Finite-Dimensional Irreducible Representations of the Very Non-Standard Quantum $\mathfrak{so}_{2N-1}$-Algebra

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Abstract

Quantum groups arose in the mid 1980s in the study of certain integrable models in mathematical physics. Fundamental objects in the theory of quantum groups, as developed by V. Drinfeld and M. Jimbo, are certain deformations of universal enveloping algebras of semisimple Lie algebras. The resulting quantised enveloping algebras depend on a deformation parameter q which, for the purpose of this thesis, is assumed *not* to be a root of unity. Crucially, quantised enveloping algebras retain the structure of a Hopf algebra from universal enveloping algebras. Moreover, the finite-dimensional representations of quantised enveloping algebras are classified in terms of highest weights, similar to the classification of finite-dimensional representations of semisimple Lie algebras.

There exist other deformations of universal enveloping algebras which are not Drinfeld-Jimbo quantum groups. One of the earliest classes of examples is that of the non-standard quantum $\mathfrak{so}_n$-algebras introduced by A. Gavrilik and A. Klimyk. These algebras appear as special examples in the theory of quantum symmetric pairs developed by G. Letzter in the late 1990s. This theory provides quantum group analogues of Lie subalgebras fixed under an involutive automorphism. Quantum symmetric pairs are given in terms of a coideal subalgebra of a quantised enveloping algebra. The representation theory of these coideal subalgebras is not known in general, having only been determined for certain classes of examples.

The present thesis is devoted to the representation theory of the coideal subalgebra corresponding to the symmetric pair of type DII in the Cartan classification. In this case, $\mathfrak{so}_{2N-1}$ is realised as a Lie subalgebra of $\mathfrak{so}_{2N}$ fixed under an involutive automorphism. The resulting coideal subalgebras are not isomorphic to the Gavrilik-Klimyk algebras, and we hence call them *very* non-standard quantum $\mathfrak{so}_{2N-1}$-algebras.

In this thesis, we classify the finite-dimensional irreducible representations of the very non-standard quantum $\mathfrak{so}_{2N-1}$-algebras. Importantly, these algebras have a very simple analogue of a Cartan subalgebra, and every finite-dimensional module is a weight module. We show that (up to a choice of signs) the irreducible representations of very non-standard quantum $\mathfrak{so}_{2N-1}$-algebras are uniquely determined by a highest weight. We construct root vectors and prove a Poincaré-Birkhoff-Witt theorem which supports a triangular decomposition. The root vectors then allow us to introduce a notion of Verma modules, and we show that every simple module is obtained as a quotient of a Verma module. The arguments to this point mimic the known approach to the representation theory of quantised enveloping algebras. The weights also need to be dominant integral for the simple quotients of Verma modules to be finite-dimensional. However, in the coideal case, it is harder to show that dominant integral weights are actually sufficient to obtain finite-dimensional simple quotients. The reason for this is a missing $\mathfrak{sl}_2$-triple which, when found, acts only on a subspace of the representation, and hence cannot be used to obtain Weyl-group invariance. Instead, we use a filtered-graded argument to show that a (possibly larger) quotient of the Verma module, which can be considered as a module for the ambient Hopf algebra, is finite-dimensional.

Contents

Chapter 1

Introduction

In the theory of quantum groups, one deals with quantum deformations $U_q(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of a complex semisimple Lie algebra $\mathfrak{g}$. The algebras $U(\mathfrak{g})$ have an additional structure, namely a coproduct, and it is crucial that the deformations are compatible with this additional structure. The algebras $U_q(\mathfrak{g})$ were introduced by V. Drinfeld and M. Jimbo in the mid-1980s, see [Dri87] and [Jim85], and are often called *quantised enveloping algebras*. They have found many applications in areas ranging from representation theory to low-dimensional topology and knot theory.

The algebra $U_q(\mathfrak{g})$ depends on a deformation parameter q and is usually written in terms of generators and relations. More precisely, the generators of $U_q(\mathfrak{g})$ are denoted by E_i, F_i, $K_i^{\pm 1}$ for each i which labels a node of the Dynkin diagram of $\mathfrak{g}$. There are automorphisms T_i of the algebra $U_q(\mathfrak{g})$ intoduced by G. Lusztig which satisfy braid relations, see [Lus94]. These Lusztig automorphisms T_i are essential for constructing a Poincaré-Birkhoff-Witt-type basis for $U_q(\mathfrak{g})$.

1.1 Quantum symmetric pairs

Let $\mathfrak{g}$ be a complex semsimple Lie algebra, and let $U_q(\mathfrak{g})$ be the corresponding Drinfeld-Jimbo quantised enveloping algebra. Let $\theta\colon \mathfrak{g} \to \mathfrak{g}$ be an involutive Lie algebra automorphism, and let

$$\mathfrak{k} = \left\{ x \in \mathfrak{g} \;\middle|\; \theta(x) = x \right\} \tag{1.1}$$

be the corresponding fixed Lie subalgebra. We call $\mathfrak{g}$ a *symmetric Lie algebra* and the pair $(\mathfrak{g}, \mathfrak{k})$ a *symmetric pair*. Up to conjugation, involutive automorphisms are parametrised by combinatorial data attached to the Dynkin diagram of $\mathfrak{g}$ known as *Satake diagrams* (X, τ). Here, X denotes a subset of the index set I which labels the nodes of the Dynkin diagram of $\mathfrak{g}$, and τ denotes a diagram automorphism. We obtain a classification for symmetric pairs through Satake diagrams, see [Ara62, p. 32/33].

Quantum symmetric pairs provide quantum group analogues of the universal enveloping algebra $U(\mathfrak{k})$. In particular, families of subalgebras $\mathcal{B}_{\mathbf{c}} \subset U_q(\mathfrak{g})$ are constructed which depend on a set of parameters $\mathbf{c}$, see [Let99], [Let02] and [Kol14]. Such subalgebras are quantum group analogues of $U(\mathfrak{k})$ in the sense that $\mathcal{B}_{\mathbf{c}}$ specialises to $U(\mathfrak{k})$ as q tends to 1. The crucial property of the algebra $\mathcal{B}_{\mathbf{c}}$ is that it is a *right coideal subalgebra* of $U_q(\mathfrak{g})$, meaning that

$$\Delta(\mathcal{B}_{\mathbf{c}}) \subseteq \mathcal{B}_{\mathbf{c}} \otimes U_q(\mathfrak{g}) \tag{1.2}$$

where Δ denotes the coproduct of $U_q(\mathfrak{g})$. We call the pair $\big(U_q(\mathfrak{g}), \mathcal{B}_{\mathbf{c}}\big)$ a *quantum symmetric pair.*

A comprehensive theory of quantum symmetric pairs was developed by G. Letzter in [Let99] and [Let02] for all finite-dimensional symmetric semisimple Lie algebras. The construction of quantum symmetric pairs in this setting only relies on the Drinfeld-Jimbo presentation of quantised enveloping algebras, see [Dri87] and [Jim85], and on involutive automorphisms of $\mathfrak{g}$.

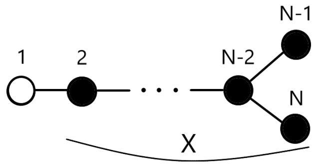
In [GK91], A. Gavrilik and A. Klimyk introduced a non-standard quantum deformation $U_q'(\mathfrak{so}_n)$ of the universal enveloping algebra of the special orthogonal Lie algebra $\mathfrak{so}_n$ for $n \geq 3$ in terms of generators and relations. These algebras have the desirable property that there are natural inclusions $U_q'(\mathfrak{so}_{n-1}) \subseteq U_q'(\mathfrak{so}_n)$ which do not exist for the Drinfeld-Jimbo quantum groups. Gavrilik and Klimyk did not observe that their algebras could be realised as coideal subalgebras of $U_q(\mathfrak{sl}_n)$.

In the early 1990s, M. Noumi, T. Sugitani and M. Dijkhuizen constructed quantum group analogues of all classical symmetric pairs with the aim to perform harmonic analysis on quantum group analogues of symmetric spaces. In his influential paper [Nou96], Noumi constructed quantum group analogues of the symmetric pairs $\big(\mathfrak{sl}_n, \mathfrak{so}_n\big)$ and $\big(\mathfrak{sl}_{2N}, \mathfrak{sp}_{2N}\big)$. In the first case, this reproduced Gavrilik and Klimyk's non-standard quantum deformation $U_q'(\mathfrak{so}_n)$. In the second case, he obtained a non-standard quantum deformation $U_q'(\mathfrak{sp}_{2N})$ of $U(\mathfrak{sp}_{2N})$. Up to some conventional differences, Noumi's non-standard quantum deformations coincide with those obtained in Letzter's theory for these symmetric pairs.

A detailed classification of finite-dimensional irreducible representations of $U_q'(\mathfrak{so}_n)$ has been obtained by N. Iorgov and Klimyk for when q is not a root of unity, see [IK05]. Similarly, A. Molev gave a classification of finite-dimensional irreducible representations of $U_q'(\mathfrak{sp}_{2N})$ in terms of their highest weights, see [Mol06]. More recently, H. Wenzl studied the representation theory of $U_q'(\mathfrak{so}_n)$ via a Verma module approach and obtained new results at roots of unity, see [Wen20].

The general classification of finite-dimensional representations of the coideal subalgebras $\mathcal{B}_{\mathbf{c}}$ is an open problem. One difficulty is to construct a suitable analogue of a Cartan subalgebra for $\mathcal{B}_{\mathbf{c}}$. This problem has essentially been solved by Letzter in [Let19]. The next difficulty is to construct suitable analogues of (simple) root vectors for $\mathcal{B}_{\mathbf{c}}$. To this end, a general approach has been developed by H. Watanabe in [Wat21]. However, Watanabe's approach hinges on a crucial construction [Wat21, Conjecture 3.3.3]. Once this conjecture is verified, a Verma module approach leads to a classification of finite-dimensional simple $\mathcal{B}_{\mathbf{c}}$-modules. Specialisation (that is, the limit $q \to 1$) forms an essential ingredient in this classification, and also allows it to determine the characters of irreducible representations and to prove complete reducibility.

In the present thesis, we study the case of the symmetric pair $(\mathfrak{so}_{2N}, \mathfrak{so}_{2N-1})$, which is the case *DII* in Araki's table [Ara62, p. 32/33]. In this case, the Satake diagram is illustrated below



and $\mathcal{B}_{\mathbf{c}}$ contains a large Hopf subalgebra $U_q(\mathfrak{g}_X)$ for $X = \{2, \ldots, N\}$. It turns out that the subalgebra generated by the elements $K_i^{\pm 1}$ for $i \in X$ is a suitable analogue of a Cartan subalgebra of $\mathfrak{so}_{2N-1}$. This case has not been studied in [Wat21]. Moreover, we wish to avoid specialisation arguments in the classification of finite-dimensional representations of $\mathcal{B}_{\mathbf{c}}$. We refer to $\mathcal{B}_{\mathbf{c}}$ as the *very non-standard quantum* $\mathfrak{so}_{2N-1}$*-algebra* to distinguish it from the non-standard quantum deformation of $U(\mathfrak{so}_{2N-1})$ that was introduced by A. Gavrilik and A. Klimyk in [GK91]. The algebra $\mathcal{B}_{\mathbf{c}}$ is generated by $U_q(\mathfrak{g}_X)$ and an additional element which we denote by B_1.

1.2 Poincaré-Birkhoff-Witt-type basis for $\mathcal{B}_{\mathbf{c}}$

By a careful investigation of the classical case, we manage to define analogues of root vectors for $\mathcal{B}_{\mathbf{c}}$. Let $\alpha_1, \ldots, \alpha_N$ denote the simple roots of the Lie algebra $\mathfrak{so}_{2N}$. Define reduced words in the Weyl group of $\mathfrak{so}_{2N}$ by

$$\sigma_i = \begin{cases} s_i \cdots s_{N-2} s_{\tau^i(N)} s_{\tau^i(N-1)} s_{N-2} \cdots s_i & \text{for } 1 \le i \le N-2, \\ s_{\tau^i(N)} s_{\tau^i(N-1)} & \text{if } i = N-1, \end{cases} \tag{1.3}$$

where the s_j for $1 \le j \le N$ denote the simple reflections for $\mathfrak{so}_{2N}$, and we have the transposition $\tau = (N-1\ N)$. Then,

$$w_0 = \sigma_1 \sigma_2 \cdots \sigma_{N-1} \tag{1.4}$$

is a reduced expression of the longest element in the Weyl group. Write this reduced expression as $w_0 = s_{i_1} \cdots s_{i_{N(N-1)}}$. We obtain the $N(N-1)$ distinct positive roots of $\mathfrak{so}_{2N}$ as

$$\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}). \tag{1.5}$$

The algebra $U_q(\mathfrak{so}_{2N})$ is generated by the elements E_i, F_i and $K_i^{\pm 1}$ for $1 \le i \le N$, and exhibits a triangular decomposition $U_q(\mathfrak{so}_{2N}) \cong \mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+$. Let $T_i \colon U_q(\mathfrak{so}_{2N}) \to U_q(\mathfrak{so}_{2N})$ be the algebra automorphism defined in [Lus94] for $1 \le i \le N$. Define the root vectors

$$E_{\beta_j} = \begin{cases} E_1 & \text{if } j = 1, \\ T_{i_1}^{-1} \cdots T_{i_{j-1}}^{-1}(E_{i_j}) & \text{for } 2 \le j \le N(N-1) \end{cases} \tag{1.6}$$

in $\mathcal{U}^+$, and

$$F_{\beta_j} = \begin{cases} F_1 & \text{if } j = 1, \\ T_{i_1} \cdots T_{i_{j-1}}(F_{i_j}) & \text{for } 2 \le j \le N(N-1) \end{cases} \tag{1.7}$$

in $\mathcal{U}^-$. By the Poincaré-Birkhoff-Witt Theorem [Jan96, 8.24], the ordered monomials

$$E_{\mathcal{I}} = E_{\beta_{N(N-1)}}^{i_{N(N-1)}} \cdots E_{\beta_1}^{i_1} \qquad \text{and} \qquad F_{\mathcal{J}} = F_{\beta_{N(N-1)}}^{j_{N(N-1)}} \cdots F_{\beta_1}^{j_1} \tag{1.8}$$

form a vector space basis of $\mathcal{U}^+$ and $\mathcal{U}^-$ respectively, for $\mathcal{I} = \big(i_1, \ldots, i_{N(N-1)}\big) \in \mathbb{N}_0^{N(N-1)}$ and $\mathcal{J} = \big(j_1, \ldots, j_{N(N-1)}\big) \in \mathbb{N}_0^{N(N-1)}$. In analogy to the root vectors in $\mathcal{U}^-$, we define root vectors of $\mathcal{B}_{\mathbf{c}}$ by

$$B_{\beta_j} = \begin{cases} B_1 & \text{if } j = 1, \\ T_j^{-1} \cdots T_2^{-1}(B_1) & \text{for } 2 \leq j \leq N-1, \\ T_{2N-j}^{-1} \cdots T_N^{-1} T_{N-2}^{-1} \cdots T_2^{-1}(B_1) & \text{for } N \leq j \leq 2(N-1), \\ F_{\beta_j} & \text{if } j > 2(N-1). \end{cases} \tag{1.9}$$

Let $\gamma_1, \ldots, \gamma_{N-1}$ be the simple roots of $\mathfrak{so}_{2N-1}$, and denote the Chevalley generators of $\mathfrak{so}_{2N-1}$ by e_{γ_i}, f_{γ_i} and h_{γ_i} for $1 \leq i \leq N-1$. It turns out that the elements E_{i+1} and F_{i+1} are positive and negative root vectors corresponding to the simple roots γ_i, respectively for $1 \leq i \leq N-2$. Moreover, a comparison with the classical case shows that the elements $B_{\beta_{N-1}}$ and B_{β_N} defined by (1.9) are suitable analogues of the positive and negative root vectors corresponding to the simple root γ_{N-1}, respectively. Finally, it turns out that the root vectors B_{β_j} are positive for $1 \leq j \leq N-1$ and are negative for $j \geq N$, see Proposition 5.7. Similarly to the monomials $F_{\mathcal{J}}$ in (1.8), we define

$$B_{\mathcal{J}} = B_{\beta_{N(N-1)}}^{j_{N(N-1)}} \cdots B_{\beta_1}^{j_1} \tag{1.10}$$

for $\mathcal{J} = \big(j_1, \ldots, j_{N(N-1)}\big) \in \mathbb{N}_0^{N(N-1)}$. Section 5.4 is dedicated to proving the following result.

Theorem A (Theorem 5.11)**.** *The ordered monomials $B_{\mathcal{J}} K_{\mathcal{D}} E_{\mathcal{I}}$ form a basis for the algebra $\mathcal{B}_{\mathbf{c}}$, where $\mathcal{I} \in \mathbb{N}_0^{(N-1)(N-2)}$, $\mathcal{J} \in \mathbb{N}_0^{N(N-1)}$, and $K_{\mathcal{D}} = K_N^{d_N} \cdots K_2^{d_2}$ for $\mathcal{D} = (d_2, \ldots, d_N) \in \mathbb{Z}^{N-1}$.*

Observe that the monomials in Theorem A are ordered in such a way that all of the positive root vectors are on the right-hand side (since the root vectors B_{β_j} for $1 \leq j \leq N-1$ commute with the K_i up to a power of q), whilst all of the negative root vectors are on the left-hand side.

1.3 Classification of finite-dimensional simple $\mathcal{B}_{\mathbf{c}}$-modules

By trying to mimic [Jan96, Chapter 5], we classify all of the finite-dimensional irreducible representations of the algebra $\mathcal{B}_{\mathbf{c}}$ (up to a choice of signs). Firstly, if V is a finite-dimensional $\mathcal{B}_{\mathbf{c}}$-module, then the subalgebra consisting of the elements K_i is always diagonalisable by the representation theory of $U_q(\mathfrak{g}_X)$. Therefore, V must decompose into weight spaces and, since there are only finitely many weights for which the weight spaces are non-zero, there must exist a weight vector $v \in V$ of highest weight λ such that $E_i v = 0$ for all $i \in X$, and $B_{\beta_{N-1}} v = 0$. Furthermore, such a weight λ is dominant integral in the sense that $\lambda(h_{\gamma_{i-1}}) \geq 0$ for all $i \in X$.

The representations of $\mathcal{B}_{\mathbf{c}}$ may be constructed by Verma modules. Similarly to [Jan96, 5.5], we are now able to define Verma modules since we have a triangular decomposition supported by Theorem A and have also defined root vectors for $\mathcal{B}_{\mathbf{c}}$. Thus, the Verma module of highest weight λ is the quotient $\mathcal{B}_{\mathbf{c}}$-module

$$M(\lambda) = \frac{\mathcal{B}_{\mathbf{c}}}{J_\lambda} \tag{1.11}$$

where J_λ is the left ideal of $\mathcal{B}_{\mathbf{c}}$ defined by

$$J_\lambda = \left(\sum_{i=2}^{N} \mathcal{B}_{\mathbf{c}} E_i + \mathcal{B}_{\mathbf{c}} \Big(K_i - q^{n_i} \Big) \right) + \mathcal{B}_{\mathbf{c}} B_{\beta_{N-1}} \tag{1.12}$$

for the non-negative integers

$$n_i = \begin{cases} \lambda\big(h_{\gamma_{i-1}}\big) & \text{for } 2 \leq i \leq N-1, \\ \lambda\big(h_{\gamma_{N-2}}\big) + \lambda\big(h_{\gamma_{N-1}}\big) & \text{if } i = N. \end{cases} \tag{1.13}$$

By definition, we know that $E_i v_\lambda = 0$ and $K_i v_\lambda = q^{n_i} v_\lambda$ for all $i \in X$, and also $B_{\beta_{N-1}} v_\lambda = 0$, where the coset $v_\lambda = 1 + J_\lambda$ generates the Verma module $M(\lambda)$. It turns out that the elements $B_{\mathcal{J}}$ in (1.10) where $j_1 = \cdots = j_{N-1} = 0$ form a basis of $M(\lambda)$ on v_λ, see Theorem 6.9.

Given any $\mathcal{B}_{\mathbf{c}}$-module V and highest weight vector $v \in V$ of weight λ, there exists a unique homomorphism of $\mathcal{B}_{\mathbf{c}}$-modules $\phi\colon M(\lambda) \to V$ such that $v_\lambda \mapsto v$. This is the universal property of the Verma module $M(\lambda)$. By this universal property, and by the existence of a highest weight vector, each finite-dimensional simple $\mathcal{B}_{\mathbf{c}}$-module must be a homomorphic image of some $M(\lambda)$ and, hence, is isomorphic to a simple quotient module of the form

$$L(\lambda) = \frac{M(\lambda)}{N(\lambda)} \tag{1.14}$$

where $N(\lambda)$ is the unique proper maximal submodule of $M(\lambda)$. It is not clear at this stage that the quotient module $L(\lambda)$ is finite-dimensional. To show finite-dimensionality, we manage to find submodules of $M(\lambda)$ using an analogue of [Jan96, Lemma 5.6] which are Verma modules themselves, and by which we may take quotients of $M(\lambda)$. More precisely, we find $N-2$ of these submodules, namely for each γ_i for $1 \leq i \leq N-2$. The root vectors $B_{\beta_{N-1}}$ and B_{β_N} do not form an $\mathfrak{sl}_2$-triple, however, for some non-zero subspace $H(V)$ of V, we show that these root vectors satisfy the relations of $U_q(\mathfrak{sl}_2)$, see Proposition 6.6. From this, we obtain another submodule of $M(\lambda)$, namely for the simple root γ_{N-1}, by which we may also take a quotient of $M(\lambda)$.

We cannot use the Weyl-group invariance argument from [Jan96, 5.9] since the last $\mathfrak{sl}_2$-triple only acts on the subspace $H(V)$. Instead, we consider an algebra $\mathcal{A}$, generated by $U_q(\mathfrak{g}_X)$ and the element F_1, which is in fact isomorphic to the associated graded algebra of $\mathcal{B}_{\mathbf{c}}$ via a filtration of the algebras $\mathcal{A}$ and $\mathcal{B}_{\mathbf{c}}$ (defined by a degree function on the generators), see Proposition 5.3. Using a filtered-graded argument (see Section 6.5) this enables us to take an even larger quotient of the Verma module $M(\lambda)$ and we show that this quotient is finite-dimensional. Since this quotient is at least as large as the simple quotient module $L(\lambda)$, the module $L(\lambda)$ must also be finite-dimensional. Hence, we obtain the following main result.

Theorem B (Theorem 6.18)**.** *For each dominant weight λ, the simple $\mathcal{B}_{\mathbf{c}}$-module $L(\lambda)$ has finite-dimension. Moreover, each finite-dimensional simple $\mathcal{B}_{\mathbf{c}}$-module is isomorphic to exactly one $L(\lambda)$ with dominant weight λ.*

We deduce that there is a one-to-one correspondence between the dominant integral weights λ and the finite-dimensional simple $\mathcal{B}_{\mathbf{c}}$-modules $L(\lambda)$. Moreover, each finite-dimensional simple $\mathcal{B}_{\mathbf{c}}$-module is uniquely determined by an $(N-1)$-tuple of non-negative integers.

1.4 Organisation

The thesis is organised as follows. In Chapter 2, we provide necessary background material on algebras, more specifically, Hopf algebras and universal enveloping algebras of Lie algebras. In particular, we recall the PBW-Theorem in Section 2.4.4. We then discuss the theory of finite-dimensional complex semisimple Lie algebras, and information such as the roots, generators and Dynkin diagram are given explicitly in Section 2.6 for the Lie algebras $\mathfrak{so}_{2N}$, $\mathfrak{so}_{2N-1}$ and $\mathfrak{sp}_{2N}$.

In Chapter 3, we review general theory of symmetric semisimple Lie algebras and then discuss symmetric pairs. The fixed Lie subalgebra is given in terms of generators in Section 3.2. We study examples of symmetric pairs and, in particular, we give an explicit construction of the symmetric pair $(\mathfrak{so}_{2N}, \mathfrak{so}_{2N-1})$ in Section 3.3.3.

Chapter 4 provides an overview of the theory of quantum groups, more specifically, quantised enveloping algebras of semisimple Lie algebras. In Section 4.2, we review the most fundamental example, the quantised enveloping algebra of the Lie algebra $\mathfrak{sl}_2$, and recall the classification of its finite-dimensional irreducible representations. In Section 4.5, we introduce notions of root vectors and state the PBW-Theorem for $U_q(\mathfrak{g})$ in general. We make this explicit in Section 4.5.1 for the Lie algebra $\mathfrak{so}_{2N}$. At the end of the chapter, we introduce quantum symmetric pairs and give an explicit general presentation of quantum symmetric pair coideal subalgebras $\mathcal{B}_{\mathbf{c}}$ in terms of generators and relations.

The main results of this thesis are contained in Chapters 5 and 6. In Chapter 5, we give a formal presentation of the very non-standard quantum $\mathfrak{so}_{2N-1}$-algebra in terms of generators and relations following [BK15]. We justify a choice of root vectors for the algebra $\mathcal{B}_{\mathbf{c}}$, and then give a PBW-type basis of $\mathcal{B}_{\mathbf{c}}$ in terms of ordered monomials of these root vectors. This proves Theorem A. We provide a list of all of the positive and negative root vectors of $\mathcal{B}_{\mathbf{c}}$ in the tables in Appendix A.

The representation theory of $\mathcal{B}_{\mathbf{c}}$ is studied in Chapter 6. We prove the existence of a highest weight vector in finite-dimensional $\mathcal{B}_{\mathbf{c}}$-modules in Section 6.1. In Section 6.2, we find the missing $\mathfrak{sl}_2$-triple acting on the subspace $H(V)$. We construct the Verma modules $M(\lambda)$ in Section 6.3, and show that the highest weights need to be dominant integral for simple quotients to be finite-dimensional. After we quotient by the submodules of $M(\lambda)$ found in Section 6.4, we use the filtered-graded argument in Section 6.5 to obtain (up to a choice of signs) a classification of the finite-dimensional irreducible representations of $\mathcal{B}_{\mathbf{c}}$ and, hence, prove Theorem B.

Chapter 2

Background

2.1 Notions from linear algebra

We begin by giving a summary of the results we need from linear algebra. We assume that the reader is familiar with the definition (and examples) of vector spaces, bases, subspaces and direct sums. Although we provide the general theory, we will only deal with finite-dimensional vector spaces over the complex field, so the main example to bear in mind here is $\mathbb{C}^n$.

2.1.1 Isomorphism theorems and dual spaces

Fix a field $\mathbb{F}$, and let V and W be finite-dimensional vector spaces over $\mathbb{F}$. From [EW06, 16.1] we recall that, for some vector $v \in V$, a **coset** of a subspace $W \subseteq V$ is a set of the form

$$v + W = \{v + w \mid w \in W\}. \tag{2.1}$$

The set of all cosets of W in V is called the **quotient space** of V by W and is denoted by V/W. This is itself a vector space, with the zero element $\underline{0} + W = W$, if we define addition by

$$(u + W) + (v + W) = u + v + W \tag{2.2}$$

for $u, v \in V$, and scalar multiplication by

$$\lambda(v + W) = \lambda v + W \tag{2.3}$$

for $v \in V$ and $\lambda \in \mathbb{F}$. From [EW06, 16.2] we also recall that, for any two finite-dimensional $\mathbb{F}$-vector spaces V and W, a map $f \colon V \to W$ is **linear** if it satisfies

$$f(\lambda u + \mu v) = \lambda f(u) + \mu f(v) \tag{2.4}$$

for all $u, v \in V$ and $\lambda, \mu \in \mathbb{F}$. Additionally if it is *bijective*, then the map f is called an **isomorphism**, and we write $V \cong W$ to indicate that the vector spaces V and W are isomorphic.

We now state (without proof) the isomorphism theorems for vector spaces.

Theorem 2.1 ([EW06, Theorem 16.1])**.** *Let V and W be vector spaces over a field $\mathbb{F}$.*

(a) *If $f\colon V \to W$ is a linear map, then* $\ker(f)$ *is a subspace of V,* $\operatorname{im}(f)$ *is a subspace of W, and*

$$V \big/ \ker(f) \;\cong\; \operatorname{im}(f) \;.$$

(b) *If V and W are subspaces of the same vector space, then*

$$(V+W) \big/ W \;\cong\; V \big/ (V \cap W) \;.$$

(c) *If V and W are subspaces of a vector space U such that $W \subseteq V$, then V/W is a subspace of U/W, and*

$$(U/W) \big/ (V/W) \;\cong\; U \big/ V \;.$$

The following result is deduced from Theorem 2.1 (b) and (c), and will be crucial later on when proving the main classification theorem (see Theorem 6.18) for finite-dimensional irreducible representations of the very non-standard quantum $\mathfrak{so}_{2N-1}$-algebra.

Corollary 2.2. *If U is a vector space and $V, W \subseteq U$ are two subspaces, then*

$$(U/V) \big/ \big(W/(V \cap W)\big) \;\cong\; U \big/ (V+W) \;\cong\; (U/W) \big/ \big(V/(V \cap W)\big) \;.$$

are isomorphisms of quotient spaces.

Proof. By symmetry, it is only necessary to prove one of the isomorphisms, so we shall check the one on the right.

Suppose that V and W are two subspaces of the vector space U. Theorem 2.1 (b) tells us that $V/(V \cap W) \cong (V+W)/W$ is an isomorphism of quotient spaces given by $v + (V \cap W) \mapsto v + W$. Additionally, since $V, W \subseteq U$ we have $W \subseteq V + W \subseteq U$. By Theorem 2.1 (c) it follows that $(V+W)/W$ is a subspace of U/W and, futhermore, we obtain a canonical isomorphism

$$(U/W) \big/ (V+W)/W \;\cong\; U \big/ (V+W)$$

by mapping the coset $(u+W) + (V+W)/W$, a coset in U/W by the subspace $(V+W)/W$, to $u + V + W$. Putting the isomorphisms together, we are done. ☐

Definition 2.3 ([EW06, 16.7.1])**.** *The **dual space** of V, denoted by V^*, is defined by*

$$V^* = \{ f\colon V \to \mathbb{F} \mid f \text{ is linear} \}.$$

If V has a vector space basis $\{v_1, \ldots, v_n\}$, then the associated **dual basis** of V consists of the linear maps $f_i\colon V \to \mathbb{F}$ defined on the basis elements by $f_i(v_j) = \delta_{i,j}$. Here, δ is the **Kronecker**

delta defined by

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.5}$$

In other words, $\{f_1, \dots, f_n\}$ is a basis for V^* and, in particular, $\dim(V) = \dim(V^*)$.

Remark 2.4. The dual space of V^*, denoted V^{**}, can be identified with V in a natural way. Given $v \in V$, we may define a linear map $\epsilon_v \colon V^* \to \mathbb{F}$ (known as the **evaluation map**) by

$$\epsilon_v(f) = f(v) \tag{2.6}$$

for all $f \in V^*$. Notice that, by definition, we have $\epsilon_v \in V^{**}$. Moreover, one may show that the map $\epsilon \colon V \to V^{**}$; $v \mapsto \epsilon_v$ is linear, and, in fact, an isomorphism, see [EW06, 16.7.1].

2.1.2 Bilinear forms

Definition 2.5 ([EW06, Definition 16.9])**.** *A **bilinear form** on an $\mathbb{F}$-vector space V is a map $\langle -,- \rangle \colon V \times V \to \mathbb{F}$ such that*

$$\langle \lambda_1 v_1 + \lambda_2 v_2, \mu_1 w_1 + \mu_2 w_2 \rangle = \lambda_1 \mu_1 \langle v_1, w_1 \rangle + \lambda_2 \mu_1 \langle v_2, w_1 \rangle + \lambda_1 \mu_2 \langle v_1, w_2 \rangle + \lambda_2 \mu_2 \langle v_2, w_2 \rangle$$

for all $v_1, v_2, w_1, w_2 \in V$ and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{F}$, that is, the map is linear in each component.

Example 2.6. If $\mathbb{F} = \mathbb{C}$ and $V = \mathbb{C}^n$, then the usual *dot product* is a bilinear form on V.

We can represent bilinear forms by matrices. Suppose that V is an n-dimensional $\mathbb{F}$-vector space with ordered basis $B = \{v_1, \dots, v_n\}$, and let $\langle -,- \rangle$ be a bilinear form on V. The $n \times n$-matrix

$$A_B = \left(a_{i,j}\right)_{i,j=1}^n = \left(\langle v_i, v_j \rangle\right)_{i,j=1}^n. \tag{2.7}$$

is referred to as the **matrix of the form** $\langle -,- \rangle$ with respect to the ordered basis B.

Remark 2.7. A bilinear form is completely determined by such a matrix of the form (2.7). Moreover, any $n \times n$-matrix over $\mathbb{F}$ is the matrix of some bilinear form on an $\mathbb{F}$-vector space V.

Proposition 2.8 ([Rom08, Theorem 11.2])**.** *Let V be an $\mathbb{F}$-vector space with ordered basis $B = \{v_1, \dots, v_n\}$, and let $\langle -,- \rangle$ be a bilinear form on V represented by the matrix A_B in* (2.7).

(a) If $u = \sum_{i=1}^n \lambda_i v_i$ and $w = \sum_{j=1}^n \mu_j v_j$ where $\lambda_i, \mu_j \in \mathbb{F}$, the form can be recovered from the matrix by the formula

$$\langle u, w \rangle = \underline{\lambda}^t A_B \, \underline{\mu} \tag{2.8}$$

where $\underline{\lambda}, \underline{\mu} \in \mathbb{F}^n$ denote the column vectors with entries λ_i, μ_j respectively.

(b) If $B' = \{v'_1, \dots, v'_n\}$ is also an ordered basis for V, then the new matrix of the form is

$$A_{B'} = P^t A_B P \tag{2.9}$$

where $P = \left(p_{i,j}\right)_{i,j=1}^n$ is the invertible $n \times n$-matrix defined by $v'_j = \sum_{i=1}^n p_{i,j} v_i$.

Proof. (a) Since $\langle -, - \rangle$ is a bilinear form over a field $\mathbb{F}$, from direct computation we obtain

$$\langle u, w \rangle = \left\langle \sum_{i=1}^{n} \lambda_i v_i, \sum_{j=1}^{n} \mu_j v_j \right\rangle = \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \mu_j \langle v_i, v_j \rangle = \sum_{i,j=1}^{n} \lambda_i \, a_{i,j} \, \mu_j = \underline{\lambda}^t A_B \, \underline{\mu}.$$

(b) We may now write $u = \sum_{i=1}^{n} \lambda'_i v'_i$ and $w = \sum_{j=1}^{n} \mu'_j v'_j$ where $\lambda'_i, \mu'_j \in \mathbb{F}$, and let $\underline{\lambda}', \underline{\mu}' \in \mathbb{F}^n$ denote the column vectors with entries λ'_i, μ'_j respectively. Then

$$\langle u, w \rangle = \underline{\lambda}^t A_B \, \underline{\mu} = \big(\underline{\lambda}'^t P^t\big) A_B \big(P \underline{\mu}'\big) = \underline{\lambda}'^t \big(P^t A_B P\big) \underline{\mu}'$$

and so $A_{B'} = P^t A_B P$ as required. □

Remark 2.9. Two matrices such as A_B and $A_{B'}$ that are related by (2.9) in Proposition 2.8 (b) are said to be **congruent**. Moreover, any two matrices are congruent if and only if they represent the *same* bilinear form on a vector space. A proof is provided in [Rom08, 11.2].

Definition 2.10 ([EW06, 16.7.2])**.** *The* ***orthogonal complement*** *of a subset U of a vector space V is the subspace*

$$U^{\perp} = \big\{ v \in V \;\big|\; \langle u, v \rangle = 0, \; \forall u \in U \big\} \subseteq V \tag{2.10}$$

given a bilinear form $\langle -, - \rangle$ on V. Furthermore, the form $\langle -, - \rangle$ is ***non-degenerate*** *if*

$$V^{\perp} = \{\underline{0}\}. \tag{2.11}$$

Lemma 2.11 ([EW06, Lemma 16.11])**.** *Suppose that $\langle -, - \rangle$ is a non-degenerate bilinear form on a finite-dimensional vector space V. For all subspaces $U \subseteq V$, we have*

$$\dim(U) + \dim(U^{\perp}) = \dim(V). \tag{2.12}$$

Additionally, if $U \cap U^{\perp} = \{\underline{0}\}$ then

$$V = U \oplus U^{\perp} \tag{2.13}$$

and, furthermore, the restrictions of $\langle -, - \rangle$ to U and to $U^{\perp}$ are non-degenerate.

Proof. For (2.12) we need to show that

$$\dim(U) = \dim(V) - \dim(U^{\perp}) = \dim\big(V/U^{\perp}\big).$$

The image of the map $U \to V^*$ given by $u \mapsto \langle u, - \rangle$ is 0 on $U^{\perp}$. Hence, we get a map

$$\varphi \colon U \to \big(V/U^{\perp}\big)^*; \qquad u \mapsto \langle -, u \rangle$$

which is an isomorphism. Indeed, injectivity of φ follows from the non-degeneracy of $\langle -,-\rangle$, and the map $\varphi\colon V \to V^*$; $v \mapsto \langle -, v\rangle = f$ is an isomorphism. We consider the restriction map $r\colon V^* \to U^*$ given by $f \mapsto f|_U$. By the isomorphism φ, observe that

$$\ker(r) = \big\{ f \in V^* \;\big|\; f|_U = 0 \big\} \cong \big\{ v \in V \;\big|\; \langle u, v\rangle = 0,\, \forall u \in U \big\} = U^\perp.$$

Theorem 2.1 (a) also implies that

$$\dim\big(\ker(r)\big) + \dim(U^*) = \dim(V^*)$$

and we deduce that this is equivalent to $\dim(U^\perp) + \dim(U) = \dim(V)$.

Now, let $\{u_1, \ldots, u_k\}$ be a basis of U and $\{w_1, \ldots, w_l\}$ be a basis of $U^\perp$. We assume that $U \cap U^\perp = \{\underline{0}\}$, and claim that $\{u_1, \ldots, u_k, w_1, \ldots, w_l\}$ is a basis of V. It suffices to show that this set is linearly independent. Indeed, suppose $u = \sum_{i=1}^k \lambda_i u_i$ and $w = \sum_{j=1}^l \mu_j w_j$ such that $u + w = 0$. This implies that

$$u = -w \in U \cap U^\perp$$

and hence, by the assumption, we get $u = w = \underline{0}$. □

Definition 2.12 ([EW06, Definition 16.12])**.** *A bilinear form $\langle -,-\rangle\colon V \times V \to \mathbb{F}$ is* ***symmetric*** *if*

$$\langle v, w\rangle = \langle w, v\rangle \tag{2.14}$$

for all $v, w \in V$, and ***skew-symmetric*** *if*

$$\langle v, w\rangle = -\langle w, v\rangle \tag{2.15}$$

for all $v, w \in V$.

Corollary 2.13. *Let V be a finite-dimensional $\mathbb{F}$-vector space with basis vectors $v_1, \ldots, v_n$, and let $\langle -,-\rangle\colon V \times V \to \mathbb{F}$ be a (skew-)symmetric, non-degenerate bilinear form. For any two vectors $u, w \in V$ with respective coefficient vectors $\underline{\lambda}, \underline{\mu} \in \mathbb{F}^n$, there exists an invertible (skew-)symmetric $n \times n$-matrix S such that*

$$\langle u, w\rangle = \underline{\lambda}^t S \underline{\mu}. \tag{2.16}$$

Proof. The formula (2.16) itself follows directly from (2.8) in Proposition 2.8 (a). Moreover, if the form $\langle -,-\rangle$ is symmetric, by definition we know that

$$\underline{\lambda}^t S \underline{\mu} = \underline{\mu}^t S \underline{\lambda} = (\underline{\mu}^t S \underline{\lambda})^t = \underline{\lambda}^t S^t \underline{\mu}$$

for $\underline{\lambda}, \underline{\mu} \in \mathbb{F}^n$ and this implies that $\underline{\lambda}^t (S - S^t)\underline{\mu} = 0$. Since $\dim(V) = n$ and V has a basis $\{v_1, \ldots, v_n\}$, there exists a canonical isomorphism $V \to \mathbb{F}^n$ given by $v_i \mapsto e_i$ for each i. Now let $M \in \mathrm{Mat}_n(\mathbb{F})$ and assume that $\underline{\lambda}^t M \underline{\mu} = 0$ for all $\underline{\lambda}, \underline{\mu} \in \mathbb{F}^n$. If we write $M = (m_{i,j})_{i,j=1}^n$, then

$$\underline{\lambda}^t M \underline{\mu} = \sum_{i,j=1}^{n} \lambda_i m_{i,j} \mu_j = 0$$

for all $\underline{\lambda}, \underline{\mu} \in \mathbb{F}^n$. However, setting $\underline{\lambda} = e_i$ and $\underline{\mu} = e_j$ with $i \neq j$ implies that $m_{i,j} = 0$, and hence $M = \mathbf{0}_n$. Therefore, we deduce that $S - S^t = 0$, or equivalently $S = S^t$ so S is indeed a symmetric matrix. The case that $\langle -, - \rangle$ is skew-symmetric is similar; instead we obtain that $\underline{\lambda}^t (S + S^t) \underline{\mu} = 0$ and therefore deduce that $S = -S^t$ as expected.

It remains to show that S is invertible. Indeed, assume that $\underline{\mu} \in \mathbb{F}^n$ satisfies $S\underline{\mu} = \underline{0}$. This implies that $\underline{\lambda}^t S \underline{\mu} = 0$ for all $\underline{\lambda} \in \mathbb{F}^n$ and, moreover, for the corresponding $u, w \in V$ we have $\langle u, w \rangle = 0$. However, since $\langle -, - \rangle$ is non-degenerate, we must have $w = \underline{0}$, and hence $\underline{\mu} = \underline{0}$. In other words, the linear equation $S\underline{\mu} = \underline{0}$ has only the trivial solution $\underline{\mu} = \underline{0}$, or equivalently, the matrix S is invertible. □

Proposition 2.14 ([EW06, Lemma 16.14])**.** *Let V be an $\mathbb{F}$-vector space with char$(\mathbb{F}) \neq 2$. If $\langle -, - \rangle$ is a non-degenerate symmetric bilinear form on V, then there is a basis $\{v_1, \ldots, v_n\}$ of V such that*

$$\langle v_i, v_j \rangle = 0$$

if $i \neq j$, and

$$\langle v_i, v_i \rangle \neq 0.$$

Proof. We use induction on $n = \dim(V)$. The result is clear if $n = 1$; if $\langle v_1, v_1 \rangle = 0$ then $v_1 \in V^\perp$, but $\langle -, - \rangle$ is non-degenerate and $v_1 \neq \underline{0}$ so we get a contradiction.

Assume now that $\dim(V) \geq 2$, and for $v, w \in V$ observe the identity

$$\langle v + w, v + w \rangle = \langle v, v \rangle + \langle w, w \rangle + 2\langle v, w \rangle \tag{2.17}$$

since $\langle -, - \rangle$ is symmetric. If $\langle v, v \rangle = 0$ for all $v \in V$, then since char$(\mathbb{F}) \neq 2$ we have that $\langle v, w \rangle = 0$ for all $v, w \in V$ which would contradict the assumption that $\langle -, - \rangle$ is non-degenerate.

We may choose $v \in V$ such that $\langle v, v \rangle \neq 0$. If we let $U = \mathrm{Span}\{v\}$, then

$$U \cap U^\perp = \{\underline{0}\} \tag{2.18}$$

by hypothesis. It follows by Lemma 2.11 that

$$V = U \oplus U^\perp$$

and, moreover, the restriction of $\langle -, - \rangle$ to $U^\perp$ is non-degenerate. Hence, by the inductive hypothesis, there exists a basis $\{v_1, \ldots, v_{n-1}\}$ of $U^\perp$ such that $\langle v_i, v_j \rangle = 0$ for $i \neq j$ and

$\langle v_i, v_i \rangle \neq 0$ for $1 \leq i \leq n-1$. Also, since

$$\langle v, v_j \rangle = 0 \tag{2.19}$$

for $j \neq n$, if we set $v_n = v$ then the basis $\{v_1, \dots, v_n\}$ has the required properties. □

Remark 2.15. We can be more precise about the diagonal entries $d_i = \langle v_i, v_i \rangle$. If $\mathbb{F} = \mathbb{C}$, then we can find $\lambda_i \in \mathbb{C}$ so that

$$\lambda_i^2 = d_i \tag{2.20}$$

and hence we may assume that

$$\langle v_i, v_i \rangle = 1 \tag{2.21}$$

for all i. In this case, the matrix representing $\langle -, - \rangle$ is the $n \times n$ identity matrix.

Proposition 2.16 ([EW06, Lemma 16.15]). *Let V be a finite-dimensional $\mathbb{F}$-vector space. If $\langle -, - \rangle$ is a non-degenerate skew-symmetric bilinear form on V, then we have $\dim(V) = 2N$ for some $N \in \mathbb{N}$ and, moreover, there is a basis of V such that*

$$\langle v_i, v_{N+i} \rangle \neq 0$$

for $1 \leq i \leq N$, and

$$\langle v_i, v_j \rangle = 0$$

if $|i - j| \neq N$.

Proof. Since $\langle -, - \rangle$ is non-degenerate, for $\underline{0} \neq v \in V$ we may find some $w \in V$ such that $\langle v, w \rangle \neq 0$. Since $\langle v, v \rangle = 0 = \langle w, w \rangle$, we know that v and w are linearly independent, so set $v_1 = v$ and $v_2 = w$. If $\dim(V) = 2$, then we are done.

Otherwise, if we let $U = \operatorname{Span}\{v_1, v_2\}$, then one can show that $U \cap U^\perp = \{\underline{0}\}$ and $V = U \oplus U^\perp$ by Lemma 2.11. Now the restriction of $\langle -, - \rangle$ to $U^\perp$ is non-degenerate (and also skew-symmetric). The result follows by induction on N. □

Remark 2.17. If $\mathbb{F} = \mathbb{C}$, one may arrange that the matrix representing $\langle -, - \rangle$ has the form

$$\begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$$

where I_N is the $N \times N$ identity matrix.

2.1.3 Tensor products

Let U, V, W be vector spaces. A map $U \times V \to W$ is **bilinear** if it is linear in each component.

