New models of higher-rank graph algebras arising from buildings, and computation of their K-theory

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Abstract

Formally, a k-rank graph is defined as a small category Λ together with a functor d which associates to each morphism $\lambda \in \mathrm{Hom}(\Lambda)$ an element of $\mathbb{N}^k$, called its *degree*. The functor d must satisfy a special factorisation property which at first glance might not seem to restrict much, but actually heralds intricate and complex properties, even in low dimensions.

In practise, it is often more instructive to regard k-rank graphs as generalisations of graphs. These manifest themselves as directed graphs with each edge painted one of k different colours, which can be decomposed into a set of *squares*: subgraphs with four edges which resemble a geometric square. Hazlewood, Raeburn, Sims and Webster demonstrated that, if G is a k-coloured graph with a decomposition into squares such that the squares can be assembled into cubes, then G induces a k-rank graph.

In this thesis, we generate two main new infinite families of higher-rank graphs, exploiting this cubical structure in order to do so. In two dimensions, we use a theorem of Vdovina to construct for each complete connected bipartite graph a so-called *tile complex* which induces two different 2-rank graphs. In higher dimensions, we define a class of groups called *domino groups* which act freely and transitively on a k-dimensional affine building. The quotient of this action on the building defines a k-dimensional cube complex, which in turn induces a k-rank graph.

Perhaps most importantly, to each k-rank graph can be associated a $C^\star$-algebra. These *higher-rank graph algebras* are extremely versatile, and have arisen in connection with quantum spheres, Yang–Baxter equations, Thompson's groups, and braid groups. We show that the $C^\star$-algebras corresponding to all of our models of k-rank graphs are separable, nuclear, unital, purely infinite, and simple, and hence that they are determined by their K-groups. Some of our main theorems are dedicated to computations of the K-theory of these infinite families of algebras, and the *Kirchberg–Phillips Classification Theorem* tells us that our examples are indeed new. Until now, there have been very few explicit computations of the K-theory of such algebras, so these results furnish a large part of this thesis. We also identify some of the relationships between the structure of k-rank graph $C^\star$-algebras and the algebras of lower-rank subgraphs, which we hope will simplify hitherto difficult computations in high dimensions.

Each of the tile complexes and domino groups we employ in this thesis can be viewed geometrically as a cube complex, and we examine their topological properties. We compute the cellular homology groups and again show how these can be retrieved from the geometry of lower-rank subgraphs. Thus we introduce a concrete link between the $C^\star$-algebra–based theory of higher-rank graphs and the geometrical theory of affine buildings.

We used the computer algebra package `MAGMA` for many of the computations, and wrote most of the algorithms for constructing the k-rank graphs in `Python`.

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Contents

Chapter 0

Introduction

§0.1 Thesis overview

Higher-rank graphs are archetypal of twenty-first century mathematics, uniting elements of category theory, K-theory, and geometric group theory. Although higher-rank graphs were initially defined in a categorical manner (1.1.4), they are primarily studied by functional analysts for their associated *graph algebras* (1.3.3). These generalise classical (1-rank) graph $C^\star$-algebras and the *Cuntz–Krieger algebras* of [CK80], and provide a rich source of intricate and exciting mathematics (not least in [BNR14; ES12]). Despite 20 years of study, however, there are remarkably few models of infinite families of k-rank graphs for arbitrary k (one example can be found in [KPS08]), and even fewer which have incidence matrices with entries in $\{0, 1\}$ (another example, in [APS06]).

Our approach is geometrical: certain groups acting on buildings and products of trees can be shown to induce higher-rank graphs, and hence can be associated to graph $C^\star$-algebras. Indeed, the prototype of a higher-rank graph (of rank two) arose in [RS99] from the action of a group on an affine building of type $\tilde{A}_1 \times \tilde{A}_1$. Then in [KP00], Kumjian and Pask solidified the status of a higher-rank graph as an abstract category, and the building-like nature was largely overlooked until [KV15]. In this thesis, we consider both the geometric and algebraic aspects of higher-rank graphs by using constructions we call *tile complexes* (Chapter 2) and *domino complexes* (Chapters 3–5). These are cube complexes which induce higher-rank graphs via functions which record the *adjacency* (2.2.1, 3.1.6) of cells—vertices in the k-graph are indexed by the k-dimensional cells of the complexes. A domino complex induces a natural *domino group* (3.2.1, introduced in [Vdo21]) which acts freely and transitively on a type $\prod_{i=1}^{k} \tilde{A}_1$ building (3.2.10), generalising the *BM-groups* introduced in [BM97; Wis96], and the concepts of [RS99]. Such a generalisation is no mean feat, owing to the associativity insisted upon by k-rank graphs, which only comes into play when $k \geq 3$. However, Rungtanapirom, Stix and Vdovina developed an elegant algorithm in [RSV19] which generates k-domino groups for arbitrary k, and which we implement in Chapter 5.

To each higher-rank graph Λ is bequeathed a universal $C^\star$-algebra $\mathcal{A}(\Lambda)$, and these are often the main focus of study (see [EFG+22; Eva08; Sim06] for a flavour). We show in 2.3.12, 2.4.6 and 3.5.4 that our new models of higher-rank graphs induce algebras which satisfy the conditions of the *Kirchberg–Phillips Classification Theorem* (2.3.11), and as such can be completely determined by invariants called their *K-groups* (1.2.11–1.2.12). Using a method of Evans, we can wholly understand the K-theory of the algebras of tile complexes, but increasing the rank k dramatically decreases the amount of information we can obtain about the K-groups of k-graph algebras. We give an analysis of what, by our methods, is knowable about *domino graph algebras* in Chapter 4, and discuss the limitations of our methods. In all cases, we learn enough about the K-theory to know that the algebras are new, and in some sense *irreducible*, in that they do not arise as a direct result of the *Künneth Theorem* for tensor products (1.2.18). We also highlight some promising ideas, including the connectedness of Evans' exact sequences with *Matui's HK-Conjecture* (4.2.2) and with the geometry of the domino complexes themselves.

Indeed, the geometry of the associated complexes provides some potential clues as to how to simplify some of the notoriously difficult K-theory computations. The *cellular homology* is easier to work out (2.6.2, 4.1.7) and is clearly linked with the K-theory (4.4.3), but a more detailed investigation between the lower homology groups and the K-groups remains a direction for future research. It should be remarked that the study of the geometry of k-graphs has often yielded notable results (see [HRSW13; KPW21]), but that our approach takes into account the geometry of a complex which is distinct from the graph itself. We provide some suggestions on how to build higher-rank graph algebras by taking the product of a domino group with a free group, and explore how this process affects the geometry and the graph algebras in §4.4.

The new families of objects which we examine in this thesis usher in a new way of thinking about higher-rank graphs as structures arising from groups acting on buildings, reminiscent of the outlook of Robertson–Steger. We frequently interchange geometric and algebraic language; since they are so intimately linked, this is very natural. We provide new details of the K-theory of k-graph $C^\star$-algebras, focussing in particular on the case where $k = 3$, which is complex enough to provide novel results, while still being tangible.

§0.2 Summary of contributions

Chapter 1, save for a few examples, is standard material.

Chapter 2 has been published as the single-author paper [Mut22].

An abbreviated form of Chapter 3 appeared in the paper [MRV20], which was joint work with Alina Vdovina and Aura-Cristiana Radu. Anything in that chapter which did not also appear in that paper is the author's own addition.

Some of the examples and computations of K-theory from §4.1–4.3 have also appeared in [MRV20], but many are new and arise directly from the programs in Chapter 5. The remainder of Chapter 4 comprises work entirely original to this thesis.

The code in the Appendix (Chapter 5) is the author's own work. Previous versions were drawn from in order to make some of the computations in [MRV20]. Other programs used in this thesis are available from the author on request.

Chapter 1

Background

Kumjian and Pask in [KP00], with motivation from [RS99], introduced the notion of a *k-rank graph* as a higher-dimensional analogue of conventional directed graphs. Although these objects can be treated as graphs, they are better considered abstractly, as categories whose morphisms possess a special *factorisation property* (1.1.4). When the number of dimensions k is less than 3, this factorisation property doesn't restrict too much, but as soon as $k \geq 3$, it becomes increasingly tricky to construct interesting k-rank graphs. Since we will be playing with these objects throughout the course of the thesis, we dedicate a whole section to their introduction.

In the following, we use the notation $\mathbb{N} = \{0, 1, 2, \ldots\}$, and we write $\mathbf{e}_1, \ldots, \mathbf{e}_k$ to denote the standard generators of $\mathbb{N}^k$ as an additive Abelian monoid: $\mathbf{e}_i$ is the k-tuple with 1 in the ith position, and 0 elsewhere. We often use $\mathbf{0}$ to denote the identity in $\mathbb{N}^k$. For $\mathbf{m}, \mathbf{n} \in \mathbb{N}^k$, we write $\mathbf{m} \leq \mathbf{n}$ whenever each co-ordinate of $\mathbf{m}$ is less than or equal to the corresponding co-ordinate of $\mathbf{n}$.

Throughout the chapter (and indeed the thesis), k is a positive integer unless otherwise specified, and is sometimes referred to as the dimension.

§1.1 Higher-rank graphs

1.1.1 We might describe a directed graph G by its vertex set G^0 and edge set G^1; these sets can be finite or countably infinite. An edge $e \in G^1$ is an arrow connecting two vertices. We write $s_G(e)$ for the arrow's origin, and $r_G(e)$ for its target. A (finite) path in G is a sequence of edges $\{e_i\}_{i=1}^n$ such that $r_G(e_i) = s_G(e_{i+1})$ for all i. Let G^* denote the set of all finite paths in G, and let $d : G^* \to \mathbb{N}$ be the function which returns the length of such a path in G^*. By writing $G^n := \{\{e_i\} \in G^* \mid d(\{e_i\}) = n\}$, we recover the notation for the vertex and edge sets of G.

We extend the origin and target maps to G^* in a natural way: for a path $\lambda = \{\lambda_i\}_{i=1}^n \in G^*$, where λ_i are edges, we define $s_G(\lambda) := s_G(\lambda_1)$ and $r_G(\lambda) := r_G(\lambda_n)$. Two paths $\mu, \nu \in G^*$ of respective lengths m and n are said to be concatenatable whenever $r_G(\mu) = s_G(\nu)$. Concatenation of paths is usually denoted by a dot and written from left to right, that is, $\mu \cdot \nu := \mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n$. Clearly $d(\mu \cdot \nu) = d(\mu) + d(\nu)$.

Now we observe a seemingly banal property of G^* and d which will turn out to have interesting consequences when generalised to higher dimensions.

1.1.2 Path factorisation property of graphs *Let G be a directed graph, and let $\lambda \in G^*$ be a path with $d(\lambda) = m + n$ for some $m, n \in \mathbb{N}$. Then we can find unique paths $\mu, \nu \in G^*$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu \cdot \nu$.*

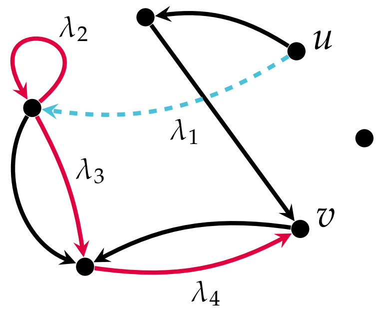


Figure 1.1: The path $\lambda = \lambda_1, \lambda_2, \lambda_3, \lambda_4$ is one of a number of paths from u to v with length 4. The path factorisation property (1.1.2) says that, for each m, n with $m + n = 4$, we can find unique paths μ, ν such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu \cdot \nu$.

In the example pictured with $m = 1$ and $n = 3$, it is clear that $\mu = \lambda_1$ (dashed blue), and $\nu = \lambda_2, \lambda_3, \lambda_4$ (pink).

1.1.3 Figure 1.1 illustrates this property. It says that, given a finite path λ and a natural number m not greater than its length, then there is exactly one place we can "snip" λ so that it splits into two paths, with the first of length m.

We will see that a directed graph is equivalent to a *category* whose morphisms have this factorisation property. Recall that a category Λ consists of a collection $\mathrm{Ob}(\Lambda)$ of *objects*, and a collection $\mathrm{Hom}(\Lambda)$ of *morphisms* between the objects. For any objects $u, v \in \mathrm{Ob}(\Lambda)$, we write $\mathrm{Hom}_\Lambda(u, v)$ for the set of morphisms from u to v. Composition of morphisms is associative, and for each object u there exists an *identity morphism* $\mathrm{id}_u \in \mathrm{Hom}_\Lambda(u, u)$ such that $\mathrm{id}_u \lambda = \lambda$ and $\lambda\, \mathrm{id}_u = \lambda$ whenever λ is in $\mathrm{Hom}_\Lambda(*, u)$ or $\mathrm{Hom}_\Lambda(u, *)$, respectively.

In what follows, it is useful to regard $\mathbb{N}^k$ as a category with one object, and morphisms $\mathbf{e}_i$. Then, with the path factorisation property 1.1.2, we are able to generalise the notion of a directed graph as follows:

1.1.4 Definition (Higher-rank graph) Let Λ be a category such that $\mathrm{Ob}(\Lambda)$ and $\mathrm{Hom}(\Lambda)$ are countable sets, that is, a *countable small category*. For a morphism $\lambda \in \mathrm{Hom}_\Lambda(u, v)$, we define domain and range maps $s(\lambda) := u$ and $r(\lambda) := v$ respectively.

Let $d : \Lambda \to \mathbb{N}^k$ be a functor, called the degree map. We call the pair (Λ, d) a k-rank graph (or simply a k-graph) if for each $\lambda \in \mathrm{Hom}(\Lambda)$, whenever $d(\lambda) = \mathbf{m} + \mathbf{n}$ for some $\mathbf{m}, \mathbf{n} \in \mathbb{N}^k$, we can find *unique* elements $\mu, \nu \in \mathrm{Hom}(\Lambda)$ such that $d(\mu) = \mathbf{m}$, $d(\nu) = \mathbf{n}$, and $\lambda = \nu\mu$. Note that for μ, ν to be composable, we must have $r(\mu) = s(\nu)$. Sometimes we just write Λ to denote a k-graph when the degree map d is obvious or not needed.

For $\mathbf{n} \in \mathbb{N}^k$, we write $\Lambda^{\mathbf{n}} := \{\lambda \in \Lambda \mid d(\lambda) = \mathbf{n}\}$. By the above property, we may identify $\Lambda^{\mathbf{0}}$ with $\mathrm{Ob}(\Lambda)$ and the identity morphisms in $\mathrm{Hom}(\Lambda)$; in this way, when we talk about Λ we only need to consider $\mathrm{Hom}(\Lambda)$. We call the elements of $\Lambda^{\mathbf{0}}$ the vertices of (Λ, d), and refer to the general elements of $\Lambda = \mathrm{Hom}(\Lambda)$ as paths.

1.1.5 Let us investigate why a directed graph defines a 1-rank graph. Indeed, given a directed graph G as in 1.1.1, then we may regard the set of finite paths G^* as a set of morphisms. Thus we construct a countable small category G^* with morphism set G^* and object set G^0; again we identify G^0 with the set of identity morphisms in G^*. We give to each morphism $\lambda \in G^*$ the domain and range maps $s(\lambda) := r_G(\lambda)$ and $r(\lambda) := s_G(\lambda)$ respectively. Take heed that, because paths in a graph are concatenated *on the right* and morphisms in a category are composed *on the left*, the domain s of a path in G^* is defined to be its target, and the range r its origin. The pair (G^*, d) is a 1-rank graph.

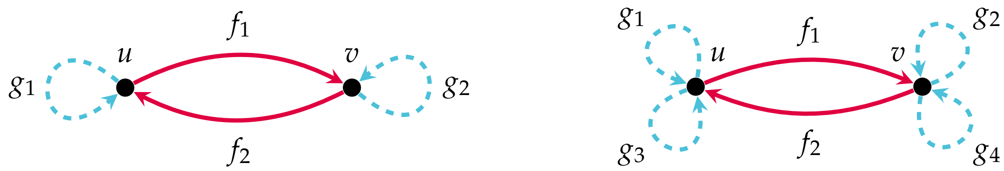
Conversely, we may regard any 1-rank graph (Λ, d) as a directed graph with vertex and edge sets Λ^0 and Λ^1 respectively, and with arrows λ pointing from $r(\lambda)$ to $s(\lambda)$. Then $d : \Lambda \to \mathbb{N}$ is just the graph path-length function from 1.1.1.

1.1.6 One might notice that a 1-rank graph is actually the *path category* generated by a directed graph, with concatenation of paths written from right to left to coincide with the convention for composition of functions. From now on we will always concatenate paths on the left.

We usually visualise a k-graph (Λ, d) as a collection of k different *coloured graphs* (described in Definition 1.1.11) on the same set of vertices—we call this the 1-skeleton of (Λ, d). Not all k-coloured graphs have the structure of a k-rank graph, since they may not have the required factorisation property. We are sometimes able to shoehorn this in (see 1.1.16).

1.1.7 **Example** Consider a category Λ with $\mathrm{Ob}(\Lambda) = \{u, v\}$, and $\mathrm{Hom}(\Lambda)$ generated by morphisms $f_1 : u \to v$, $f_2 : v \to u$, $g_1 : u \to u$, $g_2 : v \to v$ under composition wherever it makes sense. This is the path category generated by the graph in Figure 1.2a. Define a degree functor $d : \Lambda \to \mathbb{N}^2$ by $d(f_i) := (1, 0)$, $d(g_i) := (0, 1)$.

Let $\mathbf{m} = (1, 1)$, $\mathbf{n} = (0, 0)$, and $\lambda = g_2 f_1$, so that λ is a path from u to v with degree $(1, 1) = \mathbf{m} + \mathbf{n}$. Then we could set $\mu = g_2 f_1$ and $\nu = \mathrm{id}_v$ in Definition 1.1.4, such that $\lambda = \nu\mu$ and the degrees add up. But $f_1 g_1$ is another path from u to v with degree $(1, 1)$, and it may be that putting $\mu = f_1 g_1$ also results in a valid decomposition. In order for (Λ, d) to define a higher-rank graph, μ and ν *must be unique*; hence we force the equality $g_2 f_1 = f_1 g_1$ and, as a notational short-cut, refer to this quotient category as Λ.



(a) Here, $g_2 f_1$ and $f_1 g_1$ are both morphisms from u to v with degree $(1, 1)$. In order to preserve the factorisation property of 1.1.4, we must set $g_2 f_1 = f_1 g_1$. Likewise, we set $g_1 f_2 = f_2 g_2$.

(b) Here, there is a choice of commutativity relations. For example, we could require that $g_3 f_1$ equal $f_1 g_1$ or $f_1 g_2$. Different sets of such *commuting squares* define different higher-rank graphs.

Figure 1.2: Two examples of 2-rank graphs, where pink and dashed blue arrows have degrees $(1, 0)$ and $(0, 1)$, respectively. Depicted are the 1-*skeletons*, which are just 2-coloured graphs; in order to define a higher-rank graph, we also need to specify some commutativity relations.

