The Construction of Rational Tetra-Inner Functions

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Abstract

The *tetrablock* is the set

$$\mathcal{E} = \{x \in \mathbb{C}^3 : \quad 1 - x_1 z - x_2 w + x_3 zw \neq 0 \quad \textit{whenever} \quad |z| \leq 1, |w| \leq 1\}.$$

The closure of $\mathcal{E}$ is denoted by $\overline{\mathcal{E}}$. A *tetra-inner* function is an analytic map x from the unit disc $\mathbb{D}$ to $\overline{\mathcal{E}}$ whose boundary values at almost all points of the unit circle $\mathbb{T}$ belong to the distinguished boundary $b\mathcal{E}$ of $\overline{\mathcal{E}}$. There is a natural notion of *degree* of a rational tetra-inner function x; it is simply the topological degree of the continuous map $x|_{\mathbb{T}}$ from $\mathbb{T}$ to $b\overline{\mathcal{E}}$.

In this thesis we give a prescription for the construction of a general rational tetra-inner function of degree n. The prescription makes use of a known solution of an interpolation problem for finite Blaschke products of given degree in terms of a Pick matrix formed from the interpolation data. Alsalhi and Lykova proved that if $x = (x_1, x_2, x_3)$ is a rational tetra-inner function of degree n, then $x_1 x_2 - x_3$ either is equal to 0 or has exactly n zeros in the closed unit disc $\overline{\mathbb{D}}$, counted with an appropriate notion of multiplicity. It turns out that a natural choice of data for the construction of a rational tetra-inner function $x = (x_1, x_2, x_3)$ consists of the points in $\overline{\mathbb{D}}$ for which $x_1 x_2 - x_3 = 0$ and the values of x at these points.

We also give a matricial formulation of a criterion for the solvability of a μ_{Diag}-*synthesis problem*. The symbol μ_{Diag} denotes an instance of the *structured singular value* of 2×2 matrix corresponding to the subspace of diagonal matrices in $M_{2\times 2}(\mathbb{C})$. Given distinct points $\lambda_1, ..., \lambda_n \in \mathbb{D}$ and target matrices $W_1, ..., W_n \in M_{2\times 2}(\mathbb{C})$ one seeks an analytic 2×2 matrix-valued function F on $\mathbb{D}$ such that

$$F(\lambda_j) = W_j \quad \text{for } j = 1, ..., n, \quad \text{and}$$

$$\mu_{\mathrm{Diag}}(F(\lambda)) \; < \; 1, \text{ for all } \lambda \in \mathbb{D}.$$

Declaration on collaborative work

My thesis contains collaborative work with my supervisors Dr. Z. A. Lykova and Prof. N. J. Young. Lykova and I are planning to write a joint paper. The main problems and ideas how to solve these problems were provided to me by Lykova. We have had weekly meetings to discuss mathematics, methods, new ideas and research papers related to my thesis. We have done research together.

The rest of each week I have worked independently on my thesis. I did calculations which were required in each step of proofs, searched for research material related to our research project, organised all research material in thesis. I have attended and given several talks on topics of my thesis to Young Functional Analysis Workshops in Glasgow, Newcastle and Leeds.

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Contents

Chapter 1

Introduction and historical remarks

1.1 Introduction

The unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in $\mathbb{C}$ will be denoted by $\mathbb{T}$, the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ will be denoted by $\mathbb{D}$, the closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$ will be denoted by $\overline{\mathbb{D}}$, and $M_{m\times n}(\mathbb{C})$ will be the set of complex $m\times n$ matrices. The symmetrized bidisc Γ and the tetrablock $\overline{\mathcal{E}}$ have attracted considerable interest in recent years [1, 3, 2]. The symmetrized bidisc Γ is a domain in $\mathbb{C}^2$ defined as

$$\Gamma = \{(z + w, zw) : |z| \leq 1, |w| \leq 1\} \subset \mathbb{C}^2.$$

and the tetrablock $\overline{\mathcal{E}}$ is a domain in $\mathbb{C}^3$ defined as

$$\overline{\mathcal{E}} = \{x_1, x_2, x_3 \in \mathbb{C}^3 :\ 1 - x_1 z - x_2 w + x_3 zw \neq 0 \text{ whenever } |z| < 1, |w| < 1\}.$$

An $\overline{\mathcal{E}}$*-inner function* is an analytic map x from the unit disc $\mathbb{D}$ to $\overline{\mathcal{E}}$ whose boundary values at almost all points of the unit circle $\mathbb{T}$ belong to the distinguished boundary of $\overline{\mathcal{E}}$. The degree of $x = (x_1, x_2, x_3)$ is defined to be the topological degree of $x|_{\mathbb{T}}$ as a continuous map from $\mathbb{T}$ to the distinguished boundary of $\overline{\mathcal{E}}$. It was known to Nevanlinna and Pick that an n-point interpolation problem for functions in the Schur class is solvable if and only if it is solvable by a rational inner function of degree at most n. We shall consider the analogue for rational tetra-inner functions of a problem about rational inner functions φ from $\mathbb{D}$ to $\overline{\mathbb{D}}$ solved by W. Blaschke [19]. By the

Argument Principle, a rational inner function φ of degree n has exactly n zeros in $\mathbb{D}$, counted with multiplicitity. From this fact one deduces that φ is a finite Blaschke product

$$\varphi(\lambda) = c\prod_{j=1}^{n} \frac{\lambda - \alpha_j}{1 - \overline{\alpha_j}\lambda}$$

where $|c| = 1$ and $\alpha_1, ..., \alpha_n$ are the zeros of φ. In a similar way, we would like to write down the general rational $\overline{\mathcal{E}}$-inner function of degree n. It was shown in [12] that if $x = (x_1, x_2, x_3)$ is a rational $\overline{\mathcal{E}}$-inner function of degree n, then $x_1x_2 - x_3$ has exactly n zeros in the closed unit disc $\overline{\mathbb{D}}$, counted with multiplicity.

The royal variety

$$\mathcal{R}_{\bar{\mathcal{E}}} = \{(x_1, x_2, x_3) \in \overline{\mathcal{E}} : x_3 = x_1x_2\}$$

plays a special role in the function theory of $\overline{\mathcal{E}}$. For a rational $\overline{\mathcal{E}}$-inner function $x = (x_1, x_2, x_3)$, the zeros of $x_1x_2 - x_3$ in $\overline{\mathbb{D}}$ are the points $\lambda \in \overline{\mathbb{D}}$ such that $x(\lambda) \in \mathcal{R}_{\bar{\mathcal{E}}}$. We call them the royal nodes of x. If $\sigma \in \overline{\mathbb{D}}$ is a royal node of x, so that $x(\sigma) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$ for some $\eta, \tilde{\eta} \in \overline{\mathbb{D}}$, then we call $\eta, \tilde{\eta}$ the royal values of x corresponding to the royal nodes of x. In this thesis we give a prescription for the construction of a general rational tetra-inner function of degree n with the aid of a solution of an interpolation problem for finite Blaschke products (Theorem 1.2.7). The data for the construction of a rational tetra-inner function x consists of the royal nodes and royal values of x.

1.2 Main results

To describe our main results (Theorems 4.1.1, 4.2.5, 1.2.7) on the construction of a general rational tetra-inner function we need to recall some definitions and results on the Blaschke interpolation problem.

Definition 1.2.1. [6, Definition 1.2] *Let* $n \geq 1$ *and* $0 \leq k \leq n$. *By* Blaschke interpolation data *we mean a triple* (σ, η, ρ) *where*

(i) $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ *is an n-tuple of distinct points of* $\overline{\mathbb{D}}$ *such that* $\sigma_j \in \mathbb{T}$ *for* $j = 1, ..., k$ *and* $\sigma_j \in \mathbb{D}$ *for* $j = k + 1, ..., n$;

(ii) *$\eta = (\eta_1, \eta_2, ..., \eta_n)$ where $\eta_j \in \mathbb{T}$ for $j = 1, ..., k$ and $\eta_j \in \mathbb{D}$ for $j = k+1, ..., n$;*

(ii) *$\rho = (\rho_1, \rho_2, ..., \rho_k)$ where $\rho_j > 0$ for $j = 1, ..., k$.*

Problem 1.2.2. (The Blaschke interpolation problem) *For given Blaschke interpolation data (σ, η, ρ), find if possible a rational inner function φ on $\mathbb{D}$ (that is, a finite Blaschke product) of degree n with the properties*

$$\varphi(\sigma_j) = \eta_j \quad \text{for } j = 1, ..., n \tag{1.1}$$

and

$$A\varphi(\sigma_j) = \rho_j \quad \text{for } j = 1, ..., k \tag{1.2}$$

where $A\varphi(e^{i\theta})$ denotes the rate of change of the argument of $\varphi(e^{i\theta})$ with respect to θ.

There is a criterion for the existence of a solution of the Blaschke interpolation problem, Problem 1.2.2, in terms of an associated "Pick matrix", and there is a parametrization of all solutions φ by a linear fractional expression in terms of a parameter $\zeta \in \mathbb{T}$. There are polynomials a, b, c and d of degree at most n such that the general solution of Problem 1.2.2 is

$$\varphi = \frac{a\zeta + b}{c\zeta + d}$$

where the parameter ζ ranges over a cofinite subset of $\mathbb{T}$.

Definition 1.2.3. *Let $n \geqslant 1$, and $0 \leq k \leq n$. By* royal tetra-interpolation data *with n nodes and k boundary nodes we mean a four-tuple $(\sigma, \eta, \tilde{\eta}, \rho)$ where*

(i) *$\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ is an n-tuple of distinct points such that $\sigma_j \in \mathbb{T}$ for $j = 1, ..., k$ and $\sigma_j \in \mathbb{D}$ for $j = k+1, ..., n$;*

(ii) *$\eta = (\eta_1, \eta_2, ..., \eta_n)$ where $\eta_j \in \mathbb{T}$ for $j = 1, ..., k$ and $\eta_j \in \mathbb{D}$ for $j = k+1, ..., n$;*

(iii) *$\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2, ..., \tilde{\eta}_n)$ where $\tilde{\eta}_j \in \mathbb{T}$ for $j = 1, ..., k$ and $\tilde{\eta}_j \in \mathbb{D}$ for $j = k+1, ..., n$.*

(iv) *$\rho = (\rho_1, \rho_2, ..., \rho_k)$ where $\rho_j > 0$ for $j = 1, ..., k$.*

Problem 1.2.4. (The royal tetra-interpolation problem) *Given royal tetra-interpolation data* $(\sigma, \eta, \tilde{\eta}, \rho)$*, find if possible a rational* $\overline{\mathcal{E}}$*-inner function* $x = (x_1, x_2, x_3)$ *of degree* n *such that*

$$x(\sigma_j) = (\eta_j, \tilde{\eta}_j, \eta_j\tilde{\eta}_j) \quad \textit{for } j = 1, ..., n$$

and

$$Ax_1(\sigma_j) = \rho_j \quad \textit{for } j = 1, ..., k.$$

In [12] Alsalhi and Lykova gave description of rational $\overline{\mathcal{E}}$-inner function $x = (x_1, x_2, x_3)$ of degree n. They showed that if such $x = (x_1, x_2, x_3)$ of degree n are given, then there exist polynomials E_1, E_2, D such that

(i) *deg*(E_1), *deg*(E_2), *deg*$(\mathrm{D}) \leq n$,

(ii) $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$,

(iii) $E_1(\lambda) = E_2^{\sim n}(\lambda)$, for all $\lambda \in \mathbb{T}$, where $E_2^{\sim n}(\lambda) = \lambda^n \overline{E_2(\frac{1}{\overline{\lambda}})}$,

(iv) $|E_i(\lambda)| \leq |D(\lambda)|$ on $\overline{\mathbb{D}}$, $i = 1, 2$,

(v) $x_1 = \frac{E_1}{D}$ on $\overline{\mathbb{D}}$,

(vi) $x_2 = \frac{E_2}{D}$ on $\overline{\mathbb{D}}$,

(vii) $x_3 = \frac{D^{\sim n}}{D}$ on $\overline{\mathbb{D}}$, where $D^{\sim n}(\lambda) = \lambda^n \overline{D(\frac{1}{\overline{\lambda}})}$.

Definition 3.3.3. *Let* $x = (x_1, x_2, x_3)$ *be a rational tetra-inner function of degree* n*. The* royal polynomial *of* x *is*

$$R_x(\lambda) = D(\lambda)D^{\sim n}(\lambda) - E_1(\lambda)E_2(\lambda).$$

where E_1, E_2 *and* D *are as described above.*

Definition 3.3.8 . *Let* $x = (x_1, x_2, x_3)$ *be a rational* $\overline{\mathcal{E}}$*-inner function such that* $x(\overline{\mathbb{D}}) \nsubseteq \mathcal{R}_{\overline{\mathcal{E}}}$ *and let* R_x *be a royal polynomial of* x*. If* σ *is a zero of* R_x *of order* l*, we define the multiplicity* $\#\sigma$ *of* σ *(as a royal node of* x*) by*

$$\#\sigma = \begin{cases} l & \textit{if } \sigma \in \mathbb{D} \\ \frac{1}{2}l & \textit{if } \sigma \in \mathbb{T}. \end{cases}$$

We define the type *of x to be the ordered pair (n, k), where n is the sum of the multiplicities of the royal nodes of x that lie in $\overline{\mathbb{D}}$, and k is the sum of the multiplicities of the royal nodes of x that lie in $\mathbb{T}$. We denote by $\mathcal{R}^{n,k}$ the collection of rational $\overline{\mathcal{E}}$-inner functions of type (n, k).*

Definition 1.2.5. [1, Definition 2.1] *For $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ and $z \in \mathbb{C}$ we define*

$$\Psi(z, x) = \frac{x_3 z - x_1}{x_2 z - 1} \quad \text{when} \quad x_2 z - 1 \neq 0, \tag{1.3}$$

and, for $\omega \in \mathbb{T}$,

$$\Psi_\omega(x) = \frac{x_3 \omega - x_1}{x_2 \omega - 1} \quad \text{when} \quad x_2 \omega - 1 \neq 0. \tag{1.4}$$

Note that when $x_3 = x_1 x_2$, then

$$\Psi(z, x) = \frac{x_1 x_2 z - x_1}{x_2 z - 1} = \frac{x_1(x_2 z - 1)}{x_2 z - 1} = x_1.$$

Definition 1.2.6. [6, Definition 3.10] *Let (σ, η, ρ) be Blaschke interpolation data, with n distinct interpolation nodes of which k lie in $\mathbb{T}$. Suppose that* Problem 1.2.2 *is solvable. We say that*

$$\varphi = \frac{a\zeta + b}{c\zeta + d}$$

is a normalised linear fractional parametrization of the solutions *of* Problem 1.2.2 *if*

(i) *a, b, c, d are polynomials of degree at most n;*

(ii) *for all but at most k values of $\zeta \in \mathbb{T}$, the function*

$$\varphi(\lambda) = \frac{a(\lambda)\zeta + b(\lambda)}{c(\lambda)\zeta + d(\lambda)} \tag{1.5}$$

is a solution of Problem 1.2.2*;*

(iii) *for some point $\tau \in \mathbb{T} \setminus \{\sigma_1, ..., \sigma_k\}$,*

$$\begin{bmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

(iv) *every solution* φ *of* Problem 1.2.2 *has the form* (1.5) *for some* $\zeta \in \mathbb{T}$.

The main theorem of this thesis is the following.

Theorem 1.2.7. *For royal tetra-interpolation data* $(\sigma, \eta, \tilde{\eta}, \rho)$ *the following two statements are equivalent:*

(i) *The royal tetra-interpolation problem* (Problem 1.2.4) *with data* $(\sigma, \eta, \tilde{\eta}, \rho)$ *is solvable by a rational* $\overline{\mathcal{E}}$*-inner function* x *such that* $x(\overline{\mathbb{D}}) \nsubseteq \mathcal{R}_{\overline{\mathcal{E}}}$;

(ii) *The Blaschke interpolation problem* (Problem 1.2.2) *with data* (σ, η, ρ) *is solvable and there exist* $x_1^\circ, x_2^\circ, x_3^\circ \in \mathbb{C}$ *such that*

$$|x_3^\circ| = 1, \quad |x_1^\circ| < 1, \quad |x_2^\circ| < 1, \quad x_1^\circ = \overline{x_2^\circ} x_3^\circ,$$

and

$$\frac{x_3^\circ c(\sigma_j) + x_2^\circ d(\sigma_j)}{x_1^\circ c(\sigma_j) + d(\sigma_j)} = \tilde{\eta}_j \quad \text{for } j = 1, ..., n,$$

where a, b, c *and* d *are the polynomials in the normalized parametrization* $\varphi = \dfrac{a\zeta + b}{c\zeta + d}$ *of the solution of Problem* 1.2.2.

The theorem follows from Theorems 4.1.1 and 4.2.5.

Theorem 4.1.1. *Let* $x = (x_1, x_2, x_3)$ *be a rational* $\overline{\mathcal{E}}$*-inner function of type* (n, k) *having distinct royal nodes* $\sigma_1, \sigma_2, ..., \sigma_n$*, where* $\sigma_1, \sigma_2, ..., \sigma_k \in \mathbb{T}$ *and* $\sigma_{k+1}, ..., \sigma_n \in \mathbb{D}$*, and corresponding royal values* $\eta_1, .., \eta_n$ *and* $\tilde{\eta}_1, ..., \tilde{\eta}_n$*, that is,* $x(\sigma_j) = (\eta_j, \tilde{\eta}_j, \eta_j \tilde{\eta}_j)$*. Let* $\rho_j = Ax_1(\sigma_j)$ *for* $j = 1, 2, .., k$.

(1) *There exists a rational function* φ *that solves the Blaschke interpolation* Problem 1.2.2 *for* (σ, η, ρ) *that is, such that deg*$(\varphi) = n$.

$$\varphi(\sigma_j) = \eta_j \quad \text{for } j = 1, ..., n \tag{1.6}$$

and

$$A\varphi(\sigma_j) = \rho_j \quad \text{for } j = 1, ..., k. \tag{1.7}$$

Any such function φ *is expressible in the form* $\varphi = \Psi_\omega \circ x$ *for some* $\omega \in \mathbb{T}$.

(2) *There exist polynomials* a, b, c, d *of degree at most* n *such that a normalized parametrization of the solutions of* Problem 1.2.2 *is*

$$\varphi = \frac{a\zeta + b}{c\zeta + d}, \quad \text{for } \zeta \in \mathbb{T}.$$

(3) *For any polynomials a, b, c, d as in (2), there exist $x_1^\circ, x_2^\circ, x_3^\circ \in \mathbb{C}$ such that*

$$|x_3^\circ| = 1, \quad |x_1^\circ| < 1, \quad |x_2^\circ| < 1, \tag{1.8}$$

$$x_1^\circ = \overline{x_2^\circ} x_3^\circ, \tag{1.9}$$

and

$$x_1 = \frac{x_1^\circ a + b}{x_1^\circ c + d} \tag{1.10}$$

$$x_2 = \frac{x_3^\circ c + x_2^\circ d}{x_1^\circ c + d} \tag{1.11}$$

$$x_3 = \frac{x_2^\circ b + x_3^\circ a}{x_1^\circ c + d}. \tag{1.12}$$

Theorem 4.2.5. *Let (σ, η, ρ) be Blaschke interpolation data with n distinct interpolation nodes of which k lie in $\mathbb{T}$, and let $(\sigma, \eta, \tilde{\eta}, \rho)$ be royal tetra-interpolation data, where $\tilde{\eta}_j \in \mathbb{T}, j = 1, ..., k$ and $\tilde{\eta}_j \in \mathbb{D}, j = k+1, ..., n$. Suppose that Problem* 1.2.2 *with these data is solvable and the solutions φ of Problem* 1.2.2 *have normalized parametrization*

$$\varphi = \frac{a\zeta + b}{c\zeta + d}.$$

Suppose that there exist scalars $x_1^\circ, x_2^\circ, x_3^\circ \in \mathbb{C}$ such that

$$|x_3^\circ| = 1, \quad |x_1^\circ| < 1, \quad |x_2^\circ| < 1, \quad x_1^\circ = \overline{x_2^\circ} x_3^\circ,$$

and

$$\frac{x_3^\circ c(\sigma_j) + x_2^\circ d(\sigma_j)}{x_1^\circ c(\sigma_j) + d(\sigma_j)} = \tilde{\eta}_j \quad \text{for } j = 1, ..., n. \tag{1.13}$$

Then there exists a rational tetra-inner function $x = (x_1, x_2, x_3)$ given by,

$$x_1(\lambda) = \frac{x_1^\circ a(\lambda) + b(\lambda)}{x_1^\circ c(\lambda) + d(\lambda)} \tag{1.14}$$

$$x_2(\lambda) = \frac{x_3^\circ c(\lambda) + x_2^\circ d(\lambda)}{x_1^\circ c(\lambda) + d(\lambda)} \tag{1.15}$$

$$x_3(\lambda) = \frac{x_2^\circ b(\lambda) + x_3^\circ a(\lambda)}{x_1^\circ c(\lambda) + d(\lambda)}, \tag{1.16}$$

for $\lambda \in \mathbb{D}$, such that

(i) $x \in \mathcal{R}^{n,k}$, *and x is a solution of the royal tetra-interpolation problem with the data $(\sigma, \eta, \tilde{\eta}, \rho)$, that is,*

$$x(\sigma_j) = (\eta_j, \tilde{\eta}_j, \eta_j \tilde{\eta}_j) \text{ for } j = 1, ..., n,$$

and

$$Ax_1(\sigma_j) = \rho_j \quad \text{for } j = 1, ..., k,$$

(ii) *for all but finitely many $\omega \in \mathbb{T}$, the function $\Psi_\omega \circ x$ is a solution of Problem* 1.2.2*.*

The proofs of these theorems are given in Section 4.1 and Section 4.2 respectively.

The connection between the solution sets of the royal $\overline{\mathcal{E}}$-interpolation problem and the Blaschke interpolation problem can be made explicitly with the aid of Ψ_ω functions.

Corollary 4.2.6. *Let (σ, η, ρ) be Blaschke interpolation data. Suppose that x is a solution of* Problem 1.2.4 *with $(\sigma, \eta, \tilde{\eta}, \rho)$ for some $\tilde{\eta}_j \in \overline{\mathbb{D}}, j = 1, ..., n$, and that $x(\mathbb{D}) \not\subset \mathcal{R}_{\overline{\mathcal{E}}}$. For all $\omega \in \mathbb{T} \backslash \{\overline{\tilde{\eta}_1}, ..., \overline{\tilde{\eta}_k}\}$, the function $\varphi = \Psi_\omega \circ x$ is a solution of* Problem 1.2.2 *with Blaschke interpolation data (σ, η, ρ). Conversely, for every solution φ of the Blaschke interpolation problem with data (σ, η, ρ), there exists $\omega \in \mathbb{T}$ such that $\varphi = \Psi_\omega \circ x$.*

1.3 Basic materials

The complex conjugate transpose of a matrix $A = (a_{ij})_{i,j=1}^{n,m}$ will be written A^*, and so $A^* = (a^*_{ij})$, where $a^*_{ij} = \overline{a_{ji}}$ for all i, j.

Definition 1.3.1. [9, Definition 1] *A* finite Blaschke product *is a function B on $\mathbb{D}$ of the form*

$$B(z) = c \prod_{j=1}^{n} \frac{z - \alpha_j}{1 - \overline{\alpha_j} z} \qquad \text{for } z \in \mathbb{D},$$

and for some $c \in \mathbb{T}$ and $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{D}$. A Blaschke factor *is a function on $\mathbb{D}$ of the form $B_a(z) = \dfrac{a - z}{1 - \overline{a} z}$, where $a \in \mathbb{D}$.*

Definition 1.3.2. *Let Ω be an open set in $\mathbb{C}$ and $(X, ||.||_X)$ be a Banach space. Then we say a map $f : \Omega \to X$ is analytic on Ω if, for every $z_\circ \in \Omega$, there is $f'(z_\circ) \in X$ such that*

$$\lim_{z \to z_\circ} \left|\left| \frac{f(z) - f(z_\circ)}{z - z_\circ} - f'(z_\circ) \right|\right|_X = 0.$$

Definition 1.3.3. *Let Y be a domain in $\mathbb{C}^n$. For every domain Ω in $\mathbb{C}$, $\mathrm{Hol}(\Omega, \overline{Y})$ is the space of analytic functions from Ω to $\overline{Y}$.*

Definition 1.3.4. *$H^\infty(\mathbb{D})$ is the Banach space of bounded analytic functions f on $\mathbb{D}$ with supremum norm*

$$||f||_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

1.4 The μ-synthesis problem

The structured singular value $\mu(A)$ of a matrix A relative to a space of matrices was introduced by J. C. Doyle and G. Stein in 1980 [23, 24]. The μ is a refinement of the usual operator norm of a matrix. However, its behaviour is different from the operator norm; μ is not a norm in general. The μ-synthesis problem is an interpolation problem for analytic matrix functions. It is a generalization of the classical problem of Nevanlinna-Pick. For any $A \in M_{k\times l}(\mathbb{C})$ and for any subspace E of $M_{l\times k}(\mathbb{C})$ we define,

$$\mu_E(A) = (\inf \{||X|| : X \in E, 1 - AX \text{ is singular }\})^{-1},$$

where $\mu_E(A) = 0$ in the case that $1 - AX$ is nonsingular for all $X \in E$. There are two extreme examples of μ. If $E = M_{l\times k}(\mathbb{C})$, then $\mu_E(A) = ||A||$. Another example is when $k = l$ and E is chosen to be the space of the scalar multiples of the identity matrix: then $\mu_E(A) = r(A)$, where the spectral radius r of a matrix A is given by

$$r(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

Note that, for any E, $\mu_E(A) \leq ||A||$. If $k = l$ and E contains the identity matrix, then $\mu_E(A) \geqslant r(A)$ [31]. In particular, there is a special case of the

μ-synthesis problem which is *the spectral Nevanlinna-Pick problem* (see [10], [11]):

Problem SNP *Given distinct points* $\lambda_1, ..., \lambda_n \in \mathbb{D}$ *and* $k \times k$ *matrices* $W_1, ..., W_n$*, construct an analytic* $k \times k$ *matrix function* F *on* $\mathbb{D}$ *such that*

$$F(\lambda_j) = W_j, \quad 1 \le j \le n,$$

and

$$r(F(\lambda)) \le 1 \quad \text{for all } \lambda \in \mathbb{D}.$$

In this thesis we study the following μ-synthesis problem which was introduced in [1]. In this case E is the space Diag of 2×2 diagonal matrices

$$\text{Diag} = \left\{ \begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix} : z, w \in \mathbb{C} \right\},$$

and, for $A \in M_{2\times 2}(\mathbb{C})$,

$$\mu_{\text{Diag}}(A) = (\inf\{||X|| : X \in \text{Diag}, 1 - AX \textit{ is singular}\})^{-1}.$$

where $\mu_{\text{Diag}}(A) = 0$ in the case that $1 - AX$ is nonsingular for all $X \in \text{Diag}$.

The μ_{Diag}***-synthesis problem***: given distinct points $\lambda_1, ..., \lambda_n \in \mathbb{D}$ and target matrices $W_1, ..., W_n \in M_{2\times 2}(\mathbb{C})$ such that $\mu_{\text{Diag}}(W_k) < 1,\ k = 1, ..., n$, find, if possible, an analytic 2×2-matrix-valued function F on $\mathbb{D}$ such that

$$F(\lambda_j) = W_j \quad \textit{for } j = 1, ..., n, \textit{ and}$$

$$\mu_{\text{Diag}}(F(\lambda)) \ < \ 1 \text{ for all } \lambda \in \mathbb{D}.$$

1.5 Description of results by sections

In Chapter 2 we describe the results of Agler, Lykova and Young from [6]. In their paper they give an explicit construction of rational Γ-inner functions with the aid of a solution of an interpolation problem for finite Blaschke products. In Section 2.1 we state the criteria for the solvability of the Blaschke interpolation problem from [6]. In Section 2.2, we show their construction of the rational Γ-inner functions $h = (s, p)$ of degree n with n zeros of $s^2 - 4p$

prescribed. We state the royal Γ-interpolation problem and the main theorem in [6] which connects the solvability of the royal Γ-interpolation problem and the Blaschke interpolation problem.

In Chapter 3 we describe the tetrablock and its distinguished boundary $b\overline{\mathcal{E}}$. In Section 3.2, for a rational $\overline{\mathcal{E}}$-inner function $x = (x_1, x_2, x_3) : \mathbb{D} \to \overline{\mathcal{E}}$, we consider the rational functions $\psi_\omega : \mathbb{D} \to \overline{\mathbb{D}}$ and $\Upsilon_\omega : \mathbb{D} \to \overline{\mathbb{D}}$ which are given by

$$\psi_\omega(\lambda) = \Psi_\omega \circ x(\lambda) \;=\; \frac{\omega x_3 - x_1}{x_2\omega - 1}(\lambda), \quad x_2(\lambda)\omega - 1 \neq 0 \text{ for all } \lambda \in \overline{\mathbb{D}}.$$

$$\Upsilon_\omega(\lambda) = \Upsilon_\omega \circ x(\lambda) = \frac{x_3\omega - x_2}{x_1\omega - 1}(\lambda), \quad x_1(\lambda)\omega - 1 \neq 0 \text{ for all } \lambda \in \overline{\mathbb{D}}.$$

respectively. We calculate the phasar derivatives of $\Psi_\omega \circ x$ and $\Upsilon_\omega \circ x$. In Section 3.3 we define rational tetra-inner functions x and royal polynomials of x, and we introduce the notions of a royal node σ of x and royal values $\eta, \tilde{\eta}$ corresponding to the royal node.

