# Algebraic Aspects of Rational Tetra-Inner Functions

Omar Mossa Alsalhi

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*Academic Supervisor*: Dr. Zinaida Lykova
*Advisor*: Prof. Nicholas Young

School of Mathematics, Statistics and Physics
Newcastle University
Newcastle Upon Tyne
United Kingdom

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## Abstract

The tetrablock

$$\mathbb{E} = \{x \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 zw \neq 0 \quad \text{for } \ |z| \leq 1, |w| \leq 1\}$$

has very interesting complex-geometric properties. It meets $\mathbb{R}^3$ in a regular tetrahedron and its distinguished boundary is homeomorphic to $\overline{\mathbb{D}} \times \mathbb{T}$, where $\overline{\mathbb{D}}$ is the closed unit disc and $\mathbb{T}$ is the unit circle. We exploit this geometry to develop an explicit and detailed structure theory for the rational maps from the unit disc $\mathbb{D}$ to $\overline{\mathbb{E}}$, the closure of $\mathbb{E}$, that maps the boundary of the disc to the distinguished boundary of $\overline{\mathbb{E}}$. We call such maps rational $\overline{\mathbb{E}}$-inner functions or rational tetra-inner functions.

In this thesis, we provide a description of all rational inner functions $x$ from $\mathbb{D}$ to $\overline{\mathbb{E}}$ of degree $n$. Here $\deg(x)$ is the degree of $x$, defined in a natural way by means of fundamental groups. We show that, for any rational $\overline{\mathbb{E}}$-inner function $x = (x_1, x_2, x_3)$, $\deg(x)$ is equal to $\deg(x_3)$ (in the usual sense) of the finite Blaschke product $x_3$.

The variety $\mathcal{R}_{\overline{\mathbb{E}}} = \{(x_1, x_2, x_3) \in \overline{\mathbb{E}} : x_1 x_2 = x_3\}$ plays a crucial role in the function theory of $\mathbb{E}$. We prove that if $x$ is a rational $\overline{\mathbb{E}}$-inner function, then either $x(\overline{\mathbb{D}}) = \mathcal{R}_{\overline{\mathbb{E}}}$ or $x(\overline{\mathbb{D}})$ meets $\mathcal{R}_{\overline{\mathbb{E}}}$ exactly $\deg(x)$ times.
For a rational $\overline{\mathbb{E}}$-inner function $x$, we call the points $\lambda \in \overline{\mathbb{D}}$ such that $x(\lambda) \in \mathcal{R}_{\overline{\mathbb{E}}}$ the royal nodes of $x$. We describe the construction of rational $\overline{\mathbb{E}}$-inner functions $x = (x_1, x_2, x_3)$ of prescribed degree from the following interpolation data: the zeros of $x_1$ and $x_2$ in $\overline{\mathbb{D}}$ and the royal nodes of $x$.

It is easy to see that the set $\mathcal{J}$ of all rational $\overline{\mathbb{E}}$-inner functions is not convex. We prove that the subset of $\mathcal{J}$ of rational $\overline{\mathbb{E}}$-inner functions $(x_1, x_2, x_3)$ for a fixed inner function $x_3$ *is* convex. We show that a rational $\overline{\mathbb{E}}$-inner function $x$ is not an extreme point of the set $\mathcal{J}$ if the number of royal nodes of $x$ on $\mathbb{T}$, counted with multiplicity, is less than or equal to $\frac{1}{2}\deg(x)$.

**Declaration on collaborative work**

My thesis contains collaborative work with my supervisors Dr Z. A. Lykova and Prof. N. J. Young. Lykova and I have one joint paper in preparation. The main problems and ideas how to solve these problems were provided to me by Lykova. Lykova and I have had weekly meetings to discuss mathematics, methods, new ideas and research papers related to my thesis. We have done research together.

The rest of each week I have worked independently on my thesis. I did the calculations which were required in each step of the proofs, searched for research material related to our research project, organised all research material in thesis. I have given several talks on topics of my thesis to Young Functional Analysts Workshops in Newcastle and Leeds and to pure PhD workshops in Newcastle.

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# Contents

# Chapter 1

# Introduction and historical remarks

The *tetrablock* $\mathbb{E}$ was introduced by A. A. Abouhajar, M. C. White and N. J. Young in [2] which studied the complex geometry of the tetrablock. Recently, the tetrablock has also been studied in several papers (see [8, 16, 18, 27, 38, 39]). The motivation to study the tetrablock came from a $\mu$-synthesis problem.

In this thesis we study algebraic and geometric properties of rational tetra-inner functions. In Theorem 4.3.1 we give a description of rational tetra-inner functions $x = (x_1, x_2, x_3)$ with a prescribed degree $n$. A rational tetra-inner function is a rational function from the open unit disc to $\overline{\mathbb{E}}$, the closure of $\mathbb{E}$, which maps the unit circle to the distinguished boundary $b\overline{\mathbb{E}}$. The royal variety is defined by

$$\mathcal{R}_{\overline{\mathbb{E}}} = \big\{(x_1, x_2, x_3) \in \overline{\mathbb{E}} : x_1 x_2 = x_3\big\}.$$

It plays a crucial role in the geometry and the function theory of $\overline{\mathbb{E}}$. The degree of a rational tetra-inner function $x = (x_1, x_2, x_3)$ is the degree of $x_3$ in the usual sense of the finite Blaschke product (Proposition 4.2.4). In the case that $x$ is nonconstant, then either $x(\overline{\mathbb{D}}) = \mathcal{R}_{\overline{\mathbb{E}}}$ or the number of times that $x(\overline{\mathbb{D}})$ meets the royal variety is equal to the degree of $x$ (Theorem 5.2.5). One of our main results is Theorem 5.2.10. There we describe the construction of a rational tetra-inner function from certain interpolation data. The set of rational $\overline{\mathbb{E}}$-inner functions, denoted by $\mathcal{J}$, is not convex. However, we prove that, for a fixed inner function $x_3$, the set of functions in $\mathcal{J}$ with the third component $x_3$ *is* convex. We study extremality in $\mathcal{J}$ and we show that no point of $\mathcal{J}$ can be extreme if the number of its royal nodes on $\mathbb{T}$, counted with multiplicity, is at most half of its degree. To prove all the above results we adapt methods and results which were obtained in [7].

## 1.1 Relation to the $\mu$-synthesis problem

We begin by stating two known types of interpolation problems, the classical Nevanlinna-Pick problem and the two-by-two spectral Nevanlinna-Pick problem.
**The Nevanlinna-Pick problem:** *Given $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$ and $a_1, \ldots, a_n \in \mathbb{D}$. Does there exist an analytic function $F : \mathbb{D} \to \mathbb{C}$ such that $f(\lambda_j) = a_j, j = 1, \ldots, n$ and $|f(\lambda)| \leq 1$, for all $\lambda \in \mathbb{D}$?*
It was shown, by G. Pick in 1916 that this problem is solvable if and only if the Pick matrix

$$\left[\frac{1 - \overline{a_i} a_j}{1 - \overline{\lambda_i} \lambda_j}\right]_{i,j=1}^{n}$$

is positive semi-definite.
**The two-by-two spectral Nevanlinna-Pick problem:** Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$ and $A_1, \ldots, A_n \in \mathbb{C}^{2\times 2}$. Construct, if possible, an analytic function $F$ on $\mathbb{D}$ such that

(1) $F(\lambda_j) = A_j \quad j = 1, \ldots, n,$

(2) $r\big(F(\lambda)\big) \leq 1$ for every $\lambda \in \mathbb{D}$.

Here $r$ is the *spectral radius* defined, for $A \in \mathbb{C}^{2\times 2}$, by

$$r(A) = \max\{|\lambda_i| : \lambda_i \text{ are the eigenvalues of } A\}.$$

The $\mu$-synthesis problem is an interpolation problem for analytic matrix functions on the disc which are subject to a boundedness condition. It is a generalisation of the Nevanlinna-Pick problem. In order to solve the $\mu$-synthesis problem, we have to construct an analytic $m \times n$ matrix function $F$ on the open unit disc $\mathbb{D}$ which satisfies some interpolation conditions and $\mu(F(\lambda)) \leq 1$ for all $|\lambda| < 1$, where $\mu$ is a type of cost function.

**Definition 1.1.1.** *Let $E$ be a linear subspace of $\mathbb{C}^{n\times m}$ and let $A$ be an $m \times n$ matrix. The* structured singular value *of $A$ relative to $E$ is*

$$\mu_E(A) = \frac{1}{\inf\{||X|| : X \in E,\ (I - AX) \textit{ is singular}\}},$$

where $\mu_E(A) = 0$ in the event that $(I - AX)$ is nonsingular for all $X \in E$.

**The $\mu_E$-synthesis problem is the following:**
For given distinct points $\lambda_j$ in $\mathbb{D}$ and $W_j \in \mathbb{C}^{m\times n}$, $j = 1, .., \ell$, construct, if possible, an analytic $m \times n$ matrix function $F$ on $\mathbb{D}$ such that

(1) $F(\lambda_j) = W_j$ , $j = 1, .., \ell$; and

(2) $\mu_E(F(\lambda)) \leq 1$ for every $\lambda \in \mathbb{D}$.

One can see that if $n = m = 1$, the $\mu$-synthesis problem is the classical Nevanlinna-Pick problem. If $E$ is the whole space, that is, $E = \mathbb{C}^{n\times m}$ then $\mu_E(A) = ||A||$, where $||A||$ is the operator norm of the matrix $A$. In the case that $n = m$ and $E$ is the space of scalar multiples of the identity matrix $I$, in other words, $E = \{cI : c \in \mathbb{C}\}$, $\mu_E$ is the spectral radius $r$ and the $\mu$-synthesis problem is the spectral Nevanlinna-Pick problem. It is worth noting that these two special cases are extremal in the sense that, for any $E$, $\mu_E(A) \leq ||A||$ and if $n = m$ and $I$ belongs to $E$, then $\mu_E(A) \geq r(A)$.
In the attempt to solve the two-by-two spectral Nevanlinna-Pick problem, Agler and Young introduced in [10] a domain in $\mathbb{C}^2$ known as the *symmetrised bidisc* $\mathbb{G}$. It is defined by

$$\mathbb{G} = \left\{(z+w, zw) : |z| < 1, |w| < 1\right\}$$

and its closure is

$$\Gamma = \left\{(z+w, zw) : |z| \leq 1, |w| \leq 1\right\}.$$

A $\Gamma$-inner function is an analytic function $h : \mathbb{D} \to \Gamma$ with the property that $h$ maps the unit circle $\mathbb{T}$ to the distinguished boundary $b\Gamma$ of $\Gamma$. In [11], Agler and Young showed that the solvability of the two-by-two spectral Nevanlinna-Pick problem is equivalent to the solvability of the **$\Gamma$-interpolation problem**: given $\lambda_1, \ldots, \lambda_n$ distinct in $\mathbb{D}$ and $(s_j, p_j) \in \mathbb{G}$, $j = 1, \ldots, n$, find, if possible, an analytic function $h : \mathbb{D} \to \Gamma$ such that $h(\lambda_j) = (s_j, p_j)$. Since 1995, the $\Gamma$-interpolation problem and its associated domain, the symmetrised bidisc, have been studied widely, see for example, [4, 5, 7, 8, 10, 11, 12, 13].

In this thesis, we consider the $\mu_{Diag}$-synthesis problem from $\mathbb{D}$ to $\mathbb{C}^{2\times 2}$. The structured singular value in this case is defined by

$$\mu_{Diag}(A) = \frac{1}{\inf\{||X|| : X \in Diag, \det(I - AX) = 0\}}, \tag{1.1.1}$$

where

$$Diag := \left\{ \begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix} : z, w \in \mathbb{C} \right\}.$$

We set $\mu_{Diag}(A) = 0$ if $(I - AX)$ is non-singular for all $X \in Diag$. The domain associated with this problem is called the tetrablock and defined as

$$\mathbb{E} = \{x \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 zw \neq 0 \quad \text{for} \ \ |z| \leq 1, |w| \leq 1\}.$$

This interpolation problem was introduced and studied by Abouhajar, White and Young in [2]. They showed that the solvability of the $\mu_{Diag}$-synthesis interpolation problem is equivalent to the solvability of the *tetra-interpolation problem*: given $\lambda_1, \ldots, \lambda_n$ in $\mathbb{D}$ and $x^k = \big(x_1^k, x_2^k, x_3^k\big) \in \mathbb{E}$, $1 \leq k \leq n$, can we construct a tetra-inner function $x : \mathbb{D} \to \overline{\mathbb{E}}$ such that

$$x(\lambda_k) = \big(x_1(\lambda_k), x_2(\lambda_k), x_3(\lambda_k)\big) = \big(x_1^k, x_2^k, x_3^k\big),$$

(see Theorem 2.2.3).

In [18], D. C. Brown, Lykova and Young used a different strategy to give a criterion for the solvability of the $\mu_{Diag}$-synthesis problem. First, they reduced the problem to an interpolation problem in the set of analytic functions $\mathrm{Hol}(\mathbb{D}, \mathbb{E})$ from $\mathbb{D}$ to $\mathbb{E}$. Then they induced a duality between $\mathrm{Hol}(\mathbb{D}, \mathbb{E})$ and the Schur class $\mathcal{S}^2$ of the bidisc. Finally, they used Hilbert space models for the Schur class $\mathcal{S}$ to obtain a necessary and sufficient condition for the existence of a rational tetra-inner function $x : \mathbb{D} \to \overline{\mathbb{E}}$, (see Theorem 2.2.4).

## 1.2 Historical remarks

The structured singular value of a matrix was first introduced by J. C. Doyle [Caltech, USA] and G. Stein [Honeywell Laboratories, USA] in 1980s [23] and studied further in [21, 22] by Doyle. The motivation was a fundamental question which arises in $H^\infty$ control theory, the $\mu$-synthesis problem. It is a problem of robust stabilisation of a system which is subject to structured uncertainty. It is a fact that this type of problem has led to interpolation problems, see for example [21]. Although the $\mu$-synthesis problem is still unsolved, there are computational approaches: see for example [19]. Accordingly, the study of even special cases of the problem will throw the light on the difficulty of the more general cases and provide a test tool for the existing software.

The tetrablock $\mathbb{E}$ arose in connection with the study of the $\mu_{Diag}$-synthesis interpolation problem [2]. In [2] Abouhajar, White and Young [all at Newcastle University, UK], proved connections between $\mu_{Diag}$-interpolation problem and tetra-interpolation problem and presented some geometrical properties of the tetrablock. They showed that $\mathbb{E}$ is non-convex and polynomially convex. Further, the authors showed that the Carathéodory distance and the Lempert function of $\mathbb{E}$ coincide with one of the arguments fixed at the origin. They also provide a proof of a Schwarz lemma for the tetrablock and described a large group of automorphisms of $\mathbb{E}$ which they conjectured to be the group of all automorphisms of the tetrablock. Later, in [38] Young proved that this group of automorphisms is indeed the group of all automorphisms of $\overline{\mathbb{E}}$. Moreover, he showed that the tetrablock is inhomogeneous and not a holomorphic retract of the unit ball of the space of $2 \times 2$ matrices.

During the last 30 years, attempts to solve the $\mu$-synthesis problem have led to the study of several domains in $\mathbb{C}^n$. For instance, the symmetrised bidisc in $\mathbb{C}^2$, which was introduced in [10] by Agler [UC San Diego, USA] and Young; the tetrablock in $\mathbb{C}^3$ in [2], pentablock in $\mathbb{C}^3$ by Agler, Lykova and Young in [6]; the symmetrised poly-disc in [26] and the generalised tetrablock in $\mathbb{C}^n$ [39] have all been studied. These domains turned out to have rich structures, and they drew the attention of specialists from several complex variables and operator theory areas.

The pentablock is defined to be the bounded domain

$$\mathcal{P} = \left\{ (a_{21}, \text{tr } A, \det A) : A = \left[a_{ij}\right]_{i,j=1}^{2} \in \mathbb{B}_{2\times 2} \right\} \subset \mathbb{C}^3$$

where $\mathbb{B}_{2\times 2}$ is the open unit ball in the space of $2 \times 2$ complex matrices. In [6] the authors gave a number of characterisations of the domain $\mathcal{P}$. They proved some basic complex geometry of $\mathcal{P}$. In particular, it is nonconvex, polynomially convex, starlike and intersects $\mathbb{R}^3$ in a convex bounded set with five faces. They gave a description of the distinguished boundary and studied the connection between the pentablock and the symmetrised bidisc. A group of automorphisms of $\mathcal{P}$ is described. It was shown later, by L. Kosiński [Jagiellonian University, Poland] in [31], that this group forms the whole group of automorphisms of $\mathcal{P}$.

The Lempert theorem asserts that for any bounded convex domain $\Omega \subset \mathbb{C}^n$, the Carathéodory distance and Lempert function coincide. It was an open question for more than 20 years regarding whether there exists a domain which cannot be exhausted by convex domains with the property that the Carathéodory distance and Lempert function coincide. In 2004, Agler and Young [12] proved that the symmetrised bidisc is such a domain. Costara in [20] proved that $\mathbb{G}$ is not isomorphic to a convex domain. Later in [25], A. Edigarian, Kosiński and W. Zwonek [all at Jagiellonian University, Poland] proved that $\mathbb{E}$ cannot be exhausted by any convex domains and yet the Carathéodory distance and the Lempert function are equal on $\mathbb{E}$. In [5], Agler, Lykova and Young studied the 3-extremal holomorphic maps. These are the maps from $\mathbb{D}$ to $\mathbb{G}$, whose restriction to any three distinct points in $\mathbb{D}$ gives interpolation data that are extremally solvable. They describe a large class of such maps; these maps are rational of degree less than or equal to 4.

As a generalisation of the symmetrised bidisc, D. J. Ogle [Newcastle University, UK] in [32] established the study of the *symmetrised polydisc*, also known as the *symmetrised $n$-disc*. In his PhD thesis he studied the connection between the symmetrised polydisc and the solvability of the $n \times n$ spectral Nevanlinna-Pick Problem. He used an operator theoretic approach. In [32], Ogle gave an extended necessary condition for the solvability of the $n \times n$ Nevanlinna-Pick problem. He derived a necessary condition for the existence of solution for the spectral $n \times n$ Nevanlinna-Pick problem by establishing necessary conditions for $n$-tuples of commuting operators to have the symmetrised polydisc as a complete spectral set.

In [15], G. Bharali [Indian Institution of Science, Bangalore, India] introduced a large family of domains related to the $\mu$-synthesis problem, called $\mu_{1,n}$*-quotients*. This family contains some known domains, such as the symmetrised polydisc and the tetrablock. The author studied analytic interpolation from $\mathbb{D}$ into the space of $n \times n$ matrices $A$ with structured singular value $\mu_{1,n}(A)$ less than 1. He showed that such an interpolating problem is equivalent to an interpolation problem from $\mathbb{D}$ to the associated $\mu_{1,n}$-quotient domain. In addition, he introduced characterisations of $\mu_{1,n}$.

The generalised tetrablock, was introduced by P. Zapałowski [Jagiellonian University, Poland] in [39]. It contains the family of $\mu_{1,n}$-quotients which was introduced by Bharali in [15]. The paper showed that the generalised tetrablock $\overline{\mathbb{E}}_n, n \geq 2$, cannot be exhausted by domains which are biholomorphic to convex ones. It is also proved that the Carathéodory distance and Lempert function are not equal on a large subfamily of the generalised tetrablocks for $n \geq 4$. In addition, he studied the complex geometry of the generalised tetrablocks $n \geq 4$ and showed that none of them is convex or starlike about the origin.

A subset $V$ of a domain $U \in \mathbb{C}^n$ has the *norm-preserving extension property* if every bounded analytic function on $V$ has an analytic extension to $U$ with the same norm. In [8] Agler, Lykova and Young showed that an algebraic subset $V$ of the symmetrised bidisc $\mathbb{G}$ has the norm-preserving extension property if and only if $V$ is either the whole set $\mathbb{G}$, a singleton, a complex geodesic of $\mathbb{G}$, or the union of the set $\{(2\lambda, \lambda^2) : |\lambda| < 1\}$ and a complex geodesic of degree 1 in $\mathbb{G}$. They also proved that the complex geodesics in $\mathbb{G}$ coincide with the nontrivial holomorphic retracts of $\mathbb{G}$ (see Definition B.0.33).

## 1.3 Main results

The closed tetrablock is the subset of $\mathbb{C}^3$ defined by

$$\overline{\mathbb{E}} = \{x \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 z w \neq 0 \quad \text{for} \ \ |z| < 1, |w| < 1\}.$$

In this thesis we study rational $\overline{\mathbb{E}}$-inner functions. We define a rational $\overline{\mathbb{E}}$-inner function to be a rational analytic function from $\mathbb{D}$ into $\overline{\mathbb{E}}$ which maps $\mathbb{T}$ into $b\overline{\mathbb{E}}$ where $b\overline{\mathbb{E}}$ is the distinguished boundary of $\mathbb{E}$, or Shilov boundary. The distinguished boundary $b\overline{\mathbb{E}}$ of $\mathbb{E}$ is

$$b\overline{\mathbb{E}} = \{x = (x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 = \overline{x_2} x_3, \ |x_3| = 1 \text{ and } |x_2| \leq 1\},$$

see [2].

**Definition 4.2.1.** *The degree of a rational $\overline{\mathbb{E}}$-inner function $x$, denoted by* $\deg(x)$ *is defined to be $x_*(1)$, where $x_* : \mathbb{Z} = \pi_1(\mathbb{T}) \to \pi_1(b\overline{\mathbb{E}})$ is the homomorphism of fundamental groups induced by $x$ when $x$ is regarded as a continuous map from $\mathbb{T}$ to $b\overline{\mathbb{E}}$.*

**Proposition 4.2.4.** *For any rational $\overline{\mathbb{E}}$-inner function $x = (x_1, x_2, x_3)$,* $\deg(x)$ *is the degree* $\deg(x_3)$ *(in the usual sense) of the finite Blaschke product $x_3$.*

**Theorem 4.3.1.** *If $x = (x_1, x_2, x_3)$ is a rational $\overline{\mathbb{E}}$-inner function of degree $n$ then there exist polynomials $E_1, E_2, D$ such that*

(i) $\deg(E_1), \deg(E_2), \deg(D) \leq n$,

(ii) $D(\lambda) \neq 0$ *on* $\overline{\mathbb{D}}$,

(iii) $x_3 = \frac{D^{\sim n}}{D}$ *on* $\overline{\mathbb{D}}$, *where* $D^{\sim n}(\lambda) = \lambda^n \overline{D(1/\overline{\lambda})}$,

(iv) $x_1 = \frac{E_1}{D}$ *on* $\overline{\mathbb{D}}$,

(v) $x_2 = \frac{E_2}{D}$ *on* $\overline{\mathbb{D}}$,

(vi) $|E_i(\lambda)| \leq |D(\lambda)|$ *on* $\overline{\mathbb{D}}$, *for* $i = 1, 2$,

(vii) $E_1(\lambda) = E_2^{\sim n}(\lambda)$, *for* $\lambda \in \overline{\mathbb{D}}$.

*Conversely, if $E_1, E_2$ and $D$ satisfy* (i),(vi) and (vii), *$D(\lambda) \neq 0$ on $\mathbb{D}$ and $x_1, x_2$ and $x_3$ are defined by* (iii)–(v), *then $x = (x_1, x_2, x_3)$ is a rational $\overline{\mathbb{E}}$-inner function of degree at most $n$.*
*Furthermore, a triple of polynomials $E_1^1, E_2^1$ and $D^1$ satisfies* (i)–(vii) *if and only if there exists a real number $t \neq 0$ such that*

$$E_1^1 = tE_1, \quad E_2^1 = tE_2 \quad \text{and} \quad D^1 = tD.$$

**Proposition 5.2.5.** *If $x$ is a non-constant rational $\overline{\mathbb{E}}$-inner function, then either $x(\overline{\mathbb{D}}) = \mathcal{R}_{\overline{\mathbb{E}}}$ or $x(\overline{\mathbb{D}})$ meets $\mathcal{R}_{\overline{\mathbb{E}}}$ exactly* $\deg(x)$ *times.*

Let $x = \left(\dfrac{E_1}{D}, \dfrac{E_2}{D}, \dfrac{D^{\sim n}}{D}\right)$ be a rational $\overline{\mathbb{E}}$-inner function. The *royal polynomial* of $x$ is $R(\lambda) = \big(D^{\sim n}D - E_1E_2\big)(\lambda)$. We call the points $\sigma \in \overline{\mathbb{D}}$ such that $R(\sigma) = 0$ the royal nodes of $x$.

**Theorem 5.2.10.** *Suppose that* $\alpha_1^1, ..., \alpha_{k_1}^1 \in \overline{\mathbb{D}}$ *and* $\alpha_1^2, ..., \alpha_{k_2}^2 \in \overline{\mathbb{D}}$, *where* $k_1 + k_2 = n$. *Suppose that* $\sigma_1, ..., \sigma_n \in \overline{\mathbb{D}}$ *are distinct from the points of the set* $\{\alpha_j^i, j = 1, ..., k_i, i = 1, 2\} \cap \mathbb{T}$. *Then there exists a rational* $\overline{\mathbb{E}}$*-inner function* $x = (x_1, x_2, x_3) : \mathbb{D} \to \overline{\mathbb{E}}$ *such that*

(1) *the zeros of* $x_1$ *in* $\overline{\mathbb{D}}$, *repeated according to multiplicity, are* $\alpha_1^1, ..., \alpha_{k_1}^1$;

(2) *the zeros of* $x_2$ *in* $\overline{\mathbb{D}}$, *repeated according to multiplicity, are* $\alpha_1^2, ..., \alpha_{k_2}^2$;

(3) *the royal nodes of* $x$ *are* $\sigma_1, ..., \sigma_n \in \overline{\mathbb{D}}$, *with repetition according to multiplicity of the nodes.*

*Such a function* $x$ *can be constructed as follows. Let* $t_+ > 0$ *and let* $t \in \mathbb{C}\backslash\{0\}$. *Let* $R$ *be defined by*

$$R(\lambda) = t_+ \prod_{j=1}^{n} (\lambda - \sigma_j)(1 - \overline{\sigma_j}\lambda).$$

*Let* $E_1$ *be defined by*

$$E_1(\lambda) = t \prod_{j=1}^{k_1} (\lambda - \alpha_j^1) \prod_{j=1}^{k_2} (1 - \overline{\alpha}_j^2 \lambda).$$

*Then* (i) *and* (ii) *hold:*

(i) *There exists an outer function* $D$ *of degree at most* $n$ *such that*

$$\lambda^{-n} R(\lambda) + |E_1(\lambda)|^2 = |D(\lambda)|^2$$

*for all* $\lambda \in \mathbb{T}$.

(ii) *The function* $x$ *defined by*

$$x = \left( \frac{E_1}{D}, \frac{E_1^{\sim n}}{D}, \frac{D^{\sim n}}{D} \right)$$

*is a rational* $\overline{\mathbb{E}}$*-inner function such that the degree of* $x$ *is equal to* $n$ *and conditions* (1), (2) and (3) *hold. The royal polynomial of* $x$ *is* $R$.

**Proposition 6.1.3.** *The following sets are convex:*

(1) $\overline{\mathbb{E}} \cap \left(\mathbb{C}^2 \times \{x_3\}\right)$ *for any* $x_3 \in \overline{\mathbb{D}}$;

(2) $b\overline{\mathbb{E}} \cap \left(\mathbb{C}^2 \times \{x_3\}\right)$ *for any* $x_3 \in \overline{\mathbb{D}}$.

We have shown that the set of all $\overline{\mathbb{E}}$-inner functions is not convex.

**Theorem 6.2.1.** For a fixed inner function $x_3$, the set of $\overline{\mathbb{E}}$-inner functions $(x_1, x_2, x_3)$ is convex.

**Theorem 6.2.12.** Let $x \in \mathcal{R}^{n,k}$. If $2k \leq n$, then $x$ is not an extreme point of the set of rational $\overline{\mathbb{E}}$-inner functions $\mathcal{J}$.

## 1.4 Description of results by section

In Chapter 2, we recall the main properties of the tetrablock. Most of the definitions and results in this chapter are from [2]. We present a number of characterisations for $\mathbb{E}$ and $\overline{\mathbb{E}}$ in Theorems 2.1.4 and 2.1.5 respectively. We state that the solvability of the $\mu_{Diag}$-synthesis problem is equivalent to the solvability of the tetra-interpolation problem in Theorem 2.2.3 and Theorem 2.2.4. Chapter 2 concludes with a number of equivalent definitions for the distinguished boundary in Theorem 2.3.1 and for the topological boundary in Corollary 2.4.2.

In Chapter 3, we recall the definitions of the symmetrised bidisc $\mathbb{G}$ and its closure $\Gamma$. We recall characterisations of the topological and distinguished boundary of $\mathbb{G}$ in Proposition 3.1.4 from [7]. We also provide the definition of the $\Gamma$-inner functions. Finally, we recall a description of rational $\Gamma$-inner functions of prescribed degree $n$ in Proposition 3.3.4 from [7].

In Chapter 4, we define the degree of a rational $\overline{\mathbb{E}}$-inner function by the means of fundamental group $\pi_1$. In Proposition 4.2.4, we show that $\deg(x)$ is the degree of $x_3$ in the usual sense of finite Blaschke products. We also define the $\overline{\mathbb{E}}$-inner functions. Then we study the relation between $\mathbb{G}$ and $\mathbb{E}$ in Lemmas 4.1.5, 4.1.6 and 4.1.7. Specifically, in Lemma 4.1.6, for $x = (x_1, x_2, x_3) \in \mathbb{C}^3$, we show that

$$x = (x_1, x_2, x_3) \in \overline{\mathbb{E}}$$

if and only if, for every $a \in \overline{\mathbb{D}}$,

$$(ax_1 + \overline{a}x_2, x_3) \in \Gamma.$$

This result allows us to study the connection between $\Gamma$-inner functions and $\overline{\mathbb{E}}$-inner functions, see Lemma 4.1.9. In Theorem 4.3.1, we give a description of all rational tetra-inner functions of degree $n$, then give examples of rational tetra-inner functions.

In Chapter 5, we define the royal variety, the royal polynomial, the royal nodes and the multiplicity of the royal nodes. In Theorem 5.2.4, we show that if $x$ is a rational $\overline{\mathbb{E}}$-inner function such that $x$ has exactly $n$ royal nodes in $\overline{\mathbb{D}}$, where $k$ of them lie in $\mathbb{T}$, then the degree of $x$ is exactly $n$. In Theorem 5.2.5, we prove that if $x$ is a non-constant $\overline{\mathbb{E}}$-inner function, then either $x$ maps $\overline{\mathbb{D}}$ to $\mathcal{R}_{\overline{\mathbb{E}}}$ or $x(\overline{\mathbb{D}})$ meets $\mathcal{R}_{\overline{\mathbb{E}}}$ exactly $\deg(x)$ times. In Theorem 5.2.10 we construct a rational $\overline{\mathbb{E}}$-inner function from the royal nodes of $x$ and zeros of $x_1$ and $x_2$. In Example 5.2.12, we use Theorem 5.2.10 to construct a concrete rational $\overline{\mathbb{E}}$-inner function of degree 1. Theorem 5.2.14 is the converse of Theorem 5.2.10.

This thesis concludes with Chapter 6. In this chapter, we study the convexity and the extremality of certain subsets of $\overline{\mathbb{E}}$ and subsets of $\overline{\mathbb{E}}$-inner functions $\mathcal{J}$. Although $\overline{\mathbb{E}}$ and $\mathcal{J}$

are not convex, their subsets with a fixed $x_3$ are convex. Specifically, by Proposition 6.1.3, the subset

$$\overline{\mathbb{E}} \cap \left(\mathbb{C}^2 \times \{x_3\}\right) \quad \text{for any fixed } x_3 \in \mathbb{D} \text{ is convex.}$$

The subset of $\mathcal{J}$ with a fixed inner function $x_3$ is convex (Theorem 6.2.1). In Section 6.2, we present extreme points of $\mathcal{J}$. In Theorem 6.2.12, we show that whether $x \in \mathcal{R}^{n,k}$ is an extreme point of $\mathcal{J}$ depends on how many royal nodes lie on $\mathbb{T}$. In more detail, $x$ which has $n$ royal nodes where $k$ of them are in $\mathbb{T}$ cannot be an extreme point of $\mathcal{J}$ if $2k \leq n$. We provide a class of extreme functions of the set $\mathcal{J}$ in Proposition 6.2.14.

In the Appendix we give some essential supplementary material. In Section A, we provide the basic background of the fundamental group. Section B contains the basic definitions and results required throughout the thesis.

# Chapter 2

# The tetrablock $\mathbb{E}$

## 2.1 Introduction to the tetrablock

**Definition 2.1.1.** *The* tetrablock *is the domain defined as*

$$\mathbb{E} = \{x \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 zw \neq 0 \ \text{for} \ |z| \leq 1, |w| \leq 1\}.$$

The closure of the tetrablock is denoted by $\overline{\mathbb{E}}$. It is shown in [2, Theorem 2.4] that

$$\overline{\mathbb{E}} = \{x \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 zw \neq 0 \quad \text{for} \ |z| < 1, |w| < 1\}.$$

Despite the fact that $\mathbb{E}$ is not convex, its intersection with $\mathbb{R}^3$ is. It is proved in [2] that $\mathbb{E} \cap \mathbb{R}^3$ is the open tetrahedron with the vertices $(1,1,1), (1,-1,-1), (-1,1,-1)$ and $(-1,-1,1)$, see Figure 1.

The next step is to define some rational functions which play an important role in the study of the tetrablock.