Theorem 2.18 ([Kas95, Theorem II.1.1]). *Given vector spaces U and V, there exists a vector space $U \otimes V$ and a bilinear map $\varphi\colon U \times V \to U \otimes V$ such that, for every vector space W and bilinear map $\phi\colon U \times V \to W$, there is a unique linear map $h\colon U \otimes V \to W$ so that $\phi = h \circ \varphi$.*

Equivalently, the diagram

$$
\begin{array}{ccc}
U \times V & \xrightarrow{\varphi} & U \otimes V \\
 & \searrow^{\phi} & \downarrow h \\
 & & W
\end{array}
\tag{2.22}
$$

commutes. Moreover, $U \otimes V$ *is the unique vector space up to isomorphism satisfying this property.*

The vector space $U \otimes V$ characterised in Theorem 2.18 is called the **tensor product** of U and V. Tensor products are useful objects to study in algebra since the universal property (2.22) reduces the study of bilinear maps to that of linear maps. The following result, stated without proof, provides a few canonical isomorphisms for tensor products of vector spaces.

Lemma 2.19 ([Kas95, Proposition II.1.3])**.** *Let* U*,* V *and* W *be vector spaces over a field* $\mathbb{F}$*.*

(a) *There exists an isomorphism* $\big(U \otimes V\big) \otimes W \cong U \otimes \big(V \otimes W\big)$ *given by* $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$*.*

(b) *There exists a natural isomorphism* $\mathbb{F} \otimes V \cong V$ *given by* $\lambda \otimes v \mapsto \lambda v$*, also referred to as the standard scalar multiplication map on* V*. Similarly, there exists a natural isomorphism* $V \otimes \mathbb{F} \cong V$ *given by* $v \otimes \lambda \mapsto \lambda v$*.*

(c) *The linear map* $\tau_{U,V} \colon U \otimes V \to V \otimes U$ *given by* $\tau_{U,V}(u \otimes v) = v \otimes u$ *is an isomorphism.*

One can define a general element of the tensor product space as follows, see [EW06, 15.1.3]. Suppose that U and V are finite-dimensional vector spaces over a field $\mathbb{F}$. Let $\{u_1, \ldots, u_m\}$ be a basis of U and $\{v_1, \ldots, v_n\}$ be a basis of V. The tensor product $U \otimes V$ has a basis given by

$$\big\{u_i \otimes v_j \;\big|\; 1 \leq i \leq m, 1 \leq j \leq n\big\}$$

and, in particular, we have $\dim(U \otimes V) = mn = \dim(U)\dim(V)$. Then for $u = \sum_i \lambda_i u_i$ and $v = \sum_j \mu_j v_j$, with scalars $\lambda_i, \mu_j \in \mathbb{F}$ for all i, j, we define an element $u \otimes v \in U \otimes V$ by

$$u \otimes v = \sum_{i,j} \lambda_i \mu_j (u_i \otimes v_j). \tag{2.23}$$

Now suppose that U, U', V and V' are vector spaces, and let $f \colon U \to U'$ and $g \colon V \to V'$ be linear maps. The tensor product $f \otimes g \colon U \otimes V \to U' \otimes V'$ is the unique linear map given by

$$(f \otimes g)(u \otimes v) = f(u) \otimes g(v) \tag{2.24}$$

for all $u \in U$ and $v \in V$, studied further in [Kas95, II.2]. This map arises naturally from the universal property (2.22). Indeed, there is a bilinear map $f \times g \colon U \times V \to U' \otimes V'$ defined by

$$(f \times g)(u, v) = f(u) \otimes g(v)$$

which induces the map (2.24) following Theorem 2.18, by setting $W = U' \otimes V'$ and $\phi = f \times g$.

2.2 Algebras and modules

2.2.1 Algebras

By paraphrasing the definition in [Kas95, I.1], we first recall that an **algebra** over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space A that is equipped with a linear map $\mu\colon A\otimes A\to A$ known as *multiplication*. Let $id = id_A\colon A\to A$ denote the *identity* map of A. We say that an algebra A is **associative** if the diagram

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{\mu\otimes id} & A\otimes A \\ {\scriptstyle id\otimes\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A\otimes A & \xrightarrow[\mu]{} & A \end{array} \tag{2.25}$$

commutes, that is, if $\mu\circ(\mu\otimes id)=\mu\circ(id\otimes\mu)$. We also say that an algebra A is **unital** if there exists a linear map $\eta\colon\mathbb{F}\to A$ so that the diagram

$$\begin{array}{ccccc} \mathbb{F}\otimes A & \xrightarrow{\eta\otimes id} & A\otimes A & \xleftarrow{id\otimes\eta} & A\otimes\mathbb{F} \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle\cong} & \\ & & A & & \end{array} \tag{2.26}$$

commutes, that is, if $\mu\circ(\eta\otimes id)=\mu\circ(id\otimes\eta)$. The isomorphisms in diagram (2.26) are given by the standard scalar multiplication maps in Lemma 2.19(b).

Notation 2.20. *From now on, all algebras are assumed to be associative and unital. One may write the triple* (A,μ,η) *to express an algebra* A *together with its multiplication* μ *and unit* η*. Multiplication will generally be written as juxtaposition:* $\mu(a\otimes b)=a\cdot b=ab$ *for all* $a,b\in A$*. Thus, diagram* (2.25) *amounts to the associative law* $(ab)c=a(bc)$ *for all* $a,b,c\in A$*, whilst diagram* (2.26) *expresses the unit laws* $\eta(1_{\mathbb{F}})\,a=a=a\,\eta(1_{\mathbb{F}})$ *for all* $a\in A$*; so* A *has the identity element* $\eta(1_{\mathbb{F}})=1_A$*, also called a unit of* A (*and we may refer to* η *as the unit map*).

Additionally, we say that an algebra A is **commutative** if the diagram

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\tau} & A\otimes A \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & A & \end{array} \tag{2.27}$$

commutes, that is, if $\mu=\mu\circ\tau$, where $\tau=\tau_{A,A}$ denotes the *flip* map given in Lemma 2.19(c). Following Notation 2.20, diagram (2.27) amounts to the commutative law $ab=ba$ for all $a,b\in A$.

Definition 2.21 ([Lor18, 1.1.1 (1.2)])**.** *Let* (A,μ,η) *and* (A',μ',η') *be* $\mathbb{F}$*-algebras. A linear map* $f\colon A\to A'$ *is called an* ***algebra homomorphism*** *if the two diagrams*

$$\begin{array}{ccc} A\otimes A & \xrightarrow{f\otimes f} & A'\otimes A' \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu'} \\ A & \xrightarrow[f]{} & A' \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{F} & \xrightarrow{\eta} & A \\ & {\scriptstyle\eta'}\searrow & \downarrow{\scriptstyle f} \\ & & A' \end{array} \tag{2.28}$$

commute, that is, if $\mu' \circ (f \otimes f) = f \circ \mu$ *and* $\eta' = f \circ \eta$. *These diagrams are equivalent to the equations* $f(1_A) = 1_{A'}$ *and*

$$f(ab) = f(a)f(b)$$

for all $a, b \in A$. *Moreover, if* f *is also bijective, then* f *is called an* ***algebra isomorphism****.*

> *Example* 2.22 ([Kas95, I.1(7)]). The space $\mathrm{End}_{\mathbb{F}}(V)$ of linear endomorphisms of an $\mathbb{F}$-vector space V is an algebra, called the *endomorphism algebra* of V, with multiplication given by the composition of endomorphisms and unit being the identity map id_V of V. If V is n-dimensional, one may show that $\mathrm{End}_{\mathbb{F}}(V)$ is isomorphic to $\mathrm{Mat}_n(\mathbb{F})$, the $n \times n$-matrix algebra over $\mathbb{F}$.

An $\mathbb{F}$-subspace B of a given $\mathbb{F}$-algebra A that is itself an $\mathbb{F}$-algebra such that the inclusion map $B \hookrightarrow A$ is an algebra homomorphism is called a **subalgebra** of A. Moreover, B is called a (two-sided) **ideal** of A if

$$\mu_A(A \otimes B) \subset B \supset \mu_A(B \otimes A)$$

(and a *left* or *right* ideal if just the left or right inclusion above is satisfied, respectively). If B is an ideal of A, then there exists a unique algebra structure on the quotient space A/B such that the canonical projection map $A \to A/B$ is an algebra homomorphism, see [Kas95, I.1(3)].

2.2.2 Graded and filtered algebras

Definition 2.23 ([Kas95, Definition I.6.1]). *An algebra* A *is* ***graded*** *if there exist subspaces* $(A_i)_{i\in\mathbb{N}_0}$ *such that*

$$A = \bigoplus_{i\in\mathbb{N}_0} A_i \qquad \text{and} \qquad A_i \cdot A_j \subset A_{i+j} \tag{2.29}$$

for all $i, j \in \mathbb{N}_0$. *The elements of* A_i *are said to be "homogeneous of degree* i*". For convention, set* $1_A \in A_0$.

> *Example* 2.24. Free algebras are graded by the length of words; for $A = \mathbb{F}\{X\}$ where X has elements of degree 1, the subspace A_i is linearly generated by all monomials of degree i.

Proposition 2.25 ([Kas95, Proposition I.6.2]). *Let* $A = \bigoplus_{i\in\mathbb{N}_0} A_i$ *be a graded algebra, and consider a two-sided ideal* I *generated by homogeneous elements. Then*

$$I = \bigoplus_{i\in\mathbb{N}_0} I \cap A_i \tag{2.30}$$

and the quotient algebra A/I *is graded with*

$$(A/I)_i = A_i/(I \cap A_i) \tag{2.31}$$

for all i.

> *Example* 2.26. A polynomial algebra $\mathbb{F}[x_1, \ldots, x_n]$, where the generators $x_1, \ldots, x_n$ are of degree 1, is graded as the quotient of the free algebra $A = k\{x_1, \ldots, x_n\}$ by the ideal I generated by all the homogeneous elements $x_ix_j - x_jx_i$ for all $i, j \in \{1, \ldots, n\}$.

Definition 2.27 ([Kas95, Definition I.6.3])**.** *An algebra A is **filtered** if there exists an increasing sequence of subspaces* $\{0\} \subset \mathcal{F}_0(A) \subset \cdots \subset \mathcal{F}_i(A) \subset \cdots \subset A$ *such that*

$$A = \bigcup_{i \in \mathbb{N}_0} \mathcal{F}_i(A) \qquad \text{and} \qquad \mathcal{F}_i(A) \cdot \mathcal{F}_j(A) \subset \mathcal{F}_{i+j}(A). \tag{2.32}$$

The elements of $\mathcal{F}_i(A)$ are said to be "of degree at most i".

Any algebra A has a trivial filtration: $\mathcal{F}_i(A) = A$, for all i.

Example 2.28. Let A be an algebra generated by the set of elements $X = \{x_1, \ldots, x_n\}$. Let $\deg\colon X \to \mathbb{N}_0$ be a *degree* function. Then, we may define a filtration $\mathcal{F}$ on A as follows: let $\mathcal{F}_i(A)$ be the subspace generated by monomials in the generators $x_1, \ldots, x_n$ such that the sum of the degrees of the generators in each monomial is at most i.

Definition 2.29. *For any filtered algebra A, there exists a graded algebra* $\operatorname{gr}(A) = \bigoplus_{i \in \mathbb{N}_0} S_i$ *where*

$$S_i = \mathcal{F}_i(A)/\mathcal{F}_{i-1}(A) \tag{2.33}$$

for all i. We call $\operatorname{gr}(A)$ *the **associated graded algebra** of A. For each $f_i \in \mathcal{F}_i(A)$, we write the coset $x_i = f_i + \mathcal{F}_{i-1}(A)$ in S_i. Then, for each $x_i \in S_i$ and $x_j \in S_j$, we define*

$$x_i \cdot x_j = \big(f_i + \mathcal{F}_{i-1}(A)\big)\big(f_j + \mathcal{F}_{j-1}(A)\big) = f_i f_j + \mathcal{F}_{i+j-1}(A). \tag{2.34}$$

Remark 2.30. We filter any graded algebra $A = \bigoplus_{i \in \mathbb{N}_0} A_i$ by

$$\mathcal{F}_i(A) = \bigoplus_{0 \le j \le i} A_j \tag{2.35}$$

for $i, j \in \mathbb{N}_0$. Then, $\operatorname{gr}(A) = A$.

Example 2.31. Take a filtered algebra $A \supset \cdots \supset \mathcal{F}_1(A) \supset \mathcal{F}_0(A)$ and two-sided ideal I of A, then the quotient algebra A/I is filtered with

$$\mathcal{F}_i(A/I) = \mathcal{F}_i(A)/\big(I \cap \mathcal{F}_i(A)\big). \tag{2.36}$$

The associated graded algebra of A/I is then defined by

$$\operatorname{gr}(A/I) = \bigoplus_{i \in \mathbb{N}_0} \mathcal{F}_i(A)/\big(\mathcal{F}_{i-1}(A) + \mathcal{F}_i(A) \cap I\big). \tag{2.37}$$

2.2.3 Modules

Let (A, μ, η) be an $\mathbb{F}$-algebra. Recall that a (left) **A-module** is an $\mathbb{F}$-vector space V together with a linear map $\nu\colon A \otimes V \to V$ given by $a \otimes v \mapsto a \cdot v = av$ such that the two diagrams

$$\begin{array}{ccc} A \otimes A \otimes V & \xrightarrow{\mu \otimes id_V} & A \otimes V \\ {\scriptstyle id_A \otimes \nu}\downarrow & & \downarrow{\scriptstyle \nu} \\ A \otimes V & \xrightarrow[\nu]{} & V \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{F} \otimes V & \xrightarrow{\eta \otimes id_V} & A \otimes V \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \nu} \\ & & V \end{array}$$

commute, or equivalently,

$$a \cdot (b \cdot v) = (ab) \cdot v, \qquad 1_A \cdot v = v \tag{2.38}$$

for all $a, b \in A$ and $v \in V$. Then, the map $\nu \circ \varphi\colon A \times V \to V$ given by $(a, v) \mapsto av$ is called a (left) **action** of A on V, where we denote the canonical bilinear map $\varphi\colon A \times V \to A \otimes V$.

Remark 2.32. One similarly defines a *right* A-module using a linear map from $V \otimes A$ to V, however this is essentially the same as a left module over the so-called *opposite algebra* A^{op}, the algebra identical to A as a vector space but equipped with the multiplication $\mu^{\mathrm{op}} = \mu \circ \tau$. Moreover, right A^{op}-modules become left modules over $A^{\mathrm{op}\,\mathrm{op}} \cong A$, and, in particular, if A is commutative then $A = A^{\mathrm{op}}$. Therefore, we shall only consider left modules which, for simplicity, we just refer to as "modules" from now on.

Given any two A-modules V and W, an **A-module homomorphism** is a linear map $f\colon V \to W$ such that

$$f(av) = a \cdot f(v)$$

for all $a \in A$ and $v \in V$. Moreover, a subspace $U \subseteq V$ with an A-module structure is called an **A-submodule** of V if the inclusion map $u \mapsto u \in V$ is an A-module homomorphism.
Given A-modules $V_1, \ldots, V_k$, recall that the direct sum $V_1 \oplus \cdots \oplus V_k$ has an A-module structure given by

$$a(v_1, \ldots, v_k) = (av_1, \ldots, av_k)$$

where $a \in A$ and $v_i \in V_i$ for each i.

Definition 2.33 ([Kas95, Definition I.1.1])**.** *An A-module V is called **simple** if it has no other submodules than $\{0\}$ and V. Moreover, an A-module V is called **semisimple** if it is isomorphic to a direct sum of simple A-modules.*

Remark 2.34. An A-module V defines an algebra homomorphism $\rho\colon A \to \mathrm{End}(V)$ given by

$$\rho(a)(v) = av$$

which is called a *representation* of A on V. In the language of representations, simple modules are called *irreducible* representations (and semisimple modules are called *completely reducible* representations). This connection enables us to transfer the module-theoretic notions into the context of representations.

2.2.4 Graded and filtered modules

For graded algebras, there is a corresponding notion of graded modules.

Definition 2.35. *Let $A = \bigoplus_{i \in \mathbb{N}_0} A_i$ be a graded algebra. Then, a **graded A-module** is an A-module V such that*

$$V = \bigoplus_{j \in \mathbb{N}_0} V_j \qquad \text{and} \qquad A_i \cdot V_j \subseteq V_{i+j}.$$

Similarly, there is a corresponding notion of filtered modules over filtered algebras.

Definition 2.36. *Let* $A = \bigcup_{i\in\mathbb{N}_0} \mathcal{F}_i(A)$ *be a filtered algebra. Then, a* ***filtered*** A***-module*** *is an* A*-module* V *such that there exists an increasing sequence* $\{0\} \subset \mathcal{G}_0(V) \subset \cdots \subset \mathcal{G}_j(V) \subset \cdots \subset V$ *of subspaces of* V *satisfying the relations*

$$V = \bigcup_{j\in\mathbb{N}_0} \mathcal{G}_j(V) \qquad \text{and} \qquad \mathcal{F}_i(A) \cdot \mathcal{G}_j(V) \subseteq \mathcal{G}_{i+j}(V). \tag{2.39}$$

For any filtered module V over a filtered algebra A, we can form the associated graded $\mathrm{gr}(V)$ as a vector space. The vector space $\mathrm{gr}(V)$ is itself a module over $\mathrm{gr}(A)$. Its structure (2.40) is given in the proof of the following important result.

Lemma 2.37 ([KL85, Chapter 6])**.** *Let* A *be a filtered algebra, and let* V *be a filtered* A*-module. Then,* $\mathrm{gr}(V)$ *is a graded* $\mathrm{gr}(A)$*-module.*

Proof. Suppose that $A = \bigcup_i \mathcal{F}_i(A)$ is a filtered algebra and $V = \bigcup_j \mathcal{G}_j(V)$ is a filtered A-module. We denote the associated graded spaces of A and V respectively by

$$\mathrm{gr}(A) = \bigoplus_{i\in\mathbb{N}_0} \mathcal{F}_i(A)/\mathcal{F}_{i-1}(A) \qquad \text{and} \qquad \mathrm{gr}(V) = \bigoplus_{j\in\mathbb{N}_0} \mathcal{G}_j(V)/\mathcal{G}_{j-1}(V).$$

The $\mathrm{gr}(A)$-module structure on $\mathrm{gr}(V)$ is defined by

$$\big(a_i + \mathcal{F}_{i-1}(A)\big) \cdot \big(v_j + \mathcal{G}_{j-1}(V)\big) = a_i v_j + \mathcal{G}_{i+j-1}(V) \tag{2.40}$$

for any $a_i \in \mathcal{F}_i(A)$ and $v_j \in \mathcal{G}_j(V)$. By (2.39), this operation is well defined. We now need to check that this operation satisfies the properties in (2.38). Since $1_{\mathrm{gr}(A)} = 1_A \in \mathcal{F}_0(A)$, we have

$$1_{\mathrm{gr}(A)} \cdot v = 1_A \cdot \big(v_j + \mathcal{G}_{j-1}(V)\big) = (1_A \cdot v_j) + \mathcal{G}_{j-1}(V) = v_j + \mathcal{G}_{j-1}(V) = v$$

for all $v = v_j + \mathcal{G}_{j-1}(V) \in \mathrm{gr}(V)$. Additionally, consider

$$a = a_i + \mathcal{F}_{i-1}(A) \qquad \text{and} \qquad b = b_k + \mathcal{F}_{k-1}(A)$$

as elements of $\mathrm{gr}(A)$. Then, making use of either (2.34) or (2.40) in each step, we get

$$\begin{aligned}(a \cdot b) \cdot v &= \big(a_i b_k + \mathcal{F}_{i+k-1}(A)\big) \cdot \big(v_j + \mathcal{G}_{j-1}(V)\big)\\ &= a_i b_k v_j + \mathcal{G}_{i+k+j-1}(V)\\ &= \big(a_i + \mathcal{F}_{i-1}(A)\big) \cdot \big(b_k v_j + \mathcal{G}_{k+j-1}(V)\big) = a \cdot (b \cdot v).\end{aligned}$$

Hence, both relations in (2.38) are satisfied and indeed $\mathrm{gr}(V)$ is a module over $\mathrm{gr}(A)$. □

2.3 Hopf algebras

2.3.1 Coalgebras and coideals

The definition of a coalgebra is dual to that of an algebra in the sense that we systematically reverse all of the arrows in diagrams (2.25)-(2.28).

Definition 2.38 ([Kas95, Definition III.1.1(a)])**.** *A **coalgebra** over a field $\mathbb{F}$ is a triple $\big(C, \Delta, \varepsilon\big)$ where C is an $\mathbb{F}$-vector space and the maps $\Delta\colon C \to C \otimes C$ and $\varepsilon\colon C \to \mathbb{F}$ are linear such that the diagrams*

$$\begin{array}{ccc} C & \xrightarrow{\ \Delta\ } & C \otimes C \\ {\scriptstyle \Delta}\big\downarrow & & \big\downarrow{\scriptstyle id\otimes\Delta} \\ C \otimes C & \xrightarrow[\Delta\otimes id]{} & C \otimes C \otimes C \end{array} \tag{2.41}$$

and

$$\begin{array}{ccccc} \mathbb{F} \otimes C & \xleftarrow{\varepsilon\otimes id} & C \otimes C & \xrightarrow{id\otimes\varepsilon} & C \otimes \mathbb{F} \\ & {\scriptstyle \cong}\nwarrow & \big\uparrow{\scriptstyle \Delta} & \nearrow{\scriptstyle \cong} & \\ & & C & & \end{array} \tag{2.42}$$

commute, that is, we have $\big(id \otimes \Delta\big) \circ \Delta = \big(\Delta \otimes id\big) \circ \Delta$ and $\big(\varepsilon \otimes id\big) \circ \Delta = \big(id \otimes \varepsilon\big) \circ \Delta$.

The map Δ is called the **coproduct** (or *comultiplication*) of the coalgebra, while the map ε is called the **counit** of the coalgebra. The diagrams (2.41) and (2.42) express that the coproduct Δ is *coassociative* and *counital* respectively. Additionally, a coalgebra (C, Δ, ε) is **cocommutative** if the diagram

$$\begin{array}{ccc} & C & \\ {\scriptstyle \Delta}\swarrow & & \searrow{\scriptstyle \Delta} \\ C \otimes C & \xrightarrow{\ \ \tau\ \ } & C \otimes C \end{array} \tag{2.43}$$

commutes, that is, if $\Delta = \tau \circ \Delta$ where $\tau = \tau_{C,C}$ denotes the *flip* map given in Lemma 2.19(c).

Definition 2.39 ([Kas95, Definition III.1.1(b)])**.** *Let (C, Δ, ε) and $(C', \Delta', \varepsilon')$ be $\mathbb{F}$-coalgebras. A linear map $f\colon C \to C'$ is called a **coalgebra homomorphism** if the two diagrams*

$$\begin{array}{ccc} C & \xrightarrow{\ \Delta\ } & C \otimes C \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f\otimes f} \\ C' & \xrightarrow[\Delta']{} & C' \otimes C' \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\ \varepsilon\ } & \mathbb{F} \\ {\scriptstyle f}\big\downarrow & \nearrow{\scriptstyle \varepsilon'} & \\ C' & & \end{array} \tag{2.44}$$

commute, that is, if $(f \otimes f) \circ \Delta = \Delta' \circ f$ and $\varepsilon = \varepsilon' \circ f$.

Another important concept we will make use of is that of a coideal.

Definition 2.40 ([Kas95, Definition III.1.5])**.** *Let (C, Δ, ε) be an $\mathbb{F}$-coalgebra. A subspace $I \subseteq C$ is a **coideal** of C if $\varepsilon(I) = 0$, and*

$$\Delta(I) \subseteq I \otimes C + C \otimes I.$$

Given an $\mathbb{F}$-coalgebra (C, Δ, ε) and a coideal I, we can construct a new coalgebra in the following way. The coproduct Δ factors through a map $\bar{\Delta}$ from C/I to

$$C \otimes C \Big/ (I \otimes C + C \otimes I) = C\Big/I \otimes C\Big/I .$$

Similarly, the counit ε factors through a map $\bar{\varepsilon} \colon C/I \to \mathbb{F}$. Together, this gives a coalgebra structure on the quotient space C/I. We call the triple $\left(C/I, \bar{\Delta}, \bar{\varepsilon}\right)$ the **quotient coalgebra**.

Remark 2.41. We also have the notions of a *left coideal* and a *right coideal*. In particular, a subspace $I \subseteq C$ is called a right coideal if

$$\Delta(I) \subseteq I \otimes C.$$

2.3.2 Bialgebras and Hopf algebras

We now let H be an $\mathbb{F}$-vector space equipped simultaneously with an algebra structure (H, μ, η) and a coalgebra structure (H, Δ, ε).

Definition 2.42 ([Kas95, Definition III.2.2])**.** *An $\mathbb{F}$-**bialgebra** is a 5-tuple $(H, \mu, \eta, \Delta, \varepsilon)$ where the triple (H, μ, η) is an $\mathbb{F}$-algebra, the triple (H, Δ, ε) is an $\mathbb{F}$-coalgebra and the maps Δ and ε are algebra homomorphisms, that is, all of the diagrams*

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\Delta \otimes \Delta} & (H \otimes H) \otimes (H \otimes H) \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle (\mu \otimes \mu) \circ (id \otimes \tau \otimes id)} \\ H & \xrightarrow[\Delta]{} & H \otimes H \end{array} \qquad \begin{array}{ccc} \mathbb{F} & \xrightarrow{\eta} & H \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \Delta} \\ \mathbb{F} \otimes \mathbb{F} & \xrightarrow[\eta \otimes \eta]{} & H \otimes H \end{array}$$

and

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\varepsilon \otimes \varepsilon} & \mathbb{F} \otimes \mathbb{F} \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \cong} \\ H & \xrightarrow[\varepsilon]{} & \mathbb{F} \end{array} \qquad \begin{array}{ccc} \mathbb{F} & \xrightarrow{\eta} & H \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \varepsilon} \\ & & \mathbb{F} \end{array}$$

commute.

Remark 2.43. The condition that Δ and ε are algebra homomorphisms is equivalent to the dual condition that μ and η are coalgebra homomorphisms, see [Kas95, Theorem III.2.1]. Therefore the four commutative diagrams in Definition 2.42 are identical to those whose commutativity expresses the fact that μ and η are coalgebra homomorphisms. This gives a compatibility between the algebra and coalgebra structures on H.

Naturally, a **homomorphism of bialgebras** is an $\mathbb{F}$-linear map between $\mathbb{F}$-bialgebras that is a homomorphism of *both* algebras and coalgebras.

Example 2.44. The dual vector space H^* of a *finite-dimensional* bialgebra H has a natural bialgebra structure.

Example 2.45 ([Kas95, III.2, Example 2]). Let X be a set with a unital *monoid* structure, that is, an associative map $\mu\colon X \times X \to X$ having a unit e. The map μ induces an algebra structure on the coalgebra $\mathbb{F}[X]$ with unit e. The coalgebra structure of $\mathbb{F}[X]$ is given by $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1_{\mathbb{F}}$, for all $x \in X$. The maps Δ and ε for $\mathbb{F}[X]$ are algebra homomorphisms since

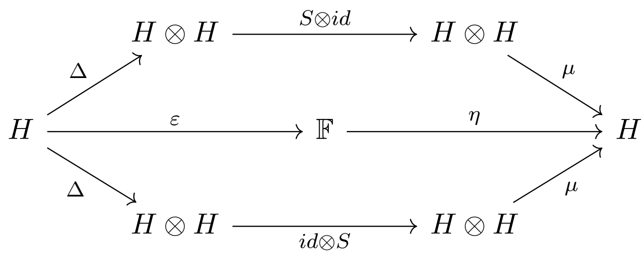
$$\Delta(xy) = xy \otimes xy = (x \otimes x)(y \otimes y) = \Delta(x)\Delta(y)$$

and

$$\varepsilon(xy) = 1_{\mathbb{F}} = \varepsilon(x)\varepsilon(y)$$

for all $x, y \in X$, and hence $\mathbb{F}[X]$ has the structure of a bialgebra.

Definition 2.46 ([Kas95, Definition III.3.2])**.** *An* $\mathbb{F}$*-**Hopf algebra** is a 6-tuple* $(H, \mu, \eta, \Delta, \varepsilon, S)$ *where the 5-tuple* $(H, \mu, \eta, \Delta, \varepsilon)$ *is an* $\mathbb{F}$*-bialgebra and* $S\colon H \to H$ *is an* $\mathbb{F}$*-linear map such that the diagram*



commutes. The map $S\colon H \to H$ *is called the **antipode** for* H*.*

Example 2.47 ([Kas95, Definition III.3, Example 2]). Recall the bialgebra $\mathbb{F}[X]$ of Example 2.45. For an antipode S to exist, by definition we must have

$$xS(x) = S(x)x = \varepsilon(x)1_{\mathbb{F}} = 1_{\mathbb{F}}$$

for any $x \in X$. Hence, the bialgebra $\mathbb{F}[X]$ has an antipode if and only if each $x \in X$ has an inverse (denoted by x^{-1}), that is, X is a *group* and then $S(x) = x^{-1}$ for all $x \in X$.

2.4 Lie algebras

2.4.1 The definition of a Lie algebra

Let $\mathbb{F}$ be a field. We begin by recalling the definition Lie algebras and some important examples.

Definition 2.48 ([Kas95, Definition V.1.1 (a)])**.** *A **Lie algebra** over* $\mathbb{F}$ *is an* $\mathbb{F}$*-vector space* $\mathfrak{g}$ *with a bilinear map*

$$[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\,,$$

*called the **Lie bracket**, satisfying the alternating law, that is,* $[X, X] = 0$ *for all* $X \in \mathfrak{g}$*, and*

$$\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0$$

for all $X, Y, Z \in \mathfrak{g}$*, also referred to as the Jacobi identity.*

Since the Lie bracket is bilinear, the alternating law for Lie algebras implies that

$$[X, Y] = -[Y, X]$$

for all $X, Y \in \mathfrak{g}$, called the *anticommutativity* rule and is an equivalent condition if $\mathrm{char}(\mathbb{F}) \neq 2$. In our setting, we will take the base field $\mathbb{F} = \mathbb{C}$ of complex numbers which has characteristic 0.

Example 2.49 ([EW06, 1.2(2)]). Any vector space V over $\mathbb{F}$ can be made into an *abelian* Lie algebra by defining $[X, Y] = 0$ for all $X, Y \in V$. Any 1-dimensional Lie algebra is abelian, by the alternativity of the Lie bracket. In particular, the field $\mathbb{F}$ may be regarded as a 1-dimensional abelian Lie algebra.

Example 2.50. On an algebra A over a field $\mathbb{F}$ with multiplication $(a, b) \mapsto ab$, a Lie bracket may be defined by the **commutator**

$$[a, b] = ab - ba \tag{2.45}$$

for all $a, b \in \mathcal{A}$. Then A together with this bracket is a *Lie algebra*. For all $a, b, c \in A$, we also have the identity

$$[a, bc] = [a, b]c + b[a, c].$$

Example 2.51 ([Kas95, V.1(5)]). For any $\mathbb{F}$-vector space V, consider the algebra $\mathrm{End}(V)$ of all endomorphisms of V as a Lie algebra with the commutator (2.45), which we denote by $\mathfrak{gl}(V)$. If V is of finite dimension n, then $\mathfrak{gl}(V)$ is isomorphic to the Lie algebra $\mathfrak{gl}_n(\mathbb{F})$ of $n \times n$-matrices with entries in $\mathbb{F}$ and *matrix commutator* $[X, Y] = XY - YX$ for $X, Y \in \mathfrak{gl}_n(\mathbb{F})$.

Let $\mathfrak{g}$ be a Lie algebra. We define a **Lie subalgebra** of $\mathfrak{g}$ to be a subspace $\mathfrak{g}' \subseteq \mathfrak{g}$ such that

$$[X, Y] \in \mathfrak{g}' \tag{2.46}$$

for all $X, Y \in \mathfrak{g}'$. In other words, $\mathfrak{g}'$ is closed under the Lie bracket of $\mathfrak{g}$. Moreover, if (2.46) holds for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}'$, then $\mathfrak{g}'$ is called an **ideal** of $\mathfrak{g}$. Recall that a **simple** Lie algebra is a Lie algebra that is non-abelian and contains no non-zero proper ideals.

Example 2.52. The commutator of two matrices with zero trace must have zero trace itself. Consequently, the vector space $\mathfrak{sl}_n(\mathbb{F})$ of traceless $n \times n$-matrices is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{F})$ defined in Example 2.51. By [EW06, 1.15(i)], for any $n \times n$-matrix $S \in \mathfrak{gl}_n(\mathbb{F})$, the subspace

$$\mathfrak{gl}_{S,n}(\mathbb{F}) = \left\{ X \in \mathfrak{gl}_n(\mathbb{F}) \;\middle|\; X^t S = -SX \right\}$$

is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{F})$. Moreover, for all invertible $S \in \mathfrak{gl}_n(\mathbb{F})$ and $X \in \mathfrak{gl}_{S,n}(\mathbb{F})$ we have

$$\mathrm{tr}(X) = \mathrm{tr}(-S^{-1} X^t S) = -\mathrm{tr}(X^t) = -\mathrm{tr}(X)$$

and so $\mathrm{tr}(X) = 0$. Hence, $\mathfrak{gl}_{S,n}(\mathbb{F})$ is also a Lie subalgebra of $\mathfrak{sl}_n(\mathbb{F})$ for all invertible $S \in \mathfrak{gl}_n(\mathbb{F})$.

Definition 2.53 ([Kas95, Definition V.1.1 (c)]). *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras. A linear map $f\colon \mathfrak{g} \to \mathfrak{g}'$ is a* ***homomorphism*** *of Lie algebras if*

$$f\big([X,Y]_{\mathfrak{g}}\big) = \big[f(X), f(Y)\big]_{\mathfrak{g}'}$$

for all $X, Y \in \mathfrak{g}$. We say that f is an ***isomorphism*** *of Lie algebras if it is also bijective.*

Example 2.54 ([EW06, 1.4]). Given a Lie algebra $\mathfrak{g}$, the *adjoint homomorphism* is the linear map $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by

$$(\mathrm{ad}X)(Y) = [X, Y]$$

for all $X, Y \in \mathfrak{g}$. Since the map $X \mapsto \mathrm{ad}X$ is also linear, to verify that ad is a homomorphism one needs that $\mathrm{ad}[X,Y] = \mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(Y) \circ \mathrm{ad}(X)$ which coincides with the Jacobi identity.

2.4.2 Representations of Lie algebras

Given any $\mathbb{F}$-vector space V, recall from Example 2.51 the Lie algebra $\mathfrak{gl}(V)$ consisting of all endomorphisms of V with Lie bracket given by the commutator $[X,Y] = XY - YX$ for all $X, Y \in \mathfrak{gl}(V)$.

Definition 2.55 ([EW06, Definition 7.1]). *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$, and let V be an $\mathbb{F}$-vector space. A* ***representation*** *of $\mathfrak{g}$ on V is a Lie algebra homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$. The vector space V, together with the representation ρ, is called a* ***$\mathfrak{g}$-module****.*

Equivalently, one can define a $\mathfrak{g}$-module as a vector space V together with a *bilinear* map $\mathfrak{g} \times V \to V$ given by $(X, v) \mapsto X \cdot v$ such that

$$[X,Y] \cdot v = X \cdot (Y \cdot v) - Y \cdot (X \cdot v)$$

for all $X, Y \in \mathfrak{g}$ and $v \in V$. This is related to Definition 2.55 by setting

$$X \cdot v = \rho(X)(v) \tag{2.47}$$

for all $X \in \mathfrak{g}$ and $v \in V$. For ease of notation, we use module-theoretic concepts from now on.

Definition 2.56. *Given any $\mathfrak{g}$-module V, a* ***$\mathfrak{g}$-submodule*** *of V is a subspace $W \subseteq V$ which is invariant under the action* (2.47) *of $\mathfrak{g}$, that is, for each $X \in \mathfrak{g}$ and for each $w \in W$, we have*

$$X \cdot w \in W.$$

Moreover, a $\mathfrak{g}$-module V is called ***simple*** *if it is non-zero and the only $\mathfrak{g}$-submodules are $\{0\}$ and V.*

To observe the representation theory of Lie algebras more explicitly, one can refer to [FH91].

2.4.3 Universal enveloping algebras

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{F}$ with vector space basis $\{X_1, ..., X_n\}$. Let c_{ijk} be the structure constants for this basis such that

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k.$$

Then, the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is the unital, associative $\mathbb{F}$-algebra generated by the elements $x_1, ..., x_n$, subject to the relations

$$x_i x_j - x_j x_i = \sum_{k=1}^{n} c_{ijk} x_k.$$

Example 2.57. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is spanned by the matrices

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which satisfy the commutation relations $[H, E] = 2E$, $[H, F] = -2F$, and $[E, F] = H$. Then, the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$ is the algebra generated by the elements e, f, h subject to the relations $he - eh = 2e$, $hf - fh = -2f$, and $ef - fe = h$.

The essence of the above technical definition of the universal enveloping algebra becomes clearer if one defines $U(\mathfrak{g})$ in terms of the tensor algebra of $\mathfrak{g}$. Every Lie algebra $\mathfrak{g}$ is in particular a vector space defined over some field $\mathbb{F}$. Thus, we can construct the **tensor algebra** $T(\mathfrak{g})$ from it, namely, the space

$$T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k} = \mathbb{F} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \dots$$

where $\otimes$ is the tensor product, and $\oplus$ is the direct sum of vector spaces. Observe that the equality $\mathfrak{g}^{\otimes k} \otimes \mathfrak{g}^{\otimes l} = \mathfrak{g}^{\otimes k+l}$ induces an associative product on $T(\mathfrak{g})$ given by the concatenation

$$(x_1 \otimes \cdots \otimes x_k)(x_{k+1} \otimes \cdots \otimes x_{k+l}) = x_1 \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_{k+l}$$

where $x_1, \dots, x_k, x_{k+1}, \dots, x_{k+l}$ are elements of $\mathfrak{g}$. The tensor algebra $T(\mathfrak{g})$ is a free algebra; it contains all possible tensor products of all possible vectors in $\mathfrak{g}$, without any restrictions on those products. The universal enveloping algebra of $\mathfrak{g}$ is then obtained by taking the quotient and imposing certain relations for elements in the embedding of $\mathfrak{g}$ in $T(\mathfrak{g})$.

Definition 2.58. *The* ***universal enveloping algebra*** *of a Lie algebra* $\mathfrak{g}$ *is the quotient space*

$$U(\mathfrak{g}) = T(\mathfrak{g})/I$$

where I *is the two-sided ideal of the tensor algebra* $T(\mathfrak{g})$ *generated by elements of the form*

$$X \otimes Y \; - \; Y \otimes X \; - \; [X, Y] \in \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \subset T(\mathfrak{g}).$$

The crucial property of the universal enveloping algebra $U(\mathfrak{g})$ is that it actually has the same representation theory as the Lie algebra $\mathfrak{g}$.

Lemma 2.59 ([EW06, Lemma 15.10])**.** *Let $\mathfrak{g}$ be a Lie algebra and let $U(\mathfrak{g})$ be its universal enveloping algebra. There is a bijective correspondence between $\mathfrak{g}$-modules and $U(\mathfrak{g})$-modules. Under this correspondence, a $\mathfrak{g}$-module is simple if and only if it is simple as a $U(\mathfrak{g})$-module.*

The proof of this Lemma, provided in [EW06, 15.2.1], demonstrates a certain *universal property* of the universal enveloping algebra $U(\mathfrak{g})$. This is stated, without proof, in the following Theorem. Recall that any *associative* algebra can be considered as a Lie algebra with the commutator bracket, see Example 2.50.

Theorem 2.60 ([Kas95, Theorem V.2.1])**.** *Let $\mathfrak{g}$ be a Lie algebra and let $U(\mathfrak{g})$ be its universal enveloping algebra. For any associative algebra A and homomorphism $f\colon \mathfrak{g} \to A$ of Lie algebras, there exists a unique homomorphism of associative algebras $\tilde{f}\colon U(\mathfrak{g}) \to A$ such that the diagram*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ f\ } & A \\ {\scriptstyle \iota}\big\downarrow & \nearrow{\scriptstyle \tilde{f}} & \\ U(\mathfrak{g}) & & \end{array}$$

commutes, that is, we have $\tilde{f} \circ \iota = f$, where we define the Lie algebra map $\iota\colon \mathfrak{g} \to U(\mathfrak{g})$; $X \mapsto X + I$ as the composition of the embedding $\mathfrak{g} \hookrightarrow T(\mathfrak{g})$ with the canonical epimorphism $T(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$.

The universal enveloping algebra $U(\mathfrak{g})$ of any Lie algebra $\mathfrak{g}$ is not only an algebra, but also a Hopf algebra. In particular, $U(\mathfrak{g})$ is given the Hopf algebra structure defined by

$$\begin{aligned} \Delta(X) &= 1 \otimes X + X \otimes 1, \\ \varepsilon(X) &= 0, \\ S(X) &= -X \end{aligned}$$

for all $X \in \mathfrak{g}$.

2.4.4 The Poincaré-Birkhoff-Witt Theorem

Given a basis of $\mathfrak{g}$, we can write down a basis of $U(\mathfrak{g})$. This is the important Poincaré-Birkhoff-Witt Theorem (or PBW-Theorem), which is stated in [Kas95, Theorem V.2.5(a)].

Theorem 2.61 (Poincaré-Birkhoff-Witt Theorem)**.** *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$, and suppose that $\{X_1, \dots, X_n\}$ is a totally ordered basis of $\mathfrak{g}$. Then the universal enveloping algebra $U(\mathfrak{g})$ has a basis*

$$\left\{x_1^{d_1} \cdots x_n^{d_n} \,\middle|\, d_1, \dots, d_n \geq 0\right\}.$$

An important corollary is that the elements $x_1, \dots, x_n$ are linearly independent, and therefore the Lie algebra $\mathfrak{g}$ can be found as a subspace of its universal enveloping algebra $U(\mathfrak{g})$.

Example 2.62 ([EW06, Lemma 15.9]). Consider again the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. From the PBW-Theorem, take $x_1 = f$, $x_2 = h$ and $x_3 = e$. Then, the universal enveloping algebra $U\big(\mathfrak{sl}_2(\mathbb{C})\big)$ has as a $\mathbb{C}$-vector space basis

$$\big\{f^{d_1}h^{d_2}e^{d_3} \,\big|\, d_1, d_2, d_3 \geq 0\big\}.$$

Note that one could equally well have taken the basis elements in a different order.

2.5 Complex semisimple Lie algebras

The classification of finite-dimensional Lie algebras proceeds in several steps. Firstly, recall from [EW06, Definition 4.2] that a Lie algebra $\mathfrak{g}$ is called **solvable** if $\mathfrak{g}^{(m)} = \{0\}$ for some $m \in \mathbb{N}$. Here $\mathfrak{g}^{(m)}$ forms part of the *derived series* defined inductively by $\mathfrak{g}^{(0)} = \mathfrak{g}$ and

$$\mathfrak{g}^{(m)} = \big[\mathfrak{g}^{(m-1)}, \mathfrak{g}^{(m-1)}\big]$$

for all $m \in \mathbb{N}$. Every finite-dimensional Lie algebra $\mathfrak{g}$ contains a unique solvable ideal $\mathrm{rad}(\mathfrak{g})$ of maximal dimension, see [EW06, Corollary 4.5]. A Lie algebra $\mathfrak{g}$ is called **semisimple** if

$$\mathrm{rad}(\mathfrak{g}) = \{0\}.$$

For any finite-dimensional Lie algebra $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is a semisimple Lie algebra. To understand finite-dimensional Lie algebras in general, one can hence proceed in three steps:

- Firstly, one needs to understand arbitrary solvable Lie algebras. Over the complex field $\mathbb{C}$, this is achieved by Lie's Theorem [EW06, Theorem 6.5] which states that every solvable Lie algebra is isomorphic to a Lie subalgebra of $\mathfrak{gl}(V)$ consisting of upper triangular matrices.
- Secondly, one needs to classify semisimple Lie algebras. For complex Lie algebras, this reduces to the classification of simple Lie algebras since every finite-dimensional semisimple Lie algebra is a direct sum of simple Lie algebras by [EW06, Theorem 9.11]. Then, the simple Lie algebras in turn are classified in terms of their root systems, see Section 2.5.2.
- The final question in the classification of finite-dimensional Lie algebras is how to extend a semisimple Lie algebra $\mathfrak{g}$ by a solvable Lie algebra $\mathfrak{b}$. In other words, one would have to describe all Lie algebras $\mathfrak{l}$ which fit into a short exact sequence

  $$0 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{l} \longrightarrow \mathfrak{g} \longrightarrow 0.$$

  This question, however, is not relevant for the present thesis.

Throughout this section, we will discuss the theory of finite-dimensional complex semisimple Lie algebras and hence understand the classification of simple Lie algebras. Several important examples of simple Lie algebras will be discussed explicitly in Section 2.6.

2.5.1 Root space decomposition

Let $\mathfrak{g}$ be a complex semisimple Lie algebra containing an *abelian* Lie subalgebra $\mathfrak{h}$ consisting of *semisimple* elements. Recall that, by definition, an element $H \in \mathfrak{g}$ is called **semisimple** if $\mathrm{ad}(H)$ is diagonalisable, see [EW06, p.87]. In fact, $\mathfrak{h}$ acts diagonalisably on the Lie algebra $\mathfrak{g}$ in the adjoint representation. We observe a decomposition of $\mathfrak{g}$ for the action of $\mathrm{ad}(H)$ for $H \in \mathfrak{h}$.