In Chapter 4 we show how to construct rational $\overline{\mathcal{E}}$-inner functions with prescribed royal nodes and values. In this chapter, with the aid of a solution of an interpolation problem for finite Blaschke products, we construct rational $\overline{\mathcal{E}}$-inner functions of degree n with the n zeros of $x_1x_2 - x_3$ prescribed. In Section 4.1 we prove Theorem 4.1.1 for the given Blaschke interpolation data (σ, η, ρ) which shows that the existence of a solution x for the royal tetra-interpolation problem allows us to construct a solution for the Blaschke interpolation problem including the support Lemma 4.1.2. In Section 4.2 we prove Theorem 4.2.5 which gives us the construction of a solution of the royal $\overline{\mathcal{E}}$-interpolation problem with data $(\sigma, \eta, \tilde{\eta}, \rho)$ for some $\tilde{\eta} = (\tilde{\eta}_1, ..., \tilde{\eta}_n)$ in terms of a normalized parametrization of solutions of the corresponding Blaschke interpolation problem with given Blaschke interpolation data (σ, η, ρ). To prove Theorem 4.2.5, we start with some important propositions in that section.

In Chapter 5 we summarize the steps in the solution of the royal $\overline{\mathcal{E}}$-interpolation problem, and in Section 5.2 we give some examples of Problem 1.2.4.

In Chapter 6 we give a matricial formulation of the solvability criteria of a μ_{Diag}-synthesis problem from [3].

1.6 Historical remarks

The original Pick problem is the following. Given n distinct points $\lambda_1, ..., \lambda_n$ in the unit disk $\mathbb{D}$ and n points $\omega_1, ..., \omega_n$ in $\overline{\mathbb{D}}$, find, if possible, an analytic function $\varphi : \mathbb{D} \to \overline{\mathbb{D}}$ such that

$$\varphi(\lambda_j) = \omega_j \quad \text{for } j = 1, ..., n. \tag{1.17}$$

A necessary and sufficient condition to solve this problem was found by G. Pick in 1916 [16], and R. Nevanlinna in 1919 independently [18].

Theorem 1.6.1 (Pick)**.** *There is a function φ in* $\mathrm{Hol}(\mathbb{D}, \overline{\mathbb{D}})$ *that satisfies the interpolation conditions* (1.17) *if and only if the Pick matrix*

$$\left(\frac{1 - \omega_i \overline{\omega_j}}{1 - \lambda_i \overline{\lambda_j}} \right)_{i,j=1}^{n}$$

is positive semi-definite. Moreover, the function φ is unique if and only if the Pick matrix has rank r strictly less than n. In this case, φ is a Blaschke product of degree r.

The tetrablock is one of domains in $\mathbb{C}^3$ which is connected to a $\mu-$synthesis problem.

In [1] Abouhajar, White and Young introduced the tetrablock $\mathcal{E}$, and they determined the distinguished boundary of $\mathcal{E}$ and some geometric properties of $\mathcal{E}$. A Schwarz lemma for the tetrablock is one of the main results of this paper. They explain the connection between $\mathcal{E}$ and $\mu_{\mathbf{Diag}}$- synthesis.

In [33], it was shown that the tetrablock $\mathcal{E}$ is inhomogeneous. In this paper, Young gave the full group of automorphisms of $\mathcal{E}$. Also, he proved a Schwarz lemma for the tetrablock.

In [26], Edigarian and Zwonek describe all complex geodesics in the tetrablock passing through the origin. Their paper includes some extremals for the Lempert function and some geodesics. The results in their paper may be recognised as a continuation of [1].

In [25], the authors talk about Lempert theorem, which is the equality between the Lempert function and the Carathéodory distance. They showed that the Lempert theorem holds in the tetrablock, a bounded hyperconvex domain that is neither $\mathbb{C}$-convex nor biholomorphic to a convex domain.

For the symmetrised bidisc and for the tetrablock, Brown, Lykova and Young study in [22] the structure of interconnections between the matricial Schur class, the Schur class of the bidisc, the set of pairs of positive kernels on the bidisc subject to a boundedness condition, and the set of analytic functions from the disc into the given inhomogeneous domain. They use the theory of reproducing kernels and Hilbert function spaces in these connections. They also give a solvability criterion for the interpolation problem that arises from the μ-synthesis problem related to the tetrablock.

In [34] N. J. Young gave a criterion for the solvability of a 2×2 spectral Nevanlinna-Pick problem with two interpolation points. The goal is to construct an analytic 2×2 matrix function F on the unit disc with a finite number of interpolation constraints and a bound on $\sup_{\lambda \in \mathbb{D}} \mu(F(\lambda))$, where μ is an instance of the structured singular value. This problem is equivalent to the interpolation problems in $\mathrm{Hol}(\mathbb{D}, \Gamma)$, where Γ is the closed symmetrised bidisc.

In [5] the authors analyze the 3-extremal analytic maps from the unit disc $\mathbb{D}$ to the open symmetrized bidisc $\mathcal{G}$. These are the maps in $\mathrm{Hol}(\mathbb{D}, \mathcal{G})$ whose restriction to any 3-point set yields interpolation data that are only just solvable. In their paper, they identify a large class of 3-extremal maps in $\mathrm{Hol}(\mathbb{D}, \mathcal{G})$; they are rational functions of degree at most 4, and they are $\mathcal{G}$-inner functions. There are two qualitatively different classes of rational $\mathcal{G}$-inner functions of degree at most 4, that they call *aligned* and *caddywhompus*. The aligned ones are 3-extremal. They give a method for the construction of aligned rational $\mathcal{G}$-inner functions. With the aid of this method, they reduce the solution of a 3-point interpolation problem for aligned analytic maps from $\mathbb{D}$ to $\mathcal{G}$ to a collection of classical Nevanlinna-Pick problems with interior and boundary interpolation nodes.

During the last ten years, several more domains in $\mathbb{C}^n$ were introduced in the connection with a various μ-synthesis problems.

The pentablock $\mathcal{P}$ is introduced in [4] by Agler, Lykova and Young. They address the complex geometry of pentablock $\mathcal{P}$. Their paper describes a lot of characterisations of $\mathcal{P}$, its distinguished boundary, and a 4-parameter group of automorphisms of the pentablock $\mathcal{P}$. They show the connections between the new case of μ-synthesis problem and the pentablock $\mathcal{P}$. They also introduced some linear fractional functions which play a significant role in the paper. In [28], L. Kosinski showed that this group of automorphisms is the full automorphism group of the pentablock.

In [35], Zapalowski studied the geometric properties of a large family of domains which is called the generalized tetrablocks, related to the μ-synthesis. It contains both the family of the symmetrized polydiscs and the family of the $\mu_{1,n}$–quotients $\mathcal{E}_n, n \geq 2$, introduced recently by G. Bharali in [15]. Zapalowski proved that the generalized tetrablock cannot be exhausted by domains biholomorphic to convex ones. Moreover, it is shown in this paper that the Carathéodory distance and the Lempert function are not equal on a large subfamily of the generalized tetrablocks $\mathcal{E}_n, n \geq 4$. This paper has also a number of geometric properties of the generalized tetrablocks $\mathcal{E}_n$.

In [8], the authors defined the norm-preserving extension property. A set V in a domain U in $\mathbb{C}^n$ has the norm-preserving extension property if every bounded analytic function on V has a analytic extension to U with the same supremum norm. They prove that an algebraic subset of the open symmetrized bidisc $\mathcal{G}$ has the norm-preserving extension property if and only if it is either a singleton, $\mathcal{G}$ itself, a complex geodesic of $\mathcal{G}$, or the union of the set $\{(2z, z^2) : |z| < 1\}$ and a complex geodesic of degree 1 in $\mathcal{G}$. They also prove that the complex geodesics in $\mathcal{G}$ coincide with the nontrivial analytic retracts in $\mathcal{G}$. They show that there exist sets in $\mathcal{G}$ which have the norm-preserving extension property but are not analytic retracts of $\mathcal{G}$. They give applications to von Neumann-type inequalities for Γ-contractions. They find three other domains that contain sets with the norm-preserving extension property which are not retracts: they are the spectral ball of 2×2 matrices, the tetrablock and the pentablock.

Chapter 2

The finite Blaschke interpolation problem

2.1 Criteria for the solvability of the Blaschke interpolation problem

The Blaschke interpolation Problem 1.2.2 (σ, η, ρ) as described in [6] is an algebraic variant of the classical Pick interpolation problem. One seeks a Blaschke product of a given degree n satisfying n interpolation conditions, rather than a Schur-class function, and one admits interpolation nodes in both the open unit disc and the unit circle. There is a criterion for the solvability of the Blaschke interpolation problem in terms of positivity of a Pick matrix formed from the interpolation data.

Definition 2.1.1. *The Schur class is the set of analytic functions S from $\mathbb{D}$ to $\bar{\mathbb{D}}$, $S : \mathbb{D} \to \bar{\mathbb{D}}$.*

Definition 2.1.2. *A function $f : \mathbb{D} \to \overline{\mathbb{D}}$ is* inner *if it is an analytic map such that the radial limit*

$$\lim_{r \to 1^-} f(r\lambda) \text{ exists and belongs to } \mathbb{T}$$

for almost all $\lambda \in \mathbb{T}$ with respect to Lebesgue measure.

Definition 2.1.3. *The* Pick matrix *associated with Blaschke interpolation* *data* (σ, η, ρ) *is defined to be the* $n \times n$ *matrix* $M = [m_{ij}]_{i,j=1}^n$ *with entries*

$$m_{ij} = \begin{cases} \rho_i, & \text{if } i = j \leq k. \\ \dfrac{1 - \overline{\eta_i}\eta_j}{1 - \overline{\sigma_i}\sigma_j}, & \text{otherwise.} \end{cases}$$

Definition 2.1.4. *The Pick matrix* $M = [m_{ij}]_{i,j=1}^n$ *is minimally positive if* $M \geq 0$ *and there is no positive diagonal* $n \times n$ *matrix* D*, other than* $D = 0$*, such that* $M \geq D$*.*

The following is a refinement of the Sarason Interpolation Theorem [32].

Theorem 2.1.5. [6, Theorem 3.3] *Let* M *be the Pick matrix associated with Blaschke interpolation data* (σ, η, ρ)*.*

(i) *There exists a function* φ *in the Schur class such that*

$$\varphi(\sigma_j) = \eta_j \quad for\ j = 1, ..., n, \tag{2.1}$$

and the phasar derivative $A\varphi$ *exists and satisfies*

$$A\varphi(\sigma_j) \leq \rho_j \quad for\ j = 1, ..., k, \tag{2.2}$$

if and only if $M \geqslant 0$*;*

(ii) *if* M *is positive semi-definite and of rank* $r < n$ *then there is a unique function* φ *in the Schur class satisfying conditions* (2.1) *and* (2.2) *above, and this function is a Blaschke product of degree* r*;*

(iii) *the unique function* φ *in statement* (ii) *satisfies*

$$A\varphi(\sigma_j) = \rho_j \quad for\ j = 1, ..., k, \tag{2.3}$$

if and only if M *is minimally positive.*

In [6] the authors described their strategy for the construction of the general solution of Blaschke interpolation problem (Problem 1.2.2). Their strategy is to adjoin an additional boundary interpolation condition $\varphi(\tau) = \zeta$ where $\tau \in \mathbb{T} \setminus \{\sigma_1, ..., \sigma_k\}$ and $\zeta \in \mathbb{T}$. This augmented problem will have a unique solution. All the solutions of Problem 1.2.2 will be obtained in terms of a unimodular parameter.

Lemma 2.1.6. [6, Lemma 3.4] *If C is an $n\times n$ positive definite matrix, u is an $n\times 1$ column, $\rho=\langle C^{-1}u,u\rangle$ and the $(n+1)\times(n+1)$ matrix B is defined by*

$$B=\begin{bmatrix} C & u \\ u^* & \rho \end{bmatrix},$$

then B is positive semi-definite, rank$(B)=n$ and

$$B\begin{bmatrix} -C^{-1}u \\ 1 \end{bmatrix}=0.$$

The Pick matrix $B_{\zeta,\tau}$ of the augmented problem is the $(n+1)\times(n+1)$ matrix,

$$B_{\zeta,\tau}=\begin{bmatrix} M & u_{\zeta,\tau} \\ u^*_{\zeta,\tau} & \rho_{\zeta,\tau} \end{bmatrix}, \tag{2.4}$$

where

$$\rho_{\zeta,\tau}=\langle M^{-1}u_{\zeta,\tau},u_{\zeta,\tau}\rangle.$$

and M is the Pick matrix associated with Problem 1.2.2, $u_{\zeta,\tau}$ is the $n\times 1$ column matrix defined by

$$u_{\zeta,\tau}=\begin{bmatrix} \frac{1-\bar{\eta}_1\zeta}{1-\bar{\sigma}_1\tau} \\ \vdots \\ \frac{1-\bar{\eta}_n\zeta}{1-\bar{\sigma}_n\tau} \end{bmatrix}. \tag{2.5}$$

Theorem 2.1.7. [6, Proposition 3.6] *If the Pick matrix M associated with* Problem 1.2.2 *is positive definite then, for any $\tau\in\mathbb{T}\setminus\{\sigma_1,...,\sigma_k\}$ and $\zeta\in\mathbb{T}$, there is at most one solution φ of* Problem 1.2.2 *for which $\varphi(\tau)=\zeta$.*

The jth standard basis vector in $\mathbb{C}^n$ will be denoted by e_j.

Theorem 2.1.8. [6, Proposition 3.7] *If the Pick matrix M associated with* Problem 1.2.2 *is positive definite, if $\tau\in\mathbb{T}\setminus\{\sigma_1,...,\sigma_k\}$ and $\zeta\in\mathbb{T}$ and*

$$\langle M^{-1}u_{\zeta,\tau},e_j\rangle \;\neq\; 0 \tag{2.6}$$

for $j=1,\ .\ .\ .\ ,k$, then there exists a unique solution φ to Problem 1.2.2 *such that $\varphi(\tau)=\zeta$.*

The *exceptional set* Z_τ for Problem 1.2.2 is defined as

$$Z_\tau = \{\zeta \in \mathbb{T} : \text{ for some } j, 1 \leq j \leq k, \langle M^{-1}u_{\zeta,\tau}, e_j\rangle = 0\} \tag{2.7}$$

Define $n \times 1$ vectors x_λ and y_λ for $\lambda \in \overline{\mathbb{D}} \backslash \{\sigma_1, ..., \sigma_k\}$ by

$$x_\lambda = \begin{bmatrix} \frac{1}{1-\bar{\sigma}_1\lambda} \\ \vdots \\ \frac{1}{1-\bar{\sigma}_n\lambda} \end{bmatrix}, \qquad y_\lambda = \begin{bmatrix} \frac{\bar{\eta}_1}{1-\bar{\sigma}_1\lambda} \\ \vdots \\ \frac{\bar{\eta}_n}{1-\bar{\sigma}_n\lambda} \end{bmatrix}, \tag{2.8}$$

so that

$$u_{\zeta,\tau} = x_\tau - \zeta y_\tau \tag{2.9}$$

Theorem 2.1.9. [6, Proposition 3.8]

(i) *For any $\tau \in \mathbb{T} \setminus \{\sigma_1, ..., \sigma_k\}$ if*

$$\langle x_\tau, M^{-1}e_j\rangle = 0 = \langle y_\tau, M^{-1}e_j\rangle \text{ for some } j, 1 \leq j \leq k,$$

then $Z_\tau = \mathbb{T}$.

(ii) *There exist uncountably many $\tau \in \mathbb{T}\setminus \{\sigma_1, ..., \sigma_k\}$ such that*

$$\langle x_\tau, M^{-1}e_j\rangle = 0 = \langle y_\tau, M^{-1}e_j\rangle$$

does not hold for any j, $1 \leq j \leq k$. Moreover, for such τ, the set Z_τ consists of at most k points.

Theorem 2.1.10. [6, Theorem 3.9] *Let the Pick matrix M for* Problem 1.2.2 *be positive definite, and let $\tau \in \mathbb{T}\backslash\{\sigma_1, ..., \sigma_k\}$ be such that the set*

$$Z_\tau = \{\zeta \in \mathbb{T} : u_{\zeta,\tau} \perp M^{-1}e_j \text{ for some } j, \ 1 \leq j \leq k\}$$

contains at most k points, where $u_{\zeta,\tau}$ is defined by equation (2.5).

(i) *If $\zeta \in \mathbb{T}\backslash Z_\tau$, then there is a unique solution φ_ζ of* Problem 1.2.2 *that satisfies $\varphi_\zeta(\tau) = \zeta$.*

(ii) *There exist unique polynomials a_τ, b_τ, c_τ, and d_τ of degree at most n such that*

$$\begin{bmatrix} a_\tau(\tau) & b_\tau(\tau) \\ c_\tau(\tau) & d_\tau(\tau) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.10}$$

and, for all $\zeta \in \mathbb{T}$, if φ is a solution of a Problem 1.2.2 *such that $\varphi_\zeta(\tau) = \zeta$, then*

$$\varphi(\lambda) = \frac{a_\tau(\lambda)\zeta + b_\tau(\lambda)}{c_\tau(\lambda)\zeta + d_\tau(\lambda)} \tag{2.11}$$

for all $\lambda \in \mathbb{D}$.

(iii) *If $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ are rational functions satisfying the equation*

$$\begin{bmatrix} \tilde{a}(\tau) & \tilde{b}(\tau) \\ \tilde{c}(\tau) & \tilde{d}(\tau) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.12}$$

and such that for three distinct points ζ in $\mathbb{T}\backslash Z_\tau$, the equation

$$\frac{a_\tau(\lambda)\zeta + b_\tau(\lambda)}{c_\tau(\lambda)\zeta + d_\tau(\lambda)} = \frac{\tilde{a}(\lambda)\zeta + \tilde{b}(\lambda)}{\tilde{c}(\lambda)\zeta + \tilde{d}(\lambda)} \tag{2.13}$$

holds for all $\lambda \in \mathbb{D}$, then there exists a rational function X such that $\tilde{a} = Xa_\tau$, $\tilde{b} = Xb_\tau$, $\tilde{c} = Xc_\tau$ and $\tilde{d} = Xd_\tau$.

In the light of Theorem 2.1.10, we can define what we mean by a parametrization of the solutions of a Blaschke interpolation problem.

Definition 2.1.11. [6, Definition 3.10] *Let (σ, η, ρ) be Blaschke interpolation data, with n distinct interpolation nodes of which k lie in $\mathbb{T}$. Suppose that* Problem 1.2.2 *is solvable. We say that*

$$\varphi = \frac{a\zeta + b}{c\zeta + d}$$

is a normalised linear fractional parametrization of the solutions *of* Problem 1.2.2 *if*

(i) *a, b, c, d are polynomials of degree at most n;*

(ii) *for all but at most k values of $\zeta \in \mathbb{T}$, the function*

$$\varphi(\lambda) = \frac{a(\lambda)\zeta + b(\lambda)}{c(\lambda)\zeta + d(\lambda)} \tag{2.14}$$

is a solution of Problem 1.2.2*;*

(iii) *for some point* $\tau \in \mathbb{T} \setminus \{\sigma_1, ..., \sigma_k\}$,

$$\begin{bmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

(iv) *every solution* φ *of* Problem 1.2.2 *has the form* (2.14) *for some* $\zeta \in \mathbb{T}$.

From Definition 2.1.11 and Theorem 2.1.10, we can obtain the following.

Corollary 2.1.12. [6, Corollary 3.12] *Let* (σ, η, ρ) *be Blaschke interpolation data, with* n *distinct interpolation nodes. Suppose the Pick matrix* M *of this problem is positive definite. There exists a normalized linear fractional parameterization*

$$\varphi = \frac{a\zeta + b}{c\zeta + d}$$

of the solutions of Problem 1.2.2. *Moreover*

(i) *at least one of the polynomials* a, b, c, d *has degree* n,

(ii) *the polynomials* a, b, c, d *have no common zero in* $\mathbb{C}$*;*

(iii) $|c| \leq |d|$ *on* $\overline{\mathbb{D}}$.

2.2 The Blaschke interpolation problem and the royal Γ-interpolation problem

Definition 2.2.1. *The* open symmetrized bidisc *is the set*

$$\mathcal{G} = \{(z + w, zw) : |z| < 1, |w| < 1\} \subset \mathbb{C}^2. \tag{2.15}$$

The closed symmetrized bidisc *is the set*

$$\Gamma = \{(z + w, zw) : |z| \leq 1, |w| \leq 1\} \subset \mathbb{C}^2. \tag{2.16}$$

Definition 2.2.2. [3, Definition 3.2] *The function* Φ *is defined for* $(z, s, p) \in \mathbb{C}^3$ *such that* $zs \neq 2$ *by*

$$\Phi(z, s, p) = \frac{2zp - s}{2 - zs} = -\frac{1}{2}s + \frac{(p - \frac{1}{4}s^2)z}{(1 - \frac{1}{2}sz)}. \tag{2.17}$$

Theorem 2.2.3. [11, Theorem 2.3] *$\mathcal{G}$ is polynomially convex but not convex.*

The proof that Γ and $\mathcal{G}$ are polynomially convex is found in [11]. Since Γ is polynomially convex, there is a distinguished boundary $b\Gamma$ of Γ. By [11, Theorem 2.4],

$$b\Gamma = \{(z+w, zw) : |z| = |w| = 1\}.$$

It is shown there that, topologically, $b\Gamma$ is a Möbius band.

Definition 2.2.4. *A* Γ-inner function *is an analytic function $h : \mathbb{D} \to \Gamma$ such that, for almost all $\lambda \in \mathbb{T}$ (with respect to Lebesgue measure) the radial limit*

$$\lim_{r \to 1-} h(r\lambda) \textit{ exists and belongs to } b\Gamma, \tag{2.18}$$

In [6] the authors explicitly constructed the rational Γ-inner functions $h = (s,p)$ of degree n with n zeros of $s^2 - 4p$ prescribed. They used a solution of the associated Blaschke interpolation problem. They explain that there is a a simple criterion for the existence of a solution of Problem 1.2.2 in terms of an associated "Pick matrix," and there is parametrization of all solutions of φ by a linear fractional expression in terms of a parameter $\zeta \in \mathbb{T}$. The general solution of Problem 1.2.2 is

$$\varphi = \frac{a\zeta + b}{c\zeta + d}\ , \tag{2.19}$$

where a, b, c *and* d are polynomials of degree at most n and $\zeta \in \mathbb{T}$.

Problem 2.2.5. (The royal Γ-interpolation problem) *Given Blaschke interpolation data (σ, η, ρ)* (Definition 1.2.1) *with n interpolation nodes of which k lie in $\mathbb{T}$, find if possible a rational Γ-inner function h=(s,p) of degree n such that*

$$h(\sigma_j) = (-2\eta_j, \eta_j^2) \quad \textit{for } j = 1, ..., n$$

and

$$Ap(\sigma_j) = 2\rho_j \quad \textit{for } j = 1, ..., k.$$

Theorem 2.2.6. [6, Theorem 1.5] *For Blaschke interpolation data (σ, η, ρ) the following two statements are equivalent.*

(i) Problem 2.2.5 *with data (σ, η, ρ) is solvable by a rational Γ-inner function h such that $h(\mathbb{D}) \not\subset \mathcal{R}_\Gamma$;*

(ii) Problem 1.2.2 *with data* (σ, η, ρ) *is solvable and there exist* $s_0, p_0 \in \mathbb{C}$ *such that*

$$|s_0| < 2, \quad |p_0| = 1, \quad s_0 = \bar{s}_0 p_0,$$

and

$$s_0 a - 2b + 2p_0 d = 0,$$

where a, b, c *and* d *are the polynomials in the normalized parametrization* (2.19) *of the solutions of* Problem 1.2.2 .

In the next chapters we will give the construction of a general rational tetra-inner function with the aid of solutions of the associated Blaschke interpolation problem.

Chapter 3

The tetrablock $\mathcal{E}$ and tetra-inner functions

3.1 Introduction to the tetrablock

Definition 3.1.1. *The* open *tetrablock is the domain*

$$\mathcal{E} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 zw \neq 0 \textit{ whenever } |z| \leq 1, |w| \leq 1\}. \tag{3.1}$$

Definition 3.1.2. *The* closed *tetrablock is the domain*

$$\overline{\mathcal{E}} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 zw \neq 0 \textit{ whenever } |z| < 1, |w| < 1\}. \tag{3.2}$$

Observe that the closed tetrablock is the closure of the open tetrablock. The tetrablock was introduced in [1], and it is related to the μ_{Diag}-synthesis problem.

Theorem 3.1.3. [1, Theorem 2.9] *$\mathcal{E} \cap \mathbb{R}^3$ is the open tetrahedron with vertices $(1,1,1), (1,-1,-1), (-1,1,-1)$ and $(-1,-1,1)$.*

The following definition is very important in the study of $\mathcal{E}$.

Definition 3.1.4. [1, Definition 2.1] *For $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ and $z \in \mathbb{C}$ we define*

$$\Psi(z, x) = \frac{x_3 z - x_1}{x_2 z - 1} \quad \text{when} \quad x_2 z - 1 \neq 0, \tag{3.3}$$

For $\omega \in \mathbb{T}$, let

$$\Psi_\omega(x) = \frac{x_3\omega - x_1}{x_2\omega - 1} \text{ when } x_2\omega - 1 \neq 0. \tag{3.4}$$

$$\Upsilon(z,x) = \frac{x_3 z - x_2}{x_1 z - 1} \quad \text{when} \quad x_1 z - 1 \neq 0, \tag{3.5}$$

$$D(x) = \sup_{z\in\mathbb{D}} |\Psi(z,x)| = ||\Psi(.,x)||_{H^\infty}. \tag{3.6}$$

Hence,

$$D(x) = \begin{cases} \dfrac{|x_1 - \overline{x}_2 x_3| + |x_1x_2 - x_3|}{1-|x_2|^2} & \text{if } \; |x_2| < 1 \\ |x_1| & \text{if } \; x_1x_2 = x_3 \\ \infty & \text{otherwise.} \end{cases} \tag{3.7}$$

Note that, when $x_3 = x_1x_2$, then

$$\Psi(z,x) = \frac{x_1x_2z - x_1}{x_2z - 1} = \frac{x_1(x_2z-1)}{x_2z-1} = x_1,$$

and

$$\Upsilon(z,x) = \frac{x_1x_2z - x_2}{x_1z - 1} = \frac{x_2(x_1z-1)}{x_1z-1} = x_2.$$

Theorem 3.1.5. [1, Theorem 2.2] *For $x \in \mathbb{C}^3$ the following are equivalent.*

(i) $x \in \mathcal{E}$*;*

(ii) $||\Psi(.,x)||_{H^\infty} \; < \; 1$ *and if* $x_1x_2 = x_3$ *then* $|x_2| \; < 1$*;*

(iii) $||\Upsilon(.,x)||_{H^\infty} \; < \; 1$ *and if* $x_1x_2 = x_3$ *then* $|x_1| \; < 1$*;*

(iv) $|x_1 - \bar{x}_2x_3| + |x_1x_2 - x_3| < 1 - |x_2|^2$*;*

(v) $|x_2 - \bar{x}_1x_3| + |x_1x_2 - x_3| < 1 - |x_1|^2$*;*

(vi) $|x_1|^2 - |x_2|^2 + |x_3|^2 + 2|x_2 - \bar{x}_1x_3| < 1$ *and* $|x_2| < 1$*;*

(vii) $-|x_1|^2 + |x_2|^2 + |x_3|^2 + 2|x_1 - \bar{x}_2x_3| < 1$ *and* $|x_1| < 1$*;*

(viii) $|x_1|^2 + |x_2|^2 - |x_3|^2 + 2|x_1x_2 - x_3| < 1$ *and* $|x_3| < 1$*;*

(ix) $|x_1 - \bar{x}_2x_3| + |x_2 - \bar{x}_1x_3| < 1 - |x_3|^2$*;*

(x) *there exists a* 2×2 *matrix* $A = [a_{ij}]$ *such that* $||A|| < 1$ *and* $x = (a_{11}, a_{22}, \det A)$;

(xi) *there exists a symmetric* 2×2 *matrix* $A = [a_{ij}]$ *such that* $||A|| < 1$ *and* $x = (a_{11}, a_{22}, \det A)$;

Theorem 3.1.6. [1, Theorem 2.4] *For* $x \in \mathbb{C}^3$ *the following are equivalent.*

(i) $1 - x_1 z - x_2 w + x_3 zw \neq 0$ *for all* $z, w \in \mathbb{D}$;

(ii) $x \in \bar{\mathcal{E}}$;

(iii) $||\Psi(., x)||_{H^\infty} \leq 1$ *and if* $x_1 x_2 = x_3$ *then* $|x_2| \leq 1$;

(iv) $||\Upsilon(., x)||_{H^\infty} \leq 1$ *and if* $x_1 x_2 = x_3$ *then* $|x_1| \leq 1$;

(v) $|x_1 - \bar{x_2} x_3| + |x_1 x_2 - x_3| \leq 1 - |x_2|^2$ *and if* $x_3 = x_1 x_2$, *then* $|x_1| \leq 1$;

(vi) $|x_2 - \bar{x_1} x_3| + |x_1 x_2 - x_3| \leq 1 - |x_1|^2$ *and if* $x_3 = x_1 x_2$, *then* $|x_2| \leq 1$;

(vii) $|x_1|^2 - |x_2|^2 + |x_3|^2 + 2|x_2 - \bar{x_1} x_3| \leq 1$ *and* $|x_2| \leq 1$;

(viii) $-|x_1|^2 + |x_2|^2 + |x_3|^2 + 2|x_1 - \bar{x_2} x_3| \leq 1$ *and* $|x_1| \leq 1$;

(ix) $|x_1|^2 + |x_2|^2 - |x_3|^2 + 2|x_1 x_2 - x_3| \leq 1$ *and* $|x_3| \leq 1$;

(x) $|x_1 - \bar{x_2} x_3| + |x_2 - \bar{x_1} x_3| \leq 1 - |x_3|^2$ *and if* $|x_3| = 1$ *then* $|x_1| \leq 1$

(xi) *there exists a* 2×2 *matrix* $A = [a_{ij}]$ *such that* $||A|| \leq 1$ *and* $x = (a_{11}, a_{22}, \det A)$;

(xii) *there exists a symmetric* 2×2 *matrix* $A = [a_{ij}]$ *such that* $||A|| \leq 1$ *and* $x = (a_{11}, a_{22}, \det A)$;

Theorem 3.1.7. [1, Theorem 2.9] $\bar{\mathcal{E}}$ *is polynomially convex.*

The *royal variety* of $\bar{\mathcal{E}}$ is $\mathcal{R}_{\bar{\mathcal{E}}} = \{(x_1, x_2, x_3) \in \bar{\mathcal{E}} : x_3 = x_1 x_2\}$.

