho(b)V_{i,j}$ is in the span of $\{V_w\rho(b) \ : \}$for all $b \in \mathcal{A}'$ and $(i,j) \in ([d],[n_i])$. Suppose that $v_i = [v_{i,j_i}]_{j_i\in[n_i]}$ is the inverse of u_i. Then we can write

$$b = \sum_{k\in[n_i]}\sum_{l\in[n_i]} u_{i,k}v_{i,k}bu_{i,l}v_{i,l} = \sum_{k\in[n_i]}\sum_{l\in[n_i]} u_{i,k}b_{i,k,l}v_{i,l}$$

where $b_{i,k,l} := v_{i,k}bu_{i,l}$. We can then appeal to Proposition 3.3.9 to give that $b_{i,k,l}$ is in $\mathcal{A}'$ since $b \in \mathcal{A}' \subseteq \alpha_i(\mathcal{A})'$. Therefore we have that

$$bu_{i,j} = \sum_{k\in[n_i]}\sum_{l\in[n_i]} u_{i,k}b_{i,k,l}v_{i,l}u_{i,j} = \sum_{k\in[n_i]} u_{i,k}b_{i,k,j},$$

which gives that

$$\begin{aligned}\rho(b)V_{i,j} &= L_i\rho(b)\rho(u_{i,j}) \\ &= \sum_{k\in[n_i]} L_i\rho(u_{i,k}b_{i,k,j}) \\ &= \sum_{k\in[n_i]} V_{i,k}\rho(b_{i,k,j}).\end{aligned}$$

Since v_i is the inverse of u_i we have that $\|\sum_{k\in F} u_{i,k}v_{i,k}\| \leq 1$ for every finite subset F of $[n_i]$, hence

$$\|\sum_{k\in F} u_{i,k}b_{i,k,j}\| = \|\sum_{k\in F} u_{i,k}v_{i,k}bu_{i,j}\| \leq \|b\|\,\|u_{i,j}\|\,.$$

Thus the net $\left(\sum_{k\in F} u_{i,k}b_{i,k,j}\right)_{\{F:\text{finite}\}}$ is bounded and the sum $\sum\limits_{k\in[n_i]} V_{i,k}\rho(b_{i,k,j})$. converges in the w*-topology. This follows because the sum is considered in the sot, so it is also in the wot. Since it is bounded it is also w*-convergent. Hence $\rho(b)V_{i,j}$ is in $\mathcal{A}' \overline{\times}_u \mathcal{L}_d$ since it is w*-closed by definition. ∎

3.3.2 Semicrossed Products over $\mathbb{Z}_+^d$

In analogy to the previous section, we can define w*-semicrossed products over $\mathbb{Z}_+^d$ in a similar manner.

Definition 3.3.12 (Dynamical System)**.** A *dynamical system*, $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ consists of an operator algebra $\mathcal{A}$ and a semigroup action $\alpha : \mathbb{Z}_+^d \to \mathrm{End}(\mathcal{A})$ such that

$$\sup\{\|\alpha_{\underline{n}}\| \ : \ \underline{n} \in \mathbb{Z}_+^d\} < \infty.$$

We define the representation,

$$\pi(a)\xi \otimes e_{\underline{n}} = \alpha_{\underline{n}}(a)\xi \otimes e_{\underline{n}}$$

and creation operators on $\mathcal{H} \otimes \ell^2(\mathbb{Z}_+^d)$,

$$L_{\mathbf{i}}\xi \otimes e_{\underline{n}} = \xi \otimes e_{\mathbf{i}+\underline{n}}.$$

This allows us to define a w*-semicrossed product in this setting.

Definition 3.3.13. Let $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ be a unital w*-dynamical system. We define the w*-semicrossed product

$$\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d := \overline{\mathrm{span}}^{\mathrm{w}^*}\{L_{\underline{n}}\pi(a) \mid a \in \mathcal{A}, \underline{n} \in \mathbb{Z}_+^d\}.$$

Again we can show that the generators satisfy the covariance relation

$$\pi(a)L_{\mathbf{i}} = L_{\mathbf{i}}\pi\alpha_{\mathbf{i}}(a).$$

Applying on elementary tensors we have that

$$\pi(a)L_{\mathbf{i}}\xi \otimes e_{\underline{n}} = \alpha_{\mathbf{i}+\underline{n}}(a)\xi \otimes e_{\mathbf{i}+\underline{n}} = L_{\mathbf{i}}\pi\alpha_{\mathbf{i}}(a)\xi \otimes e_{\underline{n}}.$$

In a similar manner to Proposition 3.3.5 we have the following

Proposition 3.3.14. *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ be a unital w*-dynamical system. Then an operator $T \in \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{Z}_+^d))$ is in $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ if and only if it is lower triangular and*

$$G_{\underline{m}}(T) = L_{\underline{m}}\pi(a_{\underline{m}}) \quad \textit{for } a_{\underline{m}} \in \mathcal{A}$$

for all $\underline{m} \in \mathbb{Z}_+^d$.

Proof. The proof follows in a similar manner to that of Proposition 3.3.5. If $T = L_{\underline{m}}\pi(a)$ then T is lower triangular since

$$\begin{aligned} \langle L_{\underline{m}}\pi(a)\xi \otimes e_{\underline{n}'}, \eta \otimes e_{\underline{n}} \rangle &= \langle L_{\underline{m}}\alpha_{\underline{n}'}(a)\xi \otimes e_{\underline{n}'}, \eta \otimes e_{\underline{n}} \rangle \\ &= \langle \alpha_{\underline{n}'}(a)\xi \otimes e_{\underline{m}+\underline{n}'}, \eta \otimes e_{\underline{n}} \rangle \\ &= \delta_{\underline{m}+\underline{n}',\underline{n}} \langle \alpha_{\underline{n}'}(a)\xi, \eta \rangle . \end{aligned}$$

Clearly this is zero whenever $\underline{m} + \underline{n}' \neq \underline{n}$. If $\underline{m} + \underline{n}' = \underline{n}$, then $|\underline{n}| > |\underline{n}'|$ and thus $\underline{n} \not< \underline{n}'$, thus T is lower triangular. Furthermore for $T = L_{\underline{m}}\pi(a)$ we have

$$\begin{aligned} \langle T_{\underline{n}+\underline{m},\underline{n}}\xi, \eta \rangle &= \langle T\xi \otimes e_{\underline{n}}, \eta \otimes e_{\underline{n}+\underline{m}} \rangle \\ &= \langle L_{\underline{m}}\pi(a)\xi \otimes e_{\underline{n}}, \eta \otimes e_{\underline{n}+\underline{m}} \rangle \\ &= \langle L_{\underline{m}}\pi(a)\xi \otimes e_{\underline{n}}, L_{\underline{m}}\eta \otimes e_{\underline{n}} \rangle \\ &= \langle \pi(a)\xi \otimes e_{\underline{n}}, \eta \otimes e_{\underline{n}} \rangle . \end{aligned}$$

Hence $\sum_{\underline{n} \in \mathbb{Z}_+^d} T_{\underline{n}+\underline{m},\underline{n}} \otimes p_{\underline{n}} = \pi(a)$. Therefore $G_{\underline{m}}(T) = L_{\underline{m}}\pi(a_{\underline{m}})$ where $a_{\underline{m}} = a$ and $a_{\underline{m}} = 0$ for $\underline{m} \neq \underline{n}$.

Conversely, if T satisfies the above conditions then we shall show every $G_{\underline{m}}(T) \in \mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$. This follows since $G_{\underline{m}}(T) = L_{\underline{m}}\pi(a_{\underline{m}})$ for $a_{\underline{m}} \in \mathcal{A}$ and so for $a_{\underline{m}} = a$, $G_{\underline{m}}(T) \in \mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ by definition. ∎

Usefully, in this setting we also have the following proposition which allows us to decompose a semicrossed product over $\mathbb{Z}_+^d$ in each direction.

Proposition 3.3.15. *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ be a unital w*-dynamical system. Then the semicrossed product $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ is unitarily equivalent to*

$$(\cdots((\mathcal{A} \overline{\times}_{\alpha_1} \mathbb{Z}_+) \overline{\times}_{\widehat{\alpha}_2} \mathbb{Z}_+) \cdots) \overline{\times}_{\widehat{\alpha}_d} \mathbb{Z}_+$$

where $\widehat{\alpha}_{\mathbf{i}} = \alpha_{\mathbf{i}} \otimes^{(i-1)} I$ *for* $i = 2, \ldots, d$.

Proof. We show how this decomposition works when $d = 2$; the general case follows by iteration. Let $\alpha_{\mathbf{1}}$ and $\alpha_{\mathbf{2}}$ be commuting endomorphisms of $\mathcal{A}$. Then $\mathcal{A} \overline{\times}_{\alpha_{\mathbf{1}}} \mathbb{Z}_+$ acts on $\mathcal{H} \otimes \ell^2$ by

$$\pi(a)\xi \otimes e_n = \alpha_{(n,0)}(a)\xi \otimes e_n \quad \text{and} \quad L_1\xi \otimes e_n = \xi \otimes e_{n+1}$$

by definition. Now we can define the w*-dynamical system $(\mathcal{A} \overline{\times}_{\alpha_{\mathbf{1}}} \mathbb{Z}_+, \widehat{\alpha}_{\mathbf{2}}, \mathbb{Z}_+)$ by setting

$$\widehat{\alpha}_{\mathbf{2}}(\pi(a)) = \pi\alpha_{\mathbf{2}}(a) \quad \text{and} \quad \widehat{\alpha}_{\mathbf{2}}(L_1) = L_1.$$

To see that $\widehat{\alpha}_{\mathbf{2}}$ defines a w*-continuous completely bounded endomorphism on $\mathcal{A} \overline{\times}_{\alpha_{\mathbf{1}}} \mathbb{Z}_+$ first note that $\mathcal{A} \overline{\times}_{\alpha_{\mathbf{1}}} \mathbb{Z}_+$ is a w*-closed subalgebra of $\mathcal{A} \otimes \mathcal{B}(\ell^2)$. Then the map $\alpha_2 \otimes \mathrm{id}$ defines a w*-endomorphism of $\mathcal{A} \otimes \mathcal{B}(\ell^2)$. Since $\alpha_{\mathbf{2}}$ is w*-continuous and completely bounded, for $X \in \mathcal{A} \otimes \mathcal{B}(\ell^2)$ we can obtain $\alpha_{\mathbf{2}} \otimes \mathrm{id}(X)$ as the limit of

$$\alpha_{\mathbf{2}} \otimes \mathrm{id}_n(P_{\mathcal{H}\otimes\ell^2(n)} X|_{\mathcal{H}\otimes\ell^2(n)}) \in \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}).$$

Hence $\alpha_{\mathbf{2}} \otimes \mathrm{id}$ defines a w*-completely bounded endomorphism of $\mathcal{A} \otimes \mathcal{B}(\ell^2)$ and $\widehat{\alpha}_{\mathbf{2}}$ is its restriction to the $\mathcal{A} \overline{\times}_{\alpha_{\mathbf{1}}} \mathbb{Z}_+$. To allow comparisons write

$$\widehat{\pi} \colon \mathcal{A} \overline{\times}_{\alpha_{\mathbf{1}}} \mathbb{Z}_+ \to \mathcal{B}(\mathcal{H} \otimes \ell^2 \otimes \ell^2)$$

for the orbit representation and

$$\widehat{L} = I_{\mathcal{H}\otimes\ell^2} \otimes \mathbf{l_i} \in \mathcal{B}(\mathcal{H} \otimes \ell^2 \otimes \ell^2)$$

for the amplification of the unilateral shift. Then let $Q \colon \mathcal{H} \otimes \ell^2(\mathbb{Z}_+^2) \to \mathcal{H} \otimes \ell^2 \otimes \ell^2$ be the unitary given by

$$Q\xi \otimes e_{(n,m)} = \xi \otimes e_n \otimes e_m.$$

We show that Q induces the required unitary equivalence between $\mathcal{A} \overline{\times}_{\alpha} \mathbb{Z}_+^2$ and $\mathcal{A} \overline{\times}_{\alpha_{\mathbf{1}}} \mathbb{Z}_+ \overline{\times}_{\widehat{\alpha}_{\mathbf{2}}} \mathbb{Z}_+$. It suffices to check that it maps generators to generators. For

$a \in \mathcal{A}$ we see that

$$\begin{aligned} Q^*\widehat{\pi}(\pi(a))Q\xi \otimes e_{(n,m)} &= Q^*\widehat{\pi}(\pi(a))\xi \otimes e_n \otimes e_m \\ &= Q^*\widehat{\alpha}_{\mathbf{2}}^m(\pi(a))(\xi \otimes e_n) \otimes e_m \\ &= Q^*\pi\alpha_{(0,m)}(a)(\xi \otimes e_n) \otimes e_m \\ &= Q^*\alpha_{(n,0)}\alpha_{(0,m)}(a)\xi \otimes e_n \otimes e_m \\ &= \alpha_{(n,m)}(a)\xi \otimes e_{(n,m)}. \end{aligned}$$

Therefore $Q^*\widehat{\pi}(\pi(\mathcal{A}))Q$ is the copy of $\mathcal{A}$ inside $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^2$. A similar computation gives that

$$\begin{aligned} Q^*\widehat{\pi}(L_1)Q\xi \otimes e_{(n,m)} &= Q^*\widehat{\pi}(L_1)\xi \otimes e_n \otimes e_m \\ &= Q^*\xi \otimes e_{n+1} \otimes e_m = \xi \otimes e_{(n+1,m)} \end{aligned}$$

so that $Q^*\widehat{\pi}(L)Q = L_{\mathbf{1}}$. Likewise we have that

$$\begin{aligned} Q^*\widehat{L}Q\xi \otimes e_{(n,m)} &= Q^*\widehat{L}\xi \otimes e_n \otimes e_m \\ &= Q^*\xi \otimes e_n \otimes e_{m+1} = \xi \otimes e_{(n,m+1)}. \end{aligned}$$

Therefore we have the required unitary equivalence and the proof is complete. ∎

Chapter 4

Examples of Dynamics Over $\mathbb{Z}_+^d$

Now we will focus on actions of $\mathbb{Z}_+^d$ implemented by a Cuntz family. There are several examples of dynamics implemented by Cuntz families in the works of Laca [40] and Kakariadis and Peters [33]. They arise naturally and form generalizations of the Cuntz-Krieger odometer (Examples 4.2.2). Our setting accommodates $\mathbb{Z}_+^d$-actions where the generators $\alpha_{\mathbf{i}}$ are implemented by unitaries but the unitaries implementing the actions may not commute. Such cases arise naturally. For example any two commuting automorphisms over $\mathcal{B}(\mathcal{H})$ are implemented by two unitaries that satisfy Weyl's relation and may not commute (see Example 4.1.3). By using results of Laca [40] we determine when an automorphism of $\mathcal{B}(\mathcal{H})$ commutes with specific endomorphisms induced by Cuntz isometries.

4.1 Automorphisms of an algebra

Proposition 4.1.1. *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be an algebra and suppose that $\alpha_i, \alpha_j \in \mathrm{Aut}(\mathcal{A})$ such that*

$$\alpha_i = \mathrm{ad}_{U_i} \quad \textit{and} \quad \alpha_j = \mathrm{ad}_{U_j},$$

where U_i and U_j are unitaries. Then α_i commutes with α_j if and only if there exists a unitary for $w \in \mathcal{A}'$ such that $U_iU_j = U_jU_iw$.

Proof. First suppose that α_i commutes with α_j. Let $W = U_iU_j$ and $Q = U_jU_i$. Since α_i commutes with α_j we have,

$$U_iU_jaU_j^*U_i^* = U_jU_iaU_i^*U_j^*,$$

That is, $WaW^* = QaQ^*$. So multiplying on the left by Q^* and the right by W we have $Q^*Wa = aQ^*W$. Therefore $Q^*W \in \mathcal{A}'$, therefore $Q^*W = w$ and therefore $W = Qw$ thus,

$$U_iU_j = U_jU_iw.$$

On the other hand, if $U_iU_j = U_jU_iw$ for $w \in \mathcal{A}'$ then we have,

$$\begin{aligned}\alpha_i\alpha_j(a) &= U_iU_jaU_j^*U_i^* \\ &= U_jU_iwaw^*U_i^*U_j^* \\ &= U_jU_iaU_i^*U_j^* = \alpha_j\alpha_i(a).\end{aligned}$$

and the proof is complete. ∎

Automorphisms of $\mathcal{B}(\mathcal{H})$ are all spatial in the sense that they have the form ad_V, [10, Example II.5.5.14]. Similarly automorphisms of a m.a.s.a. are also given by ad_V for a unitary V. [16, Theorem 17.4] In particular, we have the following examples.

Example 4.1.2. If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is $L^\infty(X, m)$, where X is a measure space and if $\alpha_i = \mathrm{ad}_{U_i}$ and $\alpha_j = \mathrm{ad}_{U_j}$ are automorphisms of $\mathcal{A}$ then α_i commutes with α_j if and only if

$$U_iU_j = U_jU_iM_f, \qquad \text{with } |f| = 1 \; a.e.$$

where M_f is the multiplication operator given by $M_fg = fg$ and $f \in L^\infty(X, m)$.