Since $\mathfrak{h}$ is abelian, the elements of $\mathrm{ad}(\mathfrak{h})$ must commute. By [EW06, Lemma 16.7], we know that $\mathfrak{g}$ has a basis of common eigenvectors for the elements of $\mathrm{ad}(\mathfrak{h})$. Given a common eigenvector $X \in \mathfrak{g}$, the eigenvalues are given by the associated *weight* $\alpha \colon \mathfrak{h} \to \mathbb{C}$ defined by

$$(\mathrm{ad}\, H)(X) = [H, X] = \alpha(H)X$$

for all $H \in \mathfrak{h}$. Note that weights are elements of the dual space $\mathfrak{h}^*$, see Definition 2.3. For each $\alpha \in \mathfrak{h}^*$, we denote the corresponding *weight space* by

$$\mathfrak{g}_\alpha = \big\{X \in \mathfrak{g} \bigm| [H, X] = \alpha(H)X, \text{ for all } H \in \mathfrak{h}\big\}.$$

Let Φ denote the set of non-zero $\alpha \in \mathfrak{h}^*$, for which $\mathfrak{g}_\alpha$ is also non-zero. For $\alpha = 0$, we have the zero weight space

$$\mathfrak{g}_0 = \big\{X \in \mathfrak{g} \bigm| [H, X] = 0, \text{ for all } H \in \mathfrak{h}\big\}.$$

Then, the Lie algebra $\mathfrak{g}$ decomposes into the direct sum of the weight spaces $\mathfrak{g}_\alpha$ for all $\alpha \in \Phi \cup \{0\}$. Consequently, if $\mathfrak{g}$ is finite-dimensional, then Φ is finite. Additionally, since $\mathfrak{h}$ is abelian, we have $\mathfrak{h} \subseteq \mathfrak{g}_0$. For the decomposition of $\mathfrak{g}$ into weight spaces to be as useful as possible, $\mathfrak{h}$ should be as large as possible. In the following definition, we describe $\mathfrak{h}$ as *maximal* in the sense that, if $\mathfrak{h}' \supseteq \mathfrak{h}$ is another abelian Lie subalgebra of $\mathfrak{g}$ consisting of semisimple elements, then $\mathfrak{h} = \mathfrak{h}'$.

Definition 2.63 ([EW06, Definition 10.2])**.** *A Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called a* ***Cartan subalgebra*** *of $\mathfrak{g}$ if $\mathfrak{h}$ is abelian and every element $H \in \mathfrak{h}$ is semisimple, and moreover $\mathfrak{h}$ is maximal with these properties.*

> *Example* 2.64. Let $\mathfrak{g} = \mathfrak{sl}_{N+1}(\mathbb{C})$. For a Cartan subalgebra $\mathfrak{h}$, we may take the algebra of all traceless $(N+1) \times (N+1)$-matrices which are *diagonal.* For example, a Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ can be
>
> $$\mathfrak{h} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \,\middle|\, a \in \mathbb{C} \right\}.$$

One can show that, for any complex semisimple Lie algebra $\mathfrak{g}$, there exists a non-zero Cartan subalgebra. Moreover, if $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then

$$\mathfrak{h} = \mathfrak{g}_0$$

by [EW06, Theorem 10.4]. Hence, $\mathfrak{g}$ has the following decomposition into weight spaces for $\mathfrak{h}$.

Definition 2.65 ([EW06, Section 10.3]). *The* ***root space decomposition*** *of a semisimple Lie algebra* $\mathfrak{g}$ *with Cartan subalgebra* $\mathfrak{h}$ *is the direct sum decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where Φ *is the set of* $\alpha \in \mathfrak{h}^*$ *such that* $\alpha \neq 0$ *and* $\mathfrak{g}_\alpha \neq 0$*. Each* $\alpha \in \Phi$ *is called a* ***root*** *of* $\mathfrak{g}$*, and* $\mathfrak{g}_\alpha$ *is called the corresponding* ***root space****.*

> *Remark* 2.66. If $\mathfrak{g}$ is finite-dimensional, then the set Φ must be finite. Moreover, the roots and root spaces of a semisimple Lie algebra $\mathfrak{g}$ depend on the choice of Cartan subalgebra $\mathfrak{h}$.

We may associate to each root $\alpha \in \Phi$ a Lie subalgebra $\mathfrak{sl}(\alpha)$ of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Suppose that $\alpha \in \Phi$ and that e_α is a non-zero element in $\mathfrak{g}_\alpha$. By [EW06, Lemma 10.5], there exists a root $-\alpha \in \Phi$ and non-zero element $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $\mathrm{Span}\big\{e_\alpha, f_\alpha, [e_\alpha, f_\alpha]\big\}$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. In particular, note that $h_\alpha = [e_\alpha, f_\alpha] \in \mathfrak{h}$. For $\alpha \in \Phi$, let $\mathfrak{sl}(\alpha)$ denote the Lie subalgebra of $\mathfrak{g}$ with basis $\{e_\alpha, f_\alpha, h_\alpha\}$ such that $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$, $h_\alpha \in \mathfrak{h}$ and $\alpha(h_\alpha) = 2$.

Proposition 2.67 ([EW06, Exercise 10.3(ii)]). *For each root* α*, the map* $\mathfrak{sl}(\alpha) \to \mathfrak{sl}_2(\mathbb{C})$ *given by*

$$e_\alpha \mapsto e, \qquad f_\alpha \mapsto f, \qquad h_\alpha \mapsto h$$

is a Lie algebra isomorphism.

We call the triple $(e_\alpha, f_\alpha, h_\alpha)$ the $\mathfrak{sl}_2(\mathbb{C})$-triple corresponding to the root α.

2.5.2 Root systems

Let E denote a *Euclidean* space, that is, a finite-dimensional $\mathbb{R}$-vector space that is equipped with an inner product $(-,-)$. For all non-zero $\alpha, \beta \in E$, the notions of length and angle are given by

$$||\alpha|| = \sqrt{(\alpha, \alpha)} \qquad \text{and} \qquad (\alpha, \beta) = ||\alpha||\, ||\beta|| \cos(\theta)$$

where θ denotes the angle between α and β. As it will be a useful convention, we shall write

$$\langle \beta, \alpha \rangle = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} = 2\frac{||\beta||}{||\alpha||} \cos(\theta) \tag{2.48}$$

for all non-zero $\alpha, \beta \in E$. Note that $\langle -, - \rangle$ defined in (2.48) is not an inner product; it is not necessarily symmetric and is only linear in the first argument. For each non-zero $\alpha \in E$, let s_α denote the reflection in the hyperplane normal to α given by

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha \tag{2.49}$$

for all $\beta \in E$. Note, in particular, that $s_\alpha(\alpha) = -\alpha$. Moreover, each reflection s_α preserves the inner product $(-,-)$, that is,

$$\big(s_\alpha(\beta), s_\alpha(\gamma)\big) = (\beta, \gamma)$$

for all $\beta, \gamma \in E$. With this notation, we can now formally define *root systems*.

Definition 2.68 ([EW06, Definition 11.1]). *A* ***root system*** *Φ is a subset of a Euclidean space E satisfying the following axioms:*

(R1) *Φ is finite, it does not contain $\underline{0}$, and it spans E;*

(R2) *If $\alpha \in \Phi$, then the only scalar multiples of α in Φ are $\pm\alpha$;*

(R3) *If $\alpha \in \Phi$, then the reflection s_α permutes the elements of Φ;*

(R4) *If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.*

Example 2.69 ([EW06, Example 11.2]). Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Recall the root space decomposition from Definition 2.65, and let Φ be the set of roots of $\mathfrak{g}$ with respect to some fixed Cartan subalgebra $\mathfrak{h}$. Let E be the $\mathbb{R}$-span of Φ. The symmetric bilinear form $(-,-)$ on E induced by the so-called *Killing form* of $\mathfrak{g}$, given explictly in [Lor18, p.331 (6.18)], is an inner product by [EW06, Proposition 10.15]. One can show that Φ is a root system in E.

If Φ is a root system, we will find it useful to define the **coroot** $\alpha^\vee$ of a root $\alpha \in \Phi$ by $\alpha^\vee = \frac{2}{(\alpha,\alpha)}\alpha$.

Definition 2.70 ([EW06, Definition 11.9]). *A* ***base*** *for a root system Φ is a subset $\Pi \subset \Phi$ which is a vector space basis for E, such that every $\beta \in \Phi$ can be written as*

$$\beta = \sum_{\alpha \in \Pi} m_\alpha \, \alpha \tag{2.50}$$

with $m_\alpha \in \mathbb{Z}$, where all of the non-zero coefficients m_α have the same sign.

By [EW06, Theorem 11.10], there exists a base Π for every root system Φ. Usually, there are many possible bases for a root system. For example, if Π is a base for Φ, then so is the set $\{-\alpha \mid \alpha \in \Pi\}$, as well as the set $\{s_\beta(\alpha) \mid \alpha \in \Pi\}$ for any $\beta \in \Phi$.

A root $\beta \in \Phi$ is **positive with respect to** Π if the coefficients m_α in (2.50) are non-negative, otherwise β is **negative with respect to** Π. Let Φ^+ and Φ^- denote the set of all positive roots and negative roots respectively in Φ with respect to Π. Then, we have the disjoint union

$$\Phi = \Phi^+ \cup \Phi^-$$

and, moreover, the set Π is contained in Φ^+. The elements of Π are called the **simple roots**, and the reflections s_α for $\alpha \in \Pi$ given by (2.49) are known as the **simple reflections**.

Consider the group of all invertible linear transformations of E generated by the reflections s_β for $\beta \in \Phi$. This is called the **Weyl group** of Φ, and is denoted by $\mathcal{W}$. One may show that $\mathcal{W}$ is finite, and by [EW06, Lemma 11.15] we may restrict the generators to the simple reflections, that is, we can write

$$\mathcal{W} = \big\langle s_\alpha \bigm| \alpha \in \Pi \big\rangle.$$

The following result, stated without proof, tells us geometrically that all of the bases of a root system are essentially the same. For a technical proof, refer to [EW06, Chapter 19].

Proposition 2.71 ([EW06, Theorem 11.16]). *Suppose that Π and Π' are two bases of a root system Φ, and let $\mathcal{W}$ be the Weyl group of Φ. Then, there exists an element $w \in \mathcal{W}$ such that*

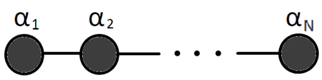
$$\Pi' = \{w(\alpha) \mid \alpha \in \Pi\}.$$

Fix an order on the simple roots of Π in Φ, say $\{\alpha_1, \ldots, \alpha_N\}$. Then, the **Cartan matrix** of Φ is defined to be the $N \times N$-matrix with ij^{th} entry $a_{i,j} = \langle \alpha_j, \alpha_i \rangle \in \mathbb{Z}$, called a *Cartan integer*. Observe that

$$(\alpha_i, \alpha_j) = d_i a_{i,j} \qquad \text{where} \qquad d_i = \frac{(\alpha_i, \alpha_i)}{2} \in \{1, 2, 3\}$$

by [Jan96, 4.1(1)]. The **Dynkin diagram** of Φ is a graph, with nodes labelled by the simple roots, and between the nodes labelled by $\alpha_i, \alpha_j \in \Pi$ we draw exactly $a_{i,j}a_{j,i} \in \{0, 1, 2, 3\}$ lines, following [EW06, Lemma 11.4]. When $a_{i,j}a_{j,i} \in \{2, 3\}$, an arrow points to the "shorter" root. The Dynkin diagram determines the root system Φ, and is independent of the choice of base.

Example 2.72. Let $\mathfrak{g} = \mathfrak{sl}_{N+1}(\mathbb{C})$. Its root system has the Dynkin diagram of type $\mathbf{A}_N$ given below.



2.5.3 The Serre presentation

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Choose a set of simple roots $\Pi = \{\alpha_i \mid i \in I\}$ with respect to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, where $I \subset \mathbb{N}$ denotes an indexing set for the nodes of the Dynkin diagram of $\mathfrak{g}$. For each $i \in I$, let (e_i, f_i, h_i) be the $\mathfrak{sl}_2(\mathbb{C})$-triple corresponding to the root α_i, see Section 2.5.1. One may show that the Lie algebra $\mathfrak{g}$ is generated by the elements e_i, f_i and h_i for all $i \in I$, which we refer to as the *Chevalley generators* for $\mathfrak{g}$, since every element of $\mathfrak{g}$ can be obtained by repeatedly taking linear combinations and Lie brackets of these elements.

Example 2.73 ([EW06, Example 14.3]). Let $\mathfrak{g} = \mathfrak{sl}_{N+1}(\mathbb{C})$. The elements $e_i = E_{i,i+1}$ and $f_i = E_{i+1,i}$ for $1 \leq i \leq N$ already generate $\mathfrak{g}$ as a Lie algebra. Indeed, a basis for the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices is obtained by taking the commutators $[E_{i,i+1}, E_{i+1,i}] = E_{i,i} - E_{i+1,i+1} = h_i$ for $1 \leq i \leq N$. For $i + 1 < j$, we have $[E_{i,i+1}, E_{i+1,j}] = E_{i,j}$, and hence we get all $E_{i,j}$ with $i < j$ by induction. Similarly, we may obtain all $E_{i,j}$ with $i > j$.

The following relations, satisfied by the Chevalley generators e_i, f_i and h_i, only involve information which can be obtained from the Cartan matrix. In particular, we write $a_{i,j} = \langle \alpha_j, \alpha_i \rangle$.

Lemma 2.74 ([EW06, Lemma 14.5]). *The elements $\{e_i, f_i, h_i \mid i \in I\}$ satisfy the relations*

(S1) $[h_i, h_j] = 0$ *for all* i, j,

(S2) $[h_i, e_j] = a_{i,j} e_j$ *and* $[h_i, f_j] = -a_{i,j} f_j$ *for all* i, j,

(S3) $[e_i, f_j] = \delta_{i,j} h_i$ *for all* i, j *where* δ *is the Kronecker delta,*

(S4) $\big(\mathrm{ad}(e_i)\big)^{1-a_{i,j}}(e_j) = 0$ *and* $\big(\mathrm{ad}(f_i)\big)^{1-a_{i,j}}(f_j) = 0$ *if* $i \neq j$.

Note that $a_{i,j} \leq 0$ for all $i \neq j$ since the angle between any two simple roots is obtuse, see [EW06, Exercise 11.3]. The relations (S1)-(S4) in Lemma 2.74 are known as the *Serre relations*.

The following Theorem says that the relations (S1)-(S4) completely determine the Lie algebra.

Theorem 2.75 (Serre's Theorem [EW06, Theorem 14.6])**.** *Let $\mathcal{A}$ be the Cartan matrix of a root system. Let $\mathfrak{g}$ be the complex Lie algebra which is generated by the elements e_i, f_i and h_i for $i \in I$, subject to the relations* (S1)-(S4)*. Then $\mathfrak{g}$ is finite-dimensional and semisimple with Cartan subalgebra $\mathfrak{h}$ spanned by all of the elements h_i, and its root system has Cartan matrix $\mathcal{A}$.*

For a detailed proof of Serre's Theorem, refer to [EW06, Section 14.2]. An immediate corollary of Theorem 2.75 is that, up to isomorphism, there is just one Lie algebra for each root system. Moreover, one can show that the root system of a semisimple Lie algebra is uniquely determined by its Cartan matrix, up to isomorphism. Hence, complex semisimple Lie algebras with different Dynkin diagrams must not be isomorphic. We therefore deduce the following important result.

Corollary 2.76 (Classification of simple Lie algebras)**.** *There is a one-to-one correspondence between the isomorphism classes of complex simple Lie algebras and the list of Dynkin diagrams.*

For any complex semisimple Lie algebra $\mathfrak{g}$, Serre's Theorem gives a description of the universal enveloping algebra $U(\mathfrak{g})$ in terms of generators and relations, also called its *Serre presentation.*

Example 2.77. Let $\mathfrak{g} = \mathfrak{sl}_{N+1}(\mathbb{C})$. The universal enveloping algebra $U\big(\mathfrak{sl}_{N+1}(\mathbb{C})\big)$ is the $\mathbb{C}$-algebra generated by elements e_i, f_i and h_i for $1 \leq i \leq N$ subject to the Serre relations (S1)-(S4) in Lemma 2.74. Explicitly, (S4) becomes

$$\begin{aligned} e_i^2 e_j - 2e_i e_j e_i + e_j e_i^2 &= 0 \qquad \text{if } |i-j| = 1, \\ e_i e_j - e_j e_i &= 0 \qquad \text{if } |i-j| > 1, \\ f_i^2 f_j - 2f_i f_j f_i + f_j f_i^2 &= 0 \qquad \text{if } |i-j| = 1, \\ f_i f_j - f_j f_i &= 0 \qquad \text{if } |i-j| > 1. \end{aligned}$$

2.6 The classical Lie algebras

With five exceptions, every finite-dimensional complex simple Lie algebra is isomorphic to one of the *classical* Lie algebras, namely the Lie algebras which, from now on, we denote by

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{sl}_n, \qquad \mathfrak{so}_n(\mathbb{C}) = \mathfrak{so}_n, \qquad \mathfrak{sp}_n(\mathbb{C}) = \mathfrak{sp}_n$$

for $n \geq 2$, see [EW06, Theorem 4.12]. The five exceptional Lie algebras are known as $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$ and $\mathfrak{g}_2$. Constructions of their corresponding root systems are described in [EW06, 13.2].

We have already discussed the family of special linear Lie algebras, $\mathfrak{sl}_{N+1}$ for $N \geq 1$, throughout Section 2.5. We define the remaining families as certain subalgebras of $\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{gl}_n$ as follows.

Let $\langle -, - \rangle$ be a (skew-)symmetric, non-degenerate bilinear form on the vector space $\mathbb{C}^n$, and consider the subspace

$$\mathfrak{gl}_{\mathcal{Q},n} = \big\{ X \in \mathfrak{gl}_n \bigm| \langle Xu, v\rangle + \langle u, Xv\rangle = 0, \ \forall u, v \in \mathbb{C}^n \big\}. \tag{2.51}$$

By Corollary 2.13, there exists an *invertible*, (skew-)symmetric matrix $S \in \mathfrak{gl}_n$ such that

$$\langle Xu, v\rangle + \langle u, Xv\rangle = (Xu)^t Sv + u^t SXv = u^t\big(X^t S + SX\big)v$$

for all $u, v \in \mathbb{C}^n$ and $X \in \mathfrak{gl}_n$. Since this holds for all $u, v \in \mathbb{C}^n$, the defining property of $\mathfrak{gl}_{Q,n}$ in (2.51) becomes $X^t S + SX = 0$, or equivalently, $X^t = -SXS^{-1}$. We deduce that $\mathfrak{gl}_{Q,n}$ is equal to the subspace

$$\mathfrak{gl}_{S,n} = \big\{X \in \mathfrak{gl}_n \;\big|\; X^t S = -SX\big\}. \tag{2.52}$$

Recall from Example 2.52 that, for any matrix $S \in \mathfrak{gl}_n$, the subspace $\mathfrak{gl}_{S,n}$ is a Lie subalgebra of $\mathfrak{sl}_n$.

Proposition 2.78. *For any matrix $S \in \mathfrak{gl}_n$ and matrix $T = P^t SP$ where $P \in \mathfrak{gl}_n$ is invertible, the map*

$$\mathfrak{gl}_{S,n} \to \mathfrak{gl}_{T,n}\,; \quad X \mapsto P^{-1}XP$$

is an isomorphism of Lie algebras.

Proof. We first need to check the defining property of the Lie algebra $\mathfrak{gl}_{T,n}$. For all $X \in \mathfrak{gl}_{S,n}$, we have

$$(P^{-1}XP)^t = P^t X^t (P^{-1})^t = -P^t SXS^{-1}(P^t)^{-1} = -P^t SX(P^t S)^{-1} = -TP^{-1}XPT^{-1}$$

and therefore the map in Proposition 2.78 is well-defined. It is clear that the map is injective by its definition, and moreover the map is surjective.

It remains to check that we have a homomorphism of Lie algebras. Indeed, for all $X, Y \in \mathfrak{gl}_n$ we have

$$[P^{-1}XP, P^{-1}YP] = P^{-1}XYP - P^{-1}YXP = P^{-1}[X, Y]P$$

as required. □

Following Proposition 2.78, suppose that we take an arbitrary *non-degenerate* matrix S with an associated bilinear form. If S is also *symmetric*, then by Proposition 2.14 (and Remark 2.15) we can find a basis of the vector space $\mathbb{C}^n$ such that the corresponding matrix of this bilinear form with respect to this basis is the identity matrix I_n. Together with Proposition 2.8(b), we now see that $\mathfrak{gl}_{S,n} \cong \mathfrak{gl}_{I_n,n}$. In other words, each Lie subalgebra $\mathfrak{gl}_{S,n}$ for any invertible symmetric matrix $S \in \mathfrak{gl}_n$ is isomorphic to $\mathfrak{gl}_{I_n,n}$, and hence they are all isomorphic subalgebras regardless of the choice of S. In particular, by taking $S = I_n$, we obtain the set of all skew-symmetric $n \times n$-matrices which is often given as the definition of the Lie algebra $\mathfrak{so}_n$. We prefer to work with a different matrix S, see (2.53) and (2.62), which we discuss in Sections 2.6.1 and 2.6.2.

Similarly, suppose that we take an arbitrary non-degenerate *skew-symmetric* $n \times n$-matrix and its associated bilinear form, where $n = 2N$ for some $N \in \mathbb{N}$. Then, by Proposition 2.16 (and Remark 2.17), we can find a basis of the vector space $\mathbb{C}^n$ such that the corresponding matrix of this bilinear form with respect to this basis is the matrix S in Section 2.6.3, see (2.70).

2.6.1 The Lie algebra $\mathfrak{so}_{2N}$

We review the construction of $\mathfrak{so}_n$ for even $n = 2N$ following [EW06, Section 12.4]. Take S to be the matrix with $N \times N$-blocks

$$S = \left(\begin{array}{c|c} \mathbf{0} & I_N \\ \hline I_N & \mathbf{0} \end{array}\right) \tag{2.53}$$

and define $\mathfrak{so}_{2N} = \mathfrak{gl}_{S,2N}$ as in (2.52). By calculation, we obtain a block matrix realisation

$$\mathfrak{so}_{2N} = \left\{ \left(\begin{array}{c|c} M & P \\ \hline Q & -M^t \end{array}\right) \,\middle|\, P = -P^t,\, Q = -Q^t \right\} \tag{2.54}$$

where M, P and Q are $N \times N$-matrices. Note that the Lie algebra $\mathfrak{so}_{2N}$ has dimension $2N^2 - N$. In particular, the Lie algebra $\mathfrak{so}_2$ is 1-dimensional, and therefore *not* semisimple. For this reason, we assume that $N \geq 2$. For a Cartan subalgebra, we choose the set $\mathfrak{h}_{2N}$ of all diagonal matrices in $\mathfrak{so}_{2N}$, namely the semisimple elements

$$H = \sum_{i=1}^{N} a_i H_i$$

where $a_i \in \mathbb{C}$ and

$$H_i = E_{i,i} - E_{N+i,N+i} \qquad \text{for } 1 \leq i \leq N \tag{2.55}$$

with the matrix entries labelled from 1 up to $2N$. The elements (2.55) form a basis of $\mathfrak{h}_{2N}$, which we extend to a basis of $\mathfrak{so}_{2N}$ with the matrices

$$\begin{aligned} M_{i,j} &= E_{i,j} - E_{N+j,N+i} && \text{for } 1 \leq i \neq j \leq N, \\ P_{i,j} &= E_{i,N+j} - E_{j,N+i} && \text{for } 1 \leq i < j \leq N, \\ Q_{j,i} = P_{i,j}^t &= E_{N+j,i} - E_{N+i,j} && \text{for } 1 \leq i < j \leq N. \end{aligned} \tag{2.56}$$

The basis elements (2.56) are in fact simultaneous eigenvectors for the action of $\mathfrak{h}_{2N}$. Moreover, for $H \in \mathfrak{h}_{2N}$, calculation shows that

$$[H, M_{i,j}] = (a_i - a_j)M_{i,j}, \quad [H, P_{i,j}] = (a_i + a_j)P_{i,j}, \quad [H, Q_{j,i}] = -(a_i + a_j)Q_{j,i}. \tag{2.57}$$

We can now list the roots. For $1 \leq i \leq N$, let $\varepsilon_i \in \mathfrak{h}_{2N}^*$ be the map sending the element H to a_i, its entry in position i. Then, (2.57) implies that the eigenvectors $M_{i,j}$, $P_{i,j}$ and $Q_{j,i}$ have corresponding roots $\varepsilon_i - \varepsilon_j$, $\varepsilon_i + \varepsilon_j$ and $-(\varepsilon_i + \varepsilon_j)$ respectively. One sees that the set $\{\alpha_i \,|\, 1 \leq i \leq N\}$ is a base for the root system of $\mathfrak{so}_{2N}$, where

$$\alpha_i = \begin{cases} \varepsilon_i - \varepsilon_{i+1} & \text{for } 1 \leq i \leq N-1 \\ \varepsilon_{N-1} + \varepsilon_N & \text{for } i = N \end{cases} \tag{2.58}$$

are refered to as the simple roots. If $N = 2$, the base has two orthogonal roots only and hence the root system is *reducible* (see [EW06, Definition 11.7]). In fact, one sees that $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

We therefore assume that $N \geq 3$. For $1 \leq i \leq N-1$, we take

$$e_{\alpha_i} = M_{i,i+1}, \quad f_{\alpha_i} = M^t_{i,i+1}, \quad h_{\alpha_i} = H_i - H_{i+1}, \tag{2.59}$$

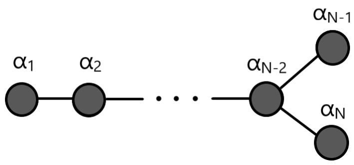
and then

$$e_{\alpha_N} = P_{N-1,N}, \quad f_{\alpha_N} = Q_{N,N-1}, \quad h_{\alpha_N} = H_{N-1} + H_N. \tag{2.60}$$

In particular, notice that $h_{\alpha_i} = \left[e_{\alpha_i}, f_{\alpha_i}\right]$ for all i. By calculating $\left[h_{\alpha_j}, e_{\alpha_i}\right]$ for each pair of simple roots, the Cartan integers $\alpha_i(H_j)$ are determined to be

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{for } i = j \\ -1 & \text{if } |i-j| = 1 \text{ for } i,j \neq N, \text{ or } \{i,j\} = \{N-2, N\} \\ 0 & \text{otherwise.} \end{cases} \tag{2.61}$$

Importantly, the Cartan integers (2.61) describe the Dynkin diagram of the root system of the Lie algebra. If $N = 3$, the Dynkin diagram is the same as that of $\mathbf{A}_3$, the root system of $\mathfrak{sl}_4$, and indeed $\mathfrak{so}_6 \cong \mathfrak{sl}_4$ as one might expect. In general, the Lie algebra $\mathfrak{so}_{2N}$ for $N \geq 4$ has the Dynkin diagram



of type $\mathbf{D}_N$. If we remove the node corresponding to (and edge connecting to) the simple root α_1, the remaining diagram is of type $\mathbf{D}_{N-1}$ with a shifted numbering. In other words, one sees that a copy of $\mathbf{D}_{N-1}$ sits inside $\mathbf{D}_N$. This justifies that $\mathfrak{so}_{2N-2} \subset \mathfrak{so}_{2N}$.

2.6.2 The Lie algebra $\mathfrak{so}_{2N+1}$

Now assume that $n = 2N+1$, and refer to [EW06, Section 12.3]. For this case, take the matrix

$$S = \left(\begin{array}{c|c|c} 1 & 0 \cdots 0 & 0 \cdots 0 \\ \hline \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \mathbf{0} & I_N \\ \hline \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & I_N & \mathbf{0} \end{array}\right) \tag{2.62}$$

and define $\mathfrak{so}_{2N+1} = \mathfrak{gl}_{S,2N+1}$ as in (2.52). One calculates that

$$\mathfrak{so}_{2N+1} = \left\{ \left(\begin{array}{c|c|c} 0 & c^t & -b^t \\ \hline b & M & P \\ \hline -c & Q & -M^t \end{array}\right) \middle| \, P = -P^t, Q = -Q^t \right\} \tag{2.63}$$

writing elements as block matrices of sizes adapted to the blocks of S from (2.62). The Lie algebra $\mathfrak{so}_{2N+1}$ has dimension $2N^2 + N$. Since $\mathfrak{so}_3 \cong \mathfrak{sl}_2$, we assume that $N \geq 2$.

It is convenient to label the matrix entries from 0 up to $2N$. Take the Cartan subalgebra $\mathfrak{h}_{2N+1}$ of diagonal matrices in $\mathfrak{so}_{2N+1}$. With our convention, the matrices H_i in (2.55) provide a basis for $\mathfrak{h}_{2N+1}$. We now extend the elements (2.55) and (2.56) to a basis of $\mathfrak{so}_{2N+1}$ by the matrices

$$b_i = E_{i,0} - E_{0,N+i}, \quad c_i = E_{0,i} - E_{N+i,0} \tag{2.64}$$

for $1 \leq i \leq N$. For $H = \sum_i a_i H_i \in \mathfrak{h}_{2N+1}$, in addition to (2.57), calculation shows that

$$[H, b_i] = a_i b_i, \quad [H, c_i] = -a_i c_i \tag{2.65}$$

and again these basis elements are simultaneous eigenvectors for the action of $\mathfrak{h}_{2N+1}$. We deduce from (2.65) that the eigenvectors b_i and c_i have corresponding roots ε_i and $-\varepsilon_i$ respectively. Moreover, the set $\{\gamma_i \mid 1 \leq i \leq N\}$ is a base for the root system of $\mathfrak{so}_{2N+1}$ if we set $\gamma_i = \alpha_i$ for $1 \leq i \leq N-1$ as given in (2.58), and define $\gamma_N = \varepsilon_N$.

In addition to the Chevalley generators e_{γ_i}, f_{γ_i}, h_{γ_i} for $1 \leq i \leq N-1$ from (2.59), we take

$$e_{\gamma_N} = \sqrt{2}\, b_N, \quad f_{\gamma_N} = \sqrt{2}\, c_N, \quad h_{\gamma_N} = 2\, H_N. \tag{2.66}$$

Then, as well as the Cartan integers from (2.61), by calculating $\left[h_{\gamma_N}, e_{\gamma_i}\right]$ and $\left[h_{\gamma_i}, e_{\gamma_N}\right]$ we get

$$\langle \gamma_i, \gamma_N \rangle = \begin{cases} -2 & \text{for } i = N-1, \\ 0 & \text{for } i \leq N-2, \end{cases} \quad \text{and} \quad \langle \gamma_N, \gamma_i \rangle = \begin{cases} -1 & \text{for } i = N-1, \\ 0 & \text{for } i \leq N-2. \end{cases} \tag{2.67}$$

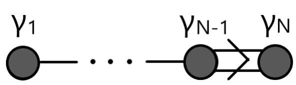
Note that

$$\left[f_{\gamma_i}, b_i\right] = b_{i+1}, \quad \left[e_{\gamma_i}, b_{i+1}\right] = b_i, \quad \left[f_{\gamma_i}, c_{i+1}\right] = -c_i, \quad \left[e_{\gamma_i}, c_i\right] = -c_{i+1} \tag{2.68}$$

for $1 \leq i \leq N-1$, and if we consider the Chevalley generators (2.60) for the copy of the Lie algebra $\mathfrak{so}_{2N}$ inside $\mathfrak{so}_{2N+1}$, then

$$\left[f_{\alpha_N}, b_{N-1}\right] = -c_N, \quad \left[e_{\alpha_N}, c_{N-1}\right] = b_N, \quad \left[f_{\alpha_N}, b_N\right] = c_{N-1}, \quad \left[e_{\alpha_N}, c_N\right] = -b_{N-1}. \tag{2.69}$$

Formulas (2.68) and (2.69) reflect the fact that the matrices (2.64) form a basis of the vector representation of the Lie subalgebra $\mathfrak{so}_{2N} \subset \mathfrak{so}_{2N+1}$ under the adjoint action. This is also evident from the block matrix representation (2.63) of $\mathfrak{so}_{2N+1}$. In general, the Lie algebra $\mathfrak{so}_{2N+1}$ for $N \geq 2$ has the Dynkin diagram of type $\mathbf{B}_N$ which is illustrated below.



2.6.3 The Lie algebra $\mathfrak{sp}_{2N}$

The Lie algebra $\mathfrak{sp}_n$ is defined only for *even* $n = 2N$, see [EW06, Section 12.5]. Take S to be the matrix with $N \times N$-blocks

$$S = \left(\begin{array}{c|c} \mathbf{0} & I_N \\ \hline -I_N & \mathbf{0} \end{array}\right) \tag{2.70}$$

and then we may define $\mathfrak{sp}_{2N} = \mathfrak{gl}_{S,2N}$ as in (2.52). By calculation, we obtain a block matrix realisation

$$\mathfrak{sp}_{2N} = \left\{ \left(\begin{array}{c|c} M & P \\ \hline Q & -M^t \end{array}\right) \,\middle|\, P = P^t,\, Q = Q^t \right\} \tag{2.71}$$

where M, P and Q are $N \times N$-matrices. Note that $\dim(\mathfrak{sp}_{2N}) = \dim(\mathfrak{so}_{2N+1}) = 2N^2 + N$, and in particular, $\mathfrak{sp}_2$ is the same as the Lie algebra $\mathfrak{sl}_2$. In what follows, we assume that $N \geq 2$.

As a Cartan subalgebra, let $\mathfrak{h}$ be the set of all diagonal matrices in $\mathfrak{sp}_{2N}$. Recall the elements H_i in (2.55) which again form a basis of $\mathfrak{h}$, and we extend to a basis of $\mathfrak{sp}_{2N}$ with the matrices in (2.56) and the additional $2N$ matrices

$$P_{i,i} = E_{i,N+i} \qquad \text{and} \qquad Q_{i,i} = P_{i,i}^t = E_{N+i,i} \tag{2.72}$$

for $1 \leq i \leq N$. The calculations in (2.57) hold, and additionally for $H = \sum_i a_i H_i$ we note that

$$\left[H, P_{i,i}\right] = 2a_i P_{i,i}, \qquad \left[H, Q_{i,i}\right] = -2a_i\, Q_{i,i}. \tag{2.73}$$

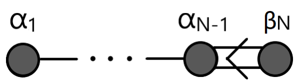
We can now list the roots. For $1 \leq i \leq N$, let $\varepsilon_i \in \mathfrak{h}^*$ be the map sending the element H to a_i, its entry in position i. Then, (2.73) implies that the eigenvectors $P_{i,i}$ and $Q_{i,i}$ have corresponding roots $2\varepsilon_i$ and $-2\varepsilon_i$ respectively. One sees that the set $\{\alpha_i \,|\, 1 \leq i \leq N-1\} \cup \{\beta_N\}$ is a base for the root system of $\mathfrak{sp}_{2N}$, where we define the simple roots α_i in (2.58) for $1 \leq i \leq N-1$ and $\beta_N = 2\varepsilon_N$. For $1 \leq i \leq N-1$ we have the Chevalley generators in (2.59), and additionally we take

$$e_{\beta_N} = P_{N,N}, \quad f_{\beta_N} = Q_{N,N}, \quad h_{\beta_N} = H_N. \tag{2.74}$$

We obtain the Cartan integers

$$\langle \alpha_i, \beta_N \rangle = \begin{cases} -1 & \text{for } i = N-1, \\ 0 & \text{otherwise,} \end{cases} \qquad \langle \beta_N, \alpha_i \rangle = \begin{cases} -2 & \text{for } i = N-1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.75}$$

In general, the Lie algebra $\mathfrak{sp}_{2N}$ for $N \geq 2$ has the Dynkin diagram of type $\mathbf{C}_N$ which is illustrated below.



Chapter 3

Symmetric Semisimple Lie Algebras

Suppose that $\mathfrak{g}$ is a Lie algebra, and let $\theta\colon \mathfrak{g} \to \mathfrak{g}$ be an automorphism of $\mathfrak{g}$ such that

$$\theta^2 = id_{\mathfrak{g}}. \tag{3.1}$$

A Lie algebra automorphism θ of $\mathfrak{g}$ satisfying this property is said to be *involutive*, and θ is often called an **involution** of $\mathfrak{g}$. Moreover, we call such a pair $(\mathfrak{g}, \theta)$ a **symmetric Lie algebra**.

The main focus of this chapter is to introduce the notion of a symmetric pair, that is, a symmetric Lie algebra $\mathfrak{g}$ together with a fixed Lie subalgebra $\mathfrak{k}$, see Section 3.1. For semisimple $\mathfrak{g}$, symmetric Lie algebras can be classified in terms of combinatorial data called Satake diagrams. In Section 3.2, we recall how to obtain a Satake diagram from an involution θ. Finally, in Section 3.3, we discuss the main examples of symmetric pairs considered in this thesis, for which we give the generators and relations explicitly. The results of this chapter follow mostly the paper [Kol14].

3.1 The fixed Lie subalgebra

Let θ be an involution of a Lie algebra $\mathfrak{g}$. Recall Equation (3.1) for the defining property of θ. Then, $\mathfrak{g}$ has the so-called **symmetric decomposition**

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{3.2}$$

where we denote

$$\mathfrak{k} = \mathfrak{g}^{\theta} = \{x \in \mathfrak{g} \mid \theta(x) = x\} \tag{3.3}$$

and similarly

$$\mathfrak{p} = \{x \in \mathfrak{g} \mid \theta(x) = -x\}.$$

For all $x, y \in \mathfrak{p}$, we have

$$\theta([x, y]) = [\theta(x), \theta(y)] = [-x, -y] = [x, y]$$

and hence $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. On the other hand, one similarly shows that $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$ and, in particular, we deduce that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$. We refer to $\mathfrak{k}$ as the **fixed Lie subalgebra** of $\mathfrak{g}$.

Lemma 3.1. *For any element $x \in \mathfrak{g}$, we have $x + \theta(x) \in \mathfrak{k}$.*

Proof. It is clear that $x + \theta(x) \in \mathfrak{g}$, since θ is an automorphism of $\mathfrak{g}$. It suffices to show that, as an element, $x + \theta(x)$ is invariant under the involution θ. Indeed, we calculate that

$$\begin{aligned} \theta\big(x + \theta(x)\big) &= \theta(x) + \theta\big(\theta(x)\big), && \text{by the linearity of } \theta \\ &= \theta(x) + \theta^2(x) \\ &= \theta(x) + x, && \text{since } \theta^2 = id \end{aligned}$$

as necessary. □

3.2 Involutions of semisimple Lie algebras

Now suppose that a Lie algebra $\mathfrak{g}$ is *semisimple*. As in Section 2.5, let $\Pi = \{\alpha_i \mid i \in I\}$ be the set of simple roots with respect to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, where $I \subset \mathbb{N}$ denotes the indexing set for the nodes of the Dynkin diagram of $\mathfrak{g}$. Recall from Section 2.5.3 the Chevalley generators for $\mathfrak{g}$ denoted by e_i, f_i, h_i for $i \in I$, where the elements h_i correspond to the generators of the Cartan subalgebra $\mathfrak{h}$. Now, we let

$$\mathfrak{n}^+ = \langle e_i \mid i \in I \rangle \qquad \text{and} \qquad \mathfrak{n}^- = \langle f_i \mid i \in I \rangle \tag{3.4}$$

denote the Lie subalgebras of $\mathfrak{g}$ generated by the elements e_i and f_i respectively. Then, the Lie algebra $\mathfrak{g}$ has the triangular decomposition [EW06, Section 15.1]

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+. \tag{3.5}$$

For any subset $X \subseteq I$, let $\mathfrak{g}_X$ be the Lie subalgebra of $\mathfrak{g}$ generated by the set $\{e_i, f_i, h_i \mid i \in X\}$. Let θ be an involution of $\mathfrak{g}$, for which the defining property is given in Equation (3.1). Following [Let02, Section 7] we may assume, up to conjugation, that

$$\theta(\mathfrak{h}) = \mathfrak{h} \tag{3.6}$$

and that there exists a subset $X \subseteq I$ such that

$$\theta\big|_{\mathfrak{g}_X} = \mathrm{id}_{\mathfrak{g}_X} \tag{3.7}$$

and

$$\theta(e_i) \in \mathfrak{n}^- \quad \text{and} \quad \theta(f_i) \in \mathfrak{n}^+ \quad \text{if } i \in I \setminus X. \tag{3.8}$$

Let Φ denote the root system of $\mathfrak{g}$. Define the **root lattice** of Φ by $Q = \mathbb{Z}\Phi$, and denote its corresponding positive part by $Q^+ = \mathbb{N}_0\Phi$. Additionally, for any root $\mu = \sum_{i \in I} n_i \alpha_i \in Q^+$, define the **height** of μ by

$$\mathrm{ht}(\mu) = \sum_{i \in I} n_i. \tag{3.9}$$

Let Q_X be the subgroup of the root lattice Q generated by the simple roots α_i for all $i \in X$. Effectively, Q_X is the *root lattice* for the Lie algebra $\mathfrak{g}_X$, and we denote its positive part by

$$Q_X^+ = Q^+ \cap Q_X. \tag{3.10}$$

For involutions of semisimple Lie algebras satisfying assumptions (3.6)-(3.8), one can give a description of the generators of the fixed Lie subalgebra $\mathfrak{k}$.

Proposition 3.2 ([Kol14, Lemma 2.8]). *Let θ be an involution of a semisimple Lie algebra $\mathfrak{g}$ satisfying assumptions (3.6)-(3.8). Then, the fixed Lie subalgebra $\mathfrak{k}$ is generated by the elements*

$$e_i, f_i \quad \text{for } i \in X, \tag{3.11}$$

$$h \in \mathfrak{h} \quad \text{with } \theta(h) = h, \tag{3.12}$$

$$f_i + \theta(f_i) \quad \text{for } i \in I \setminus X. \tag{3.13}$$

Proof. Let $\tilde{\mathfrak{k}}$ denote the Lie subalgebra of $\mathfrak{g}$ generated by the elements (3.11), (3.12) and (3.13). By assumptions (3.6) and (3.7), the generators (3.11) and (3.12) are invariant under θ. The remaining generators (3.13) are also invariant under θ following Lemma 3.1, and hence $\tilde{\mathfrak{k}} \subseteq \mathfrak{k}$.

Conversely, assume that $x \in \mathfrak{k}$. By the triangular decomposition (3.5) of $\mathfrak{g}$, we may write

$$x = x^- + x^0 + x^+$$

with $x^- \in \mathfrak{n}^-$, $x^0 \in \mathfrak{h}$, and $x^+ \in \mathfrak{n}^+$. Since $x^- \in \langle f_i \,|\, i \in I \rangle$, we can write

$$x^- = \sum_{\mathcal{I}=(i_1,\dots,i_k)} a_{\mathcal{I}} \big[f_{i_1}, [\dots, [f_{i_{k-1}}, f_{i_k}] \dots] \big] \in \bigoplus_{\alpha \in \Phi^+,\ \mathrm{ht}(\alpha) \leq m} \mathfrak{g}_{-\alpha}$$

for some coefficients $a_{\mathcal{I}}$ and index set $\mathcal{I} \in I^k$ for $k \leq m$ where $m \in \mathbb{N}$ is minimal. Let us define the element

$$u = \sum_{\mathcal{I}=(i_1,\dots,i_k)} a_{\mathcal{I}} \big[b_{i_1}, [\dots, [b_{i_{k-1}}, b_{i_k}] \dots] \big] \in \tilde{\mathfrak{k}}$$

where $b_{i_j} = f_{i_j}$ for $i_j \in X$, and $b_{i_j} = f_{i_j} + \theta(f_{i_j})$ for $i_j \in I \setminus X$. If $i_j \in X$ for $1 \leq j \leq k$, then $u = x^-$ and, hence, $u - x^- = 0 \in \mathfrak{h} + \mathfrak{n}^+$. On the other hand, assume that some $i_j \in I \setminus X$. Then, $u = x^- + y^- + y^0 + y^+$ with $y^0 \in \mathfrak{h}$, $y^+ \in \mathfrak{n}^+$, and $y^- \in \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}$ for $\mathrm{ht}(\alpha) \leq m-1$, since $\theta(f_{i_j}) \in \mathfrak{n}^+$ for $i_j \in I \setminus X$ by assumption (3.8). We may now consider the element $u - x^- = y^- + y^0 + y^+ \in \mathfrak{k}$, and by induction on $\mathrm{ht}(\alpha)$ we eventually get that $y^- = 0$. This shows that there always exists an element $u \in \tilde{\mathfrak{k}}$ such that

$$u - x^- \in \mathfrak{h} \oplus \mathfrak{n}^+.$$

Such an element $u \in \tilde{\mathfrak{k}}$ is a linear combination of elements of the form (3.11), (3.13) and all possible Lie brackets between these elements. Hence, we may assume that $x^- = 0$. Further, since $\theta(x^0) = x^0$ we have $x^0 \in \tilde{\mathfrak{k}}$, and therefore we can also assume that $x^0 = 0$.

We can write $x = x^+ \in \mathfrak{n}^+$ as a sum of weight vectors $x = \sum_{\alpha \in Q^+} x_\alpha$. Since

$$\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-w_X \circ \tau(\alpha)}$$

the relation $x_\alpha \neq 0$ implies that $\alpha = \sum_{i \in X} \mathbb{N}_0 \alpha_i$. Hence, the element x is contained in the Lie subalgebra generated by $\{e_i \mid i \in X\}$, and therefore $x \in \tilde{\mathfrak{k}}$ as required. $\square$

By assumption (3.6), the map θ induces an involution $\Theta\colon \mathfrak{h}^* \to \mathfrak{h}^*$ which leaves Φ invariant. Hence, by construction, we have

$$\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{\Theta(\alpha)} \tag{3.14}$$

for all roots $\alpha \in \Phi$. Moreover, if assumptions (3.6) to (3.8) hold, then there exists an involutive map $\tau\colon I \to I$ where $a_{i,j} = a_{\tau(i),\tau(j)}$ for all $i, j \in I$, called a **diagram automorphism** for the Dynkin diagram of $\mathfrak{g}$, such that

$$\Theta(-\alpha_i) - \alpha_{\tau(i)} \in Q_X^+ \tag{3.15}$$

for all $i \in I$, see also [Let02, (7.5)]. We can also view τ as an automorphism of $\mathfrak{g}$ by setting

$$\tau(e_i) = e_{\tau(i)}, \qquad \tau(f_i) = f_{\tau(i)}, \qquad \tau(h_i) = h_{\tau(i)} \tag{3.16}$$

for all $i \in I$. The induced map of $\mathfrak{h}^*$ further satisfies $\tau(\alpha_i) = \alpha_{\tau(i)}$ for all $i \in I$.

Let $\mathcal{W}_X$ be the parabolic subgroup of the Weyl group $\mathcal{W}$ of $\mathfrak{g}$ generated by the simple reflections s_i for all $i \in X$. Effectively, $\mathcal{W}_X$ is the *Weyl group* of the Lie algebra $\mathfrak{g}_X$. Let $w_X \in \mathcal{W}_X$ denote the longest element of $\mathcal{W}_X$. Additionally, let $\rho_X \in \mathbb{R}\Phi$ and $\rho_X^\vee \in (\mathbb{R}\Phi)^*$ denote the half sum of the positive roots and coroots for $\mathfrak{g}_X$, respectively.