**Definition 2.1.2.** *For $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ and $z \in \mathbb{C}$ we define*

$$\begin{aligned}
\Psi(z,x) &= \frac{x_3 z - x_1}{x_2 z - 1}, \qquad \text{whenever } x_2 z \neq 1,\\
\Upsilon(z,x) &= \frac{x_3 z - x_2}{x_1 z - 1}, \qquad \text{whenever } x_1 z \neq 1,\\
D(x) &= \sup_{z \in \mathbb{D}} |\Psi(z,x)|.
\end{aligned}$$

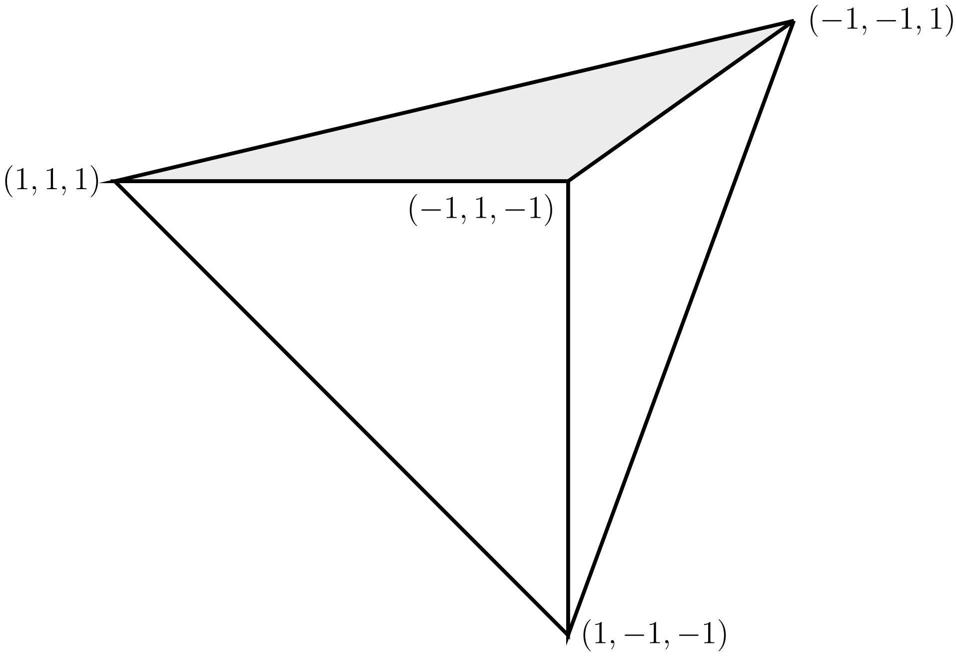



Figure 2.1: The tetrablock $\mathbb{E}$ intersects $\mathbb{R}^3$ in a regular tetrahedron

Clearly, the function $\Psi(.,x)$ is defined if either $x_2z \neq 1$ or $x_1x_2 = x_3$, while the function $\Upsilon(.,x)$ is defined in the case that either $x_1z \neq 1$ or $x_1x_2 = x_3$.

**Remark 2.1.3.** *In the case that $x_3 = x_1x_2$, $z \in \mathbb{C}$,*

$$\Psi(z,x) = \frac{x_1x_2z - x_1}{x_2z - 1} = \frac{x_1(x_2z-1)}{x_2z-1} = x_1,$$

*and*

$$\Upsilon(z,x) = \frac{x_1x_2z - x_2}{x_1z - 1} = \frac{x_2(x_1z-1)}{x_1z-1} = x_2.$$

The quantity $D(x)$ is given by:

$$D(x) = \begin{cases} \dfrac{|x_1 - \overline{x}_2x_3| + |x_1x_2 - x_3|}{1-|x_2|^2} & \text{if } |x_2| < 1 \\ |x_1| & \text{if } x_1x_2 = x_3 \\ \infty & \text{otherwise.} \end{cases}$$

Let us look at the three cases in more detail.
**Case 1**. If $x_2 \in \mathbb{D}$, applying Lemma B.0.10, we can see that the linear fractional transformation $\Psi$ maps $\mathbb{D}$ to another disc with centre and radius

$$\frac{x_1 - \overline{x}_2 x_3}{1 - |x_2|^2}, \quad \frac{|x_1 x_2 - x_3|}{1 - |x_2|^2}$$

respectively. Thus

$$D(x) = \sup_{z \in \mathbb{D}} |\Psi(z, x)| = \frac{|x_1 - \overline{x}_2 x_3| + |x_1 x_2 - x_3|}{1 - |x_2|^2}.$$

**Case 2**. If $x_1 x_2 = x_3$, we obtain the constant function $\Psi(., x) = x_1$ and hence $D(x) = |x_1|$.
**Case 3**. $D(x)$ is infinite otherwise.

One can also see that if $x_1 \in \mathbb{D}$, $\Upsilon(., x)$ maps $\mathbb{D}$ to the open disc with centre and radius

$$\frac{x_2 - \overline{x}_1 x_3}{1 - |x_1|^2}, \quad \frac{|x_1 x_2 - x_3|}{1 - |x_1|^2}$$

respectively.

**Theorem 2.1.4.** [2, Theorem 2.2] *Let $x \in \mathbb{C}^3$. The following are equivalent,*

(1) $x \in \mathbb{E}$;

(2) $||\Psi(., x)||_{H^\infty} < 1$ *and if* $x_1 x_2 = x_3$, *then* $|x_2| < 1$;

(3) $||\Upsilon(., x)||_{H^\infty} < 1$ *and if* $x_1 x_2 = x_3$, *then* $|x_1| < 1$;

(4) $|x_1 - \overline{x}_2 x_3| + |x_1 x_2 - x_3| < 1 - |x_2|^2$;

(5) $|x_2 - \overline{x}_1 x_3| + |x_1 x_2 - x_3| < 1 - |x_1|^2$;

(6) $|x_1 - \overline{x}_2 x_3| + |x_2 - \overline{x}_1 x_3| < 1 - |x_3|^2$;

(7) *there exists a* $2 \times 2$ *matrix* $A = [a_{ij}]$ *such that* $||A|| < 1$ *and* $x = (a_{11}, a_{22}, \det(A))$;

(8) *there exists a symmetric* $2 \times 2$ *matrix* $A = [a_{ij}]$ *such that* $||A|| < 1$ *and* $x = (a_{11}, a_{22}, \det(A))$;

(9) $|x_3| < 1$ *and there exist* $\beta_1, \beta_2 \in \mathbb{C}$ *such that* $|\beta_1| + |\beta_2| < 1$ *and*

$$x_1 = \beta_1 + \overline{\beta}_2 x_3, \quad x_2 = \beta_2 + \overline{\beta}_1 x_3.$$

**Theorem 2.1.5.** [2, Theorem 2.4] *Let $x \in \mathbb{C}^3$. The following are equivalent,*

(1) $x \in \overline{\mathbb{E}}$;

(2) $||\Psi(.,x)||_{H^\infty} \leq 1$ *and if* $x_1x_2 = x_3$, *then* $|x_2| \leq 1$;

(3) $||\Upsilon(.,x)||_{H^\infty} \leq 1$ *and if* $x_1x_2 = x_3$, *then* $|x_1| \leq 1$;

(4) $|x_1 - \overline{x}_2x_3| + |x_1x_2 - x_3| \leq 1 - |x_2|^2$;

(5) $|x_2 - \overline{x}_1x_3| + |x_1x_2 - x_3| \leq 1 - |x_1|^2$;

(6) $|x_1 - \overline{x}_2x_3| + |x_2 - \overline{x}_1x_3| \leq 1 - |x_3|^2$;

(7) *there exists a* $2 \times 2$ *matrix* $A = [a_{ij}]$ *such that* $||A|| \leq 1$ *and* $x = (a_{11}, a_{22}, \det(A))$;

(8) *there exists a symmetric* $2 \times 2$ *matrix* $A = [a_{ij}]$ *such that* $||A|| \leq 1$ *and* $x = (a_{11}, a_{22}, \det(A))$;

(9) $|x_3| \leq 1$ *and there exist* $\beta_1, \beta_2 \in \mathbb{C}$ *such that* $|\beta_1| + |\beta_2| \leq 1$ *and*

$$x_1 = \beta_1 + \overline{\beta}_2x_3, \quad x_2 = \beta_2 + \overline{\beta}_1x_3.$$

**Lemma 2.1.6.** [2, Theorem 6.4] *Let* $x = (x_1, x_2, x_3) \in \overline{\mathbb{E}}$. *Then* $(x_1, x_2, x_3) \longmapsto (x_2, x_1, x_3)$ *is an automorphism of* $\overline{\mathbb{E}}$.

*Proof.* This follows immediately from Theorem 2.1.5 (5) and (6). See the description of the group of automorphisms of $\mathbb{E}$ in [2, Theorem 6.4]. □

**Definition 2.1.7.** $x = (x_1, x_2, x_3) \in \mathbb{E}$ *is a* triangular point *if* $x_1x_2 = x_3$.

**Theorem 2.1.8.** [2, Theorem 2.9] $\overline{\mathbb{E}}$ *is polynomially convex.*

## 2.2 The tetrablock and the $\mu_{Diag}$-synthesis problem

**Definition 2.2.1.** *We define the map* $\pi : \mathbb{C}^{2\times2} \to \mathbb{C}^3$ *for a matrix* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ *in* $\mathbb{C}^{2\times2}$ *to be*

$$\pi(A) = (a_{11}, a_{22}, \det(A)).$$

*and* $\Sigma$ *to be*

$$\Sigma := \{A \in \mathbb{C}^{2\times2} : \mu_{Diag}(A) < 1\}$$

*where* $\mu_{Diag}(A)$ *is defined by equation* (1.1.1).

**Theorem 2.2.2.** [2, Theorem 9.1] *Let $x \in \mathbb{C}^3$. Then $x \in \mathbb{E}$ if and only if there exists $A \in \mathbb{C}^{2\times 2}$ such that*

$$x = \pi(A) \qquad \text{and} \qquad \mu_{Diag}(A) < 1.$$

*Similarly, $x$ belongs to the closure $\overline{\mathbb{E}}$ of the tetrablock if and only if there exists $A \in \mathbb{C}^{2\times 2}$ such that*

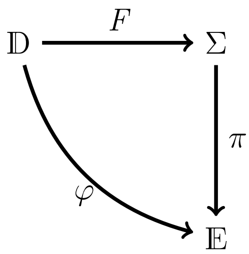
$$x = \pi(A) \qquad \text{and} \qquad \mu_{Diag}(A) \leq 1.$$

**Theorem 2.2.3.** [2, Theorem 9.2] *Suppose that $\lambda_1, ..., \lambda_n \in \mathbb{D}$ are distinct points and $A_k = [a_{ij}^k] \in \Sigma$ are such that $a_{11}^k a_{22}^k \neq \det(A_k)$, $1 \leq k \leq n$. The following conditions are equivalent.*

(1) *There exists an analytic function $F : \mathbb{D} \to \Sigma$ such that $F(\lambda_k) = A_k$, $1 \leq k \leq n$;*

(2) *There exists an analytic function $\varphi : \mathbb{D} \to \mathbb{E}$ such that $\varphi(\lambda_k) = \pi(A_k)$, that is,*

$$\varphi(\lambda_k) = (a_{11}^k, a_{22}^k, \det(A_k)), \quad k = 1, 2, ..., n.$$




In the following theorem the authors give a necessary and sufficient condition for the solvability of a $\mu_{Diag}$-synthesis problem by a rational $\overline{\mathbb{E}}$-inner function.

**Theorem 2.2.4.** [18, Theorem 1.1 and Theorem 8.1] *Let $\lambda_1, .., \lambda_n$ be distinct points in $\mathbb{D}$ and let $A_k = [a_{ij}^k] \in \mathbb{C}^{2\times 2}$ be such that $a_{11}^k a_{22}^k \neq \det(A_k)$, $1 \leq k \leq n$. Let*

$$(x_1^k, x_2^k, x_3^k) = \left(a_{11}^k, a_{22}^k, \det(A_k)\right), \quad 1 \leq k \leq n.$$

*The following two conditions are equivalent.*

(1) *There exists an analytic $2 \times 2$ matrix function $F$ in $\mathbb{D}$ such that*

$$F(\lambda_k) = A_k \quad \text{for} \quad k = 1, .., n,$$

*and*

$$\mu_{Diag}(F(\lambda)) \leq 1 \quad \text{for all} \quad \lambda \in \mathbb{D};$$

(2) *there exists a* rational $\overline{\mathbb{E}}$-*inner function* $x : \mathbb{D} \to \overline{\mathbb{E}}$ *such that*

$$x(\lambda_k) = (x_1^k, x_2^k, x_3^k) \quad \textit{for} \quad k = 1, .., n.$$

Theorem 2.2.4 shows that the solvability of the $\mu_{Diag}$-synthesis problem is equivalent to the solvability of the tetra-interpolation problem. Therefore, the understanding of rational $\overline{\mathbb{E}}$-inner functions will be useful for such $\mu$-synthesis problems.

## 2.3 The distinguished boundary of the tetrablock

Let $\mathbb{E}$ be the tetrablock. By Theorem 2.1.8, the tetrablock is polynomially convex. Therefore, there exists a distinguished boundary $b\overline{\mathbb{E}}$ of $\mathbb{E}$. Let $A(\mathbb{E})$ be the algebra of continuous scalar functions on $\overline{\mathbb{E}}$ that are holomorphic on $\mathbb{E}$ endowed with the supremum norm. If there is a function $f \in A(\mathbb{E})$ and a point $p$ in $\overline{\mathbb{E}}$ such that $f(p) = 1$ and $|f(x)| < 1$ for all $x \in \overline{\mathbb{E}} \backslash \{p\}$, then $p \in b\overline{\mathbb{E}}$ and is called a *peak point* of $\overline{\mathbb{E}}$ and the function $f$ is called *peaking function* for $p$.

**Theorem 2.3.1.** [2, Theorem 7.1] *For* $x \in \mathbb{C}^3$ *the following are equivalent.*

(1) $x_1 = \overline{x}_2 x_3, |x_3| = 1$ *and* $|x_2| \leq 1$*;*

(2) *either* $x_1 x_2 \neq x_3$ *and* $\Psi(., x)$ *is an automorphism of* $\mathbb{D}$ *or* $x_1 x_2 = x_3$ *and* $|x_1| = |x_2| = |x_3| = 1$*;*

(3) $x$ *is a peak point of* $\overline{\mathbb{E}}$*;*

(4) *there exists a* $2 \times 2$ *unitary matrix* $U$ *such that* $x = \pi(U)$*;*

(5) *there exists a symmetric* $2 \times 2$ *unitary matrix* $U$ *such that* $x = \pi(U)$*;*

(6) $x \in b\overline{\mathbb{E}}$*;*

(7) $x \in \overline{\mathbb{E}}$ *and* $|x_3| = 1$.

**Lemma 2.3.2.** *Let* $x = (x_1, x_2, x_3) \in \mathbb{C}^3$. *Then* $x \in b\overline{\mathbb{E}}$ *if and only if*

$$x_2 = \overline{x}_1 x_3, \ |x_3| = 1 \ \textit{ and } \ |x_1| \leq 1.$$

*Proof.* By Theorem 2.3.1 (1),

$$x \in b\overline{\mathbb{E}} \quad \Leftrightarrow \quad x_1 = \overline{x}_2 x_3, \ |x_3| = 1 \text{ and } |x_2| \leq 1.$$

Since $|x_3| = 1$ this implies $\overline{x}_3 x_3 = 1$. Now, since $x \in b\overline{\mathbb{E}}$,

$$\begin{aligned} x_1 &= \overline{x}_2 x_3, \text{ and so} \\ \overline{x}_1 &= x_2 \overline{x}_3. \end{aligned}$$

Thus $\overline{x}_1 x_3 = x_2 \overline{x}_3 x_3 = x_2$. Note, by Theorem 2.1.5, $|x_1| \leq 1$.

Conversely, if

$$x_2 = \overline{x}_1 x_3, \quad |x_3| = 1 \quad \text{and} \quad |x_1| \leq 1$$

then, similar to the previous steps, one can show that $x \in b\overline{\mathbb{E}}$. Therefore,

$$x \in b\overline{\mathbb{E}} \text{ if and only if } x_2 = \overline{x_1} x_3, \ |x_3| = 1 \text{ and } |x_2| \leq 1.$$

□

## 2.4 The topological boundary of the tetrablock

The topological boundary of $\mathbb{E}$ is denoted by $\partial\overline{\mathbb{E}}$ . Recall that the tetrablock is a subset of $\mathbb{C}^3$ such that, for $x = (x_1, x_2, x_3)$, $x \in \overline{\mathbb{E}}$ if and only if

$$|x_2|^2 + |x_1 - \overline{x}_2 x_3| + |x_1 x_2 - x_3| \leq 1 \quad \text{and} \quad |x_1| \leq 1.$$

In Abouhajar's PhD thesis [1], the following was shown.

**Lemma 2.4.1.** [1, Lemma 4.2.1] Let $x \in \mathbb{C}^3$. Then $x \in \partial\mathbb{E}$ if and only if

$$|x_1 - \overline{x}_2 x_3| + |x_1 x_2 - x_3| = 1 - |x_2|^2 \quad \text{and} \quad |x_1| \leq 1.$$

**Corollary 2.4.2.** [1, Corollary 4.2.7] Let $x = (x_1, x_2, x_3) \in \mathbb{C}^3$. Then the following are equivalent.

(1) $x \in \partial\mathbb{E}$.

(2) $|x_1 - \overline{x}_2 x_3| + |x_1 x_2 - x_3| = 1 - |x_2|^2 \quad \text{and} \quad |x_1| \leq 1.$

(3) $|x_2 - \overline{x}_1 x_3| + |x_1 x_2 - x_3| = 1 - |x_1|^2 \quad \text{and} \quad |x_2| \leq 1.$

(4) There exist $b, c \in \mathbb{C}$ such that $bc = x_1 x_2 - x_3$ and
$\left\| \begin{bmatrix} x_1 & b \\ c & x_2 \end{bmatrix} \right\| = 1.$

(5) There exist $b, c \in \mathbb{C}$ such that $|b| = |c| = \sqrt{|x_1 x_2 - x_3|}$, $bc = x_1 x_2 - x_3$ and
$\left\| \begin{bmatrix} x_1 & b \\ c & x_2 \end{bmatrix} \right\| = 1.$

(6) $1 - |x_1|^2 - |x_2|^2 + |x_3|^2 - 2|x_1x_2 - x_3| = 0$, and $|x_1| \leq 1, |x_2| \leq 1, |x_3| \leq 1$.

(7) $1 - |x_1|^2 + |x_2|^2 - |x_3|^2 - 2|x_1\overline{x}_3 - \overline{x}_2| = 0$, and $|x_1| \leq 1, |x_2| \leq 1$.

(8) $1 + |x_1|^2 - |x_2|^2 - |x_3|^2 - 2|x_2\overline{x}_3 - \overline{x}_1| = 0$, and $|x_1| \leq 1, |x_2| \leq 1$.

# Chapter 3

# The symmetrised bidisc and Γ-inner functions

## 3.1 Introduction to the symmetrised bidisc

We define the symmetrisation map on $\mathbb{C}^2$ by

$$\begin{aligned} \delta &: \mathbb{C}^2 \to \mathbb{C}^2 \\ &: (z, w) \longmapsto (z + w, zw) \end{aligned}$$

Consider the bidisc $\mathbb{D}^2 = \{(z_1, z_2) : |z_1| < 1, |z_2| < 1)\}$. The image of $\mathbb{D}^2$ under the symmetrisation map $\delta$ is called the symmetrised bidisc.

**Definition 3.1.1.** *The symmetrised bidisc is the set*

$$\mathbb{G} \stackrel{def}{=} \big\{(z + w, zw) : |z| < 1, |w| < 1\big\},$$

*and its closure is*

$$\Gamma \stackrel{def}{=} \big\{(z + w, zw) : |z| \leq 1, |w| \leq 1\big\}.$$

In 1995 Jim Agler and Nicholas Young started the study of the symmetrised bidisc with the aim of solving a robust control problem in $H^\infty$ control theory. Although, the aim has not yet been achieved, it turned out that the symmetrised bidisc has a rich structure and it has attracted the several complex variables and operator theory specialists' attention.

In this chapter, we review some background materials for the symmetrised bidisc and rational Γ-inner functions. Afterwards, we focus on the connection between the two-by-two spectral Nevanlinna-Pick problem and Γ-interpolation problem. Finally, we recall a description of rational Γ-inner functions of prescribed degree. Most of the material in this chapter is given in [3, 4, 7, 11, 12, 13].

We denote by Δ the *spectral unit ball*,

$$\Delta = \{A \in \mathbb{C}^{2\times 2} : r(A) \leq 1\}.$$

An equivalent definition of the closed symmetrised bidisc is

$$\Gamma \quad = \quad \big\{(\mathrm{tr}A, \det A) : A \in \Delta\big\}.$$

Define a function $\Phi : \mathbb{C}^3 \to \mathbb{C}$ by

$$\Phi(z,s,p) = \frac{2zp - s}{2 - zs}, \qquad \text{for } (z,s,p) \text{ such that } zs \neq 2.$$

This rational function, which was introduced in [13], plays an important role in the study of the symmetrised bidisc. Clearly, for $z \in \mathbb{D}$ the function Φ is defined for $(s,p) \in \Gamma$. In the special case, when $(s,p) \in \Gamma$ and $s^2 = 4p$,

$$\Phi(z,s,p) = \frac{2zp - s}{2 - zs} = \frac{2z\frac{s^2}{4} - s}{2 - zs} = \frac{\frac{-1}{2}s(2 - zs)}{2 - zs} = -\frac{1}{2}s.$$

The following lemma gives a a characterisation of points of Γ.

**Lemma 3.1.2.** [13, Lemma 1.2] *For $s, p \in \mathbb{C}$ the following conditions are equivalent:*

(1) $(s,p) \in \Gamma$*;*

(2) $|s| \leq 2$ *and, for all* $z \in \mathbb{D}$,

$$|\Phi(z,s,p)| = \left|\frac{2zp - s}{2 - zs}\right| \leq 1.$$

**Theorem 3.1.3.** [12, Theorem 2.3] $\mathbb{G}$ *is non convex, polynomially convex, and starlike about* $(0,0,0)$.

**Proposition 3.1.4.** [4, Proposition 3.2] *Let* $(s,p)$ *belong to* $\mathbb{C}^2$*. Then*

(i) $(s,p)$ *belongs to* $\mathbb{G}$ *if and only if*

$$|s - \bar{s}p| < 1 - |p|^2;$$

(ii) $(s,p)$ *belongs to* Γ *if and only if*

$$|s| \leq 2 \quad \textit{and} \quad |s - \bar{s}p| \leq 1 - |p|^2;$$

(iii) $(s,p)$ *lies in* $b\Gamma$ *if and only if*

$$|p| = 1, \quad |s| \le 2 \quad \textit{and} \quad s - \bar{s}p = 0;$$

(iv) $(s,p) \in \partial\Gamma$ *if and only if*

$$|s| \le 2 \quad \textit{and} \quad |s - \bar{s}p| = 1 - |p|^2.$$

**Definition 3.1.5.** *A* Γ-inner function *is an analytic function* $h : \mathbb{D} \to \Gamma$ *such that the radial limit*

$$\lim_{r \to 1^-} h(r\lambda) \tag{3.1.1}$$

*exists and belongs to* $b\Gamma$ *for almost all* $\lambda \in \mathbb{T}$ *with respect to Lebesgue measure.*

Γ-inner functions were defined in [4]. By Fatou's Theorem, the limit (3.1.1) exists for almost all $\lambda \in \mathbb{T}$.

## 3.2 The two-by-two spectral Nevanlinna-Pick problem and the Γ-interpolation problem

In [11] Agler and Young showed the connection between the two-by-two spectral Nevanlinna-Pick problem and the Γ-interpolation problem. Instead of studying the interpolation problem from $\mathbb{D}$ into the 4-dimensional domain of the $2 \times 2$ matrices, they studied the interpolation problem from $\mathbb{D}$ into the compact 2-dimensional set Γ.

**Theorem 3.2.1.** [11, Theorem 2.1] Let $\lambda_1, \dots, \lambda_n \in \mathbb{D}$ be distinct and $A_1, \dots, A_n \in \mathbb{C}^{2\times 2}$. Suppose that either all or none of $A_1, \dots, A_n$ are scalar matrices. The following are equivalent:

(1) there exists an analytic $2 \times 2$ matrix function $F : \mathbb{D} \to \mathbb{C}^{2\times 2}$ such that

$$r\big(F(\lambda)\big) \le 1 \quad \text{for all } \lambda \in \mathbb{D} \qquad \text{and} \qquad F(\lambda_k) = A_k, \ k = 1, \dots, n;$$

(2) there exists an analytic function $h : \mathbb{D} \to \Gamma$ such that

$$h(\lambda_k) = (\ \mathrm{tr} A_k, \det A_k), \quad k = 1, \dots, n.$$

**Theorem 3.2.2.** [3, Theorem 8.1] *Let* $\lambda_1, \dots, \lambda_n \in \mathbb{D}$ *be distinct and let* $(s_k, p_k) \in \Gamma$ *for* $k = 1, \dots, n$. *The following are equivalent:*

(1) *there exists an analytic function* $h : \mathbb{D} \to \Gamma$ *such that*

$$h(\lambda_k) = (s_k, p_k), \qquad k = 1, \dots, n.$$

(2) *there exists a rational* Γ*-inner function* $h : \mathbb{D} \to \Gamma$ *satisfying*

$$h(\lambda_k) = (s_k, p_k), \qquad k = 1, \dots, n.$$

## 3.3 Description of rational Γ-inner functions

**Definition 3.3.1.** *Let* $f$ *be a polynomial of degree less than or equal to* $n$*, where* $n \geq 0$*. Then we define the polynomial* $f^{\sim n}$ *by*

$$f^{\sim n}(\lambda) = \lambda^n \overline{f(1/\overline{\lambda})}.$$

*The polynomial* $f^{\vee}$ *is defined by*

$$f^{\vee}(\lambda) = \overline{f(\overline{\lambda})}.$$

Note: $f^{\sim n}(\lambda) = \lambda^n \overline{f(1/\overline{\lambda})} = \lambda^n f^{\vee}(1/\lambda)$.

The following result is well-known.

**Lemma 3.3.2.** *Let* $f$ *be a polynomial of degree* $k$*. For* $n \geq k$*,* $\left(f^{\sim n}\right)^{\sim n}(\lambda) = f(\lambda)$*.*

*Proof.* Let $f(\lambda) = a_0 + a_1\lambda + \cdots + a_k\lambda^k$ where $a_j \in \mathbb{C}$ for $j = 1, \dots k$ and $a_k \neq 0$. By Definition 3.3.1,

$$\begin{aligned} f^{\sim n}(\lambda) &= \lambda^n \overline{f\left(1/\overline{\lambda}\right)} \\ &= \lambda^n \left( \overline{a_0 + \frac{a_1}{\overline{\lambda}} + \dots + \frac{a_k}{\overline{\lambda}}} \right) \\ &= \lambda^n \left( \overline{a_0} + \frac{\overline{a_1}}{\lambda} + \dots + \frac{\overline{a_k}}{\lambda^k} \right) \\ &= \overline{a_0}\lambda^n + \overline{a_1}\lambda^{n-1} + \dots + \overline{a_k}\lambda^{n-k}. \end{aligned}$$

Applying the definition again yields

$$\begin{aligned} \left(f^{\sim n}\right)^{\sim n}(\lambda) = \lambda^n \overline{f^{\sim n}\left(1/\overline{\lambda}\right)} &= \lambda^n \left( \frac{a_0}{\lambda^n} + \frac{a_1}{\lambda^{n-1}} + \cdots + \frac{a_k}{\lambda^{n-k}} \right) \\ &= a_0 + a_1\lambda + \cdots + a_n\lambda^k \\ &= f(\lambda). \end{aligned}$$

□

**Corollary 3.3.3.** [4, Corollary 6.10] *If (s,p) is a rational $\Gamma$-inner function, then $s$ and $p$ can be written as a ratio of polynomials with the same denominators. Suppose that*

$$p(\lambda) = c\frac{\lambda^k D_p^{\sim}(\lambda)}{D_p(\lambda)}$$

*where $|c| = 1, k \geq 0$ and $D_p$ is a polynomial of degree $n$ such that $D_p(0) = 1$. Then $s$ can be written as*

$$s(\lambda) = \frac{\lambda^\ell N_s(\lambda)}{D_p(\lambda)}$$

*where $0 \leq \ell \leq \frac{1}{2}(n+k) = \frac{1}{2}d(p)$, and $N_s$ is a polynomial of degree $d(p) - 2\ell$ such that $N_s(0) \neq 0$. Moreover, if $N_s(\lambda) = \sum_{j=0}^{n+k-2\ell} b_j \lambda^j$ then*

$$b_j = c\bar{b}_{n+k-2\ell-j} \quad \textit{for } j = 0, 1, ..., n+k-2\ell.$$

*The degree of $s$ is at most $\max\{n+k-\ell, n\}$.*

**Proposition 3.3.4.** [7, Proposition 2.2] *If $h = (s,p)$ is a rational $\Gamma$-inner function of degree $n$ then there exist polynomials $E$ and $D$ such that*

(i) $\deg(E), \deg(D) \leq n$,

(ii) $E^{\sim n} = E$,

(iii) $D(\lambda) \neq 0$ *on* $\overline{\mathbb{D}}$,

(iv) $|E(\lambda)| \leq 2\,|D(\lambda)|$ *on* $\overline{\mathbb{D}}$,

(v) $s = \dfrac{E}{D}$ *on* $\overline{\mathbb{D}}$,

(vi) $p = \dfrac{D^{\sim n}}{D}$ *on* $\overline{\mathbb{D}}$.

*Furthermore, $E_1$ and $D_1$ is a second pair of polynomials satisfy* (i)–(vi) *if and only if there exists a nonzero $t \in \mathbb{R}$ such that*

$$E_1 = tE \qquad \textit{and} \qquad D_1 = tD.$$

*Conversely, if $E$ and $D$ are polynomials satisfies* (i), (ii), (iv), *$D(\lambda) \neq 0$ on $\mathbb{D}$, and $s$ and $p$ are defined by* (v) *and* (vi), *then $h = (s,p)$ is a rational $\Gamma$-inner function of degree less than or equal to $n$.*

The royal variety $\mathcal{R}_\Gamma$ of the symmetrised bidisc is

$$\mathcal{R}_\Gamma = \{(s,p) \in \mathbb{C}^2 : s^2 = 4p\}.$$

**Definition 3.3.5.** [7, Page 7] *Let $h = (s,p)$ be a $\Gamma$-inner function of degree $n$. Let $E$ and $D$ be as in Proposition* 3.3.4*. The royal polynomial $R_h$ of $h$ is defined by*

$$R_h(\lambda) = 4D(\lambda)D^{\sim n}(\lambda) - E(\lambda)^2.$$

**Definition 3.3.6.** [7, Definition 3.6] *Let $h$ be a rational $\Gamma$-inner function such that $h(\overline{\mathbb{D}}) \nsubseteq \mathcal{R}_\Gamma \cap \Gamma$. Let $R_h$ be the royal polynomial of $h$. If $\sigma$ is a zero of $R_h$ of order $\ell$, we define the* multiplicity $\#\sigma$ *of $\sigma$ (as a royal node of $h$) by*

$$\#\sigma = \begin{cases} \ell & \text{if } \sigma \in \mathbb{D}, \\ \frac{1}{2}\ell & \text{if } \sigma \in \mathbb{T}. \end{cases}$$

*We define the type of $h$ to be the ordered pair $(n,k)$, where $n$ is the sum of the multiplicities of the royal nodes of $h$ that lie in $\overline{\mathbb{D}}$, and $k$ is the sum of the multiplicities of the royal nodes of $h$ that lie in $\mathbb{T}$. We define $\mathcal{R}_\Gamma^{n,k}$ to be the collection of rational $\Gamma$-inner functions $h$ of type $(n,k)$.*

**Theorem 3.3.7.** [7, Theorem 3.8] *If $h \in \mathcal{R}_\Gamma^{n,k}$ is nonconstant then* $\deg(h) = n$.

# Chapter 4

# Rational $\overline{\mathbb{E}}$-inner functions

In this chapter we give a definition of the degree of a rational tetra-inner function $x$ by means of the fundamental group $\pi_1$. Recall that the rational inner functions on $\mathbb{D}$ of degree $n$ are exactly the finite Blaschke products of degree $n$. As an analogue of this description of rational inner functions on $\mathbb{D}$ we describe all rational $\overline{\mathbb{E}}$-inner functions on $\mathbb{D}$ in Theorem 4.3.1. In [7], the authors describe all rational $\Gamma$-inner functions (see Proposition 3.3.4). We use this description and the connection between $\Gamma$-inner functions and $\overline{\mathbb{E}}$-inner functions to describe all rational $\overline{\mathbb{E}}$-inner functions on $\mathbb{D}$.