Example 4.1.3. If $\alpha_i = \mathrm{ad}_{U_i}$ and $\alpha_j = \mathrm{ad}_{U_j}$ are automorphisms of $\mathcal{B}(\mathcal{H})$, for unitaries U_i and U_j then α_i commutes with α_j if and only if

$$U_iU_j = \lambda_{i,j}U_jU_i \text{ for } \lambda_{i,j} \in \mathbb{T}.$$

That is, α_i commutes with α_j if and only if they satisfy Weyl's relation.

4.2 Endomorphisms

We now recall the definition of a Cuntz family.

Definition 4.2.1. Let $\{S_i\}_{i=1}^d$ be a family of isometries on a Hilbert space $\mathcal{H}$. Then

$\{S_i\}_{i=1}^d$ is a *Cuntz family* if

$$\sum_{i=1}^{d} S_i S_i^* = I \quad \text{and} \quad S_i^* S_j = \delta_{ij} I.$$

Arveson [7] showed that every irreducible representation of $\mathcal{B}(\mathcal{H})$ is unitarily equivalent to the identity representation. We can use this to demonstrate that every endomorphism of $\mathcal{B}(\mathcal{H})$ is implemented by a Cuntz family. Suppose that we have $\alpha : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ to be a w*-continuous endomorphism and let $\alpha|_{\mathcal{K}(\mathcal{H})} : \mathcal{K}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be its restriction to $\mathcal{K}(\mathcal{H})$. By [5, Section 1.4] we can write $\alpha|_{\mathcal{K}(\mathcal{H})} \simeq \bigoplus \mathrm{id}$. Therefore there exist Cuntz isometries $\{S_1, \dots, S_d\}$ such that

$$\alpha(T) = \sum_{i=1}^{d} S_i T S_i^* \text{ for all } T \in \mathcal{K}(\mathcal{H}).$$

Since $\mathcal{K}(\mathcal{H})$ is a w*-closed ideal of $\mathcal{B}(\mathcal{H})$ we have $\overline{\mathcal{K}(\mathcal{H})}^{w^*} = \mathcal{B}(\mathcal{H})$ and therefore the map $\alpha|_{\mathcal{K}(\mathcal{H})}$ has a unique w*-continuous extension to $\mathcal{B}(\mathcal{H})$ which is namely α. Therefore $\alpha(x) = \sum_{i=1}^{d} S_i x S_i^*$ for all $x \in \mathcal{B}(\mathcal{H})$.
Laca [40] shows that the Cuntz family implementing such endomorphisms may not be unique. We have the following example.

Examples 4.2.2. We have seen that every (unital) endomorphism of $\mathcal{B}(\mathcal{H})$ is implemented by a countable Cuntz family when $\mathcal{H}$ is separable. Examples of endomorphisms of maximal abelian selfadjoint algebras implemented by a Cuntz family have been considered in [33]. In particular let $\varphi\colon X \to X$ be an onto map on a measure space (X, m) such that:

(i) φ and φ^{-1} preserve the null sets.

(ii) There are d Borel cross-sections $\psi_1, \dots, \psi_d$ of φ with $\psi_i(X) \cap \psi_j(X) = \emptyset$ such that $\cup_{i=1}^{d} \psi_i(X)$ is almost equal to X.

Then it is shown in [33, Proposition 2.2] that the endomorphism $\alpha\colon L^\infty(X, m) \to L^\infty(X, m)$ given by $f \mapsto \varphi$ is realised through a Cuntz family. Specifically, there are Cuntz isometries $S_i : L^2(X) \to L^2(X)$ such that

$$M_{f \circ \phi}|_{L^2(X,m)} = S_i M_f S_i^*$$

for all $f \in L^\infty(X,m)$. Such cases arise in the context of d-to-1 local homeomorphisms for which an appropriate decomposition of X into disjoint sets can be obtained [33, Lemma 3.1]. As long as the boundaries of the components are null sets then the requirements (i) and (ii) above are satisfied by [33, Proposition 2.2].

The prototypical example is the Cuntz-Krieger odometer, where

$$X = \prod_{k=1}^{\infty}\{1,\dots,d\} \quad \text{and} \quad m = \prod_{k=1}^{\infty} m'$$

for the averaging measure m', and the backward shift φ [33, Example 3.3]. Here φ is a local homeomorphism and X can be appropriately decomposed into the (disjoint) cylinder sets $U_i := \{(i_1,i_2,\dots) : i_1 = i\}$. Thus applying [33, Theorem 3.2] in this case, if α is an endomorphism of $L^\infty(X,m)$ given by $\alpha : M_f \to M_{f\circ\phi}$ then it admits an extension to an endomorphism α_S of $\mathcal{B}(L^2(X,m))$ which is ergodic (i.e. the von Neumann algebra $\mathcal{N}_{\alpha_S} = \{T \in \mathcal{B}(\mathcal{H}) : \alpha_s(T) = T\}$ is trivial). Further, α_S is implemented by a Cuntz family.

The results of [33] follow the work of Courtney-Muhly-Schmidt [13] on endomorphisms α of the Hardy algebra induced by a Blaschke product. Let (a_n) be a sequence of complex numbers inside the unit disk such that $\sum_n (1-|a_n|) < \infty$ then the Blaschke product is

$$B(z) = \prod_{n=1}^{\infty} B(a_n,z) = \frac{|a_n|}{a_n}\frac{a_n - z}{1-\overline{a}z},$$

where $a_n \neq 0$ for all n.
In [13, Corollary 3.5] It is shown that there is a Cuntz family implementing α if and only if there is a specific orthonormal basis $\{v_1,\dots,v_d\}$ for $H^2(\mathbb{T}) \ominus b\cdot H^2(\mathbb{T})$. In [33], further necessary and sufficient conditions are given for a Cuntz family to implement an endomorphism of $L^\infty(X,m)$.

Definition 4.2.3 (Conjugacy)**.** Two endomorphisms α_1 of $\mathcal{B}(\mathcal{H}_1)$ and α_2 of $\mathcal{B}(\mathcal{H}_2)$ are called *conjugate* if there is an isomorphism $\theta : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ such that

$$\theta \circ \alpha_1 = \alpha_2 \circ \theta.$$

Note that θ is implemented by a unitary operator $W : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\alpha_1(a) = W^*\alpha_2(WaW^*)W,$$

for $a \in \mathcal{B}(\mathcal{H}_1)$. Therefore conjugacy corresponds to α_1 and α_2 being unitarily equivalent. Let n be a positive integer and let $\{v_j\}_{j=1}^n$ be a collection of isometries which satisfy $\sum\limits_{i=1}^{n} v_j v_j^* < I$. Let $\mathcal{T}_n$ be the algebra generated by the $v_j's$. If $n < \infty$ the projection $I - \sum\limits_{j=1}^{n} v_j v_j^* \in \mathcal{T}_n$ and generates an ideal $\mathcal{J}_n$ which is isomorphic to $\mathcal{K}(\mathcal{H})$. The quotient $\mathcal{T}_n/\mathcal{J}_n$ is the Cuntz algbera $\mathcal{O}_n = C^*(\{S_i\}_{i=1}^n)$.

Remark 4.2.4. [14], Whenever $\{S_i\}_{i=1}^n$ are n isometries on a Hilbert space $\mathcal{H}$ satisfying $\sum\limits_i S_i S_i^* \leq I$ there is a unique representation π of $\mathcal{T}_n$ such that $\pi(v_i) = S_i$ for $j = 1, \ldots, n$. If $n < \infty$ and $\sum\limits_i S_i S_i^* = I$ the representation factors through $\mathcal{O}_n$ and thus can be thought of as a representation of $\mathcal{O}_n$.

Now let $\mathcal{E}$ be the Hilbert space generated by $\{v_j\}_{j=1}^n$ with the inner product $\langle x, y\rangle I = y^*x$. Then whenever U is a unitary on $\mathcal{E}$ then there is a unique automorphism γ_U of $\mathcal{T}_n$ such that $\gamma_U(x) = Ux$ for all $x \in \mathcal{E}$. In [40] Laca demonstrates a link between the representation theory of the C*-algebras $\mathcal{T}_n$ to the study of endomorphisms of $\mathcal{B}(\mathcal{H})$ by way of the following theorem.

Theorem 4.2.5. *[40, Theorem 2.1] If π is a nondegenerate representation of $\mathcal{T}_n$ on $\mathcal{H}$ then*

$$\alpha(a) = \sum_{j=1}^{n} \pi(v_j)a\pi(v_j)^* = \mathrm{ad}_\pi \ \textit{ for } a \in \mathcal{B}(\mathcal{H})$$

defines an endomorphism α of $\mathcal{B}(\mathcal{H})$. Conversely, every endomorphism of $\mathcal{B}(\mathcal{H})$ arises in this fashion for some n and some representation π.
Furthermore the set $E = \{T \in \mathcal{B}(\mathcal{H}) : \alpha(a)T = Ta,$ for all $a \in \mathcal{B}(\mathcal{H})\}$ is a Hilbert space with the inner product given by $T^*S = \langle S, T\rangle I$ and π establishes a unitary equivalence between $\mathcal{E}$ and E. In particular, $\pi(\mathcal{E}) = E$.

Therefore we see that the isometries determine the endomorphism α.

Proposition 4.2.6. *[40, Proposition 2.2] Suppose that π and σ are nondegenerate representations of $\mathcal{T}_m$ and $\mathcal{T}_n$ respectively. Then $\mathrm{ad}_\pi = \mathrm{ad}_\sigma$ if and only if $m = n$ and $\pi = \sigma \circ \gamma_U$ for some unitary operator U in $\mathcal{E}$.*

We now apply this result to examine commuting endomorphisms of $\mathcal{B}(\mathcal{H})$. Suppose that $\alpha, \beta \in \mathrm{End}(\mathcal{B}(\mathcal{H}))$ commute and are given by

$$\alpha(x) = \sum_{i\in[n]} s_i x s_i^* \quad \text{and} \quad \beta(x) = \sum_{j\in[m]} t_j x t_j^*$$

where $\{s_i\}_{i\in[n]}$ and $\{t_j\}_{j\in[m]}$ are both Cuntz families. Therefore

$$\sum_{i\in[n]} \sum_{j\in[m]} s_i t_j x t_j^* s_i^* = \sum_{j\in[m]} \sum_{i\in[n]} t_j s_i x s_i^* t_j^*.$$

On each side we have orthogonal representations of $\mathcal{B}(\mathcal{H})$ and thus we can take the limits so that

$$\sum_{(i,j)\in[n]\times[m]} s_i t_j x t_j^* s_i^* = \sum_{(i,j)\in[n]\times[m]} t_j s_i x s_i^* t_j^*.$$

We can see

$$\{s_i t_j\}_{(i,j)\in[n]\times[m]} \quad \text{and} \quad \{t_j s_i\}_{(i,j)\times[n]\times[m]}$$

both as representations of the Cuntz algebra $\mathcal{O}_{n\cdot m}$. Applying Proposition 4.2.6 gives a unitary operator $W = [w_{(k,l),(i,j)}]$ in $\mathcal{M}_{nm}(\mathbb{C})$ such that

$$t_j s_i = \sum_{(k,l)\in[n]\times[m]} w_{(k,l),(i,j)} s_k t_l. \tag{4.1}$$

We call the unitary W Laca's Unitary resolution Since [40, Proposition 2.2] works both ways this condition is also necessary for having that α and β commute. We are going to use this to study commuting endomorphisms of $\mathcal{B}(\mathcal{H})$. However, we cannot study all such representations as $\mathcal{O}_n$ does not have a nice representation space in the sense that there is no countable collection of Borel functions that distinguish the unitary invariants. So we restrict our attention to free atomic representations.

4.3 Free Atomic Representations

Recall that a d-tuple of isometries $(S_1, \ldots, S_d)$ is called *free-atomic* if $S_j^* S_i = \delta_{i,j} I$ and $\sum_{i=1}^{d} S_i S_i^* \leq I$ and there is an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$ for which there are injections $\pi_i : \mathbb{N} \to \mathbb{N}$ (for $1 \le i \le d$) and scalars $\lambda_{i,n} \in \mathbb{T}$ satisfying

$$S_i e_n = \lambda_{i,n} e_{\pi_i(n)}.$$

Davidson and Pitts [22] define the following three types of representation.

Definition 4.3.1 (Left Regular Representation)**.** The *left regular representation* of $\mathbb{F}_+^d$ acts on $\ell^2(\mathbb{F}_+^d)$ by

$$S_i e_v = e_{iv} \text{ for } v \in \mathbb{F}_+^d.$$

Definition 4.3.2 (Infinite Tail)**.** Fix $x = x_1 x_2 \dots x_n \dots$ to be an infinite word in $\mathbb{F}_+^d$. Define $z_n = x_1 x_2 \dots x_n$ for $n \ge 0$. Let $\mathbb{F}_+^d x^{-1}$ denote the collection of words of the form $v = u z_n^{-1}$ for $n \ge 0$ and $u \in \mathbb{F}_+^d$. Call two words $x = z_{i_1} \dots z_{i_m} \dots$ and $x' = z_{j_1} \dots z_{j_m} \dots$ *tail equivalent* if there are integers k, ℓ so that $i_{m+k} = j_{m+\ell}$ for all $m \ge 0$. Identify words after cancellation, i.e. $u z_n^{-1} = (u x_{n+1}) z_{n+1}^{-1}$. Let $\mathcal{H}_x$ be the Hilbert space with orthonormal basis $\{e_v : v \in \mathbb{F}_+^d x^{-1}\}$. Then define the representation S_i of $\mathbb{F}_+^d$ by: $S_i e_v = e_{iv}$ for $v \in \mathbb{F}_+^d x^{-1}$.

Example 4.3.3. Fix the aperiodic word $x = x_1 x_2 \dots x_n \dots = 01001000100001 \dots$. Then for $\mu \in \mathbb{F}_+^d$, let

$$\mathcal{H} = \left\langle e_{\mu(x_1 \dots x_n)^{-1}} : n \in \mathbb{N}, \mu(x_1 \dots x_n)^{-1} \text{ is in reduced form} \right\rangle.$$

Define $S_\alpha e_{\mu(x_1 \dots x_n)^{-1}} = e_{\alpha\mu(x_1 \dots x_n)^{-1}}$. Then this yields an infinite tail representation with the following diagram:

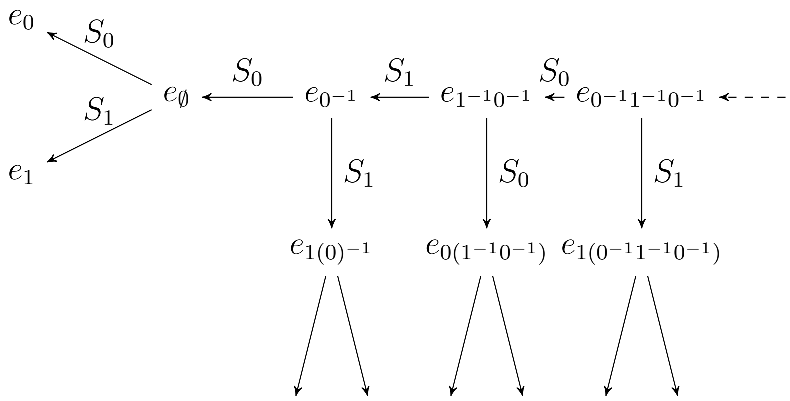

Figure 4.1

Definition 4.3.4 (Cycle Representation)**.** Fix a non-void word $g = g_t \cdots g_0 \in \mathbb{F}_+^d$. Let $\mathcal{K}$ be a Hilbert space with orthonormal basis given by

$$\{e_{j,w} : 1 \leq j \leq t \text{ and } w \in \mathbb{F}_+^d \setminus \mathbb{F}_+^d g_{j+1}\}.$$

Define a representation S_i of $\mathbb{F}_+^d$ by:

$$\begin{cases} S_i e_{j,\emptyset} = e_{j+1,\emptyset} & \text{if } i = g_{j+1},\ i \neq g_0 \\ S_i e_{t,\emptyset} = \lambda e_{0,\emptyset} & \text{if } i = g_0, \\ S_i e_{j,\emptyset} = e_{j,i} & \text{if } i \neq g_{j+1}, \\ S_i e_{j,w} = e_{j,iw} & \text{if } w \neq \emptyset \text{ and } w \notin \mathbb{F}_+^d g_{j+1}, \end{cases}$$

for $\lambda \in \mathbb{T}$. Note that $S_g e_{j,\emptyset} = \lambda e_{j,\emptyset}$ for all $j \in \{0, \ldots, t\}$. Then the word g is called the central generator for this representation. A word $g = g_t \cdots g_0 \in \mathbb{F}_+^d$ is called *primitive* if it is not the power of a smaller word.