Similarly, $f_2 g_2$ and $g_1 f_2$ both define paths of degree $(1, 1)$ from v to u, and so we insist that they be equal. The pair (Λ, d), together with the two commutativity relations $g_2 f_1 = f_1 g_1$, $f_2 g_2 = g_1 f_2$, defines a 2-rank graph. ♦

1.1.8 Example Let Λ and d be as in 1.1.7, but with two additional morphisms $g_3 : u \to u$ and $g_4 : v \to v$; Λ is the path category generated by the graph in Figure 1.2b. Now, in order to define a 2-rank graph, we may choose how to make the paths commute.

We could set $g_2 f_1 = f_1 g_1$, in which case $g_4 f_1 = f_1 g_3$, or we could set $g_2 f_1 = f_1 g_3$, in which case $g_4 f_1 = f_1 g_1$. Then, independently of the first choice, we can also choose whether to set $g_1 f_2$ equal to $f_2 g_2$ or $f_2 g_4$. Of these four sets of commutativity relations, two define *non-isomorphic* higher-rank graphs, and we will see that this may change the structure of the associated k-graph algebra. We explain now what it means for two higher-rank graphs to be isomorphic. ♦

1.1.9 Definition (Isomorphism in k-graphs) Given two k-rank graphs (Λ_1, d_1) and (Λ_2, d_2), we define a k-graph morphism to be a functor $\Lambda_1 \to \Lambda_2$ which respects the degree maps. The k-graphs (Λ_1, d_1) and (Λ_2, d_2) are said to be isomorphic if there exist k-graph morphisms $F : \Lambda_1 \to \Lambda_2$ and $F' : \Lambda_2 \to \Lambda_1$ such that $F'F = \mathrm{id}_{\Lambda_1}$ and $FF' = \mathrm{id}_{\Lambda_2}$.

1.1.10 In Example 1.1.7, there was only one possible set of commutativity relations for the arrows in the 1-skeleton (Figure 1.2a). In 1.1.8, however, different choices of path decomposition resulted in the higher-rank graphs being non-isomorphic, despite having the same 1-skeleton. Therefore when sketching k-graphs, it is necessary to include the set of commutativity relations, unless there is no choice to be made.

Constructing higher-rank graphs from coloured graphs

Conversely, given a k-coloured graph, we may define criteria under which the induced path category is a k-rank graph. For further detail, consult [HRSW13].

1.1.11 Definition (Coloured graph) A k-coloured graph is a directed graph $G = (G^0, G^1, s_G, r_G)$ equipped with a map $c : G^1 \to \{1, \ldots, k\}$ which assigns to each edge a colour. Writing $\mathbb{F}_k^+$ to denote the free semigroup on k generators, we may extend this map to a functor $c : G^* \to \mathbb{F}_k^+$ such that if $\lambda = \{e_i\}_{i=1}^n \in G^n$ is a path of length n, then $c(\lambda) = c(e_n)c(e_{n-1}) \cdots c(e_1)$. Given two k-coloured graphs G_1 and G_2, a coloured-graph morphism is a (not necessarily surjective) function $G_1 \to G_2$ which respects the origin, target, and colour maps.

1.1.12 Example (Lattice k-coloured graph) Let $\mathbf{m} \in \mathbb{N}^k$ be fixed. We write $E = E(k, \mathbf{m})$ for the k-coloured graph with vertices $E^0 = \{\mathbf{n} \in \mathbb{N}^k \mid \mathbf{n} \leq \mathbf{m}\}$, and arrows $z_i^{\mathbf{n}}$ of colour i from $\mathbf{n} + \mathbf{e}_i$ to $\mathbf{n}$, where $\mathbf{e}_i$ is the ith canonical generator of $\mathbb{N}^k$. This example of a k-coloured graph clearly defines a k-rank graph, and will make a regular appearance in what follows. See Figure 1.3 for an illustration.

We say that a coloured-graph morphism $\lambda : E(k, \mathbf{m}) \to G$ has degree $d(\lambda) := \mathbf{m}$, and define domain and range maps $s(\lambda) := \lambda(\mathbf{m})$ and $r(\lambda) := \lambda(\mathbf{0})$ respectively. ♦

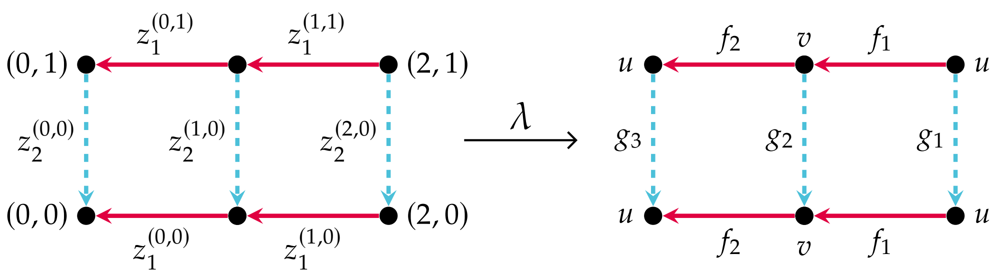


Figure 1.3: The 2-coloured graph $E(k, (2, 1))$. There is a coloured-graph morphism $\lambda : E(k, (2, 1)) \to G_\Lambda$, where G_Λ is the graph depicted in Figure 1.2b, sending $z_2^{(0,0)}$ to g_3, $z_2^{(1,0)}$ to g_2, $z_2^{(2,0)}$ to g_1, $z_1^{(1,0)}$ and $z_1^{(1,1)}$ both to f_1, and $z_1^{(0,0)}$ and $z_1^{(0,1)}$ both to f_2. This kind of diagram is useful for picturing sets of commutativity relations for k-rank graphs: for example, relations $g_2 f_1 = f_1 g_1$ and $g_3 f_2 = f_2 g_2$ are shown here.

1.1.13 Example (Commuting squares in 1.1.8) Let E be the 2-coloured graph $E(2, (2, 1))$, and let G_Λ be the 2-coloured graph depicted in Figure 1.2b, that is, the 1-skeleton of the 2-rank graph from Example 1.1.8. There are lots of coloured-graph morphisms from E to G_Λ; one example is the map $\lambda : E \to G_\Lambda$ described in Figure 1.3.

We'll see in the rest of this section that coloured-graph morphisms from the graphs $E(k, \mathbf{m})$ can help us to visualise the commutativity relations of k-rank graphs as *commuting squares*. For example, let Λ be the 2-rank graph with 1-skeleton G_Λ as above and commutativity relations

$$g_2 f_1 = f_1 g_1, \quad g_4 f_1 = f_1 g_3, \quad g_3 f_2 = f_2 g_2, \quad g_4 f_2 = f_2 g_3.$$

It is easy to extract some of these commutativity relations from Figure 1.3 which depicts the image of E under the coloured-graph morphism λ. We might choose different coloured-graph morphisms to visualise different sets of commutativity relations for Λ. ♦

1.1.14 The next definition and Theorem 1.1.16 will show us criteria on k-coloured graphs under which they define k-rank graphs, making precise the rules which determine valid sets of commutativity relations. We find it beneficial to use the geometric terminology from the above examples, and to visualise the relations as squares in the images of coloured-graph morphisms from lattices.

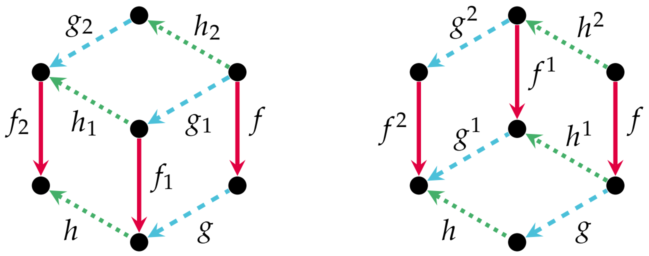


Figure 1.4: A set of squares C is *associative* if, whenever a tri-coloured path hgf defines two half-cubes as above, then the half-cubes actually form the twelve edges of a single cube, that is, $f_2 = f^2$, $g_2 = g^2$ and $h_2 = h^2$.

1.1.15 **Definition (Complete associative collection of squares)** Recall from Example 1.1.12 the k-coloured lattice graphs $E(k, \mathbf{m})$. Let G be a k-coloured graph for $k \geq 2$, and let $i, j \in \{1, \dots, k\}$ be distinct colours. We define an i-j–square (or simply a square) in G to be a pair of paths of length two, one coloured ji and one coloured ij, which have the same origin and target as each other—that is, a coloured-graph morphism $\sigma : E(k, \mathbf{e}_i + \mathbf{e}_j) \to G$. We define a complete associative collection of squares to be a set C of squares in G with the following properties:

C1 For each path of length two $gf \in G^2$ with colour ji, there is a *unique* square $\sigma \in C$ such that $\sigma\big(z_j^{\mathbf{e}_i}\big) = f$ and $\sigma\big(z_i^{\mathbf{0}}\big) = g$. In other words, for every path gf coloured ji in G^2, there is a unique path $f'g'$ coloured ij such that μ and ν label the sides of a square. We write $gf \sim f'g'$.

C2 In the case that $k \geq 3$, suppose that $hgf \in G^3$ is a path where f, g, h are different colours, and that

$$\begin{aligned} &gf \sim f_1 g_1, \quad hf_1 \sim f_2 h_1, \quad h_1 g_1 \sim g_2 h_2, \\ &hg \sim g^1 h^1, \quad h^1 f \sim f^1 h^2, \quad g^1 f^1 \sim f^2 g^2, \end{aligned}$$

for some edges $f_1, f_2, g_1, g_2, h_1, h_2$ and $f^1, f^2, g^1, g^2, h^1, h^2$, as illustrated in Figure 1.4. Then $f_2 = f^2$, $g_2 = g^2$, and $h_2 = h^2$.

If $\lambda : E(k, \mathbf{m}) \to G$ is a coloured-graph morphism and σ is an i-j–square in G, then we say that σ occurs in λ whenever there exists some $\mathbf{n} \in \mathbb{N}^k$ with $\mathbf{n} + \mathbf{e}_i + \mathbf{e}_j \le \mathbf{m}$, and

$$\sigma(z_i^{\mathbf{0}}) = \lambda(z_i^{\mathbf{n}}), \quad \sigma(z_j^{\mathbf{0}}) = \lambda(z_j^{\mathbf{n}}), \quad \sigma(z_i^{\mathbf{e}_i}) = \lambda(z_i^{\mathbf{e}_i+\mathbf{n}}), \quad \sigma(z_i^{\mathbf{e}_j}) = \lambda(z_i^{\mathbf{e}_j+\mathbf{n}}).$$

We say that λ is C-compatible for some complete associative collection of squares C if every square which occurs in λ belongs to C.

1.1.16 Theorem (Hazlewood, Raeburn, Sims and Webster, 2013) *Consider a k-coloured graph G with a complete associative set of squares C, and let $\mu : E(k, \mathbf{m}) \to G$ and $\nu : E(k, \mathbf{n}) \to G$ be C-compatible coloured-graph morphisms with $\nu(\mathbf{m}) = \mu(\mathbf{0})$. Then there is a unique C-compatible morphism $\mu\nu : E(k, \mathbf{m} + \mathbf{n}) \to G$ such that*

(i) $\mu\nu(z_i^{\mathbf{p}}) = \mu(z_i^{\mathbf{p}})$, *whenever* $\mathbf{p} + \mathbf{e}_i \le \mathbf{m}$, *and*

(ii) $\mu\nu(z_i^{\mathbf{p}}) = \nu(z_i^{\mathbf{p}-\mathbf{m}})$, *whenever* $\mathbf{m} \le \mathbf{p} \le \mathbf{n} - \mathbf{e}_i$.

Consider the category whose morphism set is the set of k-coloured–graph morphisms $\Lambda := \{\lambda : E(k, \mathbf{m}) \to G \mid \mathbf{m} \in \mathbb{N}^k\}$, with domain and range maps as defined in 1.1.12. It is convenient to use Λ to denote this category. Define the functor $d : \Lambda \to \mathbb{N}^k$ by $d(\lambda) := \mathbf{m}$ whenever the domain of λ is $E(k, \mathbf{m})$, and domain and range maps $s(\lambda) := \lambda(\mathbf{m})$ and $r(\lambda) := \lambda(\mathbf{0})$ respectively, as in Example 1.1.12. Then (Λ, d) is the unique k-rank graph such that $\Lambda^{\mathbf{e}_i} = c^{-1}(i)$ for each i, and $gf = f'g'$ in Λ if and only if $gf \sim f'g'$ in G. ■

1.1.17 So k-coloured graphs with a unique tri-coloured path factorisation property define k-rank graphs. Our main examples in Chapter 3 arise in this way. When $k = 2$, it is sufficient for a set of squares to satisfy property **C1** from 1.1.15 in order to define a 2-rank graph. We study such examples when $k = 2$ in Chapter 2.

For now, we present another fundamental example of a k-rank graph (the infinite version of 1.1.12) which is easily visualised as a k-coloured graph.

1.1.18 Example (Infinite lattice k-rank graph) Let $k \ge 1$, and let Ω_k be the countable small category defined by sets $\mathrm{Ob}(\Omega_k) := \mathbb{N}^k$ and $\mathrm{Hom}(\Omega_k) := \{(\mathbf{m}, \mathbf{n}) \in \mathbb{N}^k \times \mathbb{N}^k \mid \mathbf{m} \le \mathbf{n}\}$.

We identify $\mathrm{Ob}(\Omega_k)$ with the set of identity morphisms $\{(\mathbf{m}, \mathbf{m}) \mid \mathbf{m} \in \mathbb{N}^k\}$, and hence identify Ω_k with $\mathrm{Hom}(\Omega_k)$. Define range and domain maps $r(\mathbf{m}, \mathbf{n}) := \mathbf{m}$ and $s(\mathbf{m}, \mathbf{n}) := \mathbf{n}$, respectively. Then Ω_k together with the degree map $d(\mathbf{m}, \mathbf{n}) := \mathbf{n} - \mathbf{m}$ forms a k-rank graph. We can draw the 1-skeleton G_{Ω_k} of (Ω_k, d) as an infinite k-dimensional non-negative integer lattice with arrows of colour i from $\mathbf{n} + \mathbf{e}_i$ to $\mathbf{n}$ for all $\mathbf{n} \in \mathbb{N}^k$, as in Figure 1.5.

Any paths between the same two domain and range vertices are deemed to be equivalent in the k-rank graph, thus the elements of (Ω_k, d) can be visualised as rectangles in the lattice (as in Example 1.1.7, we are recycling the symbol Ω_k to denote the category Ω_k

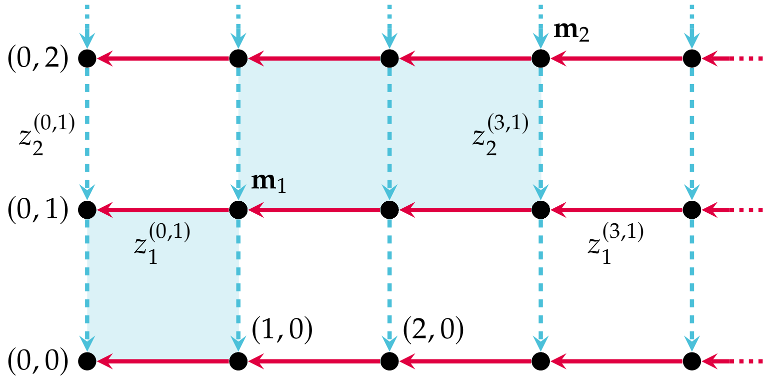


Figure 1.5: The 1-skeleton of the 2-rank graph (Ω_2, d) from 1.1.18, with some vertices and edges labelled. While elements of $\mathrm{Hom}(\Omega_2)$ are paths between vertices, a path of degree (n_1, n_2) in the 2-rank graph is represented by a 2-dimensional rectangle with shape $n_1 \times n_2$. The paths $(\mathbf{m}_1, \mathbf{0})$ and $(\mathbf{m}_2, \mathbf{m}_1)$ of respective degrees $(1, 1)$ and $(2, 1)$ are shaded above, for example. The path $(\mathbf{m}_2, \mathbf{m}_1) \in \Omega_2^{(2,1)}$ should remind the reader of the 2-coloured graph $E(2, (2, 1))$ from Figure 1.3.

together under this equivalence relation). Indeed, elements of $\Omega_k^{\mathbf{m}}$ (the paths of degree $\mathbf{m}$) will look like those k-coloured graphs $E(k, \mathbf{m})$ from Example 1.1.12, and can be indexed by the k-coloured–graph morphisms $\lambda : E(k, \mathbf{m}) \to G_{\Omega_k}$. The k-graph (Ω_k, d) could be regarded as the k-graph induced by $E(k, \mathbf{m})$ where each co-ordinate of $\mathbf{m}$ is ∞. ♦

§1.2 Introduction to operator algebras

1.2.1 We will observe in §1.3 that any directed graph and any k-graph can be assigned a *graph algebra*. These $C^\star$-algebras incorporate a good deal of interesting examples, both new and classical (see [Rae05] and 1.3.4–1.3.6), while at the same time inheriting properties from their underlying graphs.

The groups $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$ associated to a $C^\star$-algebra $\mathcal{A}$ prove to be powerful invariants, and in all of our cases actually determine the algebras themselves—this means that by computing the groups K_ϵ (where $\epsilon \in \{0, 1\}$) for each of our graph algebras, we can see at a glance whether or not they are isomorphic.

In order to understand the impact of graph algebras and their generalisation to higher-rank graph algebras, the reader ought to be familiar with some fundamentals of functional analysis. We present in this section a very brief overview and direct the reader to reference texts [Bla06; Bou07b, III–V; Rae05; WO93].

1.2.2 Definition (Banach space) Consider a vector space $\mathcal{V}$ over a field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. A norm on $\mathcal{V}$ is a function $\| * \| : \mathcal{V} \to \mathbb{R}$ such that, for all $x, y \in \mathcal{V}$ and $a \in \mathbb{K}$,

N1 $\|ax\| = |a| \cdot \|x\|$ (it is *absolutely homogeneous*),

N2 $\|x\| = 0$ if and only if $x = 0$ (it *separates points*),

N3 $\|x + y\| \leq \|x\| + \|y\|$ (it is *subadditive*, or satisfies the *triangle inequality*).

We call a vector space endowed with a norm a normed vector space. An infinite sequence of elements $x_1, x_2, \ldots$ in a normed vector space $\mathcal{V}$ is said to converge in norm to a limit $x \in \mathcal{V}$ if $\|x_i - x\| \to 0$ as i increases. Similarly, a sequence is called Cauchy if its elements eventually get arbitrarily close to each other, that is, $\|x_j - x_i\| \to 0$ as i increases, for all $j > i$.

In general, every convergent sequence in a normed vector space $\mathcal{V}$ is also a Cauchy sequence, and if $\mathcal{V} = \mathbb{C}$ or $\mathcal{V} = \mathbb{R}$ then the converse is also true: sequences in $\mathbb{C}$ or $\mathbb{R}$ converge if and only if they are Cauchy. We now generalise this by defining a type of space where this is always the case.

A normed vector space $\mathcal{V}$ is said to be complete if every Cauchy sequence in $\mathcal{V}$ converges in norm to some limit inside $\mathcal{V}$.