## 4.1 Definition of $\overline{\mathbb{E}}$-inner functions

**Definition 4.1.1.** *An* $\overline{\mathbb{E}}$-inner function *is a map $f : \mathbb{D} \to \overline{\mathbb{E}}$ that is analytic and is such that the radial limit*

$$\lim_{r \to 1^-} f(r\lambda)$$

*exists and belongs to $b\overline{\mathbb{E}}$ for almost all $\lambda \in \mathbb{T}$ with respect to Lebesgue measure.*

**Remark 4.1.2.** *Let $x : \mathbb{D} \to \overline{\mathbb{E}}$ be a rational $\overline{\mathbb{E}}$-inner function. Since $x$ is rational and bounded on $\mathbb{D}$ it has no poles in $\overline{\mathbb{D}}$ and hence $x$ is continuous on $\overline{\mathbb{D}}$. Thus one can consider the continuous function*

$$\tilde{x} : \mathbb{T} \to b\overline{\mathbb{E}}, \qquad \text{where} \quad \tilde{x}(\lambda) = \lim_{r \to 1^-} x(r\lambda) \quad \text{for all } \lambda \in \mathbb{T}.$$

*Later we will use the same notation $x$ for both continuous functions $x$ and $\tilde{x}$.*

**Lemma 4.1.3.** *Let $x = (x_1, x_2, x_3)$ be an $\overline{\mathbb{E}}$-inner function. Then*

(i) $x_1(\lambda) = \overline{x_2(\lambda)} x_3(\lambda)$, $|x_2(\lambda)| \le 1$ *and* $|x_3(\lambda)| = 1$ *for almost all* $\lambda \in \mathbb{T}$*;*

(ii) *$x_3$ is an inner function on $\mathbb{D}$.*

*Proof.* (i) By the definition of $\overline{\mathbb{E}}$-inner function

$$x(\lambda) = (x_1(\lambda), x_2(\lambda), x_3(\lambda)) \in b\overline{\mathbb{E}}, \quad \text{for almost every } \lambda \in \mathbb{T}$$

and, by Theorem 2.3.1,

$$x_1(\lambda) = \overline{x_2}(\lambda)x_3(\lambda), \quad |x_3(\lambda)| = 1 \quad \text{and} \quad |x_2(\lambda)| \leq 1 \quad \text{for almost all } \lambda \in \mathbb{T}.$$

(ii) Since

$$x_3 : \mathbb{D} \to \overline{\mathbb{D}} \quad \text{and,} \quad \text{for almost all } \lambda \in \mathbb{T},\ |x_3(\lambda)| = 1,$$

$x_3$ is an inner function. □

**Remark 4.1.4.** *Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$-inner function. By Lemma* 4.1.3*, $x_3$ is an inner function on $\mathbb{D}$, and so $x_3$ is a finite Blaschke product.*

In [16] the author shows that there is a relation between points in the symmetrised bidisc and the tetrablock as follows

**Lemma 4.1.5.** [16, Lemma 3.2] *A point $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ belongs to the tetrablock if and only if the pair $(x_1 + zx_2, zx_3)$ is in the symmetrised bidisc $\mathbb{G}$ for every $z \in \mathbb{T}$.*

*Proof.* By Proposition 3.1.4 (i), $(s, p) \in \mathbb{G}$ if and only if

$$|s - \bar{s}p| < 1 - |p|^2. \tag{4.1.1}$$

Suppose that $x = (x_1, x_2, x_3) \in \mathbb{E}$, $s_z = x_1 + zx_2$ and $p_z = zx_3$.

$$\begin{aligned} |s_z - \overline{s_z}p_z| &= |x_1 + zx_2 - \overline{(x_1 + zx_2)}zx_3| \\ &= |x_1 + zx_2 - z\overline{x_1}x_3 - \overline{x_2}x_3| \\ &= |x_1 - \overline{x_2}x_3 + z(x_2 - \overline{x_1}x_3)| \\ &\leq |x_1 - \overline{x_2}x_3| + |x_2 - \overline{x_1}x_3|, \qquad \text{since } |z| = 1, \\ &< 1 - |x_3|^2 = 1 - |p_z|^2, \quad \text{by Theorem 2.1.4 (6).} \end{aligned}$$

Hence $(s_z, p_z) \in \mathbb{G}$.

Conversely, let $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ and, for $z \in \mathbb{T}$, let

$$s_z = x_1 + zx_2 \quad \text{and} \quad p_z = zx_3. \tag{4.1.2}$$

Suppose for all $z \in \mathbb{T}$, we have $(s_z, p_z) \in \mathbb{G}$. We want to show that $x = (x_1, x_2, x_3) \in \mathbb{E}$. Let us prove that

$$|x_1 - \overline{x_2}x_3| + |x_2 - \overline{x_1}x_3| < 1 - |x_3|^2.$$

By assumption for all $z \in \mathbb{T}$, $|s_z - \overline{s_z} p_z| < 1 - |x_3|^2$. By equations (4.1.2), we have

$$|x_1 - \overline{x_2} x_3 + z\big(x_2 - \overline{x_1} x_3\big)| < 1 - |x_3|^2, \quad \text{for all } z \in \mathbb{T}. \tag{4.1.3}$$

Let

$$\begin{cases} z = e^{i\theta} & \theta \in (0, 2\pi]; \\ w_1 = x_1 - \overline{x_2} x_3 = |w_1| e^{i\theta_1} & \theta_1 \in (0, 2\pi]; \\ w_2 = x_2 - \overline{x_1} x_3 = |w_2| e^{i\theta_2} & \theta_2 \in (0, 2\pi]. \end{cases}$$

Now substitute $z$, $w_1$ and $w_2$ in inequality (4.1.3)

$$\big||w_1| e^{i\theta_1} + e^{i\theta}(|w_2| e^{i\theta_2})\big| < 1 - |x_3|^2.$$

This implies that

$$\big||w_1| e^{i\theta_1} + |w_2| e^{i(\theta + \theta_2)}\big| < 1 - |x_3|^2, \quad \text{for all } e^{i\theta}.$$

We can choose $\theta$ such that $\theta + \theta_2 = \theta_1$, that is, $\theta = \theta_1 - \theta_2$. Hence

$$\big||w_1| e^{i\theta_1} + |w_2| e^{i\theta_1}\big| = |e^{i\theta_1}||w_1| + |w_2| = |w_1| + |w_2| = |x_1 - \overline{x_2} x_3| + |x_2 - \overline{x_1} x_3| < 1 - |x_3|^2.$$

By Theorem 2.1.4 (6), $(x_1, x_2, x_3) \in \mathbb{E}$.

□

**Lemma 4.1.6.** *A point $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ belongs to the tetrablock if and only if for every $a \in \overline{\mathbb{D}}$, $(ax_1 + \overline{a} x_2, x_3) \in \Gamma$.*

*Proof.* Suppose $x = (x_1, x_2, x_3) \in \overline{\mathbb{E}}$. Consider $(s_a, p_a) = (ax_1 + \overline{a} x_2, x_3)$. By Proposition 3.1.4 (ii), $(s, p) \in \Gamma$ if and only if

$$|s - \overline{s} p| \le 1 - |p|^2. \tag{4.1.4}$$

$$\begin{aligned} |s_a - \overline{s_a} p_a| &= |ax_1 + \overline{a} x_2 - \overline{(ax_1 + \overline{a} x_2)} x_3| \\ &= |ax_1 + \overline{a} x_2 - \overline{a x_1} x_3 - a \overline{x_2} x_3| \\ &= |a(x_1 - \overline{x_2} x_3) + \overline{a}(x_2 - \overline{x_1} x_3)| \qquad (4.1.5) \\ &\le |a(x_1 - \overline{x_2} x_3)| + |\overline{a}(x_2 - \overline{x_1} x_3)|, \\ &\le |x_1 - \overline{x_2} x_3| + |x_2 - \overline{x_1} x_3|, \qquad \text{since } |a| \le 1, \\ &\le 1 - |x_3|^2, \qquad \text{by Theorem 2.1.5 (6).} \qquad (4.1.6) \end{aligned}$$

Thus, $|s_a - \overline{s_a} p_a| \le 1 - |x_3|^2 = 1 - |p_a|^2$. Hence $(s_a, p_a) \in \Gamma$.

Conversely, let $x = (x_1, x_2, x_3) \in \mathbb{C}^3$. Suppose for every $a \in \overline{\mathbb{D}}$, we have $(s_a, p_a) \in \Gamma$ where

$$s_a = ax_1 + \overline{a}x_2 \quad \text{and} \quad p_a = x_3. \tag{4.1.7}$$

By equations (4.1.5) and (4.1.7), we have

$$|s_a - \overline{s_a}p_a| = |a(x_1 - \overline{x_2}x_3) + \overline{a}\big(x_2 - \overline{x_1}x_3\big)| \leq 1 - |p_a|^2, \quad \text{for all } a \in \overline{\mathbb{D}}. \tag{4.1.8}$$

Take $a \in \mathbb{T}$, then

$$\begin{cases} a = e^{i\theta} & \theta \in (0, 2\pi]; \\ w_1 = x_1 - \overline{x_2}x_3 = |w_1|e^{i\theta_1} & \theta_1 \in (0, 2\pi]; \\ w_2 = x_2 - \overline{x_1}x_3 = |w_2|e^{i\theta_2} & \theta_2 \in (0, 2\pi]. \end{cases}$$

Substitute $a$, $w_1$ and $w_2$ into inequality (4.1.8), we get

$$\begin{aligned} |a(x_1 - \overline{x_2}x_3) + \overline{a}\big(x_2 - \overline{x_1}x_3\big)| = \big|e^{i\theta}|w_1|e^{i\theta_1} + e^{-i\theta}|w_2|e^{i\theta_2}\big| &= \big||w_1|e^{i(\theta+\theta_1)} + |w_2|e^{i(\theta_2-\theta)}\big| \\ &\leq 1 - |x_3|^2, \end{aligned}$$

for every $\theta \in (0, 2\pi]$. Now choose $\theta = \dfrac{\theta_2 - \theta_1}{2}$ to get

$$\begin{aligned} |x_1 - \overline{x_2}x_3| + |x_2 - \overline{x_1}x_3| &= |w_1| + |w_2| \\ &= |e^{i(\frac{\theta_2+\theta_1}{2})}|\big(|w_1| + |w_2|\big) \\ &= \big||w_1|e^{i(\frac{\theta_2+\theta_1}{2})} + |w_2|e^{i(\frac{\theta_2+\theta_1}{2})}\big| \\ &= \big||w_1|e^{i(\frac{\theta_2-\theta_1}{2}+\theta_1)} + |w_2|e^{i(\theta_2-\frac{\theta_2-\theta_1}{2})}\big| \leq 1 - |x_3|^2. \end{aligned}$$

Therefore $x = (x_1, x_2, x_3) \in \overline{\mathbb{E}}$.

□

**Lemma 4.1.7.** *Let $s, p \in \mathbb{C}$ be such that $|s| \leq 2$ and $|p| \leq 1$. The pair $(s, p)$ belongs to $\Gamma$ if and only if $\big(\frac{1}{2}s, \frac{1}{2}s, p\big) \in \overline{\mathbb{E}}$.*

*Proof.* By Theorem 2.1.5 (6),

$$\big(\tfrac{1}{2}s, \tfrac{1}{2}s, p\big) \in \overline{\mathbb{E}} \quad \Leftrightarrow \quad |\tfrac{1}{2}s - \tfrac{1}{2}\overline{s}p| + |\tfrac{1}{2}s - \tfrac{1}{2}\overline{s}p| \leq 1 - |p|^2.$$

Thus

$$\begin{aligned} (\tfrac{1}{2}s, \tfrac{1}{2}s, p) \in \overline{\mathbb{E}} &\Leftrightarrow 2|\tfrac{1}{2}s - \tfrac{1}{2}\overline{s}p| \leq 1 - |p|^2 \\ &\Leftrightarrow |s - \overline{s}p| \leq 1 - |p|^2. \end{aligned}$$

By assumption $|s| \leq 2$, hence by Proposition 3.1.4 (ii),

$$(\tfrac{1}{2}s, \tfrac{1}{2}s, p) \in \overline{\mathbb{E}} \Leftrightarrow (s, p) \in \Gamma.$$

$\square$

**Proposition 4.1.8.** *The symmetrised bidisc $\mathbb{G}$ is an analytic retract in the tetrablock $\mathbb{E}$.*

*Proof.* For $(s, p) \in \mathbb{G}$ the map

$$\iota : \mathbb{G} \to \mathbb{E}$$

with $\iota(s, p) = (\frac{1}{2}s, \frac{1}{2}s, p) \in \mathbb{E}$ is a holomorphic injection with left inverse

$$k : \mathbb{E} \to \mathbb{G}$$

where $k\big((z_1, z_2, z_3)\big) = (z_1 + z_2, z_3)$. Now $k \circ \iota : \mathbb{G} \to \mathbb{G}$ and

$$(k \circ \iota)(s, p) = k(\tfrac{1}{2}s, \tfrac{1}{2}s, p) = (\tfrac{1}{2}s + \tfrac{1}{2}s, p) = (s, p) = \mathrm{id}_{\mathbb{G}}.$$

Therefore $\mathbb{G}$ is an analytic retract of $\mathbb{E}$. $\square$

**Lemma 4.1.9.** *Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$-inner function. Then*

*(1) $h_1(\lambda) = \big(x_1(\lambda) + x_2(\lambda), x_3(\lambda)\big)$, for $\lambda \in \mathbb{D}$, is a rational $\Gamma$-inner function;*

*(2) $h_2(\lambda) = (ix_1(\lambda) - ix_2(\lambda), x_3(\lambda))$, for $\lambda \in \mathbb{D}$, is a rational $\Gamma$-inner function.*

*Proof.* (1) By Lemma 4.1.5, for all $\lambda \in \mathbb{D}$, $x(\lambda) \in \mathbb{E}$ implies that

$$\big(x_1(\lambda) + x_2(\lambda), x_3(\lambda)\big) \in \mathbb{G}.$$

Consider $h_1 = (s_1, p_1)$ where

$$s_1(\lambda) = x_1(\lambda) + x_2(\lambda) \quad \text{and} \quad p_1(\lambda) = x_3(\lambda), \quad \text{for } \lambda \in \mathbb{D}.$$

It is obvious that $h_1$ is a rational function from $\mathbb{D}$ to $\mathbb{G}$. By assumption, $x$ is an $\overline{\mathbb{E}}$-inner function. Thus $x(\lambda) \in b\overline{\mathbb{E}}$ for almost every $\lambda \in \mathbb{T}$. By Theorem 2.3.1 and Lemma 2.3.2, for almost all $\lambda \in \mathbb{T}$,

$$x_2(\lambda) = \overline{x_1(\lambda)}x_3(\lambda), \quad x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda), \quad |x_3(\lambda)| = 1 \quad \text{and } |x_2(\lambda)| \leq 1. \qquad (4.1.9)$$

It is clear that

$$|p_1(\lambda)| = |x_3(\lambda)| = 1 \qquad \text{for } \lambda \in \mathbb{T},$$

and, for almost all $\lambda \in \mathbb{T}$,

$$\begin{aligned} |s_1(\lambda)| &= |x_1(\lambda) + x_2(\lambda)| \\ &\leq |x_1(\lambda)| + |x_2(\lambda)| \\ &\leq 2. \end{aligned}$$

Since, for almost all $\lambda \in \mathbb{T}$, $x_2(\lambda) = \overline{x_1(\lambda)}x_3(\lambda)$, we have

$$\begin{aligned} \overline{s_1(\lambda)}p_1(\lambda) &= [\overline{x_1(\lambda)} + \overline{x_2(\lambda)}]x_3(\lambda) \\ &= \overline{x_1(\lambda)}x_3(\lambda) + \overline{x_2(\lambda)}x_3(\lambda), \quad \text{by equations (4.1.9),} \\ &= x_1(\lambda) + x_2(\lambda) \\ &= s_1(\lambda). \end{aligned}$$

Hence $s_1(\lambda) = \overline{s_1(\lambda)}p_1(\lambda)$ for almost every $\lambda \in \mathbb{T}$. Therefore, by Proposition 3.1.4 (iii), $h_1$ is a rational $\Gamma$-inner function.

(2) Following the same steps as (1), let $h_2(\lambda) = (s_2(\lambda), p_2(\lambda))$, where

$$s_2(\lambda) = ix_1(\lambda) - ix_2(\lambda) \qquad \text{and} \qquad p_2(\lambda) = x_3(\lambda), \lambda \in \mathbb{D}.$$

By Lemma 4.1.6, $h_2$ is rational function from $\mathbb{D}$ to $\mathbb{G}$. Since $x$ is an $\overline{\mathbb{E}}$-inner function, $x(\lambda) \in b\overline{\mathbb{E}}$ for almost all $\lambda \in \mathbb{T}$. By Proposition 3.1.4, to prove that $h_2$ is a rational $\Gamma$-inner function we need to show that

$$|p_2(\lambda)| = 1, \ |s_2(\lambda)| \leq 2 \text{ and } s_2(\lambda) = \overline{s_2(\lambda)}p_2(\lambda) \quad \text{for almost every } \lambda \in \mathbb{T}.$$

By Theorem 2.3.1, for almost all $\lambda \in \mathbb{T}$, $|p(\lambda)| = |x_3(\lambda)| = 1$ and

$$|s_2(\lambda)| \leq |ix_1(\lambda)| + |ix_2(\lambda)| \leq 2.$$

By Lemma 2.3.2, $x_2(\lambda) = \overline{x_1(\lambda)}x_3(\lambda)$ for almost all $\lambda \in \mathbb{T}$. Hence, for almost all $\lambda \in \mathbb{T}$,

$$\begin{aligned} \overline{s_2(\lambda)}p_2(\lambda) &= [\overline{ix_1(\lambda)} - \overline{ix_2(\lambda)}]x_3(\lambda) \\ &= i\big(\overline{x_2(\lambda)}\big)x_3(\lambda) - i\big(\overline{x_1(\lambda)}\big)x_3(\lambda), \quad \text{by equations 4.1.9,} \\ &= ix_1(\lambda) - ix_2(\lambda) \\ &= s_2(\lambda). \end{aligned}$$

Hence $s_2(\lambda) = \overline{s_2(\lambda)}p_2(\lambda)$ for almost every $\lambda \in \mathbb{T}$. Therefore, by Proposition 3.1.4, $h_2$ is a rational $\Gamma$-inner function. □

**Lemma 4.1.10.** *Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$-inner function. Then*

$$x_1(\lambda) = x_2^{\vee}(1/\lambda)x_3(\lambda) \quad \textit{for all} \quad \lambda \in \mathbb{C}.$$

*Proof.* By Theorem 2.3.1, for all $\lambda \in \mathbb{T}$,

$$x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda).$$

For $\lambda \in \mathbb{T}$, we have $|\lambda| = 1$, that is, $\lambda\overline{\lambda} = 1$, and so

$$\overline{x_2(\lambda)} = x_2^{\vee}(\overline{\lambda}) = x_2^{\vee}(\tfrac{1}{\lambda}).$$

Therefore, for all $\lambda \in \mathbb{T}$,

$$x_1(\lambda) = x_2^{\vee}(1/\lambda)x_3(\lambda).$$

Since $x_1, x_2, x_3$ are rational functions,

$$x_1(\lambda) = x_2^{\vee}(1/\lambda)x_3(\lambda) \qquad \text{for all } \lambda \in \mathbb{C}.$$

□

**Proposition 4.1.11.** *Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$-inner function*

(i) *If $a \in \mathbb{C}\cup\{\infty\}$ is a pole of $x_3$ of multiplicity $k \geq 0$ and $\frac{1}{\overline{a}}$ is a zero of $x_2$ of multiplicity $\ell \geq 0$, then $a$ is a pole of $x_1$ of multiplicity at least $k - \ell$.*

(ii) *If $a \in \mathbb{C}\cup\{\infty\}$ is a pole of $x_1$ of multiplicity $k \geq 1$, then $a$ is a pole of $x_3$ of multiplicity at least $k$.*

*Proof.* (i) By Lemma 4.1.10, we have

$$x_1(\lambda) = x_2^{\vee}(1/\lambda)x_3(\lambda) \qquad \text{for } \lambda \in \mathbb{C}. \tag{4.1.10}$$

Since $x_3$ is a rational inner function, $x_3$ cannot have any pole in $\overline{\mathbb{D}}$. Hence $|a| > 1$ and so $|\frac{1}{a}| < 1$. We know that $x_2^{\vee}$ is analytic in $\mathbb{D}$, so $\frac{1}{a}$ cannot be a pole of $x_2^{\vee}$. By equation (4.1.10),

$$(\lambda - a)^{k-\ell-1}x_1(\lambda) = (\lambda - a)^{k-\ell-1}x_2^{\vee}(1/\lambda)x_3(\lambda).$$

Take the limit for both sides as $\lambda$ goes to $a$:

$$\lim_{\lambda\to a}(\lambda - a)^{k-\ell-1}x_1(\lambda) = \lim_{\lambda\to a}(\lambda - a)^{k-\ell-1}x_2^{\vee}(1/\lambda)x_3(\lambda).$$

The right hand side goes to $\infty$, therefore $x_1$ has a pole of multiplicity at least $k - l$ at $a$.

Now suppose that $\infty$ is a pole of $x_3$ of multiplicity $k$ and 0 is a zero of $x_2$ of multiplicity $\ell$. By equation (4.1.10), for all $\lambda \in \mathbb{C}\backslash\{0\}$, we have

$$x_1(\tfrac{1}{\lambda}) = x_2^\vee(\lambda) x_3(\tfrac{1}{\lambda}).$$

Multiply both sides by $\dfrac{\lambda^{k-1}}{\lambda^\ell}$ yields

$$\frac{\lambda^{k-1}}{\lambda^\ell} x_1(\tfrac{1}{\lambda}) = \frac{\lambda^{k-1}}{\lambda^\ell} x_2^\vee(\lambda) x_3(\tfrac{1}{\lambda}). \tag{4.1.11}$$

Since $x_2^\vee$ is analytic at 0 and has a zero of multiplicity $\ell > 0$ at 0, we have

$$\lim_{\lambda\to 0} \frac{x_2^\vee(\lambda)}{\lambda^\ell} = c, \qquad \text{where } c \in \mathbb{C}\backslash\{0\}.$$

Since by assumption, $x_3(\lambda)$ has a pole of multiplicity $k$ at $\infty$,

$$\lim_{\lambda\to 0} \lambda^{k-1} x_3(\tfrac{1}{\lambda}) = \infty.$$

Hence by equation (4.1.11),

$$\lim_{\lambda\to 0} \lambda^{k-\ell-1} x_1(\tfrac{1}{\lambda}) = \infty.$$

It follows that $x_1(\frac{1}{\lambda})$ has a pole of multiplicity at least $k-\ell$ at 0. That is, $x_1(\lambda)$ has a pole of multiplicity at least $k-\ell$ at $\infty$.

(ii) Let $a \in \mathbb{C}$ be a pole of $x_1$ of multiplicity $k \geq 1$. Then $|a| > 1$. This implies $|\frac{1}{a}| < 1$. Therefore $x_2^\vee$ is analytic at $\frac{1}{a}$. Now

$$\lim_{\lambda\to a} (\lambda - a)^{k-1} x_1(\lambda) = \infty.$$

Thus $a$ is a pole of $x_3$ of multiplicity at least $k$.

If $\infty$ is a pole of $x_1$ of multiplicity $k \geq 1$. Then 0 is a pole of $x_1(\frac{1}{\lambda})$ of multiplicity $k$, that is,

$$\lim_{\lambda\to 0} \lambda^{k-1} x_1(\tfrac{1}{\lambda}) = \infty.$$

By relation (4.1.11),

$$\lambda^{k-1} x_1(\tfrac{1}{\lambda}) = \lambda^{k-1} x_2^\vee(\lambda) x_3(\tfrac{1}{\lambda}).$$

Since $x_2^\vee$ is analytic at 0, 0 cannot be a pole of $x_2^\vee$ and thus

$$\lim_{\lambda\to 0} x_2^\vee(\lambda) = x_2^\vee(0).$$

Therefore

$$\lim_{\lambda\to 0} \lambda^{k-1} x_3(\tfrac{1}{\lambda}) = \infty$$

This completes the proof that $x_3$ has a pole of multiplicity at least $k$ at $\infty$. □

## 4.2 The degree of a rational $\overline{\mathbb{E}}$-inner function

Let us consider an $\overline{\mathbb{E}}$-inner function $x : \mathbb{D} \to \overline{\mathbb{E}}$,

$$x(\lambda) = \big(x_1(\lambda), x_2(\lambda), x_3(\lambda)\big), \quad \lambda \in \mathbb{D}.$$

Since $x$ is an $\overline{\mathbb{E}}$-inner function, $x(\lambda) \in b\overline{\mathbb{E}}$ for almost all $\lambda \in \mathbb{T}$, see Section 4.1.

**Definition 4.2.1.** *The degree of a rational $\overline{\mathbb{E}}$-inner function $x$, denoted by $\deg(x)$ is defined to be $x_*(1)$, where $x_* : \mathbb{Z} = \pi_1(\mathbb{T}) \to \pi_1(b\overline{\mathbb{E}})$ is the homomorphism of fundamental groups induced by $x$ when $x$ is regarded as a continuous map from $\mathbb{T}$ to $b\overline{\mathbb{E}}$.*

**Lemma 4.2.2.** *$b\overline{\mathbb{E}}$ is homotopic to $\mathbb{T}$ and $\pi_1(b\overline{\mathbb{E}}) = \mathbb{Z}$.*

*Proof.* The maps

$$f : b\overline{\mathbb{E}} \to \mathbb{T} \qquad f(x_1, x_2, x_3) = x_3$$
$$g : \mathbb{T} \to b\overline{\mathbb{E}} \qquad g(z) = (0, 0, z)$$

satisfy

$$(g \circ f)(x_1, x_2, x_3) = g\big(f(x_1, x_2, x_3)\big) = g(x_3) = (0, 0, x_3)$$

and

$$(f \circ g)(z) = f(0, 0, z) = z,$$

that is, $f \circ g = \mathrm{id}_{\mathbb{T}}$. If $(x_1, x_2, x_3) \in b\overline{\mathbb{E}}$ and $0 \le t \le 1$, then $(tx_1, tx_2, x_3) \in b\overline{\mathbb{E}}$. Let $I = [0, 1]$. Consider the map

$$h : b\overline{\mathbb{E}} \times I \to b\overline{\mathbb{E}},$$

which is defined by

$$h(x_1, x_2, x_3, t) = (tx_1, tx_2, x_3).$$

One can see that

$$h(x_1, x_2, x_3, 0) = (0x_1, 0x_2, x_3) = (0, 0, x_3) = (g \circ f)(x_1, x_2, x_3) \text{ and}$$

$$h(x_1, x_2, x_3, 1) = (1x_1, 1x_2, x_3) = (x_1, x_2, x_3) = \mathrm{id}_{b\overline{\mathbb{E}}}.$$

Therefore $h$ defines a homotopy between $g \circ f$ and $\mathrm{id}_{b\overline{\mathbb{E}}}$, that is, $g \circ f \simeq \mathrm{id}_{b\overline{\mathbb{E}}}$. Hence $b\overline{\mathbb{E}}$ is homotopically equivalent to $\mathbb{T}$ and it follows that $\pi_1(b\overline{\mathbb{E}}) = \pi_1(\mathbb{T}) = \mathbb{Z}$. ☐

**Lemma 4.2.3.** *Let $B$ be a finite Blaschke product. Then the degree of $B$ is equal to $B_*(1)$.*

*Proof.* Since $B$ is a finite Blaschke product, it can be written as

$$B(\lambda) = e^{i\theta} \prod_{j=1}^{N} \frac{\lambda - \alpha_j}{1 - \overline{\alpha_j}\lambda}, \qquad \text{where } \alpha_j \in \mathbb{D},\ j = 1 \dots N, \text{ and } \theta \in [0, 2\pi).$$

One can consider the map, $B : \mathbb{T} \to \mathbb{T}$, and

$$B_* : \pi_1(\mathbb{T}) = \mathbb{Z} \to \pi_1(\mathbb{T}) = \mathbb{Z}.$$

Now $1 \in \pi_1(\mathbb{T})$ is the homotopy class of $\mathrm{id}_{\mathbb{T}}$ and $B_*(1)$ is equal to the homotopy class of $B \circ \mathrm{id}_{\mathbb{T}} = B$, when $B$ is regarded as a continuous map from $\mathbb{T}$ to $\mathbb{T}$. Therefore
$B_*(1) = n(\gamma, a)$, where $n(\gamma, a)$ is the winding number of $\gamma$ about $a$, which lies inside $\gamma = \{B(e^{it}) : 0 \le t \le 2\pi\}$. Thus

$$\begin{aligned} n(\gamma, a) &= \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - a} \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{dB(e^{it})}{B(e^{it}) - a} \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{B'(e^{it}) i e^{it} dt}{B(e^{it}) - a} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{B'(e^{it}) e^{it} dt}{B(e^{it}) - a} \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{B'(z) dz}{B(z) - a}. \end{aligned}$$

By the Argument Principle, [14, Theorem 18], the integral

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{B'(z)}{B(z) - a} dz,$$

is equal to the number of zeros of $B$ in $\mathbb{D}$. It is clear that $B$ has $N$ zeros, counting multiplicities, and has degree $N$. Therefore the number of zeros of $B$ is equal to the winding number of $\gamma$ about $a$, and it is equal to $N$.

□

**Proposition 4.2.4.** *For any rational $\overline{\mathbb{E}}$-inner function $x = (x_1, x_2, x_3)$, $\deg(x)$ is the degree $\deg(x_3)$ (in the usual sense) of the finite Blaschke product $x_3$.*

*Proof.* Since $x$ is a rational $\overline{\mathbb{E}}$-inner function, $x_3$ is an inner function, and so $x_3$ is a finite Blaschke product. By Definition A.0.1, two $\overline{\mathbb{E}}$-inner functions $x = (x_1, x_2, x_3)$ and $y = (0, 0, x_3)$ are homotopic if there exists a continuous mapping $x(\lambda, t) : \mathbb{T} \times I \to b\overline{\mathbb{E}}$ such that

$$x(\lambda, 0) = y \qquad \text{and} \qquad x(\lambda, 1) = x.$$

Let

$$x^t(\lambda) = \big(tx_1(\lambda), tx_2(\lambda), x_3(\lambda)\big) \text{ for } \lambda \in \mathbb{D} \text{ and } t \in [0,1].$$

Since $x(\lambda) \in b\overline{\mathbb{E}}$, for all $\lambda \in \mathbb{T}$, by Theorem 2.3.1 (1),

$$x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda) \quad \text{and} \quad |x_3(\lambda)| = 1.$$

Hence for all $\lambda \in \mathbb{T}$,

$$tx_1(\lambda) = t\overline{x_2(\lambda)}x_3(\lambda).$$

Therefore,

$$x^t(\lambda) = \big(tx_1(\lambda), tx_2(\lambda), x_3(\lambda)\big) \in b\overline{\mathbb{E}} \text{ for } \lambda \in \mathbb{T}.$$

Hence $x^t$ is a homotopy between $x = x^1$ and $(0, 0, x_3) = x^0$.
It follows that the homomorphism

$$x_* : \pi_1(\mathbb{T}) = \mathbb{Z} \to \pi_1(b\overline{\mathbb{E}}) = \mathbb{Z}$$

coincides with $(x^0)_* = (0, 0, x_3)_*$. By Lemma 4.2.3, $(x_3)_*(1) = \deg x_3$, since $x_3$ is a finite Blaschke product. Therefore $(0, 0, x_3)_*(1)$ is the degree of the finite Blaschke product $x_3$. □

## 4.3 Description of rational $\overline{\mathbb{E}}$-inner functions

**Theorem 4.3.1.** *If $x = (x_1, x_2, x_3)$ is a rational $\overline{\mathbb{E}}$-inner function of degree $n$ then there exist polynomials $E_1, E_2, D$ such that*

(i) $\deg(E_1), \deg(E_2), \deg(D) \leq n$,

(ii) $D(\lambda) \neq 0$ *on* $\overline{\mathbb{D}}$,

(iii) $x_3 = \frac{D^{\sim n}}{D}$ *on* $\overline{\mathbb{D}}$,

(iv) $x_1 = \frac{E_1}{D}$ *on* $\overline{\mathbb{D}}$,

(v) $x_2 = \frac{E_2}{D}$ *on* $\overline{\mathbb{D}}$,

(vi) $|E_i(\lambda)| \leq |D(\lambda)|$ *on* $\overline{\mathbb{D}}$, *for* $i = 1, 2$,

(vii) $E_1(\lambda) = E_2^{\sim n}(\lambda)$, *for* $\lambda \in \overline{\mathbb{D}}$.

*Conversely, if* $E_1, E_2$ *and* $D$ *satisfy* (i),(vi) and (vii), $D(\lambda) \neq 0$ *on* $\mathbb{D}$ *and* $x_1, x_2$ *and* $x_3$ *are defined by* (iii)–(v), *then* $x = (x_1, x_2, x_3)$ *is a rational* $\overline{\mathbb{E}}$*-inner function of degree at most* $n$.
*Furthermore, a triple of polynomials* $E_1^1, E_2^1$ *and* $D^1$ *satisfies* (i)–(vii) *if and only if there exists a real number* $t \neq 0$ *such that*

$$E_1^1 = tE_1, \quad E_2^1 = tE_2 \quad \text{and} \quad D^1 = tD.$$

*Proof.* By assumption $x = \big(x_1, x_2, x_3\big)$ is a rational $\overline{\mathbb{E}}$-inner function. By Lemma 4.1.9 (1), $h_1 = (s, p)$ where $s = x_1 + x_2, p = x_3$ is a rational $\Gamma$-inner function. Since $x_3 : \mathbb{D} \to \mathbb{D}$ is an inner function, it is a finite Blaschke product and, by [4, Corollary 6.10], it can be written in the form

$$x_3(\lambda) = c\frac{\lambda^k D^{\sim(n-k)}(\lambda)}{D(\lambda)},$$

where $|c| = 1, 0 \leq k \leq n$ and $D$ is a polynomial of degree $n - k$ such that $D(0) = 1$. By Proposition 3.3.4, there exist polynomials $E, D$ such that

(i) $\deg(E), \deg(D) \leq n$,

(ii) $E^{\sim n} = E$,

(iii) $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$,

(iv) $|E(\lambda)| \leq 2\,|D(\lambda)|$ on $\overline{\mathbb{D}}$,

(v) $s = \frac{E}{D}$ on $\overline{\mathbb{D}}$,

(vi) $p = \frac{D^{\sim n}}{D}$ on $\overline{\mathbb{D}}$.

Hence

$$x_1 + x_2 = s = \frac{E}{D} \quad \text{and} \quad x_3 = p = \frac{D^{\sim n}}{D}. \tag{4.3.1}$$

By Lemma 4.1.9 (2), $h_2 = (s_2, p_2)$, where $s_2 = ix_1 - ix_2$, $p_2 = x_3 = p_1$ is a rational $\Gamma$-inner function. By Proposition 3.3.4, for $h_2 = (s_2, p_2)$, there exist polynomials $G, D$ such that

(i) $\deg(G), \deg(D) \leq n$,

(ii) $G^{\sim n} = G$,

(iii) $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$,

(iv) $|G(\lambda)| \leq 2\,|D(\lambda)|$ on $\overline{\mathbb{D}}$,

(v) $s_2 = ix_1 - ix_2 = \frac{G}{D}$ on $\overline{\mathbb{D}}$,

(vi) $p_2 = x_3 = \frac{D^{\sim n}}{D}$ on $\overline{\mathbb{D}}$.