Example 4.3.5. Let $\mathcal{H} = \ell^2(\mathbb{N})$ and let

$$S_1 e_n = e_{2n} \quad \text{and} \quad S_2 e_n = e_{2n+1}.$$

Then $S_1 e_0 = e_0$ and therefore this is a cycle representation with the following diagram:

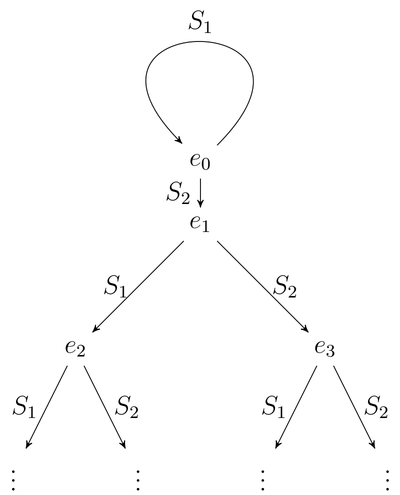

Figure 4.2

In [22] Davidson and Pitts give a classification of the free atomic representations of $\mathbb{F}_+^d$ up to unitary equivalence via the following theorem.

Theorem 4.3.6. *[22, Theorem 3.4] Every irreducible free atomic representation of $\mathbb{F}_+^d$ is unitarily equivalent to one of the following:*

1. *The left regular representation.*
2. *The infinite tail representation corresponding to an aperiodic infinite word which is unique up to tail equivalence.*
3. *The representation arising from a primitive central generator g, which is unique up to cyclic permutations and a scalar $\lambda \in \mathbb{T}$.*

The key to this result is the following split into two cases. If the free atomic representation is given by $\sum_{i=1}^{d} S_i S_i^* < I$ then it gives rise to the left regular representation. If $\sum_{i=1}^{d} S_i S_i^* = I$ then Davidson and Pitts showed that there are two possibilities. Firstly, if there is an eigenvector for the S_i then this gives rise to a cycle representation. However if there is no eigenvector then this leads to the case of an infinite tail.

4.3.1 Certain Endomorphisms of $\mathcal{B}(\mathcal{H})$

Fix $g = g_t \cdots g_0$ to be non-void primitive word in $\mathbb{F}_+^d$. Up to permutation suppose that $g_0, \ldots, g_t \in \{0, \ldots, \ell\}$ Let $\mathcal{H}$ be a Hilbert space with basis $\{e_{j,w}\}$, for $j \in \{0, \ldots, t\}$ and $w \in \mathbb{F}_+^d$. Then j gives the position in the cycle and the corresponding branches. Our aim is to identify the unitaries $U \in \mathcal{B}(\mathcal{H})$ such that the induced actions

$$\alpha(x) = \mathrm{ad}_U = UxU^* \quad \text{and} \quad \beta(x) = \mathrm{ad}_S = \sum_{i=0}^{d} S_i x S_i^*$$

commute, where ad_S is given by a cycle free atomic representation.

We shall firstly show that we can arrange the weights around the cycle to have the same value. To this end let $\{\overline{S_i}\}_{i=0}^{d}$ be the representation given by the following: For $i \neq g_0$ we have

$$\overline{S_i} e_{j,\emptyset} = \begin{cases} e_{j,i} & \text{if } i \neq g_{j+1} \\ e_{j+1,\emptyset} & \text{if } i = g_{j+1}, \end{cases}$$

and for $w \neq \emptyset$, we have $\overline{S_i} e_{j,w} = e_{j,iw}$. Then for $i = g_0$ we have

$$\overline{S_{g_0}} e_{j,w} = \begin{cases} e_{j,g_0 w} & \text{if } j \neq t \\ \lambda^{t+1} e_{0,\emptyset} & \text{if } j = t. \end{cases}$$

Which gives the diagram:

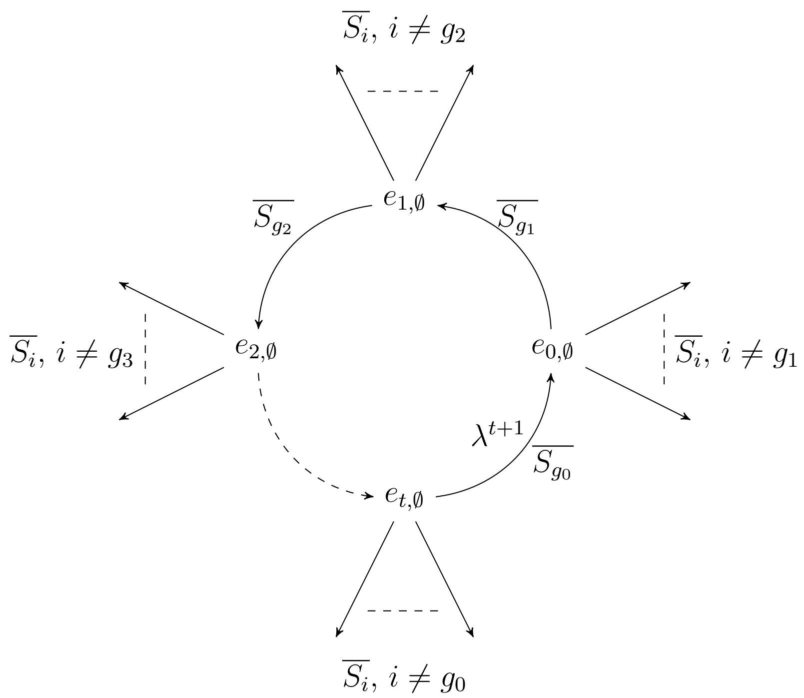

Figure 4.3

We also define the Cuntz family $\{S_i\}_{i=0}^{d}$ by:

$$S_i e_{j,\emptyset} = \begin{cases} e_{j,i} & \text{if } i \neq g_{j+1} \\ \lambda \cdot e_{j+1,\emptyset} & \text{if } i = g_{j+1} \end{cases}$$

and for $w \neq \emptyset$ we have $S_i e_{j,w} = e_{j,iw}$. This yields the diagram:

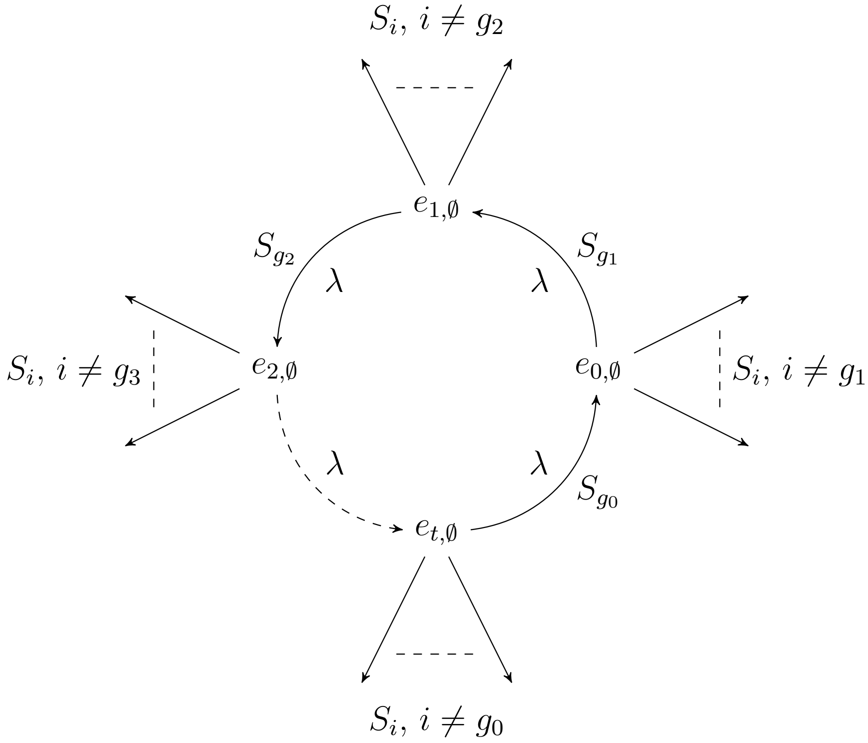

Figure 4.4

These Cuntz families are unitarily equivalent by the following lemma. x

Lemma 4.3.7. *The family* $\{S_i\}_{i=0}^d$ *is unitarily equivalent to the* $\{\overline{S_i}\}_{i=0}^d$ *via the unitary*

$$We_{j,w} = \frac{1}{\lambda^j} e_{j,w}$$

Proof. We claim that $WS_i = \overline{S_i}W$. We have the following cases

- Case 1 ($j \neq t, i = g_{j+1}$): Firstly we have that

$$WS_i e_{j,\emptyset} = \lambda W e_{j+1,\emptyset} = \lambda \cdot \frac{1}{\lambda^{j+1}} e_{j+1,\emptyset} = \frac{1}{\lambda^j} e_{j+1,\emptyset}$$

  and,

$$\overline{S_i} W e_{j,\emptyset} = \frac{1}{\lambda^j} \overline{S_i} e_{j,\emptyset} = \frac{1}{\lambda^j} e_{j+1,\emptyset}.$$

- Case 2 ($j \neq t, i \neq g_{j+1}$): We have that

$$WS_i e_{j,\emptyset} = We_{j,i} = \frac{1}{\lambda^j} e_{j,i}.$$

On the other hand

$$\overline{S_i}We_{j,\emptyset} = \frac{1}{\lambda^j}\overline{S_i}e_{j,\emptyset} = \frac{1}{\lambda^j}e_{j,i}.$$

- Case 3 ($j = t, i = g_0$): We see that

$$WS_ie_{t,\emptyset} = \lambda We_{0,\emptyset} = \lambda e_{0,\emptyset}$$

and,

$$\overline{S_i}We_{t,\emptyset} = \frac{1}{\lambda^t}\overline{S_i}e_{t,\emptyset} = \frac{1}{\lambda^t}\cdot\lambda^{t+1}e_{0,\emptyset} = \lambda e_{0,\emptyset}.$$

- Case 4 ($j = t, i \neq g_0$): We have

$$WS_ie_{t,\emptyset} = We_{t,i} = \frac{1}{\lambda^t}e_{t,i}.$$

and,

$$\overline{S_i}We_{t,\emptyset} = \frac{1}{\lambda^t}\overline{S_i}e_{t,\emptyset} = \frac{1}{\lambda^t}e_{t,i}.$$

- Case 5 ($w \neq \emptyset$): Finally we have that

$$WS_ie_{j,w} = We_{j,iw} = \frac{1}{\lambda^j}e_{j,iw} = \frac{1}{\lambda^j}\overline{S_i}e_{j,w} = \overline{S_i}We_{j,w}.$$

Thus, in each case $WS_i = \overline{S_i}W$. ∎

Therefore, without loss of generality we now fix the cycle representation to have the form of $\{S_i\}_{i=0}^d$ in Figure 4.4. For the main theorem of this section we will need to consider permutations. To this end suppose that there exists a cyclic permutation of $\{e_{j,\emptyset}\}_{i=0}^t$ which induces:

1. A permutation σ on $\{0, \ldots, t\}$
2. A permutation $\overline{\sigma}$ on $\{g_0, \ldots, g_t\} = \{0, \ldots, \ell\}$ such that $g_{\sigma(j)} = \overline{\sigma}(g_j)$ for all $j = 0, \ldots, t$.

We can extend $\overline{\sigma}$ to $\mathbb{F}_+^d$ by

$$\overline{\sigma}(w_k \cdots w_0) = \overline{\sigma}(w_k)\cdots\overline{\sigma}(w_0)$$

such that $\overline{\sigma}(w_i) = w_i$ if $w_i \notin \{g_0, \ldots, g_\ell\}$. We can now state the main theorem of this section.

Theorem 4.3.8. *Let $U \in \mathcal{B}(\mathcal{H})$ be a unitary. If ad_U commutes with ad_S, where S is the cycle representation given in Figure 4.4, then there exist permutations σ on $\{0, \dots, t\}$ and $\overline{\sigma}$ on $\{g_0, \dots, g_t\} = \{0, \dots, \ell\}$ such that $g_{\sigma(j)} = \overline{\sigma}(g_j)$. Furthermore, there exist weights $\mu_0, \dots, \mu_\ell \in \mathbb{T}$ such that*

$$Ue_{j,w} = \begin{cases} \mu_j e_{\sigma(j),\emptyset} & \text{if } w = \emptyset, \\ \sum\limits_{|\mu|=|w|} \mu_j v_{\mu,w} S_\mu e_{\sigma(j),\emptyset} & \text{if } w \notin \mathbb{F}_+^d g_{j+1}, \end{cases}$$

and Laca's unitary resolution has the form

$$V = \begin{bmatrix} A_{(\overline{\sigma},\mu_0,\dots,\mu_\ell)} & 0 \\ 0 & B \end{bmatrix},$$

where $A_{(\overline{\sigma},\mu_0,\dots,\mu_\ell)}$ is the permutation matrix for $\overline{\sigma}$ such that

$$(A_{(\overline{\sigma},\mu_0,\dots,\mu_\ell)})_{i,j} = \begin{cases} \frac{\mu_{j+1}}{\mu_j} & \text{if } i = \overline{\sigma}(j) \\ \frac{\mu_0}{\mu_\ell} & \text{if } i = \overline{\sigma}(0) \\ 0 & \text{otherwise.} \end{cases}$$

Conversely if there exist permutations σ and $\overline{\sigma}$ and if U and Laca's resolution V have the forms above then ad_U commutes with ad_S.

In order to prove this theorem we shall make use of the following preliminary lemmas.

Lemma 4.3.9. *Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ for the Hilbert spaces*

$$\mathcal{H}_1 = \left\langle e_w : w \neq \emptyset, w \in \mathbb{F}_+^d \right\rangle \quad \text{and} \quad \mathcal{H}_2 = \left\langle f_w : w \neq \emptyset, w \in \mathbb{F}_+^d \right\rangle.$$

Let the Cuntz family $\{S_i\}_{i=0}^d$ be such that

$$S_i e_w = \begin{cases} \lambda e_j & \text{if } i = j, w = j, \\ e_i & \text{if } w \neq j, |w| = 1, i \neq j, \\ e_{iw} & \text{otherwise,} \end{cases}$$

and,

$$S_i f_w = \begin{cases} \lambda f_{j'} & \text{if } i = j', w = j', \\ f_i & \text{if } w \neq j', |w| = 1, i \neq j', \\ f_{iw} & \text{otherwise.} \end{cases}$$

Let $U \in \mathcal{B}(\mathcal{H})$ be a unitary and suppose that ad_U commutes with the induced ad_S and $V = (v_{i,j})$ be Laca's resolution matrix. Then one of the following holds:

(1) $v_{j,j} = 1$ and thus $Ue_0 = \mu e_0$ for $\mu \in \mathbb{T}$; or

(2) $v_{j,j'} = 1$ and thus $Ue_0 = \mu f_0$ for $\mu \in \mathbb{T}$.

Proof. It is clear that $\{S_i\}_{i=0}^d$ is the direct sum of two one-cycle representations. We have the following picture:

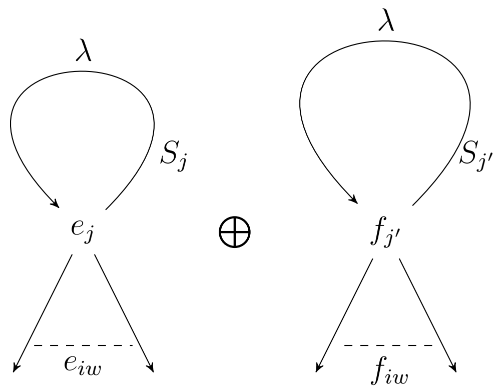

Figure 4.5

We have that

$$Ue_0 = U(\overline{\lambda} S_j e_0) = \overline{\lambda} U S_j e_0 = \sum_{i=0}^{d} \overline{\lambda} v_{i,j} S_i U e_0.$$

Then,

$$\langle Ue_0, f_0 \rangle = \sum_{i=0}^{d} \overline{\lambda} v_{i,j} \langle Ue_0, S_i^* f_0 \rangle = \overline{\lambda} v_{j,j'} \left\langle Ue_0, S_{j'}^* f_0 \right\rangle = v_{j,j'} \langle Ue_0, f_0 \rangle . \qquad (4.2)$$

Thus $\langle Ue_0, f_0 \rangle (1 - v_{j,j'}) = 0$ and there are two cases, either $\langle Ue_0, f_0 \rangle = 0$ or $v_{j,j'} = 1$.