The closure of $\mathcal{V}$ comprises the limit points of all convergent (not just Cauchy) sequences in $\mathcal{V}$. The completion of a normed vector space $\mathcal{V}$ is a complete space inside which $\mathcal{V}$ is *dense*; the completion of a space always exists and is unique (see [Yos95, I.10]).

A complete normed vector space over $\mathbb{C}$ (or over $\mathbb{R}$) is called a Banach space.

An algebra is a vector space $\mathcal{A}$ over a field $\mathbb{K}$ together with an associative multiplication operation $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, written $(t, u) \mapsto tu$, which satisfies

A1 $t(u + u') = tu + tu'$ and $(t + t')u = tu + t'u$, for all $t, t', u, u' \in \mathcal{A}$,

A2 $(at)(bu) = (ab)(tu)$ for all $t, u \in \mathcal{A}$ and $a, b \in \mathbb{K}$.

We say that $\mathcal{A}$ is unital if, in addition, there is an element $\mathrm{id} \in \mathcal{A}$ such that $\mathrm{id}\, t = t\, \mathrm{id} = t$ for all $t \in \mathcal{A}$. All of the algebras in this thesis will be unital and defined over the field of complex numbers $\mathbb{C}$. The notions of norm, convergence, and completeness are all valid if, instead of a vector space, $\mathcal{A}$ is an algebra over $\mathbb{C}$—we need only replace each instance of the words "vector space" with "algebra" above, and insist that the norm also satisfies

N4 $\|tu\| \leq \|t\| \cdot \|u\|$ (the norm $\| * \|$ is *submultiplicative*).

Thus a Banach algebra is a complete normed algebra.

1.2.3 While a norm gives us a measure of distance between two elements of a vector space or algebra, an *inner product* can express the notion of *angle*. Consider a vector space $\mathcal{V}$ over $\mathbb{C}$, and let $\langle\, * \,,\, * \,\rangle : \mathcal{V} \to \mathbb{C}$ be an inner product: a function which satisfies

IP1 $\langle x, ay + by' \rangle = a\langle x, y\rangle + b\langle x, y'\rangle$,

IP2 $\langle y, x\rangle = \overline{\langle x, y\rangle}$,

IP3 $\langle x, x\rangle \in \mathbb{R}_{\geq 0}$ and $\langle x, x\rangle = 0$ if and only if $x = 0$,

for all $x, y, y' \in \mathcal{V}$ and $a, b \in \mathbb{C}$. This function is designed to be reminiscent of the *dot product* of two vectors in $\mathbb{C}^n$. We call $\mathcal{V}$ an inner product space, and property **IP3** ensures that we can define a norm on $\mathcal{V}$ by $\|x\| := \sqrt{\langle x, x\rangle}$. An inner product space which is complete with respect to this norm is called a Hilbert space.

1.2.4 **Example (Sequence spaces ℓ^p)** Let p be a positive real number and let I be a countable index set. We write $\ell^p(I)$ to denote the set of all infinite sequences $\{x_i \in \mathbb{C}\}_{i\in I}$ such that the series $\sum_i |x_i|^p$ converges. Note that these constructions also work for sequences of real numbers. Sequences are said to be summable if they are in $\ell^1(I)$, square-summable in $\ell^2(I)$, and p-summable for all other p. In the case that $p = \infty$, the space $\ell^\infty(I)$ comprises all bounded sequences, those being sequences $\{x_i\}_{i\in I}$ such that $\sup_i |x_i| < \infty$. The index set I is most often the natural numbers $\mathbb{N}$, so we usually denote $\ell^p(\mathbb{N})$ by the shorthand ℓ^p.

The sets $\ell^p(I)$ are vector spaces under co-ordinate–wise addition, and when $1 \leq p < \infty$ there is a norm $\| * \|_p$ on each respective space called the ℓ^p-norm, defined by

$$\|\{x_i\}\|_p := \left(\sum_i |x_i|^p \right)^{1/p}$$

and $\|\{x_i\}\|_\infty := \sup_i |x_i|$ when $p = \infty$. Indeed, whenever $p \geq 1$, the space $\ell^p(I)$ is complete with respect to its ℓ^p-norm, and hence is a Banach space.

If the index set I is finite with $|I| = n$, then a sequence indexed by I is just an r-tuple with entries in $\mathbb{C}$. Clearly such a sequence is p-summable for all p, so $\ell^p(I) \cong \mathbb{C}^n$ whenever $|I| = n$. In this way, the ℓ^p spaces could be seen as a generalisation of $\mathbb{C}^n$ to infinite-dimensional space.

Given any index set I, the space $\ell^p(I)$ is strictly contained in each space $\ell^q(I)$ whenever $p < q$. In particular, all of the ℓ^p spaces are distinct from each other. Moreover, ℓ^2 is the only ℓ^p space which is a Hilbert space, with inner product defined by

$$\langle \{x_i\}, \{y_i\} \rangle := \sum_i x_i \overline{y_i}.$$

1.2.5 By comparing the inner product in a Hilbert space with the dot product in $\mathbb{C}^n$, we arrive at the generalised definition of orthogonality: two elements $x, y \in \mathcal{H}$ are orthogonal if $\langle x, y \rangle = 0$. Given a subset $S \subseteq \mathcal{H}$, we define the orthogonal complement $S^\perp$ of S to be the set of elements $y \in \mathcal{H}$ which are orthogonal to every $x \in S$. A vector subspace $\mathcal{V} \subseteq \mathcal{H}$ is dense whenever $\mathcal{V}^\perp = \{0\}$, or equivalently, if the closure (1.2.2) of $\mathcal{V}$ is equal to $\mathcal{H}$. An orthonormal basis for $\mathcal{H}$ is a sequence $\{x_i \in \mathcal{H}\}$ of mutually orthogonal elements, with $\|x_i\| = 1$ for all i, and whose span $\{\sum_i a_i x_i \mid a_i \in \mathbb{C}\}$ is dense in $\mathcal{H}$. A Hilbert space is called separable if it admits an orthonormal basis.

We have discussed above that the ℓ^p spaces are generalisations of $\mathbb{C}^n$ to infinite dimensions, and that ℓ^2 is the only one which is a Hilbert space. In fact, *any* separable Hilbert space $\mathcal{H}$ is isometrically isomorphic to $\ell^2(I)$ for some finite or countable set I. The cardinality of I is the dimension of $\mathcal{H}$, as well as the length of any orthonormal basis of $\mathcal{H}$ (as seen, for example, in [Bou07b, V, §2.4]).

1.2.6 Example (Bounded linear operators) Let $\mathcal{H} \neq \{0\}$ be a separable Hilbert space over the complex numbers and let $\mathcal{B}(\mathcal{H})$ be the space of maps $t : \mathcal{H} \to \mathcal{H}$ which are *continuous* with respect to the usual norm (1.2.3). These are precisely the bounded linear operators on $\mathcal{H}$: maps t which satisfy $t(x + ay) = t(x) + a \cdot t(y)$ and for which there exists some $c > 0$ with $\|t(x)\| \leq c \cdot \|x\|$, for all $x, y \in \mathcal{H}$ and $a \in \mathbb{C}$. We define a norm on $\mathcal{B}(\mathcal{H})$, called the operator norm, by $\|t\| := \sup\{\|t(x)\|/\|x\| \mid x \in \mathcal{H}, x \neq 0\}$.

For each bounded linear operator $t \in \mathcal{B}(\mathcal{H})$, there is a unique map $t^\star \in \mathcal{B}(\mathcal{H})$, called the adjoint to t, which satisfies $\langle x, t(y) \rangle = \langle t^\star(x), y \rangle$ for all $x, y \in \mathcal{H}$. The space $\mathcal{B}(\mathcal{H})$ is a Banach algebra with respect to the operator norm. Subalgebras of $\mathcal{B}(\mathcal{H})$ whenever $\mathcal{H}$ is a separable Hilbert space are often referred to as *operator algebras*. ♦

1.2.7 Definition ($C^\star$-Algebra) Abstractly, a $C^\star$-algebra $\mathcal{A}$ is a Banach algebra over the field $\mathbb{C}$ together with a map $\star : \mathcal{A} \to \mathcal{A}$, written $x \mapsto x^\star$, such that for all $x, y \in \mathcal{A}$ and $a \in \mathbb{C}$:

⋆1 $(x^\star)^\star = x$ ($\star$ is an *involution*),

⋆2 $(x + y)^\star = x^\star + y^\star$ and $(ax)^\star = \bar{a}x^\star$, where $\bar{a}$ is the complex conjugate of a ($\star$ is *conjugate-linear*),

⋆3 $(xy)^\star = y^\star x^\star$ ($\star$ is *antimultiplicative*),

⋆4 $\|x^\star\| = \|x\|$ ($\star$ is *isometric*),

⋆5 $\|x^\star x\| = \|xx^\star\| = \|x\|^2$ ($\star$ satisfies the $C^\star$ *Identity*),

A map with properties **⋆1–4** is sometimes called a star operation, and an algebra equipped with such a map is called a Banach $\star$-algebra.

Equivalently, a $C^\star$-algebra is a subalgebra $\mathcal{A}$ of the operator algebra $\mathcal{B}(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$, which has the following properties:

- **⋆a** $\mathcal{A}$ is *norm-closed* in the sense that every convergent sequence in $\mathcal{A}$ has its limit (with respect to the *operator norm*) in $\mathcal{A}$,
- **⋆b** $\mathcal{A}$ is closed under the *adjoint* operation (see 1.2.6).

Given two $C^\star$-algebras $\mathcal{A}$ and $\mathcal{A}'$, a bounded linear operator $\pi : \mathcal{A} \to \mathcal{A}'$ is called a ⋆-homomorphism if $\pi(xy) = \pi(x)\pi(y)$ and $\pi(x^\star) = \pi(x)^\star$ for all $x, y \in \mathcal{A}$. We call a bijective ⋆-homomorphism a ⋆-isomorphism.

1.2.8 When we are regarding a $C^\star$-algebra as an operator algebra with properties **⋆a–b**, we sometimes describe it as concrete, else we might call it abstract.

Historically, the second definition of $C^\star$-algebras was the first to appear, along with another class of operator algebras called *von Neumann algebras*. The *Gelfand–Naĭmark Theorem* of 1943 allowed the study of $C^\star$-algebras as abstract objects without considering Hilbert spaces (see [Bla06, II.2.2]). The $C^\star$-algebras in this thesis are generated by bounded linear operators, though we rarely care about the underlying Hilbert space.

1.2.9 **Definition (Some qualities of operators)** Consider a (unital) $C^\star$-algebra $\mathcal{A}$ concretely, as an algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. A projection is an element $p \in \mathcal{A}$ such that $p^2(x) = p(x) = p^\star(x)$ for all $x \in \mathcal{H}$. A partial isometry is an operator $t \in \mathcal{A}$ such that $t(x) = tt^\star t(x)$ for all $x \in \mathcal{H}$, or equivalently, one of $tt^\star$ or $t^\star t$ is a projection (in which case both are projections). An operator $t \in \mathcal{A}$ is self-adjoint if $t = t^\star$.

An isometry is an operator $t \in \mathcal{A}$ which satisfies $t^\star t(x) = x$ for all $x \in \mathcal{H}$. If, in addition, $tt^\star(x) = x$, then we say that t is unitary; we usually use u to symbolise an operator we know to be unitary. An operator u on $\mathcal{H}$ is unitary if and only if it is surjective and $\langle u(x), u(y)\rangle = \langle x, y\rangle$ for all $x, y \in \mathcal{H}$.

The K-theory of a $C^\star$-algebra

1.2.10 To every $C^\star$-algebra $\mathcal{A}$, we can assign a pair of Abelian groups $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$, called the K-groups. These groups, as we shall see in Chapters 3–4, have the potential to encode an enormous amount of data about their parent algebras, and can be used as invariants to distinguish them from one another. We offer a whistle-stop description of K-theory in the remainder of this section, and invite the reader to consult [Rae05, Chap. 7; RLL00; WO93] when in need of more exposition or rigour.

1.2.11 **The group K_0** Write $\mathbb{M}_n(\mathcal{A})$ to denote the set of all $n \times n$ matrices with entries in some *unital* concrete $C^\star$-algebra $\mathcal{A}$; this has a natural $C^\star$-algebra structure. Consider the space $\mathbb{M}_\infty(\mathcal{A}) := \bigcup_{n\geq 1} \mathbb{M}_n(\mathcal{A})$. We can define an equivalence relation on the subset of projections of $\mathbb{M}_\infty(\mathcal{A})$, calling two projections p, p' *(Murray–von Neumann) equivalent* whenever we can find a partial isometry $t \in \mathbb{M}_\infty(\mathcal{A})$ such that $t^\star t = p$ and $tt^\star = p'$; write $[p]$ to denote the class of a projection p under this equivalence.

Define the subset $V(\mathcal{A}) := \{[p] \mid p \in \mathbb{M}_\infty(\mathcal{A}) \text{ is a projection}\}$ and observe that this is an additive Abelian semigroup under the operation $[p] + [q] := [p' \oplus q']$, where $p' \in [p]$ and $q' \in [q]$ are such that $p'q' = 0$, and $p' \oplus q'$ is the block matrix with p, q on the diagonal and the zero map elsewhere.

We define $K_0(\mathcal{A})$ to be the *Grothendieck group* of $V(\mathcal{A})$, that is, the group

$$K_0(\mathcal{A}) := \{[p] - [q] \mid p, q \in V(\mathcal{A})\}$$

of "formal differences" between equivalence classes of projections in $\mathbb{M}_\infty(\mathcal{A})$. This is an Abelian group with addition defined by

$$([p_1] - [q_1]) + ([p_2] - [q_2]) := ([p_1] + [p_2]) - ([q_1] + [q_2]).$$

The identity is the class which contains the zero matrices (all of whose entries are the trivial projection 0, which sends everything to zero), since p is Murray–von Neumann equivalent to $p \oplus 0$.

1.2.12 The group K_1 As above, let $\mathcal{A}$ be a concrete unital $C^\star$-algebra, and let $\mathbb{M}_n(\mathcal{A})$ denote the set of all $n \times n$ matrices with entries from $\mathcal{A}$. Consider the groups $\mathcal{U}_n(\mathcal{A}) := \mathcal{U}(\mathbb{M}_n(\mathcal{A}))$ of unitary elements of $\mathbb{M}_n(\mathcal{A})$ for each n. We will call unitary matrices u, u' *equivalent* if they are homotopy equivalent, that is, if there is a continuous path $[0, 1] \to \mathcal{U}_n(\mathcal{A})$ valued u at 0 and u' at 1. This extends to an equivalence relation on $\mathcal{U}_\infty(\mathcal{A}) := \bigcup_{n\geq 1} \mathcal{U}_n(\mathcal{A})$, since we can embed $\mathbb{M}_n(\mathcal{A})$ in $\mathbb{M}_{n+1}(\mathcal{A})$ via the map $u \mapsto u \oplus \mathrm{id}_\mathcal{A}$. We write $[u]$ for the equivalence class of a unitary matrix $u \in \mathcal{U}_\infty(\mathcal{A})$.

Now, we set $K_1(\mathcal{A})$ as the multiplicative Abelian group

$$K_1(\mathcal{A}) := \{[u] \mid u \in \mathcal{U}_\infty(\mathcal{A})\},$$

with multiplication defined by $[u] \cdot [v] := [u \cdot v]$, where $u \cdot v$ is the block matrix with u, v on the diagonal and the zero map elsewhere. The identity is the class containing those matrices with each diagonal entry $\mathrm{id}_\mathcal{A}$ and the zero map elsewhere.

1.2.13 The groups K_0 and K_1 define functors from the category of $C^\star$-algebras to the category of Abelian groups. The significance of this functoriality cannot be overstated; indeed, the K-theory of $C^\star$-algebras can be characterised entirely by its properties as a functor (see [WO93, Chap. 11]). Moreover, it is often easier to determine the properties of the algebras by passing to the K-groups, as we shall see in the remainder of this thesis.

Modifications can be made to the above constructions which would let us associate K-groups to non-unital algebras, but Proposition 1.3.13 will allow us to get away without them.

1.2.14 Definition (Tensor product) Let $\mathcal{V}, \mathcal{V}'$ be vector spaces over a field $\mathbb{K}$, and suppose there is a map π taking $(v, v') \in \mathcal{V} \times \mathcal{V}'$ to an element $v \otimes v' \in \mathcal{V} \otimes \mathcal{V}'$, which is *multilinear*, that is, which satisfies

(i) $(v + w) \otimes w' = v \otimes w' + w \otimes w'$ and $v \otimes (v' + w') = v \otimes v' + v \otimes w'$,

(ii) $a(v \otimes v') = (av) \otimes v' = v \otimes (av')$,

for all $v, w \in \mathcal{V}$, $v', w' \in \mathcal{V}'$, and $a \in \mathbb{K}$, and which satisfies the *universal property*:

(iii) Whenever $\rho : \mathcal{V} \times \mathcal{V}' \to \mathcal{W}$ is a multilinear map into any vector space $\mathcal{W}$, then there exists a unique linear map $\overline{\rho} : \mathcal{V} \otimes \mathcal{V}' \to \mathcal{W}$ such that $\overline{\rho} \circ \pi = \rho$.

The space $\mathcal{V} \otimes \mathcal{V}'$ is called the tensor product of $\mathcal{V}$ and $\mathcal{V}'$. If $\{v_i\}_{i=1}^{n}$ and $\{v'_j\}_{j=1}^{n'}$ are respective bases for $\mathcal{V}, \mathcal{V}'$, then $\{v_i \otimes v'_j\}$ forms a basis for $\mathcal{V} \otimes \mathcal{V}'$—the dimension of a tensor product is therefore the product of the dimensions of the multiplicands.

For $C^\star$-algebras $\mathcal{A}$ and $\mathcal{A}'$ we use a slightly different notation, since their tensor product as vector spaces does not always induce a natural norm. We write $\mathcal{A} \odot \mathcal{A}'$ to denote the algebraic tensor product, that is, their tensor product as vector spaces over $\mathbb{C}$. This space may admit different norms which satisfy the $C^\star$-identity $\star$**5**, under which the algebraic tensor product $\mathcal{A} \odot \mathcal{A}'$ satisfies condition $\star$**a** of 1.2.7, turning it into a $C^\star$-algebra $\star$-algebra. We call a $C^\star$-algebra $\mathcal{A}$ nuclear if, for every $C^\star$-algebra $\mathcal{A}'$, there is a *unique* norm on the tensor product $\mathcal{A} \odot \mathcal{A}'$ which makes it a $C^\star$-algebra. We write $\mathcal{A} \otimes \mathcal{A}'$ for the completion of $\mathcal{A} \odot \mathcal{A}'$ with respect to this norm.

1.2.15 The class of nuclear $C^\star$-algebras comprises a lot of handy $C^\star$-algebras, including all matrix algebras $\mathbb{M}_n(\mathbb{C})$, the *Cuntz–Krieger algebras* $\mathcal{O}_A$ from 1.3.6 (by, for example, [EL99, §7]), and any $C^\star$-algebra which is finite or *approximately finite* (see [WO93, §12.1]). Nuclearity is also preserved under tensor products and quotients (the reader should consult [Bla06, II.9.4] for a completer picture).