Therefore, by (v),

$$x_1 - x_2 = -\frac{iG}{D}. \tag{4.3.2}$$

By relation (4.3.1),

$$x_1 + x_2 = \frac{E}{D} \ . \tag{4.3.3}$$

Add equations (4.3.2) and (4.3.3) to get

$$x_1 = \frac{\frac{1}{2}(E - iG)}{D}.$$

Substituting $x_1$ in equation (4.3.3) gives

$$x_2 = \frac{\frac{1}{2}(E + iG)}{D}.$$

Define the polynomials $E_1$ and $E_2$ by

$$E_1 = \frac{1}{2}(E - iG), \qquad E_2 = \frac{1}{2}(E + iG).$$

Since the degrees of both polynomials $E, G$ are at most $n$, $\deg(E_1), \deg(E_2) \le n$. Thus, for $\lambda \in \overline{\mathbb{D}}$,

$$x_1(\lambda) = \frac{E_1(\lambda)}{D(\lambda)} \qquad \text{and} \qquad x_2(\lambda) = \frac{E_2(\lambda)}{D(\lambda)}.$$

Since $x$ is an $\overline{\mathbb{E}}$-inner function, for $\lambda \in \overline{\mathbb{D}}$,

$$\begin{aligned} & |x_1(\lambda)| \le 1 \qquad \text{and} \qquad |x_2(\lambda)| \le 1, \\ \text{and so} \qquad & |E_1(\lambda)| \le |D(\lambda)| \qquad \text{and} \qquad |E_2(\lambda)| \le |D(\lambda)|. \end{aligned}$$

Hence $|E_i(\lambda)| \le |D(\lambda)|$ on $\overline{\mathbb{D}}$, where $i = 1, 2$. Therefore (i)–(vi) of Theorem 4.3.1 are satisfied.

By assumption, $x$ is a rational $\overline{\mathbb{E}}$-inner function. Thus, for all $\lambda \in \mathbb{T}$,

$$\begin{aligned}
x_1(\lambda) = \overline{x_2}(\lambda)x_3(\lambda) \Leftrightarrow \frac{E_1(\lambda)}{D(\lambda)} &= \frac{\overline{E_2(\lambda)}}{\overline{D(\lambda)}} \times \frac{D^{\sim n}(\lambda)}{D(\lambda)} \\
\Leftrightarrow \frac{E_1(\lambda)}{D(\lambda)} &= \frac{\overline{E_2(\lambda)}}{D^{\vee}(1/\lambda)} \times \frac{\lambda^n D^{\vee}(1/\lambda)}{D(\lambda)}, \text{ since } \overline{D}(\lambda) = D^{\vee}(\overline{\lambda}) = D^{\vee}(1/\lambda). \\
\Leftrightarrow \frac{E_1(\lambda)}{D(\lambda)} &= \frac{\lambda^n \overline{E_2(\lambda)}}{D(\lambda)} \\
\Leftrightarrow \frac{E_1(\lambda)}{D(\lambda)} &= \frac{E_2^{\sim n}(\lambda)}{D(\lambda)} \\
\Leftrightarrow E_1(\lambda) &= E_2^{\sim n}(\lambda). \qquad (4.3.4)
\end{aligned}$$

Hence $E_1(\lambda) = E_2^{\sim n}(\lambda)$ for all $\lambda \in \mathbb{T}$, and therefore on $\overline{\mathbb{D}}$. Thus (vii) is proved.

**Let us prove the converse statement**. Let $E_1, E_2$ and $D$ satisfy (i), (vi) and (vii) of Theorem 4.3.1 and $D(\lambda) \neq 0$ on $\mathbb{D}$, and $x_1, x_2, x_3$ be defined by (iii)–(v), that is,

$$x_1 = \frac{E_1}{D}, \quad x_2 = \frac{E_2}{D} \quad \text{and} \quad x_3 = \frac{D^{\sim n}}{D}.$$

Let us show that $x = (x_1, x_2, x_3)$ is a rational $\overline{\mathbb{E}}$-inner function. By Theorem 2.3.1, we have to prove that $x : \mathbb{D} \to \mathbb{E}$ and the following conditions are satisfied.

(1) $|x_3(\lambda)| = 1$ for almost all $\lambda$ on $\mathbb{T}$, that is, $x_3$ is inner,

(2) $|x_2| \leq 1$ on $\overline{\mathbb{D}}$,

(3) $x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda)$ for almost all $\lambda \in \mathbb{T}$.

(1) Firstly, if $D$ has no zeros on the unit circle, then $D$ and $D^{\sim n}$ have no common factor. Therefore, $x_3(\lambda) = \dfrac{D^{\sim n}(\lambda)}{D(\lambda)}$ maps $\mathbb{T}$ to $\mathbb{T}$. Hence, $x_3$ is inner function and

$$\deg(x_3) = \deg\left(\frac{D^{\sim n}}{D}\right) = \max\{\deg(D^{\sim n}), \deg(D)\} = n.$$

Second case: if $D$ has the zeros $a_1, ..., a_\ell$ on $\mathbb{T}$ then $D$ and $D^{\sim n}$ have the common factor $\prod_{i=1}^{\ell}(\lambda - a_i)$ and hence $x_3 = \dfrac{D^{\sim n}}{D}$ is inner and

$$\deg(x_3) = \deg\left(\frac{D^{\sim n}}{D}\right) \leq n - \ell.$$

(2) By assumption (vi),

$$|E_2(\lambda)| \leq |D(\lambda)| \quad \text{for all } \lambda \in \overline{\mathbb{D}}.$$

This implies $|\frac{E_2(\lambda)}{D(\lambda)}| \leq 1$ and hence $|x_2(\lambda)| \leq 1$.

(3) By assumption (vii), $E_1(\lambda) = E_2^{\sim n}(\lambda)$, for almost all $\lambda \in \mathbb{T}$ and by equality (4.3.4), $x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda)$, for almost all $\lambda \in \mathbb{T}$.

**Let us show that** $x = (x_1, x_2, x_3) = \left(\frac{E_1}{D}, \frac{E_2}{D}, \frac{D^{\sim n}}{D}\right)$ **maps** $\mathbb{D}$ **to** $\mathbb{E}$, that is, $x(\lambda) = (x_1(\lambda), x_2(\lambda), x_3(\lambda)) \in \mathbb{E}$ for all $\lambda \in \mathbb{D}$. By Theorem 2.1.5 (2), for $\lambda \in \mathbb{D}$,

$$x(\lambda) \in \overline{\mathbb{E}} \Leftrightarrow ||\Psi(., x(\lambda))||_{H^\infty} \leq 1,$$

where $\Psi(z, x) = \dfrac{x_3 z - x_1}{x_2 z - 1}$. Note that, for every $z \in \mathbb{D}$,

$$\begin{aligned} \Psi(z,x) : \ & \mathbb{D} \to \mathbb{C} \\ : \ & \lambda \to \Psi(z, x(\lambda)) \end{aligned}$$

is analytic on $\mathbb{D}$ because $x_i, i = 1, 2, 3$, are analytic functions on $\mathbb{D}$, and $|x_2(\lambda)| \leq 1$ and $x_2(\lambda)z \neq 1$ for all $\lambda \in \mathbb{D}$. We have shown above that, for almost all $\lambda \in \mathbb{T}$, $x(\lambda) \in b\overline{\mathbb{E}}$. Thus, by Theorem 2.3.1 (2),

$$x(\lambda) \in b\overline{\mathbb{E}} \text{ is equivalent to } \Psi(., x(\lambda)) \text{ is an automorphism of } \mathbb{D}.$$

By the maximum principle, for all $z, \lambda \in \mathbb{D}$, $|\Psi(z, x(\lambda))| < 1$. Thus by Theorem 2.3.1, $x(\lambda) \in \mathbb{E}$ for all $\lambda \in \mathbb{D}$.

Suppose that $t$ is a nonzero real number and

$$E_1^1 = tE_1, \quad E_2^1 = tE_1 \quad \text{and} \quad D^1 = tD.$$

Then it is clear that $E_1^1, E_2^1$ and $D_1$ satisfy (i)–(vii). Conversely, let $E_1^1, E_2^1$ and $D^1$ be a second triple that satisfies (i)–(vii). Then

$$x_1 = \frac{E_1}{D} = \frac{E_1^1}{D^1} \quad \text{on } \overline{\mathbb{D}}, \tag{4.3.5}$$

$$x_2 = \frac{E_2}{D} = \frac{E_2^1}{D^1} \quad \text{on } \overline{\mathbb{D}}, \tag{4.3.6}$$

$$x_3 = \frac{D^{\sim n}}{D} = \frac{D^{1\sim n}}{D^1} \quad \text{on } \overline{\mathbb{D}}. \tag{4.3.7}$$

Suppose that $D(\lambda) = a_0 + a_1\lambda + ... + a_k\lambda^k$ where $a_0 \neq 0$ and $k \leq n$. Then

$$\begin{aligned} D^{\sim n}(\lambda) &= \lambda^n \overline{D\big(1/\overline{\lambda}\big)} \\ &= \lambda^n \left(\overline{a_0 + \frac{a_1}{\overline{\lambda}} + ... + \frac{a_k}{\overline{\lambda}}}\right) \\ &= \lambda^n \left(\overline{a_0} + \frac{\overline{a_1}}{\lambda} + ... + \frac{\overline{a_k}}{\lambda^k}\right) \\ &= \overline{a_0}\lambda^n + \overline{a_1}\lambda^{n-1} + ... + \overline{a_k}\lambda^{n-k}. \end{aligned}$$

Thus, for all $\lambda \in \mathbb{D}$,

$$x_3 = \frac{D^{\sim n}(\lambda)}{D(\lambda)} = \frac{\lambda^{n-k}\big(\overline{a_0}\lambda^k + \overline{a_1}\lambda^{k-1} + ... + \overline{a_k}\big)}{a_0 + a_1\lambda + ... + a_k\lambda^k}.$$

Therefore, $x_3$ has a zero of multiplicity $(n-k)$ at 0 , has $k$ poles in $\mathbb{C}$, counting multiplicity, and has degree $n$. Hence the poles of $x_3$ in $\{z \in \mathbb{C} : |z| > 1\}$, $n$ and $k$ are determined by $x_3$. Therefore, $D$ and $D^1$ have the same degree $k$ and the same finite number of zeros in $\{z \in \mathbb{C} : |z| > 1\}$, counting multiplicity. Thus there exists $t \in \mathbb{C}, t \neq 0$ where

$$D^1 = tD \quad \textit{on } \overline{\mathbb{D}}. \tag{4.3.8}$$

By equality (4.3.7), for $\lambda \in \overline{\mathbb{D}}$

$$x_3 = \frac{D^{\sim n}}{D} = \frac{D^{1\sim n}}{D^1} = \frac{\bar{t}D^{\sim n}}{tD}$$

Thus $t = \bar{t}$, and so, $t \in \mathbb{R}\backslash\{0\}$. By the equalities (4.3.5) and (4.3.8)

$$x_1 = \frac{E_1}{D} = \frac{E_1^1}{D^1} = \frac{E_1^1}{tD}, \qquad \text{on } \overline{\mathbb{D}}.$$

This implies that $E_1^1 = tE_1$. By the equalities (4.3.6) and (4.3.8)

$$x_2 = \frac{E_2}{D} = \frac{E_2^1}{D^1} = \frac{E_2^1}{tD}, \qquad \text{on } \overline{\mathbb{D}}.$$

Thus $E_2^1 = tE_2$. □

**Lemma 4.3.2.** *Let*

$$x = (x_1, x_2, x_3) = \left(\frac{E_1}{D}, \frac{E_2}{D}, \frac{D^{\sim n}}{D}\right)$$

*be a rational $\overline{\mathbb{E}}$-inner function. Then, for $\lambda \in \mathbb{T}$,*

$$|E_1(\lambda)| = |E_2(\lambda)|, \quad \textit{and so} \quad |x_1(\lambda)| = |x_2(\lambda)|.$$

*Proof.* By Theorem 4.3.1 (vii), for all $\lambda \in \mathbb{T}$,

$$E_1(\lambda) = E_2^{\sim n}(\lambda) = \lambda^n \overline{E_2(1/\overline{\lambda})}.$$

Thus, since $\lambda\overline{\lambda} = 1$,

$$\begin{aligned} |E_1(\lambda)| &= |\lambda^n \overline{E_2(1/\overline{\lambda})}| \\ &= |E_2(1/\overline{\lambda})| \\ &= |E_2(\lambda)|. \end{aligned}$$

By Theorem 4.3.1 (iv, v),

$$x_1 = \frac{E_1}{D}, \qquad x_2 = \frac{E_2}{D} \qquad \text{on } \overline{\mathbb{D}}.$$

Therefore, for all $\lambda \in \mathbb{T}$,

$$|x_1(\lambda)| = |x_2(\lambda)|.$$

□

**Example 4.3.3.** Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$-inner function such that $x_3(\lambda) = \lambda$. Clearly,

$$\deg(x) = \deg(x_3) = 1 \qquad \text{and} \qquad D(\lambda) = 1.$$

By Theorem 4.3.1,

$$\deg(E_1) \le 1 \qquad \text{and} \qquad \deg(E_2) \le 1.$$

Thus

$$E_1(\lambda) = x_1(\lambda) = a_1 + a_2\lambda, \qquad E_2(\lambda) = x_2(\lambda) = E_1^{\sim 1}(\lambda) = \overline{a_2} + \overline{a_1}\lambda$$

where $a_1$ and $a_2$ are complex numbers such that, for all $\lambda \in \overline{\mathbb{D}}$, $|E_i(\lambda)| \le |D(\lambda)| = 1$ $i = 1, 2$. Therefore the function

$$x(\lambda) = \big(a_1 + a_2\lambda, \overline{a_2} + \overline{a_1}\lambda, \lambda\big)$$

is rational $\overline{\mathbb{E}}$-inner for $a_1, a_2 \in \overline{\mathbb{D}}$ such that

$$|a_1 + a_2\lambda| \le 1 \qquad \text{and} \qquad |\overline{a_2} + \overline{a_1}\lambda| \le 1 \qquad \text{for all } \lambda \in \overline{\mathbb{D}}.$$

In particular, one can choose $a_1 = 1$ and $a_2 = 0$ to get the rational $\overline{\mathbb{E}}$-inner function

$$x(\lambda) = (1, \lambda, \lambda).$$

**Example 4.3.4. More examples of $\overline{\mathbb{E}}$-inner functions**
Suppose that $\mathbb{B}_{2\times 2} = \{A \in \mathbb{C}^{2\times 2} : ||A|| < 1\}$. Let us construct an analytic map from the open unit disc $\mathbb{D}$ to $\mathbb{B}_{2\times 2}$. Consider nonconstant inner functions $\varphi, \psi \in H^\infty(\mathbb{D})$ and the diagonal matrix

$$h(\lambda) = \begin{bmatrix} \varphi(\lambda) & 0 \\ 0 & \psi(\lambda) \end{bmatrix} \qquad \text{for } \lambda \in \mathbb{D}.$$

Note $||h(\lambda)|| = \max\{|\varphi(\lambda)|, |\psi(\lambda)|\} < 1$ for $\lambda \in \mathbb{D}$ and $h : \mathbb{D} \to \mathbb{B}_{2\times 2}$ is analytic.
By Theorem 2.1.4, for all $\lambda \in \mathbb{D}$,

$$\big(\varphi(\lambda), \psi(\lambda), \det h(\lambda)\big) \in \mathbb{E},$$

and $\varphi(\lambda)\psi(\lambda) = \det h(\lambda)$. Recall that such points are called triangular points of $\mathbb{E}$ (see Definition 2.1.7). However, we are seeking more interesting and general examples. To get such examples we make use of the singular value decomposition.

Let $U, V$ be $2 \times 2$ unitary matrices. Then $h_1 : \mathbb{D} \to \mathbb{C}^{2\times 2}$ defined by

$$h_1 = UhV$$

maps $\mathbb{D}$ to $\mathbb{B}_{2\times 2}$.
For example, if

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad V = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then $U$ is unitary and we obtain

$$\begin{aligned} h_1(\lambda) &= Uh(\lambda)I \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \varphi(\lambda) & 0 \\ 0 & \psi(\lambda) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \varphi(\lambda) & 0 \\ 0 & \psi(\lambda) \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} \varphi(\lambda) & \psi(\lambda) \\ -\varphi(\lambda) & \psi(\lambda) \end{bmatrix}. \end{aligned}$$

Define $x : \mathbb{D} \to \overline{\mathbb{E}}$ by $x = \pi \circ h_1$ where $\pi$ defined as in Definition 2.2.1. Then for $\lambda \in \mathbb{D}$,

$$\begin{aligned} x(\lambda) &= \pi\big(h_1(\lambda)\big) \\ &= \pi\left(\frac{1}{\sqrt{2}}\begin{bmatrix} \varphi(\lambda) & \psi(\lambda) \\ -\varphi(\lambda) & \psi(\lambda) \end{bmatrix}\right) \\ &= \left(\frac{\varphi(\lambda)}{\sqrt{2}}, \frac{\psi(\lambda)}{\sqrt{2}}, \varphi(\lambda)\psi(\lambda)\right). \end{aligned}$$

Note that this $x(\lambda)$ is not a triangular point unless either $\varphi(\lambda) = 0$ or $\psi(\lambda) = 0$.

Let us show that the function $x = (x_1, x_2, x_3) = \pi \circ h_1 : \mathbb{D} \to \overline{\mathbb{E}}$ is $\overline{\mathbb{E}}$-inner where

$$x_1(\lambda) = \frac{\varphi(\lambda)}{\sqrt{2}}, \quad x_2(\lambda) = \frac{\psi(\lambda)}{\sqrt{2}}, \quad x_3(\lambda) = \varphi(\lambda)\psi(\lambda)$$

By Theorem 2.3.1(1), for $\lambda \in \mathbb{T}$, since $\varphi, \psi$ are inner functions,

$$\begin{aligned} \overline{x_2(\lambda)}x_3(\lambda) &= \overline{\left(\frac{\psi(\lambda)}{\sqrt{2}}\right)}\varphi(\lambda)\psi(\lambda) \\ &= \frac{\varphi(\lambda)}{\sqrt{2}}\overline{\psi(\lambda)}\psi(\lambda) \\ &= \frac{\varphi(\lambda)}{\sqrt{2}}|\psi(\lambda)|^2 \\ &= \frac{\varphi(\lambda)}{\sqrt{2}} = x_1(\lambda). \end{aligned}$$

Since $|\psi(\lambda)| < 1$ for $\lambda \in \mathbb{D}$, this implies that $\left|\frac{\psi(\lambda)}{\sqrt{2}}\right| < 1$. Thus $|x_2(\lambda)| < 1$. Finally, for $\lambda \in \mathbb{T}$, since $\varphi, \psi$ are inner functions,

$$\begin{aligned} |x_3(\lambda)| &= \big|\varphi(\lambda)\psi(\lambda)\big| \\ &= |\varphi(\lambda)||\psi(\lambda)| \\ &= 1. \end{aligned}$$

Therefore $x$ is an $\overline{\mathbb{E}}$-inner function.

In particular, $x$ is $\overline{\mathbb{E}}$-inner when

$$\varphi = c_1 B_\alpha, \text{where} \quad |c_1| = 1, \alpha \in \mathbb{D} \quad \text{and} \quad B_\alpha = \frac{\lambda - \alpha}{1 - \overline{\alpha}\lambda} \quad \text{is a Blaschke factor,}$$

$$\psi = c_2 B_\beta, \text{where} \quad |c_2| = 1, \beta \in \mathbb{D} \quad \text{and} \quad B_\beta = \frac{\lambda - \beta}{1 - \overline{\beta}\lambda} \quad \text{is a Blaschke factor.}$$

Now,

$$\varphi(\lambda) = c_1 \frac{\lambda - \alpha}{1 - \overline{\alpha}\lambda}, \quad \psi(\lambda) = c_2 \frac{\lambda - \beta}{1 - \overline{\beta}\lambda}.$$

This gives

$$x(\lambda) = \left( \frac{c_1}{\sqrt{2}} B_\alpha, \frac{c_2}{\sqrt{2}} B_\beta, c_1 c_2 B_\alpha B_\beta \right)(\lambda).$$

Thus

$$x(\lambda) = \left( \frac{c_1}{\sqrt{2}} \frac{\lambda - \alpha}{(1 - \overline{\alpha}\lambda)}, \frac{c_2}{\sqrt{2}} \frac{\lambda - \beta}{(1 - \overline{\beta}\lambda)}, c_1 c_2 \frac{\lambda - \alpha}{(1 - \overline{\alpha}\lambda)} \frac{\lambda - \beta}{(1 - \overline{\beta}\lambda)} \right).$$

Let us find $E_1, E_2, D$ as in Theorem 4.3.1 for this example.

Let $D(\lambda) = (1 - \overline{\alpha}\lambda)(1 - \overline{\beta}\lambda)c$, $|c| = 1$.

$$\begin{aligned} D^{\sim 2}(\lambda) &= \lambda^2 \overline{D(1/\overline{\lambda})} \\ &= \lambda^2 \overline{(1 - \overline{\alpha}/\overline{\lambda})(1 - \overline{\beta}/\overline{\lambda})c} \\ &= \lambda^2 (1 - \alpha/\lambda)(1 - \beta/\lambda)\overline{c} \\ &= (\lambda - \alpha)(\lambda - \beta)\overline{c} \,. \end{aligned}$$

Then

$$\frac{D^{\sim 2}(\lambda)}{D(\lambda)} = \frac{(\lambda - \alpha)(\lambda - \beta)\overline{c}}{(1 - \overline{\alpha}\lambda)(1 - \overline{\beta}\lambda)c}, \quad \text{where } \overline{c}^2 = c_1 c_2.$$

We have $x_1(\lambda) = \dfrac{E_1(\lambda)}{D(\lambda)}$. Hence

$$\frac{c_1}{\sqrt{2}} \frac{\lambda - \alpha}{1 - \overline{\alpha}\lambda} = \frac{E_1(\lambda)}{(1 - \overline{\alpha}\lambda)(1 - \overline{\beta}\lambda)c}$$

and therefore

$$E_1(\lambda) = \frac{c c_1}{\sqrt{2}} (\lambda - \alpha)(1 - \overline{\beta}\lambda).$$

Note that, since $\overline{c}^2 = c_1 c_2$, $|c_1| = 1$ and $|c_2| = 1$,

$$\begin{aligned} \overline{c}^2 &= c_1 c_2 \\ \overline{c}^2 \overline{c}_2 &= c_1 \end{aligned}$$

thus

$$c c_1 = \overline{c c_2}.$$

Similarly, $x_2(\lambda) = \dfrac{E_2(\lambda)}{D(\lambda)}$ implies

$$\frac{c_2}{\sqrt{2}}\frac{\lambda-\beta}{1-\overline{\beta}\lambda} = \frac{E_2(\lambda)}{(1-\overline{\alpha}\lambda)(1-\overline{\beta}\lambda)c}.$$

Hence

$$E_2(\lambda) = \frac{cc_2}{\sqrt{2}}(\lambda-\beta)(1-\overline{\alpha}\lambda).$$

(ii) Note, for $\lambda \in \mathbb{D}$, since $cc_1 = \overline{cc_2}$,

$$\begin{aligned} E_2^{\sim 2}(\lambda) = E_2^{\vee}(1/\lambda)\lambda^2 &= \overline{E_2(1/\overline{\lambda})}\lambda^2 \\ &= \overline{\frac{cc_2}{\sqrt{2}}(1/\overline{\lambda}-\beta)(1-\overline{\alpha}/\overline{\lambda})}\lambda^2 \\ &= \frac{\overline{cc_2}}{\sqrt{2}}(1/\lambda-\overline{\beta})(1-\alpha/\lambda)\lambda^2 \\ &= \frac{\overline{cc_2}}{\sqrt{2}}(\lambda-\alpha)(1-\overline{\beta}\lambda) \\ &= \frac{cc_1}{\sqrt{2}}(\lambda-\alpha)(1-\overline{\beta}\lambda) \\ &= E_1(\lambda). \end{aligned}$$

**Remark 4.3.5.** *In the previous example if we choose the functions $\varphi$ and $\psi$ to be in the Schur class but not to be inner functions then one can check that we obtain an analytic function $x : \mathbb{D} \to \overline{\mathbb{E}}$ which is not an $\overline{\mathbb{E}}$-inner function.*

**Proposition 4.3.6.** *Let $(s,p)$ be a $\Gamma$-inner function. Then $x = \left(\frac{s}{2}, \frac{s}{2}, p\right)$ is an $\overline{\mathbb{E}}$-inner function.*

*Proof.* By Lemma 4.1.7, for every $\lambda \in \mathbb{D}$, $x(\lambda) = \left(\frac{s}{2}(\lambda), \frac{s}{2}(\lambda), p(\lambda)\right) \in \overline{\mathbb{E}}$. It is easy to see that $x \in \mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$. By Proposition 3.1.4 (iii), for almost all $\lambda \in \mathbb{T}$,

$$|p(\lambda)| = 1, \qquad |s(\lambda)| \le 2 \qquad \text{and} \quad s(\lambda) - \overline{s(\lambda)}p(\lambda) = 0.$$

Thus, for almost all $\lambda \in \mathbb{T}$,

$$|p(\lambda)| = 1, \qquad \frac{s(\lambda)}{2} = \frac{\overline{s(\lambda)}}{2}p(\lambda) \qquad \text{and} \qquad \frac{|s(\lambda)|}{2} \le 1.$$

Hence $x$ is $\overline{\mathbb{E}}$-inner. □

See [4] for many examples of $\Gamma$-inner functions.

## 4.4 Superficial $\overline{\mathbb{E}}$-inner functions

In this section, we study $\overline{\mathbb{E}}$-inner functions $x = (x_1, x_2, x_3)$ such that $x(\lambda)$ lies in the topological boundary $\partial\overline{\mathbb{E}}$ of $\overline{\mathbb{E}}$ for all $\lambda \in \mathbb{D}$. For any inner function $x_3$ and $\beta_1, \beta_2 \in \mathbb{C}$ such that $|\beta_1| + |\beta_2| = 1$, the function $x = (\beta_2 + \overline{\beta}_1 x_3, \beta_1 + \overline{\beta}_2 x_3, x_3)$ is $\overline{\mathbb{E}}$-inner and has the property that it maps $\mathbb{D}$ to $\partial\overline{\mathbb{E}}$. We also consider the connection between superficial $\Gamma$-inner functions, which were studied in [4], and superficial $\overline{\mathbb{E}}$-inner functions.

**Definition 4.4.1.** *An analytic function* $h : \mathbb{D} \to \Gamma$ *is* superficial *if* $h(\mathbb{D}) \subset \partial\Gamma$ .

**Proposition 4.4.2.** [4, Proposition 8.3] *A* $\Gamma$*-inner function* $h$ *is superficial if and only if there is an* $\omega \in \mathbb{T}$ *and an inner function* $p$ *such that* $h = (\omega p + \overline{\omega}, p)$.

One can define a similar notion for functions from $\mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$.

**Definition 4.4.3.** *An analytic function* $x : \mathbb{D} \to \overline{\mathbb{E}}$ *is* superficial *if* $x(\mathbb{D}) \subset \partial\overline{\mathbb{E}}$.

**Proposition 4.4.4.** *An analytic function* $x : \mathbb{D} \to \overline{\mathbb{E}}$ *such that*

$$x(\lambda) = (\beta_1 + \overline{\beta_2} x_3(\lambda), \beta_2 + \overline{\beta_1} x_3(\lambda), x_3(\lambda)), \quad \lambda \in \mathbb{D}$$

*where* $x_3$ *is an inner function and* $|\beta_1| + |\beta_2| = 1$ *is* $\overline{\mathbb{E}}$*-inner and superficial.*

*Proof.* By Lemma 2.4.1, we need to show that, for $\lambda \in \mathbb{D}$,

$$x(\lambda) = (\beta_1 + \overline{\beta_2} x_3(\lambda), \beta_2 + \overline{\beta_1} x_3(\lambda), x_3(\lambda)).$$

is in $\partial\mathbb{E}$. Here

$$x_1(\lambda) = \beta_1 + \overline{\beta_2} x_3(\lambda), \quad x_2(\lambda) = \beta_2 + \overline{\beta_1} x_3(\lambda).$$

Note, for $\lambda \in \mathbb{D}$,

$$\begin{aligned} |(x_1 - \overline{x}_2 x_3)(\lambda)| &= \left|\beta_1 + \overline{\beta}_2 x_3(\lambda) - \left(\overline{\beta_2 + \overline{\beta_1} x_3(\lambda)}\right) x_3(\lambda)\right| \\ &= \left|\beta_1 + \overline{\beta}_2 x_3(\lambda) - \left(\overline{\beta}_2 + \beta_1 \overline{x_3(\lambda)}\right) x_3(\lambda)\right| \\ &= \left|\beta_1 + \overline{\beta}_2 x_3(\lambda) - \overline{\beta}_2 x_3(\lambda) - \beta_1 |x_3(\lambda)|^2\right| \\ &= \left|\beta_1 \left(1 - |x_3(\lambda)|^2\right)\right|. \end{aligned} \tag{4.4.1}$$

We also have

$$\begin{aligned}
|(x_2 - \overline{x}_1 x_3)(\lambda)| &= \left|\beta_2 + \overline{\beta}_1 x_3(\lambda) - \left(\overline{\beta_1 + \overline{\beta_2} x_3(\lambda)}\right) x_3(\lambda)\right| \\
&= \left|\beta_2 + \overline{\beta}_2 x_3(\lambda) - \left(\overline{\beta}_1 + \beta_2 \overline{x_3(\lambda)}\right) x_3(\lambda)\right| \\
&= \left|\beta_2 + \overline{\beta}_1 x_3(\lambda) - \overline{\beta}_1 x_3(\lambda) - \beta_2 |x_3(\lambda)|^2\right| \\
&= \left|\beta_2\left(1 - |x_3(\lambda)|^2\right)\right|. \qquad (4.4.2)
\end{aligned}$$

Note that by equations (4.4.1) and (4.4.2), for all $\lambda \in \mathbb{D}$,

$$\begin{aligned}
|(x_1 - \overline{x}_2 x_3)(\lambda)| + |(x_2 - \overline{x}_1 x_3)(\lambda)| &= \left|\beta_1\left(1 - |x_3(\lambda)|^2\right)\right| + \left|\beta_2\left(1 - |x_3(\lambda)|^2\right)\right| \\
&= |\beta_1|\left|1 - |x_3(\lambda)|^2\right| + |\beta_2|\left|1 - |x_3(\lambda)|^2\right| \\
&= \left(|\beta_1| + |\beta_2|\right)\left(1 - |x_3(\lambda)|^2\right) = 1 - |x_3(\lambda)|^2.
\end{aligned}$$

By Theorem 2.1.5 and Lemma 2.4.1, for $\lambda \in \mathbb{D}$, the point

$$x(\lambda) = \left(x_1(\lambda), x_2(\lambda), x_3(\lambda)\right)$$

lies in $\partial\mathbb{E}$. Let us check that $x$ is $\overline{\mathbb{E}}$-inner. Clearly, for almost all $\lambda \in \mathbb{T}$,

$$\begin{aligned}
\overline{x_2(\lambda)} x_3(\lambda) &= \left(\overline{\beta_2 + \overline{\beta_1} x_3(\lambda)}\right) x_3(\lambda) \\
&= \overline{\beta_2} x_3(\lambda) + \beta_1 \overline{x_3(\lambda)} x_3(\lambda) \\
&= \beta_1 + \overline{\beta_2} x_3(\lambda) = x_1(\lambda).
\end{aligned}$$

We also have, for almost all $\lambda \in \mathbb{T}$,

$$|x_2(\lambda)| = |\beta_2 + \overline{\beta_1} x_3(\lambda)| \le |\beta_2| + |\beta_1 x_3(\lambda)| = |\beta_2| + |\beta_1| = 1.$$

Since $x_3$ is inner, for almost all $\lambda \in \mathbb{T}$, $|x_3(\lambda)| = 1$. Therefore $x(\lambda) \in b\overline{\mathbb{E}}$, for almost all $\lambda \in \mathbb{T}$, and hence $x$ is $\overline{\mathbb{E}}$-inner.