Definition 3.3 ([Sat60, p.109], [Kol14, Definition 2.3])**.** *Let $X \subseteq I$ and let $\tau\colon I \to I$ be a diagram automorphism such that $\tau(X) = X$. The pair (X, τ) is called a* ***Satake diagram*** *if it satisfies the following properties:*

(1) $\tau^2 = \mathrm{id}_I$,

(2) *The action of τ on X coincides with the action of $-w_X$,*

(3) *If $j \in I \setminus X$ and $\tau(j) = j$, then $\alpha_j(\rho_X^\vee) \in \mathbb{Z}$.*

Any such Satake diagram (X, τ) determines an involution θ of $\mathfrak{g}$ uniquely up to conjugation, see [Kol14, Theorem 2.7]. The restriction of θ to $\mathfrak{h}$ is given by

$$\theta\big|_{\mathfrak{h}} = -w_X \circ \tau. \tag{3.17}$$

Graphically, the components of a Satake diagram are recorded in the Dynkin diagram of $\mathfrak{g}$. The nodes labelled by X are coloured black and we indicate the diagram automorphism τ by a double-sided arrow. A complete list of Satake diagrams for simple $\mathfrak{g}$ is found in [Ara62, p.32/33].

Remark 3.4. There exists a diagram automorphism $\tau_0\colon I \to I$ such that the longest element $w_0 \in \mathcal{W}$ of the Weyl group of $\mathfrak{g}$ satisfies $w_0(\alpha_i) = -\alpha_{\tau_0(i)}$ for all $i \in I$. It follows from this, and Definition 3.3, that the pair $(X = I, \tau = -w_X)$ is always a Satake diagram.

By inspection of the list of all Satake diagrams, one sees that the set X is always τ_0-invariant.

3.3 Symmetric pairs

For any Lie algebra $\mathfrak{g}$, let θ be an involution of $\mathfrak{g}$ with corresponding fixed Lie subalgebra $\mathfrak{k}$ as defined in Section 3.1. Then, the pair $(\mathfrak{g}, \mathfrak{k})$ is called a **symmetric pair**.

For the remainder of this chapter, we will study examples of symmetric pairs where $\mathfrak{g}$ is a complex semisimple Lie algebra. More specifically, we first consider the symmetric pair $\big(\mathfrak{sl}_n, \mathfrak{so}_n\big)$, and then give explicit constructions of the symmetric pairs $\big(\mathfrak{sl}_{2N}, \mathfrak{sp}_{2N}\big)$ and $\big(\mathfrak{so}_{2N}, \mathfrak{so}_{2N-1}\big)$, the latter of which is of main interest in this thesis.

3.3.1 The symmetric pair $\big(\mathfrak{sl}_n, \mathfrak{so}_n\big)$

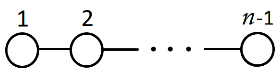
Recall the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$ of traceless $n \times n$-matrices where $n \geq 2$, and the Lie subalgebra $\mathfrak{so}_n \subset \mathfrak{sl}_n$ of skew-symmetric $n \times n$-matrices. Here, we take the involution

$$\theta \colon \mathfrak{sl}_n \to \mathfrak{sl}_n, \qquad x \mapsto -x^t.$$

In terms of the Chevalley generators, the involution θ sends $e_i \mapsto -f_i$, $f_i \mapsto -e_i$ and $h_i \mapsto -h_i$. The fixed Lie subalgebra is then

$$\mathfrak{k} = \big\{x \in \mathfrak{sl}_n \;\big|\; x^t = -x\big\} = \mathfrak{gl}_{I_n,n}$$

which we have seen is just the standard realisation of $\mathfrak{so}_n$. This means that the Satake diagram corresponding to the symmetric Lie algebra $(\mathfrak{sl}_n, \theta)$ coincides with the Dynkin diagram of $\mathfrak{sl}_n$, see below.



Indeed, for $(X, \tau) = (\emptyset, \mathrm{id})$ the assumptions (3.6) to (3.8) are satisfied, and moreover we have

$$\Theta(\alpha_i) = -\alpha_i$$

for $1 \leq i \leq n-1$. Since the fixed Lie subalgebra $\mathfrak{k} = \mathfrak{gl}_{I_n,n}$ is isomorphic to $\mathfrak{so}_n$, we obtain the symmetric pair $\big(\mathfrak{sl}_n, \mathfrak{so}_n\big)$.

3.3.2 The symmetric pair $\big(\mathfrak{sl}_{2N}, \mathfrak{sp}_{2N}\big)$

Recall the definition of the Lie algebra $\mathfrak{sp}_{2N}$ for $N \geq 2$ from Section 2.6.3. We have some freedom with the choice of the matrix S, see (2.70). Take the skew-symmetric $2N \times 2N$-matrix

$$S' = \left(\begin{array}{c|c|c|c} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \mathbf{0} & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \cdots & \mathbf{0} & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{array}\right) = \sum_{k=1}^{N} E_{2k-1,2k} - E_{2k,2k-1}. \tag{3.18}$$

Notice that the matrices S and S' can be obtained from each other by some permutation of their rows and columns. This means that there exists an invertible $2N \times 2N$-permutation matrix P such that $S' = P^{-1}SP = P^tSP$, since $P^{-1} = P^t$. Moreover, by Proposition 2.78, the map

$$\mathfrak{sp}_{2N} \to \mathfrak{sp}'_{2N} \ ; \ x \mapsto P^t x P \tag{3.19}$$

is an isomorphism of Lie algebras. Hence, we can replace S by S' in the definition of $\mathfrak{sp}_{2N}$ to alternatively define the isomorphic Lie algebra

$$\mathfrak{sp}'_{2N} = \big\{ x \in \mathfrak{sl}_{2N} \ \big| \ x^t S' = -S'x \big\}. \tag{3.20}$$

Remark 3.5. Explicitly, the permutation matrix P in the isomorphism (3.19) is, in fact, the $2N \times 2N$-matrix

$$P = \left(\begin{array}{c|c|c|c} 1\ \ 0 & 0\ \ 0 & \cdots & 0\ \ 0 \\ \hline 0\ \ 0 & 1\ \ 0 & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0\ \ 0 \\ \hline 0\ \ 0 & \cdots & 0\ \ 0 & 1\ \ 0 \\ \hline 0\ \ 1 & 0\ \ 0 & \cdots & 0\ \ 0 \\ \hline 0\ \ 0 & 0\ \ 1 & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0\ \ 0 \\ \hline 0\ \ 0 & \cdots & 0\ \ 0 & 0\ \ 1 \end{array}\right) = \sum_{k=1}^{N} E_{k,2k-1} + \sum_{l=N+1}^{2N} E_{l,2(l-N)}.$$

From now on, we choose only to work with the realisation (3.20) of $\mathfrak{sp}_{2N}$. For notational purposes, we may therefore write $\mathfrak{sp}_{2N}$ instead of $\mathfrak{sp}'_{2N}$ moving forward.

Lemma 3.6. *The map*

$$\theta \colon \mathfrak{sl}_{2N} \to \mathfrak{sl}_{2N} \, ; \quad x \mapsto -S'x^tS'^t$$

is an involution of the Lie algebra $\mathfrak{sl}_{2N}$*.*

Proof. Let $x, y \in \mathfrak{sl}_{2N}$. Firstly, for any $a, b \in \mathbb{C}$ we have

$$\theta(ax + by) = -S'(ax + by)^t S'^t = -aS'x^tS'^t - bS'y^tS'^t = a\,\theta(x) + b\,\theta(y)$$

so θ is linear. Also, for any $x, y \in \mathfrak{sl}_{2N}$ we have

$$\begin{aligned} \big[\theta(x), \theta(y)\big] &= \big[(-S'x^tS'^t), (-S'y^tS'^t)\big] \\ &= S'x^ty^tS'^t - S'y^tx^tS'^t \\ &= S'(x^ty^t - y^tx^t)S'^t \\ &= -S'\big((xy)^t - (yx)^t\big)S'^t \\ &= -S'(xy - yx)^tS'^t \\ &= -S'[x, y]^tS'^t = \theta\big([x, y]\big) \end{aligned}$$

so θ is a Lie algebra homomorphism. Furthermore, since θ is a linear map from $\mathfrak{sl}_{2N}$ to itself and has zero kernel, it is also bijective and hence θ is indeed a Lie algebra automorphism. Additionally, for any $x \in \mathfrak{sl}_{2N}$ we have

$$\theta^2(x) = \theta\big(\theta(x)\big) = \theta(-S'x^tS'^t) = -S'(-S'x^tS'^t)^tS'^t = \underbrace{S'S'}_{-I_{2N}}\,x\,\underbrace{S'^tS'^t}_{-I_{2N}} = x$$

since $S'^2 = -I_{2N}$. Hence, $\theta^2 = id_{\mathfrak{sl}_{2N}}$ and the Lie algebra automorphism θ is involutive. □

For any $x \in \mathfrak{sl}_{2N}$, observe that

$$\begin{aligned} x \in \mathfrak{sp}_{2N} &\iff -S'x = x^tS', && \text{by definition, see (3.20),} \\ &\iff S'x = x^tS'^t, && \text{since } S' \text{ is skew-symmetric,} \\ &\iff S'x = -S'^2x^tS'^t, && \text{since } -S'^2 = I_{2N}, \\ &\iff x = -S'x^tS'^t = \theta(x) \end{aligned}$$

where θ is the involution of $\mathfrak{sl}_{2N}$ from Lemma 3.6. This implies that $\mathfrak{sp}_{2N} = {\mathfrak{sl}_{2N}}^\theta$ is exactly the fixed Lie subalgebra of $\mathfrak{sl}_{2N}$, as expected. In other terms, we have shown that

$$\mathfrak{sp}_{2N} = \big\{x \in \mathfrak{sl}_{2N} \;\big|\; \theta(x) = x\big\} \tag{3.21}$$

and therefore, by definition, we obtain the symmetric pair $\big(\mathfrak{sl}_{2N}, \mathfrak{sp}_{2N}\big)$.

From Example 2.73, recall that the Chevalley generators of the Lie algebra $\mathfrak{sl}_{2N}$ are defined by

$$e_i = E_{i,i+1}\,, \quad f_i = E_{i+1,i}\,, \quad h_i = E_{i,i} - E_{i+1,i+1} \tag{3.22}$$

for $i \in \{1, \dots, 2N-1\}$. Recall that, for every $i \in I$, the triple $\big(e_i, f_i, h_i\big)$ spans a Lie subalgebra isomorphic to the Lie algebra $\mathfrak{sl}_2$. Now, for *even* $j \in \big\{2, ..., 2(N-1)\big\}$ we define the element

$$b_j = E_{j+1,j} + E_{j-1,j+2} = f_j + \big[e_{j+1}, [e_{j-1}, e_j]\big]. \tag{3.23}$$

Proposition 3.7. *The Lie algebra $\mathfrak{sp}_{2N}$ is generated by the elements* (3.22) *for odd i and* (3.23)*, that is, by the set*

$$\big\{e_i, f_i, h_i, b_j \;\big|\; 1 \le i, j \le 2N-1,\; i \text{ odd},\; j \text{ even}\big\}.$$

Proof. We first need to show that the Lie algebra $\mathfrak{sp}_{2N}$ contains the elements e_i, f_i, h_i for all odd i, as well as the elements b_j for even $j \in \{2, 4, 6, ..., 2(N-1)\}$. To this end, we check that

$$\theta(e_i) = \theta\big(E_{i,i+1}\big) = (-E_{i,i+1})E_{i+1,i}(-E_{i,i+1}) = E_{i,i+1} = e_i$$

for odd i, and similarly one calculates that $\theta(f_i) = f_i$. Since θ is a Lie algebra automorphism, we also have that

$$\theta(h_i) = \theta([e_i, f_i]) = [\theta(e_i), \theta(f_i)] = [e_i, f_i] = h_i$$

for odd i. Finally, we check that

$$\begin{aligned}\theta(b_j) &= \theta\big(E_{j+1,j} + E_{j-1,j+2}\big)\\ &= \big(-E_{j-1,j} - E_{j+1,j+2}\big)\big(E_{j,j+1} + E_{j+2,j-1}\big)\big(-E_{j+1,j+2} - E_{j-1,j}\big)\\ &= E_{j-1,j+2} + E_{j+1,j}\\ &= b_j\end{aligned}$$

for even j. It remains for us to see that $\mathfrak{sp}_{2N}$ is generated by the elements e_i, f_i, h_i and b_j. Indeed, these are exactly the elements (3.11) to (3.13) listed in Proposition 3.2, since

$$b_j - f_j = E_{j-1,j+2} = \theta(f_j)$$

for even j, taking the subset $X = \big\{i \in I \,\big|\, i \text{ odd}\big\}$ of the indexing set $I = \{1, \ldots, 2N-1\}$. □

The involution θ of the Lie algebra $\mathfrak{sp}_{2N}$ from Lemma 3.6 satisfies assumptions (3.6) to (3.8) in with $X = \{1, 3, \ldots, 2N-1\}$. Indeed, for even j we calculate that

$$\begin{aligned}\theta(e_j) = \theta(E_{j,j+1}) &= (-E_{j+2,j+1})E_{j+1,j}(-E_{j,j-1})\\ &= E_{j+2,j-1}\\ &= \big[f_{j+1}, [f_{j-1}, f_j]\big]\end{aligned}$$

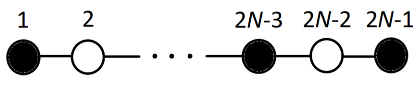
and

$$\begin{aligned}\theta(f_j) = \theta(E_{j+1,j}) &= (E_{j-1,j})E_{j,j+1}(E_{j+1,j+2})\\ &= E_{j-1,j+2}\\ &= \big[e_{j+1}, [e_{j-1}, e_j]\big].\end{aligned}$$

This shows that, for even j where $2 \leq j \leq 2N-2$, we have

$$\Theta(-\alpha_j) - \alpha_j = \alpha_{j-1} + \alpha_{j+1} \in Q_X^+$$

as expected, and the involution θ defined in Lemma 3.6 corresponds to the Satake diagram of type *AII* in [Ara62, p.32/33], illustrated below.



Example 3.8. We may give a non-standard presentation of the Lie algebra $\mathfrak{sp}_4$ in terms of generators and relations. By Proposition 3.7, we know that the Lie algebra $\mathfrak{sp}_4$ is generated by the elements

$$e_1, f_1, h_1, b_2, e_3, f_3, h_3.$$

Moreover, one can verify that such generators of the Lie algebra satisfy the defining relations

$$\begin{aligned} [e_i, b_2] &= 0 \\ [h_i, b_2] &= b_2 \\ \big[f_i, [f_i, b_2]\big] &= 0 \\ \big[b_2, [b_2, f_1]\big] &= -2e_3 \\ \big[b_2, [b_2, f_3]\big] &= -2e_1 \end{aligned}$$

together with the $\mathfrak{sl}_2$-relations of the triples (e_1, f_1, h_1) and (e_3, f_3, h_3). This presentation is useful for studying the classification of irreducible representations in the quantum setting. One can extend to a presentation of $\mathfrak{sp}_{2N}$ by exchanging the elements b_2, f_1 and f_3 for b_j, f_{j-1} and f_{j+1} (for even j) in the above relations, respectively. This case, however, is not the focus of the present thesis.

3.3.3 The symmetric pair $\big(\mathfrak{so}_{2N}, \mathfrak{so}_{2N-1}\big)$

Recall that, following Proposition 2.78 in Section 2.6, the Lie algebras $\mathfrak{gl}_{S,n}$ are all isomorphic, where S is a non-degenerate, symmetric matrix. If we take $S = I_n$, then $\mathfrak{gl}_{I_n,n}$ is just the space of skew-symmetric $n \times n$-matrices $x^t = -x$ for $x \in \mathfrak{gl}_n$. In this realisation, is it clear that $\mathfrak{so}_n \subset \mathfrak{so}_{n+1}$ because one can embed the space of skew-symmetric $n \times n$-matrices into the space of skew-symmetric $(n+1) \times (n+1)$-matrices.

Instead, we work with the different realisations (2.54) and (2.63) of the Lie algebras $\mathfrak{so}_{2N}$ and $\mathfrak{so}_{2N+1}$, respectively. One can see an immediate embedding of the Lie algebras $\mathfrak{so}_{2N} \subset \mathfrak{so}_{2N+1}$ as the lower right $2N \times 2N$ block. However, the embedding $\mathfrak{so}_{2N-1} \subset \mathfrak{so}_{2N}$ is less obvious in this realisation. In the following, we will realise $\mathfrak{so}_{2N-1} \subset \mathfrak{so}_{2N}$ as a symmetric pair for an involution θ satisfying assumptions (3.6)-(3.8). Consider the symmetric $2N \times 2N$-matrix

$$L = \left(\begin{array}{c|ccc|c|ccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & 0 & & & \\ \vdots & & \mathbf{0} & & \vdots & & I_{N-1} & \\ 0 & & & & 0 & & & \\ \hline 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & & & & 0 & & & \\ \vdots & & I_{N-1} & & \vdots & & \mathbf{0} & \\ 0 & & & & 0 & & & \end{array}\right) = \big(E_{1,1} + E_{N+1,N+1}\big) + \sum_{k=2}^{N} \big(E_{N+k,k} + E_{k,N+k}\big). \quad (3.24)$$

Observe that L and the matrix S in (2.53) commute, and recall that

$$\mathfrak{so}_{2N} = \{x \in \mathfrak{gl}_{2N} \colon x^t = -SxS^{-1}\}. \quad (3.25)$$

Lemma 3.9. *The map*

$$\theta\colon \mathfrak{so}_{2N} \to \mathfrak{so}_{2N}\,; \quad x \mapsto -Lx^tL^{-1}$$

is an involution of the Lie algebra $\mathfrak{so}_{2N}$.

Proof. Let $x \in \mathfrak{so}_{2N}$. Then, using that $L^{-1} = L^t$, $S^{-1} = S^t$ and $LS = SL$, we have

$$\theta(x)^t = (-Lx^tL^{-1})^t = (LSxS^{-1}L^{-1})^t = LSx^tS^{-1}L^{-1} = S(Lx^tL^{-1})S^{-1} = -S\,\theta(x)S^{-1},$$

hence θ is well-defined, and $\theta\big(\mathfrak{so}_{2N}\big)$ does indeed lie in $\mathfrak{so}_{2N}$. Moreover, for all $x, y \in \mathfrak{so}_{2N}$ the map θ satisfies the homomorphism property

$$\begin{aligned}\big[\theta(x), \theta(y)\big] &= \big[-Lx^tL^{-1}, -Ly^tL^{-1}\big] \\ &= L(x^ty^t - y^tx^t)L^{-1} \\ &= -L(xy - yx)^tL^{-1} = -L[x,y]^tL^{-1} = \theta\big([x,y]\big).\end{aligned}$$

Hence, θ is a Lie algebra homomorphism. It remains to verify that θ is an involution. For all $x \in \mathfrak{so}_{2N}$, one sees that

$$\theta^2(x) = \theta(-Lx^tL^{-1}) = -L(-Lx^tL^{-1})^tL^{-1} = L(L^t)^{-1}(x^t)^tL^tL^{-1} = x$$

since $L = L^t$, and hence $\theta^2 = \mathrm{id}$. □

Under the involution θ, by Lemma 3.9 we see that the fixed Lie subalgebra of $\mathfrak{so}_{2N}$ is

$$\mathfrak{so}_{2N}{}^{\theta} = \big\{x \in \mathfrak{so}_{2N} \,\big|\, x = \theta(x)\big\} = \mathfrak{gl}_{S,2N} \cap \mathfrak{gl}_{L,2N}. \tag{3.26}$$

Now, consider the $2N \times 2N$-matrix

$$\mathcal{X} = \left(\begin{array}{c|c|c|c} a & c^t & 0 & -p^t \\ \hline b & M & p & P \\ \hline 0 & -q^t & -a & -b^t \\ \hline q & Q & -c & -M^t \end{array}\right) \in \mathfrak{gl}_{S,2N} \tag{3.27}$$

for $a \in \mathbb{C}$, $b, c, p, q \in \mathbb{C}^{N-1}$, and $M, P, Q \in \mathbb{C}^{(N-1)\times(N-1)}$ where $P = -P^t$ and $Q = -Q^t$. Note that $L = L^{-1}$, since $L^2 = I_{2N}$. We already know that $\theta(\mathcal{X})$ is a general element of $\mathfrak{gl}_{S,2N}$ and, moreover, that $\theta\big(\mathfrak{gl}_{S,2N}\big) = \mathfrak{gl}_{S,2N}$, see Lemma 3.9. By definition of the involution θ, we have

$$\theta(\mathcal{X}) = -L\mathcal{X}^tL^{-1}.$$

Calculating this explicitly, we get

$$\theta(\mathcal{X}) = \left(\begin{array}{c|c|c|c} -1 & 0\cdots 0 & 0 & 0\cdots 0 \\ \hline \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \mathbf{0} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & -I_{N-1} \\ \hline 0 & 0\cdots 0 & -1 & 0\cdots 0 \\ \hline \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & -I_{N-1} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \mathbf{0} \end{array}\right) \left(\begin{array}{c|c|c|c} a & b^t & 0 & q^t \\ \hline c & M^t & -q & Q^t \\ \hline 0 & p^t & -a & -c^t \\ \hline -p & P^t & -b & -M \end{array}\right) \left(\begin{array}{c|c|c|c} 1 & 0\cdots 0 & 0 & 0\cdots 0 \\ \hline \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \mathbf{0} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & I_{N-1} \\ \hline 0 & 0\cdots 0 & 1 & 0\cdots 0 \\ \hline \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & I_{N-1} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \mathbf{0} \end{array}\right)$$

$$= \left(\begin{array}{c|c|c|c} -a & -q^t & 0 & -b^t \\ \hline p & M & b & P \\ \hline 0 & c^t & a & -p^t \\ \hline -c & Q & q & -M^t \end{array}\right) \tag{3.28}$$

and thus, under the Lie algebra automorphism θ, we obtain an element of the form

$$\theta(\mathcal{X}) = \left(\begin{array}{c|c} M' & P' \\ \hline Q' & -M'^t \end{array}\right) \tag{3.29}$$

now consisting entirely of $N \times N$-blocks where $P' = -P'^t$ and $Q' = -Q'^t$. Crucially, observe that if we take an element $x \in \mathfrak{gl}_{S,2N}$ of the form (3.27) then, following Equation (3.28), we have

$$\theta(x) = x \iff a = 0,\ b = p \text{ and } c = -q. \tag{3.30}$$

The following result gives an embedding of $\mathfrak{so}_{2N-1}$ inside $\mathfrak{so}_{2N}$ as the fixed Lie subalgebra.

Proposition 3.10. *The linear map $\eta\colon \mathfrak{so}_{2N-1} \to \mathfrak{so}_{2N}$ defined by*

$$\left(\begin{array}{c|c|c} 0 & c^t & -b^t \\ \hline b & M & P \\ \hline -c & Q & -M^t \end{array}\right) \mapsto \left(\begin{array}{c|c|c|c} 0 & \frac{1}{\sqrt{2}}c^t & 0 & -\frac{1}{\sqrt{2}}b^t \\ \hline \frac{1}{\sqrt{2}}b & M & \frac{1}{\sqrt{2}}b & P \\ \hline 0 & \frac{1}{\sqrt{2}}c^t & 0 & -\frac{1}{\sqrt{2}}b^t \\ \hline -\frac{1}{\sqrt{2}}c & Q & -\frac{1}{\sqrt{2}}c & -M^t \end{array}\right)$$

for $b, c \in \mathbb{C}^{N-1}$, and $M, P, Q \in \mathbb{C}^{(N-1)\times(N-1)}$ where $P = -P^t$ and $Q = -Q^t$, is an injective Lie algebra homomorphism with image ${\mathfrak{so}_{2N}}^{\theta}$.

Proof. Suppose $x, y \in \mathfrak{so}_{2N-1}$, that is, let

$$x = \left(\begin{array}{c|c|c} 0 & {c_1}^t & -{b_1}^t \\ \hline b_1 & M_1 & P_1 \\ \hline -c_1 & Q_1 & -{M_1}^t \end{array}\right) \qquad \text{and} \qquad y = \left(\begin{array}{c|c|c} 0 & {c_2}^t & -{b_2}^t \\ \hline b_2 & M_2 & P_2 \\ \hline -c_2 & Q_2 & -{M_2}^t \end{array}\right).$$

Since ${c_1}^t b_2 + {b_1}^t c_2 = {c_2}^t b_1 + {b_2}^t c_1$, their matrix commutator is

$$[x, y] = \left(\begin{array}{c|c|c} 0 & c^t & -b^t \\ \hline b & M & P \\ \hline -c & Q & -M^t \end{array}\right)$$

where

$$\begin{aligned} b &= \big(M_1 b_2 - M_2 b_1\big) + \big(P_2 c_1 - P_1 c_2\big), \\ c &= \big(Q_2 b_1 - Q_1 b_2\big) + \big({M_2}^t c_1 - {M_1}^t c_2\big), \\ M &= \big(b_1 {c_2}^t - b_2 {c_1}^t\big) + \big(M_1 M_2 - M_2 M_1\big) + \big(P_1 Q_2 - P_2 Q_1\big), \\ P &= \big(b_2 {b_1}^t - b_1 {b_2}^t\big) + \big(M_1 P_2 - M_2 P_1\big) + \big(P_2 {M_1}^t - P_1 {M_2}^t\big), \\ Q &= \big(c_2 {c_1}^t - c_1 {c_2}^t\big) + \big({M_2}^t Q_1 - {M_1}^t Q_2\big) + \big(Q_1 M_2 - Q_2 M_1\big) \end{aligned}$$

taking products of the entries as their appropriate vector/matrix multiplication. On the other hand, one calculates that

$$\big[\eta(x), \eta(y)\big] = \left(\begin{array}{c|c|c|c} 0 & \hat{c}^t & 0 & -\hat{b}^t \\ \hline \hat{b} & \hat{M} & \hat{b} & \hat{P} \\ \hline 0 & \hat{c}^t & 0 & -\hat{b}^t \\ \hline -\hat{c} & \hat{Q} & -\hat{c} & -\hat{M}^t \end{array}\right)$$

where $\hat{b} = \frac{1}{\sqrt{2}} b$, $\hat{c} = \frac{1}{\sqrt{2}} c$, $\hat{M} = M$, $\hat{P} = P$, and $\hat{Q} = Q$. Therefore, we obtain the equation

$$\big[\eta(x), \eta(y)\big] = \left(\begin{array}{c|c|c|c} 0 & \frac{1}{\sqrt{2}} c^t & 0 & -\frac{1}{\sqrt{2}} b^t \\ \hline \frac{1}{\sqrt{2}} b & M & \frac{1}{\sqrt{2}} b & P \\ \hline 0 & \frac{1}{\sqrt{2}} c^t & 0 & -\frac{1}{\sqrt{2}} b^t \\ \hline -\frac{1}{\sqrt{2}} c & Q & -\frac{1}{\sqrt{2}} c & -M^t \end{array}\right) = \eta\big([x, y]\big)$$

and hence the map η is a Lie algebra homomorphism. Injectivity is clear from the way η is given explicitly on the matrix elements of $\mathfrak{so}_{2N-1}$. □

The above result shows that $\big(\mathfrak{so}_{2N}, \mathfrak{so}_{2N-1}\big)$ is indeed a symmetric pair. We will now identify the Chevalley generators of $\mathfrak{so}_{2N-1}$ inside $\mathfrak{so}_{2N}$ via the embedding η. Observe first that

$$\eta(e_{\gamma_i}) = e_{\alpha_{i+1}}, \qquad \eta(f_{\gamma_i}) = f_{\alpha_{i+1}}, \qquad \eta(h_{\gamma_i}) = h_{\alpha_{i+1}} \tag{3.31}$$

for $1 \leq i \leq N-2$. Next, recall the elements $b_i \in \mathfrak{so}_{2N-1}$ defined by (2.64) for $1 \leq i \leq N-1$. We have

$$f_{\alpha_1} + \theta(f_{\alpha_1}) = \sqrt{2}\eta(b_1). \tag{3.32}$$

Hence,

$$\begin{aligned} \eta(e_{\gamma_{N-1}}) &= \sqrt{2}\eta(b_{N-1}) \\ &= \sqrt{2}\eta([f_{\gamma_{N-2}}, [f_{\gamma_{N-3}}, \dots, [f_{\gamma_1}, b_1] \dots]]) \\ &= \sqrt{2}[f_{\alpha_{N-1}}, [f_{\alpha_{N-2}}, \dots, [f_{\alpha_2}, \eta(b_1)] \dots]] \\ &= [f_{\alpha_{N-1}}, [f_{\alpha_{N-2}}, \dots, [f_{\alpha_2}, f_{\alpha_1} + \theta(f_{\alpha_1})] \dots]] \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} \eta(f_{\gamma_{N-1}}) &= \sqrt{2}\eta(c_{N-1}) \\ &= \sqrt{2}\eta([f_{\alpha_N}, b_{N-2}]) \\ &= \sqrt{2}\eta([f_{\alpha_N}, [f_{\gamma_{N-3}}, \dots, [f_{\gamma_1}, b_1] \dots]]) \\ &= \sqrt{2}[f_{\alpha_N}, [f_{\alpha_{N-2}}, \dots, [f_{\alpha_2}, \eta(b_1)] \dots]] \\ &= [f_{\alpha_N}, [f_{\alpha_{N-2}}, \dots, [f_{\alpha_2}, f_{\alpha_1} + \theta(f_{\alpha_1})] \dots]]. \end{aligned} \tag{3.34}$$

Moreover, we have

$$\eta(h_{\gamma_{N-1}}) = \eta(2(E_{N-1,N-1} - E_{2N-2,2N-2})) = h_{\alpha_N} - h_{\alpha_{N-1}}. \tag{3.35}$$

The involution θ of the Lie algebra $\mathfrak{so}_{2N}$ from Lemma 3.9 satisfies assumptions (3.6) to (3.8) in Section 3.2 with $X = \{2, \dots, N-2, N-1, N\}$. Indeed, by Equation (3.28) we have

$$\begin{aligned} \theta(e_{\alpha_1}) &= \theta\big(E_{1,2} - E_{N+2,N+1}\big) \\ &= E_{N+1,2} - E_{N+2,1} \\ &= (-1)^{N-1}\big[f_2, \dots [f_{N-2}, [f_N, [f_{N-1}, [f_{N-2}, \dots [f_2, f_1] \dots]]]] \dots\big] \end{aligned}$$

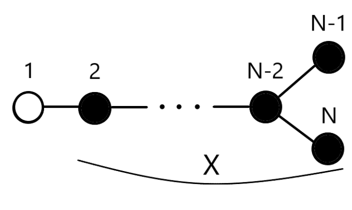
and

$$\begin{aligned} \theta(f_{\alpha_1}) &= \theta\big(E_{2,1} - E_{N+1,N+2}\big) \\ &= E_{2,N+1} - E_{1,N+2} \\ &= (-1)^{N-1}\big[e_2, \dots [e_{N-2}, [e_N, [e_{N-1}, [e_{N-2}, \dots [e_2, e_1] \dots]]]] \dots\big]. \end{aligned}$$

This shows that

$$\Theta(-\alpha_1) - \alpha_1 = 2(\alpha_2 + \dots + \alpha_{N-2}) + \alpha_{N-1} + \alpha_N \in Q_X^+$$

as expected, and the involution θ defined in Lemma 3.9 corresponds to the Satake diagram of type *DII*, see below.



Chapter 4

Quantum Groups

This chapter is dedicated to studying quantised enveloping algebras, which will ultimately enable us to introduce the notion of a quantum symmetric pair.

We begin by giving a definition of a quantised enveloping algebra in terms of generators and relations. Following [Kol14] we define the quantised enveloping algebra over the field $\mathbb{K}(q)$ of rational functions in q over a field of characteristic zero. More generally, it would suffice to assume that q is *not* a root of unity in a field of characteristic zero. We refer to [Jan96] frequently during this chapter for general theory of quantum groups, see also [CP94].

4.1 The quantised enveloping algebra $U_q(\mathfrak{g})$

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and recall the indexing set $I \subset \mathbb{N}$ for the nodes of the Dynkin diagram of $\mathfrak{g}$. Fix a base $\Pi = \{\alpha_i \mid i \in I\}$ for the root system Φ of $\mathfrak{g}$ with respect to a fixed Cartan subalgebra, and recall the Cartan integers $a_{i,j}$ $(i, j \in I)$ of Φ, see Section 2.5.2. For all $i \in I$, let

$$q_i = q^{\frac{(\alpha_i,\alpha_i)}{2}} = q^{d_i} \tag{4.1}$$

where we recall the integer d_i from Section 2.5.2. Following the notation in [Jan96, Chapter 0], define a q-**number**

$$[r]_{q_i} = \frac{{q_i}^r - {q_i}^{-r}}{q_i - {q_i}^{-1}} = \sum_{k=0}^{r-1} {q_i}^{(r-1)-2k} \tag{4.2}$$

for any $r \in \mathbb{N}$ and $i \in I$. Set $[0]_{q_i} = 0$ for convention, and notice that $[1]_{q_i} = 1$ for all $i \in I$. Naturally, a q-**factorial**, a q-analogue of the ordinary factorial, is defined recursively using q-numbers by

$$[r]^!_{q_i} = [1]_{q_i} \cdots [r]_{q_i} \tag{4.3}$$

for any $r \in \mathbb{N}$ and $i \in I$. For convention, we set $[0]^!_{q_i} = 1$. Then, for all $r \geq s \in \mathbb{N}_0$, we define the q-**binomial coefficients**

$$\begin{bmatrix} r \\ s \end{bmatrix}_{q_i} = \frac{[r]^!_{q_i}}{[s]^!_{q_i}[r-s]^!_{q_i}}. \tag{4.4}$$

Remark 4.1. If all roots $\alpha \in \Phi$ are of the same length, then we may write $[r]_q$, $[r]_q^!$ and $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]_q$. In particular, this is the case when $(\alpha, \alpha) = 2$ for all $\alpha \in \Phi$ (and hence $q_i = q$ for all $i \in I$) if the root system Φ is of type $\mathbf{A}_N$ or $\mathbf{D}_N$, that is, if $\mathfrak{g}$ is the Lie algebra $\mathfrak{sl}_{N+1}$ or $\mathfrak{so}_{2N}$.

For all $i, j \in I$, let $\mathcal{Q}_{i,j}(x, y)$ denote the *non-commutative* polynomial in two variables x and y defined by

$$\mathcal{Q}_{i,j}(x, y) = \sum_{s=0}^{1-a_{i,j}} (-1)^s \begin{bmatrix} 1 - a_{i,j} \\ s \end{bmatrix}_{q_i} x^{1-a_{i,j}-s} y x^s. \tag{4.5}$$

Recall the *Kronecker delta function* $\delta_{i,j}$ for all $i, j \in \mathbb{N}$ in Section 2.1.1. Set $\Pi^\vee = \{h_i \mid i \in I\}$, and let $Q^\vee = \mathbb{Z}\Pi^\vee$ be the **coroot lattice** of $\mathfrak{g}$. The algebra $U(\mathfrak{g})$ has the following deformation.

Definition 4.2 ([Jan96, Definition 4.3]). *The* ***quantised enveloping algebra*** $U_q(\mathfrak{g})$ *is the associative* $\mathbb{K}(q)$*-algebra with generators* E_i*,* F_i*, and* K_h *for all* $i \in I$*,* $h \in Q^\vee$ *satisfying relations*

$(U1)$ $\quad K_0 = 1$ *and* $K_h K_{h'} = K_{h+h'}$ *for all* $h, h' \in Q^\vee$,

$(U2)$ $\quad K_h E_i = q^{\alpha_i(h)} E_i K_h$ *for all* $i \in I$*,* $h \in Q^\vee$,

$(U3)$ $\quad K_h F_i = q^{-\alpha_i(h)} F_i K_h$ *for all* $i \in I$*,* $h \in Q^\vee$,

$(U4)$ $\quad E_i F_j - F_j E_i = \delta_{i,j} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ *for all* $i, j \in I$*, and where* $K_i = K_{h_i}^{d_i}$,

$(U5)$ $\quad \mathcal{Q}_{i,j}(E_i, E_j) = 0$ *for* $i, j \in I$ *where* $i \neq j$,

$(U6)$ $\quad \mathcal{Q}_{i,j}(F_i, F_j) = 0$ *for* $i, j \in I$ *where* $i \neq j$.

The relations $(U5)$ and $(U6)$ are referred to as the *quantum Serre relations*. The quantised enveloping algebra $U_q(\mathfrak{g})$ has a Hopf algebra structure which is a deformation of the Hopf algebra structure of $U(\mathfrak{g})$, see [HK02, Section 3]. The formulas for the coproduct Δ, counit ε, and antipode S of $U_q(\mathfrak{g})$ can be given explicitly.

Proposition 4.3 ([Jan96, Proposition 4.11]). *There is a unique Hopf algebra structure on* $U_q(\mathfrak{g})$ *such that*

$$\begin{aligned} \Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, & \varepsilon(E_i) &= 0, & S(E_i) &= -K_i^{-1} E_i, \\ \Delta(F_i) &= E_i \otimes K_i^{-1} + 1 \otimes F_i, & \varepsilon(F_i) &= 0, & S(F_i) &= -F_i K_i, \\ \Delta(K_h) &= K_h \otimes K_h, & \varepsilon(K_h) &= 1, & S(K_h) &= K_h^{-1} \end{aligned}$$

for all $i \in I$*,* $h \in Q^\vee$*.*

Sketch of Proof. A complete and detailed proof is provided in [Jan96, 4.8-4.11]. For this, one considers the algebra $\tilde{U}_q(\mathfrak{g})$ defined by the same generators E_i, F_i, and K_h for all $i \in I$, $h \in Q^\vee$ but where we impose only the relations $(U1)$-$(U4)$. There is a canonical surjection $\tilde{U}_q(\mathfrak{g}) \twoheadrightarrow U_q(\mathfrak{g})$. One first shows that $\tilde{U}_q(\mathfrak{g})$ has a Hopf algebra structure given by the same formulas by verifying that the relations are preserved under Δ, ε and S. To then show that $U_q(\mathfrak{g})$ has a Hopf algebra structure, one takes the quotient of $\tilde{U}_q(\mathfrak{g})$ over a suitable two-sided ideal $\mathcal{I} \subset \tilde{U}_q(\mathfrak{g})$ which adds in the quantum Serre relations $(U5)$ and $(U6)$. Such an ideal satisfies the properties

$$\Delta(\mathcal{I}) \subseteq \tilde{U}_q(\mathfrak{g}) \otimes \mathcal{I} + \mathcal{I} \otimes \tilde{U}_q(\mathfrak{g}), \qquad \varepsilon(\mathcal{I}) = 0, \qquad S(\mathcal{I}) \subseteq \mathcal{I}$$

which means that $\mathcal{I}$ is a Hopf ideal, and hence induces a Hopf structure on $U_q(\mathfrak{g})$. □

The following lemma is proved by a check using the relations $(U1)$-$(U6)$.

Lemma 4.4 ([Jan96, Lemma 4.6]).

a) *There is a unique algebra automorphism ω of $U_q(\mathfrak{g})$ such that $\omega(E_i) = F_i$, $\omega(F_i) = E_i$, and $\omega(K_h) = K_{-h}$ for all $i \in I$, $h \in Q^\vee$. Additionally, one has $\omega^2 = \mathrm{id}_{U_q(\mathfrak{g})}$.*

b) *There is a unique algebra antiautomorphism ι of $U_q(\mathfrak{g})$ such that $\iota(E_i) = E_i$, $\iota(F_i) = F_i$, and $\iota(K_h) = K_{-h}$ for all $i \in I$, $h \in Q^\vee$. Additionally, one has $\iota^2 = \mathrm{id}_{U_q(\mathfrak{g})}$.*

For all $i \in I$ and $r \in \mathbb{Z}$, set

$$\big[K_i; r\big] = \frac{q_i^r K_i - q_i^{-r} K_i^{-1}}{q_i - q_i^{-1}}. \tag{4.6}$$

Proposition 4.5 ([Jan96, 4.4(6),(7)]). *For all $i \in I$ and $r \in \mathbb{N}$, we have*

$$E_i F_i^r = F_i^r E_i + [r]_{q_i} F_i^{r-1} \big[K_i; 1-r\big], \tag{4.7}$$

and

$$F_i E_i^r = E_i^r F_i - [r]_{q_i} E_i^{r-1} \big[K_i; r-1\big]. \tag{4.8}$$

Proof. It suffices to give a proof of Equation (4.7) by induction on $r \in \mathbb{N}$. We will require a generalisation of the formula in [Jan96, 1.3(2)], that is, for all $r, s, t \in \mathbb{Z}$ and $i \in I$ one has

$$[s+t]_{q_i} \big[K_i; r\big] = [s]_{q_i} \big[K_i; r+t\big] + [t]_{q_i} \big[K_i; r-s\big]. \tag{4.9}$$

Additionally, relations $(U2)$ and $(U3)$ respectively imply that (for all $r \in \mathbb{Z}$)

$$\big[K_i; r\big] E_i = E_i \big[K_i; r+2\big] \qquad \text{and} \qquad \big[K_i; r\big] F_i = F_i \big[K_i; r-2\big].$$

Firstly, notice that Equation (4.7) holds when $r = 1$ since, from relation $(U4)$, we can write

$$E_i F_i - F_i E_i = \big[K_i; 0\big]$$

for all $i \in I$. If we now assume that Equation (4.7) holds for some $r \in \mathbb{N}$, then inductively we get

$$\begin{aligned} E_i F_i^{r+1} &= \Big(F_i^r E_i + [r]_{q_i} F_i^{r-1} \big[K_i; 1-r\big]\Big) F_i \\ &= F_i^r \Big(F_i E_i + \big[K_i; 0\big]\Big) + [r]_{q_i} F_i^{r-1} \Big(F_i \big[K_i; -(r+1)\big]\Big) \\ &= F_i^{r+1} E_i + [r+1]_{q_i} F_i^r \big[K_i; -r\big] \end{aligned}$$

using Equation (4.9), since $[r]_{q_i}\big[K_i;-(r+1)\big] = [r+1]_{q_i}\big[K_i;-r\big] - \big[K_i;0\big]$. We can then apply the automorphism ω from Lemma 4.4 a) to deduce Equation (4.8). In particular, we use that (for all $r \in \mathbb{Z}$)

$$\omega\big(\big[K_i;r\big]\big) = -\big[K;-r\big].$$

□

Let $\mathcal{U}$ be the Hopf subalgebra of $U_q(\mathfrak{g})$ generated by the elements E_i, F_i, and $K_i^{\pm 1}$ for all $i \in I$.

Remark 4.6. The Hopf algebra $\mathcal{U}$ coincides with the Hopf algebra $U_q(\mathfrak{g})$ in the case that the root system Φ is of type A_N or D_N, that is, if $\mathfrak{g}$ is the Lie algebra $\mathfrak{sl}_{N+1}$ or $\mathfrak{so}_{2N}$.

Now, let $U_q(\mathfrak{g})^+$, $U_q(\mathfrak{g})^-$ and $U_q(\mathfrak{g})^0$ denote the subalgebras of $U_q(\mathfrak{g})$ generated by the sets of elements $\{E_i \mid i \in I\}$, $\{F_i \mid i \in I\}$, and $\{K_h \mid h \in Q^\vee\}$ respectively. Moreover, let $\mathcal{U}^0$ denote the subalgebra of $\mathcal{U}$ generated by the set of elements $\{K_i^{\pm 1} \mid i \in I\}$. By [Jan96, 4.21], the algebras $U_q(\mathfrak{g})$ and $\mathcal{U}$ both have a triangular decomposition, similar to that of the Lie algebra $\mathfrak{g}$ in Equation (3.5), in the sense that the multiplication maps give isomorphisms of vector spaces

$$U_q(\mathfrak{g})^- \otimes U_q(\mathfrak{g})^0 \otimes U_q(\mathfrak{g})^+ \cong U_q(\mathfrak{g}) \qquad \text{and} \qquad U_q(\mathfrak{g})^- \otimes \mathcal{U}^0 \otimes U_q(\mathfrak{g})^+ \cong \mathcal{U}. \tag{4.10}$$

It follows from relation $(U1)$ that $\mathcal{U}^0$ is a *commutative* algebra.

Remark 4.7. Let $\mathbb{K}(q)[Q]$ be the group algebra of the root lattice $Q = \mathbb{Z}\Pi$ of $\mathfrak{g}$. By [Kol14, 3.1, p.410], there exists an algebra isomorphism $\mathbb{K}(q)[Q] \to \mathcal{U}^0$ such that $\alpha_i \mapsto K_i$. Hence, for any $\mu \in Q$, we write

$$K_\mu = \prod_{i\in I} K_i^{n_i} \tag{4.11}$$

if $\mu = \sum_{i\in I} n_i\alpha_i$ for $n_i \in \mathbb{Z}$. With this notation, the relations $(U2)$ and $(U3)$ take the form

$$K_\mu E_i = q^{(\mu,\alpha_i)} E_i K_\mu \qquad \text{and} \qquad K_\mu F_i = q^{-(\mu,\alpha_i)} F_i K_\mu \tag{4.12}$$

respectively, for all $i \in I$, $\mu \in Q$.

For each $i \in I$, let $\varpi_i \in \mathfrak{h}^*$ denote the **fundamental weight** which, by definition, is given by

$$\varpi_i(h_j) = \delta_{i,j} \tag{4.13}$$

for all $i, j \in I$. Define the **weight lattice** of Φ by $\Lambda = \sum_{i\in I} \mathbb{Z}\varpi_i$. For any $U_q(\mathfrak{g})$-module V and *weight* $\lambda \in \Lambda$, let

$$V_\lambda^\pm = \{v \in V \mid K_h v = \pm q^{\lambda(h)} v, \forall h \in Q^\vee\} \tag{4.14}$$

denote the corresponding *weight space*. In particular, with respect to the left adjoint action of $U_q(\mathfrak{g})$ on itself, one obtains a Q-grading of $U_q(\mathfrak{g})$ (and of $\mathcal{U}$). For instance, we obtain

$$U_q(\mathfrak{g})^+ = \bigoplus_{\mu\in Q^+} U_q(\mathfrak{g})^+_\mu \qquad \text{and} \qquad U_q(\mathfrak{g})^- = \bigoplus_{\mu\in Q^+} U_q(\mathfrak{g})^-_{-\mu}. \tag{4.15}$$

Using Proposition 4.3 explicitly, one calculates that

$$\operatorname{ad}(E_i)(u) = E_i u - K_i u K_i^{-1} E_i, \tag{4.16}$$

$$\operatorname{ad}(F_i)(u) = (F_i u - u F_i) K_i, \tag{4.17}$$

$$\operatorname{ad}(K_i)(u) = K_i u K_i^{-1} \tag{4.18}$$

for all $i \in I$, $u \in U_q(\mathfrak{g})$. Hence, from Equation (4.18), we can define

$$U_q(\mathfrak{g})_\mu = \left\{ u \in U_q(\mathfrak{g}) \,\middle|\, K_i u K_i^{-1} = q^{(\mu,\alpha_i)} u, \forall i \in I \right\} \tag{4.19}$$

for all $\mu \in Q$. We also write $U_q(\mathfrak{g})_\mu^\pm = U_q(\mathfrak{g})_\mu \cap U_q(\mathfrak{g})^\pm$. The algebra $\mathcal{U}_\mu$ is defined analogously.