We present at the conclusion of this section a direct sum construction for $C^\star$-algebras, and a useful feature of the K-groups of nuclear $C^\star$-algebras, as described in [WO93, §9.3].

1.2.16 A large and important category of $C^\star$-algebras introduced by Rosenberg and Schochet in [RS87] is the *bootstrap class*. The precise composition of this category is not needed; let it be sufficient to say that it is the smallest class containing those nuclear and *separable* (in the sense of 2.3.5) $C^\star$-algebras which can be obtained from $\mathbb{C}$ through countably-many "elementary K-theoretic operations" (see [Bla06, V.1.5]).

The bootstrap class contains a vast number of common $C^\star$-algebras, including all those which are commutative, and it is an open question whether it is exactly the class of all separable nuclear $C^\star$-algebras.

1.2.17 Theorem (Direct sum of $C^\star$-algebras) *Let $\mathcal{A}, \mathcal{A}'$ be unital $C^\star$-algebras. Then their direct sum $\mathcal{A} \oplus \mathcal{A}' := \{(x, x') \mid x \in \mathcal{A}, x' \in \mathcal{A}'\}$ is a $C^\star$-algebra with co-ordinate–wise operations and norm $\|(x, x')\| := \max\{\|x\|, \|x'\|\}$. Moreover, $K_\epsilon(\mathcal{A} \oplus \mathcal{A}') \cong K_\epsilon(\mathcal{A}) \oplus K_\epsilon(\mathcal{A}')$, for $\epsilon = 0, 1$.* ■

1.2.18 Theorem (Künneth Theorem for tensor products) *Let $\mathcal{A}, \mathcal{A}'$ be nuclear $C^\star$-algebras whose K-groups are finitely-generated and torsion-free (that is, isomorphic to a direct sum of copies of $\mathbb{Z}$). Then if $\mathcal{A}$ is in the bootstrap class, we have:*

(i) $K_0(\mathcal{A} \otimes \mathcal{A}') \cong K_0(\mathcal{A}) \otimes K_0(\mathcal{A}') \oplus K_1(\mathcal{A}) \otimes K_1(\mathcal{A}')$.

(ii) $K_1(\mathcal{A} \otimes \mathcal{A}') \cong K_0(\mathcal{A}) \otimes K_1(\mathcal{A}') \oplus K_1(\mathcal{A}) \otimes K_0(\mathcal{A}')$. ■

1.2.19 The construction of each $C^\star$-algebra in this thesis promises that each K-group be finitely-generated, and hence we will be able to utilise Theorem 1.2.18.

§1.3 Graph algebras

1.3.1 Definition (Properties of k-graphs) Let (Λ, d) be a k-rank graph, let $\mathbf{n} \in \mathbb{N}^k$, and let $v \in \Lambda^{\mathbf{0}}$. We write $v\Lambda^{\mathbf{n}}$ for the set of paths of degree $\mathbf{n}$ which map onto the vertex v, that is, $v\Lambda^{\mathbf{n}} := \{\lambda \in \Lambda^{\mathbf{n}} \mid r(\lambda) = v\}$.

We say that (Λ, d) is row-finite if the set $v\Lambda^{\mathbf{n}}$ is finite for each $\mathbf{n} \in \mathbb{N}^k$ and $v \in \Lambda^{\mathbf{0}}$, and that (Λ, d) has no sources if each $v\Lambda^{\mathbf{n}}$ is non-empty.

We say that (Λ, d) is locally convex if, whenever $i, j \leq k$ are distinct natural numbers and $\mu \in \Lambda^{\mathbf{e}_i}$, $\nu \in \Lambda^{\mathbf{e}_j}$ are paths with $r(\mu) = r(\nu)$, then the sets $s(\mu)\Lambda^{\mathbf{e}_j}$ and $s(\nu)\Lambda^{\mathbf{e}_i}$ are non-empty.

1.3.2 In [KPR98], a row-finite directed graph (1-graph) G with no sources was bestowed a so-called *graph algebra*. This is a concrete $C^\star$-algebra generated by a set of projections and partial isometries indexed by vertices and edges of G, respectively, and which satisfy some multiplication conditions. The graph algebra is unique up to $\star$-isomorphism, and so we have an extremely useful way of representing a graph in terms of Hilbert space operators.

1.3.3 Definition (Graph $C^\star$-algebra) Let G be a row-finite directed graph with no sources, and let $\{p_v \in \mathcal{B}(\mathcal{H}) \mid v \in G^0\}$ and $\{t_e \in \mathcal{B}(\mathcal{H}) \mid e \in G^1\}$ be, respectively, sets of projections and partial isometries on some separable Hilbert space $\mathcal{H}$. We call the collection $\{p_v, t_e\}$ a Cuntz–Krieger G-family if it satisfies

(i) $p_u p_v = 0$ for all $u \neq v$,

(ii) $t_e^\star t_e = p_{s(e)}$ for all $e \in G^1$,

(iii) $p_v = \sum_{\{e \mid r(e) = v\}} t_e t_e^\star$ for all $v \in G^0$.

Let $\mathcal{A}(G)$ denote the $C^\star$-algebra which is *universal* in the following sense: if $\{P_v, T_e\}$ is another Cuntz–Krieger G-family in some $C^\star$-algebra $\mathcal{A}'$, then there is a homomorphism $\pi : \mathcal{A}(G) \to \mathcal{A}'$ with $\pi(p_v) = P_v$ and $\pi(t_e) = T_e$ for all p_v, t_e. The algebra $\mathcal{A}(G)$ exists, and we call it the graph algebra of G, and say that $\{p_v, t_e\}$ generates $\mathcal{A}(G)$.

1.3.4 Example (Some easy graph algebras) Let G be the directed graph consisting of a single vertex v and no edges. Then a Cuntz–Krieger G-family will have one generator p_v, with $p_v^2 = p_v = p_v^\star$, and no other relations. In this instance, p_v behaves as the identity operator, and since the universal algebra generated by id is isomorphic to $\mathbb{C}$, we have $\mathcal{A}(G) \cong \mathbb{C}$.

Now, consider the directed graph O_1 which comprises one vertex v, and one directed edge e from v to itself. The graph algebra $\mathcal{A}(O_1)$ is the universal $C^\star$-algebra generated by a projection p_v and a partial isometry t_e which satisfies $t_e^\star t_e = p_v = t_e t_e^\star$. Since t_e is a partial isometry, we have $p_v t_e = t_e t_e^\star t_e = t_e$, and similarly $t_e p_v = t_e$. This means that p_v is the identity operator, and hence that t_e is unitary. Let $\mathbb{T} := \{a \in \mathbb{C} \mid |a| = 1\}$ denote the unit circle, and write $C(\mathbb{T})$ for the algebra of all functions $f : \mathbb{T} \to \mathbb{C}$ which are continuous with respect to the norm $\|f\|_\infty := \sup_{a\in\mathbb{T}}\{|f(a)|\}$—under this norm $C(\mathbb{T})$ is a $C^\star$-algebra, with complex conjugation as the involution. It is not difficult to show that the universal algebra generated by the identity and a single unitary operator is isomorphic to $C(\mathbb{T})$, using the *Continuous Functional Calculus* together with the fact that the *spectrum* of a unitary element is contained in $C(\mathbb{T})$ (consult [Bla06, II.2.3] for details). We can therefore conclude that $\mathcal{A}(O_1) \cong C(\mathbb{T})$.

Using the index in [RLL00], we find the K-groups for the above graph algebras to be as follows: $K_0(\mathcal{A}(G)) \cong K_0(\mathcal{A}(O_1)) \cong K_1(\mathcal{A}(O_1)) \cong \mathbb{Z}$, whereas $K_1(\mathcal{A}(G)) \cong 0$. ♦

1.3.5 Example (Cuntz algebras) Now consider the directed graph O_n with one vertex v and n loops around it labelled $1, \ldots, n$. We form a Cuntz–Krieger O_n-family $\{p_v, t_i \mid 1 \le i \le n\}$ with a single projection p_v and partial isometries which satisfy $t_i^\star t_i = p_v = \sum_i t_i t_i^\star$. As in Example 1.3.4, we can deduce that p_v is the identity operator in the universal $C^\star$-algebra generated by such a family, and so each t_i is an isometry (Definition 1.2.9). In [Cun77], Cuntz defined the $C^\star$-algebra $\mathcal{O}_n$ to be the one generated by n isometries t_i under the relation $\sum_i t_i t_i^\star = \text{id}$. We now call these the Cuntz algebras, and this example shows that each Cuntz algebra $\mathcal{O}_n$ is the graph algebra of a graph with one vertex and n directed edges. The K-groups of Cuntz algebras are well-understood; in fact Cuntz himself showed that $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)$ whenever $n \ge 2$ and $K_1(\mathcal{O}_n) \cong \mathbb{Z}$ for all n (from the table of K-groups in [RLL00]). ♦

1.3.6 So far, we have computed graph algebras by considering the edge-indexed partial isometries individually, and studying how the relations in 1.3.3 affect them. In [CK80], the Cuntz–Krieger algebra $\mathcal{O}_A$ was introduced for each $n \times n$ matrix A with entries in $\{0, 1\}$ and with non-zero rows and columns. This is the universal $C^\star$-algebra generated by a set

$\{t_i \mid 1 \le i \le n\}$ of partial isometries which satisfies $t_i^\star t_i = \sum_j A(i,j)t_j t_j^\star$, where $A(i,j)$ is the (i,j)-th entry in the matrix A.

Enomoto and Watatani showed in [EW80] that when G is a directed graph with a finite number of vertices, $\mathcal{A}(G)$ can be expressed as a Cuntz–Krieger algebra for some matrix A. We might naïvely consider setting A as the $|G^0| \times |G^0|$ *vertex matrix*:

$$A_G(u,v) := |\{\text{Edges } e \in G^1 \mid s(e) = u, r(e) = v\}|$$

for $u, v \in G^0$, but we quickly notice that this matrix might not have the entries in $\{0,1\}$ required by a Cuntz–Krieger algebra. Instead, we define the *edge matrix*:

$$B_G(e,f) := \begin{cases} 1 & \text{if } r(e) = s(f), \\ 0 & \text{otherwise,} \end{cases}$$

which is a $\{0,1\}$-matrix, and is such that $\mathcal{O}_{B_G} \cong \mathcal{A}(G)$ by [Rae05 2.6–2.8 and references therein]. Hence the graph algebras of finite directed graphs *with no sinks or sources* (that is, where every vertex is the target of some edge and the origin of some edge) are isomorphic to Cuntz–Krieger algebras. These methods have been adapted for graphs with sinks or sources (for example in [BPRS00]), though we don't need them here.

The Cuntz algebras $\mathcal{O}_n$ from Example 1.3.5 can be recovered as the Cuntz–Krieger algebra $\mathcal{O}_A$, where A is the $n \times n$ matrix whose entries are all 1: the edge matrix for the graph O_n.

1.3.7 Theorem (Raeburn and Szymański, 2004) *Let G be a row-finite directed graph with n vertices and no sources, and let A_G be the corresponding vertex matrix defined in 1.3.6. Write $\mathbf{1}$ to denote the $n \times n$ identity matrix, and consider the matrix $\partial := \mathbf{1} - A_G^T$ as an endomorphism of $\mathbb{Z}^n$, where the generators of $\mathbb{Z}^n$ are indexed by the vertices of G.*

Then $K_1(\mathcal{A}(G)) \cong \ker(\partial)$ and $K_0(\mathcal{A}(G)) \cong \mathbb{Z}^n/\mathrm{im}(\partial)$, the cokernel of ∂ [Rae05, 7.16]. ■

1.3.8 The above theorem was proven for graphs with a countably-infinite number of vertices in [RS04]. It gives us a taste of the deep connection between the structure of $C^\star$-algebras and the geometry of graphs, which we further explore in 2.2.6 and 3.4.7. Note that these K-theory computations use the *vertex matrix* and not the *edge matrix* (1.3.6) needed to construct the Cuntz–Krieger algebra.

A natural question to ask at this point is: "is every $C^\star$-algebra the graph algebra for some graph?" and the answer is: *no*! Theorem 1.3.7 asserts that $K_1(\mathcal{A}(G))$ be a subgroup of the free Abelian group $\mathbb{Z}^n$, and hence is a free Abelian group itself. This means that no $C^\star$-algebra whose K_1 group has torsion can be represented as a graph algebra (and algebras with such K_1 do exist). An elegant converse to this was demonstrated by Szymański in [Szy02]. We paraphrase his argument that, given countable Abelian groups $\mathcal{G}_0, \mathcal{G}_1$ where $\mathcal{G}_1$ is torsion-free, then there exists a row-finite directed graph G (with finitely-many vertices) such that $K_0(\mathcal{A}(G)) \cong \mathcal{G}_0$ and $K_1(\mathcal{A}(G)) \cong \mathcal{G}_1$. This is to say that $K_1(\mathcal{A})$ of a graph algebra $\mathcal{A}$ is destined to be free Abelian, but this is the *only* condition the K-groups need satisfy.

1.3.9 Example (K-theory of a cyclic graph) Denote by C_5 the directed graph depicted in Figure 1.6a, comprising five vertices $v_1, \ldots, v_5$ and five directed edges e_i from v_i to v_{i+1} (where e_5 points from v_5 to v_1).

The vertex matrix A_{C_5} has zeroes everywhere except in the five $(i, i+1)$-th positions. Hence the map $\partial : \mathbb{Z}^5 \to \mathbb{Z}^5$ from Theorem 1.3.7 can be expressed by the matrix

$$\partial = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

and it is straightforward to verify that $\ker(\partial) \cong \operatorname{coker}(\partial) \cong \mathbb{Z}$. Hence the graph algebra $\mathcal{A}$ of C_5 has K-groups $K_0(\mathcal{A}) \cong K_1(\mathcal{A}) \cong \mathbb{Z}$. ♦

1.3.10 As discussed in 1.2.1 and throughout this section, a wide range of useful $C^\star$-algebras arise as graph algebras for some graph; additional examples include the matrix algebras $\mathbb{M}_n(\mathbb{C})$, the algebra of *compact operators* on a Hilbert space $\mathcal{K}(\mathcal{H})$, and the *Toeplitz algebra* $\mathcal{T}$. The simplicity with which the graph algebra relations are defined allows us to paint a clearer picture of the properties of certain $C^\star$-algebras, as in §2.3, for example.

Therefore, we would be very excited if it were possible to use graph algebras to model more general algebras—namely, those without a free K_1 group. This is where Kumjian and Pask's extension of the definition of graph algebras to *higher-rank graphs* takes to the stage. Eventually (2.2.8 and beyond), we'll use their pioneering constructions to develop new classes of higher-rank graph algebras whose K_1 groups have demonstrable torsion. Since a directed graph is a 1-rank graph, it is easy to notice the parallels between the following definition from [KP00] and that of 1.3.3.

Higher-rank graph algebras

1.3.11 Definition (Higher-rank graph $C^\star$-algebra) Let (Λ, d) be a row-finite k-rank graph with no sources. We define a Cuntz–Krieger Λ-family to be a set $\{t_\lambda \mid \lambda \in \Lambda\}$ of partial isometries with the following properties:

CK1 The elements of $\left\{t_v \mid v \in \Lambda^{\mathbf{0}}\right\}$ are projections such that $t_u t_v = 0$ for all $u \neq v$,

CK2 If $r(\mu) = s(\nu)$ for some $\mu, \nu \in \Lambda$, then $t_{\nu\mu} = t_\nu t_\mu$,

CK3 For all $\lambda \in \Lambda$, we have $t_\lambda^\star t_\lambda = t_{s(\lambda)}$,

CK4 For all vertices $v \in \Lambda^{\mathbf{0}}$ and $\mathbf{n} \in \mathbb{N}^k$, we have

$$t_v = \sum_{\lambda \in v\Lambda^{\mathbf{n}}} t_\lambda t_\lambda^\star.$$

Note that without the row-finiteness condition, **CK4** is not well-defined. Similarly, if (Λ, d) has vertices which are sources, then the sets $v\Lambda^{\mathbf{n}}$ may be empty for some values of $\mathbf{n}$ and non-empty for others. All of the examples of higher-rank graphs from this point on are row-finite and without sources.

Let $\mathcal{A}(\Lambda)$ denote the $C^\star$-algebra generated by $\{t_\lambda\}$ which is *universal* in the following sense: if $\{T_\lambda\}$ is another Cuntz–Krieger Λ-family in some $C^\star$-algebra $\mathcal{A}'$, then there is a homomorphism $\pi : \mathcal{A}(\Lambda) \to \mathcal{A}'$ with $\pi(t_\lambda) = T_\lambda$ for all $\lambda \in \Lambda$. Such an algebra exists, is non-zero, and is unique up to $\star$-isomorphism by [Rae05, 1.21–1.22, 10.13]. We call $\mathcal{A}(\Lambda)$ the k-rank graph $C^\star$-algebra of (Λ, d), and say that the family $\{t_\lambda\}$ generates $\mathcal{A}(\Lambda)$.

1.3.12 We define analogues $S, R : \left\{t_\lambda, t_\lambda^\star\right\} \to \Lambda^{\mathbf{0}}$ of the domain and range maps, respectively, as:

$$S(t_\lambda) := s(\lambda), \quad S\left(t_\lambda^\star\right) := r(\lambda), \quad R(t_\lambda) := r(\lambda), \quad R\left(t_\lambda^\star\right) := s(\lambda),$$

for each $\lambda \in \Lambda$. Then we can wrap **CK2** and **CK1** up into the following equivalent condition:

$$t_\nu t_\mu = \begin{cases} t_{\nu\mu} & \text{if } R(t_\mu) = S(t_\nu), \\ 0 & \text{otherwise,} \end{cases}$$

as in [KP00, 1.6; Rae05, 1.12]. A useful fact is that $t_v t_\lambda = t_\lambda$ whenever $R(t_\lambda) = v$, and $t_\lambda t_v = t_\lambda$ whenever $S(t_\lambda) = v$. Furthermore, the sum in **CK4** need only range over the paths $\lambda \in v\Lambda^{\mathbf{e}_i}$, where $\mathbf{e}_i$ are the standard generators of $\mathbb{N}^k$.

1.3.13 Proposition *Let (Λ, d) be a k-rank graph with finite vertex set $\Lambda^{\mathbf{0}}$. Then the sum $\sum_{v \in \Lambda^{\mathbf{0}}} t_v$ is an identity in $\mathcal{A}(\Lambda)$, and so $\mathcal{A}(\Lambda)$ is unital.*

■ **Proof** Write $\Sigma := \sum_{v\in\Lambda^0} t_v$. By the observations in 1.3.12, $t_v t_\lambda = 0$ unless $v = r(\lambda)$, so $\Sigma t_\lambda = t_{r(\lambda)} t_\lambda = t_\lambda$. Likewise, $t_\lambda t_v = 0$ unless $v = s(\lambda)$, so $t_\lambda \Sigma = t_\lambda$ (see [RS99, 3.4]). In fact, the converse to this proposition is also true (see [KPR98, 1.4]). ■

1.3.14 Example (k-rank graph tori) Consider the category T^k which consists of a single object and has morphism set generated by k commuting morphisms $f_1, \ldots, f_k$. This means that each morphism $\lambda \in \mathrm{Hom}(T^k)$ will be of the form $\lambda = f_1^{n_1} f_2^{n_1} \cdots f_k^{n_k}$ for some $n_i \in \mathbb{N}$. Define a functor $d : T^k \to \mathbb{N}^k$ by $d(\lambda) := (n_1, \ldots, n_k)$. Then (T^k, d) is a k-rank graph, which can be viewed as the monoid $\mathbb{N}^k$ with generators indexed by $f_1, \ldots, f_k$. Its 1-skeleton comprises a single vertex and k distinctly-coloured loops which we label $1, \ldots, k$.