Case (1): Suppose that $\langle Ue_0, f_0\rangle = 0$, then for every $w \neq \emptyset$ we have

$$\begin{aligned}\langle Ue_0, f_w\rangle = \overline{\lambda}^{|w|}\left\langle US_j^{|w|}e_0, f_w\right\rangle &= \sum_{|g|=|w|}\overline{\lambda}^{|w|}v_{g,j^{|w|}}\left\langle Ue_0, S_g^* f_w\right\rangle \\ &= \overline{\lambda}^{|w|}v_{w,j^{|w|}}\left\langle Ue_0, f_0\right\rangle \\ &= 0.\end{aligned}$$

Thus Ue_0 is orthogonal to f_w. Applying (4.2) for $\langle Ue_0, e_0\rangle$ gives that

$$\langle Ue_0, e_0\rangle (1 - v_{j,j}) = 0.$$

Thus either $\langle Ue_0, e_0\rangle = 0$ in which case we have that $\langle Ue_0, e_w\rangle = 0$ and therefore $Ue_0 = 0$ or $v_{j,j} = 1$. The first case yields a contradiction since U is a unitary and therefore we have that $v_{j,j} = 1$. Since $v_{j,j} = 1$ we have that $v_{i,j} = 0$ if $i \neq j$ as V is a unitary. Therefore

$$Ue_0 = \overline{\lambda}US_je_0 = \overline{\lambda}\sum_{i=0}^{d} v_{i,j}S_iUe_0 = \overline{\lambda}S_j(Ue_0).$$

Hence Ue_0 is a λ-eigenvector of S_j and thus $Ue_0 \in \mathbb{C}e_0$ and so $Ue_0 = \mu e_0$. Since U is a unitary we have that $|\mu| = 1$.

Case (2): Now suppose that $v_{j,j'} = 1$ then similarly to above, we have

$$\begin{cases} v_{i,j} = 0 & \text{if } j \neq i' \\ v_{j,i'} = 0 & \text{if } j \neq i. \end{cases}$$

Then

$$Ue_0 = \overline{\lambda}US_je_0 = \overline{\lambda}\sum_{i=0}^{d} v_{i,j}S_iUe_0 = \overline{\lambda}S_{j'}(Ue_0).$$

Therefore Ue_0 is a λ-eigenvector of $S_{j'}$, thus $Ue_0 \in \mathbb{C}f_0$ and so $Ue_0 = \mu f_0$. Since U is a unitary we have that $|\mu| = 1$. ∎

We now proceed to decompose $\mathcal{H}$ as follows. For $w \in \mathbb{F}_+^{d+1}$ with $|w| = t+1$ define

$$\mathcal{F}^w(t+1) = \{S_\mu : t+1 \text{ divides } |\mu|, \mu \neq w^k, k \in \mathbb{N}\}.$$

Then set

$$\mathcal{H}_i = \left\langle \mathcal{F}^{c^i(g)}(t+1)e_{i,\emptyset} \right\rangle,$$

where c is a cyclic permutation of the word g. We shall show in the following lemma that

$$\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_t = \langle \mathcal{F}^g(t+1)e_{0,\emptyset}\rangle \oplus \cdots \oplus \left\langle \mathcal{F}^{c^t(g)}(t+1)e_{t,\emptyset} \right\rangle.$$

Therefore we must show that every vector in $\mathcal{H}$ lies in exactly one of the $\mathcal{H}_i$.

Lemma 4.3.10. *With the aforementioned notation we have that:*

(1) For all $w \in \mathbb{F}_+^d$ there exists a unique $i \in \{0,\dots,t\}$ such that $e_{j,w} \in \mathcal{H}_i$.

(2) If $w \in \mathcal{F}^{c^i(g)}(t+1)$ then:

  (a) $S_w e_{i,\emptyset} \perp S_\nu e_{i,\emptyset}$ for all $\nu \in \mathcal{F}^{c^i(g)}(t+1)$ and,

  (b) $S_w e_{i,\emptyset} \perp S_\nu e_{j,\emptyset}$ for all $\nu \in \mathcal{F}^{c^i(g)}(t+1)$.

Proof. (1) We wish to find an appropriate ν such that $e_{j,w} = S_\nu e_{i,\emptyset}$ Firstly note that the vector $e_{j,\emptyset}$ is in the cycle and satisfies

$$S_w e_{j,\emptyset} = e_{j,w},$$

and $|w| = p(k+1) + r$ where $0 \le r < k+1$. Now choose an i such that $e_{i,\emptyset}$ is connected to $e_{j,\emptyset}$ by $(k+1)-r$ steps along the cycle. That is, choose a word $w' \in \mathbb{F}_+^d$ such that,

$$S_{w'} e_{i,\emptyset} = \lambda^{(k+1)-r} e_{j,\emptyset}.$$

Let $f = \overline{\lambda}^{k+1-r} e_{i,\emptyset}$ then

$$S_w S_{w'} f = S_w S_{w'} \overline{\lambda}^{k+1-r} e_{i,\emptyset} = \overline{\lambda}^{k+1-r} \lambda^{k+1-r} S_w e_{j,\emptyset} = S_w e_{j,\emptyset} = e_{j,w}.$$

Also,

$$|w| + |w'| = p(k+1) + r + (k+1) - r = (p+1)(k+1).$$

Therefore taking $\nu = ww'$ we see that $S_\nu e_{i,\emptyset} = e_{j,w} \in \mathcal{H}_i$.

(2a) Without loss of generality suppose that $|w| > |\nu|$. There are two cases. If $w \neq \nu w'$ we have that

$$\langle S_w e_{i,\emptyset}, S_\nu e_{i,\emptyset}\rangle = \langle e_{i,w}, e_{i,\nu}\rangle = 0.$$

If $w = \nu w'$ then

$$\langle S_w e_{i,\emptyset}, S_\nu e_{i,\emptyset}\rangle = \langle S_\nu S_{w'} e_{i,\emptyset}, S_\nu e_{i,\emptyset}\rangle = \langle S_\nu e_{i,w'}, S_\nu e_{i,\emptyset}\rangle = \langle e_{i,w'}, e_{i,\emptyset}\rangle = 0.$$

(2b) Again assume that $|w| > |\nu|$. Then if $w \neq vw'$ similarly to above we have that

$$\langle S_w e_{i,\emptyset}, S_\nu e_{j,\emptyset}\rangle = 0.$$

If $w = vw'$ we have

$$\langle S_w e_{i,\emptyset}, S_\nu e_{j,\emptyset}\rangle = \langle S_\nu S_{w'} e_{i,\emptyset}, S_\nu e_{j,\emptyset}\rangle = \langle S_\nu e_{i,w'}, S_\nu e_{j,\emptyset}\rangle = \langle e_{i,w'}, e_{j,\emptyset}\rangle = 0. \qquad \blacksquare$$

Now set $(\mathrm{ad}_S)^{t+1} = \mathrm{ad}_{\tilde{S}}$ where $\tilde{S} = \{S_\mu\}_{|\mu|=t+1}$. Then since ad_U commutes with ad_S it also commutes with $(\mathrm{ad}_S)^{t+1}$. Therefore we have the following picture:

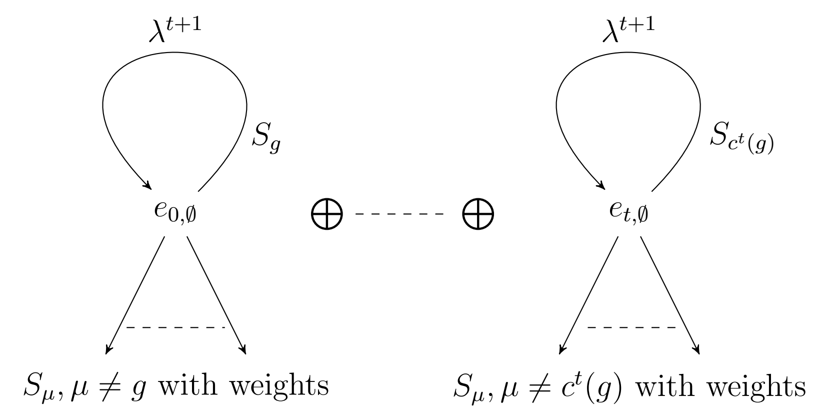

Figure 4.6

Moreover for a unitary Q, ad_{QUQ^*} commutes with $\mathrm{ad}_{Q\tilde{S}Q^*}$. We may choose a Q which makes the weights on each of the branches in the above picture equal to one. Then Q preserves the peaks of each summand and hence by Lemma 4.3.9, QUQ^* permutes the peaks. As Q is diagonal then U permutes the peaks. Therefore, there is a cyclic permutation σ of $\{0, \dots, t\}$ and $\mu_j \in \mathbb{T}$ such that

$$U e_{j,\emptyset} = \mu_j e_{\sigma(j),\emptyset}.$$

We can now turn to the proof of our main theorem.

Proof of Theorem 4.3.8. For the forward direction we have to show that $\sigma \in S_{t+1}$ defines a cyclic permutation of the word g, and induces a permutation $\overline{\sigma}$ on $\{g_0, \ldots, g_t\} = \{0, \ldots, \ell\}$ such that $\overline{\sigma}(g_i) = g_{\sigma(i)}$. That is, (for $w = \emptyset$) we have to show that if U is as above then

$$S_{g_{\sigma(i+1)}} e_{\sigma(i),\emptyset} = \lambda e_{\sigma(i+1),\emptyset}.$$

By the arguments above, the unitary U takes peaks to peaks and therefore

$$\overline{\lambda} U S_{g_{i+1}} e_{i,\emptyset} = U e_{i+1,\emptyset} = \mu_{i+1} e_{\sigma(i+1),\emptyset}.$$

On the other hand applying Laca's criterion [40], gives that

$$\begin{aligned}
\overline{\lambda} U S_{g_{i+1}} e_{i,\emptyset} &= \overline{\lambda} \sum_{k=0}^{d} v_{k,g_{i+1}} S_k U e_{i,\emptyset} \\
&= \overline{\lambda} \sum_{k=0}^{d} v_{k,g_{i+1}} \mu_i S_k e_{\sigma(i),\emptyset}. \\
&= \overline{\lambda} v_{\overline{\sigma}(g_{i+1}),g_{i+1}} \mu_i S_{\overline{\sigma}(g_{i+1})} e_{\sigma(i),\emptyset}
\end{aligned}$$

Therefore we have that

$$\mu_{i+1} e_{\sigma(i+1),\emptyset} = \overline{\lambda} v_{\overline{\sigma}(g_{i+1}),g_{i+1}} \mu_i S_{\overline{\sigma}(g_{i+1})} e_{\sigma(i),\emptyset}, \tag{4.3}$$

since by construction $e_{\sigma(i),\emptyset}$ passes to $e_{\sigma(i+1),\emptyset}$ by $S_{g_{\sigma(i+1)}}$. Therefore we must have that $g_{\sigma(i+1)} = \overline{\sigma}(g_{i+1})$. Applying this in (4.3) also gives that

$$v_{\sigma(i+1),g_{i+1}} = \frac{\mu_{i+1}}{\mu_i},$$

as required.
Now for the case where $w \notin \mathbb{F}_+^d g_{j+1}$, by applying Laca's resolution we have

$$U e_{j,w} = U S_w e_{j,\emptyset} = \sum_{|\mu|=|w|} v_{\mu,w} S_\mu U e_{j,\emptyset} = \sum_{|\mu|=|w|} \mu_j v_{\mu,w} S_\mu e_{\sigma(j),\emptyset},$$

as required.

For the reverse direction we need to consider four cases.

Case 1: $(w = \emptyset, i \neq g_{j+1})$

By direct computations we have that

$$US_i e_{j,\emptyset} = Ue_{j,i} = \sum_{k=0}^{d} \mu_j v_{k,i} S_k e_{\sigma(j),\emptyset}.$$

On the other hand by Laca's resolution we have

$$\sum_{k=0}^{d} v_{k,i} S_k U e_{j,\emptyset} = \sum_{k=0}^{d} \mu_j v_{k,i} S_k e_{\sigma(j),\emptyset}.$$

Case 2: $(w = \emptyset, i = g_{j+1})$

Here we have that

$$US_{g_{j+1}} e_{j,\emptyset} = Ue_{j+1,\emptyset} = \mu_{j+1} e_{\sigma(j+1),\emptyset}.$$

On the other hand by Laca's resolution,

$$\sum_{k=0}^{d} v_{k,g_{j+1}} S_k U e_{j,\emptyset} = \mu_j v_{\overline{\sigma}(g_{j+1}),g_{j+1}} S_{\overline{\sigma}(g_j+1)} e_{\sigma(j),\emptyset} = \mu_j \cdot \frac{\mu_{j+1}}{\mu_j} e_{\sigma(j+1),\emptyset}.$$

Case 3: $(w \notin \mathbb{F}_+^d g_{j+1},\ i \neq g_{j+1})$

Here we have, $US_i e_{j,w} = Ue_{j,iw} = \sum_{|\mu|=|iw|} \mu_j \cdot v_{\mu,iw} S_\mu e_{\sigma(j),\emptyset}$.

On the other hand,

$$\begin{aligned}
\sum_{k=0}^{d} v_{k,i} S_k U e_{j,w} &= \sum_{k=0}^{d} v_{k,i} S_k \Big(\sum_{|\nu|=|w|} \mu_j \cdot v_{\nu,w} S_\nu e_{\sigma(j),\emptyset} \Big) \\
&= \sum_{k=0}^{d} \sum_{|\nu|=|w|} \mu_j \cdot v_{k,i} v_{\nu,w} S_{k\nu} e_{\sigma(j),\emptyset} \\
&= \sum_{|\mu|=|iw|} \mu_j \cdot v_{\mu,iw} S_\mu e_{\sigma(j),\emptyset}.
\end{aligned}$$

Case 4: $(w \notin \mathbb{F}_+^d g_{j+1}\ i = g_{j+1})$

Calculating directly,

$$US_{g_{j+1}}e_{j,w} = US_{g_{j+1}w}e_{j,\emptyset} = \sum_{|\mu|=|g_{j+1}w|} \mu_j \cdot v_{\mu,g_{j+1}w}S_\mu e_{\sigma(j),\emptyset}.$$

On the other hand,

$$\begin{aligned}\sum_{k=0}^{d} v_{k,g_{j+1}}S_kUe_{j,w} &= \sum_{k=0}^{d} v_{k,g_{j+1}}S_k(\sum_{|\nu|=|w|}\mu_j \cdot v_{\nu,w}S_\nu e_{\sigma(j),\emptyset})\\ &= \sum_{k=0}^{d}\sum_{|\nu|=|w|}\mu_j \cdot v_{k,g_{j+1}}v_{\nu,w}S_{k\nu}e_{\sigma(j),\emptyset}\\ &= \sum_{|\mu|=|g_{j+1}w|}\mu_j \cdot v_{\mu,g_{j+1}w}S_\mu e_{\sigma(j),\emptyset}.\end{aligned}$$

Therefore Laca's criterion holds in each case and the proof is complete. ∎

4.3.2 Examples and Applications

As an application of Theorem 4.3.8 we have the following result when all of the weights $\mu_0, \dots, \mu_\ell$ are equal to 1. For a word $w = w_r \cdots w_0$ define

$$\mathrm{supp}_\ell(w) = \{i \in \{0, \dots, r\} : w_i \neq 0, \dots, \ell\}.$$

For $n \in \mathbb{Z}_+$ write

$$\mathcal{H}_0 = \langle e_{j,\emptyset} : j \in \{0, \dots, t\}\rangle$$

and

$$\mathcal{H}_{j,n} = \langle e_{j,w} : \mathrm{supp}_\ell(w) = \mathrm{supp}_2(n)\rangle .$$

We write $\nu \hookrightarrow w$ if:

1. $|\nu| = |w|$.
2. $\mathrm{supp}_\ell(\nu) = \mathrm{supp}_\ell(w)$.
3. $\nu_i = w_i$ for all $i \notin \mathrm{supp}_\ell(w)$.
4. $\nu_i \notin \{g_0, \dots, g_\ell\}$ for all $i \in \mathrm{supp}_\ell(\nu)$.

We then have the following corollary.

Corollary 4.3.11. *Suppose that* ad_U *commutes with* ad_S*, where* S *is the cycle representation given in Figure 4.4. Then, by Theorem 4.3.8* U *permutes the cycle by some weights* $\mu_0, \dots, \mu_t$*. In addition, suppose that* $\mu_0 = \cdots = \mu_t = 1$*. Then Laca's unitary resolution has the form*

$$V = \begin{bmatrix} A_{\overline{\sigma}} & 0 \\ 0 & B \end{bmatrix},$$

where $A_{\overline{\sigma}}$ *is the permutation matrix for the permutation* $\overline{\sigma}$ *and,*

$$Ue_{j,w} = \sum_{\nu \hookrightarrow w} v_{\overline{\sigma}(\nu),w} e_{\sigma(j),\overline{\sigma}(\nu)}$$

up to a constant of modulus one.

Proof. We proceed by induction. For $\mathcal{H}_0$, by hypothesis we have that

$$Ue_{j,\emptyset} = e_{\sigma(j),\emptyset}.$$

This gives the required result for $\mathcal{H}_0$.
For the $n = 1$ step: Let $j \in \{0, \dots, \ell\}$ and note that

$$\mathcal{H}_{j,1} = \langle e_{j,w} : \text{ for } |w| = 1, w \neq j, j \in \{0, \dots, \ell\} \rangle .$$

Then

$$\begin{aligned} Ue_{j,w} = US_w e_{j,\emptyset} = \sum_{k=0}^{d} v_{k,w} S_k U e_{j,\emptyset} &= \sum_{k=\ell+1}^{d} v_{k,w} S_k e_{\sigma(j),\emptyset} \\ &= \sum_{k=\ell+1}^{d} v_{k,w} e_{\sigma(j),k} = \sum_{k=\ell+1}^{d} v_{\sigma(k),w} e_{\sigma(j),\overline{\sigma}(k)}. \end{aligned}$$

where $k \in \{\ell+1, \dots, d\}$, $\sigma(j) \in \{0, \dots, \ell\}$ and $k \neq \sigma(j)$.