Since $\bar{\mathcal{E}}$ is polynomially convex, there is a smallest closed boundary $b\bar{\mathcal{E}}$ of $\mathcal{E}$, which is called *the distinguished boundary of* $\mathcal{E}$. If there is a function $g \in A(\mathcal{E})$ and a point $p \in \bar{\mathcal{E}}$ such that $g(p) = 1$ *and* $|g(x)| < 1$ for all $x \in \bar{\mathcal{E}} \backslash \{p\}$, then p must be in $b\bar{\mathcal{E}}$. We call p a *peak point* of $\bar{\mathcal{E}}$ and the function g is called a *peaking function* for p.

Theorem 3.1.8. [1, Theorem 7.1] *For $x \in \mathbb{C}^3$ the following are equivalent.*

(i) $x_1 = \bar{x}_2 x_3, |x_3| = 1$ *and* $|x_2| \leq 1$;

(ii) *either* $x_1 x_2 \neq x_3$ *and* $\Psi(.,x)$ *is an automorphism of* $\mathbb{D}$ *or* $x_1 x_2 = x_3$ *and* $|x_1| = |x_2| = |x_3| = 1$;

(iii) x *is a peak point of* $\bar{\mathcal{E}}$;

(iv) *there exists a* 2×2 *unitary matrix* U *such that* $x = \pi(U)$

where

$$\pi : \mathbb{C}^{2\times 2} \to \mathbb{C}^3 : U = [u_{ij}] \mapsto (u_{11}, u_{22}, \det U);$$

(v) *there exists a symmetric* 2×2 *unitary matrix* U *such that* $x = \pi(U)$;

(vi) $x \in b\bar{\mathcal{E}}$;

(vii) $x \in \bar{\mathcal{E}}$ *and* $|x_3| = 1$.

Corollary 3.1.9. [1, Corollary 7.2] $b\bar{\mathcal{E}}$ *is homeomorphic to* $\overline{\mathbb{D}} \times \mathbb{T}$.

For the map $\overline{\mathbb{D}} \times \mathbb{T} \to b\bar{\mathcal{E}} : (x_2, x_3) \mapsto (\bar{x}_2 x_3, x_2, x_3)$ is a homeomorphism.

Theorem 3.1.10. [1, Theorem 9.2] *Let* $\lambda_1, ..., \lambda_n$ *be distinct points in* $\mathbb{D}$ *and let*

$$W_j = \begin{bmatrix} w_{11}^j & w_{12}^j \\ w_{21}^j & w_{22}^j \end{bmatrix}, \quad j = 1, ..., n,$$

be 2×2 *matrices such that* $w_{11}^j w_{22}^j \neq \det W_j$ *and*

$$\mu_{\text{Diag}}(W_j) < 1, \quad j = 1, ..., n.$$

The following conditions are equivalent.

(i) *There exists an analytic* 2×2 *matrix function* F *on* $\mathbb{D}$, *such that*

$$F(\lambda_j) = W_j \quad \text{for } j = 1, ..., n$$

and

$$\sup_{\lambda \in \mathbb{D}} \mu_{\text{Diag}}(F(\lambda)) < 1; \tag{3.8}$$

(ii) *there exists an analytic function* $\phi \in \mathrm{Hol}(\mathbb{D}, \mathcal{E})$ *such that*

$$\phi(\lambda_j) = (w_{11}^j, w_{22}^j, \det W_j) \quad \textit{for } j = 1, ..., n. \tag{3.9}$$

Lemma 3.1.11. *Let* $(x_1, x_2, x_3) \in \overline{\mathcal{E}}$ *be such that* $x_1x_2 \neq x_3$. *For* $\omega \in \mathbb{T}$,

$|\Psi_\omega(x_1, x_2, x_3)| = 1$ *if and only if* $2\omega(x_2 - \overline{x_1}x_3) = 1 - |x_1|^2 + |x_2|^2 - |x_3|^2$.

Proof.

$$\begin{aligned}
|\Psi_\omega(x_1, x_2, x_3)| = 1 &\Leftrightarrow \left|\frac{x_3\omega - x_1}{x_2\omega - 1}\right| = 1 \\
&\Leftrightarrow |\omega x_3 - x_1|^2 = |x_2\omega - 1|^2 \\
&\Leftrightarrow (\omega x_3 - x_1)(\overline{\omega x_3} - \overline{x_1}) = (x_2\omega - 1)(\overline{x_2\omega} - 1) \\
&\Leftrightarrow |\omega|^2|x_3|^2 - \omega x_3\overline{x_1} - x_1\overline{\omega x_3} + |x_1|^2 = |\omega|^2|x_2|^2 - x_2\omega - \overline{x_2\omega} + 1 \\
&\Leftrightarrow |x_3|^2 - 2\mathrm{Re}(\omega x_3\overline{x_1}) + |x_1|^2 = |x_2|^2 - 2\mathrm{Re}(x_2\omega) + 1 \\
&\Leftrightarrow |x_1|^2 - |x_2|^2 + |x_3|^2 - 2\mathrm{Re}(\omega x_3\overline{x_1}) + 2\mathrm{Re}(\omega x_2) = 1 \\
&\Leftrightarrow |x_1|^2 - |x_2|^2 + |x_3|^2 + 2\mathrm{Re}(\omega(x_2 - \overline{x_1}x_3)) = 1 \\
&\Leftrightarrow 2\mathrm{Re}(\omega(x_2 - \overline{x_1}x_3)) = 1 - |x_1|^2 + |x_2|^2 - |x_3|^2. \quad (3.10)
\end{aligned}$$

Since $(x_1, x_2, x_3) \in \overline{\mathcal{E}}$, by Theorem 3.1.6 (vii),

$$2|x_2 - \bar{x}_1x_3| \leq 1 - |x_1|^2 + |x_2|^2 - |x_3|^2,$$

and $|x_2| \leq 1$. Therefore,

$$2\mathrm{Re}(\omega(x_2 - \overline{x_1}x_3)) \leq 2|x_2 - \overline{x_1}x_3| \leq 1 - |x_1|^2 + |x_2|^2 - |x_3|^2 = 2\mathrm{Re}(\omega(x_2 - \overline{x_1}x_3)).$$

Thus

$$2\mathrm{Re}(\omega(x_2 - \overline{x_1}x_3)) = 2|x_2 - \overline{x_1}x_3| = 1 - |x_1|^2 + |x_2|^2 - |x_3|^2.$$

Hence, for $\omega \in \mathbb{T}$, $|\Psi_\omega(x_1, x_2, x_3)| = 1$ if and only if

$$2\omega(x_2 - \overline{x_1}x_3) = 1 - |x_1|^2 + |x_2|^2 - |x_3|^2.$$

□

Definition 3.1.12. *An $\bar{\mathcal{E}}$-inner function is an analytic function $\varphi : \mathbb{D} \to \overline{\mathcal{E}}$ such that the radial limit*

$$\lim_{r \to 1^-} \varphi(r\lambda) \in b\overline{\mathcal{E}}$$

for almost all $\lambda \in \mathbb{T}$.

We will also use the terminology *tetra-inner function* for an $\bar{\mathcal{E}}$-inner function.

3.2 The phasar derivatives of $\Psi_\omega \circ x$ and $\Upsilon_\omega \circ x$

Recall, for $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ and $\omega \in \mathbb{T}$, we have defined

$$\Psi_\omega(x) = \frac{x_3\omega - x_1}{x_2\omega - 1} \quad \text{and} \quad \Upsilon_\omega(x) = \frac{x_3\omega - x_2}{x_1\omega - 1}.$$

For a rational $\overline{\mathcal{E}}$-inner function $x = (x_1, x_2, x_3) : \mathbb{D} \to \overline{\mathcal{E}}$, we consider the rational functions $\psi_\omega : \mathbb{D} \to \overline{\mathbb{D}}$ and $\Upsilon_\omega : \mathbb{D} \to \overline{\mathbb{D}}$ given by

$$\begin{aligned} \psi_\omega(\lambda) = \Psi_\omega \circ x(\lambda) &= \frac{\omega x_3 - x_1}{x_2\omega - 1}(\lambda), \quad x_2(\lambda)\omega - 1 \neq 0 \text{ for all } \lambda \in \overline{\mathbb{D}}. \\ \Upsilon_\omega(\lambda) = \Upsilon \circ x(\lambda) &= \frac{x_3\omega - x_2}{x_1\omega - 1}(\lambda), \quad x_1(\lambda)\omega - 1 \neq 0 \text{ for all } \lambda \in \overline{\mathbb{D}}. \end{aligned}$$

respectively. The phasar derivative is defined in Definition A.2.1.
Let us recall that $\sigma \in \mathbb{T}$ is a royal node for a rational tetra-inner function $x = (x_1, x_2, x_3)$ if $x_3(\sigma) - x_1(\sigma)x_2(\sigma) = 0$.

Lemma 3.2.1. *Let $x = (x_1, x_2, x_3)$ be a rational tetra-inner function and let $\sigma \in \mathbb{T}$ be a royal node of x. Then σ is a zero of the function $x_3 - x_1x_2$ of multiplicity at least 2.*

Proof. If $\lambda \in \mathbb{T}$, we have $x_3(\lambda) - x_1(\lambda)x_2(\lambda) = 0$ if and only if λ is a royal node of x.

For $\lambda \in \mathbb{T}$, since x is a tetra-inner function, by Theorem 3.1.8,

$$\begin{aligned} \overline{x_3(\lambda)}(x_3(\lambda) - x_1(\lambda)x_2(\lambda)) &= \overline{x_3(\lambda)}x_3(\lambda) - \overline{x_3(\lambda)}(x_1(\lambda)x_2(\lambda)) \\ &= |x_3(\lambda)|^2 - \overline{x_3(\lambda)}(\overline{x_2(\lambda)}x_3(\lambda))x_2(\lambda) \\ &= 1 - |x_3(\lambda)|^2|x_2(\lambda)|^2, \quad \text{since } |x_2(\lambda)| \leq 1 \text{ on } \mathbb{T}, \\ &= 1 - |x_2(\lambda)|^2 \geq 0. \end{aligned}$$

By assumption, $x_3(\sigma) - x_1(\sigma)x_2(\sigma) = 0$. Hence the function

$$f(\theta) = \overline{x_3(e^{i\theta})}(x_3(e^{i\theta}) - x_1(e^{i\theta})x_2(e^{i\theta}) = 1 - |x_2(e^{i\theta})|^2 \geq 0$$

has a local minimum at ξ where $\sigma = e^{i\xi}$. Therefore ξ is a critical point of f, and

$$\frac{d}{d\theta}(1 - |x_2(e^{i\theta})|^2)_{|\xi} = 0.$$

Thus,

$$\begin{aligned}
0 &= \frac{d}{d\theta}(1 - |x_2(e^{i\theta})|^2)_{|\xi} \\
&= \frac{d}{d\theta}(\overline{x_3(e^{i\theta})}(x_3(e^{i\theta}) - x_1(e^{i\theta})x_2(e^{i\theta}))_{|\xi} \\
&= \frac{d}{d\theta}(\overline{x_3(e^{i\theta})})_{|\xi}(x_3(e^{i\theta}) - x_1(e^{i\theta})x_2(e^{i\theta}))_{|\xi} + \overline{x_3(e^{i\theta})}_{|\xi}\frac{d}{d\theta}(x_3(e^{i\theta}) - x_1(e^{i\theta})x_2(e^{i\theta}))_{|\xi} \\
&= \overline{x_3(e^{i\xi})}\left[\frac{d}{d\theta}x_3(e^{i\theta})_{|\xi} - \frac{d}{d\theta}(x_1(e^{i\theta})x_2(e^{i\theta}))_{|\xi}\right] \\
&= \overline{x_3(e^{i\xi})}\left[ie^{i\xi}x_3'(e^{i\xi}) - (x_1(e^{i\xi})ie^{i\xi}x_2'(e^{i\xi}) + ie^{i\xi}x_1'(e^{i\xi})x_2(e^{i\xi}))\right].
\end{aligned}$$

Note that $|x_3(e^{i\xi})| = 1$, hence $x_3'(\sigma) = x_1(\sigma)x_2'(\sigma) + x_1'(\sigma)x_2(\sigma)$. Here we have $x_3(\sigma) - x_1(\sigma)x_2(\sigma) = 0$ and $(x_3(\sigma) - x_1(\sigma)x_2(\sigma))' = 0$. Therefore σ is a zero of $(x_3 - x_1x_2)$ of multiplicity at least 2. □

Proposition 3.2.2. *Let* $x = (x_1, x_2, x_3)$ *be a rational* $\overline{\mathcal{E}}$*- inner function. Let* $\sigma \in \mathbb{T}$ *be a royal node of* x*. Suppose* $x(\sigma) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$*,* $\omega \in \mathbb{T}$ and $\omega\tilde{\eta} \neq 1$ *Then*

$$A(\Psi_\omega \circ x)(\sigma) = Ax_1(\sigma).$$

Proof. Since x is a rational $\overline{\mathcal{E}}$-inner function, then for almost all $\lambda \in \mathbb{T}$, $x(\lambda) \in b\overline{\mathcal{E}}$, and, by Theorem 3.1.8, for almost all $\lambda \in \mathbb{T}$, $x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda)$, $|x_3(\lambda)| = 1$ and $|x_2(\lambda)| \leq 1$. By Proposition (A.2.2), for every $z \in \mathbb{T}$, and every rational inner function φ,

$$A\varphi(z) = z\frac{\varphi'(z)}{\varphi(z)}.$$

For $\sigma \in \mathbb{T}$ such that $x(\sigma) \in \mathcal{R}_{\overline{\mathcal{E}}}$ and $\omega\tilde{\eta} \neq 1$,

$$\begin{aligned}
A(\Psi_\omega \circ x)(\sigma) &= A(\omega x_3 - x_1)(\sigma) - A(x_2\omega - 1)(\sigma) \\
&= \sigma\frac{(\omega x_3 - x_1)'(\sigma)}{(\omega x_3 - x_1)(\sigma)} - \sigma\frac{(x_2\omega - 1)'(\sigma)}{(x_2\omega - 1)(\sigma)} \\
&= \frac{\sigma}{\omega\tilde{\eta} - 1}\Big(\frac{\omega x_3'(\sigma) - x_1'(\sigma)}{\eta} - \omega x_2'(\sigma)\Big)
\end{aligned}$$

Since $x_3(\sigma) \in \mathcal{R}_{\bar{\mathcal{E}}}$, we have $x_3(\sigma) = x_1(\sigma)x_2(\sigma)$, and, by Lemma 3.2.1, σ is a zero of $x_3 - x_1x_2$ of multiplicity at least 2. Thus $(x_3 - x_1x_2)'(\sigma) = 0$ and

$$x_3'(\sigma) = x_1(\sigma)x_2'(\sigma) + x_2(\sigma)x_1'(\sigma) = \eta x_2'(\sigma) + \tilde{\eta}x_1'(\sigma). \tag{3.11}$$

Thus, by equation (3.11), we have

$$\begin{aligned}
A(\Psi_\omega \circ x)(\sigma) &= \frac{\sigma}{\omega\tilde{\eta} - 1}\Big(\frac{\omega(\eta x_2'(\sigma) + \tilde{\eta}x_1'(\sigma)) - x_1'(\sigma)}{\eta} - \omega x_2'(\sigma)\Big) \\
&= \frac{\sigma}{\omega\tilde{\eta} - 1}\Big(\frac{\omega\eta x_2'(\sigma) + \omega\tilde{\eta}x_1'(\sigma) - x_1'(\sigma)}{\eta} - \omega x_2'(\sigma)\Big) \\
&= \frac{\sigma}{\omega\tilde{\eta} - 1}\Big(\frac{\omega\eta x_2'(\sigma) + x_1'(\sigma)(\omega\tilde{\eta} - 1) - \eta\omega x_2'(\sigma)}{\eta}\Big) \\
&= \sigma\Big(\frac{x_1'(\sigma)}{\eta}\Big) \\
&= \sigma\Big(\frac{x_1'(\sigma)}{x_1(\sigma)}\Big) = Ax_1(\sigma).
\end{aligned}$$

□

Proposition 3.2.3. *Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathcal{E}}$- inner function. Let $\sigma \in \mathbb{T}$ be a royal node of x. Suppose $x(\sigma) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$, $\omega \in \mathbb{T}$* and *$\omega\eta \neq 1$. Then*

$$A(\Upsilon_\omega \circ x)(\sigma) = Ax_2(\sigma).$$

Proof. Since x is a rational $\overline{\mathcal{E}}$-inner function, then for almost all $\lambda \in \mathbb{T}$, $x(\lambda) \in b\overline{\mathcal{E}}$, and, by Theorem 3.1.8 (i), for almost all $\lambda \in \mathbb{T}$, $x_1 = \bar{x}_2x_3, |x_3| = 1$ and $|x_2| \leq 1$. By Proposition (A.2.2), for every $z \in \mathbb{T}$, and every rational inner function φ,

$$A\varphi(z) = z\frac{\varphi'(z)}{\varphi(z)}.$$

For $\sigma \in \mathbb{T}$ such that $x(\sigma) \in \mathcal{R}_{\bar{\mathcal{E}}}$, and $\omega\eta \neq 1$,

$$\begin{aligned} A(\Upsilon_\omega \circ x)(\sigma) &= A(\omega x_3 - x_2)(\sigma) - A(\omega x_1 - 1)(\sigma) \\ &= \sigma \frac{(\omega x_3 - x_2)'(\sigma)}{(\omega x_3 - x_2)(\sigma)} - \sigma \frac{(x_1\omega - 1)'(\sigma)}{(x_1\omega - 1)(\sigma)} \\ &= \frac{\sigma}{\omega\eta - 1}\Big(\frac{\omega x_3'(\sigma) - x_2'(\sigma)}{\tilde{\eta}} - \omega x_1'(\sigma)\Big) \end{aligned}$$

Since $x_3(\sigma) \in \mathcal{R}_{\bar{\mathcal{E}}}$, we have $x_3(\sigma) = x_1(\sigma)x_2(\sigma)$, and, by Lemma 3.2.1, σ is a zero of $x_3 - x_1x_2$ of multiplicity at least 2. Thus $(x_3 - x_1x_2)'(\sigma) = 0$ and

$$x_3'(\sigma) = x_1(\sigma)x_2'(\sigma) + x_2(\sigma)x_1'(\sigma) = \eta x_2'(\sigma) + \tilde{\eta}x_1'(\sigma). \tag{3.12}$$

Thus, by equation (3.12), we have

$$\begin{aligned} A(\Upsilon_\omega \circ x)(\sigma) &= \frac{\sigma}{\omega\eta - 1}\Big(\frac{\omega(\eta x_2'(\sigma) + \tilde{\eta}x_1'(\sigma)) - x_2'(\sigma)}{\tilde{\eta}} - \omega x_1'(\sigma)\Big) \\ &= \frac{\sigma}{\omega\eta - 1}\Big(\frac{\omega\eta x_2'(\sigma) + \omega\tilde{\eta}x_1'(\sigma) - x_2'(\sigma)}{\tilde{\eta}} - \omega x_1'(\sigma)\Big) \\ &= \frac{\sigma}{\omega\eta - 1}\Big(\frac{\omega\eta x_2'(\sigma) + x_1'(\sigma)\omega\tilde{\eta} - x_2'(\sigma) - \tilde{\eta}\omega x_1'(\sigma)}{\tilde{\eta}}\Big) \\ &= \frac{\sigma}{\omega\eta - 1}\Big(\frac{x_2'(\sigma)(\omega\eta - 1)}{\tilde{\eta}}\Big) \\ &= \sigma\Big(\frac{x_2'(\sigma)}{\tilde{\eta}}\Big) \\ &= \sigma\Big(\frac{x_2'(\sigma)}{x_2(\sigma)}\Big) = Ax_2(\sigma). \end{aligned}$$

□

3.3 Rational tetra-inner functions and royal polynomials

In this section we will show how to construct rational $\bar{\mathcal{E}}$-inner functions with prescribed royal nodes and values. To describe this construction we need several theorems and definitions from O. M. Alsalhi's PhD thesis [12]. Detailed proofs of these statements are given in [12].

Theorem 3.3.1. [12, Theorem 4.3.1] *If $x = (x_1, x_2, x_3)$ is a rational $\bar{\mathcal{E}}$-inner function of degree n, then there exist polynomials E_1, E_2, D such that*

(i) $\deg(E_1)$, $\deg(E_2)$, $\deg(D) \le n$,

(ii) $D(\lambda) \neq 0$ *on* $\overline{\mathbb{D}}$,

(iii) $E_1(\lambda) = E_2^{\sim n}(\lambda)$, *for all* $\lambda \in \mathbb{T}$, *where* $E_2^{\sim n}(\lambda) = \lambda^n \overline{E_2(\frac{1}{\bar{\lambda}})}$,

(iv) $|E_i(\lambda)| \le |D(\lambda)|$ *on* $\overline{\mathbb{D}}$, $i = 1, 2$,

(v) $x_1 = \frac{E_1}{D}$ *on* $\overline{\mathbb{D}}$,

(vi) $x_2 = \frac{E_2}{D}$ *on* $\overline{\mathbb{D}}$,

(vii) $x_3 = \frac{D^{\sim n}}{D}$ *on* $\overline{\mathbb{D}}$, *where* $D^{\sim n}(\lambda) = \lambda^n \overline{D(\frac{1}{\bar{\lambda}})}$.

Remark 3.3.2. *Consider a rational $\bar{\mathcal{E}}$-inner function $x = (x_1, x_2, x_3)$. Let E_1, E_2, D be as in* Theorem 3.3.1*, and let $R_x(\lambda)$ be the polynomial defined by*

$$R_x(\lambda) = D(\lambda)^2(-x_1(\lambda)x_2(\lambda) + x_3(\lambda)).$$

Then, by Theorem 3.3.1*,*

$$\begin{aligned} R_x(\lambda) &= D(\lambda)^2 \left(\frac{D^{\sim n}(\lambda)}{D(\lambda)} - \frac{E_1(\lambda)}{D(\lambda)} \frac{E_2(\lambda)}{D(\lambda)} \right) \\ &= D(\lambda)D^{\sim n}(\lambda) - E_1(\lambda)E_2(\lambda). \end{aligned}$$

Definition 3.3.3. *Let $x = (x_1, x_2, x_3)$ be a rational tetra-inner function of degree n. The* royal polynomial *of x is*

$$R_x(\lambda) = D(\lambda)D^{\sim n}(\lambda) - E_1(\lambda)E_2(\lambda),$$

where E_1, E_2, D be as in Theorem 3.3.1.

Remark 3.3.4. *For a rational tetra-inner function x, since $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$, zeroes of R_x are zeroes of the function $x_3 - x_1x_2$. As we defined above they are called* the royal nodes *of x.*

Remark 3.3.5. *Let $\sigma \in \bar{\mathbb{D}}$ be a zero of $R_x(\sigma)$. By* Theorem 3.3.1, *$D(\lambda) \neq 0$ on $\bar{\mathbb{D}}$. Thus,*

$$R_x(\sigma) = D(\sigma)^2(x_3(\sigma) - x_1(\sigma)x_2(\sigma)) = 0 \text{ if and only if } x_3(\sigma) = x_1(\sigma)x_2(\sigma).$$

Let $x_1(\sigma) = \eta$ and $x_2(\sigma) = \tilde{\eta}$, then

$$x_3(\sigma) = x_1(\sigma)x_2(\sigma) = \eta\tilde{\eta}.$$

Therefore, if $\sigma \in \bar{\mathbb{D}}$ is a royal node of x, then $x(\sigma) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$ for some $\eta, \tilde{\eta} \in \bar{\mathbb{D}}$.

We call $(\eta, \tilde{\eta}, \eta\tilde{\eta})$ the *royal value* of x at σ.

Lemma 3.3.6. *Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathcal{E}}$-inner function, and $\sigma \in \bar{\mathbb{D}}$ is a royal node of x. If $\sigma \in \mathbb{T}$, then $|x_1(\sigma)| = 1$ and $|x_2(\sigma)| = 1$.*

Proof. Since x is $\overline{\mathcal{E}}$-inner function, by Definition 3.1.12, $x(\sigma) \in b\overline{\mathcal{E}}$ for $\sigma \in \mathbb{T}$. By Theorem 3.1.8, $x_1(\sigma) = \overline{x_2(\sigma)}x_3(\sigma)$ and $|x_3(\sigma)| = 1, |x_2(\sigma)| \leq 1$. By assumption σ is a royal node of x. Thus by Definition 3.3.4, $x_3(\sigma) = x_1(\sigma)x_2(\sigma)$ which implies that $|x_1(\sigma)| = 1$ and $|x_2(\sigma)| = 1$ since $|x_3(\sigma)| = 1$. □

Proposition 3.3.7. [12] *Let x be a rational $\overline{\mathcal{E}}$-inner function of degree n and let R_x be the royal polynomial of x. Then R_x is $2n$-symmetric and the zeros of R_x on $\mathbb{T}$ have even order or infinite order.*

Definition 3.3.8. *Let $x = (x_1, x_2, x_3)$ be a rational $\bar{\mathcal{E}}$-inner function such that $x(\overline{\mathbb{D}}) \nsubseteq \mathcal{R}_{\overline{\mathcal{E}}}$ and let R_x be the royal polynomial of x. If σ is a zero of R_x of order ℓ, we define the multiplicity $\#\sigma$ of σ (as a royal node of x) by*

$$\#\sigma \quad = \quad \begin{cases} \ell & \text{if } \sigma \in \mathbb{D} \\ \frac{1}{2}\ell & \text{if } \sigma \in \mathbb{T}. \end{cases}$$

We define the type *of x to be the ordered pair (n, k), where n is the sum of the multiplicities of the royal nodes of x that lie in $\bar{\mathbb{D}}$, and k is the sum of the multiplicities of the royal nodes of x that lie in $\mathbb{T}$.*

Definition 3.3.9. *We denote by $\mathcal{R}^{n,k}$ the collection of rational $\bar{\mathcal{E}}$-inner functions of type (n, k).*

Definition 3.3.10. [12] *The degree of a rational $\overline{\mathcal{E}}$-inner function x, denoted by* $\deg(x)$ *is defined to be $x_*(1)$, where $x_* : \mathbb{Z} = \pi_1(\mathbb{T}) \to \pi_1(b\overline{\mathcal{E}})$ is the homomorphism of fundamental groups induced by x when x is regarded as a continuous map from $\mathbb{T}$ to $b\overline{\mathcal{E}}$.*

Proposition 3.3.11. [12] *For any rational $\overline{\mathcal{E}}$-inner function x,* $\deg(x)$ *is the degree* $\deg(x_3)$ *(in the usual sense) of the finite Blaschke product x_3.*

Theorem 3.3.12. [12] *If $x \in \mathcal{R}^{n,k}$ is non-constant, then the degree of x is equal to n.*

Theorem 3.3.13. [12] *Let x be a non-constant rational $\overline{\mathcal{E}}$-inner function of degree n. Then, either $x(\overline{\mathbb{D}}) = \mathcal{R}_{\overline{\mathcal{E}}}$ or $x(\overline{\mathbb{D}})$ meets $\mathcal{R}_{\overline{\mathcal{E}}}$ exactly n times.*

Proposition 3.3.14. *Let $x = (x_1, x_2, x_3)$ be a non-constant rational $\overline{\mathcal{E}}$-inner function and let $\omega \in \mathbb{T}$ be such that $\omega x_2(\lambda) - 1 \neq 0$ for all $\lambda \in \mathbb{D}$. Then the rational function $\Psi_\omega \circ x = \dfrac{\omega x_3 - x_1}{x_2\omega - 1}$ has a cancellation at $\zeta \in \overline{\mathbb{D}}$ if and only if the following conditions are satisfied : $\zeta \in \mathbb{T}$, ζ is a royal node of x and $\omega = \overline{x_2(\zeta)}$.*

Proof. Let $\zeta \in \mathbb{T}$ be a royal node of x such that $x(\zeta) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$. By Lemma 3.3.6, $|\eta| = 1$ and $|\tilde{\eta}| = 1$. If $\omega = \overline{\tilde{\eta}} \in \mathbb{T}$, then

$$\omega x_3(\zeta) - x_1(\zeta) = \overline{\tilde{\eta}}\eta\tilde{\eta} - \eta = |\tilde{\eta}|^2\eta - \eta = \eta - \eta = 0,$$

and

$$x_2(\zeta)\omega - 1 = \tilde{\eta}\overline{\tilde{\eta}} - 1 = |\tilde{\eta}|^2 - 1 = 0.$$

Thus, $\Psi_\omega \circ x = \dfrac{\omega x_3(\lambda) - x_1(\lambda)}{x_2(\lambda)\omega - 1}$ has at least one cancellation at such $\zeta \in \mathbb{T}$.

Conversely, by assumption $\Psi_\omega \circ x$ has a cancellation at $\zeta \in \overline{\mathbb{D}}$, and so

$$(\omega x_3 - x_1)(\zeta) = 0 = (x_2\omega - 1)(\zeta).$$

Therefore, $x_2(\zeta)\omega = 1$ and $\omega x_3(\zeta) = x_1(\zeta)$. Since $x_2(\zeta)\omega = 1$, it implies that $x_2(\zeta) = \overline{\omega} \in \mathbb{T}$, so $|x_2(\zeta)| = 1$. Since $x_2 : \mathbb{D} \to \overline{\mathbb{D}}$ rational and analytic function with $|x_2(\zeta)| = 1$, by the maximum principle theorem, $\zeta \in \mathbb{T}$, or $x_2(\lambda) = \overline{\omega}$ for all $\lambda \in \overline{\mathbb{D}}$. By assumption, $\omega x_2(\lambda) - 1 \neq 0$ for all $\lambda \in \mathbb{D}$. Hence the function $x_2 \neq \overline{\omega}$ on $\mathbb{D}$. Therefore, $\zeta \in \mathbb{T}$.