□

**Lemma 4.4.5.** *Let $x : \mathbb{D} \to \overline{\mathbb{E}}$ be such that $x(\lambda) = \left(\beta_1 + \overline{\beta_2} x_3(\lambda), \beta_2 + \overline{\beta_1} x_3(\lambda), x_3(\lambda)\right)$, where $x_3$ is a non-constant rational inner function and $|\beta_1| + |\beta_2| = 1$. Then $\Psi_\omega(x(\lambda)) = k$ for all $\lambda \in \mathbb{D}$, where*

$$\omega = \frac{\overline{\beta_2}}{|\beta_2|}, \quad k = \frac{\beta_1}{|\beta_1|} \qquad \textit{on } \mathbb{T}.$$

*Proof.* By the definition,

$$\begin{aligned}\Psi_\omega(x(\lambda)) &= \frac{x_3(\lambda)\omega - x_1(\lambda)}{x_2(\lambda)\omega - 1}\\ &= \frac{x_3(\lambda)\omega - (\beta_1 + \overline{\beta}_2 x_3(\lambda))}{(\beta_2 + \overline{\beta}_1 x_3(\lambda))\omega - 1}\\ &= \frac{x_3(\lambda)\omega - \beta_1 - \overline{\beta}_2 x_3(\lambda)}{\beta_2\omega + \overline{\beta}_1\omega x_3(\lambda) - 1}.\end{aligned}$$

Then for all $\lambda \in \mathbb{D}$,

$$\begin{aligned}\Psi_\omega(x(\lambda)) = k &\Leftrightarrow \frac{x_3(\lambda)\omega - \beta_1 - \overline{\beta}_2 x_3(\lambda)}{\beta_2\omega + \overline{\beta}_1\omega x_3(\lambda) - 1} = k\\ &\Leftrightarrow x_3(\lambda)\omega - \beta_1 - \overline{\beta}_2 x_3(\lambda) = k[\beta_2\omega + \overline{\beta}_1\omega x_3(\lambda) - 1]\\ &\Leftrightarrow x_3(\lambda)\omega - \beta_1 - \overline{\beta}_2 x_3(\lambda) = k\beta_2\omega + k\overline{\beta}_1\omega x_3(\lambda) - k\\ &\Leftrightarrow x_3(\lambda)\omega - \beta_1 - \overline{\beta}_2 x_3(\lambda) - k\beta_2\omega - k\overline{\beta}_1\omega x_3(\lambda) + k = 0\\ &\Leftrightarrow x_3(\lambda)[\omega - \overline{\beta}_2 - k\overline{\beta}_1\omega] + [k - \beta_1 - k\beta_2\omega] = 0.\end{aligned}$$

Since $x_3$ is a nonconstant rational inner function, this implies that

$$\omega - \overline{\beta}_2 - k\overline{\beta}_1\omega = 0 \quad \text{and} \quad k - \beta_1 - k\beta_2\omega = 0.$$

Thus we get the system

$$\begin{cases}\omega - \overline{\beta}_2 - k\overline{\beta}_1\omega = 0\\ k - \beta_1 - k\beta_2\omega = 0.\end{cases}$$

Multiply both sides of the first equation by $\overline{\omega}$ and the second equation by $\overline{k}$. We get

$$\begin{cases}\overline{\beta_1}k + \overline{\beta_2}\overline{\omega} = 1\\ \beta_1\overline{k} + \beta_2\omega = 1.\end{cases} \tag{4.4.3}$$

From the equation $|\beta_1| + |\beta_2| = 1$ we deduce

$$\beta_1\frac{\overline{\beta_1}}{|\beta_1|} + \beta_2\frac{\overline{\beta_2}}{|\beta_2|} = 1. \tag{4.4.4}$$

It is easy to see that

$$\omega = \frac{\overline{\beta_2}}{|\beta_2|} \quad \text{and} \quad k = \frac{\beta_1}{|\beta_1|}$$

satisfy equation (4.4.3), and so

$$\Psi_\omega(x(\lambda)) = k \qquad \text{for all } \lambda \in \mathbb{D}.$$

□

**Lemma 4.4.6.** *For any inner function $x_3 : \mathbb{D} \to \overline{\mathbb{D}}$, there are $x_1, x_2 : \mathbb{D} \to \overline{\mathbb{D}}$ such that the function $x : \mathbb{D} \to \overline{\mathbb{E}}$ defined by $x = (x_1, x_2, x_3)$ is a superficial $\overline{\mathbb{E}}$-inner function, but $h = (x_1 + x_2, x_3) : \mathbb{D} \to \Gamma$ is not a superficial $\Gamma$-inner function.*

*Proof.* By Proposition 4.4.4, for any $\beta_1, \beta_2 \in \mathbb{C}$ such that $|\beta_1| + |\beta_2| = 1$, the function $x : \mathbb{D} \to \overline{\mathbb{E}}$ defined by

$$x = \Big(\beta_1 + \overline{\beta_2}x_3, \beta_2 + \overline{\beta_1}x_3, x_3\Big)$$

is a superficial $\overline{\mathbb{E}}$-inner function. By Proposition 4.4.2, $h : \mathbb{D} \to \Gamma$ is superficial if and only if there exists an $\omega \in \mathbb{T}$ such that $h = (\omega p + \overline{\omega}, p)$. Note that, for $x_1 = \beta_1 + \overline{\beta_2}x_3$ and $x_2 = \beta_2 + \overline{\beta_1}x_3$,

$$\begin{aligned} h(\lambda) = (x_1 + x_2, x_3)(\lambda) &= \Big(\beta_1 + \overline{\beta}_2 x_3(\lambda) + \beta_2 + \overline{\beta}_1 x_3(\lambda), x_3(\lambda)\Big) \\ &= \Big((\overline{\beta}_1 + \overline{\beta}_2)x_3(\lambda) + (\beta_1 + \beta_2), x_3(\lambda)\Big), \quad \lambda \in \mathbb{D}. \end{aligned}$$

One can see that there are some $\beta_1, \beta_2 \in \mathbb{C}$ with $|\beta_1| + |\beta_2| = 1$, but $\beta_1 + \beta_2 \notin \mathbb{T}$. For example, take

$$\beta_1 = i\frac{1}{2}, \quad |\beta_1| = \frac{1}{2},$$
$$\beta_2 = -i\frac{1}{2}, \quad |\beta_2| = \frac{1}{2}.$$

Then $|\beta_1| + |\beta_2| = \frac{1}{2} + \frac{1}{2} = 1$, but $\beta_1 + \beta_2 = i\frac{1}{2} - i\frac{1}{2} = 0 \notin \mathbb{T}$.
Thus, $h$ is not superficial for $\beta_1 = \frac{i}{2}$ and $\beta_2 = \frac{-i}{2}$. □

# Chapter 5

# The construction of rational $\overline{\mathbb{E}}$-inner functions

The formula for a Blaschke product is an explicit representation of a rational inner function in terms of its zeros. In this chapter we aim to find a comparable representation for rational $\overline{\mathbb{E}}$-inner function. The first question is: what is the tetrablock analogue of the zeros of an inner function? We shall show that one satisfactory choice consists of the royal nodes of an $\overline{\mathbb{E}}$-inner $x$ together with the zeros of $x_1$ and $x_2$. We construct a rational $\overline{\mathbb{E}}$-inner function $x$ from the royal nodes and the zeros of $x_1$ and $x_2$. We show that there exists a 3-parameter family of rational $\overline{\mathbb{E}}$-inner functions with prescribed zero sets of $x_1$, $x_2$ and prescribed royal nodes. We also prove that a nonconstant rational $\overline{\mathbb{E}}$-inner function $x$ of degree $n$ either maps $\overline{\mathbb{D}}$ to the royal variety of $\overline{\mathbb{E}}$ or $x(\overline{\mathbb{D}})$ meets the royal variety exactly $n$ times.

## 5.1 The royal polynomial of an $\overline{\mathbb{E}}$-inner function

We define the *royal variety* for $\overline{\mathbb{E}}$ to be

$$\mathcal{R}_{\overline{\mathbb{E}}} = \big\{(x_1, x_2, x_3) \in \overline{\mathbb{E}} : x_1 x_2 = x_3\big\}.$$

By Theorem 4.3.1, for a rational $\overline{\mathbb{E}}$-inner function $x = (x_1, x_2, x_3)$, there are polynomials $E_1, E_2, D$ such that

$$x_1 = \frac{E_1}{D}, \quad x_2 = \frac{E_2}{D}, \quad x_3 = \frac{D^{\sim n}}{D}.$$

Thus, for $\lambda \in \overline{\mathbb{D}}$,

$$\big(x_3 - x_1 x_2\big)(\lambda) = \left[\frac{D^{\sim n}}{D} - \frac{E_1}{D}\frac{E_2}{D}\right](\lambda).$$

The *royal polynomial* of the rational $\overline{\mathbb{E}}$-inner function $x$ is defined to be

$$\begin{aligned} R_x(\lambda) &= D^2(\lambda)\left[\frac{D^{\sim n}}{D} - \frac{E_1E_2}{D^2}\right](\lambda) \\ &= \left[D^{\sim n}D - E_1E_2\right](\lambda). \end{aligned}$$

**Definition 5.1.1.** [7, Definition 3.4] *We say a polynomial $f$ is $n$-*symmetric *if* $\deg(f) \le n$ *and* $f^{\sim n} = f$.

**Definition 5.1.2.** [7, Definition 3.4] *For any $E \subset \mathbb{C}$, the number of zeros of $f$ in $E$, counted with multiplicities, is denoted by* $\mathrm{ord}_E(f)$ *and* $\mathrm{ord}_0(f)$ *means the same as* $\mathrm{ord}_{\{0\}}(f)$.

**Proposition 5.1.3.** *Let $x$ be a rational $\overline{\mathbb{E}}$-inner function of degree $n$ and let $R_x$ be the royal polynomial of $x$. Then, for $\lambda \in \mathbb{T}$,*

(i) $\lambda^{-n}R_x(\lambda) = |D(\lambda)|^2 - |E_2(\lambda)|^2$, *and*

(ii) $\lambda^{-n}R_x(\lambda) = |D(\lambda)|^2 - |E_1(\lambda)|^2$.

*Proof.* (i) For $\lambda \in \mathbb{T}$,

$$\begin{aligned} \lambda^{-n}R_x(\lambda) &= \lambda^{-n}[D^{\sim n}D - E_1E_2](\lambda) \\ &= \lambda^{-n}[\lambda^n\overline{D(1/\overline{\lambda})}D(\lambda) - E_2^{\sim n}(\lambda)E_2(\lambda)], \quad \text{since } \ E_1(\lambda) = E_2^{\sim n}(\lambda) \text{ on } \mathbb{T} \\ &= \lambda^{-n}[\lambda^n\overline{D(\lambda)}D(\lambda) - \lambda^n\overline{E_2(1/\overline{\lambda})}E_2(\lambda)], \quad \text{since } \ E_2(1/\overline{\lambda}) = E_2(\lambda) \text{ on } \mathbb{T} \\ &= \lambda^{-n}[\lambda^n\overline{D(\lambda)}D(\lambda) - \lambda^n\overline{E_2(\lambda)}E_2(\lambda)] \\ &= |D(\lambda)|^2 - |E_2(\lambda)|^2. \end{aligned} \tag{5.1.1}$$

(ii) Since $x$ is rational $\overline{\mathbb{E}}$-inner function, by Lemma 4.3.2,

$$|E_1(\lambda)| = |E_2(\lambda)| \quad \text{for } \lambda \in \mathbb{T}. \tag{5.1.2}$$

By equations (5.1.1) and (5.1.2),

$$\lambda^{-n}R_x(\lambda) = |D(\lambda)|^2 - |E_1(\lambda)|^2 \quad \text{for } \lambda \in \mathbb{T}.$$

□

**Proposition 5.1.4.** *Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$-inner function, let $P = x_3 - x_1x_2$ and let $\sigma \in \mathbb{T}$ be a zero of $P$. Then $\sigma$ is a zero of $P$ of multiplicity at least 2.*

*Proof.* Suppose $\sigma \in \mathbb{T}$ is such that

$$P(\sigma) = x_3(\sigma) - x_1(\sigma)x_2(\sigma) = 0.$$

By Lemma 4.1.3 (i), $x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda)$, for $\lambda \in \mathbb{T}$. Hence on $\mathbb{T}$ we have

$$\begin{aligned} \overline{x_3(\lambda)}\big(x_3(\lambda) - x_1(\lambda)x_2(\lambda)\big) &= |x_3(\lambda)|^2 - \overline{x_3(\lambda)}x_1(\lambda)x_2(\lambda) \\ &= 1 - \overline{x_3(\lambda)}\big[\overline{x_2(\lambda)}x_3(\lambda)\big]x_2(\lambda) \\ &= 1 - |x_3(\lambda)|^2|x_2(\lambda)|^2, \quad \text{since } |x_2(\lambda)| \le 1 \text{ on } \mathbb{T}, \\ &= 1 - |x_2(\lambda)|^2 \ge 0. \end{aligned}$$

At $\sigma = e^{i\xi}$, the function $f(e^{i\xi}) = 1 - |x_2(\lambda)|^2$ has a local minimum. Therefore

$$\begin{aligned} 0 &= \frac{d}{d\theta}\Big(1 - |x_2(e^{i\theta})|^2\Big)_{|\xi} \\ &= \frac{d}{d\theta}\overline{x_3}\big(x_3 - x_1x_2\big)(e^{i\theta})_{|\xi} \\ &= \frac{d}{d\theta}\big(x_3 - x_1x_2\big)(e^{i\theta})_{|\xi} \cdot \overline{x_3}(e^{i\xi}) + \frac{d}{d\theta}\overline{x}_3(e^{i\theta})_{|\xi}\underbrace{\big(x_3 - x_1x_2\big)\big(e^{i\xi}\big)}_{=0} \\ &= \underbrace{\big(x_3' - x_1'x_2 - x_2'x_1\big)\big(e^{i\xi}\big)}_{=0}\underbrace{ie^{i\xi}}_{\neq 0}\underbrace{\overline{x}_3\big(e^{i\xi}\big)}_{\neq 0}. \end{aligned}$$

Since $ie^{i\xi} \neq 0$ and $\overline{x}_3\big(e^{i\xi}\big) \neq 0$,

$$\big(x_3' - x_1'x_2 - x_2'x_1\big)\big(e^{i\xi}\big) = 0,$$

and so $P'(\sigma) = 0$. We have $P(\sigma) = 0$ and $P'(\sigma) = 0$. Therefore $\sigma$ is a zero of $x$ of multiplicity at least 2. □

**Lemma 5.1.5.** *Let $E_1$ and $E_2$ be two polynomials such that $\deg E_1, \deg E_2 \le n$. Then*

$$E_1(\lambda) = E_2^{\sim n}(\lambda) \text{ for all } \lambda \in \mathbb{D} \quad \text{if and only if} \quad E_2(\lambda) = E_1^{\sim n}(\lambda) \text{ for all } \lambda \in \mathbb{D}.$$

*Proof.* Suppose that $E_1(\lambda) = E_2^{\sim n}(\lambda)$ for all $\lambda \in \mathbb{D}$. Then by definition,

$$E_1(\lambda) = E_2^{\sim n}(\lambda) = \lambda^n \overline{E_2(1/\overline{\lambda})}, \qquad \lambda \in \mathbb{D}.$$

Therefore, for all $\lambda \in \mathbb{D}$,

$$\begin{aligned} E_1(\lambda) = \lambda^n \overline{E_2(1/\overline{\lambda})} \text{ for all } \lambda \in \mathbb{D} &\Leftrightarrow (1/\lambda^n)E_1(\lambda) = \overline{E_2(1/\overline{\lambda})} \text{ for all } \lambda \in \mathbb{D} \\ &\Leftrightarrow (1/\overline{\lambda})^n\overline{E_1(\lambda)} = E_2(1/\overline{\lambda}) \text{ for all } \lambda \in \mathbb{D} \\ &\Leftrightarrow \lambda^n\overline{E_1(1/\overline{\lambda})} = E_2(\lambda) \text{ for all } \lambda \in \mathbb{D} \\ &\Leftrightarrow E_2(\lambda) = E_1^{\sim n}(\lambda) \text{ for all } \lambda \in \mathbb{D}. \end{aligned}$$

The converse is obvious. □

**Definition 5.1.6.** *A nonzero polynomial $R$ is $n$-*balanced *if*

(1) $\deg(R) \le 2n$,

(2) *$R$ is $2n$-symmetric, and*

(3) $\lambda^{-n}R(\lambda) \ge 0$ *for all* $\lambda \in \mathbb{T}$.

For completeness, we shall say that zeros of the zero polynomial have infinite order.

**Proposition 5.1.7.** *Let $x$ be a rational $\overline{\mathbb{E}}$-inner function of degree $n$ and let $R_x$ be the royal polynomial of $x$. Then $R_x$ is $2n$-symmetric, $\lambda^{-n}R_x(\lambda) \ge 0$ for all $\lambda \in \mathbb{T}$, and the zeros of $R_x$ on $\mathbb{T}$ have even order or infinite order.*

*Proof.* To show that $R_x$ is $2n$-symmetric we have to prove that $R_x^{\sim 2n}(\lambda) = R_x(\lambda)$, for $\lambda \in \mathbb{D}$. Recall that

$$R_x(\lambda) = D^{\sim n}(\lambda)D(\lambda) - E_1(\lambda)E_2(\lambda), \quad \text{where } x = \left(\frac{E_1}{D}, \frac{E_2}{D}, \frac{D^{\sim n}}{D}\right),$$

Recall also that, by Theorem 4.3.1 (vii) and Lemma 5.1.5, for $\lambda \in \mathbb{D}$,

$$E_1(\lambda) = E_2^{\sim n}(\lambda) = \lambda^n \overline{E_2(1/\overline{\lambda})}, \qquad E_2(\lambda) = E_1^{\sim n}(\lambda) = \lambda^n \overline{E_1(1/\overline{\lambda})}.$$

Now

$$\begin{aligned} R_x^{\sim 2n}(\lambda) &= \lambda^n \overline{D^{\sim n}(1\backslash\overline{\lambda})}\ \lambda^n \overline{D^{\sim n}(1\backslash\overline{\lambda})} - \lambda^n \overline{E_1(1/\overline{\lambda})}\ \lambda^n \overline{E_2(1/\overline{\lambda})} \\ &= D(\lambda)D^{\sim n}(\lambda) - E_2(\lambda)E_1(\lambda) = R_x(\lambda). \end{aligned}$$

Hence $R_x$ is $2n$-symmetric.

Clearly, if $x(\overline{\mathbb{D}}) \subseteq \mathcal{R}_{\overline{\mathbb{E}}}$, the royal polynomial $R_x$ is identically zero. Hence the zeros of $R_x$ on $\mathbb{T}$ have infinite order.
In the case $x(\overline{\mathbb{D}}) \nsubseteq \mathcal{R}_{\overline{\mathbb{E}}}$, by Proposition 5.1.3, for $\lambda \in \mathbb{T}$,

$$\lambda^{-n}R_x(\lambda) = |D(\lambda)|^2 - |E_2(\lambda)|^2. \tag{5.1.3}$$

By Theorem 4.3.1 (vi),

$$|D|^2 - |E_2|^2 \ge 0 \ \text{ on } \mathbb{T}. \tag{5.1.4}$$

By equations (5.1.3) and (5.1.4), $\lambda^{-n}R_x(\lambda) \ge 0$ on $\mathbb{T}$. By the Fejér-Riesz theorem, there exists an analytic polynomial $P(\lambda) = \sum_{i=0}^{n} b_i\lambda$ of degree $n$ such that $P$ is outer and

$$\lambda^{-n}R_x(\lambda) = |P(\lambda)|^2 \qquad \text{for all } \lambda \in \mathbb{T}.$$

Hence if $\sigma \in \mathbb{T}$ is a zero of $R_x$, then $\sigma$ is a zero of even order. Therefore in the case $x(\overline{\mathbb{D}}) \nsubseteq \mathcal{R}_{\overline{\mathbb{E}}}$, the zeros of $R_x$ that lie in $\mathbb{T}$ have even order. □

**Lemma 5.1.8.** *Let $x$ be a rational $\overline{\mathbb{E}}$-inner function of degree $n$. Then the royal polynomial $R_x$ of $x$ is either $n$-balanced or identically zero.*

*Proof.* If $x(\overline{\mathbb{D}}) \subset \mathcal{R}_{\overline{\mathbb{E}}}$ then, by the definition of the royal variety,

$$x_1(\lambda)x_2(\lambda) = x_3(\lambda) \qquad \text{for all } \lambda \in \overline{\mathbb{D}}.$$

Thus

$$D(\lambda)D^{\sim n}(\lambda) = E_1(\lambda)E_2(\lambda) \qquad \text{for all } \lambda \in \overline{\mathbb{D}}.$$

Therefore the royal polynomial $R_x$ is identically zero.

If $x(\overline{\mathbb{D}}) \nsubseteq \mathcal{R}_{\overline{\mathbb{E}}}$ then, by Proposition 5.1.7, the royal polynomial $R_x$ of $x$ is $2n$-symmetric and $\lambda^{-n}R_x(\lambda) \geq 0$ for all $\lambda \in \mathbb{T}$. Clearly, $R_x$ has degree less than or equal to $2n$. Hence $R_x$ is $n$-balanced. □

## 5.2 Rational $\overline{\mathbb{E}}$-inner functions of type $(n,k)$

**Definition 5.2.1.** *Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$-inner function such that $x(\overline{\mathbb{D}}) \nsubseteq \mathcal{R}_{\overline{\mathbb{E}}}$. Let $R_x$ be the royal polynomial of $x$. If $\sigma$ is a zero of $R_x$ of order $\ell$, we define the multiplicity $\#\sigma$ of $\sigma$ (as a royal node of $x$) by*

$$\#\sigma = \begin{cases} \ell & \text{if } \sigma \in \mathbb{D}, \\ \frac{1}{2}\ell & \text{if } \sigma \in \mathbb{T}. \end{cases}$$

*We define the type of $x$ to be the ordered pair $(n,k)$, where $n$ is the sum of the multiplicities of the royal nodes of $x$ that lie in $\overline{\mathbb{D}}$, and $k$ is the sum of the multiplicities of the royal nodes of $x$ that lie in $\mathbb{T}$.*

**Definition 5.2.2.** *Let $\mathcal{R}^{n,k}$ denote the collection of rational $\overline{\mathbb{E}}$-inner functions of type $(n,k)$.*

**Remark 5.2.3.** [7, Equations (3.2) and (3.3)] *For any $m$-symmetric polynomial $f$, the following two relations hold*

(1)

$$\deg(f) = m - \operatorname{ord}_0(f).$$

(2) *Since $f$ is $m$-symmetric, if $\alpha \in \mathbb{D}\backslash\{0\}$ is zero of $f$, then $\frac{1}{\bar{\alpha}}$ is also a zero of $f$. Thus*

$$\operatorname{ord}_0(f) + 2\operatorname{ord}_{\mathbb{D}\backslash\{0\}}(f) + \operatorname{ord}_{\mathbb{T}}(f) = \deg(f).$$

**Theorem 5.2.4.** *If $x \in \mathcal{R}^{n,k}$ is nonconstant then the degree of $x$ is equal to $n$.*

*Proof.* Let $R_x$ be the royal polynomial of $x$. By assumption $x \in \mathcal{R}^{n,k}$ and is nonconstant. Hence $n \geq 1$ and $x(\overline{\mathbb{D}}) \nsubseteq \mathcal{R}_{\overline{\mathbb{E}}}$. Thus $R_x$ is not identically zero. By Proposition 5.1.7, $R_x$ is $2\deg(x)$-symmetric. By Remark 5.2.3 (1) and (2), it follows that

$$\deg(R_x) = 2\deg(x) - \mathrm{ord}_0(R_x)$$

and

$$\mathrm{ord}_0(R_x) + 2\mathrm{ord}_{\mathbb{D}\setminus\{0\}}(R_x) + \mathrm{ord}_{\mathbb{T}}(R_x) = \deg(R_x).$$

Substitute the first equation in the second equation,

$$\mathrm{ord}_0(R_x) + 2\mathrm{ord}_{\mathbb{D}\setminus\{0\}}(R_x) + \mathrm{ord}_{\mathbb{T}}(R_x) = 2\deg(x) - \mathrm{ord}_0(R_x),$$

which implies that

$$2\mathrm{ord}_0(R_x) + 2\mathrm{ord}_{\mathbb{D}\setminus\{0\}}(R_x) + \mathrm{ord}_{\mathbb{T}}(R_x) = 2\deg(x).$$

Therefore, by Definition 5.2.2,

$$n = \mathrm{ord}_0(R_x) + \mathrm{ord}_{\mathbb{D}\setminus\{0\}}(R_x) + \frac{1}{2}\mathrm{ord}_{\mathbb{T}}(R_x) = \deg(x).$$

□

**Theorem 5.2.5.** *If $x$ is a nonconstant rational $\overline{\mathbb{E}}$-inner function, then either $x(\overline{\mathbb{D}}) = \mathcal{R}_{\overline{\mathbb{E}}}$ or $x(\overline{\mathbb{D}})$ meets $\mathcal{R}_{\overline{\mathbb{E}}}$ exactly* $\deg(x)$ *times.*

*Proof.* Suppose that $x$ is a nonconstant rational $\overline{\mathbb{E}}$-inner function. Then either, $x(\overline{\mathbb{D}}) \subseteq \mathcal{R}_{\overline{\mathbb{E}}}$ and the royal polynomial $R_x$ of $x$ is identically zero, or by Theorem 5.2.4, $x(\overline{\mathbb{D}})$ meets $\mathcal{R}_{\overline{\mathbb{E}}}$ exactly $\deg(x)$ times. □

**Lemma 5.2.6.** [7, Lemma 4.4] *For $\sigma \in \overline{\mathbb{D}}$, let the polynomial $Q_\sigma$ be defined by the formula*

$$Q_\sigma(\lambda) = (\lambda - \sigma)(1 - \overline{\sigma}\lambda).$$

*Let $n$ be a positive integer and let $R$ be a nonzero polynomial. The polynomial $R$ is $n$-balanced if and only if there exist points $\sigma_1, \sigma_2, ..., \sigma_n \in \overline{\mathbb{D}}$ and $t_+ > 0$ such that*

$$R(\lambda) = t_+ \prod_{j=1}^{n} Q_{\sigma_j}(\lambda), \quad \lambda \in \mathbb{C}.$$

**Proposition 5.2.7.** *Let the royal nodes of a rational $\overline{\mathbb{E}}$-inner function $x$ be $\sigma_1, ..., \sigma_n$, with repetition according to multiplicity of the royal nodes as described in Definition* 5.2.1*. The royal polynomial $R_x$ of $x$, up to a positive multiple, is*

$$R_x(\lambda) = \prod_{j=1}^{n} Q_{\sigma_j}(\lambda). \tag{5.2.1}$$

*Proof.* By Lemma 5.1.8, $R_x$ is $n$-balanced. This implies that, by Lemma 5.2.6, there exists $t_+ > 0$ and $\eta_1, \dots, \eta_n \in \overline{\mathbb{D}}$ such that

$$R_x(\lambda) = t_+ \prod_{j=1}^{n} Q_{\eta_j}(\lambda).$$

By Definition 5.2.1, the royal nodes of $x$ and their multiplicities are defined in terms of zeros of $R_x$ in $\overline{\mathbb{D}}$ and their multiplicities. Hence the list $\eta_1, \dots, \eta_n$ coincides, up to a permutation, with the list $\sigma_1, \dots, \sigma_n$. Therefore $R_x$ is given, up to a positive multiple, by equation (5.2.1). □

Before we proceed to the next theorem about constructing a tetra-inner function $x$ from the zeros of $x_1$ and $x_2$, let us prove the following elementary lemma.

**Lemma 5.2.8.** *Let $E_1$ and $E_2$ be polynomials of degree at most $n$ such that $E_1(\lambda) = E_2^{\sim n}(\lambda)$, for $\lambda \in \overline{\mathbb{D}}$. Let $\alpha_1^1, ..., \alpha_{k_1}^1$ be the zeros of $E_1$ in $\overline{\mathbb{D}}$, $\alpha_1^2, ..., \alpha_{k_2}^2$ be the zeros of $E_2$ in $\overline{\mathbb{D}}$, where $k_1 + k_2 = n$. Then*

$$E_1(\lambda) = t \prod_{j=1}^{k_1} (\lambda - \alpha_j^1) \prod_{j=1}^{k_2} (1 - \overline{\alpha}_j^2 \lambda),$$

*where $t \in \mathbb{C} \backslash \{0\}$.*

*Proof.* Since $\alpha_1^1, ..., \alpha_{k_1}^1 \in \overline{\mathbb{D}}$ and $\alpha_1^2, ..., \alpha_{k_2}^2 \in \overline{\mathbb{D}}$, where $k_1 + k_2 = n$, are zeros of $E_1$ and $E_2$ respectively, we have

$$E_1(\lambda) = (\lambda - \alpha_1^1)...(\lambda - \alpha_{k_1}^1).p_1(\lambda) \tag{5.2.2}$$

and

$$E_2(\lambda) = (\lambda - \alpha_1^2)...(\lambda - \alpha_{k_2}^2).p_2(\lambda).$$

where the polynomials $p_1(\lambda)$ and $p_2(\lambda)$ do not vanish in $\overline{\mathbb{D}}$.
Since $E_1(\lambda) = \lambda^n \overline{E_2(1/\overline{\lambda})}$ on $\overline{\mathbb{D}}$, we have

$$\begin{aligned} E_1(\lambda) &= \lambda^n \overline{(1/\overline{\lambda} - \alpha_1^2)}...\overline{(1/\overline{\lambda} - \alpha_{k_2}^2)}.\overline{p_2(1/\overline{\lambda})} \\ &= \lambda^{n-k_2} (1 - \overline{\alpha}_1^2 \lambda)...(1 - \overline{\alpha}_{k_2}^2 \lambda).\overline{p_2(1/\overline{\lambda})}. \end{aligned} \tag{5.2.3}$$

Since $\deg E_1 \leq n$ and $k_1 + k_2 = n$, equations (5.2.2) and (5.2.3) implies that $E_1$ can be written in the form

$$\begin{aligned} E_1(\lambda) &= t_1(\lambda-\alpha_1^1)...(\lambda-\alpha_{k_1}^1)\ \ (1-\overline{\alpha}_1^2\lambda)...(1-\overline{\alpha}_{k_2}^2\lambda) \\ &= t_1\prod_{j=1}^{k_1}(\lambda-\alpha_j^1)\prod_{j=1}^{k_2}(1-\overline{\alpha}_j^2\lambda), \qquad \lambda\in\overline{\mathbb{D}}, \end{aligned}$$

for some $t_1 \in \mathbb{C}\backslash\{0\}$, and

$$E_2(\lambda) = t_2\prod_{j=1}^{k_2}(\lambda-\alpha_j^2)\prod_{j=1}^{k_1}(1-\overline{\alpha}_j^1\lambda) \quad \lambda\in\overline{\mathbb{D}},$$

for some $t_2 \in \mathbb{C}\backslash\{0\}$. Since $E_2(\lambda) = \lambda^n\overline{E_1(1/\overline{\lambda})}$,

$$\begin{aligned} \lambda^n\overline{E_1(1/\overline{\lambda})} &= \lambda^n\overline{\left(t\prod_{j=1}^{k_1}(1/\overline{\lambda}-\alpha_j^1)\prod_{j=1}^{k_2}(1-\overline{\alpha}_j^2 1/\overline{\lambda})\right)} \\ &= \lambda^n\overline{t_1}\overline{\prod_{j=1}^{k_1}(1/\overline{\lambda}-\alpha_j^1)\prod_{j=1}^{k_2}(1-\overline{\alpha}_j^2 1/\overline{\lambda})} \\ &= \lambda^n\overline{t_1}\prod_{j=1}^{k_1}(1/\lambda-\overline{\alpha}_j^1)\prod_{j=1}^{k_2}(1-\alpha_j^2 1/\lambda) \\ &= \overline{t_1}\prod_{j=1}^{k_1}(1-\overline{\alpha}_j^1\lambda)\prod_{j=1}^{k_2}(\lambda-\alpha_j^2) \\ &= E_2(\lambda) = t_2\prod_{j=1}^{k_2}(\lambda-\alpha_j^2)\prod_{j=1}^{k_1}(1-\overline{\alpha}_j^1\lambda), \qquad \lambda\in\overline{\mathbb{D}}, \end{aligned}$$

and so $t_2 = \overline{t}_1$. □

**Remark 5.2.9.** *For the polynomials $E_1$ and $E_2$ from Lemma 5.2.8, if $\alpha \in \mathbb{D}\backslash\{0\}$ and $\alpha$ is a zero of $E_1$ then $\frac{1}{\overline{\alpha}}$ is a zero of $E_2$.*

**Theorem 5.2.10.** *Suppose that $\alpha_1^1,...,\alpha_{k_1}^1 \in \overline{\mathbb{D}}$ and $\alpha_1^2,...,\alpha_{k_2}^2 \in \overline{\mathbb{D}}$, where $k_1+k_2=n$. Suppose that $\sigma_1,...,\sigma_n \in \overline{\mathbb{D}}$ are distinct from points of the set $\{\alpha_j^i, j=1,...,k_i, i=1,2\}\cap\mathbb{T}$. Then there exists a rational $\overline{\mathbb{E}}$-inner function $x=(x_1,x_2,x_3):\mathbb{D}\to\overline{\mathbb{E}}$ such that*

(1) *the zeros of $x_1$ in $\overline{\mathbb{D}}$, repeated according to multiplicity, are $\alpha_1^1,...,\alpha_{k_1}^1$;*

(2) *the zeros of $x_2$ in $\overline{\mathbb{D}}$, repeated according to multiplicity, are $\alpha_1^2,...,\alpha_{k_2}^2$;*

(3) *the royal nodes of* $x$ *are* $\sigma_1, ..., \sigma_n \in \overline{\mathbb{D}}$, *with repetition according to multiplicity of the nodes.*

*Such a function* $x$ *can be constructed as follows. Let* $t_+ > 0$ *and let* $t \in \mathbb{C}\backslash\{0\}$. *Let* $R$ *be defined by*

$$R(\lambda) = t_+ \prod_{j=1}^{n} (\lambda - \sigma_j)(1 - \overline{\sigma}_j \lambda), \quad \textit{and}$$

*let* $E_1$ *be defined by*

$$E_1(\lambda) = t \prod_{j=1}^{k_1} (\lambda - \alpha_j^1) \prod_{j=1}^{k_2} (1 - \overline{\alpha}_j^2 \lambda).$$

*Then* (i) *and* (ii) *hold.*

(i) *There exists an outer polynomial* $D$ *of degree at most* $n$ *such that*

$$\lambda^{-n} R(\lambda) + |E_1(\lambda)|^2 = |D(\lambda)|^2 \tag{5.2.4}$$

*for all* $\lambda \in \mathbb{T}$.

(ii) *The function* $x$ *defined by*

$$x = (x_1, x_2, x_3) = \left( \frac{E_1}{D}, \frac{E_1^{\sim n}}{D}, \frac{D^{\sim n}}{D} \right)$$

*is a rational* $\overline{\mathbb{E}}$*-inner function such that the degree of* $x$ *is equal to* $n$ *and conditions* (1), (2) *and* (3) *hold. The royal polynomial of* $x$ *is* $R$.