The graph algebra $\mathcal{A}(T^k)$ is therefore the universal $C^\star$-algebra generated by partial isometries $\{t_1, \ldots, t_k\}$ which commute (we can recover t_v from the t_i via **CK4**). Notice that we are in a different situation to that of the Cuntz algebras from 1.3.5, where the partial isometries were not assumed to commute. From **CK3** and 1.3.12 we know that $t_i^\star t_i = t_v = \mathrm{id}_{\mathcal{A}(T^k)}$, that is, the t_i are unitary in $\mathcal{A}(T^k)$.

Similarly to in 1.3.4 (and as in [Bla06, II.8.3.3]), it can be shown that the universal $C^\star$-algebra generated by k-many commuting unitary operators is isomorphic to $C(\mathbb{T}^k)$, the algebra of continuous functions on the k-torus. Thus $\mathcal{A}(T^k) \cong C(\mathbb{T}^k)$.

The K-theory of this $C^\star$-algebra is also known, with $K_0(C(\mathbb{T}^k)) \cong K_1(C(\mathbb{T}^k)) \cong \mathbb{Z}^{2^{k-1}}$ for all $k \geq 1$ (as documented in [RLL00]). This aligns with our understanding of the K-theory of $\mathcal{O}_1$, the graph algebra of O_1 from 1.3.4, since O_1 and T^1 are isomorphic as 1-graphs. ♦

1.3.15 Example (1.1.7 revisited) Consider again the 2-rank graph (Λ, d) from Example 1.1.7, whose 1-skeleton is depicted in Figure 1.2a, and whose commuting squares are given by $g_2 f_1 = f_1 g_1$ and $g_1 f_2 = f_2 g_2$. Write $\mathcal{A}$ to denote the 2-rank graph $C^\star$-algebra of (Λ, d). Since Λ is generated by the morphisms f_i, g_i, the algebra $\mathcal{A}$ is generated by a family of partial isometries $\{t_u, t_v, t_{f_1}, t_{f_2}, t_{g_1}, t_{g_2}\}$ subject to relations

(i) $t_{g_2} t_{f_1} = t_{f_1} t_{g_1}$ and $t_{g_1} t_{f_2} = t_{f_2} t_{g_2}$ by **CK2** and the commuting squares,

(ii) $t_{f_1}^\star t_{f_1} = t_{g_1}^\star t_{g_1} = t_u$ and $t_{f_2}^\star t_{f_2} = t_{g_2}^\star t_{g_2} = t_v$ from **CK3**,

(iii) $t_u = t_{f_2} t_{f_2}^\star + t_{g_1} t_{g_1}^\star$ and $t_v = t_{f_1} t_{f_1}^\star + t_{g_2} t_{g_2}^\star$ from **CK4**.

We'll try to decipher these in order to get a better idea of the structure of the graph algebra. Firstly, from (i) we get $t_{f_2} t_{g_2} t_{f_2}^\star = t_{g_1} t_{f_2} t_{f_2}^\star = t_{g_1} t_v = t_{g_1}$, and so from **CK2** it follows that

$$t_{g_1}^\star t_{g_1} = \left(t_{f_2} t_{g_2} t_{f_2}^\star\right)^\star \left(t_{f_2} t_{g_2} t_{f_2}^\star\right) = t_{f_2} t_{g_2}^\star t_{f_2}^\star t_{f_2} t_{g_2} t_{f_2}^\star = t_{f_2} t_{f_2}^\star.$$

But from (ii) we know that $t_{g_1}^\star t_{g_1} = t_u$, and hence that $t_{f_2} t_{f_2}^\star = t_u$. Similarly, $t_{f_1} t_{f_1}^\star = t_v$. Now, consider the elements $x := \left(t_{f_1} + t_{g_2}\right)$ and $y := \left(t_{f_2} + t_{g_1}\right)$ in $\mathcal{A}$. We have

$$x^\star x = \left(t_{f_1} + t_{g_2}\right)^\star \left(t_{f_1} + t_{g_2}\right) = t_{f_1}^\star t_{f_1} + t_{f_1}^\star t_{g_2} + t_{g_2}^\star t_{f_1} + t_{g_2}^\star t_{g_2} = t_{f_1}^\star t_{f_1} + t_{g_2}^\star t_{g_2} = t_u + t_v,$$

by 1.3.12 and (ii). Likewise, we see that $y^\star y = t_u + t_v$. But $t_u + t_v$ is an identity for $\mathcal{A}$ by 1.3.13, so $x^\star x = xx^\star = \mathrm{id}_{\mathcal{A}} = y^\star y = yy^\star$, that is, x and y are isometries in $\mathcal{A}$. But also

$$xx^\star = t_{f_1}t_{f_1}^\star + t_{f_1}t_{g_2}^\star + t_{g_2}t_{f_1}^\star + t_{g_2}t_{g_2}^\star = t_{f_1}t_{f_1}^\star + t_{g_2}t_{g_2}^\star = t_u,$$

by 1.3.12 and (iii), so $t_u = \mathrm{id}_{\mathcal{A}}$. Similarly, we find that $t_v = \mathrm{id}_{\mathcal{A}}$. Notice that $t_{f_1} = xt_u$, $t_{f_2} = yt_v$, $t_{g_1} = yt_u$, and $t_{g_2} = xt_v$, so $\{x, y\}$ generates $\mathcal{A}$. Moreover,

$$\begin{aligned}
xyx^\star y^\star &= (t_{f_1} + t_{g_2})(t_{f_2} + t_{g_1})(t_{f_1} + t_{g_2})^\star(t_{f_2} + t_{g_1})^\star \\
&= (t_{f_1}t_{f_2} + 0 + 0 + t_{f_1}t_{g_1})(t_{f_1}^\star t_{f_2}^\star + 0 + t_{g_2}^\star t_{f_2}^\star + 0), \text{ by 1.3.12} \\
&= 0 + t_{f_1}t_{f_2}t_{g_2}^\star t_{f_2}^\star + t_{f_1}t_{g_1}t_{f_1}^\star t_{f_2}^\star + 0, \text{ again by 1.3.12} \\
&= t_{f_1}t_{f_2}t_{f_2}^\star t_{g_1}^\star + t_{g_2}t_{f_1}t_{f_1}^\star t_{f_2}^\star, \text{ by (i) and } \mathbf{CK2} \\
&= t_{f_1}t_{g_1}^\star + t_{g_2}t_{f_2}^\star, \text{ by } \mathbf{CK2} \text{ and the fact that } t_{f_2}t_{f_2}^\star = t_u \text{ and } t_{f_1}t_{f_1}^\star = t_v \\
&= xy^\star,
\end{aligned}$$

that is, $xyx^\star y^\star = xy^\star$. But then $yx^\star = \mathrm{id}_{\mathcal{A}}$, so $xy^\star = \mathrm{id}_{\mathcal{A}}$, and therefore $xyx^\star y^\star = \mathrm{id}_{\mathcal{A}}$. Since x and y are isometries, this means that $xy = yx$.

Conversely, for isometries X, Y, we can define a Cuntz–Krieger Λ-family $\{t_\lambda\}$ by setting $t_v := XX^\star$, $t_u := YY^\star$, $t_{f_1} := Xt_u$ and so on, such that the $C^\star$-algebra generated by $\{X, Y\}$ is isomorphic to $\mathcal{A}(\Lambda)$. Thus $\mathcal{A} = \mathcal{A}(\Lambda)$ is the universal $C^\star$-algebra generated by two unitary elements which commute—this is isomorphic to $C(\mathbb{T}^2)$, as we saw in Example 1.3.14. ♦

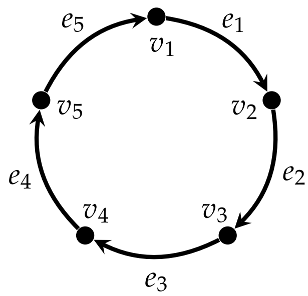
1.3.16 Proposition (Kumjian and Pask, 2000) *Let (Λ_1, d_1) and (Λ_2, d_2) be k_1- and k_2-rank graphs, respectively. Then $(\Lambda_1 \times \Lambda_2, d)$ is a $(k_1 + k_2)$-rank graph, where $\Lambda_1 \times \Lambda_2$ is the product category, and $d(\lambda_1, \lambda_2) := (d_1(\lambda_1), d_2(\lambda_2))$, for $\lambda_1 \in \Lambda_1$, $\lambda_2 \in \Lambda_2$.*

If (Λ_1, d_1), (Λ_2, d_2) are also row-finite, then $\mathcal{A}(\Lambda_1 \times \Lambda_2) \cong \mathcal{A}(\Lambda_1) \otimes \mathcal{A}(\Lambda_2)$ [KP00, 1.8, 3.5]. ■

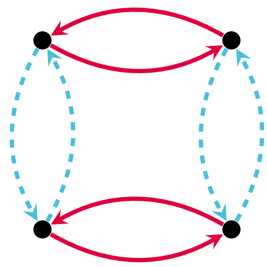
1.3.17 Example (Cyclic graphs) Denote by C_n the 1-graph comprising n vertices $v_1, \ldots, v_n$ and n morphisms $e_i : v_i \longrightarrow v_{i+1}$ (where $v_{n+1} = v_1$), as illustrated in Figure 1.6a. It is not hard to show that $\mathcal{A}(C_n) \cong \mathbb{M}_n(C(\mathbb{T}))$, the ring of $n \times n$ matrices with entries in $C(\mathbb{T})$ (consult [Rae05, 2.14] for one demonstration). According to the table in [RLL00], both K-groups of $\mathbb{M}_n(C(\mathbb{T}))$ are isomorphic to $\mathbb{Z}$, which confirms our calculations in Example 1.3.9.

Now consider the 2-graph $\Lambda = C_2 \times C_2$ as depicted in Figure 1.6b. From Proposition 1.3.16 and the above, it follows that $\mathcal{A}(\Lambda) \cong \mathbb{M}_2(C(\mathbb{T})) \otimes \mathbb{M}_2(C(\mathbb{T}))$. But there is a natural isomorphism $\mathbb{M}_2(C(\mathbb{T})) \otimes \mathbb{M}_2(C(\mathbb{T})) \cong \mathbb{M}_4(C(\mathbb{T}) \otimes C(\mathbb{T}))$ via [Bou07a, III, §4.1], and by the *Stone–Weierstraß Theorem* we also have $C(\mathbb{T}) \otimes C(\mathbb{T}) \cong C(\mathbb{T}^2)$. Hence $\mathcal{A}(\Lambda) \cong \mathbb{M}_4(C(\mathbb{T}^2))$.

Now, for any $C^\star$-algebra $\mathcal{A}$ and any natural number n, we have $K_0(\mathcal{A}) \cong K_0(\mathbb{M}_n(\mathcal{A}))$ and $K_1(\mathcal{A}) \cong K_1(\mathbb{M}_n(\mathcal{A}))$ (see [RLL00, 4.3.8, 8.2.8]). Then from 1.3.14 we are able to conclude that $K_0(\mathcal{A}(\Lambda)) \cong K_1(\mathcal{A}(\Lambda)) \cong \mathbb{Z}^2$. ♦



(a) The 1-graph C_5 has graph algebra isomorphic to $M_5(C(\mathbb{T}))$.



(b) The 2-graph $C_2 \times C_2$. We need not write the commuting squares because there is no choice.

Figure 1.6: Higher-rank graphs can be formed as the direct product of graphs of lower rank. The universal $C^\star$-algebra associated to the 2-graph in (b) is isomorphic to $\mathbb{M}_4(C(\mathbb{T}^2))$; we will come back to this in Example 2.2.9

1.3.18 **Example (1.1.8 revisited)** Let (Λ, d) be one of the 2-rank graphs from Example 1.1.8, with 1-skeleton as depicted in Figure 1.2b. Imagine firstly that the commuting squares are defined by $g_2 f_1 = f_1 g_1$ and $g_1 f_2 = f_2 g_2$, which force $g_4 f_1 = f_1 g_3$ and $g_3 f_2 = f_2 g_4$. Then (Λ, d) is isomorphic to the Cartesian product of O_1 (pink) and O_2 (dashed blue) from 1.3.14, and so $\mathcal{A}(\Lambda) \cong \mathcal{O}_1 \otimes C(\mathbb{T})$ by Proposition 1.3.16. At this point, we can use the *Künneth Theorem* (1.2.18) together with Theorem 2.3.6 and Examples 1.3.4 and 1.3.14 to deduce that $K_\epsilon(\mathcal{A}(\Lambda)) \cong \mathbb{Z}^2$ for $\epsilon = 0, 1$.

Choosing $g_4 f_1 = f_1 g_1$ and $g_1 f_2 = f_2 g_4$ results in another 2-graph isomorphic to the one above, with the isomorphism swapping g_2 and g_4.

If instead we require $g_2 f_1 = f_1 g_1$ and $g_3 f_2 = f_2 g_2$, then we can view Λ as the *crossed product* $O_2 \times_\alpha \mathbb{Z}$, where α is the automorphism which swaps g_2 and g_4. Then $\mathcal{A}(\Lambda)$ is isomorphic to the crossed-product algebra $\mathcal{O}_2 \times_{\tilde{\alpha}} \mathbb{Z}$, where $\tilde{\alpha}$ is the unique automorphism of $\mathcal{A}(\Lambda)$ with $\tilde{\alpha}(t_\lambda) = t_{\alpha(\lambda)}$ for all $\lambda \in \Lambda$ (see [FPS09, §3]).

Again, choosing $g_4 f_1 = f_1 g_1$ and $g_3 f_2 = f_2 g_4$ gives a 2-graph isomorphic to this one, which means its graph algebra will also be isomorphic to $\mathcal{O}_2 \times_{\tilde{\alpha}} \mathbb{Z}$. ♦

1.3.19 The most desirable properties a $C^\star$-algebra can have for the purposes of this thesis are summed up in Definition 2.3.5 as those of a *Kirchberg algebra*. The Kirchberg algebras in the bootstrap class are classified by their K-theory, meaning that if two such $C^\star$-algebras have the same associated K-groups, then the algebras themselves must be isomorphic.

Chapter 2

Tile systems

Our first new class of k-rank graphs are for $k = 2$. Using a result of Vdovina [Vdo02], we may associate to any complete bipartite (undirected) graph κ a 2-dimensional square complex, which we call a *tile complex*, whose *link* at each vertex is κ. We regard the tile complex in two different ways, considering the squares as both *pointed* and *unpointed*; these objects induce non-isomorphic 2-rank graphs. We compute the K-theory of each of the corresponding graph algebras.

Then, we explore extensions of these methods to *$2t$-gon systems*, constructed analogously from 2-dimensional complexes consisting entirely of $2t$-gons, where $t \geq 1$. By the end of the chapter, we will have associated 2-rank graph $C^\star$-algebras to five systems, with their K-theory computed in the following theorems:

(i) Pointed and unpointed tile systems (Theorems 2.2.8, 2.4.4),

(ii) Pointed and unpointed $2t$-gon systems, for even t (Theorems 2.5.7, 2.5.10),

(iii) Pointed $2t$-gon systems, for arbitrary t (Theorem 2.5.15).

The respective systems in (ii) directly generalise those in (i), however there is another intuitive way of building $2t$-gon systems from polyhedra in (iii). The naturality of these generalisations is discussed at the end of §2.5.

Our approach differs from that of Robertson and Steger in [RS99], who focussed on complexes with one vertex. We take advantage of higher-rank graph terminology in order to demonstrate the large intersection between the fields of k-graphs and geometry.

The majority of this chapter has appeared in [Mut22]. Throughout, α and β are positive integers, and $\kappa(\alpha, \beta)$ denotes the complete connected bipartite graph on α white and β black vertices.

§2.1 The tile system associated to a bipartite graph

2.1.1 Definition (Cube complex) An n-dimensional cube can be thought of as a product of n unit intervals $[0,1] \times \cdots \times [0,1] \subset \mathbb{R}^n$; its m-dimensional faces are the products of m intervals $[0,1]$ with $(n-m)$ copies of $\{0\}$ or $\{1\}$ or a mixture of both. We build a cube complex by "gluing together" n-dimensional cubes along their faces *isometrically* (that is, preserving distances). If $\mathcal{S}$ is a disjoint union of such cubes, and $\mathcal{R}$ is a collection of isometries between their faces, then a cube complex $\mathcal{M}$ is the quotient space $\mathcal{S}/\mathcal{R}$, and the *cubes of* $\mathcal{M}$ are the images of the cubes under this quotient map. The dimension of a cube complex is defined to be that of the highest-dimensional cube it contains.

We can form a general cell complex in the same way, where an *n-dimensional cell* is a space which is homeomorphic to an n-dimensional cube. Cells might be polygons, polyhedra, or balls, for example.

2.1.2 Definition (t-hedron) Let $t \geq 2$ be an integer, and let $A_1, \ldots, A_s$ be a sequence of solid t-gons, each with directed edges labelled from some set $\mathcal{U}$. By gluing together like-labelled edges (respecting their direction), we obtain a two-dimensional cell complex P. We call such a complex a t-hedron.

The link at a vertex z of P is the undirected graph obtained as the intersection of P with a small k-sphere centred at z.

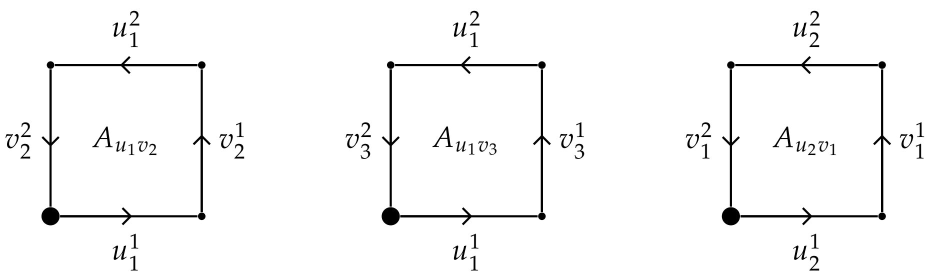


Figure 2.1: Method for constructing a $2t$-hedron from 2.1.4. For each edge $e = u_p v_q$ in a bipartite graph, draw a solid $2t$-gon A_e and select a basepoint. Label the sides anticlockwise from the basepoint by the sequence $u_p^1, v_q^1, \ldots, u_p^t, v_q^t$, as shown. Then glue together any corresponding sides, respecting their direction.