Note

$$\begin{aligned}\omega x_3(\zeta) = x_1(\zeta) &\implies \overline{x_2(\zeta)} x_3(\zeta) = x_1(\zeta) \\ &\implies x_3(\zeta) = x_1(\zeta) x_2(\zeta).\end{aligned}$$

Thus, $\zeta \in \mathbb{T}$ is a royal node for x, and $\omega = \overline{x_2(\zeta)}$. $\square$

Chapter 4

Prescribing the royal nodes and values

In this chapter we will show how to construct rational $\overline{\mathcal{E}}$-inner functions with prescribed royal nodes and values, with the aid of a solution of an interpolation theorem for finite Blaschke products. The connection between the solution sets of the royal $\overline{\mathcal{E}}$-interpolation problem and the Blaschke interpolation problem can be made explicitly with the aid of the functions Ψ_ω. The main aim of this chapter is to prove Theorem 4.1.1 and Theorem 4.2.5.

4.1 From the royal tetra-interpolation problem to the Blaschke interpolation problem

In this section we show that for the given Blaschke interpolation data (σ, η, ρ) the existence of solution x for the royal tetra-interpolation problem $(\sigma, \eta, \tilde{\eta}, \rho)$ for some $\tilde{\eta}_j \in \overline{\mathbb{D}}$ allows us to construct a solution for the Blaschke interpolation problem.

Theorem 4.1.1. *Let $x = (x_1, x_2, x_3)$ be a rational $\bar{\mathcal{E}}$-inner function of type (n, k) having distinct royal nodes $\sigma_1, \sigma_2, ..., \sigma_n$ where $\sigma_1, \sigma_2, ..., \sigma_k \in \mathbb{T}$ and $\sigma_{k+1}, ..., \sigma_n \in \mathbb{D}$ and corresponding royal values $\eta_1, .., \eta_n$ and $\tilde{\eta}_1, ..., \tilde{\eta}_n$, that is, $x(\sigma_j) = (\eta_j, \tilde{\eta}_j, \eta_j\tilde{\eta}_j)$. Let $\rho_j = Ax_1(\sigma_j)$ for $j = 1, 2, .., k$.*

(1) *There exists a rational inner function* φ *that solves the Blaschke interpolation* Problem 1.2.2 *for* (σ, η, ρ)*, that is, such that deg*$(\varphi) = n$*,*

$$\varphi(\sigma_j) = \eta_j \quad \text{for } j = 1, ..., n \tag{4.1}$$

and

$$A\varphi(\sigma_j) = \rho_j \quad \text{for } j = 1, ..., k. \tag{4.2}$$

Any such function φ *is expressible in the form* $\varphi = \Psi_\omega \circ x$ *for some* $\omega \in \mathbb{T}$.

(2) *There exist polynomials* a, b, c, d *of degree at most* n *such that a normalized parametrization of the solutions of* Problem 1.2.2 *is*

$$\varphi = \frac{a\zeta + b}{c\zeta + d}, \quad \zeta \in \mathbb{T}.$$

(3) *For any polynomials* a, b, c, d *as in* (2)*, there exist* $x_1^\circ, x_2^\circ, x_3^\circ \in \mathbb{C}$ *such that*

$$|x_3^\circ| = 1, \quad |x_1^\circ| < 1, \quad |x_2^\circ| < 1, \tag{4.3}$$

$$x_1^\circ = \overline{x_2^\circ} x_3^\circ, \tag{4.4}$$

and moreover,

$$x_1 = \frac{x_1^\circ a + b}{x_1^\circ c + d} \tag{4.5}$$

$$x_2 = \frac{x_3^\circ c + x_2^\circ d}{x_1^\circ c + d} \tag{4.6}$$

$$x_3 = \frac{x_2^\circ b + x_3^\circ a}{x_1^\circ c + d}. \tag{4.7}$$

Proof. (1) For $\omega \in \mathbb{T}$ and for a given rational $\overline{\mathcal{E}}$-inner function $x = (x_1, x_2, x_3) : \mathbb{D} \to \overline{\mathcal{E}}$, we consider the rational function $\psi_\omega : \mathbb{D} \to \overline{\mathbb{D}}$

$$\psi_\omega(\lambda) = \Psi_\omega \circ x(\lambda) \;=\; \frac{x_3\omega - x_1}{x_2\omega - 1}(\lambda). \tag{4.8}$$

Then, if $\omega \neq \overline{\tilde{\eta}_1}, ..., \overline{\tilde{\eta}_k}$,

$$\psi_\omega(\sigma_j) \;=\; \frac{x_3(\sigma_j)\omega - x_1(\sigma_j)}{x_2(\sigma_j)\omega - 1} \;=\; \frac{\eta_j\tilde{\eta}_j\omega - \eta_j}{\tilde{\eta}_j\omega - 1} \;=\; \eta_j\frac{\omega\tilde{\eta}_j - 1}{\tilde{\eta}_j\omega - 1} = \eta_j \text{ for } j = 1, ..., n. \tag{4.9}$$

We claim that, for $\omega \in \mathbb{T}\backslash\{\overline{\tilde{\eta}_1}, ..., \overline{\tilde{\eta}_k}\}$, the function $\varphi = \psi_\omega$ is a solution of Problem 1.2.2. Let us check that φ is an inner function from $\mathbb{D}$ to $\overline{\mathbb{D}}$. For any $\lambda \in \mathbb{T}$,

$$\varphi(\lambda) = \psi_\omega(\lambda) = \frac{\omega x_3(\lambda) - x_1(\lambda)}{x_2(\lambda)\omega - 1}.$$

Since x is a rational $\overline{\mathcal{E}}$-inner function, $x(\lambda) \in b\overline{\mathcal{E}}$ for almost all $\lambda \in \mathbb{T}$, and, by Theorem 3.1.8, $x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda)$ and $|x_3(\lambda)| = 1$ for almost all $\lambda \in \mathbb{T}$. Thus, for almost all $\lambda \in \mathbb{T}$,

$$\varphi(\lambda) = \psi_\omega(\lambda) = \frac{\omega x_3(\lambda) - \overline{x_2(\lambda)}x_3(\lambda)}{x_2(\lambda)\omega - 1} = \frac{x_3(\lambda)(\omega - \overline{x_2(\lambda)})}{\omega x_2(\lambda) - 1}.$$

Hence, for $\lambda \in \mathbb{T}$,

$$|\varphi(\lambda)| = |x_3(\lambda)| \left|\frac{\omega - \overline{x_2(\lambda)}}{\omega x_2(\lambda) - 1}\right|.$$

Since $|x_3(\lambda)| = 1, |\omega| = 1$ and $|\omega - \overline{x_2(\lambda)}| = |\overline{\omega} - x_2(\lambda)|$, we have, for almost all $\lambda \in \mathbb{T}$,

$$\left|\frac{\omega(\overline{\omega} - x_2(\lambda))}{\omega x_2(\lambda) - 1}\right| = \left|\frac{1 - x_2(\lambda)\omega}{-(1 - x_2(\lambda)\omega)}\right| = 1.$$

Therefore, for almost all $\lambda \in \mathbb{T}, |\varphi(\lambda)| = 1$. Hence φ is rational inner function.

The equation (4.9) shows that ψ_ω takes the required values at $\sigma_1, ..., \sigma_n$. By Proposition 3.2.2,

$$A(\Psi_\omega \circ x)(\sigma_j) = Ax_1(\sigma_j) = \rho_j \quad \text{for } j = 1, 2, ..., k. \tag{4.10}$$

It is also true that $\deg(\psi_\omega) = n$ for $\omega \neq \overline{\tilde{\eta}_1}, ..., \overline{\tilde{\eta}_k}$. By Theorem 3.3.1, for a rational $\overline{\mathcal{E}}$-inner function $x = (x_1, x_2, x_3)$ such that $\deg(x_3) = n$ and if D is the denominator when x_3 is written in its lowest terms then x_1 and x_2 can also be written with denominator D. It follows that

$$\deg(\psi_\omega) = \deg(x_3) - \#\{\text{cancellations between } \omega x_3 - x_1 \text{ and } x_2\omega - 1\} \tag{4.11}$$

By Proposition 3.3.14, such cancellations can occur only at the royal nodes $\sigma_j \in \mathbb{T}, j = 1, ..., k$, and then only when $\omega = \overline{x_2(\sigma_j)} = \overline{\tilde{\eta}_j}, j = 1, ..., k$. Hence there are no cancellations in equation (4.11), and so $\deg(\psi_\omega) = n$.

(2) Since Problem 1.2.2 is solvable, its Pick matrix is positive definite and so, by Theorem 2.1.10, there exist polynomials a, b, c, d of degree at most n which parametrise the solutions of Problem 1.2.2. Let us choose a particular such 4-tuple of polynomials, as described in Theorem 2.1.10. By Theorem 2.1.9, there exists $\tau \in \mathbb{T}\backslash\{\sigma_1, ..., \sigma_k\}$ such that the exceptional set Z_τ for Problem 1.2.2 which defined as

$$Z_\tau = \{\zeta \in \mathbb{T} : \text{ for some } j, 1 \le j \le k, \langle M^{-1}u_{\zeta,\tau}, e_j\rangle = 0\} \tag{4.12}$$

consists of at most k points. Fix such a $\tau \in \mathbb{T}\backslash\{\sigma_1, ..., \sigma_k\}$ such that Z_τ consists of at most k points, then there exist unique polynomials $a_\tau, b_\tau, c_\tau, d_\tau$ of degree at most n such that

$$\begin{bmatrix} a_\tau(\tau) & b_\tau(\tau) \\ c_\tau(\tau) & d_\tau(\tau) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.13}$$

and, for all $\zeta \in \mathbb{T} \setminus Z_\tau$, the function

$$\varphi = \frac{a_\tau\zeta + b_\tau}{c_\tau\zeta + d_\tau} \tag{4.14}$$

is the unique solution of Problem 1.2.2 that satisfies $\varphi(\tau) = \zeta$. Moreover, the general 4-tuple of polynomials that parametrises the solutions of Problem 1.2.2 is expressible in the form

$$(a, b, c, d) = (Xa_\tau, Xb_\tau, Xc_\tau, Xd_\tau) \tag{4.15}$$

for some rational function X.

(3) For $\tau \in \mathbb{T} \setminus \{\sigma_1, ..., \sigma_k\}$ as above, let $x_1^\circ = x_1(\tau), x_2^\circ = x_2(\tau), x_3^\circ = x_3(\tau)$. Since x is tetra-inner, by Theorem 3.1.8, $|x_3^\circ| = 1$ and $x_1^\circ = \overline{x_2^\circ}x_3^\circ$. Since τ is chosen not to be a royal node of x, $|x_1^\circ| < 1, |x_2^\circ| < 1$. Thus the equations (4.31) and (4.32) hold.

Lemma 4.1.2. *Let $x_1^\circ, x_2^\circ, x_3^\circ \in \mathbb{C}$ such that*

$$|x_3^\circ| = 1, \quad |x_1^\circ| < 1, \quad |x_2^\circ| < 1, \tag{4.16}$$

$$x_1^\circ = \overline{x_2^\circ}x_3^\circ. \tag{4.17}$$

Let Z_τ define as in (4.12), *let $\tau \in \mathbb{T}\backslash\{\sigma_1, ..., \sigma_k\}$ such that Z_τ consists of at most k points, and let*

$$Z_\tau^\sim = \left\{\frac{\overline{\tilde{\eta}_1}x_3^\circ - x_1^\circ}{x_2^\circ\overline{\tilde{\eta}_1} - 1}, \frac{\overline{\tilde{\eta}_2}x_3^\circ - x_1^\circ}{x_2^\circ\overline{\tilde{\eta}_2} - 1}, ..., \frac{\overline{\tilde{\eta}_k}x_3^\circ - x_1^\circ}{x_2^\circ\overline{\tilde{\eta}_k} - 1}\right\}.$$

If $\zeta \in \mathbb{T} \setminus Z_\tau^\sim$ *then the function*

$$\varphi = \frac{(x_2^\circ x_1 - x_3)\zeta + x_1^\circ x_3 - x_1 x_3^\circ}{(x_2^\circ - x_2)\zeta + x_1^\circ x_2 - x_3^\circ} \tag{4.18}$$

is a solution for Problem 1.2.2 *and satisfies* $\varphi(\tau) = \zeta$.

Proof. Observe that, by equation (4.8), for any $\omega \in \mathbb{T}$, $\psi_\omega(\tau) = \frac{\omega x_3^\circ - x_1^\circ}{x_2^\circ \omega - 1}$, which is well defined since $|x_2^\circ| < 1$. We have, for $\zeta \in \mathbb{T}$,

$$\psi_\omega(\tau) = \zeta \Leftrightarrow \frac{\omega x_3^\circ - x_1^\circ}{x_2^\circ \omega - 1} = \zeta \Leftrightarrow \omega = \frac{-\zeta + x_1^\circ}{x_3^\circ - \zeta x_2^\circ}.$$

Hence, as long as

$$\frac{-\zeta + x_1^\circ}{x_3^\circ - \zeta x_2^\circ} \neq \overline{\tilde{\eta}_1}, ..., \overline{\tilde{\eta}_k}, \tag{4.19}$$

the function

$$\begin{aligned}
\varphi(\lambda) = \psi_\omega(\lambda) &= \psi_{\frac{-\zeta + x_1^\circ}{x_3^\circ - \zeta x_2^\circ}}(\lambda) \\
&= \frac{\dfrac{x_3(\lambda)x_1^\circ - x_3(\lambda)\zeta}{x_3^\circ - x_2^\circ \zeta} - x_1(\lambda)}{\dfrac{x_2(\lambda)x_1^\circ - x_2(\lambda)\zeta}{x_3^\circ - x_2^\circ \zeta} - 1} \\
&= \frac{x_3(\lambda)x_1^\circ - x_3(\lambda)\zeta - x_1(\lambda)x_3^\circ + x_1(\lambda)x_2^\circ \zeta}{x_2(\lambda)x_1^\circ - x_2(\lambda)\zeta - x_3^\circ + x_2^\circ \zeta} \\
&= \frac{(x_1(\lambda)x_2^\circ - x_3(\lambda))\zeta + x_1^\circ x_3(\lambda) - x_1(\lambda)x_3^\circ}{(x_2^\circ - x_2(\lambda))\zeta + x_1^\circ x_2(\lambda) - x_3^\circ}.
\end{aligned}$$

is a solution of Problem 1.2.2 which satisfies $\varphi(\tau) = \zeta$. Condition (4.19) can equally be written as, for $j = 1, 2, ..., k$,

$$\zeta \neq \frac{x_1^\circ - \overline{\tilde{\eta}_j} x_3^\circ}{1 - x_2^\circ \overline{\tilde{\eta}_j}} = \frac{\overline{\tilde{\eta}_j} x_3^\circ - x_1^\circ}{x_2^\circ \overline{\tilde{\eta}_j} - 1}$$

or equivalently $\zeta \notin Z_\tau^\sim$. □

For $\zeta \in \mathbb{T} \setminus (Z_\tau \cup Z_\tau^\sim)$ where $Z_\tau^\sim$ is defined in Lemma 4.1.2, we have two expressions for the unique solution of Problem 1.2.2 for which $\varphi(\tau) = \zeta$, to wit the equations (4.14) and (4.18). Note that

$$\begin{aligned}
\begin{bmatrix} x_2^\circ x_1(\tau) - x_3(\tau) & x_1^\circ x_3(\tau) - x_1(\tau)x_3^\circ \\ x_2^\circ - x_2(\tau) & x_1^\circ x_2(\tau) - x_3^\circ \end{bmatrix} &= \begin{bmatrix} x_2^\circ x_1^\circ - x_3^\circ & x_1^\circ x_3^\circ - x_1^\circ x_3^\circ \\ x_2^\circ - x_2^\circ & x_1^\circ x_2^\circ - x_3^\circ \end{bmatrix} \\
&= (x_1^\circ x_2^\circ - x_3^\circ) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

Since the set $(Z_\tau \cup Z_\tau^\sim)$ is finite, the linear fractional transformations in equations (4.14) and (4.18) are equal at infinitely many points, hence coincide. On taking account of normalising condition we obtain

$$\begin{bmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau \end{bmatrix} = \frac{1}{x_1^\circ x_2^\circ - x_3^\circ} \begin{bmatrix} x_2^\circ x_1 - x_3 & x_1^\circ x_3 - x_1 x_3^\circ \\ x_2^\circ - x_2 & x_1^\circ x_2 - x_3^\circ \end{bmatrix}. \tag{4.20}$$

Suppose that a, b, c and d are polynomials that parametrise the solutions of Problem 1.2.2, as in Theorem 4.1.1 (2). By the observation (4.15), there exists a rational function X such that

$$Xa = x_2^\circ x_1 - x_3, \tag{4.21}$$

$$Xb = x_1^\circ x_3 - x_1 x_3^\circ, \tag{4.22}$$

$$Xc = x_2^\circ - x_2, \tag{4.23}$$

$$Xd = x_1^\circ x_2 - x_3^\circ, \tag{4.24}$$

Let us find connections between x_1, x_2, x_3 and the polynomials a, b, c, d. Equations (4.23) and (4.24) for x_2 and X could be written as

$$\begin{aligned} Xc + x_2 &= x_2^\circ \\ Xd - x_1^\circ x_2 &= -x_3^\circ. \end{aligned} \tag{4.25}$$

Then, the solution of the system (4.25) is

$$X = \frac{\begin{vmatrix} x_2^\circ & 1 \\ -x_3^\circ & -x_1^\circ \end{vmatrix}}{\begin{vmatrix} c & 1 \\ d & -x_1^\circ \end{vmatrix}} = \frac{x_1^\circ x_2^\circ - x_3^\circ}{x_1^\circ c + d} \tag{4.26}$$

and

$$x_2 = \frac{\begin{vmatrix} c & x_2^\circ \\ d & -x_3^\circ \end{vmatrix}}{\begin{vmatrix} c & 1 \\ d & -x_1^\circ \end{vmatrix}} = \frac{x_3^\circ c + x_2^\circ d}{x_1^\circ c + d}. \tag{4.27}$$

Equations (4.21) and (4.22) give us the system

$$\begin{aligned} x_2^\circ x_1 - x_3 &= Xa \\ -x_3^\circ x_1 + x_1^\circ x_3 &= Xb. \end{aligned} \tag{4.28}$$

Then, the solution of the system (4.28) is

$$x_1 = \frac{\begin{vmatrix} Xa & -1 \\ Xb & x_1^\circ \end{vmatrix}}{\begin{vmatrix} x_2^\circ & -1 \\ -x_3^\circ & x_1^\circ \end{vmatrix}} = \frac{\begin{vmatrix} \dfrac{x_1^\circ x_2^\circ a - x_3^\circ a}{x_1^\circ c + d} & -1 \\ \dfrac{x_1^\circ x_2^\circ b - x_3^\circ b}{x_1^\circ c + d} & x_1^\circ \end{vmatrix}}{\begin{vmatrix} x_2^\circ & -1 \\ -x_3^\circ & x_1^\circ \end{vmatrix}}$$

$$= \frac{\dfrac{x_1^{\circ 2} x_2^\circ a - x_1^\circ x_3^\circ a + x_1^\circ x_2^\circ b - x_3^\circ b}{x_1^\circ c + d}}{x_1^\circ x_2^\circ - x_3^\circ}$$

$$= \frac{\dfrac{x_1^\circ a(x_1^\circ x_2^\circ - x_3^\circ) + b(x_1^\circ x_2^\circ - x_3^\circ)}{x_1^\circ c + d}}{x_1^\circ x_2^\circ - x_3^\circ}$$

$$= \frac{x_1^\circ a + b}{x_1^\circ c + d}$$

and

$$x_3 = \frac{\begin{vmatrix} x_2^\circ & Xa \\ -x_3^\circ & Xb \end{vmatrix}}{\begin{vmatrix} x_2^\circ & -1 \\ -x_3^\circ & x_1^\circ \end{vmatrix}} = \frac{\begin{vmatrix} x_2^\circ & \dfrac{x_1^\circ x_2^\circ a - x_3^\circ a}{x_1^\circ c + d} \\ -x_3^\circ & \dfrac{x_1^\circ x_2^\circ b - x_3^\circ b}{x_1^\circ c + d} \end{vmatrix}}{\begin{vmatrix} x_2^\circ & -1 \\ -x_3^\circ & x_1^\circ \end{vmatrix}}$$

$$= \frac{\dfrac{x_1^\circ x_2^{\circ 2} b - x_2^\circ x_3^\circ b + x_1^\circ x_2^\circ x_3^\circ a - x_3^{\circ 2} a}{x_1^\circ c + d}}{x_1^\circ x_2^\circ - x_3^\circ}$$

$$= \frac{\dfrac{x_2^\circ b(x_1^\circ x_2^\circ - x_3^\circ) + x_3^\circ a(x_1^\circ x_2^\circ - x_3^\circ)}{x_1^\circ c + d}}{x_1^\circ x_2^\circ - x_3^\circ}$$

$$= \frac{x_2^\circ b + x_3^\circ a}{x_1^\circ c + d}.$$

Thus x_1, x_2, x_3 are given by equations (4.33), (4.34) and (4.35). The proof of Theorem 4.1.1 is complete. □

Note that we can also prove a result similar to Theorem 4.1.1, using the function Υ_ω instead of Ψ_ω, where

$$\Upsilon_\omega(x_1, x_2, x_3) = \frac{x_3\omega - x_2}{x_1\omega - 1},$$

which is defined for every (x_1, x_2, x_3) in $\mathbb{C}^3$ such that $x_1\omega - 1 \neq 0$. .

Theorem 4.1.3. *Let $x = (x_1, x_2, x_3)$ be a rational $\bar{\mathcal{E}}$-inner function of type (n, k) having distinct royal nodes $\sigma_1, \sigma_2, ..., \sigma_n$ where $\sigma_1, \sigma_2, ..., \sigma_k \in \mathbb{T}$ and $\sigma_{k+1}, ..., \sigma_n \in \mathbb{D}$ and corresponding royal values $\eta_1, .., \eta_n$ and $\tilde{\eta_1}, ..., \tilde{\eta_n}$, that is, $x(\sigma_j) = (\eta_j, \tilde{\eta}_j, \eta_j\tilde{\eta}_j)$. Let $\rho_j = Ax_2(\sigma_j)$ for $j = 1, 2, .., k$.*

(1) *There exists a rational inner function φ that solves the Blaschke interpolation* Problem 1.2.2 *for (σ, η, ρ) that is, such that $deg(\varphi) = n$.*

$$\varphi(\sigma_j) = \eta_j \quad \text{for } j = 1, ..., n \tag{4.29}$$

and

$$A\varphi(\sigma_j) = \rho_j \quad \text{for } j = 1, ..., k. \tag{4.30}$$

Any such function φ is expressible in the form $\varphi = \Upsilon_\omega \circ x$ for some $\omega \in \mathbb{T}$.

(2) *There exist polynomials a, b, c, d of degree at most n such that a normalized parametrization of the solutions of* Problem 1.2.2 *is*

$$\varphi = \frac{a\zeta + b}{c\zeta + d}, \quad \zeta \in \mathbb{T}.$$

(3) *For any polynomials a, b, c, d as in (2), there exist $x_1^\circ, x_2^\circ, x_3^\circ \in \mathbb{C}$ such that*

$$|x_3^\circ| = 1, \quad |x_1^\circ| < 1, \quad |x_2^\circ| < 1, \tag{4.31}$$

$$x_1^\circ = \overline{x_2^\circ} x_3^\circ, \tag{4.32}$$

and moreover,

$$x_1 = \frac{x_1^\circ a + b}{x_1^\circ c + d} \tag{4.33}$$

$$x_2 = \frac{x_3^\circ c + x_2^\circ d}{x_1^\circ c + d} \tag{4.34}$$

$$x_3 = \frac{x_2^\circ b + x_3^\circ a}{x_1^\circ c + d}. \tag{4.35}$$

4.2 From the Blaschke interpolation problem to the royal tetra-interpolation problem

In this section we will prove Theorem 4.2.5. This theorem shows that, if Blaschke interpolation data (σ, η, ρ) are given and the Problem 1.2.2 is solvable, then we are able to construct a solution for the royal tetra-interpolation problem $(\sigma, \eta, \tilde{\eta}, \rho)$, for some $\tilde{\eta} = (\tilde{\eta_1}, ..., \tilde{\eta_n})$. We will start with some technical lemmas.

Lemma 4.2.1. *Let $a, b, c, d, x_1^\circ, x_2^\circ, x_3^\circ \in \mathbb{C}$, and suppose that $|x_3^\circ| = 1, x_1^\circ = \overline{x_2^\circ} x_3^\circ, x_1^\circ c \neq -d$ and $|x_1^\circ| < 1, |x_2^\circ| < 1$. Let*

$$x_2 = \frac{x_3^\circ c + x_2^\circ d}{x_1^\circ c + d}.$$

Then

$$(1) \quad |x_2| \le 1 \quad \textit{if and only if } |c| \le |d|,$$

and

$$(2) \quad |x_2| < 1 \quad \textit{if and only if } |c| < |d|.$$

Proof. (1)

$$\begin{aligned}
|x_2| \le 1 \quad &\Leftrightarrow \quad \left| \frac{x_3^\circ c + x_2^\circ d}{x_1^\circ c + d} \right| \le 1 \\
&\Leftrightarrow \quad |x_3^\circ c + x_2^\circ d|^2 \le |x_1^\circ c + d|^2 \\
&\Leftrightarrow \quad (x_3^\circ c + x_2^\circ d)(\overline{x_3^\circ}\overline{c} + \overline{x_2^\circ}\overline{d}) \le (x_1^\circ c + d)(\overline{x_1^\circ}\overline{c} + \overline{d}) \\
&\Leftrightarrow \quad |x_3^\circ|^2|c|^2 + x_3^\circ c \overline{x_2^\circ}\overline{d} + x_2^\circ d \overline{x_3^\circ}\overline{c} + |x_2^\circ|^2|d|^2 \le |x_1^\circ|^2|c|^2 + x_1^\circ c\overline{d} + d\overline{x_1^\circ}\overline{c} + |d|^2 \\
&\Leftrightarrow \quad |c|^2 + 2\mathrm{Re}(x_3^\circ c \overline{x_2^\circ}\overline{d}) + |x_2^\circ|^2|d|^2 \le |x_1^\circ|^2|c|^2 + 2\mathrm{Re}(x_1^\circ c\overline{d}) + |d|^2 \\
&\Leftrightarrow \quad |c|^2 + |x_2|^2|d|^2 - |x_1^\circ|^2|c|^2 - |d|^2 \le 0 \quad (\text{since } x_1^\circ = \overline{x_2^\circ} x_3^\circ) \\
&\Leftrightarrow \quad (1 - |x_1^\circ|^2)(|c|^2 - |d|^2) \le 0 \\
&\Leftrightarrow \quad |c| \le |d| \quad (\text{since } (1 - |x_1^\circ|^2) > 0) \\
&\Leftrightarrow \quad |c| \le |d|.
\end{aligned}$$

(2) The same calculation leads to $|x_2| < 1 \Longleftrightarrow |c| < |d|$. □

Lemma 4.2.2. *Let $a, b, c, d, x_1^\circ, x_2^\circ, x_3^\circ \in \mathbb{C}$, and suppose that $|x_3^\circ| = 1, x_1^\circ = \overline{x_2^\circ} x_3^\circ$, and $x_1^\circ c \neq -d$, and $|x_1^\circ| < 1, |x_2^\circ| < 1$. Let*

$$x_1 = \frac{x_1^\circ a + b}{x_1^\circ c + d}.$$

Then

$$|x_1| < 1 \quad \text{if and only if } |x_1^\circ|^2(|a|^2 - |c|^2) + (|b|^2 - |d|^2) + 2\mathrm{Re}(x_1^\circ(a\bar{b} - c\bar{d})) < 0.$$

Proof.

$$\begin{aligned}
|x_1| < 1 \quad &\Leftrightarrow \quad \left|\frac{x_1^\circ a + b}{x_1^\circ c + d}\right| < 1 \\
&\Leftrightarrow \quad |x_1^\circ a + b|^2 < |x_1^\circ c + d|^2 \\
&\Leftrightarrow \quad (x_1^\circ a + b)(\overline{x_1^\circ}\bar{a} + \bar{b}) < (x_1^\circ c + d)(\overline{x_1^\circ}\bar{c} + \bar{d}) \\
&\Leftrightarrow \quad |x_1^\circ|^2|a|^2 + x_1^\circ a\bar{b} + \overline{x_1^\circ}\bar{a}b + |b|^2 < |x_1^\circ|^2|c|^2 + x_1^\circ c\bar{d} + d\overline{x_1^\circ}\bar{c} + |d|^2 \\
&\Leftrightarrow \quad |x_1^\circ|^2|a|^2 + 2\mathrm{Re}(x_1^\circ a\bar{b}) + |b|^2 < |x_1^\circ|^2|c|^2 + 2\mathrm{Re}(x_1^\circ c\bar{d}) + |d|^2 \\
&\Leftrightarrow \quad |x_1^\circ|^2|a|^2 + 2\mathrm{Re}(x_1^\circ a\bar{b}) + |b|^2 - |x_1^\circ|^2|c|^2 - 2\mathrm{Re}(x_1^\circ c\bar{d}) - |d|^2 < 0 \\
&\Leftrightarrow \quad |x_1^\circ|^2|a|^2 + |b|^2 - |x_1^\circ|^2|c|^2 - |d|^2 + 2\mathrm{Re}(x_1^\circ(a\bar{b} - c\bar{d})) < 0 \\
&\Leftrightarrow \quad |x_1^\circ|^2(|a|^2 - |c|^2) + (|b|^2 - |d|^2) + 2\mathrm{Re}(x_1^\circ(a\bar{b} - c\bar{d})) < 0.
\end{aligned}$$

□

Proposition 4.2.3. *Let a, b, c, d be polynomials in the indeterminate λ and suppose that $x_1^\circ, x_2^\circ, x_3^\circ \in \mathbb{C}$ satisfy $x_3^\circ \neq x_1^\circ x_2^\circ$ and $x_1^\circ c \neq -d$. Let rational functions x_1, x_2, x_3 be defined by*

$$x_1(\lambda) = \frac{x_1^\circ a(\lambda) + b(\lambda)}{x_1^\circ c(\lambda) + d(\lambda)}, \ x_2(\lambda) = \frac{x_3^\circ c(\lambda) + x_2^\circ d(\lambda)}{x_1^\circ c(\lambda) + d(\lambda)}, \ x_3(\lambda) = \frac{x_2^\circ b(\lambda) + x_3^\circ a(\lambda)}{x_1^\circ c(\lambda) + d(\lambda)}. \tag{4.36}$$

and define a rational function ζ in the indeterminate ω by

$$\zeta(\omega) = \frac{\omega x_3^\circ - x_1^\circ}{x_2^\circ \omega - 1}. \tag{4.37}$$

Then, as rational functions in (ω, λ),

$$\frac{\omega x_3(\lambda) - x_1(\lambda)}{x_2(\lambda)\omega - 1} = \frac{a(\lambda)\zeta(\omega) + b(\lambda)}{c(\lambda)\zeta(\omega) + d(\lambda)}.$$

This algebraic relation has implications for rational maps from $\mathbb{D}$ to $\overline{\mathcal{E}}$.