4.2 Representations of $U_q(\mathfrak{sl}_2)$

In this section, we restrict to the quantised enveloping algebra of the Lie algebra $\mathfrak{sl}_2$. In this case, we have $I = \{1\}$ since the root system of $\mathfrak{sl}_2$ contains only *one* simple root, namely $\alpha_1 \in \Pi$. To simplify notation, we write the elements E, F, K instead of E_1, F_1, K_{h_1} in Definition 4.2. Then, the algebra $U_q(\mathfrak{sl}_2)$ is generated by the set $\{E, F, K, K^{-1}\}$ with the relations

$$\begin{aligned} &(U1') && KK^{-1} = 1 = K^{-1}K, \\ &(U2') && KEK^{-1} = q^2 E, \\ &(U3') && KFK^{-1} = q^{-2} F, \\ &(U4') && EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \end{aligned}$$

which are special cases of the relations $(U1)$-$(U4)$. The algebra $U_q(\mathfrak{sl}_2)$ is the most fundamental example of a quantised enveloping algebra. Indeed, for any semisimple Lie algebra $\mathfrak{g}$ and $i \in I$, the subalgebra of $U_q(\mathfrak{g})$ generated by the set $\{E_i, F_i, K_i^{\pm 1}\}$ is isomorphic to $U_{q_i}(\mathfrak{sl}_2)$.

By [Jan96, Lemma 1.4], the algebra $U_q(\mathfrak{sl}_2)$ is spanned as a $\mathbb{K}(q)$-vector space by the set of monomials

$$\left\{ F^b K^d E^a \;\middle|\; a, b \in \mathbb{N}_0,\, d \in \mathbb{Z} \right\} \tag{4.20}$$

Moreover, the set (4.20) becomes a basis of $U_q(\mathfrak{sl}_2)$ by the Poincare-Birkhoff-Witt Theorem, see [Jan96, Theorem 1.5]. We will observe this Theorem for a general algebra $U_q(\mathfrak{g})$ in Section 4.5.

The weight lattice for the Lie algebra $\mathfrak{sl}_2$ is $\mathbb{Z}\varpi_1$, where ϖ_1 denotes the fundamental weight corresponding to the simple root α_1. Hence, for any $U_q(\mathfrak{sl}_2)$-module V and $m \in \mathbb{Z}$, we define the weight spaces

$$V_m^\pm = \left\{ v \in V \;\middle|\; Kv = \pm q^m v \right\} \tag{4.21}$$

of V. In particular, for all $m \in \mathbb{Z}$, the defining relations $(U2')$ and $(U3')$ of $U_q(\mathfrak{sl}_2)$ imply that

$$E\, V_m^\pm \subset V_{m+2}^\pm \qquad \text{and} \qquad F\, V_m^\pm \subset V_{m-2}^\pm. \tag{4.22}$$

Theorem 4.8 ([Jan96, Theorem 2.6]). *For each $n \in \mathbb{N}_0$, there are simple $U_q\big(\mathfrak{sl}_2\big)$-modules $V(n, \pm)$ with basis $\{v_0^\pm, \dots, v_n^\pm\}$ such that, for $0 \le i \le n$, we have*

$$K v_i^\pm = \pm q^{n-2i} v_i^\pm, \qquad F v_i^\pm = \begin{cases} v_{i+1}^\pm & \text{if } i < n, \\ 0 & \text{if } i = n, \end{cases} \qquad E v_i^\pm = \begin{cases} \pm \big[i\big]_q \big[n+1-i\big]_q v_{i-1}^\pm & \text{if } i > 0, \\ 0 & \text{if } i = 0. \end{cases}$$

Moreover, each simple $U_q\big(\mathfrak{sl}_2\big)$-module of dimension $n+1$ is isomorphic to $V(n,+)$ or $V(n,-)$.

If the sign in the first formula of Theorem 4.8 is always a plus, the $U_q\big(\mathfrak{sl}_2\big)$-module is **type-1**.

Let V be a finite-dimensional $U_q\big(\mathfrak{sl}_2\big)$-module. By [Jan96, Proposition 2.3], V is the direct sum of its weight spaces. In addition, by [Jan96, Theorem 2.9], V is a semisimple $U_q\big(\mathfrak{sl}_2\big)$-module.

4.3 Lusztig automorphisms of $U_q(\mathfrak{g})$

Definition 4.9 ([Jan96, 8.6(1)]). *The* ***divided powers*** *of the elements E_i and F_i for $i \in I$ are*

$$E_i^{(r)} = \frac{E_i{}^r}{[r]_{q_i}^!}, \qquad F_i^{(r)} = \frac{F_i{}^r}{[r]_{q_i}^!} \tag{4.23}$$

respectively, for each $r \in \mathbb{N}_0$.

For each finite-dimensional $U_q(\mathfrak{g})$-module V, there exist linear isomorphisms T_i on V for $i \in I$. These isomorphisms and their inverses are defined explicitly in [Jan96, 8.6(2),(5)]. Observe by [Jan96, Proposition 8.13] that the isomorphism T_i induces an automorphism of $U_q(\mathfrak{g})$ (which is also denoted T_i) such that, for all $u \in U_q(\mathfrak{g})$ and $v \in V$, one has

$$T_i(uv) = T_i(u)T_i(v). \tag{4.24}$$

Definition/Proposition 4.10 ([Jan96, 8.14]). *For each $i \in I$, the* ***Lusztig automorphism*** *T_i is the algebra automorphism of $\mathcal{U}_q(\mathfrak{g})$ satisfying the relations*

$$T_i(K_h) = K_{s_i(h)} = T_i^{-1}(K_h)$$

for all $h \in Q^\vee$, and

$$T_i(E_i) = -F_i K_i, \qquad T_i^{-1}(E_i) = -K_i^{-1} F_i,$$
$$T_i(F_i) = -K_i^{-1} E_i, \qquad T_i^{-1}(F_i) = -E_i K_i.$$

Additionally, for all $i, j \in I$ with $i \neq j$ we have

$$T_i(E_j) = \sum_{r=0}^{-a_{i,j}} (-1)^r q_i^{-r} E_i^{(-a_{i,j}-r)} E_j E_i^{(r)}, \qquad T_i^{-1}(E_j) = \sum_{r=0}^{-a_{i,j}} (-1)^r q_i^{-r} E_i^{(r)} E_j E_i^{(-a_{i,j}-r)},$$
$$T_i(F_j) = \sum_{r=0}^{-a_{i,j}} (-1)^r q_i^{r} F_i^{(r)} F_j F_i^{(-a_{i,j}-r)}, \qquad T_i^{-1}(F_j) = \sum_{r=0}^{-a_{i,j}} (-1)^r q_i^{r} F_i^{(-a_{i,j}-r)} F_j F_i^{(r)}.$$

Note also that the inverse satisfies

$$T_i^{-1} = \iota \circ T_i \circ \iota \tag{4.25}$$

where ι is the algebra antiautomorphism defined in Lemma 4.4 b). Lusztig shows that the algebra automorphisms $T_i \colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ satisfy certain braid relations.

Lemma 4.11 ([Jan96, 8.15(1)]). *For all distinct $i, j \in I$, the Lusztig automorphisms T_i and T_j of $U_q(\mathfrak{g})$ satisfy the equality*

$$T_i T_j \cdots = T_j T_i \cdots$$

where the number of factors on both sides is equal to the order of $s_i s_j \in \mathcal{W}$.

Remark 4.12. If $a_{i,j} = 0$, then T_i and T_j commute. For $a_{i,j} = -1$, we have $T_i T_j T_i = T_j T_i T_j$.

We may deform the Lie bracket $[-,-]$ of $\mathfrak{g}$ to the ***s*-commutator** $[-,-]_s$ in the algebra $\mathcal{U}_q(\mathfrak{g})$, for any $s \in \mathbb{K}(q)$. In most cases, we have $s \in \{q^p \mid p = -1, 0, 1\}$. For any two elements $x, y \in U_q(\mathfrak{g})$, we define

$$[x, y]_s = xy - s\, yx. \tag{4.26}$$

By Definition/Proposition 4.10, for $i, j \in I$ with $a_{i,j} = -1$, we have

$$T_i(E_j) = \big[E_i, E_j\big]_{q^{-1}} = T_j^{-1}(E_i), \tag{4.27}$$

and

$$T_i(F_j) = \big[F_j, F_i\big]_q = T_j^{-1}(F_i). \tag{4.28}$$

For any $w \in \mathcal{W}$ with reduced expression $w = s_{i_1} \cdots s_{i_t}$ for $t \in \mathbb{N}$, we obtain the well-defined algebra automorphism

$$T_w = T_{i_1} \cdots T_{i_t} \tag{4.29}$$

of $U_q(\mathfrak{g})$. It follows from the formulas in Definition/Proposition 4.10 that

$$T_w(K_h) = K_{w(h)} \tag{4.30}$$

for any $w \in \mathcal{W}$ and $h \in Q^\vee$. Now, suppose that $X \subseteq I$ is a finite subset. Consider the Lusztig automorphism T_{w_X} where w_X denotes the longest element in the parabolic subgroup $\mathcal{W}_X \subseteq \mathcal{W}$. One determines the action of T_{w_X} on the generators corresponding to the subset X as follows.

Lemma 4.13 ([Kol14, Lemma 3.4]). *Let (X, τ) be a Satake diagram, with $w_X(\alpha_i) = -\alpha_{\tau(i)}$ for all $i \in X$. Then, for all $i \in X$, one has*

$$\begin{aligned} T_{w_X}(E_i) &= -F_{\tau(i)} K_{\tau(i)}, & T_{w_X}(F_i) &= -K_{\tau(i)}^{-1} E_{\tau(i)}, & T_{w_X}(K_i) &= K_{\tau(i)}^{-1}, \\ T_{w_X}^{-1}(E_i) &= -K_{\tau(i)}^{-1} F_{\tau(i)}, & T_{w_X}(F_i) &= -E_{\tau(i)} K_{\tau(i)}, & T_{w_X}^{-1}(K_i) &= K_{\tau(i)}^{-1}. \end{aligned}$$

For any finite, strictly ascending (or descending) sequence of integers $i_1, \ldots, i_k \in I$ such that $\left|i_{j+1} - i_j\right| = 1$ for $1 \leq j < k$, it will be convenient to introduce the notation

$$T_{i_1,\ldots,i_k} = T_{i_1} \cdots T_{i_k} \qquad \text{and} \qquad T_{i_1,\ldots,i_k}^{-1} = T_{i_1}^{-1} \cdots T_{i_k}^{-1}. \tag{4.31}$$

4.4 Lusztig-Kashiwara skew derivatives

The defining relation $(U4)$ of the quantised enveloping algebra $U_q(\mathfrak{g})$ implies that, for any element $u \in U_q(\mathfrak{g})^+$, there exist elements $r_i(u), {}_ir(u) \in U_q(\mathfrak{g})^+$ for each $i \in I$ such that

$$\big[u, F_i\big] = \frac{r_i(u)K_i - K_i^{-1}\,{}_ir(u)}{q_i - q_i^{-1}}. \tag{4.32}$$

We refer to the linear maps $r_i, {}_ir \colon U_q(\mathfrak{g})^+ \to U_q(\mathfrak{g})^+$ as the *Lusztig-Kashiwara skew derivatives*, see [Lus94, 1.2.13]. As the name suggests, the maps r_i and ${}_ir$ both satisfy a skew derivation property.

Lemma 4.14 ([Jan96, Lemma 6.14 a)])**.** *For each $i \in I$, the maps $r_i, {}_ir \colon U_q(\mathfrak{g})^+ \to U_q(\mathfrak{g})^+$ are uniquely determined by $r_i(1) = 0 = {}_ir(1)$, $r_i(E_j) = \delta_{i,j} = {}_ir(E_j)$ for all $j \in I$, and the relations*

$$r_i(uu') = u\, r_i(u') + q^{(\mu',\alpha_i)} r_i(u)u' \tag{4.33}$$

and

$${}_ir(uu') = q^{(\mu,\alpha_i)} u\, {}_ir(u') + {}_ir(u)u' \tag{4.34}$$

for all $u \in U_q(\mathfrak{g})^+_\mu$ and $u' \in U_q(\mathfrak{g})^+_{\mu'}$ where $\mu, \mu' \in Q$, $\mu, \mu' \geq 0$.

Proof. The relation $r_i(1) = 0 = {}_ir(1)$ for all $i \in I$ follows immediately from the trivial commutator $[1, F_i] = 0$ and Equation (4.32). Similarly, if $u = E_j$ for some $j \in I$, then using Equation (4.32) the defining relation $(U4)$ implies that $r_i(E_j) = \delta_{i,j} = {}_ir(E_j)$ for all $i \in I$. Let $u \in U_q(\mathfrak{g})^+_\mu$ and $u' \in U_q(\mathfrak{g})^+_{\mu'}$ for some $\mu, \mu' \in Q$, $\mu, \mu' \geq 0$. For all $i \in I$, observe that

$$\big[uu', F_i\big] = uu'F_i - F_iuu' = u\big[u', F_i\big] + \big[u, F_i\big]u'$$

and, hence, using Equation (4.32) we get

$$\begin{aligned} r_i(uu')K_i - K_i^{-1}\,{}_ir(uu') &= u\Big(r_i(u')K_i - K_i^{-1}\,{}_ir(u')\Big) + \Big(r_i(u)K_i - K_i^{-1}\,{}_ir(u)\Big)u' \\ &= \Big(u\, r_i(u')K_i + r_i(u)K_iu'\Big) - \Big(uK_i^{-1}\,{}_ir(u') + K_i^{-1}\,{}_ir(u)u'\Big) \\ &= \Big(u\, r_i(u') + q^{(\alpha_i,\mu')} r_i(u)u'\Big)K_i - K_i^{-1}\Big(q^{(\alpha_i,\mu)} u\, {}_ir(u') + {}_ir(u)u'\Big) \end{aligned}$$

since $uK_i^{-1} = q^{(\mu,\alpha_i)}K_i^{-1}u$ and $K_iu' = q^{(\mu',\alpha_i)}u'K_i$ by (4.19). We then deduce Equations (4.33) and (4.34) by comparing the coefficients of the elements K_i and K_i^{-1} on both sides. □

One shows that the following result holds, by an inductive argument.

Lemma 4.15 ([Jan96, Lemma 10.1])**.** *For all $i, j \in I$ and $u \in U_q(\mathfrak{g})^+$, we have*

$$r_i \circ {}_jr(u) = {}_jr \circ r_i(u). \tag{4.35}$$

Proof. Both sides of Equation (4.35) coincide for $u = 1$ $\big($since $r_i(0) = 0 = {}_ir(0)$ for all $i \in I\big)$ and also for $u = E_k$ for any $k \in I$, by Lemma 4.14. Now, since both sides are linear in u, we only need to show that, if Equation (4.35) already holds for some $u \in U_q(\mathfrak{g})^+_\mu$ and $u' \in U_q(\mathfrak{g})^+_{\mu'}$ where $\mu, \mu' \in Q$, $\mu, \mu' \geq 0$, then it holds for the element uu'. Indeed, using Equations (4.33) and (4.34) and the induction hypothesis, for all $i, j \in I$, we get

$$\begin{aligned} r_i \circ {}_jr(uu') &= r_i\Big(q^{(\alpha_j,\mu)} u\, {}_jr(u') + {}_jr(u)u'\Big) \\ &= q^{(\alpha_j,\mu)}\Big(u(r_i \circ {}_jr)(u') + q^{(\alpha_i,\mu'-\alpha_j)} r_i(u){}_jr(u')\Big) + {}_jr(u)r_i(u') + q^{(\alpha_i,\mu')}(r_i \circ {}_jr)(u)u' \\ &= {}_jr(u)r_i(u') + q^{(\alpha_j,\mu)}u({}_jr \circ r_i)(u') + q^{(\alpha_i,\mu')}\Big(({}_jr \circ r_i)(u)u' + q^{(\alpha_j,\mu-\alpha_i)} r_i(u){}_jr(u')\Big) \\ &= {}_jr\Big(u\, r_i(u') + q^{(\alpha_i,\mu')} r_i(u)u'\Big) \\ &= {}_jr \circ r_i(uu') \end{aligned}$$

since $r_i(u) \in U_q(\mathfrak{g})^+_{\mu-\alpha_i}$ and ${}_jr(u') \in U_q(\mathfrak{g})^+_{\mu'-\alpha_j}$ when $u \in U_q(\mathfrak{g})^+_\mu$ and $u' \in U_q(\mathfrak{g})^+_{\mu'}$. □

Using the antiautomorphism ι of $U_q(\mathfrak{g})$ from Lemma 4.4 b), the skew derivatives r_i and ${}_ir$ can be related by a similar inductive argument to the one in the proof of Lemma 4.15.

Lemma 4.16 ([Jan96, Lemma 6.14 c)])**.** *The map ι intertwines the skew derivatives r_i and ${}_ir$, that is, for all $i \in I$ and $u \in U_q(\mathfrak{g})^+$ we have*

$$\iota \circ r_i(u) = {}_ir \circ \iota(u). \tag{4.36}$$

4.5 The Poincaré-Birkhoff-Witt Theorem for $U_q(\mathfrak{g})$

Recall the Weyl group $\mathcal{W}$ of $\mathfrak{g}$. Let the word $w_0 = s_{i_1} \cdots s_{i_t}$ be a reduced expression for the longest element of $\mathcal{W}$, where $t \in \mathbb{N}$ is the length of $w_0 \in \mathcal{W}$. Each subword $s_{i_1} \cdots s_{i_j}$ for $1 \leq j \leq t$ is a reduced expression in $\mathcal{W}$, since it is an interval within a reduced expression. By (4.29), and using the notation in (4.31), we have $T_{s_{i_1}\cdots s_{i_j}} = T_{i_1,\ldots,i_j}$ and $T^{-1}_{s_{i_1}\cdots s_{i_j}} = T^{-1}_{i_1,\ldots,i_j}$ for all j.

Using the subwords of $w_0 \in \mathcal{W}$, we may now write the t distinct positive roots of $\mathfrak{g}$ explicitly as

$$\beta_j = \begin{cases} \alpha_{i_1} & \text{if } j = 1, \\ s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) & \text{for } 2 \leq j \leq t. \end{cases} \tag{4.37}$$

Then, following [Jan96, Proposition 8.20], we may define *positive* root vectors $E_{\beta_j} \in U_q(\mathfrak{g})^+$ by

$$E_{\beta_j} = \begin{cases} E_{i_1} & \text{if } j = 1, \\ T^{-1}_{i_1,\ldots,i_{j-1}}(E_{i_j}) & \text{for } 2 \leq j \leq t, \end{cases} \tag{4.38}$$

and similarly, we may define *negative* root vectors $F_{\beta_j} \in U_q(\mathfrak{g})^-$ by

$$F_{\beta_j} = \begin{cases} F_{i_1} & \text{if } j = 1, \\ T_{i_1,\ldots,i_{j-1}}(F_{i_j}) & \text{for } 2 \leq j \leq t. \end{cases} \tag{4.39}$$

For all integers $m \geq 0$, one sees that $E_{\beta_j}^m \in U_q(\mathfrak{g})^+$ and $F_{\beta_j}^m \in U_q(\mathfrak{g})^-$. Moreover, for all sequences $m_1, \ldots, m_t$ of non-negative integers, there are also products

$$E_{\beta_t}^{m_t} \cdots E_{\beta_1}^{m_1} \in U_q(\mathfrak{g})^+ \qquad \text{and} \qquad F_{\beta_t}^{m_t} \cdots F_{\beta_1}^{m_1} \in U_q(\mathfrak{g})^-.$$

By the Poincaré-Birkhoff-Witt Theorem [Jan96, 8.24], such products form a PBW-type basis of $U_q(\mathfrak{g})^+$ and $U_q(\mathfrak{g})^-$ respectively. For a proof of the following Theorem, refer to [Jan96, 8.21-8.24].

Theorem 4.17 (PBW-Theorem for $U_q(\mathfrak{g})^+$ and $U_q(\mathfrak{g})^-$). *The ordered monomials*

$$E_{\mathcal{I}} = E_{\beta_t}^{i_t} \cdots E_{\beta_1}^{i_1} \tag{4.40}$$

for $\mathcal{I} = \big(i_1, \ldots, i_t\big) \in \mathbb{N}_0^t$ form a $\mathbb{K}(q)$-vector space basis of $U_q(\mathfrak{g})^+$. Analogously, the ordered monomials

$$F_{\mathcal{J}} = F_{\beta_t}^{j_t} \cdots F_{\beta_1}^{j_1} \tag{4.41}$$

for $\mathcal{J} = \big(j_1, \ldots, j_t\big) \in \mathbb{N}_0^t$ form a $\mathbb{K}(q)$-vector space basis of $U_q(\mathfrak{g})^-$.

In view of our application in Chapters 5 and 6, for the remainder of this section, we only observe the special case where $\mathfrak{g} = \mathfrak{so}_{2N}$ for fixed $N \geq 4$ and determine the root vectors of $\mathcal{U} = U_q(\mathfrak{so}_{2N})$.

4.5.1 Root vectors of $U_q(\mathfrak{so}_{2N})$

Consider the set $I = \{1, \ldots, N\}$, and let $\tau\colon I \to I$ be the non-trivial diagram automorphism given by

$$\tau(i) = \begin{cases} i & \text{for } 1 \leq i \leq N-2, \\ N & \text{if } i = N-1, \\ N-1 & \text{if } i = N. \end{cases} \tag{4.42}$$

Define reduced words σ_i for $1 \leq i \leq N-1$ in the Weyl group $\mathcal{W}$ of the Lie algebra $\mathfrak{so}_{2N}$ by

$$\sigma_i = \begin{cases} s_i \cdots s_{N-2} s_{\tau^i(N)} s_{\tau^i(N-1)} s_{N-2} \cdots s_i & \text{for } 1 \leq i \leq N-2, \\ s_{N-1} s_N & \text{if } i = N-1. \end{cases} \tag{4.43}$$

Lemma 4.18. *The word*

$$w_0 = \sigma_1 \cdots \sigma_{N-2} \sigma_{N-1} \tag{4.44}$$

is a reduced expression for the longest element in $\mathcal{W}$.

Proof. We need to show that the element $w_0 \in \mathcal{W}$ maps all of the positive roots to negative roots in $\mathfrak{so}_{2N}$. It suffices to show that all of the positive simple roots, namely α_j for $1 \leq j \leq N$, map to the negative simple roots. Firstly, we have

$$\sigma_{N-1}(\alpha_j) = \begin{cases} -\alpha_j & \text{if } j \in \{N-1, N\} \\ \alpha_{N-2} + \alpha_{N-1} + \alpha_N & \text{if } j = N-2 \\ \alpha_j & \text{if } j < N-2. \end{cases}$$

The action of each reduced word σ_i on the positive simple roots is given by

$$\sigma_i(\alpha_j) = \begin{cases} \alpha_{i-1} + 2(\alpha_i + \cdots + \alpha_{N-2}) + \alpha_{N-1} + \alpha_N & \text{if } j = i-1 \\ -\alpha_i - 2(\alpha_{i+1} + \cdots + \alpha_{N-2}) - \alpha_{N-1} - \alpha_N & \text{if } j = i \\ \alpha_{\tau(j)} & \text{if } j \notin \{i-1,\, i\}. \end{cases}$$

In particular, observe that $\sigma_j\sigma_{j+1}(\alpha_j) = -\alpha_j$ for $1 \leq j \leq N-2$. Then, on the simple roots α_j for $1 \leq j \leq N-2$, we calculate

$$\sigma_1 \cdots \sigma_{N-2}\sigma_{N-1}(\alpha_j) = \sigma_1 \cdots \sigma_j\sigma_{j+1}(\alpha_j) = \underbrace{\sigma_1 \cdots \sigma_{j-1}}_{=id \text{ if } j=1}(-\alpha_j) = -\alpha_j.$$

Alternatively, for $j \in \{N-1, N\}$ we get

$$\sigma_1 \cdots \sigma_{N-2}\sigma_{N-1}(\alpha_j) = \sigma_1 \cdots \sigma_{N-2}(-\alpha_j) = -\alpha_{\tau^{N-2}(j)}.$$

This implies that

$$w_0(\alpha_j) = -\alpha_{\tau^{N-2}(j)}$$

for the diagram automorphism τ, and hence by [Hum72, p.51, Lemma A], the reduced expression $\sigma_1 \cdots \sigma_{N-2}\sigma_{N-1}$ for w_0 is indeed the longest element of $\mathcal{W}$. □

Recall that we may write the reduced expression (4.44) as $w_0 = s_{i_1} \cdots s_{i_{N(N-1)}}$, where each $i_j \in I$ for $1 \leq j \leq N(N-1)$. Rewriting formula (4.37), we obtain the $N(N-1)$ distinct positive roots

$$\beta_j = \begin{cases} \alpha_1 & \text{if } j = 1, \\ s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) & \text{for } 2 \leq j \leq N(N-1). \end{cases}$$

of $\mathfrak{so}_{2N}$. This can now be made more explicit. For all $i \in I$, define the non-negative integers

$$M_i = (i-1)(2N-i) \qquad \text{and} \qquad P_i = M_i + (N-i). \tag{4.45}$$

In particular, notice that $M_1 = 0$, $P_1 = (N-1)$, $M_2 = 2(N-1)$, $P_{N-1} = N(N-1) - 1$, and $M_N = N(N-1) = P_N$. Additionally, for $M_i + 1 \leq j \leq M_{i+1}$ for some $i \leq N-1$, define the integers

$$n_{i,j} = \begin{cases} P_i - j & \text{for } M_i + 1 \leq j \leq P_i, \\ j - (P_i + 1) & \text{for } P_i + 1 \leq j \leq M_{i+1}. \end{cases} \tag{4.46}$$

Notice that $n_{i,j} \in \{0, \ldots, (N-1)-i\}$ for all j. Using (4.45) and (4.46), we may write the distinct positive roots β_j more explicitly as a sum of the simple roots with non-negative coefficients. Indeed, for $M_i + 1 \leq j \leq M_{i+1}$ for some $i \leq N-2$, we calculate that

$$\beta_j = \begin{cases} \alpha_i + \cdots + \alpha_{(N-1)-n_{i,j}} & \text{for } M_i + 1 \leq j \leq P_i, \\ \big(\alpha_i + \cdots + \alpha_{N-2}\big) + \big(\alpha_{N-n_{i,j}} + \cdots + \alpha_N\big) & \text{for } P_i + 1 \leq j \leq M_{i+1}, \end{cases} \tag{4.47}$$

and then we always have $\beta_{N(N-1)-1} = \alpha_{N-1}$ and $\beta_{N(N-1)} = \alpha_N$.

Example 4.19. Consider the Lie algebra $\mathfrak{so}_8$, that is, let $N = 4$. The root system of $\mathfrak{so}_8$ contains the simple roots α_1, α_2, α_3, and α_4, and has a Dynkin diagram of type $\mathbf{D}_4$. By Lemma 4.18, the longest element in $\mathcal{W}$ is the word with the reduced expression

$$w_0 = \sigma_1\sigma_2\sigma_3 = (s_1s_2s_3s_4s_2s_1)(s_2s_4s_3s_2)(s_3s_4).$$

The 12 distinct positive roots, which can be obtained directly from formula (4.47), are

$$\begin{aligned}
&\beta_1 = \alpha_1, && \beta_4 = \alpha_1+\alpha_2+\alpha_4, && \beta_7 = \alpha_2, && \beta_{10} = \alpha_2+\alpha_3+\alpha_4,\\
&\beta_2 = \alpha_1+\alpha_2, && \beta_5 = \alpha_1+\alpha_2+\alpha_3+\alpha_4, && \beta_8 = \alpha_2+\alpha_3, && \beta_{11} = \alpha_3,\\
&\beta_3 = \alpha_1+\alpha_2+\alpha_3, && \beta_6 = \alpha_1+2\alpha_2+\alpha_3+\alpha_4, && \beta_9 = \alpha_2+\alpha_4, && \beta_{12} = \alpha_4.
\end{aligned}$$

We want to rewrite the root vectors (4.38) and (4.39) in the case where β_j contains α_1 with a *non-zero* coefficient, see Corollary 4.21. To prove this, we need to apply the following Lemma.

Lemma 4.20. *For $1 \le k \le l \le N-2$, we have*

$$T^{-1}_{k,\dots,N-2}T^{-1}_{N,\dots,l+1}(E_l) = T_{l+1,\dots,N}T_{N-2,\dots,k+1}(E_k) \tag{4.48}$$

and

$$T_{k,\dots,N-2}T_{N,\dots,l+1}(F_l) = T^{-1}_{l+1,\dots,N}T^{-1}_{N-2,\dots,k+1}(F_k). \tag{4.49}$$

Proof. We prove Equation (4.49) by induction on l, and then the proof of Equation (4.48) is analogous. Firstly, since $T_iT_jT_i^{-1} = T_j^{-1}T_iT_j$ for $a_{i,j} = -1$ by Remark 4.12, observe that

$$T_{N-2}T_NT_{N-1}(F_{N-2}) = T_{N-2}T_NT^{-1}_{N-2}(F_{N-1}) = T_N^{-1}T_{N-2}(F_{N-1}) = T^{-1}_{N-1}T^{-1}_N(F_{N-2})$$

which verifies Equation (4.49) for $k = l = N-2$. Then, for $1 \le k \le N-3$, we have

$$T_{k,\dots,N-2}T_NT_{N-1}(F_{N-2}) = T_{k,\dots,N-3}T^{-1}_{N-1}T^{-1}_N(F_{N-2}) = T^{-1}_{N-1}T^{-1}_NT^{-1}_{N-2,\dots,k+1}(F_k).$$

This completes the proof of Equation (4.49) for the case $l = N-2$. Inductively, now assume that (4.49) holds for all $k \le l$, for some $l \ge 2$. Then, for $1 \le k \le l-1 \le N-3$ we get

$$\begin{aligned}
T_{k,\dots,N-2}T_{N,\dots,l}(F_{l-1}) &= T_{k,\dots,N-2}T_{N,\dots,l+1}T^{-1}_{l-1}(F_l)\\
&= T_{k,\dots,l-1}T_lT^{-1}_{l-1}T_{l+1,\dots,N}T_{N-2,\dots,l+1}(F_l)\\
&= T_l^{-1}\Big(T_{k,\dots,N-2}T_{N,\dots,l+1}(F_l)\Big)\\
&= T^{-1}_{l,\dots,N}T^{-1}_{N-2,\dots,k+1}(F_k)
\end{aligned}$$

as required, using our inductive assumption. This finishes the proof. □

Since $\sigma_i(\alpha_j) = \alpha_{\tau(j)}$ for $i < j$, one sees that the simple root α_1 is contained with a non-zero coefficient only in the roots β_j for $1 \le j \le 2(N-1)$. Moreover, the corresponding root vectors can be rewritten as a composition of (inverse) Lusztig actions on E_1 (or F_1).

Corollary 4.21. *For $2 \le j \le 2(N-1)$, the root vector E_{β_j} can be written as*

$$E_{\beta_j} = \begin{cases} T_{j,\dots,2}(E_1) & \text{for } 2 \le j \le N-1, \\ T_{2N-j,\dots,N}T_{N-2,\dots,2}(E_1) & \text{for } N \le j \le 2(N-1), \end{cases} \tag{4.50}$$

and the root vector F_{β_j} can be written as

$$F_{\beta_j} = \begin{cases} T^{-1}_{j,\dots,2}(F_1) & \text{for } 2 \le j \le N-1, \\ T^{-1}_{2N-j,\dots,N}T^{-1}_{N-2,\dots,2}(F_1) & \text{for } N \le j \le 2(N-1). \end{cases} \tag{4.51}$$

Proof. It follows from (4.27) and (4.28) that formulas (4.50) and (4.51) hold for $1 \le j \le N$. Then, by setting $k = 1$ in Lemma 4.20, formulas (4.48) and (4.49) give us the remaining root vectors for $N+1 \le j \le 2(N-1)$. □

Crucially, all of the remaining root vectors in $\mathcal{U}^+$ (and $\mathcal{U}^-$) may be expressed *without* writing E_1 (respectively F_1) or $T_1^{\pm 1}$. We generalise Corollary 4.21 to a complete list of the positive and negative root vectors in $\mathcal{U}$ as follows.

Recall the integers M_i, P_i and $n_{i,j}$ given by formulas (4.45) and (4.46), where $i \le N-1$ is fixed, and $M_i + 1 \le j \le M_{i+1}$. For each of the $2(N-i)$ distinct positive roots β_j, the root vectors E_{β_j} and F_{β_j} defined in (4.38) and (4.39) may be rewritten. For $i \le N-2$, we have

$$E_{\beta_j} = \begin{cases} E_i & \text{if } j = M_i + 1, \\ T_{N-(n_{i,j}+1),\dots,i+1}(E_i) & \text{for } M_i + 2 \le j \le P_i, \end{cases} \tag{4.52}$$

and

$$F_{\beta_j} = \begin{cases} F_i & \text{if } j = M_i + 1, \\ T^{-1}_{N-(n_{i,j}+1),\dots,i+1}(F_i) & \text{for } M_i + 2 \le j \le P_i. \end{cases} \tag{4.53}$$

Also, for $i \le N-3$ and $P_1 \le j \le M_{i+1}$, we have the root vectors

$$E_{\beta_j} = T_{N-n_{i,j},\dots,N}T_{N-2,\dots,i+1}(E_i), \tag{4.54}$$

and

$$F_{\beta_j} = T^{-1}_{N-n_{i,j},\dots,N}T^{-1}_{N-2,\dots,i+1}(F_i). \tag{4.55}$$

Finally, we always have the root vectors

$$\begin{aligned} E_{\beta_{N(N-1)-3}} &= T_N(E_{N-2}), & E_{\beta_{N(N-1)-1}} &= E_{N-1}, \\ E_{\beta_{N(N-1)-2}} &= T_{N-1}T_N(E_{N-2}), & E_{\beta_{N(N-1)}} &= E_N, \end{aligned} \tag{4.56}$$

and

$$\begin{aligned} F_{\beta_{N(N-1)-3}} &= T_N^{-1}(F_{N-2}), & F_{\beta_{N(N-1)-1}} &= F_{N-1}, \\ F_{\beta_{N(N-1)-2}} &= T^{-1}_{N-1}T_N^{-1}(F_{N-2}), & F_{\beta_{N(N-1)}} &= F_N. \end{aligned} \tag{4.57}$$

Example 4.22. We continue Example 4.19, in which we observed the 12 distinct positive roots β_j of the algebra $\mathcal{U} = U_q(\mathfrak{so}_8)$. Using formulas (4.52) - (4.57) directly, we now obtain the positive root vectors

$$\begin{aligned}
&E_{\beta_1} = E_1, && E_{\beta_4} = T_4T_2(E_1), && E_{\beta_7} = E_2, && E_{\beta_{10}} = T_3T_4(E_2),\\
&E_{\beta_2} = T_2(E_1), && E_{\beta_5} = T_3T_4T_2(E_1), && E_{\beta_8} = T_3(E_2), && E_{\beta_{11}} = E_3,\\
&E_{\beta_3} = T_3T_2(E_1), && E_{\beta_6} = T_2T_3T_4T_2(E_1), && E_{\beta_9} = T_4(E_2), && E_{\beta_{12}} = E_4,
\end{aligned}$$

and, similarly, the negative root vectors

$$\begin{aligned}
&F_{\beta_1} = F_1, && F_{\beta_4} = T_4^{-1}T_2^{-1}(F_1), && F_{\beta_7} = F_2, && F_{\beta_{10}} = T_3^{-1}T_4^{-1}(F_2),\\
&F_{\beta_2} = T_2^{-1}(F_1), && F_{\beta_5} = T_3^{-1}T_4^{-1}T_2^{-1}(F_1), && F_{\beta_8} = T_3^{-1}(F_2), && F_{\beta_{11}} = F_3,\\
&F_{\beta_3} = T_3^{-1}T_2^{-1}(F_1), && F_{\beta_6} = T_2^{-1}T_3^{-1}T_4^{-1}T_2^{-1}(F_1), && F_{\beta_9} = T_4^{-1}(F_2), && F_{\beta_{12}} = F_4.
\end{aligned}$$

4.6 Quantum symmetric pairs

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and recall that $I \subset \mathbb{N}$ denotes the indexing set for the nodes of the Dynkin diagram of $\mathfrak{g}$. Recall the notion of a Satake diagram (X, τ) from Definition 3.3, where we consider a subset $X \subseteq I$ and let $\tau\colon I \to I$ be an involutive diagram automorphism. In particular, let

$$\tau(X) = X \tag{4.58}$$

and

$$w_X(\alpha_j) = -\alpha_{\tau(j)} \tag{4.59}$$

for all $j \in X$, where $w_X \in \mathcal{W}_X$ denotes the longest element of the parabolic subgroup $\mathcal{W}_X$ of the Weyl group $\mathcal{W}$ of $\mathfrak{g}$ corresponding to the subset X.

Recall the map $\theta\big|_{\mathfrak{h}} = -w_X \circ \tau$ in (3.17) where θ is an involution of $\mathfrak{g}$ determined by the Satake diagram (X, τ). By (3.17), the involution θ maps the coroot lattice $Q^\vee$ to itself. Define

$$(Q^\vee)^\theta \coloneqq \{h \in Q^\vee \mid \theta(h) = h\} \tag{4.60}$$

and let $U_q(\mathfrak{g})^0_\theta$ denote the subalgebra of $U_q(\mathfrak{g})^0$ generated by the elements K_h for all $h \in (Q^\vee)^\theta$. Equivalently if $U_q(\mathfrak{g}) = \mathcal{U}$, for the involution Θ induced by θ in (3.14), we may denote

$$\mathcal{U}^0_\Theta = \mathbb{K}(q)\Big\langle K_iK_{\tau(i)}^{-1},\, K_j^{\pm 1} \;\Big|\; i \in I \setminus X,\, j \in X\Big\rangle. \tag{4.61}$$

Let M_X be the subalgebra of $\mathcal{U}$ generated by all $E_j, F_j, K_j^{\pm 1}$ for $j \in X$.

Definition 4.23 ([Kol14, Definition 5.1])**.** *Let (X, τ) be a Satake diagram. For any set of parameters $\mathbf{c} = (c_i)_{i \in I \setminus X} \in (\mathbb{K}(q)^\times)^{I \setminus X}$, let $\mathcal{B}_\mathbf{c}$ denote the subalgebra of $U_q(\mathfrak{g})$ generated by $\mathcal{M}_X$, $U_q(\mathfrak{g})^0_\theta$ and the elements*

$$B_i = F_i - c_iT_{w_X}(E_{\tau(i)})K_i^{-1} \tag{4.62}$$

for all $i \in I \setminus X$.

The main property of the algebra $\mathcal{B}_{\mathbf{c}}$ is that it is a *right coideal subalgebra* of $U_q(\mathfrak{g})$, which by Remark 2.41 means that

$$\Delta(\mathcal{B}_{\mathbf{c}}) \subseteq \mathcal{B}_{\mathbf{c}} \otimes U_q(\mathfrak{g}). \tag{4.63}$$

In order for $\mathcal{B}_{\mathbf{c}}$ to be a quantum analogue of $U(\mathfrak{k})$, where $\mathfrak{k} = \mathfrak{g}^\theta$ is the fixed Lie subalgebra of the Lie algebra $\mathfrak{g}$ as defined in Section 3.1, it must satisfy the property

$$\mathcal{B}_{\mathbf{c}} \cap U_q(\mathfrak{g})^0 = U_q(\mathfrak{g})^0_\theta. \tag{4.64}$$

By [Kol14, Lemma 5.3/5.4], this property holds only if the parameter $\mathbf{c}$ is contained in the set

$$\mathcal{C} = \left\{ \mathbf{c} = (c_i)_{i\in I\setminus X} \in \big(\mathbb{K}(q)^\times\big)^{I\setminus X} \,\middle|\, c_i = c_{\tau(i)} \text{ if } \Big(\alpha_i, w_X\big(\alpha_{\tau(i)}\big)\Big) = 0 \right\}. \tag{4.65}$$

The algebra $\mathcal{B}_{\mathbf{c}}$ for $\mathbf{c} \in \mathcal{C}$ is called a **quantum symmetric pair coideal subalgebra** of $U_q(\mathfrak{g})$, and the pair $\big(U_q(\mathfrak{g}), \mathcal{B}_{\mathbf{c}}\big)$ is often referred to as a **quantum symmetric pair**.

> *Remark* 4.24 ([KY21, Remark 2.8]). Quantum symmetric pairs additionally depend on a second family of parameters $\mathbf{s} = (s_i)_{i\in I\setminus X}$ in a certain subset $\mathcal{S} \subset \mathbb{K}(q)^{I\setminus X}$, as given in [Kol14, (5.11)]. The corresponding coideal subalgebras are then denoted by $\mathcal{B}_{\mathbf{c},\mathbf{s}}$. Using [Kol14, Theorem 7.1], the algebra $\mathcal{B}_{\mathbf{c},\mathbf{s}}$ is isomorphic to $\mathcal{B}_{\mathbf{c}}$ for any $\mathbf{s} \in \mathcal{S}$. It suffices to consider the case where $\mathbf{s} = \mathbf{0}$, that is, $s_i = 0$ for all $i \in I \setminus X$, and $\mathcal{B}_{\mathbf{c}} = \mathcal{B}_{\mathbf{c},\mathbf{0}}$ is often referred to as *standard*.

We extend the definition of the elements B_i in (4.62) to all $i \in I$ by writing $B_j = F_j$ for $j \in X$.

Theorem 4.25 ([Let02, Theorem 7.4]). *Let* $\mathbf{c} \in \mathcal{C}$. *The algebra* $\mathcal{B}_{\mathbf{c}}$ *is generated over* $U_q(\mathfrak{g})^0_\theta \mathcal{M}_X$ *by the elements* B_i *for* $i \in I$ *subject to the relations*

$$\begin{aligned}
K_h B_i &= q^{-\alpha_i(h)} B_i K_h && \text{for all } h \in Q^\vee,\ i \in I,\\
E_j B_i - B_i E_j &= \delta_{i,j} \frac{K_j - K_j^{-1}}{q_j - q_j^{-1}} && \text{for all } j \in X,\ i \in I,\\
\mathcal{Q}_{i,j}\big(B_i, B_j\big) &= C_{i,j}(\mathbf{c}) && \text{for all } i, j \in I,\ i \neq j.
\end{aligned}$$

The formulas for the elements $C_{i,j}(\mathbf{c})$ in Theorem 4.25 have been explicitly determined in general, see [BK15, Theorems 3.6-3.9]. In particular, if we assume that $i \in I \setminus X$ with $\tau(i) = i$, and $j \in I \setminus \{i\}$ so that $a_{i,j} = -1$, then, by [BK15, Theorem 3.9, Case 2], we have

$$C_{i,j}(\mathbf{c}) = q_i B_j c_i \mathcal{Z}_i + \frac{q_i^2 c_i r_j(\mathcal{Z}_i) K_j + q_j^2 q_i^{-2} c_i\, {}_j r(\mathcal{Z}_i) K_j^{-1}}{\big(q_i - q_i^{-1}\big)\big(q_j - q_j^{-1}\big)} \tag{4.66}$$

where we define

$$\mathcal{Z}_i = -r_{\tau(i)}\Big(T_{w_X}\big(E_{\tau(i)}\big)\Big) K_{\tau(i)} K_i^{-1}. \tag{4.67}$$

Chapter 5

Very Non-Standard Quantum $\mathfrak{so}_{2N-1}$

In Proposition 3.10, we saw that the Lie algebra $\mathfrak{so}_{2N-1}$ may be embedded into the Lie algebra $\mathfrak{so}_{2N}$ as the fixed Lie subalgebra with respect to an involution θ given in Lemma 3.9. The universal enveloping algebra of $\mathfrak{so}_{2N-1}$ already has a standard quantum analogue $U_q(\mathfrak{so}_{2N-1})$, see Section 4.1. However, this algebra is not a subalgebra of $\mathcal{U} = U_q(\mathfrak{so}_{2N})$ in any canonical way. More precisely, one may show that $U_q(\mathfrak{so}_{2N-1})$ cannot be embedded as a Hopf subalgebra into $\mathcal{U}$. We therefore look for an alternative quantum analogue inside $\mathcal{U}$.

Recall the theory of quantum symmetric pairs that we developed in Section 4.6. In particular, we obtain an algebra $\mathcal{B}_\mathbf{c}$ which is a quantum analogue of the enveloping algebra $U(\mathfrak{k})$. For $\mathfrak{g} = \mathfrak{so}_{2N}$ and $\mathfrak{k} = \mathfrak{so}_{2N-1}$, we take the subset $X = I \setminus \{1\}$ of $I = \{1, \dots, N\}$. Let $\mathcal{M}_X$ be the subalgebra of $\mathcal{U}$ generated by all E_j, F_j and $K_j^{\pm 1}$ for $j \in X$ which, by construction, is isomorphic to the algebra $U_q(\mathfrak{so}_{2N-2})$. By Definition 4.23, the algebra $\mathcal{B}_\mathbf{c}$ is the coideal subalgebra of $\mathcal{U}$ generated by $\mathcal{M}_X$ and the element

$$B_1 = F_1 - c_1 T_{w_X}(E_1) K_1^{-1} \tag{5.1}$$

depending only on a parameter $c_1 \in \mathbb{K}(q)^\times$. The algebra $\mathcal{B}_\mathbf{c}$ satisfies the coideal property (4.63).