2.1.3 Theorem (Vdovina, 2002) *Let G be a connected bipartite undirected graph on α white and β black vertices, with edge set $E(G)$. Then we can construct a $2t$-hedron $P(G)$ which has G as the link at each vertex, for each $t \geq 1$.* ∎

2.1.4 We refer the reader to [Vdo02] for the full proof, since it suffices to describe just the construction of the cell complex here:

Write $U' = \{u_1, \ldots, u_\alpha\}$ for the set of white vertices of G, and $V' = \{v_1, \ldots, v_\beta\}$ for the set of black vertices. Let U be a set with $2t\alpha$ elements, indexed $u_i^1, u_i^2, \ldots, u_i^t, \bar{u}_i^1, \bar{u}_i^2, \ldots \bar{u}_i^t$ for each $u_i \in U'$, and let V be the corresponding set with $2t\beta$ elements. Define fixed-point-free involutions $u_i^r \mapsto \bar{u}_i^r$ and $v_i^r \mapsto \bar{v}_i^r$ in U and V respectively.

Every edge of the graph G joins an element of U' to an element of V'; for each edge $e = u_p v_q$, we construct a $2t$-gon A_e with a distinguished base vertex. Label the boundary of A_e anticlockwise, starting from the base, by the sequence $u_p^1, v_q^1, u_p^2, v_q^2, \dots, u_p^t, v_q^t$, giving each side of the boundary a forward-directed arrow. We denote this pointed oriented $2t$-gon by $A_e = \left[u_p^1, v_q^1, \dots, u_p^t, v_q^t\right]$. Then, glue the A_e together in the manner of Definition 2.1.2 in order to obtain a $2t$-hedron $P(G)$ (Figure 2.1).

We equate the involution $x \mapsto \bar{x}$ with the reversion of the direction of an arrow.

2.1.5 Definition (Pointed and unpointed tiles) In this chapter, we mainly concern ourselves with 4-hedra: those polyhedra constructed by gluing together squares. We will refer to 4-hedra as tile complexes. For a connected bipartite undirected graph G, write $TC(G)$ for the tile complex $P(G)$, and define the set

$$\begin{aligned}\mathcal{S}(G) := \Big\{A_e = \left[u_p^1, v_q^1, u_p^2, v_q^2\right], \left[\bar{u}_p^1, \bar{v}_q^2, \bar{u}_p^2, \bar{v}_q^1\right],& \\ \left[u_p^2, v_q^2, u_p^1, v_q^1\right], \left[\bar{u}_p^2, \bar{v}_q^1, \bar{u}_p^1, \bar{v}_q^2\right] \;\Big|\; e = u_p v_q \in E(G)\Big\}.& \qquad (2.1)\end{aligned}$$

We call elements of $\mathcal{S}(G)$ pointed tiles, and we define an equivalence relation which, for each A_e, identifies the four corresponding pointed tiles in (2.1). We denote by $\mathcal{S}'(G)$ the quotient of $\mathcal{S}(G)$ by this relation, and we use round brackets, writing $A'_e = \left(u_p^1, v_q^1, u_p^2, v_q^2\right)$ for the equivalence class of A_e in $\mathcal{S}'(G)$. Then $\mathcal{S}'(G)$ is the set of geometric squares (that is, disregarding basepoint and orientation) of which $TC(G)$ consists. We call elements of $\mathcal{S}'(G)$ unpointed tiles.

Notice that by placing the basepoint at the bottom-left vertex, we can arrange that the horizontal sides of each pointed tile be labelled by elements of U, and the vertical sides by elements of V, such that $\mathcal{S}(G) \subseteq U \times V \times U \times V$. Indeed, the four tuples in (2.1) correspond to the four symmetries of a pointed tile which preserve this property (Figure 2.2).

Note also that by design, any two pointed tiles in $\mathcal{S}(G)$ are distinct, and any two adjacent sides of a tile uniquely determine the remaining two sides.

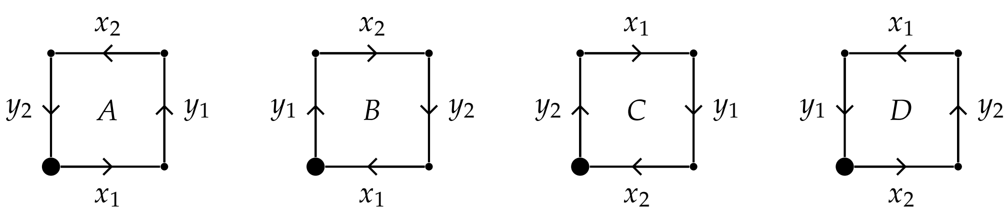


Figure 2.2: The four pointed tiles $A = [x_1, y_1, x_2, y_2]$, $B = [\bar{x}_1, \bar{y}_2, \bar{x}_2, \bar{y}_1]$, etc. are distinct in $\mathcal{S}$, but are equivalent as unpointed tiles in $\mathcal{S}'$. The labels around each of B, C, D are obtained from A through horizontal and/or vertical reflection.

2.1.6 Definition (Tile system) Let G be a connected undirected bipartite graph on α white and β black vertices. Let U, V be sets with $|U| = 4\alpha$, $|V| = 4\beta$, indexed as in 2.1.4, and let $\mathcal{S} = \mathcal{S}(G) \subseteq U \times V \times U \times V$ be the corresponding set of pointed tiles. We call the datum $(G, U, V, \mathcal{S})$ a tile system.

2.1.7 The tile system is closely related to, and indeed modelled on, the *VH-datum*, introduced in [Wis96] and developed further in [BM00]. We have stepped away from some of the terminology used in these people's work, instead making use of the language of higher-rank graphs. One reason for this is to set up some of the higher-dimensional constructions in Chapter 3, which also rely on certain properties of the *adjacency matrices* about to be discussed.

§2.2 The higher-rank graphs induced by a tile system

2.2.1 Definition (Adjacency matrices) Let $(G, U, V, \mathcal{S})$ be a tile system, and $A = [x_1, y_1, x_2, y_2]$ and $B = [x_3, y_3, x_4, y_4]$ be pointed tiles in $\mathcal{S}$. We define two $4\alpha\beta \times 4\alpha\beta$ matrices M_1, M_2 with AB-th entries as follows:

$$M_1(A, B) := \begin{cases} 1 & \text{if } y_1 = \bar{y}_4 \text{ and } x_1 \neq \bar{x}_3, \\ 0 & \text{otherwise,} \end{cases}$$

$$M_2(A, B) := \begin{cases} 1 & \text{if } x_2 = \bar{x}_3 \text{ and } y_1 \neq \bar{y}_3, \\ 0 & \text{otherwise.} \end{cases}$$

We call M_1 and M_2 the horizontal and vertical adjacency matrix respectively, and say that B is horizontally or vertically adjacent to A if $M_1(A, B) = 1$ or $M_2(A, B) = 1$, respectively (see Figure 2.3).

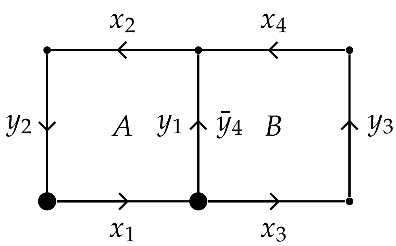
2.2.2 Definition (UCE Property) Let $(G, U, V, \mathcal{S})$ be a tile system, and let A, B, C be pointed tiles in $\mathcal{S}(G)$ such that $M_1(A, B) = 1$ and $M_2(A, C) = 1$. We say that the tile system $(G, U, V, \mathcal{S})$ has the Unique Common Extension Property, or UCE Property, if there exists a *unique* $D \in \mathcal{S}$ with $M_2(B, D) = M_1(C, D) = 1$.

2.2.3 Proposition *Consider the complete bipartite graph $\kappa = \kappa(\alpha, \beta)$, and let $(\kappa, U, V, \mathcal{S}(\kappa))$ be a tile system with corresponding adjacency matrices M_1, M_2. Then:*

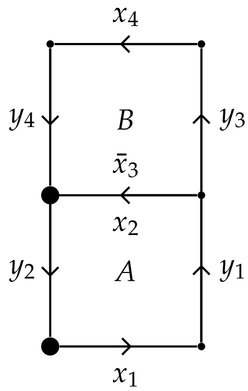
(i) M_1 and M_2 are symmetric and commute with each other.

(ii) Each row and column of M_1 and M_2 contains at least one non-zero element.

(iii) $(\kappa, U, V, \mathcal{S}(\kappa))$ has the UCE Property.



(a) The tile $B = [x_3, y_3, x_4, \bar{y}_1]$ is horizontally adjacent to the tile $A = [x_1, y_1, x_2, y_2]$, which is to say that $M_1(A, B) = 1$.



(b) Here $B = [\bar{x}_2, y_3, x_4, y_4]$, such that $M_2(A, B) = 1$.

Figure 2.3: Horizontal and vertical adjacency of pointed tiles. In (a), if $\bar{x}_3 = x_1$ or $x_4 = \bar{x}_2$, then we must set $M_1(A, B) = 0$.

■ **Proof** Firstly, it is straightforward to verify that the matrices M_1 and M_2 are symmetric. Now, without loss of generality, consider some pointed tile $A = \left[u_i^1, v_j^1, u_i^2, v_j^2\right] \in \mathcal{S}(\kappa)$, and let $D := \left[u_p^2, v_q^2, u_p^1, v_q^1\right]$ for some $p \neq i$, $q \neq j$. By the completeness of the graph κ and the fact that $\alpha, \beta \geq 2$, we can find two more tiles $B = \left[\bar{u}_p^1, \bar{v}_j^2, \bar{u}_p^2, \bar{v}_j^1\right]$ and $C = \left[\bar{u}_i^2, \bar{v}_q^1, \bar{u}_i^1, \bar{v}_q^2\right]$ in $\mathcal{S}(\kappa)$ such that $M_1(A, B) = M_2(B, D) = 1$ and $M_2(A, C) = M_1(C, D) = 1$ (see Figure 2.4). This demonstrates (ii).

Since any two adjacent sides of a tile determine the remaining two sides (2.1.5), it follows that B and C are unique, and hence that $M_2M_1(A, D) = M_1M_2(A, D) \in \{0, 1\}$. So, given $A, B, C \in \mathcal{S}(\kappa)$ as above, this means that D is the unique tile adjacent to both B and C, and so $(\kappa, U, V, \mathcal{S}(\kappa))$ has the *UCE Property*. ■

2.2.4 Proposition *Let $\kappa = \kappa(\alpha, \beta)$ be a complete bipartite graph, and let $(\kappa, U, V, \mathcal{S}(\kappa))$ be a tile system with adjacency matrices M_1, M_2. This induces a 2-rank graph $(\Lambda(\kappa), d)$, whose 1-skeleton has vertex matrices M_1, M_2.*

■ **Proof** Following the method of 2.1.4, label the elements of the sets U, V such that

$$U = \left\{u_1^1, u_1^2, \ldots, u_\alpha^1, u_\alpha^2, \bar{u}_1^1, \bar{u}_1^2, \ldots, \bar{u}_\alpha^1, \bar{u}_\alpha^2\right\},$$
$$V = \left\{v_1^1, v_1^2, \ldots, v_\beta^1, v_\beta^2, \bar{v}_1^1, \bar{v}_1^2, \ldots, \bar{v}_\beta^1, \bar{v}_\beta^2\right\},$$

where $u_1, \ldots, u_\alpha$ and $v_1, \ldots, v_\beta$ are the white and black vertices of κ, respectively. Construct the tile complex $TC(\kappa)$, and consider the set $\mathcal{S}(\kappa) \subseteq U \times V \times U \times V$ of pointed tiles

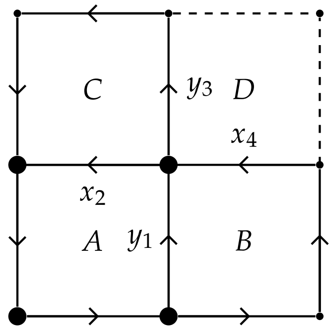


Figure 2.4: A tile system has the *Unique Common Extension Property* (2.2.2) if, given an initial pointed tile A, and horizontally and vertically adjacent tiles B, C, respectively, then there is a unique tile D adjacent to both B and C as shown.

A tile system for a complete bipartite graph has the property that, given initial tiles A and D, then B and C are uniquely determined in the above formation. This is because two adjacent sides of a tile determine the remaining two sides (2.1.5).

of $TC(\kappa)$. Since κ is complete, there is for each u_i and v_j an edge joining them, hence

$$\mathcal{S}(\kappa) = \Big\{ \big[u_i^1, v_j^1, u_i^2, v_j^2\big], \big[\bar{u}_i^1, \bar{v}_j^2, \bar{u}_i^2, \bar{v}_j^1\big], \\ \big[u_i^2, v_j^2, u_i^1, v_j^1\big], \big[\bar{u}_i^2, \bar{v}_j^1, \bar{u}_i^1, \bar{v}_j^2\big] \;\Big|\; 1 \leq i \leq \alpha, 1 \leq j \leq \beta \Big\}.$$

Consider the corresponding adjacency matrices M_1 and M_2 as described in Definition 2.2.1. We construct a 2-coloured graph G as follows:

Let $G^0 = \mathcal{S}(\kappa)$, and for each $A, B \in G^0$, draw a pink arrow from B to A whenever $M_1(A, B) = 1$, and a blue arrow whenever $M_2(A, B) = 1$. Then $A \to B \to D \in G^2$ is a pink-blue path of length two in G if and only if $M_1(A, B) = M_2(B, D) = 1$. But we know from Proposition 2.2.3 that M_1 and M_2 commute, and that two adjacent sides of a tile determine the remaining two sides. Hence this path defines a *unique* blue-pink path $A \to C \to D$ (compare with Figure 2.4), and together, all the pairs of paths $A \to B \to D$ and $A \to C \to D$ define a complete associative collection of squares for G, as in Definition 1.1.15. It then follows from Theorem 1.1.16 that G induces a 2-rank graph, which we denote by $(\Lambda(\kappa), d)$.

We have $\Lambda(\kappa)^{\mathbf{0}} = \mathcal{S}(\kappa)$, and $d(\lambda) := (1, 0)$ (resp. $(0, 1)$) if λ is a pink (resp. blue) path of length one in G. Thus elements of $\Lambda(\kappa)^{(m,n)}$ are $m \times n$ lattices of paths, each traversing m pink and n blue edges (see Figure 1.3). ∎

2.2.5 Fundamental Theorem of finitely-generated Abelian groups *Let $\mathcal{G}$ be a finitely-generated Abelian group. Then $\mathcal{G}$ is isomorphic to a direct sum of the form $\mathbb{Z}^r \oplus \bigoplus_i \mathbb{Z}/q_i\mathbb{Z}$, for some non-negative integer r and some prime powers q_i. We write $\mathrm{rk}(\mathcal{G})$ to denote the so-called torsion-free rank r, and write $\mathrm{tor}(\mathcal{G}) := \bigoplus_i \mathbb{Z}/q_i$ to denote the torsion part, or finite part of $\mathcal{G}$.*

2.2.6 Theorem (Evans, 2008) *Let (Λ, d) be a row-finite 2-graph with no sources, finite vertex set $\Lambda^{\mathbf{0}}$ with $|\Lambda^{\mathbf{0}}| = n$, and vertex matrices M_1, M_2. Then*

$$K_0(\mathcal{A}(\Lambda)) \cong \mathbb{Z}^{r_0} \oplus \operatorname{tor}\left(\operatorname{coker}\left[\mathbf{1} - M_1^T, \mathbf{1} - M_2^T\right]\right),$$
$$K_1(\mathcal{A}(\Lambda)) \cong \mathbb{Z}^{r_1} \oplus \operatorname{tor}\left(\operatorname{coker}\left[\mathbf{1} - M_1, \mathbf{1} - M_2\right]\right),$$

where $\mathbf{1}$ is the $n \times n$ identity matrix, $[\,*\,,\,*\,]$ denotes a block $n \times 2n$ matrix, and

$$r_0 = r_1 := \operatorname{rk}\left(\operatorname{coker}\left[\mathbf{1} - M_1^T, \mathbf{1} - M_2^T\right]\right) + \operatorname{rk}\left(\operatorname{coker}\left[\mathbf{1} - M_1, \mathbf{1} - M_2\right]\right).$$

2.2.7 Corollary *Let $\kappa = \kappa(\alpha, \beta)$ be a complete bipartite graph, and let $(\kappa, U, V, \mathcal{S}(\kappa))$ be a tile system with adjacency matrices M_1, M_2 as in Definition 2.2.1. Writing $\mathcal{A}(\kappa) = \mathcal{A}(\Lambda(\kappa))$, we have*

$$K_0(\mathcal{A}(\kappa)) \cong K_1(\mathcal{A}(\kappa)) \cong \operatorname{tor}\left(\operatorname{coker}\left[\mathbf{1} - M_1^T, \mathbf{1} - M_2^T\right]\right) \oplus \mathbb{Z}^r,$$

where $r := \operatorname{rk}\left(\operatorname{coker}\left[\mathbf{1} - M_1^T, \mathbf{1} - M_2^T\right]\right)$.

■ **Proof** Firstly, $\alpha, \beta < \infty$ by assumption, and by the *UCE Property* of the tile system (Proposition 2.2.3) we know that each row and column of M_1 and M_2 has at least one non-zero element. Hence $\Lambda(\kappa)$ is row-finite, has no sources, and is such that $|\Lambda(\kappa)^{\mathbf{0}}| = 4\alpha\beta$, whence the result follows from Theorem 2.2.6. ■

2.2.8 Theorem (K-theory for algebras of pointed tile systems) *Let $a, b \geq 0$, and let $\kappa(a+2, b+2)$ be the complete bipartite graph on $a+2$ white and $b+2$ black vertices. Without loss of generality, we assume that $a \leq b$. Write $l := \operatorname{lcm}(a, b)$, and $g := \gcd(a, b)$. Then, for $\epsilon = 0, 1$:*

(i) If $a = b = 0$, then $K_\epsilon(\mathcal{A}(\kappa(a+2, b+2)) = K_\epsilon(\mathcal{A}(\kappa(2,2))) \cong \mathbb{Z}^8$.

(ii) If $a \in \{0, 1\}$ and $b \geq 1$, then

$$K_\epsilon(\mathcal{A}(\kappa(a+2, b+2))) \cong (\mathbb{Z}/b)^2 \oplus \mathbb{Z}^{4(b+1)}.$$

(iii) If $a, b \geq 2$ and a, b are coprime, then

$$K_\epsilon(\mathcal{A}(\kappa(a+2, b+2))) \cong (\mathbb{Z}/a)^{b-a} \oplus (\mathbb{Z}/ab)^{a+1} \oplus \mathbb{Z}^{2(a+1)(b+1)}.$$

(iv) If $a, b \geq 2$ and a, b are not coprime, then

$$K_\epsilon(\mathcal{A}(\kappa(a+2, b+2))) \cong (\mathbb{Z}/a)^{b-a} \oplus (\mathbb{Z}/l)^{a+1} \oplus (\mathbb{Z}/g)^{a+2} \oplus \mathbb{Z}^{2(a+1)(b+1)},$$

where $(\mathbb{Z}/a)^0$ is defined to be the trivial group in the case that $a = b$.