Proof. Let x_1, x_2, x_3 be defined by equations (4.36).

Then

$$\begin{aligned}
\frac{\omega x_3(\lambda) - x_1(\lambda)}{x_2(\lambda)\omega - 1} &= \frac{\dfrac{\omega x_2^\circ b(\lambda) + \omega x_3^\circ a(\lambda)}{x_1^\circ c(\lambda) + d(\lambda)} - \dfrac{x_1^\circ a(\lambda) + b(\lambda)}{x_1^\circ c(\lambda) + d(\lambda)}}{\dfrac{\omega x_3^\circ c(\lambda) + \omega x_2^\circ d(\lambda)}{x_1^\circ c(\lambda) + d(\lambda)} - 1} \\
&= \frac{\omega x_2^\circ b(\lambda) + \omega x_3^\circ a(\lambda) - x_1^\circ a(\lambda) - b(\lambda)}{\omega x_3^\circ c(\lambda) + \omega x_2^\circ d(\lambda) - x_1^\circ c(\lambda) - d(\lambda)} \\
&= \frac{a(\lambda)(\omega x_3^\circ - x_1^\circ) + b(\lambda)(\omega x_2^\circ - 1)}{c(\lambda)(\omega x_3^\circ - x_1^\circ) + d(\lambda)(\omega x_2^\circ - 1)} \\
&= \frac{a(\lambda)\left(\dfrac{\omega x_3^\circ - x_1^\circ}{\omega x_2^\circ - 1}\right) + b(\lambda)}{c(\lambda)\left(\dfrac{\omega x_3^\circ - x_1^\circ}{\omega x_2^\circ - 1}\right) + d(\lambda)} \\
&= \frac{a(\lambda)\zeta(\omega) + b(\lambda)}{c(\lambda)\zeta(\omega) + d(\lambda)},
\end{aligned}$$

where $\zeta(\omega) = \dfrac{\omega x_3^\circ - x_1^\circ}{\omega x_2^\circ - 1}$. $\square$

Proposition 4.2.4. *Let a, b, c, d be polynomials having no common zero in $\overline{\mathbb{D}}$, and satisfying $|c| \le |d|$ on $\mathbb{D}$. Suppose that $x_1^\circ, x_2^\circ, x_3^\circ \in \mathbb{C}$ satisfy $x_1^\circ c \neq -d$, $|x_3^\circ| = 1, |x_1^\circ| < 1, |x_2^\circ| < 1$ and $x_1^\circ = \overline{x_2^\circ} x_3^\circ$. Let rational functions x_1, x_2, x_3 be defined by*

$$x_1(\lambda) = \frac{x_1^\circ a(\lambda) + b(\lambda)}{x_1^\circ c(\lambda) + d(\lambda)},\ x_2(\lambda) = \frac{x_3^\circ c(\lambda) + x_2^\circ d(\lambda)}{x_1^\circ c(\lambda) + d(\lambda)},\ x_3(\lambda) = \frac{x_2^\circ b(\lambda) + x_3^\circ a(\lambda)}{x_1^\circ c(\lambda) + d(\lambda)}, \tag{4.38}$$

and let

$$\psi_\zeta(\lambda) = \frac{a(\lambda)\zeta + b(\lambda)}{c(\lambda)\zeta + d(\lambda)}. \tag{4.39}$$

(i) *If, for all but finitely many values of $\lambda \in \mathbb{D}$,*

$$|\psi_\zeta(\lambda)| \le 1 \tag{4.40}$$

for all but finitely many $\zeta \in \mathbb{T}$, then $x_1^\circ c + d$ has no zero in $\overline{\mathbb{D}}$ and $x = (x_1, x_2, x_3)$ is an analytic map from $\mathbb{D}$ to $\overline{\mathcal{E}}$.

(ii) *If, for all but finitely many $\zeta \in \mathbb{T}$, the function ψ_ζ is inner, then either $x(\overline{\mathbb{D}}) \subseteq \mathcal{R}_{\overline{\mathcal{E}}}$ or $x = (x_1, x_2, x_3)$ is a rational tetra-inner function.*

Proof. (i) By hypothesis there is a finite subset E of $\mathbb{D}$ such that, for all $\lambda \in \mathbb{D}\backslash E$, there is a finite subset F_λ of $\mathbb{T}$ such that the inequality (4.40) holds for all $\zeta \in \mathbb{T}\backslash F_\lambda$. We claim that the denominator $x_1^\circ c + d$ of x_1, x_2, x_3 has no zeros in $\overline{\mathbb{D}}$. Suppose that $\alpha \in \overline{\mathbb{D}}$ is a zero of $(x_1^\circ c + d)$. Since $|c| \leq |d|$ on $\overline{\mathbb{D}}$,

$$\begin{aligned} |x_1^\circ c(\alpha) + d(\alpha)| &\geq |d(\alpha)| - |x_1^\circ c(\alpha)| \\ &\geq |d(\alpha)| - |x_1^\circ||d(\alpha)| \\ &= (1 - |x_1^\circ|)|d(\alpha)|. \end{aligned}$$

Thus,

$$0 = |x_1^\circ c(\alpha) + d(\alpha)| \geq (1 - |x_1^\circ|)|d(\alpha)|.$$

Since $|x_1^\circ| < 1$, $(1 - |x_1^\circ|) \neq 0$, and so $d(\alpha) = 0$, Then

$$0 = x_1^\circ c(\alpha) + d(\alpha) = x_1^\circ c(\alpha)$$

implies that $c(\alpha) = 0$.

Choose a sequence α_j in $\mathbb{D}\backslash E$ such that $\alpha_j \to \alpha$. For each j, for $\zeta \in \mathbb{T}\backslash F(\lambda_j)$, we have $|\psi_\zeta(\lambda)| \leq 1$ on $\mathbb{D}\backslash E$. Hence for all but finitely many $\zeta \in \mathbb{T}$ (that is, for $\zeta \in \mathbb{T}\backslash \cup_j F(\lambda_j)$)

$$\left| \frac{a(\alpha_j)\zeta + b(\alpha_j)}{c(\alpha_j)\zeta + d(\alpha_j)} \right| \leq 1.$$

Since $c(\alpha_j)\zeta + d(\alpha_j) \to 0$ uniformly almost everywhere for $\zeta \in \mathbb{T}$ as $j \to \infty$, the same holds for $a(\alpha_j)\zeta + b(\alpha_j)$. Hence $a(\alpha_j) \to 0$ and $b(\alpha_j) \to 0$. Thus $a(\alpha) = b(\alpha) = 0$. Hence a, b, c, d all vanish at α, contrary to hypothesis. So $x_1^\circ c + d$ has no zeros in $\overline{\mathbb{D}}$. Thus x_1, x_2, x_3 defined by equations (4.38) are rational functions having no poles in $\overline{\mathbb{D}}$.

Consider $\lambda \in \mathbb{D}\backslash E$. By Proposition 4.2.3,

$$\Psi_\omega(x_1(\lambda), x_2(\lambda), x_3(\lambda)) = \frac{\omega x_3(\lambda) - x_1(\lambda)}{x_2(\lambda)\omega - 1} = \frac{a(\lambda)\zeta(\omega) + b(\lambda)}{c(\lambda)\zeta(\omega) + d(\lambda)} \qquad (4.41)$$

whenever both sides are defined, that is, for all $\omega \in \mathbb{T}\backslash\Omega_\lambda$ where

$$\Omega_\lambda = \{\omega \in \mathbb{T} : \omega x_2(\lambda) = 1 \text{ or } c(\lambda)\zeta(\omega) = -d(\lambda)\}.$$

Ω_λ contains at most two points. On combining the relations (4.39), (4.40) and (4.41), we deduce that, for $\lambda \in \mathbb{D}\backslash E$,

$$|\Psi_\omega(x_1(\lambda), x_2(\lambda), x_3(\lambda))| \leq 1 \tag{4.42}$$

for all $\omega \in \mathbb{T}$ such that $\omega \notin \Omega_\lambda \cup \zeta^{-1}(F_\lambda)$, hence for all but finitely many $\omega \in \mathbb{T}$. By Theorem 3.1.6, $(x_1(\lambda), x_2(\lambda), x_3(\lambda)) \in \overline{\mathcal{E}}$. Sine this is true for all but finitely many $\lambda \in \mathbb{D}$, and x_1, x_2, x_3 are rational functions without poles in $\overline{\mathbb{D}}$, (x_1, x_2, x_3) maps $\mathbb{D}$ into $\overline{\mathcal{E}}$.

(ii) Suppose that for some finite subset F of $\mathbb{T}$, the function ψ_ζ is inner for all $\zeta \in \mathbb{T}\backslash F$. By part (i), (x_1, x_2, x_3) maps $\mathbb{D}$ into $\overline{\mathcal{E}}$ and therefore extends to a continuous map of $\overline{\mathbb{D}}$ into $\overline{\mathcal{E}}$. Consider $\lambda \in \mathbb{T}$. By Proposition 4.2.3 and equation (4.39),

$$\Psi_\omega(x_1(\lambda), x_2(\lambda), x_3(\lambda)) = \psi_{\zeta(\omega)}(\lambda) \tag{4.43}$$

whenever both sides are defined, that is, for all $\omega \in \mathbb{T}\backslash\Omega_\lambda$ where

$$\Omega_\lambda = \{\omega \in \mathbb{T} : \omega x_2(\lambda) = 1 \text{ or } c(\lambda)\zeta(\omega) = -d(\lambda)\}.$$

Ω_λ contains at most two points. For $\omega \in \mathbb{T}\backslash\zeta^{-1}(F)$ the function $\psi_{\zeta(\omega)}$ is inner.

Hence, for $\omega \in \mathbb{T}\backslash(\zeta^{-1}(F) \cup \Omega_\lambda)$,

$$|\Psi_\omega(x_1(\lambda), x_2(\lambda), x_3(\lambda))| = |\psi_{\zeta(\omega)}(\lambda)| = 1. \tag{4.44}$$

Case 1. Suppose that for all $\lambda \in \overline{\mathbb{D}}$, $x_1(\lambda)x_2(\lambda) = x_3(\lambda)$. Then, for all $\lambda \in \overline{\mathbb{D}}$,

$$\begin{aligned}\Psi_\omega(x_1(\lambda), x_2(\lambda), x_3(\lambda)) = \frac{\omega x_3(\lambda) - x_1(\lambda)}{x_2(\lambda)\omega - 1} &= \frac{\omega x_1(\lambda)x_2(\lambda) - x_1(\lambda)}{x_2(\lambda)\omega - 1} \\ &= \frac{x_1(\lambda)(\omega x_2(\lambda) - 1)}{x_2(\lambda)\omega - 1} = x_1(\lambda).\end{aligned}$$

Thus $x(\overline{\mathbb{D}}) \subseteq \mathcal{R}_{\overline{\mathcal{E}}}$.

Case 2. Suppose that for some $\lambda \in \overline{\mathbb{D}}$, $x_1(\lambda)x_2(\lambda) \neq x_3(\lambda)$. To prove that $x = (x_1, x_2, x_3)$ is rational $\overline{\mathcal{E}}$-inner function, by Theorem 3.1.8, we need to show that $(x_1, x_2, x_3)(\lambda) \in b\overline{\mathcal{E}}$ for almost all $\lambda \in \mathbb{T}$, that is,

(i) $|x_3(\lambda)| = 1$ for almost all $\lambda \in \mathbb{T}$,

(ii) $|x_2| \le 1$ on $\overline{\mathbb{D}}$,

(iii) $x_1(\lambda) = \overline{x_2(\lambda)} x_3(\lambda)$ for almost all $\lambda \in \mathbb{T}$.

For (ii), by Lemma 4.2.1, we showed that $|x_2(\lambda)| \le 1$ for $\lambda \in \overline{\mathbb{D}}$. By Lemma 3.1.11, for any $\omega \in \mathbb{T}$ and any point $x = (x_1, x_2, x_3) \in \overline{\mathcal{E}}$ such that $x_1 x_2 \ne x_3$,

$$|\Psi_\omega(x_1, x_2, x_3)| = 1 \text{ if and only if } \; 2\omega(x_2 - \overline{x_1} x_3)) = 1 - |x_1|^2 + |x_2|^2 - |x_3|^2.$$

Thus, for $\lambda \in \mathbb{T}$ such that $x_1(\lambda)x_2(\lambda) \ne x_3(\lambda)$, equation (4.44) implies

$$2\omega(x_2(\lambda) - \overline{x_1(\lambda)} x_3(\lambda))) = 1 - |x_1(\lambda)|^2 + |x_2(\lambda)|^2 - |x_3(\lambda)|^2.$$

Hence, for $\lambda \in \mathbb{T}$, if $|\Psi_\omega(x_1(\lambda), x_2(\lambda), x_3(\lambda))| = 1$ for two distinct $\omega \in \mathbb{T}$, say $\omega_1 \ne \omega_2$, we have the linear system

$$\begin{aligned} 2\omega_1(x_2(\lambda) - \overline{x_1(\lambda)} x_3(\lambda)) &= 1 - |x_1(\lambda)|^2 + |x_2(\lambda)|^2 - |x_3(\lambda)|^2 \\ 2\omega_2(x_2(\lambda) - \overline{x_1(\lambda)} x_3(\lambda)) &= 1 - |x_1(\lambda)|^2 + |x_2(\lambda)|^2 - |x_3(\lambda)|^2. \end{aligned} \tag{4.45}$$

Thus, for $\lambda \in \mathbb{T}$,

$$2\omega_1(x_2(\lambda) - \overline{x_1(\lambda)} x_3(\lambda)) - 2\omega_2(x_2(\lambda) - \overline{x_1(\lambda)} x_3(\lambda)) = 0$$

$$\begin{aligned} &\Longrightarrow (x_2(\lambda) - \overline{x_1(\lambda)} x_3(\lambda))(\omega_1 - \omega_2) = 0 \\ &\Longrightarrow x_2(\lambda) = \overline{x_1(\lambda)} x_3(\lambda). \end{aligned} \tag{4.46}$$

By equations (4.45), for $\lambda \in \mathbb{T}$,

$$1 - |x_1(\lambda)|^2 + |x_2(\lambda)|^2 - |x_3(\lambda)|^2 = 0. \tag{4.47}$$

Note for $\lambda \in \mathbb{T}$, since $x_2(\lambda) = \overline{x_1(\lambda)} x_3(\lambda)$,

$$\begin{aligned} (4.47) \text{ holds } &\Leftrightarrow 1 - |x_1(\lambda)|^2 + |\overline{x_1(\lambda)} x_3(\lambda)|^2 - |x_3(\lambda)|^2 = 0 \\ &\Leftrightarrow 1 - |x_1(\lambda)|^2 + |\overline{x_1(\lambda)}|^2 |x_3(\lambda)|^2 - |x_3(\lambda)|^2 = 0 \\ &\Leftrightarrow 1 - |x_1(\lambda)|^2 - |x_3(\lambda)|^2 (1 - |x_1(\lambda)|^2) = 0 \\ &\Leftrightarrow (1 - |x_1(\lambda)|^2)(1 - |x_3(\lambda)|^2) = 0 \\ &\Leftrightarrow |x_3(\lambda)| = 1 \text{ or } |x_1(\lambda)| = 1. \end{aligned}$$

Case 1. If $|x_1(\lambda)| = 1$ and $x_2(\lambda) = \overline{x_1(\lambda)}x_3(\lambda)$, we have $x_3(\lambda) = x_1(\lambda)x_2(\lambda)$ for almost all $\lambda \in \mathbb{T}$. Then since x_i are rational functions for $i = 1, 2, 3$, and $x_3(\lambda) = x_1(\lambda)x_2(\lambda)$ for $\lambda \in \mathbb{T}$, it imples that

$$x_3(\lambda) = x_1(\lambda)x_2(\lambda) \quad \text{for all } \lambda \in \overline{\mathbb{D}}.$$

Then, for all $\lambda \in \overline{\mathbb{D}}$,

$$\begin{aligned}\Psi_\omega(x_1(\lambda), x_2(\lambda), x_3(\lambda)) = \frac{\omega x_3(\lambda) - x_1(\lambda)}{x_2(\lambda)\omega - 1} &= \frac{\omega x_1(\lambda)x_2(\lambda) - x_1(\lambda)}{x_2(\lambda)\omega - 1} \\ &= \frac{x_1(\lambda)(\omega x_2(\lambda) - 1)}{x_2(\lambda)\omega - 1} = x_1(\lambda).\end{aligned}$$

Thus $x(\overline{\mathbb{D}}) \subseteq \mathcal{R}_{\overline{\mathcal{E}}}$.

Case 2. If for almost all $\lambda \in \mathbb{T}, |x_3(\lambda)| = 1$, then

$$\begin{aligned}x_2(\lambda) = \overline{x_1(\lambda)}x_3(\lambda) &\implies x_2(\lambda)\overline{x_3(\lambda)} = \overline{x_1(\lambda)} \\ &\implies x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda).\end{aligned}$$

Thus, for almost all $\lambda \in \mathbb{T}$, $|x_3(\lambda)| = 1$ and $x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda)$ that proves (i) and (iii) respectively. Therefore, the point $(x_1(\lambda), x_2(\lambda), x_3(\lambda))$ for almost all $\lambda \in \mathbb{T}$ is in the distinguished boundary $b\overline{\mathcal{E}}$ of $\overline{\mathcal{E}}$. Hence $x = (x_1, x_2, x_3)$ is a rational $\overline{\mathcal{E}}$-inner function in this case. □

Theorem 4.2.5. *Let (σ, η, ρ) be Blaschke interpolation data with n distinct interpolation nodes of which k lie in $\mathbb{T}$, and let $(\sigma, \eta, \tilde{\eta}, \rho)$ be royal tetra-interpolation data where $\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2, ..., \tilde{\eta}_n)$, $\tilde{\eta}_j \in \mathbb{T}, j = 1, ..., k$ and $\tilde{\eta}_j \in \mathbb{D}, j = k+1, ..., n$. Suppose that Problem* 1.2.2 *with (σ, η, ρ) is solvable and the solutions φ of Problem* 1.2.2 *have normalized parametrization*

$$\varphi = \frac{a\zeta + b}{c\zeta + d}.$$

Suppose that there exist scalars $x_1^\circ, x_2^\circ, x_3^\circ$ in $\mathbb{C}$ such that

$$|x_3^\circ| = 1, \quad |x_1^\circ| < 1, \quad |x_2^\circ| < 1, \quad x_1^\circ = \overline{x_2^\circ}x_3^\circ,$$

and

$$\frac{x_3^\circ c(\sigma_j) + x_2^\circ d(\sigma_j)}{x_1^\circ c(\sigma_j) + d(\sigma_j)} = \tilde{\eta}_j, \;\; j = 1, ..., n. \tag{4.48}$$

Then there exists a rational tetra-inner function $x = (x_1, x_2, x_3)$ *given by*

$$x_1(\lambda) = \frac{x_1^\circ a(\lambda) + b(\lambda)}{x_1^\circ c(\lambda) + d(\lambda)} \tag{4.49}$$

$$x_2(\lambda) = \frac{x_3^\circ c(\lambda) + x_2^\circ d(\lambda)}{x_1^\circ c(\lambda) + d(\lambda)} \tag{4.50}$$

$$x_3(\lambda) = \frac{x_2^\circ b(\lambda) + x_3^\circ a(\lambda)}{x_1^\circ c(\lambda) + d(\lambda)}, \tag{4.51}$$

for $\lambda \in \mathbb{D}$*, such that*

(i) $x \in \mathcal{R}^{n,k}$*, and* x *is a solution of the royal tetra-interpolation problem with the data* $(\sigma, \eta, \tilde{\eta}, \rho)$*, that is,*

$$x(\sigma_j) = (\eta_j, \tilde{\eta}_j, \eta_i\tilde{\eta}_j) \text{ for } j = 1, ..., n,$$

and

$$Ax_1(\sigma_j) = \rho_j \text{ for } j = 1, ..., k,$$

(ii) *for all but finitely many* $\omega \in \mathbb{T}$*, the function* $\Psi_\omega \circ x$ *is a solution of Problem* 1.2.2.

Proof. By Corollary 2.1.12 (3), $|c| \le |d|$ on $\overline{\mathbb{D}}$. Hence $\left|\frac{d(\lambda)}{c(\lambda)}\right| \ge 1$ for $\lambda \in \overline{\mathbb{D}}$. By assumption $|x_1^\circ| < 1$. We claim that $x_1^\circ c \ne -d$ on $\overline{\mathbb{D}}$. Suppose that

$$\begin{aligned} x_1^\circ c = -d \Longrightarrow & |x_1^\circ c| = |d| \\ \Longrightarrow & |x_1^\circ||c| = |d| \\ \Longrightarrow & |x_1^\circ| = \frac{|d|}{|c|}, \end{aligned}$$

which is a contradiction since $\left|\frac{d(\lambda)}{c(\lambda)}\right| \ge 1$ for all $\lambda \in \overline{\mathbb{D}}$, and $|x_1^\circ| < 1$ on $\overline{\mathbb{D}}$. Therefore, $x_1^\circ c \ne -d$ on $\overline{\mathbb{D}}$. By Proposition 4.2.4, either $x(\overline{\mathbb{D}}) \subseteq \mathcal{R}_{\overline{\mathcal{E}}}$ or x is a rational $\overline{\mathcal{E}}$-inner function. Since a, b, c, d are polynomials of degree at most n, the rational function x has degree at most n.

By Definition 2.1.11 of normalised linear fractional parametrization of the solutions of Problem 1.2.2, for some point $\tau \in \mathbb{T}\backslash\{\sigma_1, ..., \sigma_k\}$,

$$\begin{bmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus it is easy to see that

$$x_1(\tau) = \frac{x_1^\circ a(\tau) + b(\tau)}{x_1^\circ c(\tau) + d(\tau)} = \frac{x_1^\circ}{1} = x_1^\circ, \tag{4.52}$$

$$x_2(\tau) = \frac{x_3^\circ c(\tau) + x_2^\circ d(\tau)}{x_1^\circ c(\tau) + d(\tau)} = \frac{x_2^\circ}{1} = x_2^\circ, \tag{4.53}$$

$$x_3(\tau) = \frac{x_2^\circ b(\tau) + x_3^\circ a(\tau)}{x_1^\circ c(\tau) + d(\tau)} = \frac{x_3^\circ}{1} = x_3^\circ. \tag{4.54}$$

By assumption, $|x_3^\circ| = 1, |x_1^\circ| < 1$ and $|x_2^\circ| < 1$, and hence $x_3(\tau) \neq x_1(\tau)x_2(\tau)$. Therefore, $x(\overline{\mathbb{D}})$ is not in the royal variety $\mathcal{R}_{\overline{\mathcal{E}}}$.

By assumption, x_2 is defined by (4.53). Hence

$$x_2(\sigma_j) = \frac{x_3^\circ c(\sigma_j) + x_2^\circ d(\sigma_j)}{x_1^\circ c(\sigma_j) + d(\sigma_j)} = \tilde{\eta}_j \text{ for } j = 1, ..., n.$$

We want to show that x satisfies the interpolation conditions

$$x(\sigma_j) = (\eta_j, \tilde{\eta}_j, \eta_i\tilde{\eta}_j) \quad \text{for } j = 1, ..., n, \tag{4.55}$$

which is to say that $\sigma_j,\ j = 1, ..., n$, is a royal node of x with corresponding royal value $(\eta_j, \tilde{\eta}_j)$. By hypothesis, there is a finite set $F \subset \mathbb{T}$ such that, for all $\zeta \in \mathbb{T}\backslash F$, the function

$$\varphi(\lambda) = \psi_\zeta(\lambda) = \frac{a(\lambda)\zeta + b(\lambda)}{c(\lambda)\zeta + d(\lambda)}$$

is a solution of Problem 1.2.2, and so

$$\psi_\zeta(\sigma_j) = \eta_j \quad \text{for } j = 1, ..., n \tag{4.56}$$

and

$$A\psi_\zeta(\sigma_j) = \rho_j \quad \text{for } j = 1, ..., k \tag{4.57}$$

for all $\zeta \in \mathbb{T}\backslash F$. By Proposition 4.2.3,

$$\psi_{\zeta(\omega)}(\lambda) = \frac{a(\lambda)\zeta(\omega) + b(\lambda)}{c(\lambda)\zeta(\omega) + d(\lambda)} = \frac{\omega x_3(\lambda) - x_1(\lambda)}{\omega x_2(\lambda) - 1} = \Psi_\omega \circ x(\lambda) \tag{4.58}$$

whenever both sides are defined, that is, for all $\omega \in \mathbb{T}\backslash\Omega_\lambda$ where

$$\Omega_\lambda = \{\omega \in \mathbb{T} : \omega x_2(\lambda) = 1 \text{ or } c(\lambda)\zeta(\omega) = -d(\lambda)\}.$$

Ω_λ contains at most two points. Thus, (4.58) holds as rational functions in (ω, λ), where $\zeta(\omega) = \dfrac{\omega x_3^\circ - x_1^\circ}{x_2^\circ \omega - 1}$. Hence, for $\omega \in \mathbb{T}\backslash(\zeta^{-1}(F) \cup \Omega_\lambda)$, $\Psi_\omega \circ x$ is a solution of Problem 1.2.2, so this proves statement (ii).

For any $\lambda \in \overline{\mathbb{D}}$, equation (4.58) holds whenever both denominators are nonzero, hence for all but at most two values of $\omega \in \mathbb{T}$. On combining equations (4.56) and (4.58) (with $\lambda = \sigma_j$) we infer that, for $j = 1, ..., n$ and for all but finitely many $\omega \in \mathbb{T}$,

$$\frac{\omega x_3(\sigma_j) - x_1(\sigma_j)}{\omega x_2(\sigma_j) - 1} = \psi_{\zeta(\omega)}(\sigma_j) = \eta_j.$$

Therefore, for almost all $\omega \in \mathbb{T}$,

$$\omega x_3(\sigma_j) - x_1(\sigma_j) = \eta_j(\omega x_2(\sigma_j) - 1). \tag{4.59}$$

Recall that $x_2(\sigma_j) = \tilde{\eta}_j$ for $j = 1, ..., n$. Hence from (4.59) it follows that $x_1(\sigma_j) = \eta_j$ and $x_3(\sigma_j) = \eta_j\tilde{\eta}_j,\ j = 1, ..., n$, and so the interpolation conditions (4.55) hold.

We have already observed that x is a rational $\overline{\mathcal{E}}$-inner function, $\deg(x) \leq n$ and that $x(\overline{\mathbb{D}})$ is not in $\mathcal{R}_{\overline{\mathcal{E}}}$. Thus by Theorem 3.3.12, the number of royal nodes of x is equal to the degree of x. Therefore x has at most n royal nodes. Since the points $\sigma_j, j = 1, ..., n$ are royal nodes, they contain all the royal nodes of x and $\deg(x) = n$. Precisely k of the σ_j lie in $\mathbb{T}$, and so x has exactly k royal nodes in $\mathbb{T}$. Thus $x \in \mathcal{R}^{n,k}$.

Next we show that $Ax_1(\sigma_j) = \rho_j$ for $j = 1, ..., k$. Fix $j \in \{1, ..., k\}$. By Proposition 3.2.2 , for $\omega \in \mathbb{T},\ \omega\tilde{\eta}_j \neq 1$,

$$A(\Psi_\omega \circ x)(\sigma_j) = Ax_1(\sigma_j). \tag{4.60}$$

There is also a set Ω_j containing at most one $\omega \in \Omega_j$ such that $c(\sigma_j)\zeta(\omega) + d(\sigma_j) = 0$ for $\omega \in \Omega_j$. Hence if $\omega \in \mathbb{T}\backslash(\{\overline{\tilde{\eta}_j}\} \cup \Omega_j)$, it follows from equation

(4.58) that $\psi_{\zeta(\omega)} = \Psi_\omega \circ x$ in a neighbourhood of σ_j, and consequently, for such ω,

$$A\psi_{\zeta(\omega)}(\sigma_j) = A(\Psi_\omega \circ x)(\sigma_j). \tag{4.61}$$

Each of the equations (4.60), (4.61) and (4.57) hold for ω in cofinite subset of $\mathbb{T}$. Hence, for ω in the intersection of these cofinite subsets,

$$Ax_1(\sigma_j) = A(\Psi_\omega \circ x)(\sigma_j) = A\psi_{\zeta(\omega)}(\sigma_j) = \rho_j$$

Thus, (i) holds. □

Corollary 4.2.6. *Let (σ, η, ρ) be Blaschke interpolation data. Suppose that x is a solution of* Problem 1.2.4 *with $(\sigma, \eta, \tilde{\eta}, \rho)$ for some $\tilde{\eta}_j \in \overline{\mathbb{D}}, j = 1, ..., n$, and that $x(\overline{\mathbb{D}}) \not\subset \mathcal{R}_{\bar{\mathcal{E}}}$. For all $\omega \in \mathbb{T}\backslash\{\overline{\tilde{\eta}_1}, ..., \overline{\tilde{\eta}_k}\}$, the function $\varphi = \Psi_\omega \circ x$ is a solution of* Problem 1.2.2 *with Blaschke interpolation data (σ, η, ρ) . Conversely, for every solution φ of the Blaschke interpolation problem with data (σ, η, ρ), there exists $\omega \in \mathbb{T}$ such that $\varphi = \Psi_\omega \circ x$.*

Proof. ($\Longrightarrow$) Consider Blaschke interpolation data (σ, η, ρ). If $x = (x_1, x_2, x_3)$ is a solution of Problem 1.2.4 with $(\sigma, \eta, \tilde{\eta}, \rho)$ for some $\tilde{\eta}_j \in \overline{\mathbb{D}}, j = 1, ..., n$, and that $x(\overline{\mathbb{D}}) \not\subset \mathcal{R}_{\bar{\mathcal{E}}}$, then, by Theorem 4.1.1 (1), for all $\omega \in \mathbb{T}\backslash\{\overline{\tilde{\eta}_1}, ..., \overline{\tilde{\eta}_k}\}$, there exists a rational $\varphi = \Psi_\omega \circ x$ that solves Blaschke interpolation problem (Problem 1.2.2) with Blaschke interpolation data (σ, η, ρ).