*Proof.* (i) By Lemma 5.2.6, $R$ is $n$-balanced, and so $\lambda^{-n} R(\lambda) \geq 0$ for all $\lambda \in \mathbb{T}$. Therefore

$$\lambda^{-n} R(\lambda) + |E_1(\lambda)|^2 \geq 0 \quad \text{for all } \lambda \in \mathbb{T}.$$

By Fejér-Riesz theorem, there exists an outer polynomial $D$ of degree at most $n$ such that

$$\lambda^{-n} R(\lambda) + |E_1(\lambda)|^2 = |D(\lambda)|^2 \quad \text{for all } \lambda \in \mathbb{T}. \tag{5.2.5}$$

(ii) Let $D$ be an outer polynomial of degree at most $n$ such that equality (5.2.5) holds for all $\lambda \in \mathbb{T}$. By hypothesis

$$\{\sigma_j : 1 \leq j \leq n\} \cap \left( \{\alpha_j^i : 1 \leq j \leq k_i, i = 1, 2\} \cap \mathbb{T} \right) = \emptyset.$$

Then $\lambda^{-n} R(\lambda)$ and $|E_1(\lambda)|^2$ are non-negative trigonometric polynomials on $\mathbb{T}$ with no common zero. Thus

$$\lambda^{-n} R(\lambda) + |E_1(\lambda)|^2 > 0 \quad \text{on } \mathbb{T}.$$

By equality (5.2.5), $D$ has no zero on $\mathbb{T}$, and so $D$ and $D^{\sim n}$ have no common factor. Hence

$$\deg(x_3) = \deg\left(\frac{D^{\sim n}}{D}\right) = \max\{\deg(D), \deg(D^{\sim n})\} = n.$$

Since

$$\lambda^{-n}R(\lambda) \geq 0 \quad \text{for all } \lambda \in \mathbb{T},$$

$$|D(\lambda)|^2 = \lambda^{-n}R(\lambda) + |E_1(\lambda)|^2 \geq |E_1(\lambda)|^2$$

for all $\lambda \in \mathbb{T}$. It follows that

$$|D(\lambda)| \geq |E_1(\lambda)|, \qquad \text{for all } \lambda \in \mathbb{T}.$$

Since $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$, the function $\frac{E_1}{D}$ is analytic in a neighbourhood of $\overline{\mathbb{D}}$. By the Maximum Modulus Principle, we have

$$\frac{|E_1(\lambda)|}{|D(\lambda)|} \leq 1 \quad \text{for all } \lambda \in \overline{\mathbb{D}}.$$

Therefore, by the converse of Theorem 4.3.1, since conditions (i),(ii), (vi) and (vii) are satisfied, the function

$$x(\lambda) = \left(\frac{E_1(\lambda)}{D(\lambda)}, \frac{E_1^{\sim n}(\lambda)}{D(\lambda)}, \frac{D^{\sim n}(\lambda)}{D(\lambda)}\right) \quad \text{for } \lambda \in \mathbb{D},$$

is a rational $\overline{\mathbb{E}}$-inner function such that $\deg(x) = n$. The royal polynomial of $x$ is defined by

$$R_x(\lambda) = D(\lambda)D^{\sim n}(\lambda) - E_1(\lambda)E_2(\lambda), \quad \lambda \in \mathbb{D},$$

where $E_2(\lambda) = E_1^{\sim n}(\lambda), \lambda \in \mathbb{D}$. By Proposition 5.1.3, for all $\lambda \in \mathbb{T}$,

$$\lambda^{-n}R_x(\lambda) \;\; = \;\; |D(\lambda)|^2 - |E_1(\lambda)|^2.$$

Therefore, by equation (5.2.4),

$$\lambda^{-n}R_x(\lambda) = \lambda^{-n}R(\lambda) \qquad \text{for all } \; \lambda \in \mathbb{T},$$

where $E_2(\lambda) = E_1^{\sim n}(\lambda)$ for $\lambda \in \mathbb{D}$. Thus $R_x = R$, that is, the royal polynomial of $x$ is equal to $R$. □

**Remark 5.2.11.** (1) *For large $n$ the task of finding an outer polynomial $D$ satisfying equation* (5.2.4) *cannot be solved algebraically.*

(2) *The solution $D$ is only identified up to a multiplication by $\overline{\omega} \in \mathbb{T}$. Thus if we replace $D$ by $\overline{\omega}D$ we obtain a new solution*

$$x = \left(\omega \frac{E_1}{D}, \omega \frac{E_1^{\sim n}}{D}, \omega^2 \frac{D^{\sim n}}{D}\right).$$

**Example 5.2.12.** Let $n = 1$, $\alpha_1^2 = \frac{1}{2}$ and $\sigma_1 = 0$. Let us construct a rational $\overline{\mathbb{E}}$-inner function $x = (x_1, x_2, x_3) : \mathbb{D} \to \overline{\mathbb{E}}$ such that $\alpha_1^2$ is a zero of $x_2$ and $\sigma_1$ is a royal node of $x$.

As in Theorem 5.2.10, for $\lambda \in \mathbb{T}$, let

$$\begin{aligned} R(\lambda) &= t_+\lambda, && t_+ \text{ is a positive real number.} \\ E_1(\lambda) &= t(1 - \tfrac{1}{2}\lambda), && t \in \mathbb{C}\backslash\{0\}. \end{aligned}$$

The equation (5.2.4) for the polynomial $D$ is the following, for all $\lambda \in \mathbb{T}$,

$$\begin{aligned} |D(\lambda)|^2 &= \lambda^{-1}R(\lambda) + |E_1(\lambda)|^2 \\ &= \overline{\lambda}t_+\lambda + |t(1 - \tfrac{1}{2}\lambda)|^2 \\ &= t_+ + |t(1 - \tfrac{1}{2}\lambda)|^2 \\ &= t_+ + |t|^2(1 - \tfrac{1}{2}\lambda)(1 - \tfrac{1}{2}\overline{\lambda}) \\ &= t_+ + |t|^2(1 + \tfrac{1}{4} - \tfrac{1}{2}\lambda - \tfrac{1}{2}\overline{\lambda}) \\ &= t_+ + \tfrac{5}{4}|t|^2 - \tfrac{|t|^2}{2}\lambda - \tfrac{|t|^2}{2}\overline{\lambda}. \end{aligned} \tag{5.2.6}$$

Since the degree of $D$ is at most 1, $D(\lambda) = a_1 + a_2\lambda$, where $a_1, a_2 \in \mathbb{C}$ and $\lambda \in \mathbb{T}$,

$$\begin{aligned} D(\lambda)\overline{D(\lambda)} &= |a_1 + a_2\lambda|^2 \\ &= (a_1 + a_2\lambda)(\overline{a}_1 + \overline{a}_2\overline{\lambda}) = |a_1|^2 + |a_2|^2 + a_1\overline{a}_2\overline{\lambda} + \overline{a}_1 a_2\lambda. \end{aligned} \tag{5.2.7}$$

Compare equations (5.2.6) and (5.2.7). We have

$$\begin{cases} \overline{a}_1 a_2 = -\dfrac{|t|^2}{2}, \\ a_1\overline{a}_2 = -\dfrac{|t|^2}{2}, \\ |a_1|^2 + |a_2|^2 = t_+ + \frac{5}{4}|t|^2. \end{cases} \tag{5.2.8}$$

Finally the function $x$ can be written in the form

$$x(\lambda) = \left(\frac{E_1}{D}, \frac{E_1^{\sim 1}}{D}, \frac{D^{\sim 1}}{D}\right)$$

where

$$\begin{cases} x_1(\lambda) = \dfrac{E_1}{D}(\lambda) = \dfrac{t(1-\frac{1}{2}\lambda)}{a_1 + a_2\lambda}, \\ x_2(\lambda) = \dfrac{E_1^{\sim 1}}{D}(\lambda) = \dfrac{\bar{t}(\lambda - \frac{1}{2})}{a_1 + a_2\lambda}, \\ x_3(\lambda) = \dfrac{D^{\sim 1}}{D}(\lambda) = \dfrac{\bar{a}_1\lambda + \bar{a}_2}{a_1 + a_2\lambda}, \end{cases}$$

where $|a_2| < |a_1|$ and $a_1$, $a_2$ are given by solving equations (5.2.8) as functions of $t_+$ and $t$. These formulas gives a parametrization of solutions for the above problem.

For example, for the given $t = \sqrt{2}$ and $t_+ = \dfrac{7}{4}$, the system

$$\begin{cases} \bar{a}_1 a_2 = -1, \\ a_1 \bar{a}_2 = -1, \\ |a_1|^2 + |a_2|^2 = \dfrac{7}{4} + \dfrac{10}{4} = \dfrac{17}{4}. \end{cases}$$

has a solution

$$a_1 = -2i, a_2 = \tfrac{1}{2}i \quad \text{or} \quad a_1 = -2, a_2 = \tfrac{1}{2}.$$

Thus, for $\omega \in \mathbb{T}$,

$$D(\lambda) = \omega(-2 + \tfrac{1}{2}\lambda) = 2\omega(-1 + \tfrac{1}{4}\lambda) = -2\omega(1 - \tfrac{1}{4}\lambda).$$

Therefore

$$D^{\sim 1}(\lambda) = \lambda\overline{D(1/\overline{\lambda})} = -\lambda\big(2\overline{\omega}(1 - \tfrac{1}{4}\overline{\lambda})\big) = (-2\overline{\omega})(\lambda - \tfrac{1}{4}).$$

Hence the functions $x = (x_1, x_2, x_3) : \mathbb{D} \to \overline{\mathbb{E}}$, where $\omega \in \mathbb{T}$ and

$$\begin{cases} x_1(\lambda) = \dfrac{\sqrt{2}(1 - \frac{1}{2}\lambda)}{-2\omega(1 - \frac{1}{4}\lambda)}, \\ x_2(\lambda) = \dfrac{\sqrt{2}(\lambda - \frac{1}{2})}{-2\omega(1 - \frac{1}{4}\lambda)}, \\ x_3(\lambda) = \dfrac{\overline{\omega}(\lambda - \frac{1}{4})}{\omega(1 - \frac{1}{4}\lambda)}, \end{cases}$$

are rational $\overline{\mathbb{E}}$-inner functions such that $\dfrac{1}{2}$ is a zero of $x_2$ and $0$ is a royal node of $x$.

**Remark 5.2.13.** *Theorem* 5.2.10 *gives a* 3*-parameter family of rational* $\overline{\mathbb{E}}$*-inner functions with prescribed royal nodes and prescribed zeros of* $x_1$ *and* $x_2$*. It appears at first sight* *that the construction in Theorem* 5.2.10 *gives us a* 4*-parameter family of rational* $\overline{\mathbb{E}}$*-inner functions with the given data. However, the choice of* $t_+, t, D$ *and* $\omega$ *leads to the same* $x$ *as the choice* $1, t/\sqrt{t_+}, D/\sqrt{t_+}$ *and* $\omega$*. Theorem* 5.2.14 *tells us that the construction yields all solutions of the problem, and so the family of functions* $x$ *with the required properties is indeed a* 3*-parameter family.*

**Theorem 5.2.14.** *Let* $x = (x_1, x_2, x_3)$ *be a rational* $\overline{\mathbb{E}}$*-inner function of degree* $n$ *such that*

(1) *the zeros of* $x_1$*, repeated according to multiplicity, are* $\alpha_1^1, \ldots \alpha_{k_1}^1 \in \overline{\mathbb{D}}$*,*

(2) *the zeros of* $x_2$*, repeated according to multiplicity, are* $\alpha_1^2, \ldots \alpha_{k_2}^2 \in \overline{\mathbb{D}}$*, where* $k_1 + k_2 = n$*, and*

(3) *the royal nodes of* $x$ *are* $\sigma_1, \ldots, \sigma_n \in \overline{\mathbb{D}}$*, repeated according to multiplicity.*

*There exists some choice of* $t_+ > 0$*,* $t \in \mathbb{C}\backslash\{0\}$ *and* $w \in \mathbb{T}$ *such that the recipe in Theorem* 5.2.10 *with these choices produces the function* $x$*.*

*Proof.* By Theorem 4.3.1, there exist polynomials $E_1^1, E_2^1$ and $D^1$ such that

(1) $\deg(E_1^1), \deg(E_2^1)$ and $\deg(D^1) \leq n$,

(2) $D^1(\lambda) \neq 0 \quad$ on $\overline{\mathbb{D}}$,

(3) $E_2^1(\lambda) = (E_1^1)^{\sim n}(\lambda)$

(4) $|E_i^1(\lambda)| \leq |D^1(\lambda)|$ on $\overline{\mathbb{D}}, i = 1, 2$ and

(5) $x_1 = \dfrac{E_1^1}{D^1}, \quad x_2 = \dfrac{E_2^1}{D^1} \quad$ and $\quad x_3 = \dfrac{(D^1)^{\sim n}}{D^1} \quad$ on $\overline{\mathbb{D}}$.

By hypothesis, the zeros of $x_1$, repeated according to multiplicity, are $\alpha_1^1, ..., \alpha_{k_1}^1$, and the zeros of $x_2$, repeated according to multiplicity, are $\alpha_1^2, ..., \alpha_{k_2}^2$ where $k_1 + k_2 = n$.
By Lemma 5.2.8,

$$E_1^1(\lambda) = t \prod_{j=1}^{k_1} (\lambda - \alpha_j^1) \prod_{j=1}^{k_2} (1 - \overline{\alpha}_j^2 \lambda) \quad \text{for some } t \in \mathbb{C}\backslash\{0\} \text{ and all } \lambda \in \mathbb{D}.$$

By hypothesis, $\sigma_1, \ldots, \sigma_n$ are the royal nodes of $x$. Thus, by Proposition 5.2.7, for the royal polynomial $R^1$ of $x$, there exists $t_+ > 0$ such that

$$R^1(\lambda) = t_+ \prod_{j=1}^{n} (\lambda - \sigma_j)(1 - \overline{\sigma}_j \lambda).$$

By Proposition 5.1.3, for $\lambda \in \mathbb{T}$,

$$\lambda^{-n} R^1(\lambda) = |D^1(\lambda)|^2 - |E_1^1(\lambda)|^2.$$

By Theorem 4.3.1, $D^1(\lambda) \neq 0$ on $\overline{\mathbb{D}}$. Hence, for $\lambda \in \mathbb{T}$

$$\lambda^{-n} R^1(\lambda) + |E_1^1(\lambda)|^2 = |D^1(\lambda)|^2 \neq 0.$$

This implies that $\alpha_1^1, \dots, \alpha_{k_1}^1$ and $\alpha_1^2, \dots, \alpha_{k_2}^2$ which are on $\mathbb{T}$ are distinct from $\sigma_i$, $i = 1, \dots, n$. By the construction in Theorem 5.2.10, for $\sigma_i$, $i = 1, \dots, n$ and $\alpha_1^1, \dots, \alpha_{k_1}^1$ and $\alpha_1^2, \dots, \alpha_{k_2}^2$, the rational $\overline{\mathbb{E}}$-inner function $x = (x_1, x_2, x_3)$ can be defined by

$$\left( \frac{E_1}{D}, \frac{E_1^{\sim n}}{D}, \frac{D^{\sim n}}{D} \right)$$

for a suitable choice of $t_+ > 0$, $t \in \mathbb{C} \backslash \{0\}$ and $w \in \mathbb{T}$. Since $E_1^1$ and $R^1$ coincide with $E_1$ and $R$ in the construction of Theorem 5.2.10 for a suitable choice of $t_+ > 0$ and $t \in \mathbb{C} \backslash \{0\}$, $D^1$ is a permissible choice for $wD$ for some choice $w \in \mathbb{T}$, as a solution for equation (5.2.4). Hence the construction of Theorem 5.2.10 yields $x = (x_1, x_2, x_3)$ for the appropriate choices of $t_+ > 0$, $t \in \mathbb{C} \backslash \{0\}$ and $w \in \mathbb{T}$. □

# Chapter 6

# Convex subsets of $\overline{\mathbb{E}}$ and extremality

In this chapter we study convex subsets of $\overline{\mathbb{E}}$. We show that, for a fixed $x_3 \in \overline{\mathbb{D}}$, the subset $\overline{\mathbb{E}} \cap \left(\mathbb{C}^2 \times \{x_3\}\right)$ is convex. Recall that the distinguished boundaries of the tridisc $\mathbb{D}^3$ and the ball $\mathbb{B}_3$ contain no line segments. Thus every inner function in $\text{Hol}(\mathbb{D}, \mathbb{D}^3)$ is an extreme point of $\text{Hol}(\mathbb{D}, \mathbb{D}^3)$ and every inner function in $\text{Hol}(\mathbb{D}, \mathbb{B}_3)$ is an extreme point of $\text{Hol}(\mathbb{D}, \mathbb{B}_3)$. However, this property contrasts sharply with the situation in the tetrablock. Despite the fact that the set $\mathcal{J}$ of rational tetra-inner functions is not convex, the conventional notion of extreme point of $\mathcal{J}$ is well defined and fruitful. In Theorem 6.2.12, we prove that for $x \in \mathcal{R}^{n,k}$ with $2k \leq n$, $x$ is not an extreme point. A class of extreme points of the set $\mathcal{J}$ is introduced in Proposition 6.2.14.

## 6.1 Convex subsets in the tetrablock

**Definition 6.1.1.** *A domain $\Omega$ is* convex *if for all $z, w \in \Omega$ and all $t$ such that $0 \leq t \leq 1$, the point $tz + (1-t)w$ belongs to $\Omega$.*

**Proposition 6.1.2.** [2, page 8] $\overline{\mathbb{E}}$ *is not convex.*

*Proof.* Take $x = (i, 1, i)$ and $y = (-1, i, -i)$. Let us first show that $x, y \in \overline{\mathbb{E}}$. By Theorem 2.1.5 (6), the point $w = (w_1, w_2, w_3) \in \overline{\mathbb{E}}$ if and only if

$$|w_1 - \overline{w_2} w_3| + |w_2 - \overline{w_1} w_3| \leq 1 - |w_3|^2. \tag{6.1.1}$$

For $x = (i, 1, i)$, inequality (6.1.1)

$$|i - i| + |1 - 1| \quad = \quad 1 - 1 = 0$$

is satisfied, hence $x \in \overline{\mathbb{E}}$.
Similarly, for $y = (-1, i, -i)$, inequality (6.1.1)

$$|-1+1| + |i-i| \;=\; 1 - |-i|^2 = 0$$

is satisfied, hence $y \in \overline{\mathbb{E}}$.
Now if we take $t = 1/2$, then the point $w = tx + (1-t)y$ is equal to

$$\begin{aligned} w = \tfrac{1}{2}x + (1-\tfrac{1}{2})y &= \tfrac{1}{2}(i,1,i) + \tfrac{1}{2}(-1,i,-i) \\ &= \tfrac{1}{2}(-1+i, 1+i, 0). \end{aligned}$$

Note that $w$ is not in $\overline{\mathbb{E}}$, since

$$\tfrac{1}{2}|i-1-0| + \tfrac{1}{2}|1+i-0| = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \;=\; \sqrt{2} > 1.$$

Therefore $\overline{\mathbb{E}}$ is not convex. □

We show that the set $\overline{\mathbb{E}}$ is convex in $(x_1, x_2)$ for a fixed $x_3 \in \overline{\mathbb{D}}$, that is, the set

$$\overline{\mathbb{E}} \cap \big(\mathbb{C}^2 \times \{x_3\}\big) = \{x = (x_1, x_2, x_3) \in \mathbb{C}^3 : |x_1 - \overline{x_2}x_3| + |x_2 - \overline{x_1}x_3| \le 1 - |x_3|^2\}$$

is convex for every $x_3 \in \overline{\mathbb{D}}$. In consequence, some associated sets have a similar property.

**Proposition 6.1.3.** *The following sets are convex:*

(1) $\overline{\mathbb{E}} \cap \big(\mathbb{C}^2 \times \{x_3\}\big)$ *for any* $x_3 \in \overline{\mathbb{D}}$*;*

(2) $b\overline{\mathbb{E}} \cap \big(\mathbb{C}^2 \times \{x_3\}\big)$ *for any* $x_3 \in \overline{\mathbb{D}}$*;*

*Proof.* (1) Let $x, y \in \overline{\mathbb{E}} \cap \big(\mathbb{C}^2 \times \{x_3\}\big)$, and so, by Theorem 2.1.5 (6) $x$ and $y$ satisfy the inequalities

$$|x_1 - \overline{x}_2 x_3| + |x_2 - \overline{x}_2 x_3| \le 1 - |x_3|^2 \tag{6.1.2}$$

and

$$|y_1 - \overline{y}_2 x_3| + |y_2 - \overline{y}_1 x_3| \le 1 - |x_3|^2 \tag{6.1.3}$$

respectively. For all $t \in [0, 1]$,

$$\begin{aligned} w = tx + (1-t)y &= t(x_1, x_2, x_3) + (1-t)(y_1, y_2, x_3) \\ &= \big(tx_1 + (1-t)y_1, tx_2 + (1-t)y_2, tx_3 + (1-t)x_3\big) \\ &= \big(tx_1 + (1-t)y_1, tx_2 + (1-t)y_2, x_3\big). \end{aligned}$$

Let us check that the point $w \in \mathbb{C}^3$ is in the set $\overline{\mathbb{E}} \cap \{\mathbb{C}^2 \times \{x_3\}\}$. By Theorem 2.1.5 (6), $w \in \overline{\mathbb{E}}$ if and only if

$$\underbrace{|w_1 - \overline{w}_2 x_3|} + \underbrace{|w_2 - \overline{w}_1 x_3|} \leq 1 - |x_3|^2.$$

Let us consider the first term on the left hand side

$$\begin{aligned} |w_1 - \overline{w}_2 x_3| &= \big|tx_1 + (1-t)y_1 - \big(t\overline{x}_2 + (1-t)\overline{y}_2\big)x_3\big| \\ &= \big|t(x_1 - \overline{x}_2 x_3) + (1-t)(y_1 - \overline{y}_2 x_3)\big| \\ &\leq \big|t(x_1 - \overline{x}_2 x_3)\big| + \big|(1-t)(y_1 - \overline{y}_2 x_3)\big| \\ &= t\big|x_1 - \overline{x}_2 x_3\big| + (1-t)\big|y_1 - \overline{y}_2 x_3\big|. \end{aligned} \tag{6.1.4}$$

For the second term of the left hand side we have

$$\begin{aligned} |w_2 - \overline{w}_1 x_3| &= \big|tx_2 + (1-t)y_2 - \big(t\overline{x}_1 + (1-t)\overline{y}_1\big)x_3\big| \\ &= \big|t(x_2 - \overline{x}_1 x_3) + (1-t)(y_2 - \overline{y}_1 x_3)\big| \\ &\leq \big|t(x_2 - \overline{x}_1 x_3)\big| + \big|(1-t)(y_2 - \overline{y}_1 x_3)\big| \\ &= t\big|x_2 - \overline{x}_1 x_3\big| + (1-t)\big|y_2 - \overline{y}_1 x_3\big|. \end{aligned} \tag{6.1.5}$$

Add inequalities (6.1.4) and (6.1.5) we get

$$\begin{aligned} &|w_1 - \overline{w}_2 x_3| + |w_2 - \overline{w}_1 x_3| = \\ &\qquad t\big|x_1 - \overline{x}_2 x_3\big| + (1-t)\big|y_1 - \overline{y}_2 x_3\big| + t\big|x_2 - \overline{x}_1 x_3\big| + (1-t)\big|y_2 - \overline{y}_1 x_3\big|. \end{aligned}$$

Take $t$ and $(1-t)$ as common factors we have

$$\begin{aligned} &|w_1 - \overline{w}_2 x_3| + |w_2 - \overline{w}_1 x_3| = \\ &\qquad t\big(\big|x_1 - \overline{x}_2 x_3\big| + \big|x_2 - \overline{x}_1 x_3\big|\big) + (1-t)\big(\big|y_1 - \overline{y}_2 x_3\big| + \big|y_2 - \overline{y}_1 x_3\big|\big). \end{aligned}$$

Therefore, by inequalities (6.1.2) and (6.1.3),

$$\begin{aligned} |w_1 - \overline{w}_2 x_3| + |w_2 - \overline{w}_1 x_3| &= t\big(\underbrace{\big|x_1 - \overline{x}_2 x_3\big| + \big|x_2 - \overline{x}_1 x_3\big|}\big) \\ &+ (1-t)\big(\underbrace{\big|y_1 - \overline{y}_2 x_3\big| + \big|y_2 - \overline{y}_1 x_3\big|}\big) \\ &\leq t(1 - |x_3|^2) + (1-t)(1 - |x_3|^2) \\ &= (t + 1 - t)\big(1 - |x_3|^2\big) \\ &= 1 - |x_3|^2. \end{aligned}$$

Hence for all $t \in [0,1]$, $w = tx + (1-t)y \in \overline{\mathbb{E}} \cap \big(\mathbb{C}^2 \times \{x_3\}\big)$. Therefore $\overline{\mathbb{E}} \cap \big(\mathbb{C}^2 \times \{x_3\}\big)$ is convex for any fixed $x_3 \in \overline{\mathbb{D}}$.

(2) Let $x_3 \in \overline{\mathbb{D}}$ and $x, y \in b\overline{\mathbb{E}} \cap \big(\mathbb{C}^2 \times \{x_3\}\big)$ where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, x_3)$. Note that, by Theorem 2.3.1 (1),

$$w \in \mathbb{C}^3 \text{ belongs to } b\overline{\mathbb{E}} \quad \text{if and only if} \quad w_1 = \overline{w}_2 w_3, \quad |w_2| \le 1 \quad \text{and} \quad |w_3| = 1. \tag{6.1.6}$$

Thus we have

$$x_1 = \overline{x}_2 x_3, \quad |x_2| \le 1 \quad \text{and} \quad |x_3| = 1,$$

and

$$y_1 = \overline{y}_2 x_3, \quad |y_2| \le 1 \quad \text{and} \quad |x_3| = 1.$$

For $t : 0 \le t \le 1$, let

$$w = \big(w_1, w_2, w_3\big) = tx + (1-t)y = \big(tx_1 + (1-t)y_1, tx_2 + (1-t)y_2, x_3\big).$$

To prove the convexity of $b\overline{\mathbb{E}} \cap \{\mathbb{C}^2 \times \{x_3\}\}$, we need to check that, for all $t$ such that $0 \le t \le 1$, $w$ lies in $b\overline{\mathbb{E}} \cap \big(\mathbb{C}^2 \times \{x_3\}\big)$, that is, it satisfies condition (6.1.6).
Note that

$$\begin{aligned}
\overline{w}_2 w_3 &= \big(\overline{tx_2 + (1-t)y_2}\big) x_3 \\
&= t\overline{x}_2 x_3 + (1-t)\overline{y}_2 x_3 \\
&= tx_1 + (1-t)y_1 = w_1
\end{aligned}$$

and

$$\begin{aligned}
|w_2| &= |tx_2 + (1-t)y_2| \\
&\le |tx_2| + |(1-t)y_2| \\
&= t|x_2| + (1-t)|y_2| \\
&\le t + 1 - t = 1.
\end{aligned}$$

Obviously, $|w_3| = |x_3| = 1$. Therefore the set $b\overline{\mathbb{E}} \cap \{\mathbb{C}^2 \times \{x_3\}\}$ is convex for any fixed $x_3 \in \overline{\mathbb{D}}$. □

**Lemma 6.1.4.** *Let $x = (x_1, x_2, x_3)$, $x^1 = (x_1^1, x_2^1, x_3^1)$ and $x^1 = (x_1^2, x_2^2, x_3^2)$ be in $b\overline{\mathbb{E}}$ such that $x = tx^1 + (1-t)x^2$ for $t \in (0,1)$. Then $x_3 = x_3^1 = x_3^2$.*

*Proof.* Since $x$, $x^1, x^2 \in b\overline{\mathbb{E}}$, by Theorem 2.3.1,

$$|x_3| = 1, \qquad |x_3^1| = 1 \qquad \text{and} \qquad |x_3^2| = 1.$$

By assumption $x_3 = tx_3^1 + (1-t)x_3^2$. Since every point of $\mathbb{T}$ is an extreme point of $\overline{\mathbb{D}}$, $x_3 = x_3^1 = x_3^2$. □

## 6.2 Extremality in the set of $\overline{\mathbb{E}}$-inner functions

In this section we show that, for a fixed inner function $x_3$, the set of rational $\overline{\mathbb{E}}$-inner functions $x = (x_1, x_2, x_3)$ with third component $x_3$ is a convex set. We prove that an $\overline{\mathbb{E}}$-inner function $x$ is not an extreme point of the set $\mathcal{J}$ if the number of the royal nodes of $x$ on $\mathbb{T}$, counted with multiplicity, is less than or equal to half of the degree of $x$. In Proposition 6.2.14 we give a class of extreme rational $\overline{\mathbb{E}}$-inner functions $x \in \mathcal{R}^{n,k}$ of the set $\mathcal{J}$ for which $2k > n$.

**Theorem 6.2.1.** *For a fixed inner function $x_3$, the set of $\overline{\mathbb{E}}$-inner functions $(x_1, x_2, x_3)$ is convex.*

*Proof.* For the fixed inner function $x_3$, let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, x_3)$ be $\overline{\mathbb{E}}$-inner functions. For $0 \leq t \leq 1$ and $\lambda \in \mathbb{D}$,

$$\big(tx + (1-t)y\big)(\lambda) = \big(tx_1 + (1-t)y_1, tx_2 + (1-t)y_2, x_3\big)(\lambda).$$

The function

$$w(\lambda) = \big(w_1, w_2, w_3\big)(\lambda) = \big(tx_1 + (1-t)y_1, tx_2 + (1-t)y_2, x_3\big)(\lambda), \quad \lambda \in \mathbb{D}$$

is analytic on $\mathbb{D}$ and by Proposition 6.1.3 (1), $w(\mathbb{D}) \subseteq \overline{\mathbb{E}}$. By Proposition 6.1.3 (2), since for almost all $\lambda \in \mathbb{T}$, $x(\lambda)$ and $y(\lambda)$ are in $b\overline{\mathbb{E}}$, $w(\lambda)$ has also to be in $b\overline{\mathbb{E}}$. Thus $w$ is an $\overline{\mathbb{E}}$-inner function. Therefore the set of $\overline{\mathbb{E}}$-inner functions $x = (x_1, x_2, x_3)$ is convex for any fixed inner function $x_3$. □

**Definition 6.2.2.** *A rational $\overline{\mathbb{E}}$-inner function $x$ is an* extreme point *of $\mathcal{J}$ if whenever $x$ has a representation of the form $x = tx^1 + (1-t)x^2$ for $t \in (0,1)$ and $x^1, x^2$ are rational $\overline{\mathbb{E}}$-inner functions, $x^1 = x^2$.*

We will show below that $\mathcal{J}$ is not convex, however the notion of extreme points still has the usual sense.

**Lemma 6.2.3.** *Let $x = (x_1, x_2, x_3)$, $x^1 = (x_1^1, x_2^1, x_3^1)$ and $x^2 = (x_1^2, x_2^2, x_3^2)$ be rational $\overline{\mathbb{E}}$-inner functions. If $x = tx^1 + (1-t)x^2$ for some $t \in (0,1)$ then $x_3 = x_3^1 = x_3^2$.*

*Proof.* Since $x = tx^1 + (1-t)x^2$, we have

$$\big(x_1, x_2, x_3\big) = \big(tx_1^1, tx_2^1, tx_3^1\big) + \Big((1-t)x_1^2, (1-t)x_2^2, (1-t)x_3^2\Big).$$

Thus $x_3 = tx_3^1 + (1-t)x_3^2$. Hence, for every point $\lambda \in \mathbb{T}$,

$$x_3(\lambda) = tx_3^1(\lambda) + (1-t)x_3^2(\lambda).$$

By assumption, $x^1$ and $x^2$ are rational $\overline{\mathbb{E}}$-inner functions, and so, by Lemma 4.1.3 (ii), $x_3^1$ and $x_3^2$ are rational inner functions, that is, for all $\lambda \in \mathbb{T}$,

$$|x_3^1(\lambda)| = 1 \qquad \text{and} \qquad |x_3^2(\lambda)| = 1.$$

Every point of $\mathbb{T}$ is an extreme point of $\overline{\mathbb{D}}$, and therefore,

$$x_3(\lambda) = x_3^1(\lambda) = x_3^2(\lambda)$$

for all $\lambda \in \mathbb{T}$. Since $x^1$ and $x^2$ are rational functions, $x_3 = x_3^1 = x_3^2$. □

**Lemma 6.2.4.** *The set of rational $\overline{\mathbb{E}}$-inner functions $\mathcal{J}$ is not convex.*

*Proof.* Suppose that $x^1 = \left(x_1^1, x_2^1, x_3^1\right) \in \mathcal{J}$ and $x^2 = (x_1^2, x_2^2, x_3^2) \in \mathcal{J}$ such that $x_3^1 \neq x_3^2$. Then by Lemma 6.2.3, $x = tx^1 + (1-t)x^2$ is not in $\mathcal{J}$ for all $t \in (0,1)$. Therefore $\mathcal{J}$ is not convex. □

**Definition 6.2.5.** *A real or complex-valued function $f$ on a real interval $I$ is said to take a value $y$ to order $m \geq 1$ at a point $t_0 \in I$ if $f \in C^m(I)$, $f(t_0) = y$, $f^{(j)}(t_0) = 0$ for $j = 1, \dots m-1$ and $f^{(m)}(t_0) \neq 0$. We say that $f$ vanishes to order $m \geq 1$ at a point $t_0 \in I$ if $f$ takes the value $0$ to order $m$ at $t_0$.*

**Remark 6.2.6.** *Let $f \in C^m(I)$, $f(t_0) = y$ at $t_0 \in I$, and let $y \neq 0$. If $f^2$ takes the value $y^2$ to order $m \geq 1$ at $t_0$, then $f$ takes the value $y$ to order $m$ at $t_0$.*

*Proof.* Let $I$ be a real interval and let $f \in C^m(I)$. Suppose that $f^2$ takes the value $y^2$ to order $m$ at $t_0$. Then, by Definition 6.2.5

$$f^2(t_0) = y^2, \qquad [f^2]^{(1)}(t_0) = [f^2]^{(2)}(t_0) = \cdots = [f^2]^{(m-1)}(t_0) = 0, \qquad [f^2]^{(m)}(t_0) \neq 0. \tag{6.2.1}$$

For $x \in I$,

$$[f^2]^{(1)}(x) = 2f(x)f^{(1)}(x).$$

At the point $t_0$, by equations (6.2.1),

$$[f^2]^{(1)}(t_0) = 2f(t_0)f^{(1)}(t_0) = 0.$$

Since $f(t_0) \neq 0$, $f^{(1)}(t_0) = 0$.
Similarly, for $x \in I$,

$$[f^2]^{(2)}(x) = 2[f^{(1)}]^2(x) + 2f(x)f^{(2)}(x).$$

At $x = t_0$, by relation (6.2.1),

$$[f^2]^{(2)}(t_0) = 2[f^{(1)}]^2(t_0) + 2f(t_0)f^{(2)}(t_0) = 0.$$

Since $f(t_0) \neq 0$ and $f^{(1)}(t_0) = 0$,

$$f^{(2)}(t_0) = 0.$$

The same way, one can check that

$$f^{(j)}(t_0) = 0 \qquad \text{for } j = 1, \dots, m-1.$$

For the $m$th derivative we have

$$[f^2]^{(m)}(t_0) = 2f(t_0)f^{[m]}(t_0) \neq 0.$$

Hence we get $f^{(m)}(t_0) \neq 0$. Therefore

$$f(t_0) = y, \qquad f^{(j)}(t_0) = 0, \quad \text{for } j = 1, \dots, m-1 \qquad \text{and} \quad f^{(m)}(t_0) \neq 0.$$

□

**Definition 6.2.7.** *A function $f$ is analytic on $\mathbb{T}$ if there exists a function $g$ analytic in a neighbourhood $U_{\mathbb{T}}$ of $\mathbb{T}$ such that $f = g|_{\mathbb{T}}$.*