■ **Proof** We begin by proving (iii) and (iv), since (i) and (ii) are special cases thereof.

So, let $a, b \geq 2$, write $\alpha := a+2$ and $\beta := b+2$, and for $i \in \{1,\dots,\alpha\}$ and $j \in \{1,\dots,\beta\}$, let A_{ij} denote the pointed tile $[u_i^1, v_j^1, u_i^2, v_j^2] \in \mathcal{S}(\kappa)$. Similarly, write $B_{ij} := [\bar{u}_i^1, \bar{v}_j^2, \bar{u}_i^2, \bar{v}_j^1]$, $C_{ij} := [\bar{u}_i^2, \bar{v}_j^1, \bar{u}_i^1, \bar{v}_j^2]$, $D_{ij} := [u_i^2, v_j^2, u_i^1, v_j^1]$ for the tiles with the same edge labels as the horizontal reflection, vertical reflection, and rotation by π of A_{ij}, respectively; then $\mathcal{S}(\kappa) = \{A_{ij}, B_{ij}, C_{ij}, D_{ij} \mid 1 \leq i \leq \alpha, 1 \leq j \leq \beta\}$. In order to use Theorem 2.2.6, we need to find coker ∂_1, where $\partial_1 : \mathbb{Z}\Lambda^{\mathbf{0}} \oplus \mathbb{Z}\Lambda^{\mathbf{0}} \to \mathbb{Z}\Lambda^{\mathbf{0}}$ is defined by the block matrix $\partial_1 = [\mathbf{1} - M_1^T, \mathbf{1} - M_2^T]$, and $\mathbb{Z}\Lambda^{\mathbf{0}}$ represents the group of linear combination of the vertices in $\Lambda^{\mathbf{0}}$ with coefficients in $\mathbb{Z}$ (explored in higher generality in 3.4.7). We represent elements of $\mathbb{Z}\Lambda^{\mathbf{0}}$ as formal sums of elements of $\Lambda^{\mathbf{0}}$. Then

$$\begin{aligned}\text{coker} = \text{coker}\,[\mathbf{1} - M_1^T, \mathbf{1} - M_2^T] = \Bigg\langle S \in \mathcal{S}(\kappa) \;\Bigg|\; S &= \sum_{T \in \mathcal{S}(\kappa)} M_1(S,T) \cdot T \\ &= \sum_{T \in \mathcal{S}(\kappa)} M_2(S,T) \cdot T \Bigg\rangle. \end{aligned} \quad (2.2)$$

Now fix $p \in \{1,\dots,\alpha\}$, $q \in \{1,\dots,\beta\}$, and notice the following:

- $M_1(A_{pq}, T) = 1$ iff $T = B_{iq}$; $M_1(B_{pq}, T) = 1$ iff $T = A_{iq}$, for some $i \neq p$.
- $M_1(C_{pq}, T) = 1$ iff $T = D_{iq}$; $M_1(D_{pq}, T) = 1$ iff $T = C_{iq}$, for some $i \neq p$.
- $M_2(A_{pq}, T) = 1$ iff $T = C_{pj}$; $M_2(B_{pq}, T) = 1$ iff $T = D_{pj}$, for some $j \neq q$.
- $M_2(C_{pq}, T) = 1$ iff $T = A_{pj}$; $M_2(D_{pq}, T) = 1$ iff $T = B_{pj}$, for some $j \neq q$.

Hence the relations of (2.2) are given by equations of the form $A_{pq} = \sum_{i \neq p} B_{iq} = \sum_{j \neq q} C_{pj}$, and so on for each B_{pq}, C_{pq}, and D_{pq}.

In particular, we can write $B_{pq} = \sum_{i \neq p} A_{iq}$ and $C_{pq} = \sum_{j \neq q} A_{pj}$ so that

$$A_{pq} = (\alpha - 1)A_{pq} + (\alpha - 2)\sum_{i \neq p} A_{iq} \quad \text{and} \quad A_{pq} = (\beta - 1)A_{pq} + (\beta - 2)\sum_{j \neq q} A_{pj}.$$

Define $J_q := \sum_{i=1}^{\alpha} A_{iq}$, and $I_p := \sum_{j=1}^{\beta} A_{pj}$. Then $(\alpha - 2)J_q = (\beta - 2)I_p = 0$, and viewing the sum of all the tiles A_{ij} both as the sum of all the I_i and of the J_j, we conclude also that $g\Sigma = 0$, where $\Sigma := \sum_{i,j} A_{ij}$.

Now, we can also write D_{pq} (and all of the relevant relations) in terms of the A_{ij}, namely $D_{pq} = \sum_{i \neq p} \sum_{j \neq q} A_{ij}$. Hence we can remove all the B_{pq}, C_{pq}, and D_{pq} from the list of generators of coker, such that

$$\begin{aligned}\text{coker} = \big\langle A_{pq} \;\big|\; (\alpha - 2)J_q = (\beta - 2)I_p = 0,\ J_q = \textstyle\sum_i A_{iq},\ I_p = \sum_j A_{pj}, \\ \text{for } p \in \{1,\dots,\alpha\}, q \in \{1,\dots,\beta\}\big\rangle. \end{aligned} \quad (2.3)$$

We have the following equalities:

$$A_{p1} = I_p - \sum_{j=2}^{\beta} A_{pj}, \quad A_{1q} = J_q - \sum_{i=2}^{\alpha} A_{iq}, \quad I_1 = \Sigma - \sum_{i=2}^{\alpha} I_i, \quad J_1 = \Sigma - \sum_{j=2}^{\beta} J_j.$$

Furthermore, A_{11} may be expressed in terms of Σ, I_p, J_q, and A_{pq} for $p, q \geq 2$, and so after a sequence of Tietze transformations on (2.3), we find that

$$\begin{aligned} \operatorname{coker} = \langle \Sigma, I_p, J_q, A_{pq} \mid (\alpha - 2)J_q = (\beta - 2)I_p = g\Sigma = 0, \\ \text{for } p \in \{2, \dots, \alpha\}, q \in \{2, \dots, \beta\}\rangle, \end{aligned} \tag{2.4}$$

where $g := \gcd(\alpha - 2, \beta - 2)$. This, after substituting $a = \alpha - 2$ and $b = \beta - 2$, gives a presentation for $(\mathbb{Z}/b)^{a+1} \oplus (\mathbb{Z}/a)^{b+1} \oplus (\mathbb{Z}/g) \oplus \mathbb{Z}^{(a+1)(b+1)}$. In particular, we have $a+1$ copies of $(\mathbb{Z}/b) \oplus (\mathbb{Z}/a)$. It is well-known that if a and b are not coprime, $(\mathbb{Z}/b) \oplus (\mathbb{Z}/a) \cong (\mathbb{Z}/l) \oplus (\mathbb{Z}/g)$; in case (iv), this together with Corollary 2.2.7 immediately gives the desired result.

In case (iii), where a and b are coprime, we instead have that $(\mathbb{Z}/b) \oplus (\mathbb{Z}/a) \cong (\mathbb{Z}/ab)$, and we are done.

Now consider case (i), where $\alpha = \beta = 2$. Following the method above, coker is generated by $\{A_{pq} \mid p, q \in \{1, 2\}\}$ with trivial relations, so coker $\cong \mathbb{Z}^4$. Thus by 2.2.7, $K_\epsilon(C^\star(\kappa)) \cong \mathbb{Z}^8$.

Similarly, when $\alpha = 2$ and $\beta \geq 3$, it is straightforward to show that

$$\operatorname{coker} = \langle I_p, A_{pq} \mid (\beta - 2)I_p = 0, \text{ for } p \in \{1, 2\} \text{ and } q \in \{2, \dots, \beta\}\rangle,$$

and when $\alpha = 3$ and $\beta \geq 3$, we have

$$\operatorname{coker} = \langle \Sigma, I_p, J_q, A_{pq} \mid J_q = (\beta - 2)I_p = \Sigma = 0, \text{ for } p \in \{2, 3\} \text{ and } q \in \{2, \dots, \beta\}\rangle,$$

both of which are presentations for the group $(\mathbb{Z}/(\beta - 2))^2 \oplus \mathbb{Z}^{2(\beta-1)}$; hence by 2.2.7, we have proved (ii). ■

2.2.9 Example (The tile system $\kappa(2, 2)$) Consider the tile system corresponding to $\kappa(2, 2)$, as illustrated in Figure 2.5. We see from the diagram that the (1-skeleton of the) 2-rank graph $\Lambda = \Lambda(\kappa(2, 2))$ has four connected components—each component is the Cartesian product $C_2 \times C_2$ discussed in Example 1.3.17 and Figure 1.6b, so we can view Λ as the direct sum of four copies of $(C_2 \times C_2)$.

From 1.3.17, we know that $\mathcal{A}(\Lambda) \cong (M_4(C(\mathbb{T}^2)))^4$. Together with 1.2.17, this implies that this algebra's K-groups are both isomorphic to $\mathbb{Z}^8$, in agreement with Theorem 2.2.8. ♦

2.2.10 Theorem (Order of identity in K_0 for pointed tile systems) *Let $\alpha, \beta \geq 3$, let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph on α white and β black vertices, and let $\mathcal{A} = \mathcal{A}(\Lambda(\kappa))$ be the*

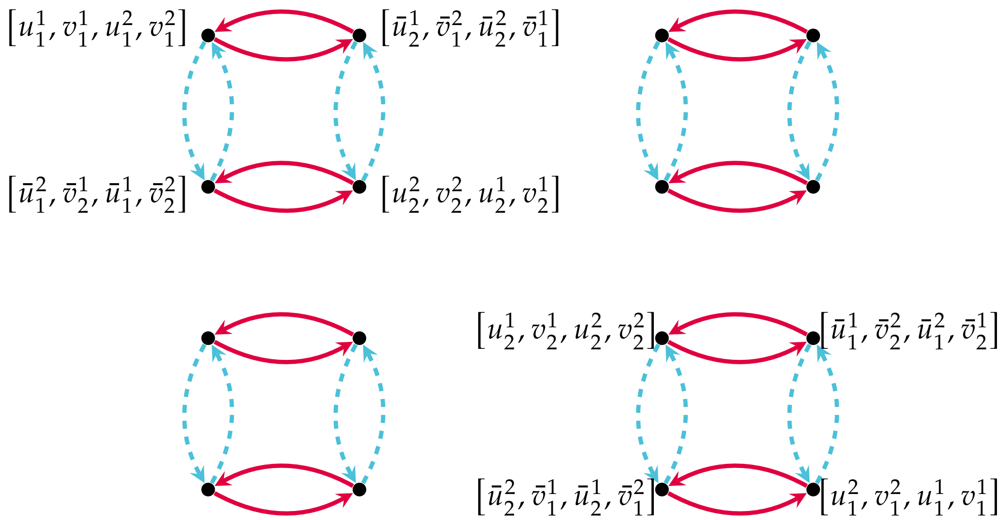


Figure 2.5: The 2-graph $\Lambda = \Lambda(\kappa(2,2))$. Each vertex is labelled by an element of $\mathcal{S}(\kappa)$; a handful have been shown here. A pink (resp. dashed blue) arrow connects vertex A to B if and only if $M_1(A,B) = 1$ (resp. $M_2(A,B) = 1$). We omit the commuting squares because there is only one choice.

By comparing this with Figure 1.6b, we see that Λ consists of four copies of $C_2 \times C_2$. The 1-skeleton of $\Lambda(\kappa(\alpha,\beta))$ is strongly connected only when $\alpha, \beta \geq 3$.

induced graph algebra. Then the order of the class of the identity $\mathrm{id}_{\mathcal{A}}$ *in* $K_0(\mathcal{A})$ *is equal to* $g := \gcd(\alpha - 2, \beta - 2)$.

■ **Proof** From [KR02], it follows that the order of $[\mathrm{id}_{\mathcal{A}}]$ in $K_0(\mathcal{A}(\kappa))$ is equal to the order of the sum of pointed tiles in $\mathcal{S}(\kappa)$; it follows from (2.4) in the proof of Theorem 2.2.8 that this is g. ■

§2.3 Aperiodicity and the *Kirchberg–Phillips Classification*

2.3.1 Definition (The Aperiodicity Condition) Recall from 1.1.18 the k-graph (Ω_k, d), an infinite k-dimensional lattice with arrows of degree $\mathbf{e}_i$ from $\mathbf{n} + \mathbf{e}_i$ to $\mathbf{n}$ for all $\mathbf{n} \in \mathbb{N}^k$.

For an arbitrary k-rank graph Λ, we define the infinite path space Λ^∞ as the set of all k-graph morphisms $\varphi : \Omega_k \to \Lambda$. Given a vertex $v \in \Lambda^{\mathbf{0}}$, we write $v\Lambda^\infty$ for the set of infinite paths which begin at v, that is, $v\Lambda^\infty := \{\varphi \in \Lambda^\infty \mid \varphi(\mathbf{0}) = v\}$.

Let $\mathbf{p} \in \mathbb{Z}^k$, and let $\varphi \in \Lambda^\infty$. We say that $\mathbf{p}$ is a period for φ if, for each $(\mathbf{m}, \mathbf{n}) \in \Omega_k$ with $\mathbf{m} + \mathbf{p} \geq \mathbf{0}$, we have $\varphi(\mathbf{m} + \mathbf{p}, \mathbf{n} + \mathbf{p}) = \varphi(\mathbf{m}, \mathbf{n})$. We call the path φ periodic if it has a non-zero period.

For a path $\varphi \in \Lambda^\infty$ and some $\mathbf{q} \in \mathbb{N}^k$, define $\varphi_{\mathbf{q}}(\mathbf{m}, \mathbf{n}) := (\mathbf{m} + \mathbf{q}, \mathbf{n} + \mathbf{q})$. We say that φ is eventually periodic if we can find some non-zero $\mathbf{q} \in \mathbb{N}^k$ such that $\varphi_{\mathbf{q}}$ is periodic. We say that an infinite path φ is aperiodic if it is neither periodic nor eventually periodic, and we say that Λ satisfies the Aperiodicity Condition—also referred to in the literature as *Condition (A)*—if, for every vertex $v \in \Lambda^0$, we can find an aperiodic path $\varphi \in v\Lambda^\infty$.

We say that Λ is cofinal if, for every vertex $v \in \Lambda^0$ and every infinite path $\varphi \in \Lambda^\infty$, we can find $\lambda \in \Lambda$ and $\mathbf{n} \in \mathbb{N}^k$ such that $r(\lambda) = v$ and $s(\lambda) = \varphi(\mathbf{n})$.

2.3.2 The *Aperiodicity Condition* is a generalisation of the condition on 1-graphs that every cycle have an exit. Similarly, cofinality is a generalisation of the property that every vertex in a 1-graph be reachable from somewhere on every infinite path.

Kumjian and Pask in [KP00] have developed conditions under which the $C^\star$-algebra of a k-rank graph is both *simple* and *purely infinite* (2.3.5–2.3.8). Shortly, we show that the conditions are satisfied by the algebras $\mathcal{A}(\kappa)$, and thus, by Kirchberg and Phillips' work in [Kir95; Phi00], that the $\mathcal{A}(\kappa)$ are completely classified by their K-theory (Corollary 2.3.12).

2.3.3 **Lemma** *The* 2*-rank graph* $\Lambda(\kappa)$ *induced by the complete bipartite graph* $\kappa(\alpha, \beta)$ *satisfies the Aperiodicity Condition whenever* $\alpha, \beta \geq 3$.

2.3.4 In order to get a feeling as to why this is true, consider Figure 2.6, which shows the 1-skeleton of $\Lambda(\kappa(3,3))$. Each vertex is labelled by a pointed tile from $\mathcal{S}(\kappa(3,3))$, and since each tile is vertically adjacent to two others (and horizontally adjacent to two others), there are two dashed blue and two pink arrows emanating from each vertex of $\Lambda(\kappa(3,3))$. This suggests that, analogously to the 1-graph condition, we can always find an exit to some cycle in $\Lambda(\kappa(3,3))$, namely by stopping mid-cycle at a vertex, and diverting the path down the second of the two available edges. Hence, as long as $\alpha, \beta \geq 3$, there will be enough choice at each vertex to be able to exit a cycle.

■ **Proof** Firstly, write $\Lambda = \Lambda(\kappa)$ and let $A \in \Lambda^0$ be an arbitrary vertex. We construct an aperiodic infinite path beginning from A in the following way:

Let $x : \Omega_1 \to \bigcup_{m \geq 0} \Lambda^{(m,0)}$ be a 1-graph morphism such that $x(0) = A$. The vertex A represents a pointed tile in $\mathcal{S}(\kappa)$, which is horizontally adjacent to $(\beta - 1)$-many other pointed tiles. Hence A is connected by bidirectional pink arrows to $(\beta - 1)$ other vertices in Λ. Choose two of these vertices, B_1 and B_2, say, and let x be such that

$$x(m,m) = \begin{cases} A & \text{if } m \text{ is even,} \\ B_1 & \text{if } m = r^2 + r + 1, \text{ for some } r \geq 1, \\ B_2 & \text{otherwise,} \end{cases}$$

for all $m \in \mathbb{N}$. Since this forms an aperiodic sequence, there is no $p \in \mathbb{Z}$ such that $x(m,m) = x(m+p, m+p)$ for all m, nor any $q \in \mathbb{N}$ such that x_q is periodic; hence x is an aperiodic path. Similarly, define $y : \Omega_1 \to \bigcup_{n \geq 0} \Lambda^{(0,n)}$ by

$$y(n,n) = \begin{cases} A & \text{if } n \text{ is even,} \\ C_1 & \text{if } n = s^2 + s + 1, \text{ for some } s \geq 1, \\ C_2 & \text{otherwise,} \end{cases}$$

for some vertices labelled by pointed tiles C_1, C_2 which are vertically adjacent to A. Then y is also an aperiodic path. By the *UCE Property* (2.2.2), x and y uniquely determine an infinite path $\varphi : \Omega_2 \to \Lambda$ with $\varphi((m,0),(m,0)) = x(m,m)$ and $\varphi((0,n),(0,n)) = y(n,n)$.

Let D denote the unique pointed tile (other than A) adjacent to both B_1 and C_1. This cannot also be adjacent to B_2, nor to C_2, and so $\varphi((m,n),(m,n)) = D$ precisely when $m = r^2 + r + 1$ and $n = s^2 + s + 1$, for some $r, s \geq 1$. As above, there is no $\mathbf{p} \in \mathbb{Z}^2$ such that $\varphi((m,n),(m,n)) = \varphi((m,n) + \mathbf{p}, (m,n) + \mathbf{p})$, nor any $\mathbf{q} \in \mathbb{N}^2$ such that $\varphi_{\mathbf{q}}$ is periodic. Since our initial vertex A was arbitrary, we are done. ■

2.3.5 Definition (Important properties of $C^\star$-algebras) Consider an arbitrary unital algebra $\mathcal{A}$, and let $\mathcal{I} \subseteq \mathcal{A}$ be a subgroup of $\mathcal{A}$ under the addition operation. We call $\mathcal{I}$ a two-sided ideal if, whenever $x \in \mathcal{A}$ and $i \in \mathcal{I}$, then $xi \in \mathcal{I}$ and $ix \in \mathcal{I}$. If $\mathcal{A}$ is a $C^\star$-algebra and $\mathcal{I} \subseteq \mathcal{A}$ is a two-sided ideal which is *closed* in the sense of 1.2.2, then $\mathcal{I}$ is also a $\star$-algebra under the star operation of $\mathcal{A}$. We say that a $C^\star$-algebra $\mathcal{A}$ is simple if it has no non-trivial closed two-sided ideals (see [Bla06]).