($\Longleftarrow$) Let φ be a solution of the Blaschke interpolation problem (Problem 1.2.2) with data (σ, η, ρ). Then, by Theorem 4.1.1 and Theorem 4.2.5 (ii), there exists $\omega \in \mathbb{T}$ such that $\varphi = \Psi_\omega \circ x$. □

Chapter 5

The Algorithm and Examples

5.1 The algorithm

In this section we summarize the steps in the solution of the royal $\overline{\mathcal{E}}$-interpolation problem in the form of a concrete algorithm.

Let $(\sigma, \eta, \tilde{\eta}, \rho)$ be royal interpolation data for the tetrablock as in Definition 1.2.3. Here there are n prescribed royal nodes σ_j, of which the first k lie in $\mathbb{T}$ and the remaining $n-k$ are in $\mathbb{D}$. One can consider the associated Blaschke interpolation data (σ, η, ρ) as in Definition 1.2.1. To construct a rational $\overline{\mathcal{E}}$-inner function $x : \mathbb{D} \to \overline{\mathcal{E}}$ of degree n having royal nodes σ_j for $j = 1, ..., n$, royal values $\eta_j, \tilde{\eta}_j$, and phasar derivatives ρ_j at σ_j for $j = 1, ..., k$, we proceed as follows.

(1) Form the Pick matrix $M = [m_{i,j}]_{i,j=1}^{n}$ for the data (σ, η, ρ), with entries

$$m_{i,j} = \begin{cases} \rho_i & \text{if } i = j \leq k \\ \dfrac{1 - \overline{\eta_i}\eta_j}{1 - \overline{\sigma_i}\sigma_j} & \text{otherwise.} \end{cases} \tag{5.1}$$

If M is not positive definite then the interpolation problem 1.2.2 is not solvable. Otherwise, we introduce the notation

$$x_\lambda = \begin{bmatrix} \frac{1}{1-\bar{\sigma}_1\lambda} \\ \vdots \\ \frac{1}{1-\bar{\sigma}_n\lambda} \end{bmatrix}, \qquad y_\lambda = \begin{bmatrix} \frac{\bar{\eta}_1}{1-\bar{\sigma}_1\lambda} \\ \vdots \\ \frac{\bar{\eta}_n}{1-\bar{\sigma}_n\lambda} \end{bmatrix}, \tag{5.2}$$

as in equations (2.8)

(2) Choose a point $\tau \in \mathbb{T} \backslash \{\sigma_1, \sigma_2, ..., \sigma_k\}$ such that the set of $\zeta \in \mathbb{T}$ for which

$$\langle M^{-1}x_\tau, e_j\rangle = \zeta \langle M^{-1}y_\tau, e_j\rangle \quad \text{for some } j \in \{1, ..., n\}$$

(where e_j is the jth standard basis vector in $\mathbb{C}^n$) is finite.

(3) Let

$$g(\lambda) = \prod_{j=1}^{n} \frac{1 - \overline{\sigma_j}\lambda}{1 - \overline{\sigma_j}\tau}\ , \tag{5.3}$$

and let polynomials a, b, c, d be given by

$$\begin{aligned} a(\lambda) &= g(\lambda)(1 - (1 - \overline{\tau}\lambda)\langle x_\lambda, M^{-1}x_\tau\rangle), & (5.4)\\ b(\lambda) &= g(\lambda)(1 - \overline{\tau}\lambda)\langle x_\lambda, M^{-1}y_\tau\rangle, & (5.5)\\ c(\lambda) &= -g(\lambda)(1 - \overline{\tau}\lambda)\langle y_\lambda, M^{-1}x_\tau\rangle & (5.6)\\ d(\lambda) &= g(\lambda)(1 + (1 - \overline{\tau}\lambda)\langle y_\lambda, M^{-1}y_\tau\rangle). & (5.7) \end{aligned}$$

Note that

$$\begin{bmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.8}$$

(See Theorem 3.9 in [6]).

(4) Find $x_1^\circ, x_2^\circ, x_3^\circ \in \mathbb{C}$ such that

$$|x_3^\circ| = 1, \quad |x_1^\circ| < 1, \quad |x_2^\circ| < 1, \text{ and } x_1^\circ = \overline{x_2^\circ}x_3^\circ,$$

and

$$\frac{x_3^\circ c(\sigma_j) + x_2^\circ d(\sigma_j)}{x_1^\circ c(\sigma_j) + d(\sigma_j)} = \tilde{\eta}_j, \;\; j = 1, ..., n.$$

If there is no $(x_1^\circ, x_2^\circ, x_3^\circ)$ satisfying these conditions, then by Theorem 4.2.5, the royal $\overline{\mathcal{E}}$-interpolation problem is not solvable.

(5) If there are such $(x_1^\circ, x_2^\circ, x_3^\circ) \in \mathbb{C}$, we define

$$\begin{aligned} x_1(\lambda) &= \frac{x_1^\circ a + b}{x_1^\circ c + d}(\lambda),\\ x_2(\lambda) &= \frac{x_3^\circ c + x_2^\circ d}{x_1^\circ c + d}(\lambda),\\ x_3(\lambda) &= \frac{x_2^\circ b + x_3^\circ a}{x_1^\circ c + d}(\lambda), \text{ for } \lambda \in \mathbb{D}. \end{aligned}$$

It is easy to see that, since the equation (5.8) is satisfied,

$$x_1(\tau) = x_1^\circ,\ x_2(\tau) = x_2^\circ \text{ and } x_3(\tau) = x_3^\circ.$$

Then, by Theorem 4.2.5, $x = (x_1, x_2, x_3)$ is a rational $\overline{\mathcal{E}}$-inner function of degree at most n such that $x(\sigma_j) = (\eta_j, \tilde{\eta}_j, \eta_j\tilde{\eta}_j)$ for $j = 1, ..., n$, and $Ax_1(\sigma_j) = \rho_j$ for $j = 1, ..., k$. By assumption, $|x_3^\circ| = 1, |x_1^\circ| < 1$ and $|x_2^\circ| < 1$, and hence $x_3(\tau) \neq x_1(\tau)x_2(\tau)$. Therefore, $x(\overline{\mathbb{D}})$ is not in the royal variety $\mathcal{R}_{\overline{\mathcal{E}}}$ and the degree of x is exactly n.

The following comments relate the steps of algorithm to results in the report.

(i) If the royal $\overline{\mathcal{E}}$-interpolation problem with data $(\sigma, \eta, \tilde{\eta}, \rho)$ for some $\tilde{\eta}_j \in \overline{\mathbb{D}}$ is solvable, then by Theorem 4.1.1, the Blaschke interpolation problem with data (σ, η, ρ) is solvable. By [6, Proposition 3.2], $M > 0$.

(ii) The conditions that $|x_3^\circ| = 1,\ |x_1^\circ| < 1,\ |x_2^\circ| < 1,$ and $x_1^\circ = \overline{x_2^\circ}x_3^\circ$ are equivalent to $(x_1^\circ, x_2^\circ, x_3^\circ) \in b\overline{\mathcal{E}}$ and $|x_2^\circ| < 1$.

(iii) The equations for x_1, x_2 and x_3 are equations (4.49), (4.50) and (4.51) respectively.

5.2 Examples

Lemma 5.2.1. *Let $\sigma_1 \in \mathbb{D}$, and $\eta, \tilde{\eta} \in \mathbb{C}$. Let $m \in \mathrm{Aut}(\mathbb{D})$ be such that $m(\sigma_1) = 0$. Suppose there exists a rational $\overline{\mathcal{E}}$-inner $y : \mathbb{D} \to \overline{\mathcal{E}}$ such that $y(0) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$. Then $x = y \circ m$ is a rational $\overline{\mathcal{E}}$- inner function such that $x(\sigma_1) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$.*

Proof. By assumption, the function $y : \mathbb{D} \to \overline{\mathcal{E}}$ is such that $y(0) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$. The Blaschke factor $m : \mathbb{D} \to \mathbb{D}$ such that $m(z) = \dfrac{z - \sigma_1}{1 - \overline{\sigma_1}z}$ moves σ_1 to 0. Note that $(y \circ m)(\sigma_1) = y(m(\sigma_1)) = y(0) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$.

It is easy to see that the composition $x = y \circ m$ is a rational $\overline{\mathcal{E}}$- inner function, $x : \mathbb{D} \to \overline{\mathcal{E}}$ such that $x(\sigma_1) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$. $\square$

Example 5.2.1. Consider the case $n = 1, k = 0$ of Problem 1.2.4. There are prescribed a single royal node $\sigma_1 \in \mathbb{D}$ and a royal value $(\eta, \tilde{\eta}, \eta\tilde{\eta})$, where

$\eta, \tilde{\eta} \in \mathbb{D}$, and we seek a $\overline{\mathcal{E}}$-inner function x of degree 1 such that $x(\sigma_1) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$. By composition with an automorphism on $\mathbb{D}$, we may reduce our problem to the case that $\sigma_1 = 0$.

Step 1. Choose an arbitrary $\tau \in \mathbb{T}$. The normalized parametrization of the solution set of the associated Blaschke interpolation problem is given by

$$\varphi(\lambda) = \frac{a(\lambda)\zeta + b(\lambda)}{c(\lambda)\zeta + d(\lambda)}, \quad \lambda \in \mathbb{D}, \text{and some } \zeta \in \mathbb{T},$$

where a, b, c, d are given by equations

$$\begin{aligned} a(\lambda) &= g(\lambda)(1 - (1 - \overline{\tau}\lambda)\langle x_\lambda, M^{-1}x_\tau\rangle), \\ b(\lambda) &= g(\lambda)(1 - \overline{\tau}\lambda)\langle x_\lambda, M^{-1}y_\tau\rangle, \\ c(\lambda) &= -g(\lambda)(1 - \overline{\tau}\lambda)\langle y_\lambda, M^{-1}x_\tau\rangle \\ d(\lambda) &= g(\lambda)(1 + (1 - \overline{\tau}\lambda)\langle y_\lambda, M^{-1}y_\tau\rangle), \end{aligned}$$

and x_λ, y_λ, g and M are given by equations (5.2), (5.3) and (5.1) respectively. Note that since $\sigma_1 = 0$, $g(\lambda) = \dfrac{1 - \overline{\sigma_1}\lambda}{1 - \overline{\sigma_1}\tau} = 1$, and $M = \dfrac{1 - \overline{\eta}\eta}{1 - \overline{\sigma_1}\sigma_1} = 1 - |\eta|^2$. Thus, $M^{-1} = \dfrac{1}{1 - |\eta|^2}$. Recall that, for $\lambda \in \overline{\mathbb{D}}$, we define x_λ and y_λ by

$$x_\lambda = \frac{1}{1 - \overline{\sigma}_1\lambda}, \qquad y_\lambda = \frac{\overline{\eta}_1}{1 - \overline{\sigma}_1\lambda}.$$

Here $x_\lambda = \dfrac{1}{1 - 0\lambda} = 1$ and $y_\lambda = \dfrac{\overline{\eta}}{1 - 0\lambda} = \overline{\eta}$. Thus, polynomials a, b, c and d are defined by

$$\begin{aligned} a(\lambda) &= g(\lambda)(1 - (1 - \overline{\tau}\lambda)\langle x_\lambda, M^{-1}x_\tau\rangle) \\ &= 1 - \frac{1 - \overline{\tau}\lambda}{1 - |\eta|^2} = \frac{1 - |\eta|^2 - 1 + \overline{\tau}\lambda}{1 - |\eta|^2} \\ &= \frac{\overline{\tau}\lambda - |\eta|^2}{1 - |\eta|^2}, \end{aligned} \tag{5.9}$$

$$\begin{aligned} b(\lambda) &= g(\lambda)(1 - \overline{\tau}\lambda)\langle x_\lambda, M^{-1}y_\tau\rangle \\ &= 1(1 - \overline{\tau}\lambda)\frac{\eta}{1 - |\eta|^2} \\ &= \frac{\eta(1 - \overline{\tau}\lambda)}{1 - |\eta|^2}, \end{aligned} \tag{5.10}$$

$$\begin{aligned}
c(\lambda) &= -g(\lambda)(1-\overline{\tau}\lambda)\langle y_\lambda, M^{-1}x_\tau\rangle \\
&= -1(1-\overline{\tau}\lambda)\frac{\overline{\eta}}{1-|\eta|^2} \\
&= \frac{-\overline{\eta}(1-\overline{\tau}\lambda)}{1-|\eta|^2},
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
d(\lambda) &= g(\lambda)(1+(1-\overline{\tau}\lambda)\langle y_\lambda, M^{-1}y_\tau\rangle) \\
&= 1(1+(1-\overline{\tau}\lambda)\frac{\eta\overline{\eta}}{1-|\eta|^2}) \\
&= 1+\frac{(1-\overline{\tau}\lambda)|\eta|^2}{1-|\eta|^2} \\
&= \frac{1-|\eta|^2+|\eta|^2-\overline{\tau}\lambda|\eta|^2}{1-|\eta|^2} \\
&= \frac{1-|\eta|^2\overline{\tau}\lambda}{1-|\eta|^2}.
\end{aligned} \tag{5.12}$$

Step 2. The next step is to determine whether there exist $x_1^\circ, x_2^\circ, x_3^\circ \in \mathbb{C}$ such that

$$|x_3^\circ| = 1, \quad |x_1^\circ| < 1, \quad |x_2^\circ| < 1, \text{ and } x_1^\circ = \overline{x_2^\circ}x_3^\circ,$$

and

$$\frac{x_3^\circ c(0) + x_2^\circ d(0)}{x_1^\circ c(0) + d(0)} = \tilde{\eta}.$$

Here,

$$a(0) = \frac{-|\eta|^2}{1-|\eta|^2}, \qquad b(0) = \frac{\eta}{1-|\eta|^2},$$

$$c(0) = \frac{-\overline{\eta}}{1-|\eta|^2}, \qquad d(0) = \frac{1}{1-|\eta|^2}.$$

Let $x_3^\circ = \omega$ for $\omega \in \mathbb{T}$. Now

$$\begin{aligned}
\frac{x_3^\circ c(0) + x_2^\circ d(0)}{x_1^\circ c(0) + d(0)} = \tilde{\eta} \quad &\Leftrightarrow \quad \frac{\omega\left[\dfrac{-\overline{\eta}}{1-|\eta|^2}\right] + x_2^\circ\left[\dfrac{1}{1-|\eta|^2}\right]}{x_1^\circ\left[\dfrac{-\overline{\eta}}{1-|\eta|^2}\right] + \left[\dfrac{1}{1-|\eta|^2}\right]} = \tilde{\eta} \\
&\Leftrightarrow \quad \frac{\dfrac{-\omega\overline{\eta}}{1-|\eta|^2} + \dfrac{x_2^\circ}{1-|\eta|^2}}{\dfrac{-x_1^\circ\overline{\eta}}{1-|\eta|^2} + \dfrac{1}{1-|\eta|^2}} = \tilde{\eta} \\
&\Leftrightarrow \quad \frac{-\omega\overline{\eta} + x_2^\circ}{-x_1^\circ\overline{\eta} + 1} = \tilde{\eta} \\
&\Leftrightarrow \quad -\omega\overline{\eta} + x_2^\circ = -x_1^\circ\overline{\eta}\tilde{\eta} + \tilde{\eta} \\
&\Leftrightarrow \quad x_2^\circ = -x_1^\circ\overline{\eta}\tilde{\eta} + \tilde{\eta} + \omega\overline{\eta}.
\end{aligned}$$

Since $x_1^\circ = \overline{x_2^\circ} x_3^\circ$ and $x_3^\circ = \omega$, we have the system

$$\begin{cases} x_3^\circ &= \omega \\ x_1^\circ &= \overline{x_2^\circ}\omega \\ x_2^\circ &= -x_1^\circ\overline{\eta}\tilde{\eta} + \tilde{\eta} + \omega\overline{\eta}. \end{cases} \tag{5.13}$$

For given $\eta, \tilde{\eta} \in \mathbb{D}$, we want to find a solution $x_1^\circ, x_2^\circ, x_3^\circ$ of the above system such that $|x_1^\circ| < 1, |x_2^\circ| < 1$. Thus we want to find $\omega \in \mathbb{T}$ and $x_1^\circ \in \mathbb{C} : |x_1^\circ| < 1$ such that

$$x_1^\circ = \omega(-\overline{x_1^\circ}\eta\overline{\tilde{\eta}} + \overline{\tilde{\eta}} + \overline{\omega}\eta),$$

that is,

$$x_1^\circ + \overline{x_1^\circ}\omega\eta\overline{\tilde{\eta}} = \omega\overline{\tilde{\eta}} + \eta. \tag{5.14}$$

One can show that there are $\omega \in \mathbb{T}$ and $x_1^\circ \in \mathbb{D}$ such that equation (5.14) is satisfied.

Step 3. For the given data, $0 \to (\eta, \tilde{\eta}, \eta\tilde{\eta})$, take $x_1^\circ \in \mathbb{C}, |x_1^\circ| < 1$, $x_3^\circ = \omega \in \mathbb{T}$ such that the equation (5.14) satisfied, and $x_2^\circ \in \mathbb{C}$ given by $x_2^\circ = -x_1^\circ\overline{\eta}\tilde{\eta} +$

$\tilde{\eta} + \omega\overline{\eta}$, the solution of the problem will be, define, for $\lambda \in \mathbb{D}$,

$$\begin{aligned}
x_1(\lambda) &= \frac{x_1^\circ a + b}{x_1^\circ c + d}(\lambda) \\
&= \frac{x_1^\circ \left[\dfrac{\overline{\tau}\lambda - |\eta|^2}{1-|\eta|^2}\right] + \left[\dfrac{\eta(1-\overline{\tau}\lambda)}{1-|\eta|^2}\right]}{x_1^\circ \left[\dfrac{-\overline{\eta}(1-\overline{\tau}\lambda)}{1-|\eta|^2}\right] + \left[\dfrac{1-|\eta|^2\overline{\tau}\lambda}{1-|\eta|^2}\right]} \\
&= \frac{x_1^\circ(\overline{\tau}\lambda - |\eta|^2) + \eta(1-\overline{\tau}\lambda)}{-x_1^\circ\overline{\eta}(1-\overline{\tau}\lambda) + 1 - |\eta|^2\overline{\tau}\lambda},
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
x_2(\lambda) &= \frac{x_3^\circ c + x_2^\circ d}{x_1^\circ c + d}(\lambda) \\
&= \frac{x_3^\circ \left[\dfrac{-\overline{\eta}(1-\overline{\tau}\lambda)}{1-|\eta|^2}\right] + x_2^\circ\left[\dfrac{1-|\eta|^2\overline{\tau}\lambda}{1-|\eta|^2}\right]}{x_1^\circ \left[\dfrac{-\overline{\eta}(1-\overline{\tau}\lambda)}{1-|\eta|^2}\right] + \left[\dfrac{1-|\eta|^2\overline{\tau}\lambda}{1-|\eta|^2}\right]} \\
&= \frac{-x_3^\circ\overline{\eta}(1-\overline{\tau}\lambda) + x_2^\circ(1-|\eta|^2\overline{\tau}\lambda)}{-x_1^\circ\overline{\eta}(1-\overline{\tau}\lambda) + 1 - |\eta|^2\overline{\tau}\lambda},
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
x_3(\lambda) &= \frac{x_2^\circ b + x_3^\circ a}{x_1^\circ c + d}(\lambda) \\
&= \frac{x_2^\circ \left[\dfrac{\eta(1-\overline{\tau}\lambda)}{1-|\eta|^2}\right] + [x_3^\circ]\left[\dfrac{\overline{\tau}\lambda - |\eta|^2}{1-|\eta|^2}\right]}{[x_1^\circ] \left[\dfrac{-\overline{\eta}(1-\overline{\tau}\lambda)}{1-|\eta|^2}\right] + \left[\dfrac{1-|\eta|^2\overline{\tau}\lambda}{1-|\eta|^2}\right]} \\
&= \frac{x_2^\circ\eta(1-\overline{\tau}\lambda) + x_3^\circ(\overline{\tau}\lambda - |\eta|^2)}{-x_1^\circ\overline{\eta}(1-\overline{\tau}\lambda) + 1 - |\eta|^2\overline{\tau}\lambda}.
\end{aligned} \tag{5.17}$$

Let us check that $x(0) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$. By equation (5.15), we have

$$\begin{aligned}
x_1(0) &= \frac{-x_1^\circ|\eta|^2 + \eta}{-x_1^\circ\overline{\eta} + 1} = \frac{\eta(-x_1^\circ\overline{\eta} + 1)}{-x_1^\circ\overline{\eta} + 1} \\
&= \eta.
\end{aligned}$$

By equation (5.16), since $x_2^\circ = -x_1^\circ\overline{\eta}\tilde{\eta} + \tilde{\eta} + \omega\overline{\eta}$, we have

$$\begin{aligned} x_2(0) &= \frac{-\omega\overline{\eta} + x_2^\circ}{-x_1^\circ\overline{\eta} + 1} \\ &= \frac{-\omega\overline{\eta} - x_1^\circ\overline{\eta}\tilde{\eta} + \tilde{\eta} + \omega\overline{\eta}}{-x_1^\circ\overline{\eta} + 1} \\ &= \frac{-x_1^\circ\overline{\eta}\tilde{\eta} + \tilde{\eta}}{-x_1^\circ\overline{\eta} + 1} \\ &= \frac{\tilde{\eta}(-x_1^\circ\overline{\eta} + 1)}{-x_1^\circ\overline{\eta} + 1} \\ &= \tilde{\eta}. \end{aligned}$$

By equation (5.17), since $x_2^\circ = -x_1^\circ\overline{\eta}\tilde{\eta} + \tilde{\eta} + \omega\overline{\eta}$, we have

$$\begin{aligned} x_3(0) &= \frac{(-x_1^\circ\overline{\eta}\tilde{\eta} + \tilde{\eta} + \omega\overline{\eta})\eta - \omega|\eta|^2}{-x_1^\circ\overline{\eta} + 1} \\ &= \frac{-x_1^\circ\overline{\eta}\tilde{\eta}\eta + \tilde{\eta}\eta + \omega|\eta|^2 - \omega|\eta|^2}{-x_1^\circ\overline{\eta} + 1} \\ &= \frac{\eta\tilde{\eta}(-x_1^\circ\overline{\eta} + 1)}{-x_1^\circ\overline{\eta} + 1} = \eta\tilde{\eta}. \end{aligned}$$

One can easily check that $x = (x_1, x_2, x_3)$ defined by equations (5.15), (5.16) and (5.17) is a $\overline{\mathcal{E}}$-inner function of degree 1 satisfying $x(0) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$.

Example 5.2.2. Let $n = 1, k = 0$. Let $\sigma_1 = \frac{1}{2}, \eta = 0, \tilde{\eta} = \dfrac{1}{2}i$. There are prescribed a single royal node $\sigma_1 = \frac{1}{2}$ and a royal value $(0, \frac{1}{2i}, 0)$, and we seek a $\overline{\mathcal{E}}$-inner function y of degree 1 such that $y(\sigma_1) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$. Let $m \in \mathrm{Aut}(\mathbb{D})$ such that $m(\frac{1}{2}) = 0$, that is, for $\sigma_1 = \frac{1}{2}$, the Blaschke factor $m : \mathbb{D} \to \mathbb{D}$ such that $m(z) = \dfrac{z - \sigma_1}{1 - \overline{\sigma_1}z}$ moves $\sigma_1 = \frac{1}{2}$ to 0. By Lemma 5.2.1, the solution y is equal to $y = x \circ m$, where x is the solution to the problem with data $\sigma_1 = 0$ and the royal value $(0, \frac{1}{2}i, 0)$ as in Example 5.2.1.

Let us follow steps of Example 5.2.1.

Step 1. Let $\tau \in \mathbb{T}$. By equations (5.9), (5.10), (5.11) and (5.12) since $\eta = 0$,

$$a(\lambda) = \frac{\overline{\tau}\lambda - |\eta|^2}{1 - |\eta|^2} = \frac{\overline{\tau}\lambda}{1} = \overline{\tau}\lambda.$$

$$b(\lambda) = \frac{\eta(1-\overline{\tau}\lambda)}{1-|\eta|^2} = 0.$$

$$c(\lambda) = \frac{-\overline{\eta}(1-\overline{\tau}\lambda)}{1-|\eta|^2} = 0.$$

$$d(\lambda) = \frac{1-|\eta|^2\overline{\tau}\lambda}{1-|\eta|^2} = 1.$$

Step 2. Let us determine whether there exist $x_1^\circ, x_2^\circ, x_3^\circ \in \mathbb{C}$ such that

$$|x_3^\circ| = 1, \quad |x_1^\circ| < 1, \quad |x_2^\circ| < 1, \text{ and } x_1^\circ = \overline{x_2^\circ}x_3^\circ, \tag{5.18}$$

and

$$\frac{x_3^\circ c(0) + x_2^\circ d(0)}{x_1^\circ c(0) + d(0)} = \tilde{\eta}. \tag{5.19}$$

For $\omega \in \mathbb{T}$ the equation (5.14) will be

$$x_1^\circ + 0 = \omega(-\frac{1}{2}i) + 0 \implies x_1^\circ = (-\frac{1}{2}i\omega).$$

Thus, we have the one-parameter family of $(x_1^\circ, x_2^\circ, x_3^\circ)$ such that equations (5.18) and (5.19) hold, given by

$$\begin{cases} x_3^\circ &= \omega \\ x_1^\circ &= -\frac{1}{2}i\omega \\ x_2^\circ &= \frac{1}{2}i \end{cases} \tag{5.20}$$

Thus, by equations (5.15), (5.16) and (5.17), the solution of the problem of the finding a $\overline{\mathcal{E}}$-inner function $x = (x_1, x_2, x_3)$ with the single royal node $\sigma = 0$ and the royal value $(0, \frac{1}{2}i, 0)$ will be, for $\eta = 0, \tilde{\eta} = \frac{1}{2}i$,

$$\begin{aligned} x_1(\lambda) &= \frac{x_1^\circ(\overline{\tau}\lambda - |0|^2) + 0(1-\overline{\tau}\lambda)}{-x_1^\circ 0(1-\overline{\tau}\lambda) + 1 - |0|^2\overline{\tau}\lambda} \\ &= x_1^\circ\overline{\tau}\lambda, \end{aligned} \tag{5.21}$$

$$\begin{aligned} x_2(\lambda) &= \frac{-x_3^\circ 0(1-\overline{\tau}\lambda) + x_2^\circ(1-|0|^2\overline{\tau}\lambda)}{-x_1^\circ 0(1-\overline{\tau}\lambda) + 1 - |0|^2\overline{\tau}\lambda} \\ &= x_2^\circ, \end{aligned} \tag{5.22}$$

$$\begin{aligned} x_3(\lambda) &= \frac{x_2^\circ 0(1-\overline{\tau}\lambda) + x_3^\circ(\overline{\tau}\lambda - |0|^2)}{-x_1^\circ 0(1-\overline{\tau}\lambda) + 1 - |0|^2\overline{\tau}\lambda} \\ &= x_3^\circ\overline{\tau}\lambda. \end{aligned} \tag{5.23}$$

Note that at 0,

$$x_1(0) = 0,$$
$$x_2(0) = x_2^\circ = \frac{1}{2}i,$$
$$x_3(0) = 0.$$

The solution to the problem of the finding a $\overline{\mathcal{E}}-$inner function y of degree 1 such that $y(\frac{1}{2}) = (0, \frac{1}{2}i, 0)$ is a one-parameter family of rational $\overline{\mathcal{E}}-$inner function $y(\lambda) = x \circ m(\lambda)$:

$$\begin{aligned} y(\lambda) &= x \circ m(\lambda) \\ &= (x_1(m(\lambda)), x_2(m(\lambda)), x_3(m(\lambda))) \\ &= \Big(-\frac{1}{2}i\omega\overline{\tau}m(\lambda), \frac{1}{2}i, \omega\overline{\tau}m(\lambda)\Big) \\ &= \Big(-\frac{1}{2}i\omega\overline{\tau}\frac{\lambda - \frac{1}{2}}{1-\frac{1}{2}\lambda}, \frac{1}{2}i, \omega\overline{\tau}\frac{\lambda - \frac{1}{2}}{1-\frac{1}{2}\lambda}\Big) \\ &= \Big(-\frac{1}{2}i\kappa\frac{\lambda - \frac{1}{2}}{1-\frac{1}{2}\lambda}, \frac{1}{2}i, \kappa\frac{\lambda - \frac{1}{2}}{1-\frac{1}{2}\lambda}\Big), \end{aligned} \tag{5.24}$$

where $\kappa = \omega\overline{\tau} \in \mathbb{T}$. Note that $y(\frac{1}{2}) = (0, \frac{1}{2}i, 0)$. Therefore, since $\kappa = \omega\overline{\tau}$ is a general point of $\mathbb{T}$, we obtain a one-parameter family of $\overline{\mathcal{E}}-$inner function y of degree 1 satisfying $y(\frac{1}{2}) = (0, \frac{1}{2}i, 0)$.