> *Definition/Remark* 5.1. We refer to the algebra $\mathcal{B}_\mathbf{c}$ as the **very non-standard quantum deformation** of $U(\mathfrak{so}_{2N-1})$, distinguishing it from the *non-standard* quantum deformation of $U(\mathfrak{so}_{2N-1})$ that was introduced by A. Gavrilik and A. Klimyk in [GK91]. Indeed, the Gavrilik-Klimyk algebra is also a quantum symmetric pair coideal subalgebra, but inside $U_q(\mathfrak{sl}_N)$. It corresponds to the Chevalley involution on $\mathfrak{sl}_N$.

5.1 Generators and relations of $\mathcal{B}_\mathbf{c}$

In terms of its generators, the coideal subalgebra $\mathcal{B}_\mathbf{c}$ of $\mathcal{U}$ can be written as the algebra

$$\mathcal{B}_\mathbf{c} := \left\langle B_1, E_j, F_j, K_j^{\pm 1} \;\middle|\; j \in X \right\rangle. \tag{5.2}$$

Recall, from Section 4.4, the Lusztig-Kashiwara skew derivatives ${}_ir, r_i \colon \mathcal{U}^+ \to \mathcal{U}^+$ for each $i \in I$, and from (4.67) define

$$\mathcal{Z}_1 = -r_1\big(T_{w_X}(E_1)\big). \tag{5.3}$$

By Theorem 4.25, the algebra $\mathcal{B}_{\mathbf{c}}$ is generated over $\mathcal{M}_X$ by the element B_1 subject only to the relations $\mathfrak{X}_j B_1 = B_1 \mathfrak{X}_j$ for $j \in X \setminus \{2\}$ where $\mathfrak{X} \in \{E, F, K^{\pm 1}\}$ and, more interestingly,

$$E_2 B_1 = B_1 E_2, \tag{5.4}$$

$$K_2 B_1 = q B_1 K_2, \tag{5.5}$$

$$F_2^2 B_1 - (q + q^{-1}) F_2 B_1 F_2 + B_1 F_2^2 = 0, \tag{5.6}$$

$$B_1^2 F_2 - (q + q^{-1}) B_1 F_2 B_1 + F_2 B_1^2 = q c_1 \left(F_2 \mathcal{Z}_1 + \frac{q r_2(\mathcal{Z}_1) K_2 + q^{-1}{}_2 r(\mathcal{Z}_1) K_2^{-1}}{(q - q^{-1})^2} \right). \tag{5.7}$$

Note that relation (5.7) is obtained from formula (4.66) since it is equal to the element $C_{1,2}(c_1)$. One can show that the algebra $\mathcal{B}_{\mathbf{c}}$ is invariant under the Lusztig automorphisms T_j for all $j \in X$. Additionally, the elements B_1 and $\mathcal{Z}_1$ defined in (5.1) and (5.3) can be given more explicitly by observing that

$$T_{w_X}(E_1) = T_{2,\dots,N} T_{N-2,\dots,2}(E_1). \tag{5.8}$$

Lemma 5.2 ([BW18, Theorem 4.2]). *For $j \in X$, we have*

$$T_j(B_1) = \begin{cases} [B_1, F_2]_q & \text{if } j = 2, \\ B_1 & \text{otherwise,} \end{cases} \quad \text{and} \quad T_j^{-1}(B_1) = \begin{cases} [F_2, B_1]_q & \text{if } j = 2, \\ B_1 & \text{otherwise.} \end{cases} \tag{5.9}$$

Proof. Using (5.1) and (5.8), we write the element

$$B_1 = F_1 - c_1 T_{2,\dots,N} T_{N-2,\dots,2}(E_1) K_1^{-1}.$$

Using Lemma 4.11 (and Remark 4.12), for all $j \in X \setminus \{2\}$, we have

$$T_j^{\pm 1} \circ T_{2,\dots,N} T_{N-2,\dots,2} = T_{2,\dots,N} T_{N-2,\dots,2} \circ T_{\tau(j)}^{\pm 1}.$$

Then, since $T_j^{\pm 1}(\mathfrak{X}_1) = \mathfrak{X}_1$ for all $j \in X \setminus \{2\}$ where $\mathfrak{X} \in \{E, F, K^{\pm 1}\}$, it follows that

$$T_j^{\pm 1}(B_1) = B_1.$$

This proves the formulas in (5.9) for $j \neq 2$. Now, it remains to consider the case for $j = 2$. By direct calculation, we have

$$\begin{aligned}
T_2(B_1) &= T_2\big(F_1 - c_1 T_{2,\dots,N} T_{N-2,\dots,2}(E_1) K_1^{-1}\big) \\
&= T_2(F_1) - c_1 T_2\Big(\big[E_2, T_{3,\dots,N} T_{N-2,\dots,2}(E_1)\big]_{q^{-1}} K_1^{-1}\Big) \\
&= \big[F_1, F_2\big]_q - c_1\big[-F_2 K_2, T_{2,\dots,N} T_{N-2,\dots,2}(E_1)\big]_{q^{-1}} K_2^{-1} K_1^{-1} \\
&= \big(F_1 F_2 - q F_2 F_1\big) - c_1\big(q^{-1} T_{2,\dots,N} T_{N-2,\dots,2}(E_1) F_2 - q F_2 T_{2,\dots,N} T_{N-2,\dots,2}(E_1)\big) K_1^{-1} \\
&= \big(F_1 - c_1 T_{2,\dots,N} T_{N-2,\dots,2}(E_1) K_1^{-1}\big) F_2 - q F_2 \big(F_1 - c_1 T_{2,\dots,N} T_{N-2,\dots,2}(E_1) K_1^{-1}\big) \\
&= B_1 F_2 - q F_2 B_1
\end{aligned}$$

and hence $T_2(B_1) = [B_1, F_2]_q$. A similar calculation proves that $T_2^{-1}(B_1) = [F_2, B_1]_q$. □

5.2 The standard filtration on $\mathcal{B}_{\mathbf{c}}$

Let $\mathcal{A}$ be the subalgebra of $\mathcal{U}$ generated by $\mathcal{M}_X$ and the element F_1, that is, define the algebra

$$\mathcal{A} := \left\langle E_j, F_i, K_j^{\pm 1} \;\middle|\; i \in I,\, j \in X \right\rangle. \tag{5.10}$$

The algebra $\mathcal{A}$ is generated over $\mathcal{M}_X$ by F_1 subject to $\mathfrak{X}_j F_1 = F_1 \mathfrak{X}_j$ for $j \in X \setminus \{2\}$ where $\mathfrak{X} \in \{E, F, K^{\pm 1}\}$ and, similar to (5.4)-(5.7),

$$E_2 F_1 = F_1 E_2, \tag{5.11}$$

$$K_2 F_1 = q F_1 K_2, \tag{5.12}$$

$$F_2^2 F_1 - (q + q^{-1}) F_2 F_1 F_2 + F_1 F_2^2 = 0, \tag{5.13}$$

$$F_1^2 F_2 - (q + q^{-1}) F_1 F_2 F_1 + F_2 F_1^2 = 0. \tag{5.14}$$

Recall graded and filtered algebras in Section 2.2.2. The algebra $\mathcal{A}$ is $\mathbb{N}_0$-graded by the degree function deg given by

$$\deg(F_1) = 1, \qquad \text{and} \qquad \deg(\mathfrak{X}_j) = 0$$

for all $j \in X$ where $\mathfrak{X} = \{E, F, K^{\pm 1}\}$. On the other hand, the algebra $\mathcal{B}_{\mathbf{c}}$ has an $\mathbb{N}_0$-filtration $\mathcal{F}$, which we often refer to as the *standard* filtration, defined by the degree function on the generators given by

$$\deg(B_1) = 1, \qquad \text{and} \qquad \deg(\mathfrak{X}_j) = 0$$

for all $j \in X$ where $\mathfrak{X} = \{E, F, K^{\pm 1}\}$. Let $\mathrm{gr}(\mathcal{B}_{\mathbf{c}})$ denote the associated graded algebra of $\mathcal{B}_{\mathbf{c}}$, see Definition 2.29. For any $x \in \mathcal{F}_k(\mathcal{B}_{\mathbf{c}})$ where $k \in \mathbb{N}$, we denote the image of x under the canonical projection by the element

$$\overline{x} \in \frac{\mathcal{F}_k(\mathcal{B}_{\mathbf{c}})}{\mathcal{F}_{k-1}(\mathcal{B}_{\mathbf{c}})}.$$

Now, similar to the triangular decomposition of $\mathcal{U}$ given in (4.10), the algebra $\mathcal{A}$ also exhibits a triangular decomposition via the vector space isomorphism

$$\mathcal{A} \cong \mathcal{U}^- \otimes \mathcal{U}_\Theta^0 \otimes \mathcal{M}_X^+ \tag{5.15}$$

for the subalgebras $\mathcal{U}_\Theta^0 := \langle K_i^{\pm 1} \,|\, i \in \{2, \ldots, N\} \rangle$ and $\mathcal{M}_X^+ := \langle E_i \,|\, i \in \{2, \ldots, N\} \rangle$ of $\mathcal{A}$. By (5.15) we are able to write a basis of $\mathcal{A}$ using the following notation.

For any multi-index $L = (l_1, \ldots, l_m) \in I^m$, we denote the corresponding elements

$$F_L := F_{l_1} \cdots F_{l_m} \qquad \text{and} \qquad B_L := B_{l_1} \cdots B_{l_m}$$

where we set $B_j = F_j$ for all $j \in X$. Now, fix a subset

$$\mathcal{L} \subseteq \bigcup_{m \geq 0} I^m = \emptyset \cup I \cup I^2 \cup \ldots$$

such that $\{F_L \,|\, L \in \mathcal{L}\}$ is a basis of $\mathcal{U}^-$. Note that there is some freedom when choosing $\mathcal{L}$. By the triangular decomposition (5.15) of the algebra $\mathcal{A}$, the set $\{F_L \,|\, L \in \mathcal{L}\}$ forms a basis of $\mathcal{A}$ as a right $\mathcal{U}_\Theta^0 \mathcal{M}_X^+$-module. Moreover, by [Kol14, Proposition 6.2], the set $\{B_L \,|\, L \in \mathcal{L}\}$ forms a basis of $\mathcal{B}_{\mathbf{c}}$ as a right $\mathcal{U}_\Theta^0 \mathcal{M}_X^+$-module. The defining relations of $\mathcal{B}_{\mathbf{c}}$ imply the following result.

Proposition 5.3. *There exists an isomorphism of algebras* $\varphi\colon \mathcal{A} \to gr(\mathcal{B}_{\mathbf{c}})$ *such that*

$$\varphi(F_1) = \overline{B_1}, \qquad \text{and} \qquad \varphi(\mathfrak{X}_j) = \overline{\mathfrak{X}_j}$$

for $j \in X$ *where* $\mathfrak{X} \in \{E, F, K^{\pm 1}\}$.

Proof. We first verify that the defining relations (5.11)-(5.14) of $\mathcal{A}$ are preserved for the associated graded algebra $\mathrm{gr}(\mathcal{B}_{\mathbf{c}})$ via the map φ. Indeed, we see that

$$E_2\overline{B_1} = \overline{E_2B_1} = \overline{B_1E_2} = \overline{B_1}E_2, \qquad K_2\overline{B_1} = \overline{K_2B_1} = \overline{qB_1K_2} = q\overline{B_1}K_2,$$

and

$$F_2^2\overline{B_1} - (q+q^{-1})F_2\overline{B_1}F_2 + \overline{B_1}F_2^2 = \overline{F_2^2B_1 - (q+q^{-1})F_2B_1F_2 + B_1F_2^2} = \overline{0} = 0.$$

Finally, we have

$$\begin{aligned}\overline{B_1}^2F_2 - (q+q^{-1})\overline{B_1}F_2\overline{B_1} + F_2\overline{B_1}^2 &= \overline{B_1^2F_2 - (q+q^{-1})B_1F_2B_1 + F_2B_1^2} \\ &= \overline{qc_1\left(F_2\mathcal{Z}_1 + \frac{qr_2(\mathcal{Z}_1)K_2 + q^{-1}{}_2r(\mathcal{Z}_1)K_2^{-1}}{(q-q^{-1})^2}\right)} \\ &= \overline{0} = 0.\end{aligned}$$

Since the map φ is defined on the generators of $\mathcal{A}$ and extends multiplicatively, the defining relations of $\mathcal{A}$ being preserved implies that φ is a well-defined algebra morphism. Moreover, φ is surjective since its image contains all of the generators of the algebra $\mathrm{gr}(\mathcal{B}_{\mathbf{c}})$. We now claim that the set of all elements $\overline{B_L}$ where the multi-index $L \in \mathcal{L}$ contains a "1" k times is a basis of the k^{th} component of $\mathrm{gr}(\mathcal{B}_{\mathbf{c}})$ as a module over $\mathcal{U}_\Theta^0\mathcal{M}_X^+$. It is sufficient to show that the set

$$\mathfrak{B}_k = \{B_L \mid L \in \mathcal{L},\ L \text{ contains a "1" at most } k \text{ times}\}$$

is a basis of the filtered subspace $\mathcal{F}_k(\mathcal{B}_{\mathbf{c}})$. Equivalently, for every $k \in \mathbb{N}_0$, we must have

$$\mathcal{F}_k(\mathcal{B}_{\mathbf{c}}) = \mathrm{span}_{\mathcal{U}_\Theta^0\mathcal{M}_X^+}\mathfrak{B}_k. \tag{5.16}$$

By definition of the subspace $\mathcal{F}_k(\mathcal{B}_{\mathbf{c}})$, one sees that the right side of (5.16) must be contained in the left. We prove the opposite inclusion using the following induction argument. By definition, the case $k = 0$ is true, so suppose now that $k \geq 1$ and assume that the inclusion holds for $0, \ldots, k-1$ as an induction hypothesis. Now, consider an arbitrary monomial which contains the element B_1 at most k times. This monomial can be written as a linear combination of the B_L for $L \in \mathcal{L}$, which consists of the element B_1 at most k times, plus an error term which must contain the B_1 *fewer* than k times, by the quantum Serre relations. By the induction hypothesis, we can already write the error term as a linear combination, and thus $\mathcal{F}_k(\mathcal{B}_{\mathbf{c}})$ must be contained in the right side. This proves our claim.

Then, since $\mathrm{gr}(\mathcal{B}_{\mathbf{c}})$ is a direct sum of all of its components, the set $\{\overline{B_L} \mid L \in \mathcal{L}\}$ is a basis of $\mathrm{gr}(\mathcal{B}_{\mathbf{c}})$ over $\mathcal{U}_\Theta^0\mathcal{M}_X^+$. Moreover, since the bases of $\mathcal{A}$ and $\mathrm{gr}(\mathcal{B}_{\mathbf{c}})$ have a one to one correspondence, they must be isomorphic. Hence, the algebra homomorphism φ is injective and, therefore, an isomorphism. □

By Lemma 5.2, the algebra isomorphisms $T_i\colon \mathcal{B}_{\mathbf{c}} \to \mathcal{B}_{\mathbf{c}}$ for $i \in X$ restrict to linear isomorphisms $\mathcal{F}_m(\mathcal{B}_{\mathbf{c}}) \to \mathcal{F}_m(\mathcal{B}_{\mathbf{c}})$ for any $m \in \mathbb{N}_0$. Hence, the maps $T_i\colon \mathcal{B}_{\mathbf{c}} \to \mathcal{B}_{\mathbf{c}}$ induce algebra isomorphisms $\mathrm{gr}(T_i)\colon \mathrm{gr}(\mathcal{B}_{\mathbf{c}}) \to \mathrm{gr}(\mathcal{B}_{\mathbf{c}})$. Equation (4.25) and Lemma 5.2 now imply the following result.

Corollary 5.4. *For any $i \in X$, the algebra isomorphism $\varphi\colon \mathcal{A} \to gr(\mathcal{B}_{\mathbf{c}})$ is compatible with the Lusztig automorphism T_i in the sense that*

$$\varphi \circ T_i = gr(T_i) \circ \varphi.$$

5.3 Root vectors for $\mathcal{B}_{\mathbf{c}}$

Recall from Lemma 5.2 that the Lusztig automorphisms T_i for $i \in X$ leave the algebra $\mathcal{B}_{\mathbf{c}}$ invariant. In analogy to formula (4.51) for the root vectors of $\mathcal{U}^-$, we define root vectors of $\mathcal{B}_{\mathbf{c}}$ for $j \in \{1, \dots, N(N-1)\}$ by

$$B_{\beta_j} = \begin{cases} B_1 & \text{if } j = 1, \\ T^{-1}_{j,\dots,2}(B_1) & \text{if } 2 \le j \le N-1, \\ T^{-1}_{2N-j,\dots,N} T^{-1}_{N-2,\dots,2}(B_1) & \text{if } N \le j \le 2(N-1), \\ F_{\beta_j} & \text{if } j > 2(N-1). \end{cases} \tag{5.17}$$

By construction, the root vectors B_{β_j} have lowest weight component F_{β_j} with respect to the left adjoint action of $\mathcal{U}^0$. Moreover, the terminology *root vector for B_{β_j}* is justified by the following Proposition and comparison with the classical case in Section 3.3.3.

Proposition 5.5. *For $j \in \{2, \dots, 2(N-1)\}$, the root vectors B_{β_j} in (5.17) can be written as*

$$B_{\beta_j} = \begin{cases} \big[F_j, \dots [F_2, B_1]_q \dots \big]_q & \text{if } 2 \le j \le N-1, \\ \big[F_N, [F_{N-2}, \dots [F_2, B_1]_q \dots]_q\big]_q & \text{if } j = N, \\ \big[F_N, [F_{N-1}, [F_{N-2}, \dots [F_2, B_1]_q \dots]_q]_q\big]_q & \text{if } j = N+1, \\ \big[T_{N,\dots,(2N-j)+1}(F_{2N-j}), T^{-1}_{N-2,\dots,2}(B_1)\big]_q & \text{if } N+2 \le j \le 2(N-1). \end{cases} \tag{5.18}$$

Proof. For $2 \le j \le N+1$, Equation (5.18) follows from Equation (4.25) and Lemma 5.2. In the case that $N+2 \le j \le 2(N-1)$, we get

$$\begin{aligned} B_{\beta_j} &= T^{-1}_{2N-j,\dots,N-2} T^{-1}_{N,\dots,2}(B_1) \\ &= T^{-1}_{2N-j,\dots,N-2}\Big(\big[T^{-1}_{N,N-1}(F_{N-2}), \underbrace{T^{-1}_{N-3,\dots,2}}_{=id \text{ if } N=4}(B_1)\big]_q\Big) \\ &= \underbrace{T^{-1}_{2N-j,\dots,N-3}}_{=id \text{ when } j=N+2}\Big(\big[T_{N,N-1}(F_{N-2}), T^{-1}_{N-2,\dots,2}(B_1)\big]_q\Big) \\ &= \big[T_{N,\dots,(2N-j)+1}(F_{2N-j}), T^{-1}_{N-2,\dots,2}(B_1)\big]_q \end{aligned}$$

making repeated use of Equation (4.25). □

Proposition 5.5 allows us to identify the limit of the root vectors B_{β_j} for $q \to 1$. We will only provide an informal discussion, but the mathematical arguments can be made precise using the notion of non-restrictive specialisation [CK90], [HK02], see also [Kol14, Section 10].

Recall the notation from Sections 2.6.1, 2.6.2, and 3.3.3. By Equation (3.32), the generator $B_1 \in \mathcal{B}_{\mathbf{c}}$ (with $c_1 \to 1$) is a q-analogue of the root vector $\sqrt{2}b_1 \in \mathfrak{so}_{2N-1}$ corresponding to the root ε_1. Similarly, the generators $F_i \in \mathcal{B}_{\mathbf{c}}$ for $i \in X$ are q-analogues of the Chevalley generators $f_{\alpha_i} \in \mathfrak{so}_{2N}$, and hence of $f_{\gamma_{i-1}} \in \mathfrak{so}_{2N-1}$, if $i \leq N-1$. Therefore, the first equation in (2.68) shows that B_{β_j} for $1 \leq j \leq N-1$ are q-analogues of the elements $\sqrt{2}b_j \in \mathfrak{so}_{2N-1}$. In particular,

$$B_{\beta_{N-1}} = T^{-1}_{N-1,\dots,2}(B_1) \tag{5.19}$$

is a q-analogue of the Chevalley generator $e_{\gamma_{N-1}} \in \mathfrak{so}_{2N-1}$, see also Equation (3.33). Similarly, by Equation (3.34),

$$B_{\beta_N} = T_N^{-1}T^{-1}_{N-2,\dots,2}(B_1) \tag{5.20}$$

is a q-analogue of the Chevalley generator $f_{\gamma_{N-1}} \in \mathfrak{so}_{2N-1}$. Then, by the third equation in (2.68), the root vector $B_{\beta_{N+1}}$ has the limit

$$\big[f_{\gamma_{N-2}}, f_{\gamma_{N-1}}\big] = -\sqrt{2}c_{N-2} \in \mathfrak{so}_{2N-1}$$

for $q \to 1$. Moreover, for $2 \leq k \leq N-2$ the element $T_{N,\dots,k+1}(F_k)$ specialises for $q \to 1$ to

$$\begin{aligned}\big[\dots[f_{\alpha_k}, f_{\alpha_{k+1}}]\dots, f_{\alpha_N}\big] &= (-1)^{N-(k+1)}\big[E_{N,k} - E_{N+k,2N}, f_{\alpha_N}\big]\\ &= (-1)^{N-k}\big(E_{N+k,N-1} - E_{2N-1,k}\big)\\ &= (-1)^{N-k}\eta\big(E_{N+(k-2),N-2} - E_{2N-3,k-1}\big)\end{aligned}$$

in $\mathfrak{so}_{2N}$. Hence, the root vector B_{β_j} for $N+2 \leq j \leq 2(N-1)$ specialises for $q \to 1$ in $\mathfrak{so}_{2N-1}$ to

$$(-1)^{N-k}\big[E_{N+(k-2),N-2} - E_{2N-3,k-1}, \sqrt{2}b_{N-2}\big] = (-1)^{N-(k-1)}\sqrt{2}c_{k-1}$$

where we take $k = 2N-j$. The above discussion is now summarised in the following Corollary.

Corollary 5.6. *Let $l \in \{1,\dots,N-1\}$. In the limit $q \to 1$, one sees that:*

(i) *the root vector B_{β_l} specialises to the element $\sqrt{2}b_l$ in $\mathfrak{so}_{2N-1}$, and;*

(ii) *the root vector $B_{\beta_{(N-1)+l}}$ specialises to the element $(-1)^{l-1}\sqrt{2}c_{N-l}$ in $\mathfrak{so}_{2N-1}$.*

Let $Q^\vee_{2N} = \mathbb{Z}\big\{h_{\alpha_i} \mid 1 \leq j \leq N\big\}$ and $Q^\vee_{2N-1} = \mathbb{Z}\big\{h_{\gamma_j} \mid 1 \leq j \leq N-1\big\}$ be the coroot lattices of $\mathfrak{so}_{2N}$ and $\mathfrak{so}_{2N-1}$, respectively. By the third formula in (3.31) and also by (3.35), the embedding η induces a group homomorphism $\eta\colon Q^\vee_{2N-1} \to Q^\vee_{2N}$ where $h_{\gamma_j} \mapsto h_{\alpha_{j+1}}$ for $1 \leq j \leq N-2$, and $h_{\gamma_{N-1}} \mapsto h_{\alpha_N} - h_{\alpha_{N-1}}$. Corollary 5.6 (i) shows that the element $B_{\beta_{N-1}} \in \mathcal{B}_{\mathbf{c}}$ is a q-analogue of the *positive* simple root vector of $\mathfrak{so}_{2N-1}$ corresponding to the root γ_{N-1}. Similarly, Corollary 5.6 (ii) shows that the element $B_{\beta_N} \in \mathcal{B}_{\mathbf{c}}$ is a q-analogue of the *negative* simple root vector of $\mathfrak{so}_{2N-1}$ corresponding to the root $-\gamma_{N-1}$. This interpretation is confirmed by the following Proposition, which also identifies q-analogues of the remaining simple root vectors.

Proposition 5.7. *For $i \in \{2, \dots, N-1\}$ and for any $h \in Q^\vee_{2N-1}$, we have the relations*

$$K_{\eta(h)} E_i K^{-1}_{\eta(h)} = q^{\gamma_{i-1}(h)} E_i, \qquad K_{\eta(h)} B_{\beta_{N-1}} K^{-1}_{\eta(h)} = q^{\gamma_{N-1}(h)} B_{\beta_{N-1}}, \tag{5.21}$$

$$K_{\eta(h)} F_i K^{-1}_{\eta(h)} = q^{-\gamma_{i-1}(h)} F_i, \qquad K_{\eta(h)} B_{\beta_N} K^{-1}_{\eta(h)} = q^{-\gamma_{N-1}(h)} B_{\beta_N}, \tag{5.22}$$

and, moreover, we have

$$K_{\eta(h)} E_N K^{-1}_{\eta(h)} = q^{(\gamma_{N-2}+2\gamma_{N-1})(h)} E_N, \tag{5.23}$$

$$K_{\eta(h)} F_N K^{-1}_{\eta(h)} = q^{-(\gamma_{N-2}+2\gamma_{N-1})(h)} F_N. \tag{5.24}$$

Proof. We first check the relations (5.21), and the relations (5.22) will follow analogously. It suffices to verify the relations (5.21) for $h = h_{\gamma_j}$ for $1 \le j \le N-1$. Indeed, for $1 \le j \le N-2$, we have

$$K_{\eta(h_{\gamma_j})} E_i K^{-1}_{\eta(h_{\gamma_j})} = q^{\alpha_i\left(\eta(h_{\gamma_j})\right)} E_i = q^{\alpha_i(h_{\alpha_{j+1}})} E_i = q^{\gamma_{i-1}(h_{\gamma_j})} E_i$$

and, moreover, we have

$$K_{\eta(h_{\gamma_{N-1}})} E_i K^{-1}_{\eta(h_{\gamma_{N-1}})} = q^{\alpha_i\left(\eta(h_{\gamma_{N-1}})\right)} E_i = q^{\alpha_i(h_{\alpha_N} - h_{\alpha_{N-1}})} E_i = q^{\gamma_{i-1}(h_{\gamma_{N-1}})} E_i.$$

This proves the first relation in (5.21). Next, using the definition of the root vector $B_{\beta_{N-1}}$ in Equation (5.19), we calculate

$$K_{\eta(h)} B_{\beta_{N-1}} K^{-1}_{\eta(h)} = q^{-s_1 \cdots s_{N-2}(\alpha_{N-1})\left(\eta(h)\right)} B_{\beta_{N-1}} = q^{-(\alpha_1 + \cdots + \alpha_{N-1})\left(\eta(h)\right)} B_{\beta_{N-1}} \tag{5.25}$$

where, for $1 \le j \le N-1$, we have

$$\begin{aligned} -(\alpha_1 + \cdots + \alpha_{N-1})\big(\eta(h_{\gamma_j})\big) &= \begin{cases} 2 & \text{if } j = N-1 \\ -1 & \text{if } j = N-2 \\ 0 & \text{otherwise} \end{cases} \\ &= \gamma_{N-1}(h_{\gamma_j}). \end{aligned}$$

Inserting this into Equation (5.25), we obtain the second relation in (5.21). Finally, we need to verify Equation (5.23), and then (5.24) follows from $(U3)$. For $1 \le j \le N-2$, we have

$$K_{\eta(h_{\gamma_j})} E_N K^{-1}_{\eta(h_{\gamma_j})} = q^{\alpha_N\left(\eta(h_{\gamma_j})\right)} E_N = q^{\alpha_N(h_{\alpha_{j+1}})} E_N = q^{-\delta_{j,N-3}} E_N \tag{5.26}$$

and

$$K_{\eta(h_{\gamma_{N-1}})} E_N K^{-1}_{\eta(h_{\gamma_{N-1}})} = q^{\alpha_N\left(\eta(h_{\gamma_{N-1}})\right)} E_N = q^{\alpha_N(h_{\alpha_N} - h_{\alpha_{N-1}})} E_N = q^2 E_N \tag{5.27}$$

On the other hand, we obtain

$$(\gamma_{N-2} + 2\gamma_{N-1})(h_{\gamma_j}) = \begin{cases} -1 & \text{if } j = N-3 \\ 2 & \text{if } j = N-1 \\ 0 & \text{otherwise} \end{cases}$$

and comparing this to Equations (5.26) and (5.27) proves Equation (5.23). $\square$

5.4 Ordered monomials in $\mathcal{B}_\mathbf{c}$

Recall from (4.41) in Theorem 4.17 that, for any multi-index $\mathcal{J} = (j_1, \dots, j_{N(N-1)}) \in \mathbb{N}_0{}^{N(N-1)}$, the set of all $F_\mathcal{J} = F_{\beta_{N(N-1)}}^{j_{N(N-1)}} \cdots F_{\beta_1}^{j_1}$ is a basis of $\mathcal{U}^-$, where each of the root vectors F_{β_j} for $1 \le j \le N(N-1)$ can be given explicitly in terms of *inverse* Lusztig actions. Moreover, by the triangular decomposition (4.10) of the algebra $\mathcal{U}$, the set $\{F_\mathcal{J} \mid \mathcal{J} \in \mathbb{N}_0{}^{N(N-1)}\}$ also forms a basis of $\mathcal{U}$ as a right $\mathcal{U}^0\mathcal{U}^+$-module.

Similarly, for any multi-index $\mathcal{J} = (j_1, \dots, j_{N(N-1)}) \in \mathbb{N}_0{}^{N(N-1)}$, we now define the elements

$$B_\mathcal{J} \coloneqq B_{\beta_{N(N-1)}}^{j_{N(N-1)}} \cdots B_{\beta_1}^{j_1} \tag{5.28}$$

where in (5.17) we defined $B_{\beta_1} \coloneqq B_1 = F_1 - c_1 T_{w_X}(E_1) K_1^{-1}$ and, for $2 \le j \le N(N-1)$,

$$B_{\beta_j} \coloneqq \begin{cases} T_{j,\dots,2}^{-1}(B_1) & \text{if } j \le N-1 \\ T_{2N-j,\dots,N}^{-1} T_{N-2,\dots,2}^{-1}(B_1) & \text{if } N-1 < j \le 2(N-1) \\ F_{\beta_j} & \text{if } j > 2(N-1) \end{cases}$$

The difference between the elements $F_\mathcal{J}$ and $B_\mathcal{J}$ depends on the choice of $\mathcal{J}$.

Lemma 5.8. *For any multi-index $\mathcal{J} \in \mathbb{N}_0{}^{N(N-1)}$, we have*

$$F_\mathcal{J} = B_\mathcal{J} + \epsilon_\mathcal{J} \quad \textit{with} \quad \epsilon_\mathcal{J} \in \sum_{|\mu|<|\mathcal{J}|} \mathcal{U}_{-\mu}^- \mathcal{U}^0 \mathcal{U}^+.$$

Proof. For each $1 \le k \le 2(N-1)$, let the element $w_{(k)} \in \mathcal{W}_X$ be the word

$$w_{(k)} \coloneqq \begin{cases} id & \text{for } k = 1 \\ s_k \cdots s_2 & \text{for } 2 \le k \le N-1 \\ s_{2N-k} \cdots s_N s_{N-2} \cdots s_2 & \text{for } N \le k \le 2(N-1). \end{cases}$$

Then for all $k \le 2(N-1)$, the elements B_{β_k} defined in (5.17) may be rexpressed as

$$B_{\beta_k} = T_{w_{(k)}}^{-1}(B_1)$$

where conventionally we let $T_{id}^{\pm 1} = id$ for the case $k = 1$. In particular, notice that $T_{w_X}(E_1) = T_{w_{(2(N-1))}}(E_1)$, and moreover $T_{w_{(k)}}^{-1} T_{w_{(2(N-1))}} = T_{w_{(2N-(k+1))}}$ for all $k \le 2(N-1)$. This implies that

$$\begin{aligned} B_{\beta_k} &= T_{w_{(k)}}^{-1}\big(F_1 - c_1 T_{w_X}(E_1) K_1^{-1}\big) \\ &= F_{\beta_k} - c_1 T_{w_{(2N-(k+1))}}(E_1) T_{w_{(k)}}^{-1}(K_1^{-1}) \\ &= F_{\beta_k} + \epsilon_k \end{aligned}$$

where $\epsilon_k \in \mathcal{U}^0\mathcal{U}^+$ for all $k \le 2(N-1)$. Note that $B_{\beta_k} = F_{\beta_k}$ for all $k \ge 2N-1$.

Let $\mathcal{J} = (j_1, \ldots, j_{N(N-1)}) \in {\mathbb{N}_0}^{N(N-1)}$. Recall that $\left|\mathcal{J}\right| = \sum_{l=1}^{N(N-1)} j_l \left|\beta_l\right|$ and assume $|\mathcal{J}| > 0$. Then,

$$\begin{aligned} B_{\beta_{N(N-1)}}^{j_{N(N-1)}} \cdots B_{\beta_1}^{j_1} &= F_{\beta_t}^{j_t} \cdots F_{\beta_{2N-1}}^{j_{2N-1}} (F_{\beta_{2(N-1)}} + \epsilon_{2(N-1)})^{j_{2(N-1)}} \cdots (F_{\beta_1} + \epsilon_1)^{j_1} \\ &= F_{\beta_t}^{j_t} \cdots F_{\beta_1}^{j_1} + \epsilon_{\mathcal{J}} \end{aligned}$$

where $\epsilon_{\mathcal{J}} \in \sum_{|\mu|<|\mathcal{J}|} \mathcal{U}^-_{-\mu} \mathcal{U}^0 \mathcal{U}^+$. Hence, we have $B_{\mathcal{J}} = F_{\mathcal{J}} + \epsilon_{\mathcal{J}}$ as required. □

Define a filtration $\mathcal{F}^*$ of $\mathcal{U}^-$ by

$$\mathcal{F}^n(\mathcal{U}^-) = \operatorname{span}\left\{ F_{\mathcal{J}} \;\middle|\; \mathcal{J} \in \mathbb{N}_0^{N(N-1)},\ |\mathcal{J}| \leq n \right\} \tag{5.29}$$

for all $n \in \mathbb{N}_0$. Observe that, since the quantum Serre relations (U5) and (U6) are homogeneous (see Section 4.1), the set

$$\left\{ F_{\mathcal{J}} \;\middle|\; \mathcal{J} \in \mathbb{N}_0^{N(N-1)},\ \left|\mathcal{J}\right| \leq n \right\}$$

in fact forms a basis of $\mathcal{F}^n(\mathcal{U}^-)$. The following two results are special cases of [Kol14, Propositions 6.1, 6.2].

Proposition 5.9. *The set $\{B_{\mathcal{J}} \mid \mathcal{J} \in \mathbb{N}_0^{N(N-1)}\}$ is a basis of $\mathcal{U}$ as a right $\mathcal{U}^0\mathcal{U}^+$-module.*

Proof. Let $\mathcal{J} \in \mathbb{N}_0^{N(N-1)}$ and assume $\left|\mathcal{J}\right| = n \in \mathbb{N}$. We first show by induction on n that $F_{\mathcal{J}}$ is contained in the right $\mathcal{U}^0\mathcal{U}^+$-module generated by the set $\left\{B_{\mathcal{J}} \,|\, \mathcal{J} \in \mathbb{N}_0^{N(N-1)}\right\}$. Indeed, by Lemma 5.8 we have $F_{\mathcal{J}} - B_{\mathcal{J}} \in \mathcal{F}^{n-1}(\mathcal{U}^-)\mathcal{U}^0\mathcal{U}^+$. Using the quantum Serre relations for $\mathcal{U}^-$, we hence obtain that $F_{\mathcal{J}} - B_{\mathcal{J}}$ is contained in the right $\mathcal{U}^0\mathcal{U}^+$-submodule of $\mathcal{U}$ generated by the set $\left\{F_{\mathcal{I}} \;\middle|\; \mathcal{I} \in \mathbb{N}_0^{N(N-1)},\ \left|\mathcal{I}\right| \leq n-1\right\}$. The induction hypothesis implies the desired result.

It remains to show that the set $\left\{B_{\mathcal{J}} \;\middle|\; \mathcal{J} \in \mathbb{N}_0^{N(N-1)}\right\}$ is linearly independent over $\mathcal{U}^0\mathcal{U}^+$. To this end, assume that

$$\sum_{\mathcal{J} \in \mathbb{N}_0^{N(N-1)}} B_{\mathcal{J}} a_{\mathcal{J}} = 0$$

for some $a_{\mathcal{J}} \in \mathcal{U}^0\mathcal{U}^+$ where all but finitely many $a_{\mathcal{J}}$ are zero. Lemma 5.8 tells us that $B_{\mathcal{J}} = F_{\mathcal{J}} + \epsilon_{\mathcal{J}}$ with

$$\epsilon_{\mathcal{J}} \in \sum_{|\mu|<|\mathcal{J}|} \mathcal{U}^-_{-\mu} \mathcal{U}^0 \mathcal{U}^+.$$

Choose m maximal such that $a_{\mathcal{J}} \neq 0$ for some $\mathcal{J} \in \mathbb{N}_0^{N(N-1)}$ with $\left|\mathcal{J}\right| = m$. Then,

$$\sum_{\mathcal{J} \in \mathbb{N}_0^{N(N-1)}} B_{\mathcal{J}} a_{\mathcal{J}} = \sum_{|\mathcal{J}|=m} B_{\mathcal{J}} a_{\mathcal{J}} + \sum_{|\mathcal{J}|<m} B_{\mathcal{J}} a_{\mathcal{J}}$$

and therefore by our assumption, we get

$$0 = \sum_{|\mathcal{J}|=m} F_{\mathcal{J}} a_{\mathcal{J}} + \underbrace{\sum_{|\mathcal{J}|=m} \epsilon_{\mathcal{J}} a_{\mathcal{J}} + \sum_{|\mathcal{J}|<m} B_{\mathcal{J}} a_{\mathcal{J}}}_{\in \sum_{|\mu|<|\mathcal{J}|} \mathcal{U}^-_{-\mu} \mathcal{U}^0 \mathcal{U}^+} .$$

However, since the set $\{F_{\mathcal{J}} \mid |\mathcal{J}| = m\}$ is a basis of $\sum_{|\mu|<|\mathcal{J}|} \mathcal{U}^-_{-\mu} \mathcal{U}^0 \mathcal{U}^+$, this implies that

$$a_{\mathcal{J}} = 0$$

if $|\mathcal{J}| = m$. This contradicts our choice of m, and hence we are done. □

By Proposition 5.9, we know that any element in the algebra $\mathcal{B}_{\mathbf{c}}$ can be written as a linear combination of elements in the set $\{B_{\mathcal{J}} \mid \mathcal{J} \in \mathbb{N}_0^{N(N-1)}\}$ with coefficients in $\mathcal{U}^0\mathcal{U}^+$. We now see that it is actually sufficient to permit coefficients from $\mathcal{U}^0_\Theta \mathcal{M}^+_X$ where

$$\mathcal{U}^0_\Theta \coloneqq \langle K_i^{\pm 1} \mid i \in \{2, \dots, N\} \rangle \qquad \text{and} \qquad \mathcal{M}^+_X \coloneqq \langle E_i \mid i \in \{2, \dots, N\} \rangle .$$

Proposition 5.10. *The set $\{B_{\mathcal{J}} \mid \mathcal{J} \in \mathbb{N}_0^{N(N-1)}\}$ is a basis of $\mathcal{B}_{\mathbf{c}}$ as a right $\mathcal{U}^0_\Theta \mathcal{M}^+_X$-module.*

Proof. The set $\{B_{\mathcal{J}} \mid \mathcal{J} \in \mathbb{N}_0^{N(N-1)}\}$ is linearly independent over $\mathcal{U}^0\mathcal{U}^+$ by Proposition 5.9. Hence, it is also linearly independent over $\mathcal{U}^0_\Theta \mathcal{M}^+_X$. It remains to prove that the set $\{B_{\mathcal{J}} \mid \mathcal{J} \in \mathbb{N}_0^{N(N-1)}\}$ spans $\mathcal{B}_{\mathbf{c}}$ as a right $\mathcal{U}^0_\Theta \mathcal{M}^+_X$-module.

Let $\mathcal{I} \in I^N$, and then define $\tilde{B}_{\mathcal{I}} \coloneqq B_{i_1} \cdots B_{i_N}$. We can write

$$\mathcal{B}_{\mathbf{c}} = \sum_{\mathcal{I} \in I^N} \tilde{B}_{\mathcal{I}} \mathcal{U}^0_\Theta \mathcal{M}^+_X$$

and we get that $\{\tilde{B}_{\mathcal{I}} \mid \mathcal{I} \in I^N\}$ is a spanning set of the right $\mathcal{U}^0_\Theta \mathcal{M}^+_X$-module $\mathcal{B}_{\mathbf{c}}$. It suffices to show that

$$\tilde{B}_{\mathcal{I}} \in \sum_{\mathcal{J} \in \mathbb{N}_0^{N(N-1)}} B_{\mathcal{J}} \mathcal{U}^0_\Theta \mathcal{M}^+_X .$$

Observe that $\tilde{B}_{\mathcal{I}} = \tilde{F}_{\mathcal{I}} + \tilde{\epsilon}_{\mathcal{I}}$ with $\tilde{\epsilon}_{\mathcal{I}} \in \sum_{|\mu|<|\mathcal{I}|} \mathcal{U}^-_{-\mu} \mathcal{U}^0 \mathcal{U}^+$, and

$$\tilde{F}_{\mathcal{I}} = \sum_{|\mathcal{J}|=|\mathcal{I}|} F_{\mathcal{J}} b_{\mathcal{J}}$$

for some $b_{\mathcal{J}} \in \mathbb{K}(q)$. Then,

$$\tilde{B}_{\mathcal{I}} = \sum_{|\mathcal{J}|=|\mathcal{I}|} B_{\mathcal{J}} b_{\mathcal{J}} + \tilde{\tilde{\epsilon}}_{\mathcal{I}}$$

with $\tilde{\tilde{\epsilon}}_{\mathcal{I}} \in \sum_{|\mu|<|\mathcal{I}|} \mathcal{U}^{-}_{-\mu}\mathcal{U}^0\mathcal{U}^+ \cap \mathcal{B}_{\mathbf{c}}$. Hence, one obtains that

$$\tilde{\tilde{\epsilon}}_{\mathcal{I}} \in \sum_{|\mathcal{J}|<|\mathcal{I}|} B_{\mathcal{J}}\mathcal{U}^0_{\Theta}\mathcal{M}^+_X$$

and thus, by induction on $|\mathcal{I}| = \sum_{k=1}^{N} i_k$, we are done. □

Notice that the elements $B_{\mathcal{J}}$ for $\mathcal{J} \in \mathbb{N}_0^{N(N-1)}$ defined in (5.28) are ordered monomials in the root vectors B_{β_j} given explicitly in (5.17). Specifically, these root vectors are ordered so that all of the negative root vectors are on the left (sending the positive root vectors to the right). In fact, the order in which each of the root vectors appear within these ordered monomials can be obtained directly from the order of the corresponding root using the reduced expression for the longest word $w_0 \in \mathcal{W}$ written in Lemma 4.18. Thus, the order of the root vectors (5.17) is

$$F_{\beta_t}, \dots, F_2, \underbrace{T^{-1}_{2,\dots N,N-2,\dots,2}(B_1), \dots, T^{-1}_{N,N-2,\dots,2}(B_1)}_{N-1 \text{ negative root vectors}}, \underbrace{T^{-1}_{N-1,\dots,2}(B_1) \dots, T^{-1}_2(B_1), B_1}_{N-1 \text{ positive root vectors}}.$$

Moreover, we can write any element of the algebra $\mathcal{B}_{\mathbf{c}}$ as a linear combination of ordered monomials in the root vectors of $\mathcal{B}_{\mathbf{c}}$. Proposition 5.10 then tells us that the elements $B_{\mathcal{J}}$ for $\mathcal{J} \in \mathbb{N}_0^{N(N-1)}$ defined in (5.28) are a basis over the algebra $\mathcal{U}^0_{\Theta}\mathcal{M}^+_X$. Now, we consider the multi-indices $\mathcal{D} = (d_2, \dots, d_N) \in \mathbb{Z}^{N-1}$ and $\mathcal{I} = (i_{2N-1}, \dots, i_{N(N-1)}) \in \mathbb{N}_0^{(N-1)(N-2)}$, and define the monomials

$$K_{\mathcal{D}} := K_2^{d_2} \cdots K_N^{d_N} \qquad \text{and} \qquad E_{\mathcal{I}} := E_{\beta_{N(N-1)}}^{i_{N(N-1)}} \cdots E_2^{i_{2N-1}}. \tag{5.30}$$

By the PBW-Theorem for $\mathcal{U}$, the elements $E_{\mathcal{I}}$ form a basis of $\mathcal{M}^+_X$, and the $K_{\mathcal{D}}$ form a basis of $\mathcal{U}^0_{\Theta}$. Hence, Proposition 5.10 implies the following important theorem.

Theorem 5.11 (PBW-Theorem for $\mathcal{B}_{\mathbf{c}}$)**.** *The ordered monomials $B_{\mathcal{J}}K_{\mathcal{D}}E_{\mathcal{I}}$ constructed by the elements* (5.28) *and* (5.30) *form a basis for the algebra $\mathcal{B}_{\mathbf{c}}$.*

5.5 Commutation of root vectors

For use in the final chapter, we shall determine some specific q-commutators of root vectors. Recall the subalgebra $\mathcal{M}^+_X$ of $\mathcal{M}_X$ generated by the set $\{E_i \mid i \in X\}$. Now, let $\mathcal{M}^+_{X,+}$ denote the *augmentation ideal* of $\mathcal{M}^+_X$, that is, the ideal generated by the set $\{E_i \mid i \in X\}$.