Now for the inductive step, assume that $Ue_{j,w} = \sum_{\nu \hookrightarrow w} v_{\overline{\sigma}(\nu),w} e_{\sigma(j),\overline{\sigma}(\nu)}$ for all $1 \leq m \leq n$. Then we have two preliminary cases.

Case 1: Suppose that $n = * \cdots * 0$ is the binary expansion of n written in reverse order. Then $n+1 = * \cdots * 1$ and

$$\mathrm{supp}_2(n) = \{r = i_r > \cdots > i_1 : i_1 \neq 0\},$$

thus

$$\mathrm{supp}_2(n+1) = \{r = i_r > \cdots > i_1 > 0\}.$$

First let $w = w_r \dots w_0$ such that $\mathrm{supp}_\ell(w) = \mathrm{supp}_2(n+1)$ with $w_r \notin \{g_0, \dots, g_\ell\}$ and

$$Ue_{j,w} = US_{w_r} e_{j,w'} = \sum_{\nu_r=\ell+1}^{d} v_{\nu_r,w_r} S_{v_r} U e_{j,w'}. \tag{4.4}$$

Now, note that $\mathrm{supp}_\ell(w') = \mathrm{supp}_2((n+1) - 2^r)$ where $n+1-2^r < n$ as $r > 0$, therefore

$$Ue_{j,w'} = \sum_{\nu' \hookrightarrow w'} v_{\overline{\sigma}(\nu'),w'} e_{\sigma(j),\overline{\sigma}(\nu')}$$

and by the inductive hypothesis, (4.4) becomes

$$Ue_{j,w} = \sum_{\nu_r=\ell+1}^{d} \sum_{\nu' \hookrightarrow w'} v_{\nu_r,w_r} v_{\overline{\sigma}(\nu'),w'} e_{\sigma(j),\nu_r\overline{\sigma}(\nu')}.$$

We see that $\nu \hookrightarrow w = w_r w'$ with $w_r \notin \{0, \dots, \ell\}$ if and only if $\nu' \hookrightarrow w'$ with $v_r \notin \{0, \dots, \ell\}$. Also if we have $\nu \hookrightarrow w$, $\nu = \nu_r \nu'$, $\nu' \hookrightarrow w'$ and $\nu_r \in \{0, \dots \ell\}$ then:

$$e_{\sigma(j),\overline{\sigma}(\nu)} = e_{\sigma(j),\overline{\sigma}(\nu_r)\overline{\sigma}(\nu')} = e_{\sigma(j),\nu_r\overline{\sigma}(\nu')}.$$

Therefore

$$v_{\nu_r,w_r} \cdot v_{\overline{\sigma}(\nu'),w'} = v_{\nu_r\overline{\sigma}(\nu'),w_r\nu'} = v_{\overline{\sigma}(\nu),w}.$$

Hence we see that we have the required equality as

$$Ue_{j,w} = \sum_{\nu_r=\ell+1}^{d} \sum_{\nu' \hookrightarrow w'} v_{\nu_r,w_r} v_{\overline{\sigma}(\nu'),w'} e_{\sigma(j),\nu_r\overline{\sigma}(\nu')} = \sum_{\nu' \hookrightarrow w'} v_{\overline{\sigma}(\nu),w} e_{\sigma(j),\overline{\sigma}(\nu)}.$$

Case 2: Now suppose that $n = * \cdots * 1$ is the binary expansion of n written in reverse order.

We consider three sub-cases.

Case 2(a): First suppose that $\operatorname{supp}_2(n) = \{i_p > \cdots > i_1\}$ such that

$$\operatorname{supp}_2(n+1) = \{i_p + 1 > \cdots > i_2 + 1 > i_1 + 1 : i_2 - i_1 = \cdots = i_p - i_{p-1} = 1\}.$$

Set $x := i_p + 1$ then we have $w = w_x w'$ with $\operatorname{supp}_\ell(w) = \operatorname{supp}_2(n+1)$ and $w_x \notin \{g_0, \ldots, g_\ell\}$. So we have:

$$Ue_{j,w} = US_{w_x} e_{j,w'} = \sum_{\nu_x = \ell+1}^{d} v_{\nu_x, w_x} S_{\nu_x} U e_{j,w'}$$

and,

$$\operatorname{supp}_\ell(w') = \{i_{p-1} + 1 > \cdots > i_2 + 1 > i_1 + 1\} = \operatorname{supp}_2(n + 1 - 2^x),$$

where $n + 1 - 2^x < n$. So by the inductive hypothesis we have:

$$Ue_{j,w} = \sum_{\nu_x = \ell+1}^{d} \sum_{\nu' \hookrightarrow w'} v_{\nu_x, w_x} v_{\overline{\sigma}(\nu'), w'} e_{\sigma(j), \nu_x \overline{\sigma}(v')}.$$

Again, we see that $\nu \hookrightarrow w = w_x w'$ with $w_x \notin \{0, \ldots, \ell\}$ if and only if $\nu' \hookrightarrow w'$ with $v_x \notin \{0, \ldots, \ell\}$. Also if we have $\nu \hookrightarrow w$, $\nu = \nu_x \nu'$, $\nu' \hookrightarrow w'$ and $\nu_x \in \{0, \ldots \ell\}$ then:

$$e_{\sigma(j), \overline{\sigma}(\nu)} = e_{\sigma(j), \overline{\sigma}(\nu_x) \overline{\sigma}(\nu')} = e_{\sigma(j), \nu_x \overline{\sigma}(\nu')}.$$

Therefore

$$v_{\nu_x, w_x} \cdot v_{\overline{\sigma}(\nu'), w'} = v_{\nu_x \overline{\sigma}(\nu'), w_x \nu'} = v_{\overline{\sigma}(\nu), w}.$$

Hence we have

$$Ue_{j,w} = \sum_{\nu_x = \ell+1}^{d} \sum_{\nu' \hookrightarrow w'} v_{\nu_x, w_x} v_{\overline{\sigma}(\nu'), w'} e_{\sigma(j), \nu_x \overline{\sigma}(v')} = \sum_{\nu \hookrightarrow w} v_{\overline{\sigma}(\nu), w} e_{\sigma(j), \overline{\sigma}(\nu)}.$$

Case 2(b): Suppose that

$$\operatorname{supp}_2(n) = \{i_r > \cdots > i_{p+1} > i_p > \cdots > i_1\}$$

with $i_{p+1} - i_p \geq 2$ and $i_2 - i_1 = \cdots = i_p - i_{p-1} = 1$. Then

$$\mathrm{supp}_2(n+1) = \{i_r > \cdots > i_{p+1} > i_p + 1 > \cdots > i_1 + 1\}.$$

Set $x = i_r \neq 0$. Let $w = w_x w'$ with $\mathrm{supp}_\ell(w) = \mathrm{supp}_2(n+1)$ and compute

$$Ue_{j,w} = US_{w_x} e_{j,w'} = \sum_{\nu_x=\ell+1}^{d} v_{\nu_x,w_x} S_{\nu_x} U e_{j,w'}$$

and,

$$\mathrm{supp}_\ell(w') = \{i_{r-1} > \cdots > i_{p+1} > i_p + 1 > \cdots > i_2 + 1 > i_1 + 1\} = \mathrm{supp}_2(n+1-2^x),$$

where $n + 1 - 2^x < n$. So by the inductive hypothesis we have:

$$Ue_{j,w} = \sum_{\nu_x=\ell+1}^{d} \sum_{\nu' \hookrightarrow w'} v_{\nu_x,w_x} v_{\overline{\sigma}(\nu'),w'} e_{\sigma(j),\nu_x \overline{\sigma}(v')}.$$

Then, performing similar computations as before we have that

$$e_{\sigma(j),\overline{\sigma}(\nu)} = e_{\sigma(j),\overline{\sigma}(\nu_x)\overline{\sigma}(\nu')} = e_{\sigma(j),\nu_x\overline{\sigma}(\nu')}$$

and

$$v_{\nu_x,w_x} \cdot v_{\overline{\sigma}(\nu'),w'} = v_{\nu_x\overline{\sigma}(\nu'),w_x\nu'} = v_{\overline{\sigma}(\nu),w},$$

which gives the required equality.

Case 2(c): Now suppose that $w = zw_r \cdots w_0$ with $z_i \in \{0, \ldots, \ell\}$ and $\mathrm{supp}_\ell(w_r \cdots w_0) = \mathrm{supp}_2(n+1)$. Hence

$$\mathrm{supp}_\ell(w) = \mathrm{supp}_\ell(w_r \cdots w_0)$$

and w_r is the first occurrence where $w_r \notin \{0, \ldots, \ell\}$. Set $w_r \cdots w_0 = w''$ then

$$\begin{aligned} Ue_{j,w} = US_z e_{j,w''} = S_{\overline{\sigma}(z)} U e_{j,w''} &= S_{\overline{\sigma}(z)} \sum_{\nu \hookrightarrow w''} v_{\overline{\sigma}(\nu),w''} e_{\sigma(j),\overline{\sigma}(\nu)} \\ &= \sum_{\nu \hookrightarrow w''} v_{\overline{\sigma}(\nu),w''} e_{\sigma(j),\overline{\sigma}(z)\overline{\sigma}(\nu)} = \sum_{\nu \hookrightarrow w''} v_{\overline{\sigma}(\nu),\nu} e_{\sigma(j),\overline{\sigma}(z\nu)}. \end{aligned}$$

Now notice that $v \hookrightarrow w''$ if and only if $z\nu \hookrightarrow w$ and $v_{\overline{\sigma}(z),z} = 1$. Hence

$$v_{\overline{\sigma}(\nu),w''} = v_{\overline{\sigma}(z\nu),w}$$

and we have

$$Ue_{j,w} = \sum_{\nu \hookrightarrow w''} v_{\overline{\sigma}(\nu),w''} e_{\sigma(j),\overline{\sigma}(w)} = \sum_{z\nu \hookrightarrow w} v_{\overline{\sigma}(z\nu),w} e_{\sigma(j),\overline{\sigma}(z\nu)} = \sum_{\nu \hookrightarrow w} v_{\overline{\sigma}(\nu),w} e_{\sigma(j),\overline{\sigma}(\nu)}.$$

So we have the required form for U in each case and the proof is complete.

∎

We end this section by noting the following examples.

Example 4.3.12. If $\sigma = \mathrm{id}$ then Corollary 4.3.11 connects with binary weights in the following way. Let $\phi(n)$ be the binary weight of n and make the identification $e_{s,w} = e_{s,w_r} \otimes \cdots \otimes e_{s,w_0}$ then define

$$B(\phi(n))e_{j,w} = e_{j,w_r} \otimes \cdots \otimes e_{j,w_0} = f_{j,r} \otimes \cdots \otimes f_{j,0}$$

such that

$$f_{j,i} = \begin{cases} \sum\limits_{\nu_i=\ell+1}^{d} v_{\nu_i,w_i} e_{j,\nu_i} & \text{if } i \in \mathrm{supp}_\ell(w), \\ e_{j,w_i} & \text{if } i \notin \mathrm{supp}_\ell(w). \end{cases}$$

Then $Ue_{j,w} = B(\phi(n))e_{j,w}$ on $\mathcal{H}_{j,n}$. That is, U is the block diagonal of the $\phi(n)$.

Example 4.3.13. Fix $\mathcal{H} = \ell^2(\mathbb{Z}_+)$ and let the Cuntz family

$$S_1 e_n = e_{2n} \quad \text{and} \quad S_2 e_n = e_{2n+1}.$$

Let $U \in \mathcal{B}(\mathcal{H})$ be a unitary and fix the induced actions

$$\alpha(x) = UxU^* \quad \text{and} \quad \beta(x) = S_1 x S_1^* + S_2 x S_2^*.$$

Then α and β commute if and only if

$$U = \lambda \, \mathrm{diag}\{\mu^{\phi(n)} \mid n \in \mathbb{Z}_+\} \quad \text{for } \lambda, \mu \in \mathbb{T},$$

where $\phi(n)$ is the sequence of the binary weights of n.

Example 4.3.14. Let $\mathcal{H} = \ell^2(\mathbb{Z})$ and the Cuntz family

$$S_1 e_n = e_{2n} \quad \text{and} \quad S_2 e_n = e_{2n+1}.$$

Let $U \in \mathcal{B}(\mathcal{H})$ be a unitary and write $\ell^2(\mathbb{Z}) = H_1 \oplus H_2$ for

$$H_1 = \langle e_n \mid n \geq 0 \rangle \quad \text{and} \quad H_2 = \langle e_n \mid n \leq -1 \rangle .$$

Therefore we have the direct sum of two cycle representations. Then either $\{S_1, S_2\}$ act as in Lemma 4.3.9 or they interchange the summands. Then we see that the actions induced by U and $\{S_1, S_2\}$ commute if and only if U has one of the forms

$$U = \lambda I_{H_1} \oplus \mu I_{H_2} \ \text{ or } \ U = \begin{bmatrix} 0 & \mu w^* \\ \lambda w & 0 \end{bmatrix} \tag{4.5}$$

where $\lambda, \mu \in \mathbb{T}$ and $w \in \mathcal{B}(H_1, H_2)$ is the unitary with $we_n = e_{-n-1}$.

Example 4.3.15. For $n \in \mathbb{Z}_+$ let $\mathcal{H} = \ell^2(\mathbb{Z}_+) = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \oplus \cdots$ such that $\mathcal{H}_0 = \langle e_0 \rangle$ and

$$\mathcal{H}_n = \left\langle e_n : n = n_0 \cdot d^0 + n_1 \cdot d^{i_1} + \cdots + n_k \cdot d^{i_k} \right\rangle ,$$

for $n_1, \ldots, n_k \in \{1, \ldots, d-1\}$ and $i_1, \ldots, i_k \in \operatorname{supp}_2(n)$. Let the Cuntz family $S_0, \cdots, S_{d-1}$ be given by

$$S_k e_n = e_{dn+k}$$

for $k = 1, \cdots, d-1$ and $\{e_n : n = 0, 1, \cdots\}$. Let $U \in \mathcal{B}(\mathcal{H})$ be a unitary and fix the actions

$$\alpha(x) = UxU^* \quad \text{and} \quad \beta(x) = \sum_{k=0}^{d-1} S_k x S_k^*.$$

For $n = n_0 \cdot d^0 + n_1 \cdot d^{i_1} + \cdots + n_k \cdot d^{i_k}$, let $\phi(n)$ be the binary weight of n and make the identification $e_n = e_{n_0} \otimes \cdots \otimes e_{n_k}$. Then define the unitary

$$V = \begin{bmatrix} I & 0 \\ 0 & W^{\phi(n)} \end{bmatrix},$$

where

$$W^{\phi(n)} = W_0 \otimes \cdots \otimes W_k,$$

with

$$W_i = \begin{cases} W & \text{when } n_i \neq 0, \\ I & \text{when } n_i = 0. \end{cases}$$

Then, the actions α and β commute if and only if $Ue_n = \lambda \sum_{j=1}^{d-1} W^{\phi(n)} e_n$ on $\mathcal{H}_n$, for $\lambda \in \mathbb{T}$.

Chapter 5

Bicommutant Property

As stated previously, we are interested in examining the bicommutant property and reflexivity of our semicrossed products. In this section we detail our results regarding the former, when the dynamics come from a uniformly bounded spatial action. We will deal with the $\mathbb{F}_+^d$ and $\mathbb{Z}_+^d$ cases separately.