Example 5.2.3. Consider the case $n = 1, k = 1$. Suppose $\sigma = 1$. The points $\eta, \tilde{\eta} \in \mathbb{T}$ and $\rho > 0$ are prescribed, and we seek a $\overline{\mathcal{E}}$-inner function $x = (x_1, x_2, x_3)$ of degree 1 such that $x(1) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$ and $Ax_1(1) = \rho$.

Step 1. Choose $\tau \in \mathbb{T} \setminus \{1\}$. The normalized parametrization of the solution set of the associated Blaschke interpolation problem is given by

$$\varphi(\lambda) = \frac{a(\lambda)\zeta + b(\lambda)}{c(\lambda)\zeta + d(\lambda)} \quad \text{for } \lambda \in \mathbb{D}, \text{and some } \zeta \in \mathbb{T}, \tag{5.25}$$

where a, b, c, d are given by the equations

$$\begin{aligned} a(\lambda) &= g(\lambda)(1 - (1 - \overline{\tau}\lambda)\langle x_\lambda, M^{-1}x_\tau\rangle), \\ b(\lambda) &= g(\lambda)(1 - \overline{\tau}\lambda)\langle x_\lambda, M^{-1}y_\tau\rangle, \\ c(\lambda) &= -g(\lambda)(1 - \overline{\tau}\lambda)\langle y_\lambda, M^{-1}x_\tau\rangle \\ d(\lambda) &= g(\lambda)(1 + (1 - \overline{\tau}\lambda)\langle y_\lambda, M^{-1}y_\tau\rangle). \end{aligned}$$

and x_λ, y_λ, g and M are given by equations (5.2), (5.3) and (5.1) respectively. Note that since $\sigma = 1$, $g(\lambda) = \dfrac{1-\overline{\sigma}\lambda}{1-\overline{\sigma}\tau} = \dfrac{1-\lambda}{1-\tau}$.

Here $M = \rho$, since $k = 1$. Thus, $M^{-1} = \dfrac{1}{\rho}$, and $x_\lambda = \dfrac{1}{1-\lambda}$ and $y_\lambda = \dfrac{\overline{\eta}}{1-\lambda}$.

Therefore, polynomials a, b, c and d are defined by

$$\begin{aligned}
a(\lambda) &= g(\lambda)\Big(1-(1-\overline{\tau}\lambda)\langle x_\lambda, M^{-1}x_\tau\rangle\Big)\\
&= \frac{1-\lambda}{1-\tau}\Big(1-(1-\overline{\tau}\lambda)\langle\frac{1}{1-\lambda},\frac{1}{\rho(1-\tau)}\rangle\Big)\\
&= \frac{1-\lambda}{1-\tau}\Big(1-(1-\overline{\tau}\lambda)(\frac{1}{\rho(1-\overline{\tau})}\frac{1}{1-\lambda})\Big)\\
&= \frac{1-\lambda}{1-\tau}\Big(1-\frac{1-\overline{\tau}\lambda}{\rho(1-\overline{\tau})(1-\lambda)}\Big)\\
&= \frac{1-\lambda}{1-\tau}-\frac{(1-\lambda)(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2(1-\lambda)}\\
&= \frac{1-\lambda}{1-\tau}-\frac{(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}
\end{aligned}$$

$$\begin{aligned}
b(\lambda) &= g(\lambda)(1-\overline{\tau}\lambda)\langle x_\lambda, M^{-1}y_\tau\rangle\\
&= \frac{1-\lambda}{1-\tau}(1-\overline{\tau}\lambda)\langle\frac{1}{1-\lambda},\frac{\overline{\eta}}{\rho(1-\tau)}\rangle\\
&= \frac{1-\lambda}{1-\tau}(1-\overline{\tau}\lambda)(\frac{\eta}{\rho(1-\overline{\tau})(1-\lambda)})\\
&= \frac{\eta(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}
\end{aligned}$$

$$\begin{aligned}
c(\lambda) &= -g(\lambda)(1-\overline{\tau}\lambda)\langle y_\lambda, M^{-1}x_\tau\rangle\\
&= -(\frac{1-\lambda}{1-\tau})(1-\overline{\tau}\lambda)\langle\frac{\overline{\eta}}{1-\lambda},\frac{1}{\rho(1-\tau)}\rangle\\
&= -(\frac{1-\lambda}{1-\tau})(1-\overline{\tau}\lambda)(\frac{1}{\rho(1-\overline{\tau})}\frac{\overline{\eta}}{1-\lambda})\\
&= -\frac{\overline{\eta}(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}
\end{aligned}$$

$$\begin{aligned} d(\lambda) &= g(\lambda)\Big(1+(1-\overline{\tau}\lambda)\langle y_\lambda, M^{-1}y_\tau\rangle\Big) \\ &= \frac{1-\lambda}{1-\tau}\Big(1+(1-\overline{\tau}\lambda)\langle\frac{\overline{\eta}}{1-\lambda},\frac{\overline{\eta}}{\rho(1-\tau)}\rangle\Big) \\ &= \frac{1-\lambda}{1-\tau}\Big(1+(1-\overline{\tau}\lambda)(\frac{|\eta|^2}{\rho(1-\overline{\tau})(1-\lambda)})\Big) \\ &= \frac{1-\lambda}{1-\tau}+\frac{1-\overline{\tau}\lambda}{\rho|1-\tau|^2}. \end{aligned}$$

Step 2. The next step is to determine whether there exist $x_1^\circ, x_2^\circ, x_3^\circ \in \mathbb{C}$ such that

$$|x_3^\circ| = 1, \quad |x_1^\circ| < 1, \quad |x_2^\circ| < 1, \text{ and } x_1^\circ = \overline{x_2^\circ}x_3^\circ,$$

and

$$\frac{x_3^\circ c(1) + x_2^\circ d(1)}{x_1^\circ c(1) + d(1)} = \tilde{\eta}.$$

Here,

$$a(1) = \frac{1-\overline{\tau}}{\rho|1-\overline{\tau}|^2}, \qquad b(1) = \frac{\eta - \eta\overline{\tau}}{\rho|1-\overline{\tau}|^2},$$

$$c(1) = \frac{-\overline{\eta}+\overline{\eta\tau}}{\rho|1-\overline{\tau}|^2}, \qquad d(1) = \frac{1-\overline{\tau}}{\rho|1-\overline{\tau}|^2}.$$

Let $x_3^\circ = \omega$ for $\omega \in \mathbb{T}$. Now

$$
\begin{aligned}
\frac{x_3^\circ c(1) + x_2^\circ d(1)}{x_1^\circ c(1) + d(1)} = \tilde{\eta} \quad & \Leftrightarrow \quad \frac{\omega \left[\dfrac{-\overline{\eta} + \overline{\eta\tau}}{\rho|1-\overline{\tau}|^2}\right] + x_2^\circ \left[\dfrac{1-\overline{\tau}}{\rho|1-\overline{\tau}|^2}\right]}{x_1^\circ \left[\dfrac{-\overline{\eta} + \overline{\eta\tau}}{\rho|1-\overline{\tau}|^2}\right] + \left[\dfrac{1-\overline{\tau}}{\rho|1-\overline{\tau}|^2}\right]} = \tilde{\eta} \\
& \Leftrightarrow \quad \frac{\dfrac{-\overline{\eta}\omega + \overline{\eta\tau}\omega}{\rho|1-\overline{\tau}|^2} + \dfrac{x_2^\circ - x_2^\circ\overline{\tau}}{\rho|1-\overline{\tau}|^2}}{\dfrac{-\overline{\eta}x_1^\circ + \overline{\eta\tau}x_1^\circ}{\rho|1-\overline{\tau}|^2} + \dfrac{1-\overline{\tau}}{\rho|1-\overline{\tau}|^2}} = \tilde{\eta} \\
& \Leftrightarrow \quad \frac{-\overline{\eta}\omega + \overline{\eta\tau}\omega + x_2^\circ - x_2^\circ\overline{\tau}}{-\overline{\eta}x_1^\circ + \overline{\eta\tau}x_1^\circ + 1 - \overline{\tau}} = \tilde{\eta} \\
& \Leftrightarrow \quad -\overline{\eta}\omega + \overline{\eta\tau}\omega + x_2^\circ - x_2^\circ\overline{\tau} = -\overline{\eta}x_1^\circ\tilde{\eta} + \overline{\eta\tau}x_1^\circ\tilde{\eta} + \tilde{\eta} - \overline{\tau}\tilde{\eta} \\
& \Leftrightarrow \quad x_2^\circ(1-\overline{\tau}) = -\overline{\eta}x_1^\circ\tilde{\eta} + \overline{\eta\tau}x_1^\circ\tilde{\eta} + \tilde{\eta} - \overline{\tau}\tilde{\eta} + \overline{\eta}\omega - \overline{\eta\tau}\omega \\
& \Leftrightarrow \quad x_2^\circ = \frac{-\overline{\eta}x_1^\circ\tilde{\eta} + \overline{\eta\tau}x_1^\circ\tilde{\eta} + \tilde{\eta} - \overline{\tau}\tilde{\eta} + \overline{\eta}\omega - \overline{\eta\tau}\omega}{1-\overline{\tau}} \\
& \Leftrightarrow \quad x_2^\circ = \frac{-\overline{\eta}x_1^\circ\tilde{\eta}(1-\overline{\tau}) + \tilde{\eta}(1-\overline{\tau}) + \overline{\eta}\omega(1-\overline{\tau})}{1-\overline{\tau}} \\
& \Leftrightarrow \quad x_2^\circ = -\overline{\eta}x_1^\circ\tilde{\eta} + \tilde{\eta} + \overline{\eta}\omega.
\end{aligned}
$$

Since $x_1^\circ = \overline{x_2^\circ} x_3^\circ$ and $x_3^\circ = \omega$, we have

$$
\begin{cases}
x_3^\circ &= \omega \\
x_1^\circ &= \overline{x_2^\circ}\omega \\
x_2^\circ &= -\overline{\eta}x_1^\circ\tilde{\eta} + \tilde{\eta} + \overline{\eta}\omega.
\end{cases} \tag{5.26}
$$

For given $\eta, \tilde{\eta} \in \mathbb{T}$, we want to find a solution $x_1^\circ, x_2^\circ, x_3^\circ$ of the above system such that $|x_1^\circ| < 1, |x_2^\circ| < 1$. Thus we want to find $\omega \in \mathbb{T}$ and $x_1^\circ \in \mathbb{C} : |x_1^\circ| < 1$ such that

$$x_1^\circ = \omega(\overline{-\overline{\eta}x_1^\circ\tilde{\eta} + \tilde{\eta} + \overline{\eta}\omega}),$$

that is

$$x_1^\circ + \overline{x_1^\circ}\omega\eta\overline{\tilde{\eta}} = \omega\overline{\tilde{\eta}} + \eta. \tag{5.27}$$

Step 3. For the given data, $1 \to (\eta, \tilde{\eta}, \eta\tilde{\eta})$, in the case that there are $x_1^\circ, x_2^\circ, x_3^\circ \in \mathbb{C}$ such that

(i) $|x_1^\circ| < 1, x_3^\circ = \omega \in \mathbb{T}$,

(ii) $x_1^\circ = \overline{x_2^\circ}\omega$,

(iii) $x_2^\circ = -\overline{\eta}x_1^\circ\tilde{\eta} + \tilde{\eta} + \overline{\eta}\omega$.

the solution of the problem will be an $\overline{\mathcal{E}}-$inner function $x = (x_1, x_2, x_3)$ where, for $\lambda \in \mathbb{D}$,

$$\begin{aligned}
x_1(\lambda) &= \frac{x_1^\circ a + b}{x_1^\circ c + d}(\lambda) \\
&= \frac{x_1^\circ \left[\frac{1-\lambda}{1-\tau} - \frac{(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}\right] + \left[\frac{\eta(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}\right]}{x_1^\circ \left[-\frac{\overline{\eta}(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}\right] + \left[\frac{1-\lambda}{1-\tau} + \frac{1-\overline{\tau}\lambda}{\rho|1-\tau|^2}\right]} \\
&= \frac{x_1^\circ[(1-\lambda)\rho(1-\overline{\tau}) - (1-\overline{\tau}\lambda)] + \eta(1-\overline{\tau}\lambda)}{x_1^\circ[-\overline{\eta}(1-\overline{\tau}\lambda)] + \rho(1-\overline{\tau})(1-\lambda) + (1-\overline{\tau}\lambda)},
\end{aligned} \tag{5.28}$$

$$\begin{aligned}
x_2(\lambda) &= \frac{x_3^\circ c + x_2^\circ d}{x_1^\circ c + d}(\lambda) \\
&= \frac{x_3^\circ \left[-\frac{\overline{\eta}(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}\right] + x_2^\circ \left[\frac{1-\lambda}{1-\tau} + \frac{1-\overline{\tau}\lambda}{\rho|1-\tau|^2}\right]}{x_1^\circ \left[-\frac{\overline{\eta}(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}\right] + \left[\frac{1-\lambda}{1-\tau} + \frac{1-\overline{\tau}\lambda}{\rho|1-\tau|^2}\right]} \\
&= \frac{x_3^\circ[-\overline{\eta}(1-\overline{\tau}\lambda)] + x_2^\circ[\rho(1-\overline{\tau})(1-\lambda) + (1-\overline{\tau}\lambda)]}{x_1^\circ[-\overline{\eta}(1-\overline{\tau}\lambda)] + \rho(1-\overline{\tau})(1-\lambda) + (1-\overline{\tau}\lambda)},
\end{aligned} \tag{5.29}$$

$$\begin{aligned}
x_3(\lambda) &= \frac{x_2^\circ b + x_3^\circ a}{x_1^\circ c + d}(\lambda) \\
&= \frac{x_2^\circ \left[\frac{\eta(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}\right] + x_3^\circ \left[\frac{1-\lambda}{1-\tau} - \frac{(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}\right]}{x_1^\circ \left[-\frac{\overline{\eta}(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}\right] + \left[\frac{1-\lambda}{1-\tau} + \frac{1-\overline{\tau}\lambda}{\rho|1-\tau|^2}\right]} \\
&= \frac{x_2^\circ\eta(1-\overline{\tau}\lambda) + x_3^\circ[\rho(1-\overline{\tau})(1-\lambda) - (1-\overline{\tau}\lambda)]}{x_1^\circ[-\overline{\eta}(1-\overline{\tau}\lambda)] + \rho(1-\overline{\tau})(1-\lambda) + (1-\overline{\tau}\lambda)}.
\end{aligned} \tag{5.30}$$

Let us check that $x(1) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$. By equation (5.28), we have

$$\begin{aligned} x_1(1) &= \frac{x_1^\circ(-1+\overline{\tau}) + \eta(1-\overline{\tau})}{-x_1^\circ\overline{\eta}(1-\overline{\tau}) + (1-\overline{\tau})} \\ &= \frac{-x_1^\circ + \eta}{-x_1^\circ\overline{\eta} + 1} = \frac{\eta(-x_1^\circ\overline{\eta} + 1)}{-x_1^\circ\overline{\eta} + 1} = \eta. \end{aligned}$$

By equation (5.29), since $x_2^\circ = -x_1^\circ\overline{\eta}\tilde{\eta} + \tilde{\eta} + \omega\overline{\eta}$, we have

$$\begin{aligned} x_2(1) &= \frac{-x_3^\circ\overline{\eta}(1-\overline{\tau}) + x_2^\circ(1-\overline{\tau})}{-x_1^\circ\overline{\eta}(1-\overline{\tau}) + (1-\overline{\tau})} \\ &= \frac{-\omega\overline{\eta} + x_2^\circ}{-x_1^\circ + 1} = \frac{-\omega\overline{\eta} - \overline{\eta}x_1^\circ\tilde{\eta} + \tilde{\eta} + \overline{\eta}\omega}{-x_1^\circ + 1} \\ &= \frac{\tilde{\eta}(-x_1^\circ + 1)}{-x_1^\circ + 1} = \tilde{\eta}. \end{aligned}$$

By equation (5.30), since $x_2^\circ = -x_1^\circ\overline{\eta}\tilde{\eta} + \tilde{\eta} + \omega\overline{\eta}$, we have

$$\begin{aligned} x_3(1) &= \frac{x_2^\circ\eta(1-\overline{\tau}) + \omega(-1+\overline{\tau})}{-x_1^\circ\overline{\eta}(1-\overline{\tau}) + (1-\overline{\tau})} \\ &= \frac{x_2^\circ\eta - \omega}{-x_1^\circ\overline{\eta} + 1} = \frac{-\overline{\eta}x_1^\circ\tilde{\eta}\eta + \tilde{\eta}\eta + \overline{\eta}\omega\eta}{-x_1^\circ\overline{\eta} + 1} \\ &= \frac{\eta\tilde{\eta}(-x_1^\circ\overline{\eta} + 1)}{-x_1^\circ\overline{\eta} + 1} = \eta\tilde{\eta}. \end{aligned}$$

One can easily check that $x = (x_1, x_2, x_3)$ defined by equations (5.28), (5.29) and (5.30) is a $\overline{\mathcal{E}}$-inner function of degree 1 satisfying $x(1) = (\eta, \tilde{\eta}, \eta\tilde{\eta})$.
Let us check that $Ax_1(1) = \rho$. By Proposition 3.2.2 and Proposition 4.2.3,

$$Ax_1(1) = A(\Psi_\omega \circ x)(1) = A\varphi(1).$$

By Proposition (A.2.2), for every $\lambda \in \mathbb{T}$ and for every rational inner function φ,

$$A\varphi(\lambda) = \lambda\frac{\varphi'(\lambda)}{\varphi(\lambda)}.$$

Recall that

$$\varphi(\lambda) = \frac{a(\lambda)\zeta + b(\lambda)}{c(\lambda)\zeta + d(\lambda)}.$$

We aim to show that $A\varphi(1) = 1\dfrac{\varphi'(1)}{\varphi(1)} = \rho$. One can easily check that, for $\lambda \in \overline{\mathbb{D}}$,

$$a(\lambda) = \frac{\rho(1-\overline{\tau})(1-\lambda) - (1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}, \quad a'(\lambda) = \frac{-\rho(1-\overline{\tau}) + \overline{\tau}}{\rho|1-\tau|^2}, \tag{5.31}$$

and so

$$a(1) = \frac{-1+\overline{\tau}}{\rho|1-\tau|^2}, \qquad a'(1) = \frac{-\rho(1-\overline{\tau}) + \overline{\tau}}{\rho|1-\tau|^2}; \tag{5.32}$$

$$b(\lambda) = \frac{\eta(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}, \qquad b'(\lambda) = \frac{-\eta\overline{\tau}}{\rho|1-\tau|^2}, \tag{5.33}$$

and so

$$b(1) = \frac{\eta(1-\overline{\tau})}{\rho|1-\tau|^2}, \qquad b'(1) = \frac{-\eta\overline{\tau}}{\rho|1-\tau|^2}; \tag{5.34}$$

$$c(\lambda) = \frac{-\overline{\eta}(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}, \qquad c'(\lambda) = \frac{\overline{\eta}\overline{\tau}}{\rho|1-\tau|^2}, \tag{5.35}$$

and so

$$c(1) = \frac{-\overline{\eta}(1-\overline{\tau})}{\rho|1-\tau|^2}, \qquad c'(1) = \frac{\overline{\eta}\overline{\tau}}{\rho|1-\tau|^2}; \tag{5.36}$$

$$d(\lambda) = \frac{\rho(1-\overline{\tau})(1-\lambda) + (1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}, \qquad d'(\lambda) = \frac{-\rho(1-\overline{\tau}) - \overline{\tau}}{\rho|1-\tau|^2}, \tag{5.37}$$

and so

$$d(1) = \frac{1-\overline{\tau}}{\rho|1-\tau|^2}, \qquad d'(1) = \frac{-\rho(1-\overline{\tau}) - \overline{\tau}}{\rho|1-\tau|^2}. \tag{5.38}$$

By the equations (5.32), (5.34), (5.36) and (5.38),

$$\begin{aligned}
\varphi(1) &= \frac{a(1)\zeta + b(1)}{c(1)\zeta + d(1)} \\
&= \frac{\left[\dfrac{-1+\overline{\tau}}{\rho|1-\tau|^2}\right]\zeta + \dfrac{\eta(1-\overline{\tau})}{\rho|1-\tau|^2}}{\left[\dfrac{-\overline{\eta}(1-\overline{\tau})}{\rho|1-\tau|^2}\right]\zeta + \dfrac{1-\overline{\tau}}{\rho|1-\tau|^2}} \\
&= \frac{(-1+\overline{\tau})\zeta + \eta(1-\overline{\tau})}{-\overline{\eta}(1-\overline{\tau})\zeta + (1-\overline{\tau})} \\
&= \frac{-\zeta+\eta}{-\overline{\eta}\zeta+1} = \frac{\eta(1-\overline{\eta}\zeta)}{1-\overline{\eta}\zeta} = \eta.
\end{aligned} \tag{5.39}$$

Let us find $\varphi'(\lambda)$. Note that, for $\lambda \in \overline{\mathbb{D}}$,

$$\varphi'(\lambda) = \left(\frac{u}{v}\right)'(\lambda) = \frac{u'v - uv'}{v^2}(\lambda),$$

where $u(\lambda) = a(\lambda)\zeta + b(\lambda)$ and $v(\lambda) = c(\lambda)\zeta + d(\lambda)$.
It is easy to see that $u'(\lambda) = a'(\lambda)\zeta + b'(\lambda)$ and $v'(\lambda) = c'(\lambda)\zeta + d'(\lambda)$. By equations (5.32) and (5.34),

$$u'(1) = a'(1)\zeta + b'(1) = \frac{-\rho(1-\overline{\tau})\zeta + \overline{\tau}\zeta - \eta\overline{\tau}}{\rho|1-\tau|^2}, \tag{5.40}$$

and

$$\begin{aligned} u(1) &= a(1)\zeta + b(1) \\ &= \frac{-\zeta + \overline{\tau}\zeta + \eta(1-\overline{\tau})}{\rho|1-\tau|^2} \\ &= \frac{(1-\overline{\tau})(-\zeta+\eta)}{\rho|1-\tau|^2} = \frac{\eta - \zeta}{\rho(1-\overline{\tau})}. \end{aligned} \tag{5.41}$$

By equations (5.36) and (5.38),

$$v'(1) = c'(1)\zeta + d'(1) = \frac{\overline{\eta\tau}\zeta - \rho(1-\overline{\tau}) - \overline{\tau}}{\rho|1-\tau|^2}, \tag{5.42}$$

and

$$\begin{aligned} v(1) &= c(1)\zeta + d(1) \\ &= \frac{-\overline{\eta}\zeta(1-\overline{\tau}) + (1-\overline{\tau})}{\rho|1-\tau|^2} \\ &= \frac{(-\overline{\eta}\zeta + 1)}{\rho(1-\tau)} = \frac{\overline{\eta}(-\zeta+\eta)}{\rho(1-\tau)}. \end{aligned} \tag{5.43}$$

Therefore,

$$u'(1)v(1) = \left[\frac{-\rho(1-\overline{\tau})\zeta + \overline{\tau}\zeta - \eta\overline{\tau}}{\rho|1-\tau|^2}\right]\left[\frac{\overline{\eta}(\eta-\zeta)}{\rho(1-\tau)}\right], \tag{5.44}$$

and

$$u(1)v'(1) = \left[\frac{\eta-\zeta}{\rho(1-\overline{\tau})}\right]\left[\frac{\overline{\eta\tau}\zeta - \rho(1-\overline{\tau}) - \overline{\tau}}{\rho|1-\tau|^2}\right], \tag{5.45}$$

and

$$v^2(1) = \left(\frac{\overline{\eta}(\eta-\zeta)}{\rho(1-\tau)}\right)^2 = \frac{\overline{\eta}^2(\eta-\zeta)^2}{\rho^2(1-\tau)^2}. \tag{5.46}$$

Thus by equations (5.44) and (5.45), we have

$$\begin{aligned} u'(1)v(1) - u(1)v'(1) &= \frac{1}{\rho(1-\tau)\rho|1-\tau|^2}(\eta-\zeta)\Big[\overline{\eta}(-\rho(1-\overline{\tau})\zeta + \overline{\tau}\zeta - \eta\overline{\tau}) \\ &\quad - (\overline{\eta}\overline{\tau}\zeta - \rho(1-\overline{\tau}) - \overline{\tau})\Big] \\ &= \frac{(\eta-\zeta)}{\rho(1-\tau)\rho|1-\tau|^2}\Big[(-\overline{\eta}\zeta+1)\rho(1-\overline{\tau})\Big] \\ &= \frac{(\eta-\zeta)(-\overline{\eta}\zeta+\eta\overline{\eta})}{\rho(1-\tau)^2} \\ &= \frac{\overline{\eta}(\eta-\zeta)^2}{\rho(1-\tau)^2}. \end{aligned} \tag{5.47}$$

Thus, by equations (5.47) and (5.46),

$$\begin{aligned} \varphi'(1) &= \frac{u'(1)v(1) - u(1)v'(1)}{v^2(1)} \\ &= \frac{\overline{\eta}(\eta-\zeta)^2}{\rho(1-\tau)^2}\frac{\rho^2(1-\tau)^2}{\overline{\eta}^2(\eta-\zeta)^2} \\ &= \frac{\rho}{\overline{\eta}} = \rho\eta. \end{aligned} \tag{5.48}$$

Hence

$$A\varphi(1) = 1\frac{\varphi'(1)}{\varphi(1)} = \frac{\rho\eta}{\eta} = \rho.$$

By Proposition 3.2.2, $Ax_1(1) = A\varphi(1) = \rho$.

Example 5.2.4. Let $n = 1, k = 1$. Let $\eta = i, \tilde{\eta} = -i$. Suppose $\sigma_1 = 1$. The points $\eta = i, \tilde{\eta} = -i \in \mathbb{T}$ and a $\rho > 0$ are prescribed, and we seek a $\overline{\mathcal{E}}$-inner function $x = (x_1, x_2, x_3)$ of degree 1 such that $x(1) = (i, -i, 1)$, and $Ax_1(1) = \rho$.

Let us follow steps of Example 5.2.3.

Step 1. Let $\tau \in \mathbb{T}\backslash\{1\}$. By equations (5.26), (5.26), (5.26) and (5.26) since $\eta = i$ and $\tilde{\eta} = -i$,

$$a(\lambda) = \frac{1-\lambda}{1-\tau} - \frac{(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}$$

$$b(\lambda) = \frac{i(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}$$

$$c(\lambda) = -\frac{-i(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}$$

$$d(\lambda) = \frac{1-\lambda}{1-\tau} + \frac{(1-\overline{\tau}\lambda)}{\rho|1-\tau|^2}.$$

Step 2. Let us determine whether there exist $x_1^\circ, x_2^\circ, x_3^\circ \in \mathbb{C}$ such that

$$|x_3^\circ| = 1, \quad |x_1^\circ| < 1, \quad |x_2^\circ| < 1, \text{ and } x_1^\circ = \overline{x_2^\circ} x_3^\circ, \tag{5.49}$$

and

$$\frac{x_3^\circ c(0) + x_2^\circ d(0)}{x_1^\circ c(0) + d(0)} = \tilde{\eta}. \tag{5.50}$$

By the above equations (5.26) and (5.27), we need to find a solution of the following system:

$$\begin{cases} x_3^\circ &=& \omega \\ x_1^\circ &=& \overline{x_2^\circ}\omega \\ x_2^\circ &=& -\overline{\eta} x_1^\circ \tilde{\eta} + \tilde{\eta} + \overline{\eta}\omega \end{cases} \tag{5.51}$$

such that $|x_1^\circ| < 1, |x_2^\circ| < 1, \omega \in \mathbb{T}$.
For the given data $1 \to (i, -i, 1)$, the system (5.51) is equivalent to

$$\begin{cases} x_3^\circ &=& \omega \\ x_1^\circ &=& \overline{x_2^\circ}\omega \\ x_2^\circ &=& i x_1^\circ(-i) + (-i) + (-i)\omega \end{cases} \tag{5.52}$$

with $|x_1^\circ| < 1, |x_2^\circ| < 1, \omega \in \mathbb{T}$.
Let take $\omega = -1 \in \mathbb{T}$. Then

$$\begin{cases} x_3^\circ &=& -1 \\ x_1^\circ &=& -\overline{x_2^\circ} \\ x_2^\circ &=& x_1^\circ \end{cases} \tag{5.53}$$

where $|x_1^\circ| < 1, |x_2^\circ| < 1$.
Thus, $x_1^\circ = -\overline{x_1^\circ}$, and so $x_1^\circ = ib$, where $b \in \mathbb{R}$ and $|b| < 1$.
Therefore,

$$\begin{cases} x_1^\circ &= ib \\ x_2^\circ &= ib \\ x_3^\circ &= -1, \end{cases} \tag{5.54}$$

where $b \in \mathbb{R}$ and $|b| < 1$, satisfy equations (5.49) and (5.50).
Hence, by equations (5.28), (5.29) and (5.30), the solution of the problem of finding a degree 1 $\overline{\mathcal{E}}-$inner function such that $x(1) = (i, -i, 1)$ and $Ax_1(1) = \rho$ will be,

$$x_1(\lambda) = \frac{ib[(1-\lambda)\rho(1-\overline{\tau}) - (1-\overline{\tau}\lambda)] + i(1-\overline{\tau}\lambda)}{ib[i(1-\overline{\tau}\lambda)] + \rho(1-\overline{\tau})(1-\lambda) + (1-\overline{\tau}\lambda)}, \tag{5.55}$$

$$x_2(\lambda) = \frac{(-1)[i(1-\overline{\tau}\lambda)] + ib[\rho(1-\overline{\tau})(1-\lambda) + (1-\overline{\tau}\lambda)]}{ib[i(1-\overline{\tau}\lambda)] + \rho(1-\overline{\tau})(1-\lambda) + (1-\overline{\tau}\lambda)}, \tag{5.56}$$

$$x_3(\lambda) = \frac{ib(i)(1-\overline{\tau}\lambda) + (-1)[\rho(1-\overline{\tau})(1-\lambda) - (1-\overline{\tau}\lambda)]}{ib[i(1-\overline{\tau}\lambda)] + \rho(1-\overline{\tau})(1-\lambda) + (1-\overline{\tau}\lambda)}. \tag{5.57}$$

Hence at 1,

$$x_1(1) = \frac{ib(-1+\overline{\tau}) + i(1-\overline{\tau}))}{ib(i)(1-\overline{\tau}) + (1-\overline{\tau})} = \frac{-x_1^\circ + i}{x_1^\circ i + 1} = \frac{i(x_1^\circ i + 1)}{x_1^\circ i + 1} = i,$$

$$x_2(1) = \frac{-i(1-\overline{\tau}) + ib(1-\overline{\tau})}{ibi(1-\overline{\tau}) + 1 - \overline{\tau}} = \frac{-i + ib}{ib(i) + 1} = \frac{-i(ib(i) + 1)}{ib(i) + 1} = -i,$$

$$x_3(1) = \frac{ib(i)(1-\overline{\tau}) + (1-\overline{\tau})}{ib(i)(1-\overline{\tau}) + (1-\overline{\tau})} = \frac{ib(i) + 1}{ib(i) + 1} = 1.$$

We have shown for the general case in Example 5.2.3, that $Ax_1(1) = \rho$.