Lemma 5.12. *For $1 \le k < j \le N-1$, we have $\left[B_{\beta_k}, B_{\beta_j}\right]_q \in \mathcal{M}_X\mathcal{M}^+_{X,+}$.*

Proof. We assume that $k \ge 2$. The argument for $k = 1$ is similar (and only requires that we omit the expressions $T^{-1}_{k,\dots,2}$ and $s_k \cdots s_2$). By definition of the root vectors, we have

$$\begin{aligned}\left[B_{\beta_k}, B_{\beta_j}\right]_q &= \left[T^{-1}_{k,\dots,2}(B_1), T^{-1}_{j,\dots,2}(B_1)\right]_q \\ &= T^{-1}_{k,\dots,2} T^{-1}_{j,\dots,3}\Big(\left[B_1, T_2^{-1}(B_1)\right]_q\Big) \end{aligned} \tag{5.31}$$

using the braid relations $T_i T_{i+1}^{-1} T_i^{-1} = T_{i+1}^{-1} T_i^{-1} T_{i+1}$ inductively for $2 \leq i \leq k$ to show that

$$T_{2,\dots,k} T^{-1}_{j,\dots,2}(B_1) = T^{-1}_{j,\dots,i+1} \underbrace{T_{2,\dots,i-1}}_{=\mathrm{id},\,\mathrm{if}\, i=2} T^{-1}_{i,\dots,2}(B_1) = T^{-1}_{j,\dots,2}(B_1).$$

By relation (5.7) and Lemma 5.2, we know that

$$\left[B_1, T_2^{-1}(B_1)\right]_q \in \mathbb{K}(q) F_2 \mathcal{Z}_1 + \mathbb{K}(q) K_2 r_2(\mathcal{Z}_1) + \mathbb{K}(q) K_2^{-1} {}_2r(\mathcal{Z}_1)$$

where $\mathcal{Z}_1 \in (\mathcal{M}_X)_{w_X(\alpha_1)-\alpha_1}$ and $r_2(\mathcal{Z}_1), {}_2r(\mathcal{Z}_1) \in (\mathcal{M}_X)_{w_X(\alpha_1)-\alpha_1-\alpha_2}$. Combining this with Equation (5.31) shows that

$$\left[B_{\beta_k}, B_{\beta_j}\right]_q \in \mathcal{M}_X (\mathcal{M}_X)_\mu + \mathcal{M}_X (\mathcal{M}_X)_\nu$$

where, since $w_X s_i = s_{\tau(i)} w_X$ for all $i \in X$, we calculate that

$$\begin{aligned}\mu &= s_k \cdots s_2 s_j \cdots s_3\big(w_X(\alpha_1) - \alpha_1\big) \\ &= w_X\big(s_k \cdots s_2(\alpha_1)\big) - s_k \cdots s_2(\alpha_1) \\ &= 2(\alpha_{k+1} + \dots + \alpha_{N-2}) + \alpha_{N-1} + \alpha_N\end{aligned}$$

and

$$\begin{aligned}\nu &= s_k \cdots s_2 s_j \cdots s_3\big(w_X(\alpha_1) - \alpha_1 - \alpha_2\big) \\ &= \mu - s_k \cdots s_2 s_j \cdots s_3(\alpha_2) \\ &= \mu - (\alpha_{k+1} + \dots + \alpha_j).\end{aligned}$$

Since $\mu, \nu \in Q_X^+$ are non-zero, the triangular decomposition of $\mathcal{M}_X$ implies that $(\mathcal{M}_X)_\mu$ and $(\mathcal{M}_X)_\nu$ are contained in $\mathcal{M}_X \mathcal{M}^+_{X,+}$. Hence, we are done. □

As an immediate consequence of Lemma 5.12, we obtain the following important result.

Corollary 5.13. *For each $j \in \{N-1, N\}$, we have $\left[B_{\beta_{N-2}}, B_{\beta_j}\right]_q \in \mathcal{M}_X \mathcal{M}^+_{X,+}$.*

Proof. The case $j = N-1$ is a special case of Lemma 5.12. The case $j = N$ then holds by symmetry between $N-1$ and N. □

The commutators in the next Lemma follow immediately from the defining relations (5.4)-(5.7) of the algebra $\mathcal{B}_\mathbf{c}$ and from Proposition 5.5, hence we omit the proof.

Lemma 5.14. *For $i \in X$ and $1 \leq j \leq N-1$, we have*

$$\left[E_i, B_{\beta_j}\right] = \begin{cases} B_{\beta_{j-1}} K_j^{-1} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

By symmetry between $N-1$ and N, Lemma 5.14 gives $\left[E_i, B_{\beta_N}\right] = \delta_{i,N} B_{\beta_{N-2}} K_N^{-1}$. Similarly, we need commutation relations between the root vectors B_{β_j} and the generators F_i for $i \in X$.

Proposition 5.15. *(i) For $j \in X$, we have*

$$\big[B_{\beta_j}, F_j\big]_q = 0.$$

(ii) For $1 \le j \le N-2$, we have

$$\big[B_{\beta_j}, F_{j+1}\big]_{q^{-1}} = -q^{-1} B_{\beta_{j+1}}.$$

(iii) Let $i \in X$ and $1 \le j \le N$ where $j \notin \{i-1, i\}$. Additionally, let $j \neq N$ if $i = N-1$, or let $j \neq N-2$ if $i = N$. Then, we have

$$\big[B_{\beta_j}, F_i\big] = 0.$$

Proof. By symmetry between $N-1$ and N, we may assume that $j \neq N$. To verify (i), we calculate

$$\begin{aligned}\big[B_{\beta_j}, F_j\big]_q &= T^{-1}_{j,\dots,2}\Big(\big[B_1, T_{2,\dots,j}(F_j)\big]_q\Big)\\ &= -T^{-1}_{j,\dots,2}\Big(\big[B_1, K_2^{-1}\cdots K_j^{-1} T_{2,\dots,j-1}(E_j)\big]_q\Big)\\ &= -q\, T^{-1}_{j,\dots,2}\Big(K_2^{-1}\cdots K_j^{-1}\big[B_1, T_{2,\dots,j-1}(E_j)\big]\Big)\\ &= -qK_j\, T_j^{-1}\Big(\big[T^{-1}_{j-1,\dots,2}(B_1), E_j\big]\Big) = 0\end{aligned}$$

$\big($taking $T_{2,\dots,j-1} = id = T^{-1}_{j-1,\dots,2}$ above for the case $j=2\big)$ which vanishes by Lemma 5.14. The relation in (ii) follows from Proposition 5.5. For $j < i-1$, the relation in (iii) follows from the defining relations of $\mathcal{B}_{\mathbf{c}}$. On the other hand, for $j > i$, we calculate

$$\big[B_{\beta_j}, F_i\big] = T^{-1}_{j,\dots,2}\Big(\big[B_1, T_{2,\dots,i,i+1}(F_i)\big]\Big) = T^{-1}_{j,\dots,2}\Big(\big[B_1, F_{i+1}\big]\Big) = 0$$

which vanishes by (iii) for the case $j < i-1$ we have already justified. ☐

By symmetry between $N-1$ and N, Proposition 5.15 (ii) gives $\big[B_{\beta_{N-2}}, F_N\big]_{q^{-1}} = -q^{-1}B_{\beta_N}$. More generally, the following Lemma, given by induction over m, will be useful in the next Chapter, in particular, for the proof of Proposition 6.12 in Section 6.4.

Lemma 5.16. *For all $m \in \mathbb{N}$, for each $j \in \{N-1, N\}$, we have*

$$B_{\beta_{N-2}} F_j^m = q^{-m} F_j^m B_{\beta_{N-2}} - q^{-1}[m]_q F_j^{m-1} B_{\beta_j}.$$

Proof. We know that the case $m=1$ follows directly from Proposition 5.15 (ii). Inductively, now assuming that the relation holds for some $m \in \mathbb{N}$, we calculate

$$\begin{aligned}B_{\beta_{N-2}} F_j^{m+1} &= \big(B_{\beta_{N-2}} F_j^m\big) F_j\\ &= q^{-m} F_j^m B_{\beta_{N-2}} F_j - q^{-1}[m]_q F_j^{m-1} B_{\beta_j} F_j\\ &= q^{-(m+1)} F_j^m \big(F_j B_{\beta_{N-2}} - B_{\beta_j}\big) - [m]_q F_j^m B_{\beta_j}\\ &= q^{-(m+1)} F_j^{(m+1)} B_{\beta_{N-2}} - q^{-1}[m+1]_q F_j^m B_{\beta_j}\end{aligned}$$

since $[m+1]_q = q^{-m} + q[m]_q$ for all $m \in \mathbb{N}$. Hence, the relation holds for all $m \in \mathbb{N}$. ☐

5.5.1 The pivot commutator

We now investigate the commutator

$$\Omega = \big[B_{\beta_{N-1}}, B_{\beta_N}\big] \tag{5.32}$$

which we shall refer to as the **pivot commutator** of $\mathcal{B}_{\mathbf{c}}$, as it plays a pivotal role in our classification. By the discussion in Section 5.3, the pivot commutator is a quantum analogue of the Cartan element $h_{\gamma_{N-1}} = [e_{\gamma_{N-1}}, f_{\gamma_{N-1}}]$ in $\mathfrak{so}_{2N-1}$. However, $\Omega \notin \mathcal{U}^0$. In this section, we determine the image of Ω under a natural projection map $U_q(\mathfrak{so}_{2N}) \to \mathcal{U}^0$ with respect to the triangular decomposition (4.10). Recall the root vectors E_{β_j} and F_{β_j} of $\mathcal{U}$ defined in Section 4.5.1, for $1 \leq j \leq N(N-1)$. For each $i \in I$, we denote the commutator

$$\Omega_i = \big[E_{\beta_i}, F_{\beta_i}\big]. \tag{5.33}$$

Lemma 5.17. *The pivot commutator* (5.32) *can be given in terms of commutators* (5.33) *by*

$$\Omega = c_1\big(\Omega_{N-1}K_N^{-1} - \Omega_N K_{N-1}^{-1}\big)K_1^{-1}\cdots K_{N-2}^{-1}.$$

Proof. From Definition 4.23 and Equation (5.1), since $E_{\beta_{N+1}} = T_{N,\ldots,2}(E_1)$, we may write the element $B_1 = F_1 - c_1 T_{2,\ldots,N-2}(E_{\beta_{N+1}})K_1^{-1}$, and hence we get

$$B_{\beta_{N-2}} = F_{\beta_{N-2}} - c_1 E_{\beta_{N+1}} K_1^{-1}\cdots K_{N-2}^{-1}.$$

Before we can determine the pivot commutator, we need to make a few observations. Firstly, note that

$$\big[F_{\beta_{N-1}}, F_{\beta_N}\big] = 0 = \Big[\big[F_{N-1}, E_{\beta_{N+1}}\big]_q, \big[F_N, E_{\beta_{N+1}}\big]_q\Big].$$

Additionally, for each $j \in \{N-1, N\}$, using the relations of $\mathcal{U}$ in Definition 4.2 we have

$$\Big[F_{\beta_j}, \big[F_{\tau(j)}, E_{\beta_{N+1}}K_1^{-1}\cdots K_{N-2}^{-1}\big]_q\Big] = \Big[F_{\beta_j}, \big[F_{\tau(j)}, E_{\beta_{N+1}}\big]\Big]_q K_1^{-1}\cdots K_{N-2}^{-1}$$

and, moreover,

$$\Big[F_{\beta_j}, \big[F_{\tau(j)}, E_{\beta_{N+1}}\big]\Big]_q = \big[F_{\beta_j}, E_{\beta_j}\big]K_{\tau(j)}^{-1}.$$

Putting everything together, we calculate

$$\begin{aligned}
\Omega &= \Big[\big[F_{N-1}, B_{\beta_{N-2}}\big]_q, \big[F_N, B_{\beta_{N-2}}\big]_q\Big] \\
&= \Big[\big[F_{N-1}, F_{\beta_{N-2}} - c_1E_{\beta_{N+1}}K_1^{-1}\cdots K_{N-2}^{-1}\big]_q, \big[F_N, F_{\beta_{N-2}} - c_1E_{\beta_{N+1}}K_1^{-1}\cdots K_{N-2}^{-1}\big]_q\Big] \\
&= -c_1\Big(\Big[F_{\beta_{N-1}}, \big[F_N, E_{\beta_{N+1}}K_1^{-1}\cdots K_{N-2}^{-1}\big]_q\Big] - \Big[F_{\beta_N}, \big[F_{N-1}, E_{\beta_{N+1}}K_1^{-1}\cdots K_{N-2}^{-1}\big]_q\Big]\Big) \\
&= -c_1\Big(\Big[F_{\beta_{N-1}}, \big[F_N, E_{\beta_{N+1}}\big]\Big]_q - \Big[F_{\beta_N}, \big[F_{N-1}, E_{\beta_{N+1}}\big]\Big]_q\Big)K_1^{-1}\cdots K_{N-2}^{-1} \\
&= -c_1\Big(\big[F_{\beta_{N-1}}, E_{\beta_{N-1}}\big]K_N^{-1} - \big[F_{\beta_N}, E_{\beta_N}\big]K_{N-1}^{-1}\Big)K_1^{-1}\cdots K_{N-2}^{-1} \\
&= c_1\big(\Omega_{N-1}K_N^{-1} - \Omega_N K_{N-1}^{-1}\big)K_1^{-1}\cdots K_{N-2}^{-1}
\end{aligned}$$

as required, and this completes the proof. □

Observe that $\Omega_j \in \mathcal{U}_0$ for all $i \in I$. Following Lemma 5.17, this implies that

$$\Omega \in \mathcal{B}_{\mathbf{c}} \cap \mathcal{U}_0 = (\mathcal{M}_X)_0.$$

Recall the triangular decomposition of $\mathcal{U}$ given in (4.10). Consider the projection map π onto the subalgebra $\mathcal{U}^0$. The goal is now to calculate $\pi(\Omega)$, or equivalently, to prove Proposition 5.20.

In the rest of this section, we make repeated use of the following notation. For any $i \in I$, define an algebra homomorphism $\big|_{K_i \to qK_i} : \mathcal{U}^0 \to \mathcal{U}^0\,;\, u \mapsto u\big|_{K_i \to qK_i}$ by

$$K_i\big|_{K_i \to qK_i} = qK_i \qquad \text{and} \qquad K_j\big|_{K_i \to qK_i} = K_j, \text{ for all } j \neq i.$$

Lemma 5.18. *For $1 \leq i \leq N-2$ we have the iterative formula*

$$\pi(\Omega_{i+1}) = \Big(\pi(\Omega_i) - q\pi(\Omega_i)\big|_{K_i \to qK_i}\Big)\frac{K_{i+1} - K_{i+1}^{-1}}{q - q^{-1}} - q^{-1}\pi(\Omega_i)K_{i+1}^{-1}.$$

Proof. For $1 \leq i \leq N-2$, we may write

$$\Omega_{i+1} = \Big[\big[E_{i+1}, E_{\beta_i}\big]_{q^{-1}}, \big[F_{i+1}, F_{\beta_i}\big]_q\Big]$$

and notice that $\big[E_{i+1}, F_{\beta_i}\big] = 0 = \big[E_{\beta_i}, F_{i+1}\big]$. Hence, under the projection map π, we have

$$\begin{aligned}
\pi(\Omega_{i+1}) &= \pi\Big(\big[E_{i+1}, E_{\beta_i}\big]_{q^{-1}}\big[F_{i+1}, F_{\beta_i}\big]_q\Big) \\
&= \pi\Big(E_{i+1}F_{i+1}E_{\beta_i}F_{\beta_i} - q^{-1}E_{\beta_i}E_{i+1}F_{i+1}F_{\beta_i} - qE_{i+1}E_{\beta_i}F_{\beta_i}F_{i+1} + E_{\beta_i}F_{\beta_i}E_{i+1}F_{i+1}\Big) \\
&= \pi\Big(\big[E_{i+1}, F_{i+1}\big]\pi(\Omega_i) - q^{-1}E_{\beta_i}\big[E_{i+1}, F_{i+1}\big]F_{\beta_i} - qE_{i+1}\pi(\Omega_i)F_{i+1} + \pi(\Omega_i)\big[E_{i+1}, F_{i+1}\big]\Big) \\
&= \pi\Big(\frac{K_{i+1} - K_{i+1}^{-1}}{q - q^{-1}}\pi(\Omega_i) - q^{-1}E_{\beta_i}\frac{K_{i+1} - K_{i+1}^{-1}}{q - q^{-1}}F_{\beta_i} - qE_{i+1}\pi(\Omega_i)F_{i+1} + \pi(\Omega_i)\frac{K_{i+1} - K_{i+1}^{-1}}{q - q^{-1}}\Big) \\
&= 2\pi(\Omega_i)\frac{K_{i+1} - K_{i+1}^{-1}}{q - q^{-1}} - \pi\Big(q^{-1}\pi(\Omega_i)\frac{qK_{i+1} - q^{-1}K_{i+1}^{-1}}{q - q^{-1}} + q\pi(\Omega_i)\big|_{K_i \to qK_i}\big[E_{i+1}, F_{i+1}\big]\Big) \\
&= \pi(\Omega_i)\frac{K_{i+1} - (2 - q^{-2})K_{i+1}^{-1}}{q - q^{-1}} - q\pi(\Omega_i)\big|_{K_i \to qK_i}\frac{K_{i+1} - K_{i+1}^{-1}}{q - q^{-1}}
\end{aligned}$$

and, by simplifying the coefficients, we obtain the required result. ∎

Lemma 5.19. *For $1 \leq i \leq N-1$, we have the formula*

$$\pi(\Omega_i) - q\pi(\Omega_i)\big|_{K_i \to qK_i} = (-1)^i q^i K_1 \cdots K_i.$$

Proof. Firstly, we verify that

$$\pi(\Omega_1) - q\pi(\Omega_1)\big|_{K_1 \to qK_1} = \frac{K_1 - K_1^{-1}}{q - q^{-1}} - q\frac{qK_1 - q^{-1}K_1^{-1}}{q - q^{-1}} = \frac{(1 - q^2)K_1}{q - q^{-1}} = -qK_1$$

as necessary. Now suppose that the formula holds for some $i \leq N-2$. Inductively, using the iterative formula in Lemma 5.18, we know that

$$\pi(\Omega_{i+1})\big|_{K_{i+1}\to qK_{i+1}} = \Big(\pi(\Omega_i) - q\pi(\Omega_i)\big|_{K_i\to qK_i}\Big)\frac{qK_{i+1} - q^{-1}K_{i+1}^{-1}}{q-q^{-1}} - q^{-2}\pi(\Omega_i)K_{i+1}^{-1}$$

and, hence, we calculate

$$\begin{aligned}\pi(\Omega_{i+1}) - q\pi(\Omega_{i+1})\big|_{K_{i+1}\to qK_{i+1}} &= \Big(\pi(\Omega_i) - q\pi(\Omega_i)\big|_{K_i\to qK_i}\Big)\frac{(1-q^2)K_{i+1}}{q-q^{-1}} \\ &= \big((-1)^i q^i K_1\cdots K_i\big)\cdot(-qK_{i+1}) \\ &= (-1)^{i+1}q^{i+1}K_1\cdots K_{i+1}\end{aligned}$$

as required. By the induction hypothesis, this completes the proof. □

By symmetry between $N-1$ and N, Lemma 5.18 gives

$$\pi(\Omega_N) = \Big(\pi(\Omega_{N-2}) - q\pi(\Omega_{N-2})\big|_{K_{N-2}\to qK_{N-2}}\Big)\frac{K_N - K_N^{-1}}{q-q^{-1}} - q^{-1}\pi(\Omega_{N-2})K_N^{-1} \tag{5.34}$$

and, moreover, Lemma 5.19 gives

$$\pi(\Omega_N) - q\pi(\Omega_N)\big|_{K_N\to qK_N} = (-1)^{N-2}q^N K_1\cdots K_{N-2}K_N.$$

The following Proposition will be useful later, particularly in the proof of Proposition 6.6.

Proposition 5.20. *The image of the pivot commutator Ω under the projection map π is given by the formula*

$$\pi(\Omega) = (-1)^{N-1}c_1q^{N-2}\frac{K_{N-1}^{-1}K_N - K_{N-1}K_N^{-1}}{q-q^{-1}}.$$

Proof. Following Lemma 5.18 and formula (5.34), for each $j \in \{N-1, N\}$, observe that

$$\pi\big(\Omega_j\big)K_{\tau(j)}^{-1} - \pi\big(\Omega_{\tau(j)}\big)K_j^{-1} = \Big(\pi(\Omega_{N-2}) - q\pi(\Omega_{N-2})\big|_{K_{N-2}\to qK_{N-2}}\Big)\frac{K_jK_{\tau(j)}^{-1} - K_{\tau(j)}K_j^{-1}}{q-q^{-1}}.$$

Hence, applying Lemmas 5.17 and 5.19, we calculate

$$\begin{aligned}\pi(\Omega) &= c_1\Big(\pi(\Omega_{N-1})K_N^{-1} - \pi(\Omega_N)K_{N-1}^{-1}\Big)K_1^{-1}\cdots K_{N-2}^{-1} \\ &= c_1\Big(\pi(\Omega_{N-2}) - q\pi(\Omega_{N-2})\big|_{K_{N-2}\to qK_{N-2}}\Big)\frac{K_{N-1}K_N^{-1} - K_NK_{N-1}^{-1}}{q-q^{-1}}K_1^{-1}\cdots K_{N-2}^{-1} \\ &= -c_1\Big((-1)^{N-2}q^{N-2}K_1\cdots K_{N-2}\Big)\frac{K_NK_{N-1}^{-1} - K_{N-1}K_N^{-1}}{q-q^{-1}}\big(K_1\cdots K_{N-2}\big)^{-1} \\ &= (-1)^{N-1}c_1q^{N-2}\frac{K_{N-1}^{-1}K_N - K_{N-1}K_N^{-1}}{q-q^{-1}}\end{aligned}$$

as required. □

We now end this section with an example which gives an explicit computation of the pivot commutator Ω for small dimension, specifically in the algebra $\mathcal{B}_{\mathbf{c}}$ when $N = 4$.

Example 5.21. Consider the coideal subalgebra $\mathcal{B}_{\mathbf{c}}$ of $U_q(\mathfrak{so}_8)$ which is generated by the elements E_j, F_j, $K_j^{\pm 1}$ for $j \in \{2,3,4\}$ and the element (5.1)

$$B_1 = F_1 - c_1 T_2 T_4 T_3 T_2(E_1) K_1^{-1}.$$

Taking the simple root vectors (5.19) and (5.20), the pivot commutator (5.32) of $\mathcal{B}_{\mathbf{c}}$ is defined

$$\Omega = \big[T_3^{-1}T_2^{-1}(B_1), T_4^{-1}T_2^{-1}(B_1)\big]$$

which, by Lemma 5.17, can be given in terms of commutators of the form (5.33) by

$$\Omega = c_1\Big(\big[T_3T_2(E_1), T_3^{-1}T_2^{-1}(F_1)\big]K_4^{-1} - \big[T_4T_2(E_1), T_4^{-1}T_2^{-1}(F_1)\big]K_3^{-1}\Big)K_1^{-1}K_2^{-1}.$$

For each $i \in \{3,4\}$, observe that

$$\begin{aligned}
\big[T_i^{-1}T_2^{-1}(F_1), T_iT_2(E_1)\big] &= q^{-1}\Big((q-q^{-1})\big[T_i^{-1}T_2^{-1}(F_1), E_iT_2(E_1)\big] - T_i^{-1}\big(\big[T_2^{-1}(F_1), T_2(E_1)\big]\big)\Big)\\
&= q^{-2}\Bigg((q-q^{-1})\Big((q-q^{-1})\big[T_i^{-1}T_2^{-1}(F_1), E_iE_2E_1\big] - \big[T_i^{-1}T_2^{-1}(F_1), E_iT_2^{-1}(E_1)\big]\Big)\\
&\qquad - T_i^{-1}\Big((q-q^{-1})\big[T_2^{-1}(F_1), E_2E_1\big] - T_2^{-1}\big(\big[F_1, E_1\big]\big)\Big)\Bigg)\\
&= q^{-2}\Big(\Psi_i^{(1)} - (q-q^{-1})\Big(\Psi_i^{(2)} + \Psi_i^{(3)} - (q-q^{-1})\Psi_i^{(4)}\Big)\Big)
\end{aligned}$$

where

$$\begin{aligned}
\Psi_i^{(1)} &= T_i^{-1}T_2^{-1}\big([F_1, E_1]\big), & \Psi_i^{(3)} &= \big[T_i^{-1}T_2^{-1}(F_1), E_iT_2^{-1}(E_1)\big],\\
\Psi_i^{(2)} &= T_i^{-1}\Big(\big[T_2^{-1}(F_1), E_2E_1\big]\Big), & \Psi_i^{(4)} &= \big[T_i^{-1}T_2^{-1}(F_1), E_iE_2E_1\big].
\end{aligned}$$

Now, if

$$\begin{aligned}
\Psi^{(1)} &= q^{-2}\Big(\Psi_4^{(1)}K_3^{-1} - \Psi_3^{(1)}K_4^{-1}\Big)K_1^{-1}K_2^{-1},\\
\Psi^{(2)} &= q^{-2}(q-q^{-1})\Big(\Psi_3^{(2)}K_4^{-1} - \Psi_4^{(2)}K_3^{-1}\Big)K_1^{-1}K_2^{-1},\\
\Psi^{(3)} &= q^{-2}(q-q^{-1})\Big(\Psi_3^{(3)}K_4^{-1} - \Psi_4^{(3)}K_3^{-1}\Big)K_1^{-1}K_2^{-1},\\
\Psi^{(4)} &= q^{-2}(q-q^{-1})^2\Big(\Psi_4^{(4)}K_3^{-1} - \Psi_3^{(4)}K_4^{-1}\Big)K_1^{-1}K_2^{-1},
\end{aligned}$$

then

$$\Omega = c_1\sum_{k=1}^{4}\Psi^{(k)}. \tag{5.35}$$

To compute Ω explicitly, we consider each $\Psi^{(k)}$ individually. Firstly, for each $i \in \{3,4\}$ we have

$$\Psi_i^{(1)} = T_i^{-1}T_2^{-1}\big([F_1, E_1]\big) = \frac{T_i^{-1}T_2^{-1}(K_1^{-1} - K_1)}{q-q^{-1}} = \frac{K_i^{-1}K_2^{-1}K_1^{-1} - K_iK_2K_1}{q-q^{-1}}.$$

and therefore

$$\begin{aligned}
\Psi^{(1)} &= q^{-2}\left(\frac{\big(K_4^{-1}K_2^{-1}K_1^{-1} - K_4K_2K_1\big)K_3^{-1} - \big(K_3^{-1}K_2^{-1}K_1^{-1} - K_3K_2K_1\big)K_4^{-1}}{q-q^{-1}}\right)K_1^{-1}K_2^{-1}\\
&= q^{-2}\left(\frac{K_3K_4^{-1} - K_3^{-1}K_4}{q-q^{-1}}\right).
\end{aligned}$$

Next, one can check the commutation relation

$$\big[T_2^{-1}(F_1), E_2E_1\big] = q^2F_2K_1E_2 + \frac{q(K_2 - K_2^{-1})K_1}{q-q^{-1}} - F_1K_2^{-1}E_1$$

and using this, for each $i \in \{3,4\}$ we have

$$\begin{aligned}\Psi_i^{(2)} &= T_i^{-1}\Big(\big[T_2^{-1}(F_1), E_2E_1\big]\Big)\\ &= q^2T_i^{-1}(F_2)K_1T_i^{-1}(E_2) + \frac{q(K_iK_2 - K_i^{-1}K_2^{-1})K_1}{q-q^{-1}} - F_1K_i^{-1}K_2^{-1}E_1.\end{aligned}$$

Then,

$$\begin{aligned}\Psi^{(2)} &= (q-q^{-1})\big(T_3^{-1}(F_2)K_1T_3^{-1}(E_2)K_4^{-1} - T_4^{-1}(F_2)K_1T_4^{-1}(E_2)K_3^{-1}\big)K_1^{-1}K_2^{-1}\\ &\quad + q^{-1}\big((K_3K_2 - K_3^{-1}K_2^{-1})K_1K_4^{-1} - (K_4K_2 - K_4^{-1}K_2^{-1})K_1K_3^{-1}\big)K_1^{-1}K_2^{-1}\\ &\quad - q^{-2}(q-q^{-1})\big(\underbrace{F_1K_3^{-1}K_2^{-1}E_1K_4^{-1} - F_1K_4^{-1}K_2^{-1}E_1K_3^{-1}}_{=0}\big)K_1^{-1}K_2^{-1}\end{aligned}$$

$$\begin{aligned}&= q^{-1}\Big((q-q^{-1})\big(T_3^{-1}(F_2)K_2^{-1}K_4^{-1}T_3^{-1}(E_2) - T_4^{-1}(F_2)K_2^{-1}K_3^{-1}T_4^{-1}(E_2)\big) + K_3K_4^{-1} - K_3^{-1}K_4\Big)\\ &= (1-q^{-2})\bigg(\frac{K_3K_4^{-1} - K_3^{-1}K_4}{q-q^{-1}} + T_3^{-1}(F_2)K_2^{-1}K_4^{-1}T_3^{-1}(E_2) - T_4^{-1}(F_2)K_2^{-1}K_3^{-1}T_4^{-1}(E_2)\bigg),\end{aligned}$$

and similarly the remaining components can be calculated as

$$\Psi^{(3)} = (1-q^{-2})\bigg(\frac{K_3K_4^{-1} - K_3^{-1}K_4}{q-q^{-1}} + F_3K_4^{-1}E_3 - F_4K_3^{-1}E_4\bigg)$$

and

$$\begin{aligned}\Psi^{(4)} = (q^2 - 2 + q^{-2})\bigg(&\frac{K_3K_4^{-1} - K_3^{-1}K_4}{q-q^{-1}} + \Big(F_3K_4^{-1}E_3 - F_4K_3^{-1}E_4\Big)\\ &+ \Big(T_4^{-1}(F_2)K_2^{-1}K_3^{-1}E_2E_4 - T_3^{-1}(F_2)K_2^{-1}K_4^{-1}E_2E_3\Big)\\ &+ \Big(T_3^{-1}(F_2)K_2^{-1}K_4^{-1}T_3^{-1}(E_2) - T_4^{-1}(F_2)K_2^{-1}K_3^{-1}T_4^{-1}(E_2)\Big)\bigg).\end{aligned}$$

Hence, using (5.35) we compute

$$\begin{aligned}\Omega &= \big(q^{-2} + 2(1-q^{-2}) + (q^2 - 2 + q^{-2})\big)\bigg(\frac{K_3K_4^{-1} - K_3^{-1}K_4}{q-q^{-1}}\bigg)\\ &\quad + \big((1-q^{-2}) + (q^2-2+q^{-2})\big)\big(T_3^{-1}(F_2)K_2^{-1}K_4^{-1}T_3^{-1}(E_2) - T_4^{-1}(F_2)K_2^{-1}K_3^{-1}T_4^{-1}(E_2)\big)\\ &\quad + \big((1-q^{-2}) + (q^2-2+q^{-2})\big)\big(F_3K_4^{-1}E_3 - F_4K_3^{-1}E_4\big)\\ &\quad + (q^2-2+q^{-2})\big(T_4^{-1}(F_2)K_2^{-1}K_3^{-1}E_2E_4 - T_3^{-1}(F_2)K_2^{-1}K_4^{-1}E_2E_3\big)\end{aligned}$$

which simplifies to

$$\begin{aligned}\Omega &= q^2\bigg(\frac{K_3K_4^{-1} - K_3^{-1}K_4}{q-q^{-1}}\bigg) + q(q-q^{-1})\big(F_3K_4^{-1}E_3 - F_4K_3^{-1}E_4\big)\\ &\quad + q(q-q^{-1})\big(T_3^{-1}(F_2)K_2^{-1}K_4^{-1}T_3^{-1}(E_2) - T_4^{-1}(F_2)K_2^{-1}K_3^{-1}T_4^{-1}(E_2)\big)\\ &\quad + (q-q^{-1})^2\big(T_4^{-1}(F_2)K_2^{-1}K_3^{-1}E_2E_4 - T_3^{-1}(F_2)K_2^{-1}K_4^{-1}E_2E_3\big).\end{aligned}$$

Chapter 6

Representation Theory of $\mathcal{B}_{\mathbf{c}}$

We now have all of the ingredients to perform the classification of finite-dimensional simple modules for the algebra $\mathcal{B}_{\mathbf{c}}$. To do this, we will define the Verma module and show that it has a highest weight and is dominant integral for simple quotients to be finite-dimensional. Ultimately, we prove Theorem 6.18 using a filtered-graded argument in Section 6.5.

6.1 Highest weight vectors

Let V be a finite-dimensional $\mathcal{B}_{\mathbf{c}}$-module, and denote the weight lattice for the Lie algebra $\mathfrak{so}_{2N-1}$ by Λ_{2N-1}. We begin by formulating the notion of a weight vector for the algebra $\mathcal{B}_{\mathbf{c}}$.

Definition 6.1. *For any weight $\lambda \in \Lambda_{2N-1}$, the corresponding* ***weight space*** *of V is the subspace*

$$V_\lambda = \left\{ v \in V \,\middle|\, K_{\eta(h)} v = q^{\lambda(h)} v, \ \forall\, h \in Q^\vee_{2N-1} \right\} \subseteq V \tag{6.1}$$

and any $v \in V_\lambda \setminus \{0\}$ is called a ***weight vector*** *of weight λ.*

Remark 6.2. Recall the definition of a weight space (4.14) for a $\mathcal{U}$-module in Section 4.1, in comparison to Definition 6.1 for a $\mathcal{B}_{\mathbf{c}}$-module. Notice that in (6.1) we have assumed that the elements $K_{\eta(h)}$ act *only* as $q^{\lambda(h)}$ on V, and *not* as $-q^{\lambda(h)}$. In general, for any $\mathcal{B}_{\mathbf{c}}$-module V and group homomorphisms $\sigma\colon Q^\vee_{2N-1} \to \{\pm 1\}$, we may actually write

$$V_{\lambda,\sigma} = \left\{ v \in V \,\middle|\, K_{\eta(h)} v = \sigma(h) q^{\lambda(h)} v, \ \forall\, h \in Q^\vee_{2N-1} \right\} \subseteq V$$

for the corresponding weight space of any weight $\lambda \in \Lambda_{2N-1}$. One may show that, for any finite-dimensional $\mathcal{B}_{\mathbf{c}}$-module V, the sum of all of the weight spaces $V_{\lambda,\sigma}$ is direct. Then, we say that a $\mathcal{B}_{\mathbf{c}}$-module V is of **type-σ** if $V = \bigoplus_{\lambda \in \Lambda_{2N-1}} V_{\lambda,\sigma}$. Moreover, we say that a $\mathcal{B}_{\mathbf{c}}$-module V is of **type-1** if V is of type-σ and, additionally, $\sigma(h) = 1$ for all $h \in Q^\vee_{2N-1}$.

In Definition 6.1, we assume that all $\mathcal{B}_{\mathbf{c}}$-modules are of type-**1**, in which case we write $V_{\lambda,\sigma} = V_\lambda$. For convention, we disregard modules whose weight spaces are defined so that $\sigma(h) = -1$ for some $h \in Q^\vee_{2N-1}$, as the classification of these modules is similar to type-**1** modules. Indeed, we can write

$$V = \bigoplus_{\lambda \in \Lambda_{2N-1}} V_\lambda. \tag{6.2}$$

For justification, V can be regarded as a $U_q(\mathfrak{sl}_2)$-module, and there is a $U_q(\mathfrak{sl}_2)$-triple for each $K_{\eta(h)}$ which acts as some $q^{\lambda(h)}$ on V for $h \in Q^\vee_{2N-1}$, and so can be diagonalised. The other $K_{\eta(h)}$ commute, so are simultaneously diagonalisable on V. Therefore, V must have a basis which consists of weight vectors. The group algebra of the coroot lattice of $\mathfrak{so}_{2N-1}$ is just the subalgebra generated by the $K_i^{\pm 1}$ for $i \in I$.

Lemma 6.3. *Let V be any $\mathcal{B}_{\mathbf{c}}$-module. For each $i \in \{2, \ldots, N-1\}$, and for all $\lambda \in \Lambda_{2N-1}$, we have the inclusions*

$$E_i V_\lambda \subset V_{\lambda+\gamma_{i-1}}, \qquad B_{\beta_{N-1}} V_\lambda \subset V_{\lambda+\gamma_{N-1}}, \tag{6.3}$$

$$F_i V_\lambda \subset V_{\lambda-\gamma_{i-1}}, \qquad B_{\beta_N} V_\lambda \subset V_{\lambda-\gamma_{N-1}}, \tag{6.4}$$

and, futhermore,

$$E_N V_\lambda \subset V_{\lambda+\gamma_{N-2}+2\gamma_{N-1}}, \tag{6.5}$$

$$F_N V_\lambda \subset V_{\lambda-\gamma_{N-2}-2\gamma_{N-1}}. \tag{6.6}$$

Proof. The inclusions (6.3)-(6.6) follow directly from relations (5.21)-(5.24) in Proposition 5.7, respectively. Indeed, for all $h \in Q^\vee_{2N-1}$ and $\lambda \in \Lambda_{2N-1}$, and for each $i \in \{2, \ldots, N-1\}$, we have

$$K_{\eta(h)} E_i V_\lambda = q^{\gamma_{i-1}(h)} E_i K_{\eta(h)} V_\lambda = q^{(\lambda+\gamma_{i-1})(h)} E_i V_\lambda$$

using (6.1) in Definition 6.1 of the weight space V_λ. Hence, by definition, this implies that $E_i V_\lambda \subset V_{\lambda+\gamma_{i-1}}$. The remaining inclusions are shown using a similar calculation. □

We may now use Lemma 6.3 to prove the existence of a so-called *highest weight vector* in V.

Proposition 6.4. *Let V be a finite-dimensional $\mathcal{B}_{\mathbf{c}}$-module. There exists a weight vector $v \in V$ such that, for all $i \in \{2, \ldots, N\}$, we have*

$$E_i v = 0 = B_{\beta_{N-1}} v.$$

Proof. Suppose that V is a finite-dimensional $\mathcal{B}_{\mathbf{c}}$-module. By (6.2), we know that V can be written as a direct sum of its weight spaces V_λ for $\lambda \in \Lambda_{2N-1}$. Furthermore, the set of λ with $V_\lambda \neq \{0\}$ is finite. This means that for $V \neq \{0\}$, there exists some weight $\lambda \in \Lambda_{2N-1}$ such that $V_\lambda \neq \{0\}$ and $V_{\lambda'} = \{0\}$ for all $\lambda' > \lambda$. In particular, for such λ, this implies that $V_{\lambda+\gamma} = \{0\}$ for all $\gamma \in \Pi$.

For any $\lambda \in \Lambda_{2N-1}$, we have $E_i V_\lambda \subset V_{\lambda+\gamma_{i-1}}$ for $i \in \{2, \ldots, N-1\}$, $B_{\beta_{N-1}} V_\lambda \subset V_{\lambda+\gamma_{N-1}}$, and $E_N V_\lambda \subset V_{\lambda+\gamma_{N-2}+2\gamma_{N-1}}$ by the inclusions (6.3) and (6.5) in Lemma 6.3. Therefore, there exists some $\lambda \in \Lambda_{2N-1}$ such that

$$E_i V_\lambda = \{0\} = B_{\beta_{N-1}} V_\lambda,$$

for all $i \in \{2, \ldots, N\}$ and, hence, $E_i v = 0 = B_{\beta_{N-1}} v$ for some weight vector $v \in V_\lambda \subset V$. □

Typically, for a highest weight vector, it suffices to know that the simple root vectors vanish. However, it is not obvious how to deduce, from the fact that $E_{N-1}v = 0$ and $B_{\beta_{N-1}}v = 0$, that $E_N v = 0$. Hence, we include (and prove) that E_N does indeed vanish on a highest weight vector in Proposition 6.4.

6.2 Induced $U_q(\mathfrak{sl}_2)$-module on an invariant subspace

In Section 4.2, we saw that $(E_i, F_i, K_i^{\pm 1})$ is a $U_q(\mathfrak{sl}_2)$-triple corresponding to the simple root γ_{i-1} for $i \in \{2, \dots, N-1\}$. A sensible question now would be to ask whether we also have such a triple for the remaining simple root γ_{N-1}. We will discover that this is indeed the case, however the $U_q(\mathfrak{sl}_2)$-triple for γ_{N-1} satisfies the $U_q(\mathfrak{sl}_2)$ relations only on certain subspaces of finite-dimensional $U_q(\mathfrak{so}_{2N})$-modules.

Let V be a finite-dimensional $\mathcal{B}_{\mathbf{c}}$-module, and consider the subspace

$$H(V) = \left\{ v \in V \,\middle|\, E_i v = 0 = B_{\beta_{N-2}} v,\ \forall i \in \{2, \dots, N\} \right\} \subset V. \tag{6.7}$$

Firstly, observe that $H(V) \neq \{0\}$ by Proposition 6.4, since $B_{\beta_{N-2}} = T_{N-1}(B_{\beta_{N-1}})$. Moreover, in the subspace (6.7), any vector $v \in H(V)$ is annihilated by all of the positive root vectors *except* for the root vector $B_{\beta_{N-1}}$ which corresponds to the simple root γ_{N-1}. Motivating the decision to consider the subspace $H(V)$, we now observe the action of both $B_{\beta_{N-1}}$ and its corresponding negative root vector B_{β_N} on $H(V)$.

Lemma 6.5. *The subspace $H(V)$ of a finite-dimensional $\mathcal{B}_{\mathbf{c}}$-module V, defined in* (6.7)*, is invariant under the root vectors $B_{\beta_{N-1}}$ and B_{β_N}.*

Proof. Let $j \in \{N-1, N\}$. Firstly, observe that for all $v \in H(V)$, we know that

$$B_{\beta_j} v = F_j \underbrace{B_{\beta_{N-2}} v}_{=0} - q B_{\beta_{N-2}} F_j v$$

and hence it suffices to check for each j that, for all $i \in \{2, \dots, N\}$, we get

$$E_i B_{\beta_{N-2}} F_j v = 0, \tag{6.8}$$

and

$$B_{\beta_{N-2}}^2 F_j v = 0. \tag{6.9}$$

We first consider the equations of the form (6.8). Notice that, for $i \in \{2, \dots, N\}$, we get the commutation relations

$$\left[E_i, B_{\beta_{N-2}}\right] = \begin{cases} q K_{N-2}^{-1} B_{\beta_{N-3}} & \text{if } i = N-2, \\ 0 & \text{if } i \neq N-2. \end{cases}$$

For the case $i = N-2$, this implies that

$$E_{N-2}B_{\beta_{N-2}}F_j v = B_{\beta_{N-2}}F_j \underbrace{E_{N-2}v}_{=0} + qK_{N-2}^{-1}B_{\beta_{N-3}}F_j v = qK_{N-2}^{-1}F_j \underbrace{B_{\beta_{N-3}}v}_{=0} = 0$$

and similarly, if $i \leq N-3$, then

$$E_i B_{\beta_{N-2}}F_j v = B_{\beta_{N-2}}F_j \underbrace{E_i v}_{=0} = 0.$$

Now, if we suppose that $i = j \in \{N-1, N\}$, then

$$\begin{aligned} E_j B_{\beta_{N-2}}F_j v &= B_{\beta_{N-2}}E_j F_j v \\ &= B_{\beta_{N-2}}F_j \underbrace{E_j v}_{=0} + B_{\beta_{N-2}}\left(\frac{K_j - K_j^{-1}}{q-q^{-1}}\right)v \\ &= \left(\frac{q^{-1}K_j - qK_j^{-1}}{q-q^{-1}}\right)\underbrace{B_{\beta_{N-2}}v}_{=0} = 0. \end{aligned}$$

It remains to consider the equations of the form (6.9). For each $j \in \{N-1, N\}$, it follows from Corollary 5.13 that $\left[B_{\beta_{N-2}}, B_{\beta_j}\right]_q v = 0$. Therefore,

$$\begin{aligned} B_{\beta_{N-2}}^2 F_j v &= B_{\beta_{N-2}}\big(B_{\beta_{N-2}}F_j v\big) \\ &= q^{-1}B_{\beta_{N-2}}\big(F_j \underbrace{B_{\beta_{N-2}}v}_{=0} - B_{\beta_j}v\big) \\ &= -B_{\beta_j}\underbrace{B_{\beta_{N-2}}v}_{=0} - q^{-1}\underbrace{[B_{\beta_{N-2}}, B_{\beta_j}]_q v}_{=0} = 0. \end{aligned}$$

Hence, we have shown that $B_{\beta_j} \cdot H(V) \subseteq H(V)$ for $j \in \{N-1, N\}$ as necessary. □

The following result shows that, for suitable coefficients $a_{\mathcal{X}}, a_{\mathcal{Y}} \in \mathbb{K}(q)^{\times}$, the operators

$$\mathcal{X} = a_{\mathcal{X}}B_{\beta_{N-1}} \qquad \text{and} \qquad \mathcal{Y} = a_{\mathcal{Y}}B_{\beta_N} \tag{6.10}$$

provide a representation of $U_q\big(\mathfrak{sl}_2\big)$ on the subspace $H(V) \subset V$.