5.1 Systems over $\mathbb{F}_+^d$

Our first result regarding the bicommutant property is in the following theorem

Theorem 5.1.1. *Let $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ be a w*-dynamical system of a uniformly bounded spatial action implemented by $\{u_i\}_{i\in[d]}$. Then we have that*

$$(\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d)' = \mathcal{A}' \overline{\times}_u \mathcal{R}_d \quad \textit{and} \quad (\mathcal{A}' \overline{\times}_u \mathcal{L}_d)' = \mathcal{A}'' \overline{\times}_\alpha \mathcal{R}_d$$

and that

$$(\mathcal{A} \overline{\times}_\alpha \mathcal{R}_d)' = \mathcal{A}' \overline{\times}_u \mathcal{L}_d \quad \textit{and} \quad (\mathcal{A}' \overline{\times}_u \mathcal{R}_d)' = \mathcal{A}'' \overline{\times}_\alpha \mathcal{L}_d.$$

Proof. We shall show the first two equalities, the others follow in a similar manner. For the first equality we begin by demonstrating that $\mathcal{A}' \overline{\times}_u \mathcal{R}_d$ is in the commutant of $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$. Recall that $\mathcal{A}' \overline{\times}_u \mathcal{R}_d$ is generated by $\rho(b) = b \otimes I$ and $W_{i,j} = u_{i,j} \otimes r_i$ for $b \in \mathcal{A}'$. $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ is generated by $\overline{\pi}(a)$ and L_i. By direct calculation we have;

$$\rho(b)L_i = (b \otimes I)(I \otimes \mathbf{l}_i) = b \otimes \mathbf{l}_i = (I \otimes \mathbf{l}_i)(b \otimes I) = L_i\rho(b),$$

$$W_{i,j}L_i = (u_{i,j} \otimes \mathbf{r}_i)(I \otimes \mathbf{l}_i) = u_{i,j} \otimes I = (I \otimes \mathbf{l}_i)(u_{i,j} \otimes \mathbf{r}_i) = L_iW_{i,j}.$$

Similarly,

$$\begin{aligned}\rho(b)\overline{\pi}(a)\xi \otimes e_w &= \rho(b)\alpha_{\overline{w}}(a)\xi \otimes e_w \\ &= b\alpha_{\overline{w}}(a)\xi \otimes e_w \\ &= \alpha_{\overline{w}}(a)b\xi \otimes e_w, \quad \text{for all } a \in \mathcal{A}, b \in \mathcal{A}'.\end{aligned}$$

and

$$\begin{aligned}W_{i,j}\overline{\pi}(a)\xi \otimes e_w &= (u_{i,j} \otimes \mathbf{r}_i)\alpha_{\overline{w}}(a)\xi \otimes e_w \\ &= u_{i,j}\alpha_{\overline{w}}(a)\xi \otimes e_{wi} \\ &= \alpha_i\alpha_{\overline{w}}(a)u_{i,j}\xi \otimes e_{wi} \\ &= \alpha_{\overline{wi}}(a)u_{i,j}.\xi \otimes e_w \\ &= \overline{\pi}(a)W_{i,j}\xi \otimes e_w, \quad \text{for all } a \in \mathcal{A}.\end{aligned}$$

Therefore the generators of $\mathcal{A}' \overline{\times}_u \mathcal{R}_d$ commute with the generators of $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ and thus $\mathcal{A}' \overline{\times}_u \mathcal{R}_d \subseteq (\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d)'$.

For the reverse inclusion let T be in the commutant of $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$. As the Fourier transform respects the commutant it suffices to show that $G_m(T)$ is in $\mathcal{A}' \overline{\times}_u \mathcal{R}_d$ for all $m \in \mathbb{Z}_+$, and it is zero for all $m < 0$. For $\mu, \nu \in \mathbb{F}_+^d$ and by using the commutant property we get that

$$\begin{aligned}\langle T_{\mu,\nu}\xi, \eta\rangle &= \langle T\xi \otimes e_\nu, \eta \otimes e_\mu\rangle \\ &= \langle TL_\nu\xi \otimes e_\emptyset, \eta \otimes e_\mu\rangle \\ &= \langle L_\nu T\xi \otimes e_\emptyset, \eta \otimes e_\mu\rangle \\ &= \langle T\xi \otimes e_\emptyset, \eta \otimes \mathbf{l}_\nu^* e_\mu\rangle .\end{aligned}$$

However we have that $(\mathbf{l}_\nu)^* e_\mu = 0$ whenever $\nu \not\leq_r \mu$. Therefore T is right lower triangular and thus

$$G_m(T) = \begin{cases} \sum_{|\mu|=m} R_\mu T_{(\mu)} & \text{if } m \geq 0, \\ 0 & \text{if } m < 0, \end{cases}$$

for $T_{(\mu)} = \sum_{w \in \mathbb{F}_+^d} T_{w\overline{\mu},w} \otimes p_w = R_\mu^* G_m(T)$. As the Fourier transform respects the commutant we also have that $G_m(T) \in (\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d)$ as well. Moreover we have that

$$\begin{aligned}\sum_{|\mu|=m} T_{w\overline{\mu},w}\xi \otimes e_{w\overline{\mu}} &= G_m(T)L_w\xi \otimes e_\emptyset \\ &= L_w G_m(T)\xi \otimes e_\emptyset \\ &= \sum_{|\mu|=m} T_{\overline{\mu},\emptyset}\xi \otimes e_{w\overline{\mu}}\end{aligned}$$

and therefore $T_{(\mu)} = \rho(T_{\overline{\mu},\emptyset})$ for all μ of length m. In addition we have that

$$\begin{aligned}\sum_{|\mu|=m} T_{\overline{\mu},\emptyset}a\xi \otimes e_{\overline{\mu}} &= G_m(T)\overline{\pi}(a)\xi \otimes e_\emptyset \\ &= \overline{\pi}(a)G_m(T)\xi \otimes e_\emptyset \\ &= \sum_{|\mu|=m} \overline{\pi}(a)T_{\overline{\mu},\emptyset}\xi \otimes e_{\overline{\mu}} \\ &= \sum_{|\mu|=m} \alpha_\mu(a)T_{\overline{\mu},\emptyset}\xi \otimes e_{\overline{\mu}}\end{aligned}$$

and therefore $T_{\overline{\mu},\emptyset}a = \alpha_\mu(a)T_{\overline{\mu},\emptyset}$ for all $a \in \mathcal{A}$. Now, for $\mu = \mu_m \dots \mu_1$ and $j_i \in [n_{\mu_i}]$ we set

$$b_{\mu,j_1,\dots,j_m} := v_{\mu_1,j_1} \cdots v_{\mu_m,j_m} T_{\overline{\mu},\emptyset}$$

where v_i is the inverse of u_i. Then $b_{\mu,j_1,\dots,j_m}$ is in $\mathcal{A}'$ since

$$\begin{aligned}a \cdot v_{\mu_1,j_1} \cdots v_{\mu_m,j_m} T_{\overline{\mu},\emptyset} &= v_{\mu_1,j_1} \cdots v_{\mu_m,j_m} \alpha_{\mu_m} \cdots \alpha_{\mu_1}(a) T_{\overline{\mu},\emptyset} \\ &= v_{\mu_1,j_1} \cdots v_{\mu_m,j_m} \alpha_\mu(a) T_{\overline{\mu},\emptyset} \\ &= v_{\mu_1,j_1} \cdots v_{\mu_m,j_m} T_{\overline{\mu},\emptyset} \cdot a\end{aligned}$$

for all $a \in \mathcal{A}$. Therefore we can write

$$\begin{aligned}R_\mu T_{(\mu)} &= \sum_{j_m \in [n_{\mu_m}]} \cdots \sum_{j_1 \in [n_{\mu_1}]} R_\mu \rho(u_{\mu_m,j_m} \cdots u_{\mu_1,j_1}) \rho(b_{\mu,j_1,\dots,j_m}) \\ &= \sum_{j_m \in [n_{\mu_m}]} \cdots \sum_{j_1 \in [n_{\mu_1}]} W_{\mu_m,j_m} \cdots W_{\mu_1,j_1} \rho(b_{\mu,j_1,\dots,j_m}).\end{aligned}$$

Then if F is a finite subset of $[n_{\mu_1}]$ we have

$$\begin{aligned}
\| \sum_{j_1 \in F} & W_{\mu_m, j_m} \cdots W_{\mu_1, j_1} \rho(b_{\mu, j_1, \dots, j_m}) \| = \\
& = \| \sum_{j_1 \in F} u_{\mu_m, j_m} \cdots u_{\mu_1, j_1} v_{\mu_1, j_1} \cdots v_{\mu_m, j_m} T_{\overline{\mu}, \emptyset} \| \\
& \leq \| u_{\mu_m, j_m} \cdots u_{\mu_2, j_2} \| \, \| \sum_{j_1 \in F} u_{\mu_1, j_1} v_{\mu_1, j_1} \| \, \| v_{\mu_2, j_2} \cdots v_{\mu_m, j_m} \| \, \| T_{\overline{\mu}, \emptyset} \| \\
& \leq K^2 \, \| T_{\overline{\mu}, \emptyset} \| ,
\end{aligned}$$

where K is the uniform bound for $\{\widehat{u}_\mu\}_\mu$ and $\{\widehat{v}_\mu\}_\mu$. Inductively we have that the sums above converge in the w*-topology and therefore each $R_\mu T_{(\mu)}$ is in $\mathcal{A}' \overline{\times}_u \mathcal{R}_d$. As in Proposition 3.2.5 an application of Fejér's Lemma induces that T is in $\mathcal{A}' \overline{\times}_u \mathcal{R}_d$.

We now move on to show the equality $(\mathcal{A}' \overline{\times}_u \mathcal{L}_d)' = \mathcal{A}'' \overline{\times}_\alpha \mathcal{R}_d$. Performing similar calculations to those above we have that $\mathcal{A}'' \overline{\times}_\alpha \mathcal{R}_d \subseteq (\mathcal{A}' \overline{\times}_u \mathcal{L}_d)'$. For the reverse inclusion let T be in the commutant. Then T commutes with all $L_i \rho(u_{i,j_i})$. First let $\nu \not\leq_r \mu$ with $\nu = \nu_k \dots \nu_1$; then

$$\begin{aligned}
\langle T_{\mu,\nu} u_{\nu_k, j_k} \dots u_{\nu_1, j_1} \xi, \eta \rangle & = \langle T \rho(u_{\nu_k, j_k} \dots u_{\nu_1, j_1}) \xi \otimes e_\nu, \eta \otimes e_\mu \rangle \\
& = \langle T L_\nu \rho(u_{\nu_k, j_k} \dots u_{\nu_1, j_1}) \xi \otimes e_\emptyset, \eta \otimes e_\mu \rangle \\
& = \langle L_\nu \rho(u_{\nu_k, j_k} \dots u_{\nu_1, j_1}) T \xi \otimes e_\emptyset, \eta \otimes e_\mu \rangle \\
& = \langle \rho(u_{\nu_k, j_k} \dots u_{\nu_1, j_1}) T \xi \otimes e_\emptyset, (L_\nu)^* \eta \otimes e_\mu \rangle \\
& = 0.
\end{aligned}$$

Therefore by summing over the j_i we obtain

$$T_{\mu,\nu} = \sum_{j_k \in [n_{\nu_k}]} \cdots \sum_{j_1 \in [n_{\nu_1}]} T_{\mu,\nu} u_{\nu_k, j_k} \dots u_{\nu_1, j_1} v_{\nu_1, j_1} \dots v_{\nu_k, j_k} = 0$$

so that T is right lower triangular. We can check the non-negative Fourier coefficients. For $m = 0$ we have that $T_{(0)}$ commutes with $\rho(\mathcal{A}')$ and therefore every

$T_{w,w}$ is in $\mathcal{A}''$. Now for $w \in \mathbb{F}_+^d$ with $w = w_k \dots w_1$ we have that

$$\begin{aligned} T_{w,w} u_{w_k,j_k} \cdots u_{w_1,j_1} \xi \otimes e_w &= G_0(T) L_w \rho(u_{w_k,j_k}) \cdots \rho(u_{w_1,j_1}) \xi \otimes e_\emptyset \\ &= L_w \rho(u_{w_k,j_k}) \cdots \rho(u_{w_1,j_1}) G_0(T) \xi \otimes e_\emptyset \\ &= u_{w_k,j_k} \cdots u_{w_1,j_1} T_{\emptyset,\emptyset} \xi \otimes e_w. \end{aligned}$$

Consequently we obtain

$$\begin{aligned} \alpha_w(T_{\emptyset,\emptyset}) &= \alpha_{w_k} \cdots \alpha_{w_1}(T_{\emptyset,\emptyset}) \\ &= \sum_{j_k \in [n_{w_k}]} \cdots \sum_{j_1 \in [n_{w_1}]} u_{w_k,j_k} \cdots u_{w_1,j_1} T_{\emptyset,\emptyset} v_{w_1,j_1} \cdots v_{w_k,j_k} \\ &= T_{w,w} \sum_{j_k \in [n_{w_k}]} \cdots \sum_{j_1 \in [n_{w_1}]} u_{w_k,j_k} \cdots u_{w_1,j_1} v_{w_1,j_1} \cdots v_{w_k,j_k} = T_{w,w}. \end{aligned}$$

Thus we have that $G_0(T) = \pi(T_{\emptyset,\emptyset})$. Now let $m > 0$ then since $G_m(T)$ commutes with $L_i \rho(u_{i,j_i})$ we have that

$$T_{(\mu)} L_i \rho(u_{i,j_i}) = R_\mu^* G_m(T) L_i \rho(u_{i,j_i}) = R_\mu^* L_i \rho(u_{i,j_i}) G_m(T).$$

However for $\xi \otimes e_\nu \in \mathcal{K}$ we have that

$$(R_\mu)^* L_i \rho(u_{i,j_i}) G_m(T) \xi \otimes e_\nu = u_{i,j_i} T_{\nu\overline{\mu},\nu} \xi \otimes (\mathbf{r}_\mu)^* e_{i\nu\mu} = L_i \rho(u_{i,j_i}) T_{(\mu)} \xi \otimes e_\nu,$$

therefore $T(\mu)$ commutes with $L_i \rho(u_{i,j_i})$ for all i. Furthermore for $b \in \mathcal{A}'$ we get that

$$\begin{aligned} T_{(\mu)} \rho(b) &= (R_\mu)^* G_m(T) \rho(b) = (R_\mu)^* \rho(b) G_m(T) \\ &= \rho(b) (R_\mu)^* G_m(T) = \rho(b) T_{(\mu)}. \end{aligned}$$

Therefore $T_{(\mu)}$ is a diagonal operator in $(\mathcal{A}' \overline{\times}_\alpha \mathcal{L}_d)'$ and thus $T_{(\mu)} = \pi(T_{\overline{\mu},\emptyset})$ by what we have shown for the zero Fourier co-efficients. Therefore we have that $G_m(T)$ is in $\mathcal{A}'' \overline{\times}_\alpha \mathcal{R}_d$ for all $m \in \mathbb{Z}_+$. ∎

From this we have the following corollaries. Note that the equivalence between items (i) and (ii) follows by using Theorem 5.1.1 to write $(\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d)'' = \mathcal{A}'' \overline{\times}_\alpha \mathcal{L}_d$ then applying the compression to the $(\emptyset, \emptyset)$-entry.

Corollary 5.1.2. *Let $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ be a w*-dynamical system of a uniformly bounded spatial action. Then the following are equivalent*

(i) $\mathcal{A}$ *has the bicommutant property;*

(ii) $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ *has the bicommutant property;*

(iii) $\mathcal{A} \overline{\times}_\alpha \mathcal{R}_d$ *has the bicommutant property;*

(iv) $\mathcal{A} \otimes \mathcal{L}_d$ *has the bicommutant property;*

(v) $\mathcal{A} \otimes \mathcal{R}_d$ *has the bicommutant property.*

If any of the items above hold then all algebras above are inverse closed.

It is known that commutants are inverse closed algebras, therefore we have the following application.

Corollary 5.1.3. *(i) Let $\{\alpha_i\}_{i\in[d]}$ be a uniformly bounded spatial action on $\mathcal{B}(\mathcal{H})$. Then the w*-semicrossed products $\mathcal{B}(\mathcal{H}) \overline{\times}_\alpha \mathcal{L}_d$ and $\mathcal{B}(\mathcal{H}) \overline{\times}_\alpha \mathcal{R}_d$ are inverse closed.* *(ii) Let $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ be an automorphic system over a maximal abelian selfadjoint algebra (m.a.s.a.) $\mathcal{A}$. Then the w*-semicrossed products $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ and $\mathcal{A} \overline{\times}_\alpha \mathcal{R}_d$ are inverse closed.*

Proof. In both cases we can write $\mathcal{A} = \mathcal{B}'$ for a suitable $\mathcal{B}$ and then $\mathcal{B} \overline{\times}_u \mathcal{L}_d$ and $\mathcal{B} \overline{\times}_u \mathcal{R}_d$ are well defined. The result then follows by applying Theorem 5.1.1. ∎

5.2 Systems over $\mathbb{Z}_+^d$

The main result regarding the bicommutant property in this setting involves applying the decomposition developed in Proposition 3.3.15. We can apply Theorem 5.1.1 recursively to each separate factor to obtain the following.

Theorem 5.2.1. *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ be a unital w*-dynamical system. Suppose that each $\alpha_{\mathbf{i}}$ is implemented by a uniformly bounded row operator $u_{\mathbf{i}}$. Then*

$$(\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d)' \simeq (\cdots((\mathcal{A}' \overline{\times}_{u_{\mathbf{1}}} \mathbb{Z}_+) \overline{\times}_{\widehat{u}_{\mathbf{2}}} \mathbb{Z}_+) \cdots) \overline{\times}_{\widehat{u}_{\mathbf{d}}} \mathbb{Z}_+$$

where $\widehat{u}_{\mathbf{i}} = u_{\mathbf{i}} \otimes^{(i-1)} I_{\ell^2}$ for $i = 2, \dots, d$.