Chapter 6

Matricial formulations of the solvability criterion for tetra-interpolation problems

6.1 Introduction

Recall that *the μ_{Diag}-synthesis problem*: given distinct points $\lambda_1, ..., \lambda_n \in \mathbb{D}$ and target matrices $W_1, ..., W_n \in M_{2\times 2}(\mathbb{C})$ such that $\mu_{\text{Diag}}(W_k) < 1,\ k = 1, ..., n$, find if possible an analytic 2×2 matrix-valued function F on $\mathbb{D}$ such that

$$F(\lambda_j) = W_j \quad \text{for } j = 1, ..., n, \text{ and}$$
$$\mu_{\text{Diag}}(F(\lambda)) \ < \ 1, \text{ for all } \lambda \in \mathbb{D}.$$

Abouhajar, White and Young showed in [1] that the solvability of μ_{Diag}-synthesis problem is equivalent to the solvability of an interpolation problem from $\mathbb{D}$ to $\overline{\mathcal{E}}$. In 2016, Brown, Lykova and Young [22] proved the following theorem, see also [21].

Theorem 6.1.1. [22, Theorems 1.1 and 8.1] *Let $\lambda_1, ..., \lambda_n$ be distinct points in $\mathbb{D}$, and let*

$$W_j = \begin{bmatrix} w_{11}^j & w_{12}^j \\ w_{21}^j & w_{22}^j \end{bmatrix} \in M_{2\times 2}(\mathbb{C}).$$

be such that $\mu_{Diag}(W_j) \le 1$ and $w_{11}^j, w_{22}^j \neq det\ W_j$ for $j = 1, ..., n$. Let $(x_{1j}, x_{2j}, x_{3j}) = (w_{11}^j, w_{22}^j, det\ W_j) \in \overline{\mathcal{E}}$ for $j = 1, ..., n$. Then the following *are equivalent.*

(i) *There exists an analytic function* $F : \mathbb{D} \to M_{2\times 2}(\mathbb{C})$ *such that* $F(\lambda_j) = W_j$ *for* $j = 1, ..., n$, *and* $\mu_{Diag}(F(\lambda)) \le 1$ *for all* $\lambda \in \mathbb{D}$;

(ii) *There is an* $x \in \mathrm{Hol}(\mathbb{D}, \overline{\mathcal{E}})$ *such that* $x(\lambda_j) = (x_{1j}, x_{2j}, x_{3j})$ *for* $j = 1, ..., n$;

(iii) *for every distinct points* $z_1, z_2, z_3 \in \mathbb{D}$, *there exist positive* $3n$*-square matrices* $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *of rank at most* 1, *and* $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *such that, for* $1 \le i, j \le n$ *and* $1 \le l, k \le 3$,

$$1 - \overline{\frac{z_l x_{3i} - x_{1i}}{x_{2i} z_l - 1}} \frac{z_k x_{3j} - x_{1j}}{x_{2j} z_k - 1} = (1 - \overline{z_l} z_k) N_{il,jk} + (1 - \overline{\lambda_i} \lambda_j) M_{il,jk}; \tag{6.1}$$

(iv) *for some distinct points* $z_1, z_2, z_3 \in \mathbb{D}$, *there exist positive* $3n$*-square matrices* $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *of rank at most* 1, *and* $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *such that*

$$\left[1 - \overline{\frac{z_l x_{3i} - x_{1i}}{x_{2i} z_l - 1}} \frac{z_k x_{3j} - x_{1j}}{x_{2j} z_k - 1}\right] \ge [(1 - \overline{z_l} z_k) N_{il,jk}] + [(1 - \overline{\lambda_i} \lambda_j) M_{il,jk}]. \tag{6.2}$$

Theorem 6.1.2. [27, Theorem 7.5.2] *If* $A \in M_{n\times n}(\mathbb{C})$ *is positive semidefinite matrix of rank* k, *then* A *may be written in the form*

$$A = v_1 v_1^* + v_2 v_2^* + ... + v_k v_k^*$$

where each $v_i \in \mathbb{C}^n$ *and the set* $\{v_1, ..., v_k\}$ *is an orthogonal set of nonzero vectors.*

6.2 Matricial formulations of the solvability criterion for tetra-interpolation problems

A matricial formulation of a solvability criterion for the spectral Nevanlinna-Pick problem was given in [3]. The next theorem presents a matricial formulation of a criterion for the solvability of a μ_{Diag}-synthesis problem.

Theorem 6.2.1. *Let $\lambda_1, ..., \lambda_n$ be distinct points in $\mathbb{D}$, and let*

$$W_j = \begin{bmatrix} w_{11}^j & w_{12}^j \\ w_{21}^j & w_{22}^j \end{bmatrix} \in M_{2\times 2}(\mathbb{C}).$$

be such that $\mu_{Diag}(W_j) \leq 1$ and $w_{11}^j, w_{22}^j \neq \det W_j$ for $j = 1, ..., n$. Let $x_{1j} = w_{11}^j$, $x_{2j} = w_{22}^j$ and $x_{3j} = \det W_j$ for each j. Let the $3n$-square matrix Λ be defined by

$$\Lambda = \mathrm{diag}\{\lambda_i\}_{i=1,\ell=1}^{n,3}, \tag{6.3}$$

The following conditions are equivalent.

(i) *There exists an analytic function $F : \mathbb{D} \to M_{2\times 2}(\mathbb{C})$ such that*

$$F(\lambda_j) = W_j \ for \ j = 1, ..., n$$

and

$$\mu_{Diag}(F(\lambda)) \leq 1 \text{ for all } \lambda \in \mathbb{D};$$

(ii) *For some distinct points $z_1, z_2, z_3 \in \mathbb{D}$, there exist positive $3n$-square matrices $N = [N_{i\ell,jk}]_{i,j=1,\ell,k=1}^{n,3}$, $M = [M_{i\ell,jk}]_{i,j=1,\ell,k=1}^{n,3}$ such that rank $N \leq 1$ and*

$$X \geq N - Z^*NZ + M - \Lambda^* M \Lambda, \tag{6.4}$$

where $3n$-square matrices X and Z are defined by

$$X = \left[1 - \overline{\frac{z_\ell x_{3i} - x_{1i}}{x_{2i} z_\ell - 1}} \frac{z_k x_{3j} - x_{1j}}{x_{2j} z_k - 1}\right]_{i,j=1,\ell,k=1}^{n,3}, \tag{6.5}$$

$$Z = \mathrm{diag}\{z_\ell\}_{i=1,\ell=1}^{n,3}; \tag{6.6}$$

(iii) *For every choice of distinct points $z_1, z_2, z_3 \in \mathbb{D}$, there exist positive $3n$-square matrices $N = [N_{i\ell,jk}]_{i,j=1,\ell,k=1}^{n,3}$, $M = [M_{i\ell,jk}]_{i,j=1,\ell,k=1}^{n,3}$ such that rank $N \leq 1$ and*

$$X = N - Z^*NZ + M - \Lambda^* M \Lambda, \tag{6.7}$$

where X and Z are defined by equations (6.5) and (6.6) ;

(iv) *For some distinct points $z_1, z_2, z_3 \in \mathbb{D}$, there exist a positive $3n$-square matrix M, a $1 \times 3n$ vector γ, and a matrix P of type $3n \times 2$ such that*

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & X \end{bmatrix} \geq \begin{bmatrix} I_2 & 0 \\ P & I_{3n} \end{bmatrix} \begin{bmatrix} -1 & 0 & \gamma \\ 0 & 1 & \gamma Z \\ \gamma^* & Z^*\gamma^* & M - \Lambda^* M \Lambda \end{bmatrix} \begin{bmatrix} I_2 & P^* \\ 0 & I_{3n} \end{bmatrix}, \tag{6.8}$$

where X and Z are defined by equations (6.5) *and* (6.6) *;*

(v) *For every distinct points $z_1, z_2, z_3 \in \mathbb{D}$, there exist a positive $3n$-square matrix M, a $1 \times 3n$ vector γ, and a matrix P of type $3n \times 2$ such that*

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & X \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ P & I_{3n} \end{bmatrix} \begin{bmatrix} -1 & 0 & \gamma \\ 0 & 1 & \gamma Z \\ \gamma^* & Z^*\gamma^* & M - \Lambda^* M \Lambda \end{bmatrix} \begin{bmatrix} I_2 & P^* \\ 0 & I_{3n} \end{bmatrix}, \tag{6.9}$$

where X and Z are defined by equations (6.5) *and* (6.6) *.*

Note that in N, M, Λ and Z the rows are indexed by the pair (i, ℓ) and the columns by the pair (j, k), where i and j run from 1 to n, and ℓ and k run from 1 to 3.

Proof. It is easy to see that (6.2) and (6.1) can be written as equations (6.4) and (6.7) respectively. If $X = [X_{i\ell,jk}]_{i,j=1,\ell,k=1}^{n,3}$, then by (6.1),

$$X_{i\ell,jk} = N_{i\ell,jk} - \overline{z_\ell} N_{i\ell,jk} z_k + M_{i\ell,jk} - \overline{\lambda_i} M_{i\ell,jk} \lambda_j$$

where $1 \leq i, j \leq n$ and $1 \leq l, k \leq 3$.
The proof has the following structure:

$$\begin{array}{ccccc} (i) & \Longleftrightarrow & (ii) & \Longleftrightarrow & (iii) \\ & & \Updownarrow & & \Updownarrow \\ & & (iv) & & (v) \end{array}$$

The equivalences $(i) \Longleftrightarrow (ii) \Longleftrightarrow (iii)$ follow from Theorem 6.1.1.
$(ii) \Longrightarrow (iv)$. Suppose (ii). Since N has rank 1 and $N \geq 0$, by Theorem 6.1.2, there exists a $1 \times 3n$ vector γ such that $N = \gamma^*\gamma$. Consider the Schur complement identity

$$\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ B^* A^{-1} & I_{3n} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - B^* A^{-1} B \end{bmatrix} \begin{bmatrix} I_2 & A^{-1}B \\ 0 & I_{3n} \end{bmatrix} \tag{6.10}$$

where A, D are of types 2×2, $3n \times 3n$, respectively. Now choose

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \gamma \\ \gamma Z \end{bmatrix}, \quad D = M - \Lambda^* M \Lambda.$$

We can write identity (6.10) as

$$\begin{bmatrix} -1 & 0 & \gamma \\ 0 & 1 & \gamma Z \\ \gamma^* & Z^*\gamma^* & M - \Lambda^* M \Lambda \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\gamma^* & Z^*\gamma^* & I_{3n} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & M - \Lambda^* M \Lambda + \gamma^*\gamma - Z^*\gamma^*\gamma Z \end{bmatrix} \begin{bmatrix} 1 & 0 & -\gamma \\ 0 & 1 & \gamma Z \\ 0 & 0 & I_{3n} \end{bmatrix}. \tag{6.11}$$

Let

$$P = -B^* A^{-1} = \begin{bmatrix} \gamma^* & -Z^*\gamma^* \end{bmatrix} \in \mathbb{C}^{3n \times 2}.$$

Thus (6.11) becomes

$$\begin{bmatrix} -1 & 0 & \gamma \\ 0 & 1 & \gamma Z \\ \gamma^* & Z^*\gamma^* & M - \Lambda^* M \Lambda \end{bmatrix}$$
$$= \begin{bmatrix} I_2 & 0 \\ -P & I_{3n} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & M - \Lambda^* M \Lambda + \gamma^*\gamma - Z^*\gamma^*\gamma Z \end{bmatrix} \begin{bmatrix} I_2 & -P^* \\ 0 & I_{3n} \end{bmatrix}. \tag{6.12}$$

By pre- and post-multiplying by the inverses of the first and third matrices on the right-hand side and using the relation (6.4) we obtain the relation (6.9):

$$\begin{bmatrix} I_2 & 0 \\ P & I_{3n} \end{bmatrix} \begin{bmatrix} -1 & 0 & \gamma \\ 0 & 1 & \gamma Z \\ \gamma^* & Z^*\gamma^* & M - \Lambda^* M \Lambda \end{bmatrix} \begin{bmatrix} I_2 & P^* \\ 0 & I_{3n} \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & M - \Lambda^* M \Lambda + \gamma^*\gamma - Z^*\gamma^*\gamma Z \end{bmatrix} \leq \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & X \end{bmatrix}. \tag{6.13}$$

Now, we prove $(iv)\Longrightarrow(ii)$.
The inequality (6.8) can be expressed as

$$\begin{bmatrix} I_2 & 0 \\ -P & I_{3n} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} I_2 & -P^* \\ 0 & I_{3n} \end{bmatrix} \geq \begin{bmatrix} A & B \\ B^* & M-\Lambda^* M\Lambda \end{bmatrix}.$$

It follows that

$$\begin{aligned} 0 &\leq \begin{bmatrix} A & -AP^* \\ -PA & PAP^*+X \end{bmatrix} - \begin{bmatrix} A & B \\ B^* & M-\Lambda^* M\Lambda \end{bmatrix} \\ &= \begin{bmatrix} 0 & -AP^*-B \\ -PA-B^* & PAP^*+X-M-\Lambda^* M\Lambda \end{bmatrix} \end{aligned}$$

Hence $P^* = -AB$ and

$$\begin{aligned} 0 &\leq PAP^*+X-M-\Lambda^* M\Lambda \\ &= B^*A^3B+X-M-\Lambda^* M\Lambda \\ &= \begin{bmatrix} \gamma^* & Z^*\gamma^* \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma \\ \gamma Z \end{bmatrix} + X-M-\Lambda^* M\Lambda \\ &= -\gamma^*\gamma + Z^*\gamma^*\gamma Z + X - M - \Lambda^* M\Lambda. \end{aligned}$$

Thus,

$$X \geq \gamma^*\gamma - Z^*\gamma^*\gamma Z + M + \Lambda^* M\Lambda.$$

So, (ii) holds with $N = \gamma^*\gamma$.
In the similar way as we have shown that $(ii) \Longleftrightarrow (iv)$, we can prove that $(iii) \Longleftrightarrow (v)$. □

Let $\lambda_1, ..., \lambda_n$ be distinct points in $\mathbb{D}$, and let

$$W_j = \begin{bmatrix} w_{11}^j & w_{12}^j \\ w_{21}^j & w_{22}^j \end{bmatrix} \in M_{2\times 2}(\mathbb{C}).$$

be such that $\mu_{Diag}(W_j) \leq 1$ and $w_{11}^j, w_{22}^j \neq \det W_j$ for $j = 1, ..., n$, and let $x_{1j} = w_{11}^j$, $x_{2j} = w_{22}^j$ and $x_{3j} = \det W_j$ for each j. By [1, Theorem 9.2], μ_{Diag}-synthesis problem reduces to the solution of the $\mathcal{E}$-interpolation problem to find

$$x \in \mathrm{Hol}(\mathbb{D}, \overline{\mathcal{E}}) \;\text{ such that }\; x(\lambda_j) = (x_{1j}, x_{2j} x_{3j}) \text{ for } j = 1, ..., n.$$

To determine with the aid of Theorem 6.2.1 whether the μ_{Diag}-interpolation problem with these data is solvable, we may test conditions (ii) of Theorem 6.2.1. That is, we must ascertain whether there exist positive matrices N of rank 1 and M satisfying (6.4). The following theorem shows that a search over a compact set of pairs of matrices (N, M) suffices.

Theorem 6.2.2. [22, Theorems 9.2] *Let $\lambda_1, ..., \lambda_n$ be distinct points in $\mathbb{D}$, and let $(x_{1j}, x_{2j}, x_{3j}) \in \overline{\mathcal{E}}$ be such that $x_{1j}x_{2j} \neq x_{3j}$ for $j = 1, ..., n$. Let z_1, z_2, z_3 be distinct points in $\mathbb{D}$. The $\overline{\mathcal{E}}$-interpolation problem*

$$\lambda_j \in \mathbb{D} \mapsto (x_{1j}, x_{2j}, x_{3j}) \in \overline{\mathcal{E}}$$

for $j = 1, ..., n$, is solvable if and only if there exist positive $3n$-square matrices $N = [N_{i\ell,jk}]_{i,j=1,\ell,k=1}^{n,3}$ of rank 1 and $M = [M_{i\ell,jk}]_{i,j=1,\ell,k=1}^{n,3}$ that satisfy

$$\left[1 - \overline{\frac{z_l x_{3i} - x_{1i}}{x_{2i} z_l - 1}} \frac{z_k x_{3j} - x_{1j}}{x_{2j} z_k - 1}\right] \geq [(1 - \overline{z_l} z_k) N_{il,jk}] + [(1 - \overline{\lambda_i}\lambda_j) M_{il,jk}], \quad (6.14)$$

and

$$|N_{i\ell,jk}| \leq \frac{1}{(1 - |x_{2i}|)(1 - |x_{2j}|)}$$

and

$$|M_{i\ell,jk}| \leq \frac{2}{|1 - \overline{\lambda_j}\lambda_j|} \sqrt{1 + \frac{1}{(1 - |x_{2i}|)^2}} \sqrt{1 + \frac{1}{(1 - |x_{2j}|)^2}}.$$

Appendix A

Background Materials

A.1 Basic definition

Definition A.1.1. [9, Definition 1] *A* Blaschke factor *is a Möbius transformation that is positive at 0,*

$$B_a(z) = \frac{\overline{a}}{|a|}\frac{a-z}{1-\overline{a}z}\ ,$$

where $a \in \mathbb{D}$.

Definition A.1.2. *The* polynomially convex hull *of a compact subset* S *of* $\mathbb{C}^N$*, denoted by* $\hat{S}$*, is defined as*

$$\hat{S} = \{z \in \mathbb{C}^N : |p(z)| \leq \max_{s\in S} |p(s)| \textit{ for all polynomials } p\}.$$

Definition A.1.3. *A domain* Ω *is said to be* polynomially convex *if for each compact subset* S *of* Ω *, the polynomial hull* $\hat{S}$ *of* S *is contained in* Ω*.*

Definition A.1.4. *Let* Ω *be a domain in* $\mathbb{C}^n$ *with closure* $\overline{\Omega}$*, and let* $A(\Omega)$ *be the algebra of continuous scalar functions on* $\overline{\Omega}$ *which are analytic on* Ω*. A subset* C *of* $\overline{\Omega}$ *is a* boundary *for* Ω *if every function in* $A(\Omega)$ *attains its maximum modulus on* C*.*

From the theory of uniform algebras [20, Corollary 2.2.10], it follows that when $\overline{\Omega}$ is polynomially convex, there is a smallest closed boundary of Ω, contained in all the closed boundaries of Ω and called the *distinguished boundary* of Ω or *Shilov boundary* of $A(\Omega)$. When the distinguished boundary of Ω exists, we denote it by $b\Omega$.

A.2 The phasar derivatives

Definition A.2.1. [6, Definition 2.3] *For any differentiable function* $f : \mathbb{T} \to \mathbb{C} \setminus \{0\}$ *the phasar derivative of* f *at* $z = e^{i\theta} \in \mathbb{T}$ *is the derivative with respect to* θ *of the argument of* $f(e^{i\theta})$ *at* θ. *We denote it by* $Af(z)$.

Here are some useful elementary properties of phasar derivatives from [2].

Proposition A.2.2. (i) *For differentiable functions* $\psi, \varphi : \mathbb{T} \to \mathbb{C} \setminus \{0\}$ *and for any* $c \in \mathbb{C} \setminus \{0\}$,

$$A(\psi\varphi) = A\psi + A\varphi \quad \textit{and} \quad A(c\psi) = A\psi. \tag{A.1}$$

(ii) *For any rational inner function* φ *and for all* $z \in \mathbb{T}$,

$$A\varphi(z) = z\frac{\varphi'(z)}{\varphi(z)}. \tag{A.2}$$

(iii) *If* $\alpha \in \mathbb{D}$ *and*

$$B_a(z) = \frac{z - \alpha}{1 - \bar{\alpha}z},$$

then

$$AB_\alpha(z) = \frac{1 - |\alpha|^2}{|z - \alpha|^2} \ > \ 0 \quad for \ z \in \mathbb{T}.$$

(iv) *For any rational inner function* p,

$$Ap(z) \ > \ 0 \quad for \ all \ z \in \mathbb{T}.$$

A.3 Positive definite matrices

Definition A.3.1. *A matrix* $A = (a_{ij}) \in M_{n\times n}(\mathbb{C})$ *is said to be* Hermitian *if* $A = A^*$.

Definition A.3.2. *A matrix* A *is said to be* positive semi-definite *if* $\langle x, Ax\rangle \geq 0$ *for all* $x \in \mathbb{C}^n$, *and* positive definite *if* $\langle x, Ax\rangle > 0$ *for all vectors* $x \neq 0,\ x \in \mathbb{C}^n$.

Note: A positive semi-definite matrix is positive definite if and only if it is invertible. There are some conditions that characterize positive matrices. They are proved in [17].

- A is positive if and only if it is Hermitian and all its eigenvalues are nonnegative. A is strictly positive if and only if all its eigenvalues are positive.
- A is positive if and only if it is Hermitian and all its principal minors are nonnegative. A is strictly positive if and only if all its principal minors are positive.
- A is positive if and only if $A = T^*T$ for some upper triangular matrix T. Further, T can be chosen to have nonnegative diagonal entries. If A is strictly positive, then T is unique. A is positive if and only if T is nonsingular.

Definition A.3.3. *A matrix A is* minimally positive *if $A \geq 0$ and there is no positive diagonal $n \times n$ matrix D, other than $D = 0$, such that $A \geq D$.*

Definition A.3.4. *The spectral radius of a square matrix A, which is denoted by $r(A)$, is the nonnegative real number*

$$r(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

A.3.1 Automorphisms of $\mathbb{D}$

Let Ω be a domain in $\mathbb{C}^n$. An automorphism is an analytic bijective map from a domain Ω to itself. The automorphism group of Ω will be denoted by $\mathrm{Aut}(\Omega)$.
For any point $a \in \mathbb{D}$, there is an automorphism of the disc

$$h_a(z) := \frac{a - z}{1 - \overline{a}z},$$

a conformal bijection of $\mathbb{D}$ that interchanges a and 0.

All automorphisms f of $\mathbb{D}$ have the form

$$f(z) = e^{i\theta} h_a(z)$$

for some point a in $\mathbb{D}$ and some real number $\theta \in [0, 2\pi)$. Automorphisms of $\mathbb{D}$ are called Mobius transformation [9].

Appendix B

Construction of kernels N and M for the tetrablock

Here are some well known definitions and results from [9] and [13].

Definition B.0.1. [13, p. 344] *Let X be a set and $k : X \times X \to \mathbb{C}$ be a function. Then k is a* positive semidefinite *function if for all $x_1, ..., x_n \in X$ and $c_1, ..., c_n \in \mathbb{C}$,*

$$\sum_{i,j=1}^{n} \overline{c_j} c_i k(x_j, x_i) \geq 0.$$

Definition B.0.2. [9, Definition 2.22] *A* kernel *on a set X is a hermitian symmetric positive semidefinite function $k : X \times X \to \mathbb{C}$, where hermitian symmetric means $k(x, y) = \overline{k(y, x)}$ for all $x, y \in X$.*

Theorem B.0.3. [22, Theorems 1.1 and 8.1] *Let $\lambda_1, ..., \lambda_n$ be distinct points in $\mathbb{D}$, and let*

$$W_j = \begin{bmatrix} w_{11}^j & w_{12}^j \\ w_{21}^j & w_{22}^j \end{bmatrix} \in M_{2\times 2}(\mathbb{C}).$$

be such that $\mu_{Diag}(W_j) \leq 1$ and $w_{11}^j, w_{22}^j \neq det\ W_j$ for $j = 1, ..., n$. Set $(x_{1j}, x_{2j}, x_{3j}) = (w_{11}^j, w_{22}^j, det\ W_j) \in \overline{\mathcal{E}}$ for $j = 1, ..., n$. Then the following are equivalent.

(i) *There exists an analytic function $F : \mathbb{D} \to M_{2\times 2}(\mathbb{C})$ such that $F(\lambda_j) = W_j$ for $j = 1, ..., n$, and $\mu_{Diag}(F(\lambda)) \leq 1$ for all $\lambda \in \mathbb{D}$;*

(ii) *for every distinct points* $z_1, z_2, z_3 \in \mathbb{D}$*, there exist positive* $3n$*-square matrices* $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *of rank at most* 1*, and* $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *such that, for* $1 \le i, j \le n$ *and* $1 \le l, k \le 3$*,*

$$1 - \overline{\frac{z_l x_{3i} - x_{1i}}{x_{2i} z_l - 1}} \frac{z_k x_{3i} - x_{1j}}{x_{2j} z_k - 1} = (1 - \overline{z_l} z_k) N_{il,jk} + (1 - \overline{\lambda_i} \lambda_j) M_{il,jk}; \quad \text{(B.1)}$$

Proof. (i) $\Longrightarrow$ (ii): Suppose there is an analytic function $x = (x_1, x_2, x_3) : \mathbb{D} \to \overline{\mathcal{E}}$ such that $x_{\lambda_j} = (x_{1j}, x_{2j}, x_{3j})$ for all $j = 1, ..., n$. By [22, Theorem 7.1], since $x_{1j} x_{2j} \neq x_{3j}$ for $j = 1, ..., n$, there is an analytic function

$$F = \begin{bmatrix} x_1 & f_1 \\ f_2 & x_2 \end{bmatrix} : \mathbb{D} \to M_{2\times 2}(\mathbb{C})$$

such that $f_2 \neq 0, ||F(\lambda)|| \le 1$ for all $\lambda \in \mathbb{D}$, and

$$1 - \overline{\Psi(\omega, x(\mu))} \Psi(z, x(\lambda)) = (1 - \overline{\omega} z) \overline{\gamma(\mu, \omega)} \gamma(\lambda, z) + (1 - \overline{\mu} \lambda) \eta(\mu, \omega)^* \frac{I - F(\mu)^* f(\lambda)}{1 - \overline{\mu} \lambda} \eta(\lambda, z) \quad \text{(B.2)}$$

for all $z, \lambda, \omega, \mu \in \mathbb{D}$, where

$$\gamma(\lambda, z) = (1 - x_2(\lambda) z)^{-1} f_2(\lambda) \text{ and } \eta(\lambda, z) = \begin{bmatrix} 1 \\ \gamma(\lambda, z) z \end{bmatrix}.$$

Let z_1, z_2, z_3 be any distinct points in $\mathbb{D}$. Then, in particular, for $1 \le i, j \le n$ and $1 \le l, k \le 3$ we have

$$\begin{aligned} &1 - \overline{\Psi(z_l, x_{1i}, x_{2i}, x_{3i})} \Psi(z_k, x_{1j}, x_{2j}, x_{3j}) = \\ &(1 - \overline{z_l} z_k) \overline{\gamma(\lambda_i, z_l)} \gamma(\lambda_j, z_k) + (1 - \overline{\lambda_i} \lambda_j) \eta(\lambda_i, z_l)^* \frac{I - F(\lambda_i)^* F(\lambda_j)}{1 - \overline{\lambda_i} \lambda_j} \eta(\lambda_j, z_k). \end{aligned} \quad \text{(B.3)}$$

Since $F \in \mathcal{S}^{2\times 2}$ with $f_2 \neq 0$, by [22, Proposition 5.1],

$$\overline{\gamma(\mu, \omega)} \gamma(\lambda, z) \text{ and } \eta(\mu, \omega)^* \frac{I - F(\mu)^* f(\lambda)}{1 - \overline{\mu} \lambda} \eta(\lambda, z)$$

are kernels on $\mathbb{D}^2$. Hence the $3n$-square matrices

$$N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3} := \left[\overline{\gamma(\lambda_i, z_l)} \gamma(\lambda_j, z_k) \right]_{i,j=1,l,k=1}^{n,3}$$

and

$$M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3} := \left[\eta(\lambda_i, z_l)^* \frac{I - F(\lambda_i)^* F(\lambda_j)}{1 - \overline{\lambda_i}\lambda_j} \eta(\lambda_j, z_k)\right]_{i,j=1,l,k=1}^{n,3}$$

are positive for all $1 \le i, j \le n$ and $1 \le l, k \le 3$. Moreover, N is of rank 1 and for all $1 \le i, j \le n$ and $1 \le l, k \le 3$,

$$1 - \overline{\Psi(z_l, x_{1i}, x_{2i}, x_{3i})}\Psi(z_k, x_{1j}, x_{2j}, x_{3j}) = (1 - \overline{z_l} z_k) N_{il,jk} + (1 - \overline{\lambda_i}\lambda_j) M_{il,jk}.$$

Thus, (i) $\Longrightarrow$ (ii).

The proof of $(ii) \Longrightarrow (i)$ can be found in [22, Theorem 8.1]. $\square$

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