**Lemma 6.2.8.** *Let $\tau = e^{it_0}$ and let $f(t) = (e^{it} - \tau)^{2v}G(e^{it})$ in a neighbourhood of $t_0$ where $G(z)$ is analytic on $\mathbb{T}$ and $G(\tau) \neq 0$. Then*

$$f^{(j)}(t_0) = 0 \;\; \textit{for} \;\; j = 0, 1, \dots, 2v-1 \quad \textit{and} \;\; f^{(2v)}(t_0) \neq 0. \tag{6.2.2}$$

*Proof.* Since $G$ is analytic on $\mathbb{T}$, by Definition 6.2.7, there exists $U_{\mathbb{T}}$ a neighbourhood of $\mathbb{T}$ and there exists $\tilde{G}$ analytic on $U_{\mathbb{T}}$ such that $G = \tilde{G}|_{\mathbb{T}}$. Let $z = e^{it}$, $\phi(z) = (z-\tau)^{2v}G(z)$ and $\tilde{\phi}(z) = (z-\tau)^{2v}\tilde{G}(z)$. Define $\gamma(\tau, r)$ to be an anticlockwise circle centred at $\tau$ with radius $r$

$$\gamma(\tau, r) = \{z \in \mathbb{C} : |z - \tau| = r\},$$

where $r$ is taken sufficiently small such that $\gamma \subset U_{\mathbb{T}}$. Hence the function $\tilde{\phi}$ is analytic inside the curve $\gamma$. Therefore, by Cauchy's integral formula,

$$\begin{aligned} \tilde{\phi}^{(j)}(\tau) &= \frac{j!}{2\pi i}\int_{\gamma} \frac{\tilde{\phi}(z)}{(z-\tau)^{j+1}} dz, \\ &= \frac{j!}{2\pi i}\int_{\gamma} \frac{(z-\tau)^{2v}\tilde{G}(z)}{(z-\tau)^{j+1}} dz \\ &= \frac{j!}{2\pi i}\int_{\gamma} (z-\tau)^{2v-j-1}\tilde{G}(z) dz. \end{aligned} \tag{6.2.3}$$

For $0 \leq j \leq 2v-1$, the function $(z-\tau)^{2v-j-1}\tilde{G}(z)$ is analytic on $U_{\mathbb{T}}$. Therefore, by Cauchy's Theorem,

$$\tilde{\phi}^{(j)}(\tau) = \frac{j!}{2\pi i}\int_{\gamma} (z-\tau)^{2v-j-1}\tilde{G}(z) dz = 0. \tag{6.2.4}$$

If $j = 2v$, then equation (6.2.3) becomes

$$\tilde{\phi}^{(2v)}(\tau) = \frac{(2v)!}{2\pi i} \int_\gamma \frac{\tilde{G}(z)}{(z-\tau)} dz.$$

By Cauchy's integral formula,

$$\tilde{\phi}^{(2v)}(\tau) = \frac{(2v)!}{2\pi i} \int_\gamma \frac{\tilde{G}(z)}{(z-\tau)} dz = (2v)!\ G(\tau) \neq 0. \tag{6.2.5}$$

Hence $\phi^{(2v)}(\tau) \neq 0$ because $\tilde{\phi}$ agrees with $\phi$ on $\mathbb{T}$. Note that,

$$f(t) = (e^{it} - e^{it_0})^{(2v)} G(e^{it}) = \phi(e^{it}).$$

By the chain rule,

$$\begin{aligned}
\frac{df}{dt} &= \frac{d\phi}{dz} \frac{dz}{dt} \\
\frac{d^2 f}{dt^2} &= \left( \frac{d^2\phi}{dz^2} \frac{dz}{dt} \right) \frac{dz}{dt} + \frac{d\phi}{dz} \frac{d^2 z}{dt^2} \\
&= \frac{d^2\phi}{dz^2} \left( \frac{dz}{dt} \right)^2 + \frac{d\phi}{dz} \frac{d^2 z}{dt^2} \\
\frac{d^3 f}{dt^3} &= \left( \frac{d^3\phi}{dz^3} \frac{dz}{dt} \right) \left( \frac{dz}{dt} \right)^2 + \frac{d^2\phi}{dz^2} \left( 2 \frac{dz}{dt} \frac{d^2 z}{dt^2} \right) + \left( \frac{d^2\phi}{dz} \frac{dz}{dt} \right) \frac{d^2 z}{dt} + \frac{d\phi}{dz} \frac{d^3 z}{dt^3} \\
&= \frac{d^3\phi}{dz^3} \left( \frac{dz}{dt} \right)^3 + 3 \frac{d^2\phi}{dz^2} \frac{d^2 z}{dt^2} \frac{dz}{dt} + \frac{d\phi}{dz} \frac{d^3 z}{dt^3}.
\end{aligned}$$

Similarly, one can see that

$$\frac{d^{2v-1} f}{dt^{2v-1}} = \frac{d^{2v-1}\phi}{dz^{2v-1}} \left( \frac{dz}{dt} \right)^{2v-1} + \cdots + \frac{d\phi}{dz} \frac{d^{2v-1} z}{dt^{2v-1}}.$$

By equation 6.2.4 and since $\tilde{\phi}$ and $\phi$ agree on $\mathbb{T}$,

$$\frac{d^j \tilde{\phi}}{dz^j}(\tau) = 0, \qquad \text{for } j = 1, \ldots, 2v-1,$$

and so,

$$\frac{d^j \phi}{dz^j}(\tau) = 0, \qquad \text{for } j = 1, \ldots, 2v-1.$$

That is, $f^{(j)}(t_0) = 0$ for $j = 1, \ldots, 2v-1$.
Now for the $(2v)$th derivative we have

$$\frac{d^{2v} f}{dt^{2v}} = \frac{d^{2v}\phi}{dz^{2v}} \left( \frac{dz}{dt} \right)^{2v} + \cdots + \frac{d\phi}{dz} \frac{d^{2v} z}{dt^{2v}}.$$

By equations (6.2.4) and (6.2.5),

$$\frac{d^j\phi}{dz^j}(\tau) = 0, \qquad \text{for } j = 1, \dots, 2v-1 \qquad \text{and} \qquad \frac{d^{2v}\phi}{dz^{2v}}(\tau) \neq 0.$$

Hence $f^{(2v)}(t_0) = \dfrac{d^{2v}\phi}{dz^{2v}}(\tau)\left(\dfrac{dz}{dt}\right)^{2v}(t_0) \neq 0.$

Therefore $f^{(j)}(t_0) = 0$ for $j = 0, \dots, 2v-1$ and $f^{(2v)}(t_0) \neq 0$. □

**Lemma 6.2.9.** *Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$-inner function. For $\tau \in \mathbb{T}$,*

(i) $|x_1(\tau)| = 1 \Leftrightarrow$ *$\tau$ is a royal node of $x$.*

(ii) $|x_2(\tau)| = 1 \Leftrightarrow$ *$\tau$ is a royal node of $x$.*
*Moreover,*

(iii) *$\tau = e^{it_0}$ is a royal node of $x$ of multiplicity $v$ if and only if $|x_1(e^{it})| = 1$ to order $2v$ at $t = t_0$.*

(iv) *$\tau = e^{it_0}$ is a royal node of $x$ of multiplicity $v$ if and only if $|x_2(e^{it})| = 1$ to order $2v$ at $t = t_0$.*

*Proof.* (i) If $\tau = e^{it_0}$ is a royal node of $x$ of multiplicity $v$, by Definition 5.2.1,

$$(x_3 - x_1x_2)(\lambda) = (\lambda - \tau)^{2v}F(\lambda) \tag{6.2.6}$$

where $F$ is a rational function, analytic on $\mathbb{T}$ and $F(\tau) \neq 0$ on $\mathbb{T}$. By Lemma 2.3.2, since $x$ is $\overline{\mathbb{E}}$-inner function, $x_2 = \overline{x_1}x_3$ on $\mathbb{T}$. Therefore, for $\lambda \in \mathbb{T}$,

$$\begin{aligned}\Big(x_3 - x_1x_2\Big)(\lambda) &= \Big(x_3 - x_1\overline{x_1}x_3\Big)(\lambda)\\ &= x_3(\lambda) - x_3(\lambda)|x_1(\lambda)|^2\\ &= x_3(\lambda)(1 - |x_1(\lambda)|^2).\end{aligned} \tag{6.2.7}$$

Therefore, for any $\tau \in \mathbb{T}$,

$$|x_1(\tau)| = 1 \qquad \Longleftrightarrow \qquad (x_3 - x_1x_2)(\tau) = 0,$$

that is, if and only if $\tau$ is a royal node of $x$.

(ii) Since $x$ is rational $\overline{\mathbb{E}}$-inner function, by Theorem 2.3.1, $x_1 = \overline{x_2}x_3$ on $\mathbb{T}$. The rest of the proof is similar to the above proof of (i).

(iii) Suppose that $\tau = e^{it_0}$ is a royal node of $x$ of multiplicity $v \geq 1$ then on combining equations (6.2.6) and (6.2.7), for all $t \in \mathbb{R}$,

$$x_3(e^{it})(1 - |x_1(e^{it})|) = (e^{it} - \tau)^{2v} F(e^{it}).$$

This gives

$$1 - |x_1(e^{it})|^2 = (e^{it} - \tau)^{2v} \frac{F(e^{it})}{x_3(e^{it})}.$$

The rational function $G(e^{it}) = \dfrac{F(e^{it})}{x_3(e^{it})}$ is analytic on $\mathbb{T}$ and is not equal to zero at $\tau = e^{it_0}$. Thus we have

$$1 - |x_1(e^{it})|^2 = (e^{it} - \tau)^{2v} G(e^{it}).$$

Since $x$ is rational and $|x_1(e^{it_0})| = 1$, the function $f(t) = 1 - |x_1(e^{it})|^2$ is $C^\infty$ on a neighbourhood of $t_0$. By Lemma 6.2.8,

$$f^{(j)}(t_0) = 0 \quad \text{for } j = 0, 1, \ldots, 2v - 1 \quad \text{and} \quad f^{(2v)}(t_0) \neq 0.$$

Therefore $f$ takes the value 0 to order $2v$ at $t_0$, which implies, by Remark 6.2.6, $|x_1(e^{it})| = 1$ to order $2v$ at $t_0$.

(iv) The proof of this statement follows from (ii) and is similar to the above proof of (iii). □

**Lemma 6.2.10.** *Let $n \geq 1$. Any $x = (x_1, x_2, x_3) \in \mathcal{R}^{n,0}$ is not an extreme point of $\mathcal{J}$.*

*Proof.* Since $x$ has no royal nodes on $\mathbb{T}$, by Lemma 6.2.9, for all $\lambda \in \mathbb{T}$ ,

$$|x_1(\lambda)| < 1 \qquad \text{and} \qquad |x_2(\lambda)| < 1.$$

Since $\mathbb{T}$ is compact, the supremum of $x_1$ and $x_2$ is attained on $\mathbb{T}$, that is, there exist $\lambda_1, \lambda_2 \in \mathbb{T}$ such that

$$\sup_{\lambda \in \mathbb{T}} |x_1(\lambda)| = |x_1(\lambda_1)| < 1 \qquad \text{and} \qquad \sup_{\lambda \in \mathbb{T}} |x_2(\lambda)| = |x_2(\lambda_2)| < 1. \tag{6.2.8}$$

Choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$|x_1(\lambda_1)|(1 + \varepsilon_1) < 1 \qquad \text{and} \qquad |x_2(\lambda_2)|(1 + \varepsilon_2) < 1. \tag{6.2.9}$$

Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. If $x_1(\lambda_1) = 0$, then

$$x_1(\lambda) = 0 \quad \text{for all } \lambda \in \mathbb{T}.$$

Likewise, if $x_2(\lambda_2) = 0$ then

$$x_2(\lambda) = 0, \quad \text{for all } \lambda \in \mathbb{T}.$$

Define $x^1$ and $x^2$ to be

$$x^1 = \big((1+\varepsilon)x_1, (1+\varepsilon)x_2, x_3\big) \qquad \text{and} \qquad x^2 = \big((1-\varepsilon)x_1, (1-\varepsilon)x_2, x_3\big).$$

Since $x = (x_1, x_2, x_3)$ is a rational $\overline{\mathbb{E}}$-inner function, for almost all $\lambda \in \mathbb{T}$,

$$x_1(\lambda) = \overline{x}_2(\lambda)x_3(\lambda), \quad |x_2(\lambda)| \le 1 \quad \text{and} \quad |x_3(\lambda)| = 1. \tag{6.2.10}$$

Let us check that $x^1$ and $x^2$ are rational $\overline{\mathbb{E}}$-inner functions. By Theorem 2.3.1 (1), this will follow if we show that

$$(1+\varepsilon)x_1(\lambda) = (1+\varepsilon)\overline{x}_2(\lambda)x_3(\lambda), \quad (1+\varepsilon)|x_2(\lambda)| \le 1 \quad \text{and} \quad |x_3(\lambda)| = 1,$$

and $x^1(\mathbb{D}) \subset \mathbb{E}$. By equations (6.2.10), we have to show only that

$$(1+\varepsilon)|x_2(\lambda)| \le 1 \quad \text{on } \mathbb{T}.$$

This statement follows from inequalities (6.2.8) and (6.2.9). Thus $x^1(\mathbb{T}) \subset b\overline{\mathbb{E}}$. By Theorem 2.3.1 (2), for almost all $\lambda \in \mathbb{T}$,

$$x^1(\lambda) \in b\overline{\mathbb{E}} \Leftrightarrow \Psi(., x^1(\lambda)) \text{ is an automorphism of } \mathbb{D}.$$

By the maximum principle, for all $\lambda \in \mathbb{D}$, $||\Psi(., x^1(\lambda))||_{H^\infty} < 1$. Therefore, by Theorem 2.1.4, for all $\lambda \in \mathbb{D}$, $x^1(\lambda) \subseteq \mathbb{E}$. This completes the proof that $x^1$ is a rational $\overline{\mathbb{E}}$-inner function.

In a similar way we can show that $x^2$ is a rational $\overline{\mathbb{E}}$-inner function. Moreover, by Lemma 6.2.9, $x^1, x^2$ have no royal nodes on $\mathbb{T}$ and therefore $x^1, x^2 \in \mathcal{R}^{n,0}$. However $x = \frac{1}{2}x^1 + \frac{1}{2}x^2$, which implies that $x$ cannot be an extreme point of $\mathcal{J}$ since $x^1 \neq x^2$. □

**Proposition 6.2.11.** *Let $x = (x_1, x_2, x_3)$ be superficial and $x = tx^1 + (1-t)x^2$ for some $0 < t < 1$, where $x^1 = (x_1^1, x_2^1, x_3^1)$ and $x^2 = (x_1^2, x_2^2, x_3^2)$ are rational $\overline{\mathbb{E}}$-inner functions. Then $x^1$ and $x^2$ are superficial and $x_3 = x_3^1 = x_3^2$.*

*Proof.* By Lemma 6.2.3, $x_3 = x_3^1 = x_3^2$. Suppose, for a contradiction, $x^1$ is not superficial. Then there exists $\lambda_0 \in \mathbb{D}$ such that $x^1(\lambda_0) \in \mathbb{E}$. Let us show that in this case $x(\lambda_0) \in \mathbb{E}$, and so is not superficial.
By Theorem 2.1.4 (6), it is enough to show that

$$|x_1(\lambda_0) - \overline{x}_2(\lambda_0)x_3(\lambda_0)| + |x_2(\lambda_0) - \overline{x}_1(\lambda_0)x_3(\lambda_0)| < 1 - |x_3(\lambda_0)|^2. \tag{6.2.11}$$

Since $x^1(\lambda_0) \in \mathbb{E}$ and $x^2$ is rational $\overline{\mathbb{E}}$-inner function, this implies that

$$|x_1^1(\lambda_0) - \overline{x}_2^1(\lambda_0)x_3^1(\lambda_0)| + |x_2^1(\lambda_0) - \overline{x}_1^1(\lambda_0)x_3^1(\lambda_0)| < 1 - |x_3^1(\lambda_0)|^2$$

and

$$|x_1^2(\lambda_0) - \overline{x}_2^2(\lambda_0)x_3^2(\lambda_0)| + |x_2^2(\lambda_0) - \overline{x}_1^2(\lambda_0)x_3^2(\lambda_0)| \leq 1 - |x_3^2(\lambda_0)|^2.$$

Let us begin with the first term on the left hand side of inequality (6.2.11).

$$\begin{aligned}
& |x_1(\lambda_0) - \overline{x}_2(\lambda_0)x_3(\lambda_0)| \\
= & \left|tx_1^1(\lambda_0) + (1-t)x_1^2(\lambda_0) - \big(t\overline{x}_2^1(\lambda_0) + (1-t)\overline{x}_2^2(\lambda_0)\big)x_3(\lambda_0)\right| \\
= & \left|tx_1^1(\lambda_0) + (1-t)x_1^2(\lambda_0) - t\overline{x}_2^1(\lambda_0)x_3(\lambda_0) - (1-t)\overline{x}_2^2(\lambda_0)x_3(\lambda_0)\right| \\
= & \left|t\big(x_1^1(\lambda_0) - \overline{x}_2^1(\lambda_0)x_3(\lambda_0)\big) + (1-t)\big(x_1^2(\lambda_0) - \overline{x}_2^2(\lambda_0)x_3(\lambda_0)\big)\right| \\
\leq & \left|t\big(x_1^1(\lambda_0) - \overline{x}_2^1(\lambda_0)x_3(\lambda_0)\big)\right| + \left|(1-t)\big(x_1^2(\lambda_0) - \overline{x}_2^2(\lambda_0)x_3(\lambda_0)\big)\right| \\
= & t\left|x_1^1(\lambda_0) - \overline{x}_2^1(\lambda_0)x_3(\lambda_0)\right| + (1-t)\left|x_1^2(\lambda_0) - \overline{x}_2^2(\lambda_0)x_3(\lambda_0)\right|.
\end{aligned} \tag{6.2.12}$$

The second term on inequality (6.2.11)

$$\begin{aligned}
& |x_2(\lambda_0) - \overline{x}_1(\lambda_0)x_3(\lambda_0)| \\
= & \left|tx_2^1(\lambda_0) + (1-t)x_2^2(\lambda_0) - \big(t\overline{x}_1^1(\lambda_0) + (1-t)\overline{x}_1^2(\lambda_0)\big)x_3(\lambda_0)\right| \\
= & \left|tx_2^1(\lambda_0) + (1-t)x_2^2(\lambda_0) - t\overline{x}_1^1(\lambda_0)x_3 - (1-t)\overline{x}_1^2(\lambda_0)x_3(\lambda_0)\right| \\
= & \left|t\big(x_2^1(\lambda_0) - \overline{x}_1^1(\lambda_0)x_3(\lambda_0)\big) + (1-t)\big(x_2^2(\lambda_0) - \overline{x}_1^2(\lambda_0)x_3(\lambda_0)\big)\right| \\
\leq & \left|t\big(x_2^1(\lambda_0) - \overline{x}_1^1(\lambda_0)x_3(\lambda_0)\big)\right| + \left|(1-t)\big(x_2^2(\lambda_0) - \overline{x}_1^2(\lambda_0)x_3(\lambda_0)\big)\right| \\
= & t\left|x_2^1(\lambda_0) - \overline{x}_1^1(\lambda_0)x_3(\lambda_0)\right| + (1-t)\left|x_2^2(\lambda_0) - \overline{x}_1^2(\lambda_0)x_3(\lambda_0)\right|.
\end{aligned} \tag{6.2.13}$$

Add inequalities (6.2.12) and (6.2.13) gives

$$\begin{aligned}
& t\left|x_1^1(\lambda_0) - \overline{x}_2^1(\lambda_0)x_3(\lambda_0)\right| + (1-t)\left|x_1^2(\lambda_0) - \overline{x}_2^2(\lambda_0)x_3(\lambda_0)\right| + \\
& t\left|x_2^1(\lambda_0) - \overline{x}_1^1(\lambda_0)x_3(\lambda_0)\right| + (1-t)\left|x_2^2(\lambda_0) - \overline{x}_1^2(\lambda_0)x_3(\lambda_0)\right| \\
= & t\Big(\left|x_1^1(\lambda_0) - \overline{x}_2^1(\lambda_0)x_3(\lambda_0)\right| + \left|x_2^1(\lambda_0) - \overline{x}_1^1(\lambda_0)x_3(\lambda_0)\right|\Big) \\
& +(1-t)\Big(\left|x_1^2(\lambda_0) - \overline{x}_2^2(\lambda_0)x_3(\lambda_0)\right| + \left|x_2^2(\lambda_0) - \overline{x}_1^2(\lambda_0)x_3(\lambda_0)\right|\Big) \\
< & t(1 - |x_3(\lambda_0)|^2) + (1-t)(1 - |x_3(\lambda_0)|^2) \\
= & (t + 1 - t)(1 - |x_3(\lambda_0)|^2) = 1 - |x_3(\lambda_0)|^2.
\end{aligned} \tag{6.2.14}$$

By equations (6.2.12), (6.2.13) and (6.2.14), this proves relation (6.2.11).

□

**Theorem 6.2.12.** *Let $x \in \mathcal{R}^{n,k}$. If $2k \leq n$, then $x$ is not an extreme point of the set $\mathcal{J}$ of rational $\overline{\mathbb{E}}$-inner functions.*

*Proof.* Let $x \in \mathcal{R}^{n,k}$. By Definition 5.2.1, $x$ has $n$ royal nodes in $\overline{\mathbb{D}}$ and $k$ royal nodes that lie in $\mathbb{T}$. By Theorem 4.3.1, there exist polynomials $E_1$, $E_2$ and $D$ of degree at most $n$ such that

$$x = \left(\frac{E_1}{D}, \frac{E_1^{\sim n}}{D}, \frac{D^{\sim n}}{D}\right)$$

where, for all $\lambda \in \overline{\mathbb{D}}$, $D(\lambda) \neq 0$ and $E_2(\lambda) = E_1^{\sim n}(\lambda)$. Let $\tau_1, \ldots, \tau_k \in \mathbb{T}$ and $\alpha_{k+1}, \ldots, \alpha_n \in \mathbb{D}$ be the royal nodes of $x$ in $\overline{\mathbb{D}}$ repeated according to multiplicity. By Proposition 5.2.7, the royal polynomial of $x$ is

$$R(\lambda) = r \prod_{j=1}^{k} Q_{\tau_j} \prod_{j=k+1}^{n} Q_{\alpha_j},$$

for some $r > 0$. Thus for all $\lambda \in \mathbb{T}$,

$$\begin{aligned} \lambda^{-n} R(\lambda) &= r\lambda^{-n} \left\{ \prod_{j=1}^{k} (\lambda - \tau_j)(1 - \overline{\tau}_j \lambda) \prod_{j=k+1}^{n} (\lambda - \alpha_j)(1 - \overline{\alpha_j} \lambda) \right\} \\ &= r \prod_{j=1}^{k} |\lambda - \tau_j|^2 \prod_{j=k+1}^{n} |\lambda - \alpha_j|^2. \end{aligned} \tag{6.2.15}$$

By Proposition 5.1.3 and equation (6.2.15), for all $\lambda \in \mathbb{T}$,

$$|D(\lambda)|^2 - |E_1(\lambda)|^2 = \lambda^{-n} R(\lambda) = r \prod_{j=1}^{k} |\lambda - \tau_j|^2 \prod_{j=k+1}^{n} |\lambda - \alpha_j|^2. \tag{6.2.16}$$

Assume first that $n$ is even and write $n = 2m$. This implies that $k \leq m$. Define a polynomial $g$ by

$$g(\lambda) = \overline{\tau}_1 \ldots \overline{\tau}_k \lambda^{m-k} \prod_{j=1}^{k} (\lambda - \tau_j)^2.$$

Clearly, the polynomial $g$ has degree $m + k \leq n$. Moreover, $g$ is $n$-symmetric since

$$\begin{aligned} g^{\sim n}(\lambda) = \lambda^n \overline{g}\left(\frac{1}{\overline{\lambda}}\right) &= \lambda^n \overline{\left\{ \overline{\tau}_1 \ldots \overline{\tau}_k \frac{1}{(\overline{\lambda})^{m-k}} \prod_{j=1}^{k} \left(\frac{1}{\overline{\lambda}} - \tau_j\right)^2 \right\}} \\ &= \lambda^{2m} \left\{ \tau_1 \ldots \tau_k \frac{1}{\lambda^{m-k}} \prod_{j=1}^{k} \left(\frac{1}{\lambda} - \overline{\tau_j}\right)^2 \right\} \\ &= \tau_1 \ldots \tau_k \lambda^{m+k} \prod_{j=1}^{k} \overline{\tau}_j^2 \left(\frac{\tau_j}{\lambda} - 1\right)^2 \\ &= \overline{\tau}_1 \ldots \overline{\tau}_k \lambda^{m-k} \prod_{j=1}^{k} (\lambda - \tau_j)^2 = g(\lambda). \end{aligned}$$

Let

$$E_1^t = E_1 + tg \qquad \text{and} \qquad E_2^t = E_1^{\sim n} + tg \qquad \text{for } t \in \mathbb{R}.$$

The polynomial $E_1^t$ has degree at most $n$. We also have, for all $\lambda \in \overline{\mathbb{D}}$,

$$\begin{aligned} \big(E_2^t\big)^{\sim n}(\lambda) &= \big(E_1^{\sim n} + tg\big)^{\sim n}(\lambda) \\ &= \big(E_1^{\sim n}\big)^{\sim n}(\lambda) + \big(tg\big)^{\sim n}(\lambda) = \big(E_1^t + tg\big)(\lambda) = E_1^t(\lambda). \end{aligned} \tag{6.2.17}$$

Note that, on $\mathbb{T}$,

$$\begin{aligned} |D|^2 - |E_1^t|^2 &= |D|^2 - |E_1 + tg|^2 \\ &= |D|^2 - (E_1 + tg)\overline{(E_1 + tg)} \\ &= |D|^2 - |E_1|^2 - t^2|g|^2 - 2\mathrm{Re}(tg\overline{E}_1). \end{aligned} \tag{6.2.18}$$

Let $||E_1||_\infty$ denote the supremum of $|E_1|$ on $\mathbb{T}$. Then, for all $\lambda \in \mathbb{T}$,

$$\begin{aligned} \mathrm{Re}(tg(\lambda)\overline{E_1}(\lambda) \le |tg(\lambda)E_1(\lambda)| &= |tE_1(\lambda)| \left|\overline{\tau}_1 \dots \overline{\tau}_j \lambda^{m-k} \prod_{j=1}^k (\lambda - \tau_j)^2\right| \\ &= |tE_1(\lambda)| \prod_{j=1}^k |\lambda - \tau_j|^2 \\ &\le |t|||E_1||_\infty \prod_{j=1}^k |\lambda - \tau_j|^2. \end{aligned} \tag{6.2.19}$$

Note that, for all $\lambda \in \mathbb{T}$,

$$|g(\lambda)|^2 = \left|\overline{\tau}_1 \dots \overline{\tau}_k \lambda^{m-k} \prod_{j=1}^k (\lambda - \tau_j)^2\right|^2 = \Big|\prod_{j=1}^k (\lambda - \tau_j)^2\Big|^2.$$

Combine equations (6.2.16) and (6.2.18) and inequality (6.2.19), for all $\lambda \in \mathbb{T}$, to get

$$\begin{aligned}
& |D(\lambda)|^2 - |E_1^t(\lambda)|^2 \\
= \; & |D(\lambda)|^2 - |E_1(\lambda)|^2 - |t|^2|g(\lambda)|^2 - 2\mathrm{Re}\big(tg(\lambda)\overline{E}_1(\lambda)\big), \text{ (by equation (6.2.18))} \\
= \; & r\prod_{j=1}^{k}|\lambda-\tau_j|^2 \prod_{j=k+1}^{n}|\lambda-\alpha_j|^2 - |t|^2|g(\lambda)|^2 - 2\mathrm{Re}\big(tg(\lambda)\overline{E}_1(\lambda)\big), \text{ (by equation (6.2.16))} \\
\geq \; & r\prod_{j=1}^{k}|\lambda-\tau_j|^2 \prod_{j=k+1}^{n}|\lambda-\alpha_j|^2 - |t|^2|g(\lambda)|^2 - 2|t|||E_1||_\infty \prod_{j=1}^{k}|\lambda-\tau_j|^2, \text{ (by inequality (6.2.19))} \\
= \; & \prod_{j=1}^{k}|\lambda-\tau_j|^2 r\prod_{j=k+1}^{n}|Q_{\alpha_j}(\lambda)| - |t|^2|g(\lambda)|^2 - 2|t|||E_1||_\infty \prod_{j=1}^{k}|\lambda-\tau_j|^2 \\
= \; & \prod_{j=1}^{k}|\lambda-\tau_j|^2 r\prod_{j=k+1}^{n}|Q_{\alpha_j}(\lambda)| - |t|^2\Big|\prod_{j=1}^{k}|\lambda-\tau_j|^2\Big|^2 - 2|t|||E_1||_\infty \prod_{j=1}^{k}|\lambda-\tau_j|^2 \\
\geq \; & \prod_{j=1}^{k}|\lambda-\tau_j|^2\left\{rM - \left(|t|^2\prod_{j=1}^{k}|\lambda-\tau_j|^2 + 2|t|||E_1||_\infty\right)\right\} \\
\geq \; & \prod_{j=1}^{k}|\lambda-\tau_j|^2\left\{rM - |t|\big(|t|\;||g||_\infty + 2||E_1||_\infty\big)\right\}
\end{aligned}$$

where $M = \inf_{\mathbb{T}}\prod|Q_{\alpha_j}| > 0$.

Let us show that for $|t|$ sufficiently small $|D(\lambda)|^2 - |E_1^t(\lambda)|^2 \geq 0$ on $\mathbb{T}$. It suffices to find $|t|$ such that

$$rM - |t|\big(|t|\;||g||_\infty + 2||E_1||_\infty\big) > 0,$$

or equivalently,

$$|t|\left(|t| + 2\frac{||E_1||_\infty}{||g||_\infty}\right) - \frac{rM}{||g||_\infty} < 0.$$

If we take $|t| \leq \min\left\{\frac{2||E_1||_\infty}{||g||_\infty}, \frac{rM}{8||E_1||_\infty}\right\}$, then

$$\begin{aligned}
|t|\left(|t| + 2\frac{||E_1||_\infty}{||g||_\infty}\right) - \frac{rM}{||g||_\infty} \;&\leq\; |t|\left(2\frac{||E_1||_\infty}{||g||_\infty} + 2\frac{||E_1||_\infty}{||g||_\infty}\right) - \frac{rM}{||g||_\infty} \\
&=\; |t|\left(4\frac{||E_1||_\infty}{||g||_\infty}\right) - \frac{rM}{||g||_\infty} \\
&\leq\; \frac{rM}{8||E_1||_\infty}\left(4\frac{||E_1||_\infty}{||g||_\infty}\right) - \frac{rM}{||g||_\infty} \\
&=\; \frac{rM}{2||g||_\infty} - \frac{rM}{||g||_\infty} = -\frac{rM}{2||g||_\infty} < 0. \qquad (6.2.20)
\end{aligned}$$

Therefore

$$|D|^2 - |E_1^t|^2 \geq 0 \qquad \text{on } \mathbb{T},$$

and by Theorem 5.2.10, the functions

$$x_{+t} = \left(\frac{E_1^{+t}}{D}, \frac{E_2^{+t}}{D}, \frac{D^{\sim n}}{D}\right)$$

and

$$x_{-t} = \left(\frac{E_1^{-t}}{D}, \frac{E_2^{-t}}{D}, \frac{D^{\sim n}}{D}\right)$$

are rational $\overline{\mathbb{E}}$-inner function. Obviously,

$$\begin{aligned}
\frac{1}{2}x_{+t} + \frac{1}{2}x_{-t} &= \frac{1}{2}\left(\frac{E_1^{+t}}{D}, \frac{E_2^{+t}}{D}, \frac{D^{\sim n}}{D}\right) + \frac{1}{2}\left(\frac{E_1^{-t}}{D}, \frac{E_2^{-t}}{D}, \frac{D^{\sim n}}{D}\right) \\
&= \left(\frac{E_1^{+t} + E_1^{-t}}{2D}, \frac{E_2^{+t} + E_2^{-t}}{2D}, \frac{D^{\sim n}}{D}\right) \\
&= \left(\frac{E_1 + tg + E_1 - tg}{2D}, \frac{E_1^{\sim n} + tg + E_1^{\sim n} - tg}{2D}, \frac{D^{\sim n}}{D}\right) \\
&= \left(\frac{E_1}{D}, \frac{E_1^{\sim n}}{D}, \frac{D^{\sim n}}{D}\right) = x.
\end{aligned}$$

Hence $x$ is not an extreme point of $\mathcal{J}$.