Proof. We show the case when $d = 2$, the result then follows by iterating. By Proposition 3.3.15 we have that

$$\mathcal{A}\,\overline{\times}_\alpha\,\mathbb{Z}_+^2 = (\mathcal{A}\,\overline{\times}_{\alpha_1}\,\mathbb{Z}_+)\overline{\times}_{\widehat{\alpha_2}}\,\mathbb{Z}_+$$

where $\widehat{\alpha_2} = \alpha_2 \otimes I_{\ell^2(n))}$. By Theorem 5.1.1 we also have that

$$(\mathcal{A}\,\overline{\times}_{\alpha_1}\,\mathbb{Z}_+)' = \mathcal{A}'\,\overline{\times}_{u_1}\,\mathbb{Z}_+.$$

Hence,

$$\begin{aligned}(\mathcal{A}\,\overline{\times}_\alpha\,\mathbb{Z}_+^2)' &= ((\mathcal{A}\,\overline{\times}_{\alpha_1}\,\mathbb{Z}_+)\overline{\times}_{\widehat{\alpha_2}}\,\mathbb{Z}_+)' \\ &= (\mathcal{A}\,\overline{\times}_{\alpha_1}\,\mathbb{Z}_+)'\,\overline{\times}_{\widehat{u_2}}\,\mathbb{Z}_+ \\ &= \mathcal{A}'\,\overline{\times}_{u_1}\,\mathbb{Z}_+\overline{\times}_{\widehat{u_2}}\,\mathbb{Z}_+,\end{aligned}$$

where $\widehat{u_2} = u_2 \otimes I_{\ell^2(n)}$. ∎

Theorem 5.2.1 and Theorem 5.1.1 now imply the following corollary.

Corollary 5.2.2. *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ be a unital w*-dynamical system. Suppose that each $\alpha_{\mathbf{i}}$ is implemented by a uniformly bounded row operator $u_{\mathbf{i}}$. Then the following are equivalent*

(i) $\mathcal{A}$ has the bicommutant property;

(ii) $\mathcal{A}\,\overline{\times}_\alpha\,\mathbb{Z}_+^d$ has the bicommutant property;

(iii) $\mathcal{A} \otimes \mathbb{H}^\infty(\mathbb{Z}_+^d)$ has the bicommutant property.

If any of the items above hold then all algebras above are inverse closed.

Corollary 5.2.3. *(i) Let $(\mathcal{B}(\mathcal{H}), \alpha, \mathbb{Z}_+^d)$ be a w*-dynamical system such that each $\alpha_{\mathbf{i}}$ is implemented by a uniformly bounded row operator $u_{\mathbf{i}}$. Then the w*-semicrossed product $\mathcal{B}(\mathcal{H})\,\overline{\times}_\alpha\,\mathbb{Z}_+^d$ is inverse closed.*
(ii) Let $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ be an automorphic system over a maximal abelian selfadjoint algebra (m.a.s.a) $\mathcal{A}$. Then the w-semicrossed product $\mathcal{A}\,\overline{\times}_\alpha\,\mathbb{Z}_+^d$ is inverse closed.*

The proofs of each of these follows from the same reasoning as in the case for $\mathbb{F}_+^d$ and are omitted.

Chapter 6

Reflexivity of Semicrossed Products

The purpose of this section is to develop some results regarding reflexivity for each of the w*-semicrossed products that we have defined previously. Once again, we split our consideration to semicrossed products over $\mathbb{F}_+^d$ and those over $\mathbb{Z}_+^d$.

6.1 Semicrossed Products over $\mathbb{F}_+^d$

Let $(\mathcal{B}(\mathcal{H}), \{\alpha_i\}_{i\in[d]})$ be a unital w*-dynamical system of a uniformly bounded spatial action such that each α_i is implemented by

$$u_i = [u_{i,j_i}]_{j_i \in [n_i]}.$$

We shall obtain our reflexivity results by showing that $\mathcal{B}(\mathcal{H}) \overline{\times}_\alpha \mathcal{L}_d$ is similar to $\mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_N$ for $N = \sum_i n_i$, where N is the capacity of the system. To this end we define the operator

$$U \colon \mathcal{H} \otimes \ell^2(\mathbb{F}_+^N) \to \mathcal{H} \otimes \ell^2(\mathbb{F}_+^d),$$

by $U(\xi \otimes e_\emptyset) = \xi \otimes e_\emptyset$ and,

$$U(\xi \otimes e_{(\mu_k,j_k)\dots(\mu_1,j_1)}) = (u_{\mu_1,j_1} \cdots u_{\mu_k,j_k}\xi) \otimes e_{\mu_k\dots\mu_1}.$$

For words of length k we can define

$$\mathcal{K}_k := \overline{\mathrm{span}}\{\xi \otimes e_{(\mu_k,j_k)\dots(\mu_1,j_1)} \mid \xi \in \mathcal{H}, (\mu_i, j_i) \in ([d], [n_{\mu_i}])\}.$$

By construction $U|_{\mathcal{K}_k} = \bigoplus\limits_{|\mu|=k} \widehat{u_\mu}$ and so

$$\|U|_{\mathcal{K}_k}\| \leq \sup_{|\mu|=k} \|\widehat{u_\mu}\| = \left\| u_{\mu_1} \cdot (u_{\mu_2} \otimes I_{[n_{\mu_1}]}) \cdots (u_{\mu_k} \otimes I_{[n_{\mu_1} \cdots n_{\mu_{k-1}}]}) \right\| \leq K,$$

where K is the uniform bound for $\{u_i\}_{i\in[d]}$. Additionally the ranges of $\mathcal{K}_k$ under U are orthogonal and so $\|U\| = \sup\limits_{|\mu|=k} \|U|_{\mathcal{K}_k}\| \leq K$, therefore U is bounded. Also note that U is invertible with

$$U^{-1} \colon \mathcal{H} \otimes \ell^2(\mathbb{F}_+^d) \to \mathcal{H} \otimes \ell^2(\mathbb{F}_+^N)$$

given by $U^{-1}(\xi \otimes e_\emptyset) = \xi \otimes e_\emptyset$ and

$$U^{-1}(\xi \otimes e_{\mu_k\ldots\mu_1}) = \left(\sum_{j_1 \in [n_{\mu_1}]} \cdots \sum_{j_k \in [n_{\mu_k}]} v_{\mu_k,j_k} \cdots v_{\mu_1,j_1} \xi \right) \otimes e_{(\mu_k,j_k)\ldots(\mu_1,j_1)}$$

where v_i is the inverse of u_i. We can see this since

$$\begin{aligned} UU^{-1}\xi \otimes e_{\mu_k\ldots\mu_1} &= U \left(\sum_{j_1 \in [n_{\mu_1}]} \cdots \sum_{j_k \in [n_{\mu_k}]} v_{\mu_k,j_k} \cdots v_{\mu_1,j_1} \xi \otimes e_{(\mu_k,j_k)\ldots(\mu_1,j_1)} \right) \\ &= (u_{\mu_1,j_1} \cdots u_{\mu_k,j_k}) \left(\sum_{j_1 \in [n_{\mu_1}]} \cdots \sum_{j_k \in [n_{\mu_k}]} v_{\mu_k,j_k} \cdots v_{\mu_1,j_1} \xi e_{\mu_k\ldots\mu_1} \right). \end{aligned}$$

We can see that each term cancels here and thus we have that $UU^{-1} = I$. Similarly,

$$\begin{aligned} U^{-1}U\xi \otimes e_{(\mu_k,j_k)\ldots(\mu_1,j_1)} &= U^{-1}(u_{\mu_1,j_1} \cdots u_{\mu_k,j_k} \xi \otimes e_{\mu_k\ldots\mu_1}) \\ &= \left(\sum_{j_1 \in [n_{\mu_1}]} \cdots \sum_{j_k \in [n_{\mu_k}]} v_{\mu_k,j_k} \cdots v_{\mu_1,j_1} \right) (u_{\mu_1,j_1} \cdots u_{\mu_k,j_k}) \xi \otimes e_{(\mu_k,j_k)\ldots(\mu_1,j_1)}. \end{aligned}$$

Again, each term passes through the sum and cancels and thus, U^{-1} is indeed the inverse of U.

Theorem 6.1.1. *Let $(\mathcal{B}(\mathcal{H}), \{\alpha_i\}_{i\in[d]})$ be a w*-dynamical system of a uniformly bounded spatial action. Suppose that every α_i is given by an invertible row operator $u_i = [u_{i,j_i}]_{j_i \in [n_i]}$ and set $N = \sum_{i\in[d]} n_i$. Then the w*-semicrossed product $\mathcal{B}(\mathcal{H}) \overline{\times}_\alpha \mathcal{L}_d$ is similar to $\mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_N$.*

Proof. We will show that U as constructed above yields the required similarity. Recall that $\alpha_{\mu_i}(x)u_{\mu_i,j_i} = u_{\mu_i,j_i}x$. Therefore applying for $x \in \mathcal{B}(\mathcal{H})$ we have

$$\begin{aligned}\overline{\pi}(x)U\xi \otimes e_{(\mu_k,j_k)\dots(\mu_1,j_1)} &= \alpha_{\mu_1}\cdots\alpha_{\mu_k}(x)u_{\mu_1,j_1}\cdots u_{\mu_k,j_k}\xi \otimes e_{\mu_k\dots\mu_1}\\ &= u_{\mu_1,j_1}\cdots u_{\mu_k,j_k}x\xi \otimes e_{\mu_k\dots\mu_1}\\ &= U\rho(x)\xi \otimes e_{(\mu_k,j_k)\dots(\mu_1,j_1)}.\end{aligned}$$

On the other hand we have that

$$\begin{aligned}L_iU\xi \otimes e_{(\mu_k,j_k)\dots(\mu_1,j_1)} &= L_iu_{\mu_1,j_1}\cdots u_{\mu_k,j_k}\xi \otimes e_{\mu_k\dots\mu_1}\\ &= u_{\mu_1,j_1}\cdots u_{\mu_k,j_k}\xi \otimes e_{i\mu_k\dots\mu_1}.\end{aligned}$$

Now applying on the second generator we have that

$$\begin{aligned}&U\sum_{j_i\in[n_i]} L_{i,j_i}\rho(v_{i,j_i})\xi \otimes e_{(\mu_k,j_k)\dots(\mu_1,j_1)} =\\ &\qquad = U\sum_{j_i\in[n_i]} v_{i,j_i}\xi \otimes e_{(i,j_i)(\mu_k,j_k)\dots(\mu_1,j_1)}\\ &\qquad = \sum_{j_i\in[n_i]} u_{\mu_1,j_1}\dots u_{\mu_k,j_k}u_{i,j_i}v_{i,j_i}\xi \otimes e_{i\mu_k\dots\mu_1}\\ &\qquad = u_{\mu_1,j_1}\dots u_{\mu_k,j_k}\xi \otimes e_{i\mu_k\dots\mu_1}\end{aligned}$$

since $\sum_{j_i\in[n_i]} u_{i,j_i}v_{i,j_i} = I$. Hence we have

$$U^{-1}L_iU = \sum_{j_i\in[n_i]} L_{i,j_i}\rho(v_{i,j_i}) \text{ for all } i \in [d].$$

Therefore the generators of $\mathcal{B}(\mathcal{H}) \overline{\times}_\alpha \mathcal{L}_d$ are mapped into $\mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_N$. To complete the proof we need to show that the elements $\rho(x)$ and $U^{-1}L_iU$ also generate

$$L_{i,j_i} \text{ for all } (i,j_i) \in ([d],[n_i]).$$

Since every u_{i,j_i} is in $\mathcal{B}(\mathcal{H})$ we have that

$$U^{-1}L_iU\rho(u_{i,j_i}) = \sum_{j_i'\in[n_i]} L_{i,j_i'}\rho(v_{i,j_i'})\rho(u_{i,j_i}) = L_{i,j_i}$$

as required. ∎

Theorem 6.1.2. *Let $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ be a w*-dynamical system of a uniformly bounded spatial action. Suppose that every α_i is given by an invertible row operator $u_i = [u_{i,j_i}]_{j_i\in[n_i]}$ and set $N = \sum_{i\in[d]} n_i$.*

(i) If $N \geq 2$ then every w-closed subspace of $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ or $\mathcal{A} \overline{\times}_\alpha \mathcal{R}_d$ is hyperreflexive. If K is the uniform bound related to $\{u_i\}$ then the hyperreflexivity constant is at most $3 \cdot K^4$.*

(ii) If $N = 1$ and $\mathcal{A}$ is reflexive then $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d = \mathcal{A} \overline{\times}_\alpha \mathcal{R}_d = \mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+$ is reflexive.

Proof. We remarked previously in Section 3.3.1 that since every α_i implemented by an invertible row operator u_i can be extended to all of $\mathcal{B}(\mathcal{H})$ we have that $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ extends to $(\mathcal{B}(\mathcal{H}), \{\alpha_i\}_{i\in[d]})$. Therefore

$$\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d \subseteq \mathcal{B}(\mathcal{H}) \overline{\times}_\alpha \mathcal{L}_d \simeq \mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_N$$

by Theorem 6.1.1. If $N \geq 2$ then every w*-closed subspace of $\mathcal{B}(\mathcal{H}) \otimes \mathcal{L}_N$ is hyperreflexive with distance constant at most 3 by [9]. As hyperreflexivity is preserved under taking similarities, by Corollary 2.4.7 we have that the hyperreflexivity constant is $3 \cdot K^4$ and the proof of item (i) is complete. Part (ii) is shown in [30]. ∎

Corollary 6.1.3. *Let $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ be a w*-dynamical system so that every α_i is given by a Cuntz family $[s_{i,j_i}]_{j_i\in[n_i]}$. If $N = \sum_{i\in[d]} n_i \geq 2$ then every w*-closed subspace of $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ or $\mathcal{A} \overline{\times}_\alpha \mathcal{R}_d$ is hyperreflexive with distance constant at most 3.*

Proof. If $d \geq 2$ then choose $W_{1,1}$ and $W_{2,1}$. If $d = 1$ then $n_1 \geq 2$ and choose $W_{1,1}$ and $W_{1,2}$. In both cases these are isometries with orthogonal ranges, in the commutant of $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ by Theorem 5.1.1, and we can apply Bercovici's result [9] to get the constant 3. ∎

Corollary 6.1.4. *Let $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ be a system of w*-continuous automorphisms on a maximal abelian selfadjoint algebra (m.a.s.a) $\mathcal{A}$. Then $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ and $\mathcal{A} \overline{\times}_\alpha \mathcal{R}_d$ are reflexive.*

The reflexivity results discussed above can be extended to systems over any factor. In [27] arguments were developed which covered the cases for Type II and Type III factors. We now follow the arguments of Helmer in [27] and treat dynamical systems over Type II and Type III factors. Again, we treat cases of dynamical systems over both $\mathbb{F}_+^d$ and $\mathbb{Z}_+^d$. We begin with the following definitions.

Definition 6.1.5. An algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is *injectively reducible* if there is a non-trivial reducing subspace M of $\mathcal{A}$ such that the representations

$$a \mapsto a|_M \quad \text{and} \quad a \mapsto a|_{M^\perp}$$

are both injective.

Definition 6.1.6. A w*-dynamical system $(\mathcal{A}, \{\alpha_i\}_{i\in[d]})$ is *injectively reflexive* if:

(i) $\mathcal{A}$ is reflexive.

(ii) $\mathcal{A}$ is injectively reducible by some M.

(iii) $\beta_\nu(\mathcal{A})$ is reflexive for all $\nu \in \mathbb{F}_+^d$ with

$$\beta_\nu(a) = \begin{bmatrix} a|_M & 0 \\ 0 & \alpha_\nu(a)|_{M^\perp} \end{bmatrix}.$$

Recall that the m-th Fourier coefficient is given by

$$G_m(T) := \frac{1}{2\pi}\int_{-\pi}^{\pi} U_s T U_s^* e^{-ims}\,\mathrm{ds},$$

Also, we have seen that if $T \in \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{F}_+^d))$ then

$$G_m(T) = \begin{cases} \sum_{|\mu|=m} \sum_{w\in\mathbb{F}_+^d} L_\mu(T_{\mu w,w} \otimes p_w) & \text{if } m \geq 0, \\ 0 & \text{if } m < 0, \end{cases}$$

Thus we have the following.

Theorem 6.1.7. *Let $(\mathcal{A}, \alpha, \mathbb{F}_+^d)$ be a unital w*-dynamical system. If $\mathcal{A}$ is injectively reflexive then the semicrossed products $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ and $\mathcal{A} \overline{\times}_\alpha \mathcal{R}_d$ are reflexive.*

Proof. Here, we show the left version, the right is provided in [8]. The crux of the argument is a translation from the language of w*-correspondences in [27].
Fix $T \in \mathrm{Ref}(\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d)$ and without loss of generality assume that $T = G_m(T)$. If $m < 0$ then $G_m(T) = 0$. If $m \geq 0$ then $T_{\mu,\emptyset} \in \mathcal{A}$ and it suffices to show that $T_{\mu\nu,\nu} = \alpha_{\overline{\nu}}(T_{\mu,\emptyset})$.