Proposition 6.6. *Choose the operators $\mathcal{X}$ and $\mathcal{Y}$ in* (6.10) *such that $a_{\mathcal{X}}a_{\mathcal{Y}} = (-1)^{N-1}c_1^{-1}q^{2-N}$. Then, the subspace $H(V)$ carries a representation of $U_q\big(\mathfrak{sl}_2\big)$ via $\mathcal{X}$ and $\mathcal{Y}$ by mapping*

$$E \mapsto \mathcal{X}; \quad F \mapsto \mathcal{Y}; \quad K \mapsto K_{N-1}^{-1}K_N.$$

Proof. Recall the pivot commutator (5.32) which is denoted by

$$\Omega = \left[B_{\beta_{N-1}}, B_{\beta_N}\right]$$

in Section 5.5. We saw that $\Omega \in \mathcal{B}_{\mathbf{c}} \cap \mathcal{U}_0 = (\mathcal{M}_X)_0$, meaning that the terms of the commutator Ω have weight 0 and lie in the algebra $\mathcal{M}_X$. Moreover, we know that this algebra exhibits a triangular decomposition, $\mathcal{M}_X \cong \mathcal{M}_X^- \otimes \mathcal{M}_X^0 \otimes \mathcal{M}_X^+$. Let π denote the projection map onto the subalgebra $\mathcal{M}_X^0$. Then, for all $v \in H$, we see that

$$\Omega v = \pi(\Omega)v$$

and, moreover, we have an explicit formula for $\pi(\Omega)$, see Proposition 5.20. Now observe that $\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X} = (-1)^{N-1}c_1^{-1}q^{2-N}\Omega$. Putting this all together, we get

$$\begin{aligned}(\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X})v &= (-1)^{N-1}c_1^{-1}q^{2-N}\pi(\Omega)v \\ &= \left(\frac{K_{N-1}^{-1}K_N - K_{N-1}K_N^{-1}}{q-q^{-1}}\right)v\end{aligned}$$

and this corresponds exactly to the $U_q(\mathfrak{sl}_2)$-relation $(U4')$ in Section 4.2, namely

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

It remains to check the $U_q(\mathfrak{sl}_2)$-relations $(U1')$-$(U3')$ which we recall are given by

$$KK^{-1} = 1 = K^{-1}K, \qquad KEK^{-1} = q^2E, \qquad KFK^{-1} = q^{-2}F.$$

For the relation $(U1')$, we immediately see that

$$K_{N-1}^{-1}K_N(K_{N-1}K_N^{-1}) = (K_{N-1}^{-1}K_{N-1})(K_NK_N^{-1}) = 1$$

and similarly one gets the other side. Finally, for the relations $(U2')$ and $(U3')$, we deduce respectively from

$$\left[B_{\beta_{N-1}}, K_{N-1}\right]_q = 0 = \left[B_{\beta_{N-1}}, K_N\right]_{q^{-1}}$$

and

$$\left[B_{\beta_N}, K_{N-1}\right]_{q^{-1}} = 0 = \left[B_{\beta_N}, K_N\right]_q$$

that

$$(K_{N-1}^{-1}K_N)\mathcal{X}(K_{N-1}K_N^{-1}) = q^2\mathcal{X} \quad \text{and} \quad (K_{N-1}^{-1}K_N)\mathcal{Y}(K_{N-1}K_N^{-1}) = q^{-2}\mathcal{Y}$$

and we are done. □

Theorem 6.7. *The weight $\lambda \in \Lambda_{2N-1}$ of a highest weight vector in any finite-dimensional simple $\mathcal{B}_{\mathbf{c}}$-module is dominant integral, that is, $\lambda(h_{\gamma_i}) \geq 0$ for $1 \leq i \leq N-1$.*

Proof. Recall the representation theory for $U_q(\mathfrak{sl}_2)$ in Section 4.2. By the first formula in Theorem 4.8, we have that

$$\lambda(h_{\gamma_i}) \geq 0$$

for $1 \leq i \leq N-2$. For the root γ_{N-1}, Proposition 6.6 now implies that $\lambda(h_{\gamma_{N-1}}) \geq 0$ and we are done. □

By Theorem 6.7, we have that each finite-dimensional simple $\mathcal{B}_{\mathbf{c}}$-module has a highest weight vector of dominant integral weight. Conversely, we would now like to show that, for each dominant integral weight, there exists a finite-dimensional irreducible $\mathcal{B}_{\mathbf{c}}$-module.

6.3 Constructing Verma modules

We now want to construct a representation of the algebra $\mathcal{B}_{\mathbf{c}}$ that is generated by a single non-zero vector of highest weight. One particular such highest-weight module is the *Verma module*, that is, the one that is maximal in the sense that every other highest-weight module with the same highest weight is a quotient of the Verma module. Guided by the classification of finite-dimensional irreducible representations of quantised enveloping algebras $U_q(\mathfrak{g})$ in [Jan96, Chapter 5], we now construct a Verma module for $\mathcal{B}_{\mathbf{c}}$.

Let $\lambda \in \Lambda_{2N-1}$. For all $i \in \{2, \ldots, N-1\}$, we now denote the integers

$$n_i = \lambda(h_{\gamma_{i-1}}), \qquad \text{and} \qquad m_N = \lambda(h_{\gamma_{N-1}}). \tag{6.11}$$

We consider the left ideal of $\mathcal{B}_{\mathbf{c}}$ generated by the elements E_i and $K_i - q^{n_i}$ for $i \in \{2, \ldots, N-1\}$, and also the elements $B_{\beta_{N-1}}$ and $K_{N-1}^{-1} K_N - q^{m_N}$. More precisely, this is the left ideal

$$I_\lambda := \left(\sum_{i=2}^{N-1} \mathcal{B}_{\mathbf{c}} E_i + \mathcal{B}_{\mathbf{c}} \Big(K_i - q^{n_i} \Big) \right) + \mathcal{B}_{\mathbf{c}} B_{\beta_{N-1}} + \mathcal{B}_{\mathbf{c}} \Big(K_{N-1}^{-1} K_N - q^{m_N} \Big). \tag{6.12}$$

It is not clear from (6.12) if the ideal I_λ contains the root vector E_N, however the rest of the positive root vectors are indeed contained in I_λ. Since we want to construct a Verma module and hence a highest-weight module, it would be desirable to extend this left ideal so that the element E_N is also contained in the ideal. One sees that this is entirely possible, since by Proposition 6.4 every finite-dimensional $\mathcal{B}_{\mathbf{c}}$-module has a highest weight vector. In addition to the integers in (5.11), let us denote the integer

$$n_N := n_{N-1} + m_N. \tag{6.13}$$

Then, we may consider a more unnatural left ideal of $\mathcal{B}_{\mathbf{c}}$, written as

$$J_\lambda := \left(\sum_{i=2}^{N} \mathcal{B}_{\mathbf{c}} E_i + \mathcal{B}_{\mathbf{c}} \Big(K_i - q^{n_i} \Big) \right) + \mathcal{B}_{\mathbf{c}} B_{\beta_{N-1}}. \tag{6.14}$$

Observe that the ideal J_λ is essentially the same as the ideal I_λ given in (5.12) with the addition of the element E_N as a generator (and notice that, for neatness, we also replace the generator involving the $K_{N-1}^{-1} K_N$ with that of the K_N). We may consider J_λ as a left $\mathcal{B}_{\mathbf{c}}$-module.

Definition 6.8. *The* ***Verma module*** *(or* ***universal highest weight module****) of highest weight $\lambda \in \Lambda_{2N-1}$ is the quotient module*

$$M(\lambda) := \frac{\mathcal{B}_{\mathbf{c}}}{J_\lambda}.$$

Since J_λ is a left ideal, the natural left action of $\mathcal{B}_{\mathbf{c}}$ on itself carries over to the quotient. Therefore, the Verma module $M(\lambda)$ is a $\mathcal{B}_{\mathbf{c}}$-module. In fact, $M(\lambda)$ is an infinite-dimensional $\mathcal{B}_{\mathbf{c}}$-module generated by the coset of 1; we denote this coset by $v_\lambda := 1 + J_\lambda$. By definition, we have

$$E_i v_\lambda = 0 \qquad \text{and} \qquad K_i v_\lambda = q^{n_i} v_\lambda \tag{6.15}$$

for all $i \in \{2, \dots, N\}$, and

$$B_{\beta_{N-1}} v_\lambda = 0 \qquad \text{and} \qquad K_{N-1}^{-1} K_N v_\lambda = q^{m_N} v_\lambda. \tag{6.16}$$

The Verma module $M(\lambda)$ should be spanned by elements obtained by "lowering" v_λ by the action of the negative root vectors. In particular, the weights of $M(\lambda)$ will consist only of those that can be obtained from the highest weight λ by subtracting integer combinations of positive roots. If we choose an ordering of all the roots, and hence of the negative root vectors of $\mathcal{B}_{\mathbf{c}}$, then by the PBW-Theorem for $\mathcal{B}_{\mathbf{c}}$ one can show that every element of $M(\lambda)$ can be written as a linear combination of ordered monomials in the negative root vectors on the highest weight vector v_λ.

Theorem 6.9 (PBW-Theorem for $M(\lambda)$)**.** *The ordered monomials*

$$F_{\beta_{N(N-1)}}^{j_{N(N-1)}} \cdots F_{\beta_{2N-1}}^{j_{2N-1}} B_{\beta_{2(N-1)}}^{j_{2(N-1)}} \cdots B_{\beta_N}^{j_N} \tag{6.17}$$

where $(j_N, \dots, j_t) \in \mathbb{N}_0^{(N-1)^2}$ *form a basis of* $M(\lambda)$ *on* v_λ*.*

Proof. The PBW-Theorem for $\mathcal{B}_{\mathbf{c}}$ (Theorem 5.11) tells us that the algebra $\mathcal{B}_{\mathbf{c}}$ has a basis of PBW-type monomials of the form $B_{\mathcal{J}} K_{\mathcal{D}} E_{\mathcal{I}}$ for appropriate multi-indices $\mathcal{J}$, $\mathcal{D}$ and $\mathcal{I}$. We want to show that the ordered monomials (6.17) are *not* elements of the left ideal J_λ. Equivalently, it suffices to show that any element of J_λ rewritten in terms of the PBW-basis of $\mathcal{B}_{\mathbf{c}}$ cannot be of the form in (6.17), and therefore must have at least one positive root vector on the right-hand side. To do this, let us assume that such a monomial is written as a linear combination of terms in J_λ. For any element $b \in \mathcal{B}_{\mathbf{c}}$, terms of the form bE_i for $i \in \{2, \dots, N\}$ are already given as PBW-type monomials (subject to commuting the E_i with themselves) so will maintain a positive root vector on the right-hand side. Similarly, terms of the form $b(K_i - q^{n_i})$ are also suitable since one can always bring any E_i to the right (and the K_i commute with themselves).

It remains to consider the final term of the form $b\,B_{\beta_{N-1}}$ rewritten in the PBW-basis, for any $b \in \mathcal{B}_{\mathbf{c}}$. Firstly, observe that the $K_{\mathcal{D}} E_{\mathcal{I}}$-part of $b \in \mathcal{B}_{\mathbf{c}}$ always (q-)commutes with the element B_1, and hence bB_1 in the PBW-basis is always an element of J_λ. Inductively, now suppose that, for some fixed $i \in \{2, \dots, N-1\}$, we have shown that $K_{\mathcal{D}} E_{\mathcal{I}} B_{\beta_{i-1}}$ in the PBW-basis is an element of J_λ. Then, $K_{\mathcal{D}} E_{\mathcal{I}}$ also moves to the right-hand side of B_{β_i}, but if it contains the element E_i we must use the commutation relation

$$\left[E_i, B_{\beta_i}\right] = B_{\beta_{i-1}} K_i^{-1}.$$

Therefore, it is now necessary to consider terms of the form $B_{\mathcal{J}} B_{\beta_i}$ for all $i \in \{2, \dots, N-1\}$. It suffices to look at the monomials $B_{\beta_k} B_{\beta_i}$ for $k < i$. Ultimately, we want to show that, for $2 \le k < i \le N-1$, the commutator

$$\left[B_{\beta_k}, B_{\beta_i}\right]_q = T_{k,\dots,2}^{-1} T_{i,\dots,3}^{-1} \left[B_1, T_2^{-1}(B_1)\right]_q$$

involves some positive root vectors on the right-hand side when rewritten in the PBW-basis. Recall the element $\mathcal{Z}_1 = r_1\big(T_{w_X}(E_1)\big)$ from (5.3) in Section 5.1. By the properties of the Lusztig actions in Section 4.3, we know that $T_{w_X}(E_1) \in \mathcal{U}^+_{w_X(\alpha_1)}$, where we calculate

$$w_X(\alpha_1) = \sigma_2(\alpha_1) = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{N-2}) + \alpha_{N-1} + \alpha_N,$$

and, moreover, in (5.8) we saw that $T_{w_X}(E_1) = T_{2,\ldots,N}T_{N-2,\ldots,2}(E_1)$. Therefore, we deduce that

$$\mathcal{Z}_1 \in \mathcal{U}^+_{w_X(\alpha_1)-\alpha_1} \quad \text{and} \quad r_2(\mathcal{Z}_1), {}_2r(\mathcal{Z}_1) \in \mathcal{U}^+_{w_X(\alpha_1)-\alpha_1-\alpha_2}.$$

Returning to the commutator in question, firstly observe that

$$\big[B_1, T_2^{-1}(B_1)\big]_q \in \mathcal{U}^-\mathcal{Z}_1 + \mathcal{U}^0\, r_2(\mathcal{Z}_1) + \mathcal{U}^0\, {}_2r(\mathcal{Z}_1).$$

Hence the commutator $T^{-1}_{k,\ldots,2}T^{-1}_{i,\ldots,3}\big[B_1, T_2^{-1}(B_1)\big]_q$ must involve terms where the right-hand side is either an element of $\mathcal{M}_{X,\mu}$ where

$$\begin{aligned}\mu = s_k\cdots s_2 s_i\cdots s_3\big(w_X(\alpha_1)-\alpha_1\big) &= w_X\big(s_k\cdots s_2(\alpha_1)\big) - s_k\cdots s_2(\alpha_1)\\ &= w_X(\alpha_1+\cdots+\alpha_k) - (\alpha_1+\cdots+\alpha_k)\\ &= w_X(\alpha_1) - \alpha_1 - 2(\alpha_2+\cdots+\alpha_k)\\ &= 2(\alpha_{k+1}+\cdots+\alpha_{N-2}) + \alpha_{N-1} + \alpha_N\end{aligned}$$

or an element of $\mathcal{M}_{X,\nu}$ where

$$\begin{aligned}\nu = s_k\cdots s_2 s_i\cdots s_3\big(w_X(\alpha_1)-\alpha_1-\alpha_2\big) &= \mu - s_k\cdots s_2 s_i\cdots s_3(\alpha_2)\\ &= \mu - (\alpha_{k+1}+\cdots+\alpha_i)\\ = \begin{cases}(\alpha_{k+1}+\cdots+\alpha_i) + 2(\alpha_{i+1}+\cdots+\alpha_{N-2}) + \alpha_{N-1} + \alpha_N & \text{if } i \le N-3\\ (\alpha_{k+1}+\cdots+\alpha_{N-2}) + \alpha_{N-1} + \alpha_N & \text{if } i = N-2\\ (\alpha_{k+1}+\cdots+\alpha_{N-2}) + \alpha_N & \text{if } i = N-1.\end{cases}\end{aligned}$$

Crucially, we see that both $\mu > 0$ and $\nu > 0$. This proves the statement. □

The ordered monomials (6.17) in Theorem 6.9 are equivalent to the elements in the set

$$\Big\{B_{\mathcal{J}} \mid \mathcal{J} \in \mathbb{N}_0^{N(N-1)},\, j_1,\ldots,j_{N-1} = 0\Big\}.$$

The Verma module $M(\lambda)$ of highest weight $\lambda \in \Lambda_{2N-1}$ has the following universal property.

Lemma 6.10 ([Jan96, 5.5])**.** *Let V be any $\mathcal{B}_{\mathbf{c}}$-module and let $v \in V_\lambda\backslash\{0\}$ be a weight vector such that $E_i v = 0$ for all $i \in \{2,\ldots,N\}$ and $B_{\beta_{N-1}}v = 0$. Then, there exists a unique homomorphism of $\mathcal{B}_{\mathbf{c}}$-modules*

$$\phi\colon M(\lambda) \to V; \qquad v_\lambda \mapsto v.$$

We have seen that, for every weight $\lambda \in \Lambda_{2N-1}$, we can construct a Verma module $M(\lambda)$ over $\mathcal{B}_{\mathbf{c}}$. Each submodule of $M(\lambda)$ is the direct sum of its weight spaces; a submodule of $M(\lambda)$ which is proper must be contained in the direct sum of all of the weight spaces $M(\lambda)_{\lambda-\mu}$ with $\mu \neq 0$.

This means that there exists a unique proper maximal submodule of $M(\lambda)$, which we denote by $N(\lambda)$. From this, we may define a simple quotient module for each Verma module $M(\lambda)$, namely the **unique simple factor module**

$$L(\lambda) \coloneqq \frac{M(\lambda)}{N(\lambda)}. \tag{6.18}$$

By Proposition 6.4 and the universal property of $M(\lambda)$ in Lemma 6.10, each finite-dimensional simple $\mathcal{B}_{\mathbf{c}}$-module is a homomorphic image of some $M(\lambda)$ and hence is isomorphic to some simple quotient module $L(\lambda)$. Note that the weight $\lambda \in \Lambda_{2N-1}$ must be unique, since it is the largest weight of the module. In order to complete the classification of all finite-dimensional simple $\mathcal{B}_{\mathbf{c}}$-modules, and thus prove Theorem 6.18 in Section 6.5, it remains for us to show that

$$\dim\big(L(\lambda)\big) < \infty \tag{6.19}$$

for any *dominant integral* weight $\lambda \in \Lambda_{2N-1}$ (by Theorem 6.7), that is, where $n_i \geq 0$ for all $i \in \{2, \dots, N\}$ and $m_N \geq 0$. This is our goal from now on.

6.4 Proper submodules of $M(\lambda)$

At this stage, it is unclear whether the unique simple factor module $L(\lambda)$ is finite-dimensional or not; it is difficult to tell if we have found the maximal submodule $N(\lambda)$ exactly. Instead, we can find an approximate of this module $L(\lambda)$ by constructing another sufficiently large submodule which is contained in $N(\lambda)$ and hence must be proper. By the way the Verma module $M(\lambda)$ is defined, this new proper submodule must be generated by monomials consisting only of the negative root vectors. Indeed, we first consider submodules of $M(\lambda)$ generated by powers of the root vectors corresponding to the negative simple roots.

Recall the integers $n_2, \dots, n_{N-1}$, and m_N defined in (6.11), which are non-negative since we assume now that the weight $\lambda \in \Lambda_{2N-1}$ is dominant integral.

Proposition 6.11 ([Jan96, Lemma 5.6])**.** *Let $\lambda \in \Lambda_{2N-1}$ be a dominant integral weight. Then, there exist $\mathcal{B}_{\mathbf{c}}$-module homomorphisms $\psi_i \colon M\big(\lambda - (n_i+1)\gamma_{i-1}\big) \to M(\lambda)$ for $i \in \{2, \dots, N-1\}$ such that*

$$v_{\lambda-(n_i+1)\gamma_{i-1}} \mapsto F_i^{n_i+1} v_\lambda,$$

and $\psi_{m_N} \colon M\big(\lambda - (m_N+1)\gamma_{N-1}\big) \to M(\lambda)$ such that

$$v_{\lambda-(m_N+1)\gamma_{N-1}} \mapsto B_{\beta_N}^{m_N+1} v_\lambda.$$

Proof. Let $\lambda \in \Lambda_{2N-1}$ be dominant integral. Then, since $n_i \geq 0$ for all $i \in \{2, \dots, N-1\}$, we have

$$F_i^{n_i+1} v_\lambda \in M(\lambda)_{\lambda-(n_i+1)\gamma_{i-1}}$$

for $v_\lambda \in M(\lambda)$. Hence, by the universal property of $M\big(\lambda - (n_i+1)\gamma_{i-1}\big)$ for each i, it is sufficient to show that

$$E_j F_i^{n_i+1} v_\lambda = 0 = B_{\beta_{N-1}} F_i^{n_i+1} v_\lambda \tag{6.20}$$

for all $j \in \{2, \dots, N\}$. Note that $E_j F_i^{n_i+1} v_\lambda = 0$ for all $j \neq i$, since E_j and F_i commute and $E_j v_\lambda = 0$ for all j. For $j = i$, using [Jan96, 4.4(6)] observe that

$$\begin{aligned} E_i F_i^{n_i+1} v_\lambda &= \Big(F_i^{n_i+1} E_i + [n_i+1]_q F_i^{n_i} \big[K_i; -n_i\big] \Big) v_\lambda \\ &= F_i^{n_i+1} \underbrace{E_i v_\lambda}_{=0} + \frac{[n_i+1]_q}{q - q^{-1}} F_i^{n_i} \big(q^{-n_i} K_i - q^{n_i} K_i^{-1} \big) v_\lambda \\ &= 0, \quad \text{since } K_i v_\lambda = q^{n_i} v_\lambda. \end{aligned}$$

For the right side of formula (6.20), we know that $\big[B_{\beta_{N-1}}, F_{N-1}\big]_q = 0$ and $\big[B_{\beta_{N-1}}, F_i\big] = 0$ when $i \neq N-1$. Since $B_{\beta_{N-1}} v_\lambda = 0$, this implies that, for all $i \in \{2, \dots, N-1\}$, we have

$$B_{\beta_{N-1}} F_i^{n_i+1} v_\lambda = 0.$$

Now, see that $m_N \geq 0$, again since $\lambda \in \Lambda$ is dominant integral. Then, similarly, for $v_\lambda \in M(\lambda)$ we have

$$B_{\beta_N}^{m_N+1} v_\lambda \in M(\lambda)_{\lambda - (m_N+1)\gamma_{N-1}}$$

This time, using the universal property of $M\big(\lambda - (m_N+1)\gamma_{N-1}\big)$, it is now sufficient to show that

$$E_j B_{\beta_N}^{m_N+1} v_\lambda = 0 = B_{\beta_{N-1}} B_{\beta_N}^{m_N+1} v_\lambda \tag{6.21}$$

for all $j \in \{2, \dots, N\}$. Note that E_j and B_{β_N} commute for $j \neq N$, whilst

$$\big[E_N, B_{\beta_N}\big] = q K_N^{-1} B_{\beta_{N-2}}.$$

We know that $E_j v_\lambda = 0$ for all j and $T_{N-2,\dots,2}^{-1}(B_1) v_\lambda = 0$. Therefore, using Corollary 5.13, it remains only for us to prove the right side of formula (6.21). However, it is necessary that we restrict to the subspace $H(V)$ here, which we defined in (6.7). This is so we can use Proposition 6.6, which states that the subspace $H(V)$ carries a $U_q\big(\mathfrak{sl}_2\big)$-representation via operators the $\mathcal{X} = a_{\mathcal{X}} B_{\beta_{N-1}}$ and $\mathcal{Y} = a_{\mathcal{Y}} B_{\beta_N}$, for suitable coefficients $a_{\mathcal{X}}, a_{\mathcal{Y}} \in \mathbb{K}(q)^\times$. Then, applying [Jan96, 4.4(6)] again, we obtain

$$\begin{aligned} a_{\mathcal{X}} a_{\mathcal{Y}}^{m_N+1} B_{\beta_{N-1}} B_{\beta_N}^{m_N+1} v_\lambda &= \mathcal{X} \mathcal{Y}^{m_N+1} v_\lambda \\ &= \mathcal{Y}^{m_N+1} \mathcal{X} v_\lambda \; + \; [m_N+1]_q \, \mathcal{Y}^{m_N} \big[K_{N-1}^{-1} K_N; -m_N\big] v_\lambda \\ &= a_{\mathcal{X}} a_{\mathcal{Y}}^{m_N+1} B_{\beta_N}^{m_N+1} \underbrace{B_{\beta_{N-1}} v_\lambda}_{=0} + a_{\mathcal{Y}}^{m_N} [m_N+1]_q B_{\beta_N}^{m_N} \big[K_{N-1}^{-1} K_N; -m_N\big] v_\lambda \end{aligned}$$

and since $\big(K_{N-1}^{-1} K_N\big) v_\lambda = q^{m_N} v_\lambda$, we get

$$\big(q^{-m_N} (K_{N-1}^{-1} K_N) - q^{m_N} (K_{N-1}^{-1} K_N)^{-1} \big) v_\lambda = 0.$$

This implies that $\left[K_{N-1}^{-1}K_N; -m_N\right]v_\lambda = 0$, and therefore

$$B_{\beta_{N-1}}B_{\beta_N}^{m_N+1}v_\lambda = 0$$

as required. This finishes the proof. □

Following Proposition 6.11, we have found a proper submodule of $M(\lambda)$ that is generated by ${F_i}^{n_i+1}v_\lambda$ for $i \in \{2, \dots, N-1\}$ and $B_{\beta_N}^{m_N+1}v_\lambda$. Unfortunately, due to our restriction to the subspace $H(V)$ in the proof, this submodule is not yet sufficiently large enough for us to use to construct a new simple quotient which contains the unique simple factor module $L(\lambda)$ and show that it has finite dimension using [Jan96, Proposition 5.9]. For this, we additionally require a proper submodule generated by some power of the negative root vector F_N.

Our strategy for the remainder of this section is as follows. Consider the submodule of $M(\lambda)$ generated by $F_N^{n_N+1}v_\lambda$, where we recall the integer $n_N \geq 0$ defined in (6.13). Note that the weight of the root vector F_N is $-\gamma_{N-2}-2\gamma_{N-1}$, so the weight of $F_N^{n_N+1}v_\lambda$ is $\lambda-(n_N+1)(\gamma_{N-2}+2\gamma_{N-1})$. To show that the submodule generated by $F_N^{n_N+1}v_\lambda$ is in fact a *proper* submodule, it suffices to show that, for any $b \in \mathcal{B}_{\mathbf{c}}^+$ of weight $(n_N+1)(\gamma_{N-2}+2\gamma_{N-1})$, we get

$$bF_N^{n_N+1}v_\lambda = 0.$$

Observing the tables of roots and root vectors for $\mathcal{B}_{\mathbf{c}}$ in Appendix A, we see that the necessary weight for the element $b \in \mathcal{B}_{\mathbf{c}}^+$ can only be reached with a non-negative linear combination of the positive roots γ_{N-2}, γ_{N-1}, $\gamma_{N-2}+\gamma_{N-1}$ and $\gamma_{N-2}+2\gamma_{N-1}$. By this inspection, such an element $b \in \mathcal{B}_{\mathbf{c}}^+$ must therefore be a monomial only in the root vectors E_{N-1}, $B_{\beta_{N-1}}$, $B_{\beta_{N-2}}$ and E_N. Using the PBW-Theorem for $M(\lambda)$ (see Theorem 6.9), we may choose an ordering for the root vectors in b; we choose to send the E_{N-1} and E_N to the right. Note that E_{N-1} commutes with F_N, whilst E_N also moves past by using the $U_q(\mathfrak{sl}_2)$-relation $(U4')$, and both E_{N-1} and E_N annihilate the highest weight vector v_λ. This means that the element b can now only consist of specific powers of the other two root vectors, namely $B_{\beta_{N-1}}$ and $B_{\beta_{N-2}}$ which are of weight γ_{N-1} and $\gamma_{N-2}+\gamma_{N-1}$ respectively.

Crucially, we need to prove the following result for our highest weight vector $v_\lambda \in M(\lambda)$.

Proposition 6.12. *In $M(\lambda)$, we have*

$$B_{\beta_{N-1}}^{n_N+1}B_{\beta_{N-2}}^{n_N+1}F_N^{n_N+1}v_\lambda = 0. \tag{6.22}$$

This is the goal for the rest of the section. We begin by observing what happens when we try to move one of the root vectors $B_{\beta_{N-2}}$ to the right side. We first need the following Lemma.

Lemma 6.13. *For all $r \in \mathbb{N}_0$, we have*

$$B_{\beta_{N-2}}B_{\beta_N}^{r}v_\lambda = 0. \tag{6.23}$$

Proof. Assume that $v_\lambda \in M(\lambda)$ is the highest weight vector of weight $\lambda \in \Lambda_{2N-1}$. Then, for all $r \in \mathbb{N}_0$, we know that $B^r_{\beta_N} v_\lambda$ must be of weight $\lambda - r\gamma_{N-1}$. However, we also know that the root vector $B_{\beta_{N-2}}$ is of weight $\gamma_{N-2} + \gamma_{N-1}$. Therefore, we deduce that the weight of $B_{\beta_{N-2}} B^r_{\beta_N} v_\lambda$ is

$$\lambda - \big((r-1)\gamma_{N-1} - \gamma_{N-2}\big) \neq \lambda - \mu$$

where μ is a non-negative sum of the positive simple roots of $\mathfrak{so}_{2N-1}$. The result follows. □

Using both Lemma 5.16 and Lemma 6.13, we can now prove the following crucial formula.

Lemma 6.14. *For all $r, s \in \mathbb{N}$ where $r \geq s$, we have*

$$B^s_{\beta_{N-2}} F^r_N v_\lambda = (-1)^s q^{-s} \left(\prod_{k=1}^{s} \big[(r+1) - k\big]_q \right) F^{r-s}_N B^s_{\beta_N} v_\lambda. \tag{6.24}$$

Proof. We give a proof by induction on s. For $s = 1$, and for any $r \in \mathbb{N}$, using Lemma 5.16 we get

$$B_{\beta_{N-2}} F^r_N v_\lambda = q^{-r} F^r_N \underbrace{B_{\beta_{N-2}} v_\lambda}_{=0} - q^{-1} [r]_q F^{r-1}_N B_{\beta_N} v_\lambda$$

as required. Assume that the result now holds for some $s < r$ as an induction hypothesis. Then,

$$\begin{aligned}
B^{s+1}_{\beta_{N-2}} F^r_N v_\lambda &= B_{\beta_{N-2}} \big(B^s_{\beta_{N-2}} F^r_N v_\lambda\big) \\
&= (-1)^s q^{-s} \left(\prod_{k=1}^{s} \big[(r+1) - k\big]_q \right) \underbrace{B_{\beta_{N-2}} F^{r-s}_N}_{\text{use Lemma 5.16}} B^s_{\beta_N} v_\lambda \\
&= (-1)^s q^{-r} \left(\prod_{k=1}^{s} \big[(r+1) - k\big]_q \right) F^{r-s}_N \underbrace{B_{\beta_{N-2}} B^s_{\beta_N} v_\lambda}_{=0,\ \text{by Lemma 6.13}} \\
&\quad + (-1)^{(s+1)} q^{-(s+1)} \left(\prod_{k=1}^{(s+1)} \big[(r+1) - k\big]_q \right) F^{r-(s+1)}_N B^{(s+1)}_{\beta_N} v_\lambda
\end{aligned}$$

again, as required. Hence, the result holds for all $r, s \in \mathbb{N}$ where $r \geq s$. □

Observe from Lemma 6.14 that, if we set $r = s = n_N + 1$, then this becomes

$$B^{n_N+1}_{\beta_{N-2}} F^{n_N+1}_N v_\lambda = (-1)^{n_N+1} q^{-(n_N+1)} \left(\prod_{k=1}^{n_N+1} [n_N + 2 - k]_q \right) B^{n_N+1}_{\beta_N} v_\lambda \tag{6.25}$$

and we know that $B^{m_N+1}_{\beta_N} v_\lambda = 0$. Clearly $n_N + 1 \geq m_N + 1$ by definition of n_N, and hence we deduce the following for the highest weight vector $v_\lambda \in M(\lambda)$ which is sufficient to prove Proposition 6.12.

Corollary 6.15. *In $M(\lambda)$, we have*

$$B_{\beta_{N-2}}^{n_N+1} F_N^{n_N+1} v_\lambda = 0. \tag{6.26}$$

Following Corollary 6.15, we see that the submodule generated by ${F_N}^{n_N+1} v_\lambda$ is indeed a proper submodule and does *not* contain the highest weight vector. We are now ready to define a new simple quotient module which contains, and is a good approximation for, the unique simple factor module $L(\lambda)$.

6.5 Classifying the finite-dimensional irreducible $\mathcal{B}_{\mathbf{c}}$-modules

Let $\lambda \in \Lambda_{2N-1}$ be a dominant integral weight, and again recall the non-negative integers n_i for each $i \in \{2, \dots, N\}$ and m_N given in (6.11) and (6.13). Directly from the results in Section 6.4, we now know that the subspace

$$\tilde{N}(\lambda) = \left(\sum_{i=2}^{N} \mathcal{B}_{\mathbf{c}} F_i^{n_i+1} \right) + \mathcal{B}_{\mathbf{c}} B_{\beta_N}^{m_N+1} \tag{6.27}$$

is a proper submodule of the Verma module $M(\lambda)$. Indeed, if $N(\lambda)$ is the unique maximal proper submodule, then $\tilde{N}(\lambda) \subseteq N(\lambda)$. Consider the simple factor module

$$\tilde{L}(\lambda) := \frac{M(\lambda)}{\tilde{N}(\lambda)} \tag{6.28}$$

as an approximation of the unique simple factor module $L(\lambda)$, the quotient module (6.18) which we aim to show is finite-dimensional, see (6.19). However, since the submodule $\tilde{N}(\lambda)$ is contained in the submodule $N(\lambda)$, the quotient $L(\lambda)$ is a quotient module of $\tilde{L}(\lambda)$. Importantly, this also means that $\dim\big(L(\lambda)\big) \leq \dim\big(\tilde{L}(\lambda)\big)$ and therefore it is now sufficient to only show that the module $\tilde{L}(\lambda)$ is finite-dimensional. This is much easier since we can define $\tilde{L}(\lambda)$ explicitly.

Remark 6.16. Equivalently, we may define this quotient module as the $\mathcal{B}_{\mathbf{c}}$-module

$$\tilde{L}(\lambda) = \frac{\mathcal{B}_{\mathbf{c}}}{\tilde{J}_\lambda} \tag{6.29}$$

where we define the left ideal

$$\tilde{J}_\lambda := J_\lambda + \tilde{N}(\lambda) \tag{6.30}$$

of $\mathcal{B}_{\mathbf{c}}$, and recall the ideal J_λ from the definition of $M(\lambda)$ in Section 6.3.

To determine the dimension of this new module (6.28), it will be necessary for us to look at its associated graded. We first want to consider a filtration on the module $\tilde{L}(\lambda)$. Specifically, for

$m \geq 0$ we define the filtered subspaces

$$\mathcal{G}_m\big(\tilde{L}(\lambda)\big) := \frac{\mathcal{F}_m(\mathcal{B}_{\mathbf{c}})}{\tilde{J}_\lambda \cap \mathcal{F}_m(\mathcal{B}_{\mathbf{c}})} \subseteq \tilde{L}(\lambda) \tag{6.31}$$

where $\mathcal{F}$ is the standard filtration on the algebra $\mathcal{B}_{\mathbf{c}}$ given by a degree function defined in Section 5.2. Then, the module $\tilde{L}(\lambda)$ becomes a filtered module over the algebra $\mathcal{B}_{\mathbf{c}}$ by the filtration $\mathcal{G}$. By definition, the associated graded of the module $\tilde{L}(\lambda)$ is

$$\operatorname{gr}\big(\tilde{L}(\lambda)\big) := \bigoplus_{m\geq 0} \mathcal{R}_m, \qquad \text{where } \mathcal{R}_i := \frac{\mathcal{G}_i\big(\tilde{L}(\lambda)\big)}{\mathcal{G}_{i-1}\big(\tilde{L}(\lambda)\big)} \text{ for } i \geq 0 \tag{6.32}$$

and with $\mathcal{G}_{-1}\big(\tilde{L}(\lambda)\big) := \{0\}$ for convention. For the following result, we now denote

$$\mathcal{L} := \bigoplus_{m\geq 0} \mathcal{L}_m, \qquad \text{where } \mathcal{L}_i := \frac{\tilde{J}_\lambda \cap \mathcal{F}_i(\mathcal{B}_{\mathbf{c}})}{\tilde{J}_\lambda \cap \mathcal{F}_{i-1}(\mathcal{B}_{\mathbf{c}})} \text{ for } i \geq 0. \tag{6.33}$$

Lemma 6.17. *The subspace $\mathcal{L}$ is a left ideal of the associated graded algebra $gr\big(\mathcal{B}_{\mathbf{c}}\big)$. Moreover, we have*

$$\operatorname{gr}\big(\tilde{L}(\lambda)\big) \cong \frac{\operatorname{gr}\big(\mathcal{B}_{\mathbf{c}}\big)}{\mathcal{L}}. \tag{6.34}$$

Proof. Since $\tilde{J}_\lambda$ is a left ideal of the algebra $\mathcal{B}_{\mathbf{c}}$, it must also be a left ideal of each of its filtered subspaces. Recall that the associated graded algebra $\operatorname{gr}\big(\mathcal{B}_{\mathbf{c}}\big)$ is the direct sum of the quotient spaces $\mathcal{S}_i := \frac{\mathcal{F}_i(\mathcal{B}_{\mathbf{c}})}{\mathcal{F}_{i-1}(\mathcal{B}_{\mathbf{c}})}$. Following the property of these filtered subspaces, for every $i, j \geq 0$ we must have that

$$\mathcal{S}_i \cdot \mathcal{L}_j \subset \mathcal{L}_{i+j}$$

and therefore $\operatorname{gr}(\mathcal{B}_{\mathbf{c}}) \cdot \mathcal{L} \subset \mathcal{L}$. Hence, $\mathcal{L}$ is a left ideal of $\operatorname{gr}\big(\mathcal{B}_{\mathbf{c}}\big)$. Furthermore, since $\mathcal{F}_{i-1}(\mathcal{B}_{\mathbf{c}})$ and $\tilde{J}_\lambda \cap \mathcal{F}_i(\mathcal{B}_{\mathbf{c}})$ are subspaces of $\mathcal{F}_i(\mathcal{B}_{\mathbf{c}})$ for all $i \geq 0$, and we have

$$\Big(\tilde{J}_\lambda \cap \mathcal{F}_i(\mathcal{B}_{\mathbf{c}})\Big) \cap \mathcal{F}_{i-1}(\mathcal{B}_{\mathbf{c}}) = \tilde{J}_\lambda \cap \mathcal{F}_{i-1}(\mathcal{B}_{\mathbf{c}})$$

for all $i \geq 0$, one can apply Corollary 2.2 directly (setting $U := \mathcal{F}_i(B_c)$, $V := \mathcal{F}_{i-1}(\mathcal{B}_{\mathbf{c}})$ and $W := \tilde{J}_\lambda \cap \mathcal{F}_i(\mathcal{B}_{\mathbf{c}})$ in the formula of the corollary) to get

$$\frac{\mathcal{G}_i\big(\tilde{L}(\lambda)\big)}{\mathcal{G}_{i-1}\big(\tilde{L}(\lambda)\big)} \cong \frac{\mathcal{S}_i}{\mathcal{L}_i}$$

for all $i \geq 0$. Hence,

$$\operatorname{gr}\big(\tilde{L}(\lambda)\big) = \bigoplus_{m\geq 0} \mathcal{R}_m \cong \bigoplus_{m\geq 0} \frac{\mathcal{S}_m}{\mathcal{L}_m} = \frac{\operatorname{gr}\big(\mathcal{B}_{\mathbf{c}}\big)}{\mathcal{L}}.$$

□

Since $\tilde{L}(\lambda)$ is a filtered module over the filtered algebra $\mathcal{B}_{\mathbf{c}}$, it follows that the associated graded $\operatorname{gr}\big(\tilde{L}(\lambda)\big)$ is a graded module over the associated graded algebra $\operatorname{gr}(\mathcal{B}_{\mathbf{c}})$. By the isomorphism φ

in Proposition 5.3, this means that we can consider it as a module over the algebra $\mathcal{A}$ defined in (5.10) in Section 5.2, and hence

$$\operatorname{gr}\big(\tilde{L}(\lambda)\big) \cong \frac{\mathcal{A}}{\mathcal{L}}. \tag{6.35}$$

By the definitions of both the ideal J_λ in (6.14) and the proper subspace $\tilde{N}(\lambda)$ in (6.27), we know that

$$E_i, (K_i - q^{n_i}), B_{\beta_{N-1}}, B_1, F_i^{n_i+1}, B_{\beta_N}^{m_N+1} \in \tilde{J}_\lambda \tag{6.36}$$

for all $i \in \{2, \ldots, N\}$ by (6.30). Recall that the isomorphism φ sends the element F_1 to B_1. If we set $n_1 := 0$, then we see that

$$E_i, (K_i - q^{n_i}), T^{-1}_{N-1,\ldots,2}(F_1), F_j^{n_j+1}, T_N^{-1}T^{-1}_{N-2,\ldots,2}(F_1)^{m_N+1} \in \mathcal{L} \tag{6.37}$$

for all $i \in \{2, \ldots, N\}$ and $j \in \{1, \ldots, N\}$. Let us now define a subspace $\mathcal{L}' \subseteq \mathcal{L}$ by

$$\begin{aligned} \mathcal{L}' := \left(\sum_{i=2}^{N} \mathcal{A}E_i + \mathcal{A}\big(K_i - q^{n_i}\big)\right) + \mathcal{A}\,T^{-1}_{N-1,\ldots,2}(F_1) \\ + \left(\sum_{j=1}^{N} \mathcal{A}F_j^{n_j+1}\right) + \mathcal{A}\,T_N^{-1}T^{-1}_{N-2,\ldots,2}(F_1)^{m_N+1}. \end{aligned} \tag{6.38}$$

Recall that the algebra $\mathcal{A}$ exhibits a triangular decomposition $\mathcal{A} \cong \mathcal{U}^- \otimes \mathcal{M}_X^0 \otimes \mathcal{M}_X^+$. From this we see that the quotient space $A/\mathcal{L}'$ is isomorphic to the quotient space

$$\frac{\mathcal{U}^-}{\mathcal{U}^- T^{-1}_{N-1,\ldots,2}(F_1) + \sum_{j=1}^{N} \mathcal{U}^- F_j^{n_j+1} + \mathcal{U}^- T_N^{-1}T^{-1}_{N-2,\ldots,2}(F_1)^{m_N+1}}. \tag{6.39}$$

In particular here, notice that (to some positive integer power) all of the F_j for $j \in \{1, \ldots, N\}$ are contained in the factor of (6.39). By [Jan96, Proposition 5.9], this means that this quotient is, in fact, finite-dimensional. We now only need to work backwards; since $A/\mathcal{L}'$ is isomorphic to (6.39) it must also have finite dimension and, since $\mathcal{L}' \subseteq \mathcal{L}$, this implies that $A/\mathcal{L}$ is finite-dimensional as well. By the isomorphism (6.35), the graded module $\operatorname{gr}\big(\tilde{L}(\lambda)\big)$, too, has finite-dimension, and since this is the associated graded of the $\mathcal{B}_\mathbf{c}$-module $\tilde{L}(\lambda)$ we finally see that $\tilde{L}(\lambda)$ is indeed finite-dimensional. This was the last step necessary to prove the following classification theorem for the finite-dimensional irreducible representations of the algebra $\mathcal{B}_\mathbf{c}$.

Theorem 6.18 (Classification Theorem of finite-dimensional simple $\mathcal{B}_\mathbf{c}$-modules of type-1)**.** *For each dominant weight $\lambda \in \Lambda_{2N-1}$, the simple $\mathcal{B}_\mathbf{c}$-module $L(\lambda)$ has finite-dimension. Moreover, each finite-dimensional simple $\mathcal{B}_\mathbf{c}$-module is isomorphic to exactly one $L(\lambda)$ with dominant weight $\lambda \in \Lambda_{2N-1}$.*

Corollary 6.19. *There is a one-to-one correspondence between the dominant weights $\lambda \in \Lambda_{2N-1}$ and the finite-dimensional simple $\mathcal{B}_\mathbf{c}$-modules $L(\lambda)$. Moreover, each finite-dimensional simple $\mathcal{B}_\mathbf{c}$-module is uniquely determined by an $(N-1)$-tuple of non-negative integers $(n_2, \ldots, n_{N-1}, m_N)$.*

Appendix A

Tables of Root Vectors for $\mathcal{B}_{\mathbf{c}} \subset \mathcal{U}$

The following two tables provide a complete list of all of the positive and negative roots of the Lie algebra $\mathfrak{so}_{2N}$ respectively, and the corresponding root vectors for the algebra $\mathcal{B}_{\mathbf{c}}$, depending on a fixed $N \geq 3$. The asterisk $*$ appears in the third column to distinguish simple root vectors.

positive root of $\mathfrak{so}_{2N-1}$	corresponding root vector of $\mathcal{B}_{\mathbf{c}}$		N	total
γ_{i-1} $(2 \leq i \leq N-1)$	E_i		$*$	$N-2$
γ_{N-1}	$T^{-1}_{N-1,\ldots,2}(B_1)$		$*$	1
$\sum_{k=i}^{N} \gamma_{k-1}$ $(2 \leq i \leq N-1)$	B_1	if $i=2$		1
	$T^{-1}_{i-1,\ldots,2}(B_1)$	if $i \geq 3$		$N-3$
$\sum_{k=i}^{j} \gamma_{k-1}$ $(2 \leq i < j \leq N-1)$	$T_{j,\ldots,i+1}(E_i)$			$\frac{(N-3)(N-2)}{2}$
$\sum_{k=i}^{N-1} \gamma_{k-1} + 2\gamma_{N-1}$ $(2 \leq i \leq N-1)$	E_N	if $i=N-1$		1
	$T_N(E_{N-2})$	if $i=N-2$		1
	$T_N T_{N-2,\ldots,i+1}(E_i)$	if $i \leq N-3$	≥ 5	$N-4$
$\sum_{k=i}^{j-1} \gamma_{k-1} + 2\sum_{l=j}^{N} \gamma_{l-1}$ $(2 \leq i < j \leq N-1)$	$T_{N,\ldots,i+1}(E_i)$	if $j=N-1$		$N-3$
	$T_{j,\ldots,N-2}T_{N,\ldots,i+1}(E_i)$	if $j \leq N-2$	≥ 5	$\frac{(N-4)(N-3)}{2}$

negative root of $\mathfrak{so}_{2N-1}$	corresponding root vector of $\mathcal{B}_{\mathbf{c}}$	N	total
$-\gamma_{i-1}$ $(2 \leq i \leq N-1)$	F_i	$*$	$N-2$
$-\gamma_{N-1}$	$T_N^{-1}T_{N-2,\dots,2}^{-1}(B_1)$	$*$	1
$-\sum_{k=i}^{N}\gamma_{k-1}$ $(2 \leq i \leq N-1)$	$T_{N,\dots,2}^{-1}(B_1)$ if $i=N-1$ $T_{i,\dots,N-2}^{-1}T_{N,\dots,2}^{-1}(B_1)$ if $i \leq N-2$		1 $N-3$
$-\sum_{k=i}^{j}\gamma_{k-1}$ $(2 \leq i < j \leq N-1)$	$T_{j,\dots,i+1}^{-1}(F_i)$		$\frac{(N-3)(N-2)}{2}$
$-\sum_{k=i}^{N-1}\gamma_{k-1} - 2\gamma_{N-1}$ $(2 \leq i \leq N-1)$	F_N if $i=N-1$ $T_N^{-1}(F_{N-2})$ if $i=N-2$ $T_N^{-1}T_{N-2,\dots,i+1}^{-1}(F_i)$ if $i \leq N-3$	 ≥ 5	1 1 $N-4$
$-\sum_{k=i}^{j-1}\gamma_{k-1} - 2\sum_{l=j}^{N}\gamma_{l-1}$ $(2 \leq i < j \leq N-1)$	$T_{N,\dots,i+1}^{-1}(F_i)$ if $j=N-1$ $T_{j,\dots,N-2}^{-1}T_{N,\dots,i+1}^{-1}(F_i)$ if $j \leq N-2$	 ≥ 5	$N-3$ $\frac{(N-4)(N-3)}{2}$

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