If $n$ is odd, assume $n = 2m+1$. This case requires a slight modification. By assumption, $2k \leq n$ thus $2k \leq 2m+1$. This implies that $k \leq m$. Choose $w \in \mathbb{T}$ such that

$$w^2 = -\overline{\tau}_1 \prod_{j=1}^{k} \overline{\tau}_j^2.$$

Let

$$g(\lambda) = w\lambda^{m-k}(\lambda - \tau_1)\prod_{j=1}^{k}(\lambda - \tau_j)^2, \quad \lambda \in \mathbb{C}.$$

Clearly, the polynomial $g$ has degree $m+k+1 \leq n$. Let us check that the $g$ is $n$-symmetric

$$\begin{aligned} g^{\sim n}(\lambda) = \lambda^n g\big(\overline{1/\overline{\lambda}}\big) &= \lambda^n \overline{\Big( w \frac{1}{\overline{\lambda}^{m-k}} \big(\frac{1}{\overline{\lambda}} - \tau_1\big) \prod_{j=1}^{k} \big(\frac{1}{\overline{\lambda}} - \tau_j\big)^2 \Big)} \\ &= \lambda^n \Big( \overline{w} \frac{1}{\lambda^{m-k}} \big(\frac{1}{\lambda} - \overline{\tau}_1\big) \prod_{j=1}^{k} \big(\frac{1}{\lambda} - \overline{\tau}_j\big)^2 \Big) \\ &= \overline{w} \lambda^{m+k+1} \big(\frac{1}{\lambda} - \overline{\tau}_1\big) \prod_{j=1}^{k} \big(\frac{1}{\lambda} - \overline{\tau}_j\big)^2 \\ &= \overline{w} \lambda^{m-k} \big(1 - \overline{\tau}_1 \lambda\big) \prod_{j=1}^{k} \big(1 - \overline{\tau}_j \lambda\big)^2 \\ &= \overline{w} \lambda^{m-k} \overline{\tau}_1 \big(\tau_1 - \lambda\big) \prod_{j=1}^{k} \overline{\tau}_j^2 \big(\tau_j - \lambda\big)^2 \\ &= \overline{w} \Big( -\overline{\tau}_1 \prod_{j=1}^{k} \overline{\tau}_j^2 \Big) \lambda^{m-k} \big(\lambda - \tau_1\big) \prod_{j=1}^{k} \big(\lambda - \tau_j\big)^2 \\ &= \overline{w}\, w^2\, \lambda^{m-k} \big(\lambda - \tau_1\big) \prod_{j=1}^{k} \big(\lambda - \tau_j\big)^2 \\ &= w\, \lambda^{m-k} \big(\lambda - \tau_1\big) \prod_{j=1}^{k} \big(\lambda - \tau_j\big)^2 = g(\lambda). \end{aligned}$$

As in the even case, define the polynomials on $\overline{\mathbb{D}}$

$$E_1^t = E_1 + tg \qquad \text{and} \qquad E_2^t = E_1^{\sim n} + tg \qquad \text{for } t \in \mathbb{R}.$$

Similar to equation (6.2.17), for all $\lambda \in \overline{\mathbb{D}}$, $E_1^t(\lambda) = \big(E_2^t\big)^{\sim n}(\lambda)$ and similar to equation (6.2.18), for all $\lambda \in \mathbb{T}$,

$$|D(\lambda)|^2 - |E_1^t(\lambda)|^2 = |D(\lambda)|^2 - |E_1(\lambda)|^2 - t^2|g(\lambda)|^2 - 2\mathrm{Re}\big(tg(\lambda)\overline{E_1(\lambda)}\big). \tag{6.2.21}$$

For all $\lambda \in \mathbb{T}$,

$$\begin{aligned} \mathrm{Re}(tg\overline{E_1(\lambda)} \leq |tgE_1(\lambda)| &= |tE_1(\lambda)| \Big| w\lambda^{m-k}(\lambda - \tau_1) \prod_{j=1}^{k} (\lambda - \tau_j)^2 \Big| \\ &= |tE_1(\lambda)||\lambda - \tau_1| \prod_{j=1}^{k} |\lambda - \tau_j|^2 \\ &\leq |t|||E_1||_\infty |\lambda - \tau_1| \prod_{j=1}^{k} |\lambda - \tau_j|^2. \end{aligned} \tag{6.2.22}$$

Combine equations (6.2.16), (6.2.21) and inequality (6.2.22) gives, for all $\lambda \in \mathbb{T}$,

$$\begin{aligned}
& |D(\lambda)|^2 - |E_1^t(\lambda)|^2 \\
= \ & |D(\lambda)|^2 - |E_1(\lambda)|^2 - |t|^2|g(\lambda)|^2 - 2\mathrm{Re}\big(tg(\lambda)\overline{E}_1(\lambda)\big) \\
= \ & r\prod_{j=1}^{k}|\lambda-\tau_j|^2 \prod_{j=k+1}^{n}|\lambda-\alpha_j|^2 - |t|^2|g(\lambda)|^2 - 2\mathrm{Re}\big(tg(\lambda)\overline{E}_1(\lambda)\big), \text{ by equation (6.2.16)} \\
\geq \ & r\prod_{j=1}^{k}|\lambda-\tau_j|^2 \prod_{j=k+1}^{n}|\lambda-\alpha_j|^2 - |t|^2|g(\lambda)|^2 - 2|t|||E_1||_\infty|\lambda-\tau_1|\prod_{j=1}^{k}|\lambda-\tau_j|^2, \text{ by inequality(6.2.22)} \\
= \ & \prod_{j=1}^{k}|\lambda-\tau_j|^2 r\prod_{j=k+1}^{n}|Q_{\alpha_j}(\lambda)| - |t|^2|g(\lambda)|^2 - 2|t|||E_1||_\infty|\lambda-\tau_1|\prod_{j=1}^{k}|\lambda-\tau_j|^2 \\
= \ & \prod_{j=1}^{k}|\lambda-\tau_j|^2 r\prod_{j=k+1}^{n}|Q_{\alpha_j}(\lambda)| - |t|^2|\lambda-\tau_1|^2\prod_{j=1}^{k}|\lambda-\tau_j|^4 - 2|t|||E_1||_\infty|\lambda-\tau_1|\prod_{j=1}^{k}|\lambda-\tau_j|^2 \\
\geq \ & \prod_{j=1}^{k}|\lambda-\tau_j|^2\left\{rM - \underbrace{|\lambda-\tau_1|}_{\leq 2}\left(\underbrace{|t|^2|\lambda-\tau_1|\prod_{j=1}^{k}|\lambda-\tau_j|^2}_{\leq |t|^2||g||_\infty} + 2|t|||E_1||_\infty\right)\right\} \\
\geq \ & \prod_{j=1}^{k}|\lambda-\tau_j|^2\left\{rM - 2\big(|t|^2||g||_\infty + 2|t|||E_1||_\infty\big)\right\}
\end{aligned}$$

where $M = \inf_{\mathbb{T}}\prod|Q_{\alpha_j}| > 0$. By similar arguments as in equations (6.2.20), one can find $|t|$ such that

$$rM - 2\big(|t|^2||g||_\infty + 2|t|||E_1||_\infty\big) > 0$$

Therefore,

$$|D|^2 - |E_1^t|^2 \geq 0, \qquad \text{on } \mathbb{T}.$$

Hence, by Theorem 5.2.10, the functions

$$x_{\pm t} = \left(\frac{E_1^{\pm t}}{D}, \frac{\big(E_1^{\sim n}\big)^{\pm t}}{D}, \frac{D^{\sim n}}{D}\right)$$

are rational $\overline{\mathbb{E}}$-inner functions. One can check that $x = \frac{1}{2}x_{+t} + \frac{1}{2}x_{-t}$ and therefore $x$ is not an extreme point of $\mathcal{J}$. □

**Theorem 6.2.13.** [7, Theorem 5.13] *A rational $\Gamma$-inner function $h \in \mathcal{R}_\Gamma^{n,k}$ is extreme in the set of rational $\Gamma$-inner functions if and only if $2k > n$.*

**Proposition 6.2.14.** *Let $x = (x_1, x_2, x_3) \in \mathcal{R}^{n,k}$ be a rational $\overline{\mathbb{E}}$-inner function such that $x_1 = x_2$ and $2k > n$. Then $x$ is an extreme point of the set $\mathcal{J}$ of rational $\overline{\mathbb{E}}$-inner functions.*

*Proof.* By Lemma 4.1.9 (1), the function $h = (s, p) = (2x_1, x_3)$ is $\Gamma$-inner. By Theorem 4.3.1, there are polynomials $E_1, E_2, D$ such that $x = \left(\frac{E_1}{D}, \frac{E_2}{D}, \frac{D^{\sim n}}{D}\right)$. Here, since $x_1 = x_2$, necessarily $E_1 = E_2$. By Definition 3.3.5, the royal polynomial $R_h$ of $h$ is

$$\begin{aligned} R_h(\lambda) &= D^2(\lambda)\Big(4x_3 - 4x_1^2\Big)(\lambda) \\ &= D^2(\lambda)\Big(4\frac{D^{\sim n}}{D} - 4\frac{E_1^2}{D^2}\Big)(\lambda) \\ &= 4\big(DD^{\sim n} - E_1^2\big)(\lambda) = 4R_x(\lambda). \end{aligned}$$

It is clear that if $x \in \mathcal{R}^{n,k}$, then $h$ has degree $n$ and $k$ royal nodes on $\mathbb{T}$, counted with multiplicities, such that $2k > n$. Thus, by Theorem 6.2.13, $h$ is an extreme point of the set of rational $\Gamma$-inner functions. That is, if $h^1 = (s^1, p^1)$ and $h^2 = (s^2, p^2)$ are $\Gamma$-inner functions such that

$$h = th^1 + (1-t)h^2 \qquad \text{for some } t \in (0,1),$$

then $h = h^1 = h^2$. Note that, in this case, we have

$$\begin{cases} s = ts^1 + (1-t)s^2 & \Rightarrow s = s^1 = s^2 \\ p = tp^1 + (1-t)p^2 & \Rightarrow p = p^1 = p^2. \end{cases} \tag{6.2.23}$$

Suppose

$$x = tx^1 + (1-t)x^2, \quad \text{for some } t \in (0,1)$$

and for rational $\overline{\mathbb{E}}$-inner functions $x^1 = (x_1^1, x_2^1, x_3^1)$ and $x^2 = (x_1^2, x_2^2, x_3^2)$. This implies that

$$\begin{cases} x_1 = tx_1^1 + (1-t)x_1^2 \\ x_1 = tx_2^1 + (1-t)x_2^2 \\ x_3 = p = tx_3^1 + (1-t)x_3^2. \end{cases}$$

Recall that $(s, p) = (2x_1, x_3)$, hence

$$\begin{cases} s = 2tx_1^1 + 2(1-t)x_1^2 \\ s = 2tx_2^1 + 2(1-t)x_2^2 \\ p = tx_3^1 + (1-t)x_3^2. \end{cases} \tag{6.2.24}$$

Therefore

$$(s, p) = t\big(2x_1^1, x_3^1\big) + (1-t)\big(2x_1^2, x_3^2\big)$$

and

$$(s, p) = t\big(2x_2^1, x_3^1\big) + (1-t)\big(2x_2^2, x_3^2\big).$$

Since $h$ is an extreme rational $\Gamma$-inner function, we have

$$\begin{cases} 2x_1^1 = 2x_1^2 = s \\ 2x_2^1 = 2x_2^2 = s \\ x_3^1 = x_3^2 = p. \end{cases}$$

Therefore $x = x^1 = x^2$. Hence $x$ is extreme in the set $\mathcal{J}$. $\square$

# Appendix A

# The fundamental group of a topological space

**Definition A.0.1.** [30, Definition, page 150] *Let $I = [0, 1]$ be the unit interval. Two mappings $f$ and $g$ of a topological space $X$ into a topological space $Y$ are* homotopic, *denoted $f \simeq g$, if there is a continuous mapping $h : X \times I \to Y$ such that for each $x \in X$*

$$h(x, 0) = f(x) \quad \text{and} \quad h(x, 1) = g(x).$$

*such a map $h$ is called a* homotopy between $f$ and $g$.

**Definition A.0.2.** [30, Definition, page 157] *Two spaces $X$ and $Y$ are* homotopically equivalent (or of the same type) *if there are mappings $f : X \to Y$ and $g : Y \to X$ such that the composite mappings*

$$f \circ g : Y \to Y \quad \text{and} \quad g \circ f : X \to X$$

*are homotopic to the identity mappings*

$$\mathrm{id} : Y \to Y \quad \text{and} \quad \mathrm{id} : X \to X$$

*respectively.*

Let $Y$ be a topological space and let $y_0 \in Y$. Let $C(Y, y_0)$ be the collection of all continuous mappings $f : I \to Y$ such that

$$f(0) = y_0 = f(1).$$

**Definition A.0.3.** [30, Definition, page 159] *Suppose that $f$ and $g$ are two mappings in $C(Y, y_0)$. Then $f$ is* homotopic to $g$ modulo $y_0$, *denoted by $f \underset{y_0}{\simeq} g$ if there exists a continuous map $h : I \times I \to Y$ such that*

$$h(x,0) = f(x) \quad \textit{and} \quad h(x,1) = g(x) \text{ for all } x \in I.$$
$$\text{and} \quad h(0,t) = y_0 = h(1,t) \qquad \text{for all } t \in I.$$

**Theorem A.0.4.** [30, Theorem 4-2] *Let $A$ be any set, and let $R$ be an equivalence relation on $A$. Then $A$ is decomposed by $R$ into disjoint subsets called* equivalence classes.

**Lemma A.0.5.** [30, Lemma 4-16] *Homotopy modulo $y_0$ is an equivalence relation on $C(Y, y_0)$.*

By Theorem A.0.4, $C(Y, y_0)$ can be decomposed by the relation $\underset{y_0}{\simeq}$ into disjoint equivalence classes, namely the arcwise-connected components of $C(Y, y_0)$. We denote the collection of such classes $\pi_1(Y, y_0)$. Now, let $[f]$ be the homotopy class such that $f$ is in $C(Y, y_0)$, that is, $[f]$ denote the collection of all $g$ in $C(Y, y_0)$ such that $f \underset{y_0}{\simeq} g$. Define the *juxtaposition* $f * g$ of $f$ and $g$ on $\pi_1(Y, y_0)$ by

$$(f * g)(x) = \begin{cases} f(2x) & 0 \leq x \leq \frac{1}{2} \\ g(2x-1) & \frac{1}{2} \leq x \leq 1. \end{cases}$$

One can see that $f * g$ is also an element in $C(Y, y_0)$, since $(f * g)(\frac{1}{2}) = f(1) = g(0) = y_0$. Finally, we define the product of $[f]$, $[g]$ in $\pi_1(Y, y_0)$ by

$$[f] \circ [g] = [f * g].$$

The set $\pi_1(Y, y_0)$ is called the *fundamental group* and it is indeed a group under the $\circ$ operation which we shall consider in this thesis.

**Theorem A.0.6.** [30, Theorem 4-20] *A continuous mapping $h : (X, x_0) \to (Y, y_0)$ induces a homomorphism $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.*

*Proof.* Define a mapping

$$\begin{aligned} h_\# &: C(X, x_0) \to C(Y, y_0) \\ &: f \longmapsto h\big(f(x)\big) \end{aligned}$$

that is, $\big(h_\# f\big)(t) = h\big(f(t)\big)$. First we need to show that $h_\#$ is continuous. Suppose that $f_0$ in $C(X, x_0)$ and let $U \ni h_\# f_0$ be any basis element in the compact-open topology of $C(Y, y_0)$. By definition, $U$ is the collection of all continuous functions in $C(Y, y_0)$ which map a compact set $K$ into an open set $O$. Now consider the basis $U^{-1}$ of the collection

of all continuous functions in $C(X, x_0)$ that map $K$ into $h^{-1}(O)$. $[h_\# f](K)$ belongs to $O$, thus $h\big(f(K)\big)$ lies in $O$ and $f(K)$ lies in $h^{-1}(O)$, therefore $f_0$ belongs to $U^{-1}$. If $g$ lies in $U^{-1}$, then $g(K)$ lies in $h^{-1}(O)$ and $[h_\# g](K) = h\big(g(K)\big)$ lies in $O$, and hence $h_\# g \in U$. Therefore, $h_\#$ is continuous.
Define $h_*\big([f]\big) = [h_\# f]$. Clearly, $h_*$ is well-defined, since $h_\#$ maps $C(X, x_0)$ into $C(Y, y_0)$. It remains to show that $h_*$ is homomorphism, that is,

$$h_*\big([f] \circ [g]\big) = h_*\big([f]\big) \circ h_*\big([g]\big).$$

We only need to show that

$$h_\#(f * g) = h_\# f * h_\# g.$$

One can see that

$$\begin{aligned} [h_\#(f * g)](x) &= \begin{cases} h\big(f(2x)\big) = [h_\# f](2x) & \text{for } 0 \le x \le \frac{1}{2} \\ h\big(g(2x-1)\big) = [h_\# g](2x-1) & \text{for } \frac{1}{2} \le x \le 1 \end{cases} \\ &= [h_\# f * h_\# g](x). \end{aligned}$$

□

**Theorem A.0.7.** [30, Theorem 4-21] *Suppose that the mappings $f$ and $g$ from $(X, x_0)$ into $(Y, y_0)$ are homotopic. Then the induced homomorphisms coincide. If $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$, then $(gf)_* = g_* f_*$.*

**Theorem A.0.8.** [30, Theorem 4-3] *Let $Y^X$ denote the space of all continuous functions from $X$ into $Y$. Then the homotopy classes of $Y^X$ are precisely the arcwise-connected components of $Y^X$.*

# Appendix B

# Basic definitions

**Definition B.0.1.** *A domain* $\Omega \subset \mathbb{C}^n, n \geq 1$, *is called* starlike *about a fixed point* $a \in \Omega$ *if, for any point* $z$ *in* $\Omega$, *the line segment between* $a$ *and* $z$ *lies entirely in* $\Omega$.

**Definition B.0.2.** *A subset* $\Omega$ *of* $\mathbb{C}^n, n \geq 1$, *is called* starlike *if it is starlike about some point.*

**Definition B.0.3.** *Let* $\Omega$ *be a domain in* $\mathbb{C}^N$. *We say that* $\overline{\Omega}$ *is* polynomially convex if for every point $z \in \mathbb{C}^N \backslash \overline{\Omega}$, there is a polynomial $p$ such that

$$\sup\{|p(w)| : w \in \overline{\Omega}\} \leq |p(z)|.$$

**Definition B.0.4.** *The* polynomially convex hull *of a compact subset* $S$ *of* $\mathbb{C}^N$, *denoted by* $\widehat{S}$, *is defined as*

$$\widehat{S} = \Big\{ z \in \mathbb{C}^N : |p(z)| \leq \max_{s \in S} |p(s)| \text{ for all polynomials} \Big\}.$$

$S$ *is said to be* polynomially convex *if* $S = \widehat{S}$.

**Definition B.0.5.** *A domain* $\Omega$ *is* polynomially convex *if for every compact subset* $S$ *of* $\Omega$, $\widehat{S} \subset \Omega$.

**Definition B.0.6.** *Let* $\Omega$ *be a domain and let* $\overline{\Omega}$ *be its closure. We denote by* $A(\Omega)$ *the algebra of continuous scalar functions on* $\overline{\Omega}$ *that are holomorphic on* $\Omega$.

**Remark B.0.7.** *Let* $\Omega$ *be a domain. A subset* $C$ *of* $\overline{\Omega}$ *is called a* boundary *if every function in* $A(\Omega)$ *attains its maximum modulus on* $C$. *By the theory of uniform algebras* [17, Corollary 2.2.10], *if* $\overline{\Omega}$ *is polynomially convex, there is a smallest closed boundary of* $\Omega$ *contained in all the closed boundaries of* $\Omega$. *This boundary is called the* distinguished boundary, or Shilov boundary, of $\Omega$ *and denoted by* $b\Omega$.

**Lemma B.0.8 (Schwarz Lemma).** [14, Theorem 13] *Let $f : \mathbb{D} \to \mathbb{D}$ be a analytic function such that $f(0) = 0$. Then*

(i) $|f(z)| \leq |z|$ *for every* $z \in \mathbb{D}$;

(ii) $|f'(0)| \leq 1$.

*Moreover, if either $|f(w)| = |w|$ for some $w \in \mathbb{D}\backslash\{0\}$, or $|f'(0)| = 1$, then $f$ is a rotation, that is, $f(z) = cz$ for some $c \in \mathbb{T}$.*

**Definition B.0.9 (Möbius transformation).** [33, Page 23] *The function*

$$f(z) = \frac{az+b}{cz+d} \tag{B.0.1}$$

*where $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$ is called Möbius transformation. In the case that $c = 0$ we have $f(z) = \frac{a}{d}z + \frac{b}{d}$, thus the Möbius transformation is linear. We extend the definition to the Riemann sphere as follows:*

$$f(-\frac{d}{c}) = \infty \qquad \text{and} \qquad f(\infty) = \frac{a}{c}.$$

*The inverse of the Möbius transformation* (B.0.1) *is given by $f^{-1}(w) = \frac{dw-b}{-cw+a}$. One can see that $f(z)$ maps the extended complex plane onto itself.*

**Lemma B.0.10.** *Let $a, b, c, d \in \mathbb{C}$ be such that $ad - bc \neq 0$ and $c \neq 0$. Suppose that $cz + d \neq 0$ for all $z \in \overline{\mathbb{D}}$. Then the linear transformation*

$$S(z) = \frac{az+b}{cz+d}$$

*maps the open unit disc $\mathbb{D}$ into the set $S(\mathbb{D}) = \{z \in \mathbb{C} : |z - C| < R\}$, where $C$ and $R$ are the centre and the radius of $S(\mathbb{D})$ respectively where*

$$C = \frac{b\overline{d} - a\overline{c}}{|d|^2 - |c|^2} \quad \text{and} \quad R = \left|\frac{|ad-bc|}{|d|^2 - |c|^2}\right|.$$

*Proof.* Let $S(z) = \dfrac{az+b}{cz+d}$ be a linear transformation such that $ad \neq bc$, and let $w = S(z)$

then we have

$$\begin{aligned} w &= \frac{az+b}{cz+d} \\ w(cz+d) &= az+b \\ wcz+wd &= az+b \\ wcz-az &= -wd+b \\ z(wc-a) &= -wd+b \\ z &= \frac{-dw+b}{cw-a} \\ z &= S^{-1}(w) = \frac{-dw+b}{cw-a}. \end{aligned}$$

So for $w = S(z)$ we have $z = S^{-1}(w) = \dfrac{-dw+b}{cw-a}$. The limit of $S^{-1}(w)$ as $w \to \infty$ is $\dfrac{-d}{c}$. The preimages of the centre $C$ and $\infty$ are conjugate with respect to the open unit circle $\mathbb{T}$, that is, $\overline{S^{-1}(C)} \cdot S^{-1}(\infty) = 1$ and so $\overline{S^{-1}(C)} \cdot \frac{-d}{c} = 1$. Therefore, $S^{-1}(C) = \frac{-\overline{c}}{\overline{d}}$.

Now,

$$\begin{aligned} S \circ S^{-1}(w) &= S\Big(\frac{dw-b}{-cw+a}\Big) \\ &= \frac{a\Big(\frac{dw-b}{-cw+a}\Big)+b}{c\Big(\frac{dw-b}{-cw+a}\Big)+d} \\ &= \frac{\frac{adw-ab-cbw+ab}{-cw+a}}{\frac{cdw-cb-cdw+ad}{-cw+a}} \\ &= \frac{adw-cbw}{-cb+ad} \\ &= \frac{w(ad-cb)}{ad-cb} = w. \end{aligned} \tag{B.0.2}$$

Similarly, $S^{-1} \circ S = \mathrm{id}$.
We have $C = S(S^{-1}(C))$ this gives

$$C = S\Big(\frac{-\overline{c}}{\overline{d}}\Big) = \frac{a(\frac{-\overline{c}}{\overline{d}})+b}{c(\frac{-\overline{c}}{\overline{d}})+d} = \frac{b\overline{d}-a\overline{c}}{|d|^2-|c|^2}.$$

The radius $R = |S(1) - C|$ is equal to

$$\begin{aligned}
R &= \left| \frac{a+b}{c+d} - \frac{b\overline{d} - a\overline{c}}{|d|^2 - |c|^2} \right| \\
&= \left| \frac{(a+b)(|d|^2 - |c|^2) - (c+d)(b\overline{d} - a\overline{c})}{(c+d)(|d|^2 - |c|^2)} \right| \\
&= \left| \frac{a|d|^2 - a|c|^2 + b|d|^2 - b|c|^2 - bc\overline{d} + a|c|^2 - b|d|^2 + ad\overline{c}}{(c+d)(|d|^2 - |c|^2)} \right| \\
&= \left| \frac{a|d|^2 - b|c|^2 - bc\overline{d} + ad\overline{c}}{(c+d)(|d|^2 - |c|^2)} \right| \\
&= \left| \frac{ad(\overline{c} + \overline{d}) - bc(\overline{c} + \overline{d})}{(c+d)(|d|^2 - |c|^2)} \right| \\
&= \left| \frac{(\overline{c} + \overline{d})(ad - bc)}{(c+d)(|d|^2 - |c|^2)} \right| \\
&= \left| \frac{ad - bc}{|d|^2 - |c|^2} \right|.
\end{aligned}$$

□

**Definition B.0.11.** *The Schur class is the class of analytic functions which map the open unit disc* $\mathbb{D}$ *to its closure* $\overline{\mathbb{D}}$. *The Schur class is denoted by* $\mathrm{Hol}(\mathbb{D}, \overline{\mathbb{D}})$.

**Definition B.0.12.** $H^\infty(\mathbb{D})$ *is the Banach space of bounded analytic functions on the open unit disc* $\mathbb{D}$ *with supreme norm* $||f||_\infty = \sup_{\lambda \in \mathbb{D}} |f(\lambda)|$.

**Definition B.0.13.** [37, Definition 13.1] $L^\infty(\mathbb{T})$ *denotes the Banach space of essentially bounded Lebesgue-measurable* $\mathbb{C}$*-valued functions on* $\mathbb{T}$ *with pointwise algebraic operations and essential supremum norm:*

$$||f||_\infty = \operatorname{ess\,sup}_{|z|=1} |f(z)|.$$

**Theorem B.0.14** (Fatou's Theorem)**.** [36, Theorem 11.32] *To every* $f \in H^\infty(\mathbb{D})$ *there corresponds a function* $\tilde{f} \in L^\infty(\mathbb{T})$ *defined almost every where by*

$$\tilde{f}(e^{it}) = \lim_{r \to 1} f(re^{it}).$$

*The equality* $||f||_\infty = ||\tilde{f}||_\infty$ *holds, where* $||\tilde{f}||_\infty = \sup_{\lambda \in \mathbb{T}} |\tilde{f}(\lambda)|$.

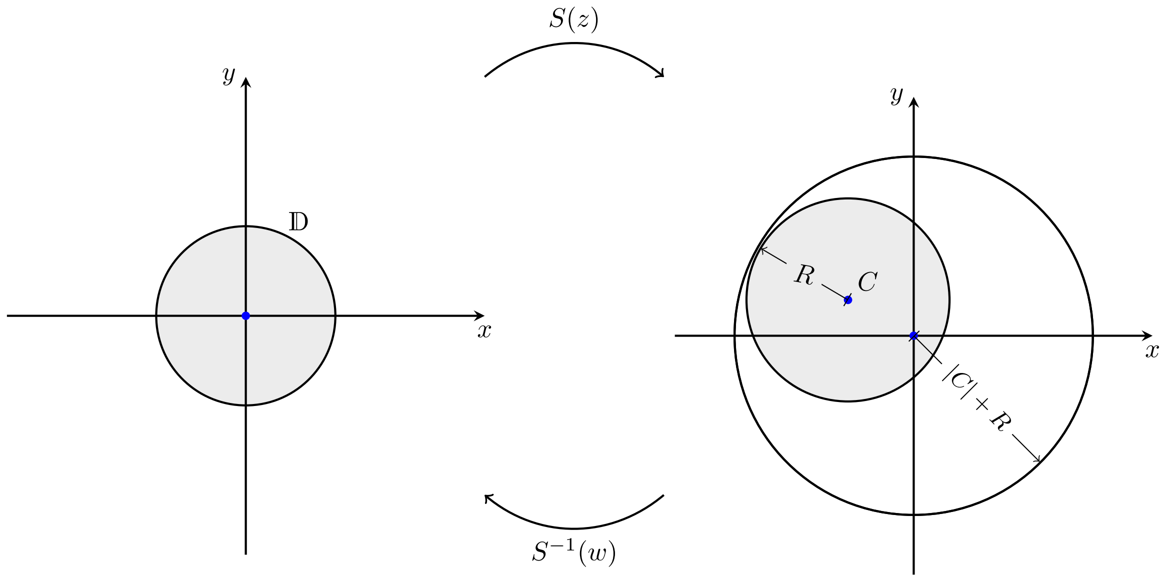



Figure B.1: The linear fractional $S$ maps the open unit disc $\mathbb{D}$.

**Definition B.0.15.** *An* inner function *is an analytic map* $f : \mathbb{D} \to \overline{\mathbb{D}}$ *such that the radial limit*

$$\lim_{r \to 1^-} f(r\lambda) \tag{B.0.3}$$

*exists and belongs to* $\mathbb{T}$ *for almost all* $\lambda \in \mathbb{T}$ *with respect to Lebesgue measure.*

**Definition B.0.16.** [9, page 2] *A Finite Blaschke product is a function of the form*

$$B(z) = c \prod_{i=1}^{n} \frac{z - \alpha_i}{1 - \overline{\alpha_i} z} \qquad \text{for } z \in \mathbb{C} \backslash \{1/\overline{\alpha_1}, \ldots, 1/\overline{\alpha_n}\},$$

*where* $|c| = 1$ *and* $\alpha_1, \ldots, \alpha_n \in \mathbb{D}$. *The function* $B_\alpha(z) = \dfrac{z - \alpha}{1 - \overline{\alpha} z}$ *is called a* Blaschke factor*.*

**Theorem B.0.17.** [9, page 2] *Let* $B$ *be a finite Blaschke product. Then the function* $B$ *has the following properties:*

(1) $B$ *is analytic in* $\mathbb{D}$ *and continuous in* $\overline{\mathbb{D}}$.

(2) $B$ *is inner.*

(3) $B(z) = 0$ *at* $\alpha_1, .., \alpha_n$ *only.*

(4) $B$ *has poles at* $\frac{1}{\overline{\alpha}_1}, .., \frac{1}{\overline{\alpha}_n}$ *only.*

**Remark B.0.18.** [29, Theorem 3] *The rational inner functions on* $\mathbb{D}$ *are precisely the finite Blaschke products.*

**Theorem B.0.19 (The maximum modulus principle.).** [14, Theorem 12] *If* $f(z)$ *is an analytic and non-constant function in a domain* $\Omega$, *then* $|f(z)|$ *has no maximum in* $\Omega$.

**Theorem B.0.20.** [14, Theorem 12′] *Let* $S$ *be the closure of a bounded domain. If* $f$ *is defined and continuous on* $S$ *and analytic on the interior of* $S$, *then* $|f|$ *attains its maximum on the boundary of* $S$.

**Lemma B.0.21.** (Fejér-Riesz theorem) [34, Section 53] *If* $f(\lambda) = \sum_{i=-n}^{n} a_i \lambda^i$ *is a trigonometric polynomial of degree* $n$ *such that* $f(\lambda) \geq 0$ *for all* $\lambda \in \mathbb{T}$ *then there exists an analytic polynomial* $D(\lambda) = \sum_{i=0}^{n} b_i \lambda^i$ *of degree* $n$ *such that* $D$ *is outer (that is,* $D(\lambda) \neq 0$ *for all* $\lambda \in \mathbb{D}$*) and*

$$f(\lambda) = |D(\lambda)|^2$$

*for all* $\lambda \in \mathbb{T}$.

**Definition B.0.22.** *Let* $\Omega$ *be an open set in* $\mathbb{C}$ *and* $X$ *a Banach space. Then we say a map* $f : \Omega \to X$ *is* analytic *if for every* $z_0 \in \Omega$ *there exists* $f'(z_0) \in X$ *such that*

$$\lim_{z \to z_0} \left|\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right|\right|_X = 0.$$

**Definition B.0.23.** *Let* $X$ *be a domain in* $\mathbb{C}^N$. *We denote by* $\mathrm{Hol}(\mathbb{D}, \overline{X})$ *the space of analytic functions from* $\mathbb{D}$ *to* $\overline{X}$.

**Definition B.0.24.** *Let* $\mathbb{C}^n = \{x : x = (x_1, ..., x_n) : \ x_i \in \mathbb{C}\}$. *The inner product of two vectors* $x, y \in \mathbb{C}^n$ *is defined by*

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y}_i.$$

Define the norm of the vector $x$ in $\mathbb{C}^n$ by

$$\begin{aligned} ||x||_{\mathbb{C}^n} &= \langle x, x \rangle^{\frac{1}{2}} \\ &= \left( \sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

**Definition B.0.25.** [37, Definition 3.4] *A* Hilbert space *is an inner product space which is a complete metric space with respect to the metric induced by its inner product.*

**Definition B.0.26.** [37, Page 23] *A* Banach space *is a normed space which is a complete metric space with respect to the metric induced by its norm.*

**Remark B.0.27 (Operator norm of a matrix).** *Let* $W \in \mathbb{C}^{m \times n}$,

$$W = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \dots & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \qquad a_{ij} \in \mathbb{C}.$$

*Then W defines a bounded linear operator*

$$\begin{aligned} W &: \mathbb{C}^n \to \mathbb{C}^m \\ &: x \longmapsto Wx, \quad \textit{where} \end{aligned}$$

$$Wx = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \dots & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \Sigma_{j=1}^n a_{1j} x_j \\ \Sigma_{j=1}^n a_{2j} x_j \\ \vdots \\ \Sigma_{j=1}^n a_{mj} x_j \end{bmatrix}.$$

*The* operator norm *of W is given by*

$$||W|| = \sup_{||x||_{\mathbb{C}^n} \leq 1} ||Wx||_{\mathbb{C}^m}.$$

**Definition B.0.28.** [14, page 115] *Let $\gamma$ be a closed curve that does not pass through a point a. Then the* winding number*, or the* index *of the point a with respect to $\gamma$ is an integer given by*

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - a}.$$

**Theorem B.0.29. (Cauchy's integral formula)** [14, Theorem 6] *Suppose that $f(z)$ is analytic in an open disc $\Omega$, and let $\gamma$ be a closed curve in $\Omega$. For any point a not on $\gamma$*

$$n(\gamma, a) \cdot f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a} dz$$

*The higher derivatives of the function f at the point a are given by*

$$n(\gamma, a) \cdot f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z) dz}{(z - a)^{n+1}}.$$

**Theorem B.0.30. (Cauchy's Theorem)** [14, Theorem 4] *If $f(z)$ is analytic in an open disc B, then*

$$\int_\gamma f(z) dz = 0$$

*for every closed curve $\gamma$ in B.*

**Theorem B.0.31.** [14, Theorem 18] *If $f(z)$ is meromorphic in a domain $\Omega$ with the zeros $a_j$ and the poles $b_k$ then*

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)dz}{f(z)} = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k).$$

**Definition B.0.32.** *A domain $\Omega_1$ is said to be an analytic retract of a domain $\Omega_2$ if there exist analytic maps $\iota : \Omega_1 \to \Omega_2$ and $k : \Omega_2 \to \Omega_1$ such that $k \circ \iota = \mathrm{id}_{\Omega_1}$.*

**Definition B.0.33.** [8, Page 1] *A* holomorphic retraction *is a holomorphic map $\iota : U \to U$ such that $\iota \circ \iota = \iota$, and a* holomorphic retract *in $U$ is a set which is the range of a holomorphic retraction of $U$.*

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