By assumption let $\mathcal{H}_0$ and $\mathcal{H}_1 = \mathcal{H}_0^\perp$ be the subspaces that injectively reduce $\mathcal{A}$. Now fix a word $\nu \in \mathbb{F}_+^d$ and define the subspace

$$E = \{\xi \otimes e_w + \eta \otimes e_{w\nu} : \xi \in \mathcal{H}_0,\ \eta \in \mathcal{H}_1,\ w \in \mathbb{F}_+^d\}$$

of $\mathcal{K} = \mathcal{H} \otimes \ell^2(\mathbb{F}_+^d)$. It is clear that E is an invariant subspaces of $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ since

$$\overline{\pi}(a)(\xi \otimes e_w + \eta \otimes e_{w\nu}) = \alpha_{\overline{w}}(a)\xi \otimes e_w + \alpha_{\overline{w\nu}}(a)\eta \otimes e_{w\nu} \in E$$

and,

$$L_i(\xi \otimes e_w + \eta \otimes e_{w\nu}) = \xi \otimes e_{iw} + \eta \otimes e_{iw\nu} \in E.$$

If p is the projection on E, since this is invariant for $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ we have that

$$G_m(T)p \in \mathrm{Ref}((\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d)p).$$

Now, define the unitary

$$U_\nu : E \to \mathcal{H} \otimes \ell^2(\mathbb{F}_+^d) : \xi \otimes e_w + \eta \otimes e_{w\nu} \mapsto (\xi + \eta) \otimes e_w.$$

Performing the following computations we have that

$$\begin{aligned} U_\nu \overline{\pi}(a) p U_\nu^* k &= U_\nu \overline{\pi}(a) p(\xi \otimes e_w + \eta \otimes e_{w\nu}) \\ &= U_\nu(\alpha_{\overline{w}}(a)|_{\mathcal{H}_0}\xi \otimes e_w + \alpha_{\overline{(w\nu)}}(a)|_{\mathcal{H}_1})\xi \otimes e_{w\nu} \\ &= \sum_{w \in \mathbb{F}_+^d} (\alpha_{\overline{w}}(a)|_{\mathcal{H}_0} + \alpha_{\overline{(w\nu)}}(a)|_{\mathcal{H}_1})\xi \otimes e_w. \end{aligned}$$

and similarly that

$$\begin{aligned} U_\nu L_i p U_\nu^* k &= U_\nu L_i p(\xi \otimes e_w + \eta \otimes e_{w\nu}) \\ &= U_\nu(\xi \otimes e_{iw} + \eta \otimes e_{iw\nu}) \\ &= (\xi + \eta)\xi \otimes e_{iw}. \end{aligned}$$

Hence we have that

$$U_\nu \overline{\pi}(a) p U_\nu^* = \sum_{w \in \mathbb{F}_+^d} (\alpha_{\overline{w}}(a)|_{\mathcal{H}_0} + \alpha_{\overline{(w\nu)}}(a)|_{\mathcal{H}_1}) \otimes p_w$$

and

$$U_\nu L_i p U_\nu^* = L_i.$$

Moreover,

$$U_\nu T p U_\nu^* \sum_{|\mu|=m} \sum_{w \in \mathbb{F}_+^d} L_\mu (T_{\mu w, w}|_{\mathcal{H}_0} + T_{\nu\mu w, w\nu}|_{\mathcal{H}_1}) \otimes p_w.$$

Taking compressions, we have that the $(\mu, \emptyset)$-entry of $U_\nu T p U_\nu^*$ is in the reflexive cover of the $(\mu, \emptyset)$-block of the algebra $\operatorname{Ref}(U_\nu(\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d) p U_\nu^*)$. However the latter coincides with the reflexive cover of, and hence equals

$$\beta_{\overline{\nu}}(\mathcal{A}) = \left\{ \begin{bmatrix} a|_{\mathcal{H}_0} & 0 \\ 0 & \alpha_{\overline{\nu}}(a)|_{\mathcal{H}_1} \end{bmatrix} \mid a \in \mathcal{A} \right\}.$$

This follows since

$$U_\nu (L_\mu \overline{\pi}(a))_{(\mu,\emptyset)} U_\nu^* k = U_\nu (L_\mu \overline{\pi}(a))_{(\mu,\emptyset)} (\xi \otimes e_w + \eta \otimes e_{w\nu}) = (a|_{\mathcal{H}_0} + \alpha_{\overline{\nu}}(a)|_{\mathcal{H}_1}) \otimes e_{\mu w}.$$

Therefore there exists an $a \in \mathcal{A}$ such that

$$T_{\mu,\emptyset}|_{\mathcal{H}_0} + T_{\mu\nu,\nu}|_{\mathcal{H}_1} = a|_{\mathcal{H}_0} + \alpha_{\overline{\nu}}(a)|_{\mathcal{H}_1}.$$

Consequently we have that $T_{\mu,\emptyset}|_{\mathcal{H}_0} = a|_{\mathcal{H}_0}$ and $T_{\mu\nu,\nu}|_{\mathcal{H}_1} = \alpha_\nu(a)|_{\mathcal{H}_1}$. Since the restrictions to $\mathcal{H}_0$ and $\mathcal{H}_1$ are injective we derive that

$$T_{\mu,\emptyset} = a \quad \text{and} \quad T_{\mu\nu,\nu} = \alpha_{\overline{\nu}}(a) = \alpha_{\overline{\nu}}(T_{\mu,\emptyset})$$

which completes the proof. ∎

From this we obtain the following corollaries.

Corollary 6.1.8. *Let $(\mathcal{A}, \{\alpha_i\}_{i \in [d]})$ be a unital w^*-dynamical system on a factor $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$. Then $\mathcal{A} \overline{\times}_\alpha \mathcal{L}_d$ and $\mathcal{A} \overline{\times}_\alpha \mathcal{R}_d$ are reflexive.*

Corollary 6.1.9. *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+)$ be a unital w^*-dynamical system on a factor $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$. Then $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+$ is reflexive.*

6.2 Semicrossed Products over $\mathbb{Z}_+^d$

Again, we develop similar results to the previous section for semicrossed products over $\mathbb{Z}_+^d$. So, consider the dynamical system $(\mathcal{A}, \{\alpha_i\}_{i\in[d]}, \mathbb{Z}_+^d)$ where each α_i is implemented by an invertible row operator $u_i = [u_{i,j_i}]_{j_i\in[n_i]}$ where we write $M = \prod_{i\in[d]} n_i$ for the capacity of the system. Then we have the following theorem.

Theorem 6.2.1. *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ be a unital w*-dynamical system. Suppose that every $\alpha_{\mathbf{i}}$ is uniformly bounded spatial, given by an invertible row operator $u_{\mathbf{i}} = [u_{i,j_i}]_{j_i\in[n_i]}$, and set $M = \prod_{i\in[d]} n_i$.*

(i) If $M \geq 2$ then every w-closed subspace of $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ is hyperreflexive. If K_i is the uniform bound associated to $u_{\mathbf{i}}$ (and its inverse) then the hyperreflexivity constant is at most $3 \cdot K^4$ for $K = \min\{K_i \mid n_i \geq 2\}$.*

(ii) If $M = 1$ and $\mathcal{A}$ is reflexive then $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ is reflexive.

Proof. For item (i), suppose without loss of generality that $n_d \geq 2$ with $K_d = \min\{K_i \mid n_i \geq 2\}$. Then we can apply Proposition 3.3.15 and write $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d \simeq \mathcal{B} \overline{\times}_{\widehat{\alpha}_{\mathbf{d}}} \mathbb{Z}_+$ for an appropriate w*-closed algebra $\mathcal{B}$. Hence we can therefore apply Theorem 6.1.2 for the system $(\mathcal{B}, \widehat{\alpha}_{\mathbf{d}}, \mathbb{Z}_+)$, as its capacity is greater than 2.
For part (ii) we can again apply Proposition 3.3.15 and write $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ as successive w*-semicrossed products. We can then apply Theorem 6.1.2(ii) recursively to each factor. ∎

Corollary 6.2.2. *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ be a unital w*-dynamical system. Suppose that at least one $\alpha_{\mathbf{i}}$ is implemented by a Cuntz family $[s_{i,j_i}]_{j_i\in[n_i]}$ with $n_i \geq 2$. Then every w*-closed subspace of $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ is hyperreflexive with distance constant 3.*

Proof. Suppose without loss of generality that $\alpha_{\mathbf{d}}$ is defined by a Cuntz family with $n_{\mathbf{d}} \geq 2$. Then $\widehat{\alpha}_{\mathbf{d}}$ is also given by the Cuntz family $\{s_j \otimes^{d-1} I\}$ of size $n_{\mathbf{d}}$. By Proposition 3.3.15 we can write $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d \simeq \mathcal{B} \overline{\times}_{\widehat{\alpha}_{\mathbf{d}}} \mathbb{Z}_+$ for some w*-closed algebra $\mathcal{B}$. Applying then Corollary 6.1.3 completes the proof. ∎

This also shows that $\mathcal{B}(\mathcal{H}) \otimes \mathbb{H}^\infty(\mathbb{Z}_+^d)$ is reflexive.

Corollary 6.2.3. *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ be a unital automorphic system over a maximal abelian selfadjoint algebra $\mathcal{A}$. Then $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ is reflexive.*

In a similar manner to the previous section we can also define injectively reflexive systems in the $\mathbb{Z}_+^d$ case.

Definition 6.2.4. A w*-dynamical system $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ is *injectively reflexive* if

(i) $\mathcal{A}$ is reflexive.

(ii) $\mathcal{A}$ is injectively reducible by M.

(iii) $\beta_{\underline{n}}(\mathcal{A})$ is reflexive for all $\underline{n} \in \mathbb{Z}_+^d$ with

$$\beta_{\underline{n}}(a) = \begin{bmatrix} a|_M & 0 \\ 0 & \alpha_{\underline{n}}(a)|_{M^\perp} \end{bmatrix}.$$

In analogy to the $\mathbb{F}_+^d$ case we have the following theorem.

Theorem 6.2.5. *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ be a unital w*-dynamical system. If the system is injectively reflexive then $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ is reflexive.*

Proof. The proof follows in a similar manner to Theorem 6.1.7. If T is in $\mathrm{Ref}(\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d)$ then T is lower triangular and $T_{\underline{m},\underline{0}} \in \mathcal{A}$ for every $\underline{m} \in \mathbb{Z}_+^d$. Thus we need to show that $T_{\underline{m}+\underline{n},\underline{n}} = \alpha_{\underline{n}}(T_{\underline{m},\underline{0}})$ for every $\underline{n} \in \mathbb{Z}_+^d$. Let $M, M^\perp$ be the subspaces that injectively reduce $\mathcal{A}$. For a fixed $\underline{n}$ define the space

$$E = \{\xi \otimes e_{\underline{w}} + \eta \otimes e_{\underline{n}+\underline{w}} : \xi \in M,\ \eta \in M^\perp,\ \underline{w} \in \mathbb{Z}_+^d\}.$$

It is clear that E is an invariant subspace of $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ since

$$\pi(a)(\xi \otimes e_{\underline{w}} + \eta \otimes e_{\underline{n}+\underline{w}}) = \alpha_{\underline{w}}(a)\xi \otimes e_{\underline{w}} + \alpha_{\underline{n}+\underline{w}}(a)\eta \otimes e_{\underline{n}+\underline{w}} \in E$$

and,

$$L_{\underline{m}}(\xi \otimes e_{\underline{w}} + \eta \otimes e_{\underline{n}+\underline{w}}) = \xi \otimes e_{\underline{m}+\underline{w}} + \eta \otimes e_{\underline{m}+\underline{n}+\underline{w}} \in E.$$

Let p be the projection onto E, since this is invariant for $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ we have that $G_{\underline{m}}(T)p \in \mathrm{Ref}((\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d)p)$. Let the unitary

$$U \colon E \to \mathcal{H} \otimes \ell^2(\mathbb{Z}_+^d) \quad \text{with} \quad U(\xi \otimes e_{\underline{w}} + \eta \otimes e_{\underline{n}+\underline{w}}) = (\xi + \eta) \otimes e_{\underline{w}}.$$

Performing the following computations we have that

$$\begin{aligned} UG_{\underline{m}}(T)pU^*(\xi+\eta)\otimes e_{\underline{w}} &= UG_{\underline{m}}(T)(\xi\otimes e_{\underline{w}}+\eta\otimes e_{\underline{n}+\underline{w}}) \\ &= U(L_{\underline{m}}\sum_{\underline{w}'\in\mathbb{Z}_+^d} T_{\underline{m}+\underline{w}',\underline{m}}\otimes p_{\underline{w}'})(\xi\otimes e_{\underline{w}}+\eta\otimes e_{\underline{n}+\underline{w}}) \\ &= U[(T_{\underline{m}+\underline{w},\underline{m}}\xi\otimes e_{\underline{m}+\underline{w}})+(T_{\underline{m}+\underline{n}+\underline{w},\underline{n}+\underline{w}}\eta\otimes e_{\underline{m}+\underline{n}+\underline{w}})] \\ &= ((T_{\underline{m}+\underline{w},\underline{m}}\xi+T_{\underline{m}+\underline{n}+\underline{w},\underline{n}+\underline{w}}\eta)\otimes e_{\underline{m}+\underline{w}}. \end{aligned}$$

On the other hand,

$$L_{\underline{m}}[\sum_{\underline{w}'\in\mathbb{Z}_+^d}(T_{\underline{m}+\underline{w}',\underline{m}}|_M+T_{\underline{m}+\underline{n}+\underline{w},\underline{n}+\underline{w}}|M^{\perp})\otimes p_{\underline{w}'}][(\xi+\eta)\otimes e_{\underline{w}}] =$$

$$\begin{aligned} &= L_{\underline{m}}[(T_{\underline{m}+\underline{w}',\underline{m}}|_M+T_{\underline{m}+\underline{n}+\underline{w},\underline{n}+\underline{w}}|M^{\perp})][(\xi+\eta)\otimes e_{\underline{w}}] \\ &= [T_{\underline{m}+\underline{w}',\underline{m}}|_M+T_{\underline{m}+\underline{n}+\underline{w},\underline{n}+\underline{w}}|M^{\perp}](\xi+\eta)\otimes e_{\underline{m}+\underline{w}} \\ &= (T_{\underline{m}+\underline{w},\underline{m}}\xi+T_{\underline{m}+\underline{n}+\underline{w},\underline{n}+\underline{w}}\eta)\otimes e_{\underline{m}+\underline{w}}. \end{aligned}$$

Therefore we see that

$$UG_{\underline{m}}(T)pU^* = L_{\underline{m}}\sum_{\underline{w}\in\mathbb{Z}_+^d}(T_{\underline{m}+\underline{w},\underline{w}}|_M+T_{\underline{n}+\underline{m}+\underline{w},\underline{n}+\underline{w}}|_{M^\perp})\otimes p_{\underline{w}}.$$

Indeed,

$$U\pi(a)pU^* = \sum_{\underline{w}\in\mathbb{Z}_+^d}(\alpha_{\underline{w}}(a)|_M+\alpha_{\underline{n}+\underline{w}}(a)|_{M^\perp})\otimes p_{\underline{w}}.$$

and,

$$UL_{\mathbf{i}}pU^*k = (\xi+\eta)\otimes e_{\mathbf{i}+w},$$

so $UL_{\mathbf{i}}pU^* = L_{\mathbf{i}}$.
Taking compressions to the $(\underline{m},0)$-block of $\mathcal{A}\overline{\times}_\alpha\mathbb{Z}_+^d$ we have that the $(\underline{m},0)$-entry of $UG_{\underline{m}}(T)U^*$ is in the reflexive cover of the $(\underline{m},0)$-block of the algebra $\mathrm{Ref}(\mathcal{A}\overline{\times}_\alpha\mathbb{Z}_+^d)$. However the latter coincides with the reflexive cover of $\beta_{\underline{n}}(\mathcal{A})$ and hence equals $\beta_{\underline{n}}(\mathcal{A})$. Therefore there exists an $a\in\mathcal{A}$ such that

$$T_{\underline{m},0}|_M+T_{\underline{n}+\underline{m},\underline{n}}|_{M^\perp} = a|_M+\alpha_{\underline{n}+\underline{m}}(a)|_{M^\perp}.$$

Therefore $T_{\underline{m}+\underline{n},\underline{n}} = \alpha_{\underline{n}}(a) = \alpha_{\underline{n}}(T_{\underline{m},\underline{0}})$ and the proof is complete. ∎

Remark 6.2.6. Type I factors are hyperreflexive. For the capacity $N = 1$ this is shown in [30], when the capacity $N > 1$ this is shown in [8]. In [27, Corollary 3.2] Helmer establishes that Type III factors have infinite multiplicity and therefore, by invoking a result of Davidson and Pitts in [22], Type III factors are hyperreflexive.

We now end by noting we have reflexivity for w*-semicrossed products of factors in this case also.

Corollary 6.2.7. *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+^d)$ be a unital w*-dynamical system on a factor $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$. Then $\mathcal{A} \overline{\times}_\alpha \mathbb{Z}_+^d$ is reflexive.*

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List of Symbols

Index