Now suppose that $\mathcal{A}$ is a unital $C^\star$-algebra, and consider an arbitrary element $x \in \mathcal{A}$. We define x to be positive whenever $x = y^\star y$ for some element $y \in \mathcal{A}$ (or equivalently, if $x = yy^\star$). A $C^\star$-subalgebra $\mathcal{A}' \subseteq \mathcal{A}$ is called hereditary if, whenever $x' \in \mathcal{A}'$ is an element such that $x' - x$ is positive, then $x \in \mathcal{A}'$.

If $\mathcal{A}$ is a simple $C^\star$-algebra, then we say that it is purely infinite if every non-zero hereditary $C^\star$-subalgebra contains a projection p such that $p = t^\star t$ and $\mathrm{id}_{\mathcal{A}} = tt^\star$ for some partial isometry $t \in \mathcal{A}$ (this is *Murray–von Neumann equivalence*, which we first saw in 1.2.11).

A Kirchberg algebra is a unital $C^\star$-algebra which is nuclear (1.2.14), purely infinite, and *separable* as a vector space—that is, there exists a countable subset $S \subseteq \mathcal{A}$ whose closure is equal to $\mathcal{A}$ (see 1.2.2, and compare with 1.2.5).

2.3.6 Theorem (Kumjian and Pask, 2000) *Let Λ be a k-rank graph. Then the associated universal $C^\star$-algebra $\mathcal{A}(\Lambda)$ lies in the bootstrap class from 1.2.16, and hence is separable and nuclear.* ■

2.3.7 Theorem (Kumjian and Pask, 2000) *Let Λ be a k-rank graph which satisfies the Aperiodicity Condition. Then the associated universal $C^\star$-algebra $\mathcal{A}(\Lambda)$ is simple if and only if Λ is cofinal.* ■

2.3.8 Theorem (Kumjian and Pask, 2000; Sims, 2006) *Let Λ be a k-rank graph which is cofinal and which satisfies the Aperiodicity Condition. Suppose that for every $v \in \Lambda^{\mathbf{0}}$, we can find $\lambda \in \Lambda$ with*

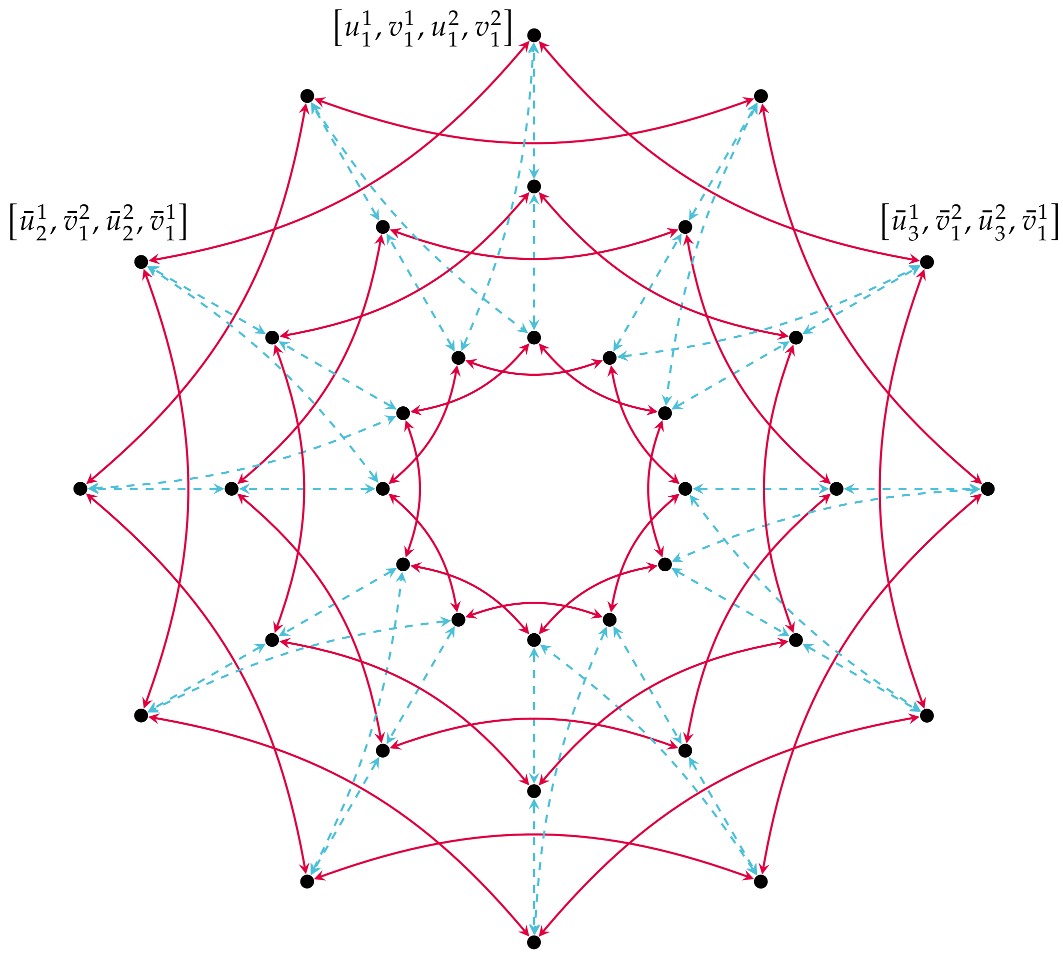


Figure 2.6: The 2-graph $\Lambda(\kappa(3,3))$. Again, we omit the commuting squares since they are determined by the 1-skeleton. Note that it is always possible to exit a cycle.

$r(\lambda) = v$, and some cycle $\mu \in \Lambda$ with an entrance, such that $d(\mu) \neq \mathbf{0}$ and $s(\lambda) = r(\mu) = s(\mu)$. Then $\mathcal{A}(\Lambda)$ is purely infinite [KP00, 4.9; Sim06, 8.8]. ■

2.3.9 Proposition *The 2-rank graph $\Lambda(\kappa)$ induced by the complete bipartite graph $\kappa(\alpha, \beta)$ is simple and purely infinite whenever $\alpha, \beta \geq 3$.*

■ **Proof** Firstly, we observe that $\Lambda(\kappa)$ is cofinal, since the 1-skeleton of $\Lambda(\kappa)$ is strongly connected. Hence from Theorem 2.3.7 it follows that $C^\star(\kappa)$ is simple.

Now, let $A \in \Lambda(\kappa)^{\mathbf{0}}$ be an arbitrary vertex. Since each edge of the 1-skeleton of $\Lambda(\kappa)$ is bidirectional, we can set μ to be a path which begins at A and traverses a single pink edge to some vertex B, before immediately returning to A. Then $d(\mu) = (2, 0)$, and since $\alpha, \beta \geq 3$, then B is the range of some other blue edge, and so μ is a cycle with an entrance. Then by strong-connectedness, the conditions of Theorem 2.3.8 are satisfied, and so $\Lambda(\kappa)$ is purely infinite. ■

2.3.10 We put all of the above theorems together in order to make use of the *Classification Theorem* explained in [Kir95; Phi00]. Given a row-finite k-rank graph Λ with no sources, the $C^\star$-algebra $\mathcal{A}(\Lambda)$ is a separable and nuclear algebra which lies in the bootstrap class of [RS87], by 2.3.6. Furthermore, we have shown in 2.3.9 that $\mathcal{A}(\kappa(\alpha,\beta))$ is simple and purely infinite whenever $\alpha,\beta \geq 3$. Hence, from Theorem 2.3.11 we immediately have Corollary 2.3.12.

2.3.11 Theorem (Kirchberg–Phillips Classification) *Let $\mathcal{A}$ be a Kirchberg algebra which is in the bootstrap class (1.2.16). Then $\mathcal{A}$ is completely determined by its K-theory and the class of $\mathrm{id}_{\mathcal{A}}$ in $K_0(\mathcal{A})$, up to isomorphism [Kir95; Phi00].* ■

2.3.12 Corollary (Classification of graph algebras for pointed tile systems) *Let $\alpha,\beta \geq 3$, and let $\kappa = \kappa(\alpha,\beta)$ be the complete bipartite graph, which induces the 2-rank graph $\Lambda(\kappa)$. Then the isomorphism class of the associated $C^\star$-algebra $\mathcal{A}(\Lambda(\kappa))$ is completely determined by the groups $K_0(\mathcal{A}(\kappa)) = K_1(\mathcal{A}(\kappa))$ and the position of the class of the identity $\mathrm{id}_{\mathcal{A}}$ in $K_0(\mathcal{A}(\kappa))$ [Mut22].* ■

§2.4 Unpointed tile systems

There is an alternative way in which we could have defined the adjacency matrices above, which will lead to a different 2-rank graph structure.

2.4.1 Definition (Unpointed tile system) Define an unpointed tile system $(G,U,V,\mathcal{S}')$ in the same way as 2.1.6, but replacing $\mathcal{S} = \mathcal{S}(G)$ with the set of unpointed tiles $\mathcal{S}' = \mathcal{S}'(G)$. We will see that analogues of the results in §2.2 also hold for unpointed tile systems.

Let $(G,U,V,\mathcal{S}')$ be an unpointed tile system, and let $A', B' \in \mathcal{S}'$ be unpointed tiles, that is, equivalence classes of some pointed tiles $A, B \in \mathcal{S}$, respectively (under the equivalence of 2.1.5, which we'll write as $\sim$). Recall the matrices M_1, M_2 from Definition 2.2.1. We define analogous functions $M_1', M_2' : \mathcal{S}' \times \mathcal{S}' \to \{0,1\}$ as follows:

$$M_1'(A',B') := \begin{cases} 1 & \text{if } M_1(A_\bullet, B_\bullet) = 1 \text{ for some } A_\bullet \sim A,\, B_\bullet \sim B, \\ 0 & \text{otherwise,} \end{cases}$$

$$M_2'(A',B') := \begin{cases} 1 & \text{if } M_2(A_\bullet, B_\bullet) = 1 \text{ for some } A_\bullet \sim A,\, B_\bullet \sim B, \\ 0 & \text{otherwise.} \end{cases}$$

We write adjacency matrices M_1', M_2' accordingly.

2.4.2 Proposition *Consider the complete bipartite graph $\kappa = \kappa(\alpha,\beta)$, and let $(\kappa,U,V,\mathcal{S}'(\kappa))$ be an unpointed tile system. Then the corresponding adjacency matrices M_1' and M_2' commute, and $(\kappa,U,V,\mathcal{S}'(\kappa))$ has the UCE Property.*

Hence $(\kappa,U,V,\mathcal{S}'(\kappa))$ induces a 2-rank graph $(\Lambda'(\kappa),d)$.

■ **Proof** Given two unpointed tiles $A', B' \in \mathcal{S}'(\kappa)$, consider their respective sets of pointed tiles $\mathcal{S}_A, \mathcal{S}_B \in \mathcal{S}(\kappa)$ as defined in 2.4.1. Notice that $M_1'(A', B') = 1$ if and only if, for *some* $A_\bullet \in \mathcal{S}_A$, we can find some $B_\bullet \in \mathcal{S}_B$ such that $M_1(A_\bullet, B_\bullet) = 1$. The same is true for M_2'. Write $A' = \left(u_i^1, v_j^1, u_i^2, v_j^2\right)$, and define sets

$$X_A := \left\{T \in \mathcal{S}'(\kappa) \mid M_1'(A, T) = 1\right\}, \quad Y_A := \left\{T \in \mathcal{S}'(\kappa) \mid M_2'(A, T) = 1\right\}.$$

Then X_A contains precisely those tiles of the form $\left(u_k^1, v_j^1, u_k^2, v_j^2\right)$, where $k \neq i$, and Y_A only those of the form $\left(u_i^1, v_l^1, u_i^2, v_l^2\right)$, where $l \neq j$. The proof then proceeds in a similar fashion to that of Proposition 2.2.3, and the 2-rank graph structure follows immediately from [KP00, §6] as in Proposition 2.2.4. ■

2.4.3 We write $\Lambda'(\kappa)$ for the 2-rank graph induced by the adjacency matrices M_1' and M_2'. It is not difficult to verify that $\Lambda'(\kappa)$ is row-finite, with finite vertex set and no sources. Hence we can apply *Evans' Theorem* 2.2.6, and we derive the following result:

2.4.4 Theorem (K-theory for algebras of unpointed tile systems) *For $a, b \geq 0$, let $\kappa(a+2, b+2)$ be the complete bipartite graph on $a+2$ white and $b+2$ black vertices. Without loss of generality, we can assume that $a \leq b$. Write $\mathcal{A}(\kappa) = \mathcal{A}(\Lambda'(\kappa))$. Then, for $\epsilon = 0, 1$:*

(i) If $a = b = 0$, then $K_\epsilon(\mathcal{A}(\kappa(a+2, b+2)) = K_\epsilon(\mathcal{A}(\kappa(2, 2)) \cong \mathbb{Z}^2$.

(ii) If $a = 0$ and $b \geq 1$, then

$$K_\epsilon(\mathcal{A}(\kappa(a+2, b+2))) \cong (\mathbb{Z}/2)^b \oplus (\mathbb{Z}/(2b)).$$

(iii) If $a, b \geq 1$, then

$$K_\epsilon(\mathcal{A}(\kappa(a+2, b+2))) \cong (\mathbb{Z}/2)^{(a+1)(b+1)-1} \oplus (\mathbb{Z}/2g),$$

where $g := \gcd(a, b)$.

■ **Proof** Again, we start with (iii), as (i) and (ii) follow. Write $\alpha := a+2$, $\beta := b+2$, and let $\alpha, \beta \geq 3$. For $i \in \{1, \ldots, \alpha\}$ and $j \in \{1, \ldots, \beta\}$, let A_{ij}' be the unpointed tile $\left(u_i^1, v_j^1, u_i^2, v_j^2\right) \in \mathcal{S}'(\kappa)$. Then

$$\operatorname{coker} = \operatorname{coker}\left[\mathbf{1} - (M_1')^T, \mathbf{1} - (M_2')^T\right] = \left\langle A_{ij}' \in \mathcal{S}'(\kappa) \;\middle|\; A_{ij}' = \sum_{T' \in \mathcal{S}'(\kappa)} M_1'(A_{ij}', T') \cdot T' = \sum_{T' \in \mathcal{S}'(\kappa)} M_2'(A_{ij}', T') \cdot T' \right\rangle. \quad (2.5)$$

Fix $p \in \{1, \ldots, \alpha\}$, $q \in \{1, \ldots, \beta\}$, and notice that:

- $M'_1(A'_{pq}, T') = 1$ if and only if $T' = A'_{iq}$, for some $i \neq p$,
- $M'_2(A'_{pq}, T') = 1$ if and only if $T' = A'_{pj}$, for some $j \neq q$.

Hence the relations of (2.5) are given by $A'_{pq} = \sum_{i\neq p} A'_{iq} = \sum_{j\neq q} A'_{pj}$. Define

$$J_{pq} := \left(\sum_{i=2}^{\alpha} A'_{iq}\right) - A'_{pq} \quad \text{and} \quad I_{pq} := \left(\sum_{j=2}^{\beta} A'_{pj}\right) - A'_{pq},$$

for $p, q \geq 2$. Then

$$\begin{aligned}
2J_{pq} &= 2\left(\sum_{i=2}^{\alpha} A'_{iq}\right) - 2A'_{pq} \\
&= 2(A'_{2q} + \cdots + A'_{\alpha q} - A'_{pq}) + A'_{1q} - A'_{1q} \\
&= (A'_{1q} + A'_{2q} + \cdots + A'_{\alpha q} - A'_{pq}) + (-A'_{1q} + A'_{2q} + \cdots + A'_{\alpha q}) - A'_{pq} \\
&= A'_{pq} + 0 - A'_{pq} = 0,
\end{aligned}$$

and similarly $2I_{pq} = 0$. Now, $J_{pq} = 0$ or $I_{pq} = 0$ only if $A'_{pq} = A'_{1q}$ or $A'_{pq} = A'_{p1}$ respectively. But since $\alpha, \beta \geq 3$, these equivalences are not relations of (2.5), and so $\operatorname{ord}(J_{pq}) = \operatorname{ord}(I_{pq}) = 2$. Notice that we can write each A'_{1q} and A'_{p1} in terms of the other A'_{ij}, for $p, q \geq 2$; hence we can remove these from the list of generators by a sequence of Tietze transformations.

Also notice that we can write $A'_{2q} = J_{2q} - \sum_{i=3}^{\alpha} A'_{iq}$. Proceeding inductively, we can write each A'_{pq} in terms of the J_{iq} and the A'_{iq} for $i > p$. Similarly, we can express each A'_{pq} in terms of the I_{pj} and the A'_{pj} for $j > q$. Hence we can rewrite the generators of coker as A'_{11}, I_{pq}, J_{pq}, for $p, q \geq 2$. But $A'_{11} = -(A'_{p1} + J_{p1}) = -(A'_{1q} + I_{1q})$ for all $p, q \geq 2$, so

$$(\alpha - 2)A'_{11} = -\sum_{i=3}^{\alpha}(A'_{i1} + J_{i1}) = -\left(J_{21} + \sum_{i=3}^{\alpha} J_{i1}\right),$$

and so $2(\alpha - 2)A'_{11} = 0$. Similarly, we find that $2(\beta - 2)A'_{11} = 0$, and hence that $2gA'_{11} = 0$, where $g := \gcd(\alpha - 2, \beta - 2)$.

Observe that, since I_{pq} is defined in terms of the A'_{pj}, and each A'_{pj} can be written in terms of the J_{ij}, we can remove the I_{pq} from the list of generators of coker. Finally, we can rewrite (2.5) as

$$\begin{aligned}
\text{coker} = \langle J_{2q}, J_{p2}, J_{pq}, A'_{11} \mid 2J_{2q} = 2J_{p2} = 2J_{pq} = 2gA'_{11} = 0,& \\
\text{for } p \in \{3, \ldots, \alpha\} \text{ and } q \in \{3, \ldots, \beta\}\rangle,&
\end{aligned}$$

and after substituting $a = \alpha - 2$ and $b = \beta - 2$, this gives a presentation for $(\mathbb{Z}/2)^{(a+1)(b+1)-1} \oplus (\mathbb{Z}/2g)$; since there is no torsion-free part, this proves (iii).

If $\alpha = 2$, then $A'_{1q} = A'_{2q}$ for all $q \in \{1, \ldots, \beta\}$, so we can write

$$\operatorname{coker} = \left\langle A'_{1q} \,\middle|\, A'_{1q} = \sum_{j \neq q} A'_{1j},\ \text{for } q \in \{1, \ldots, \beta\} \right\rangle.$$

We adjust the proof above accordingly to obtain the result of (ii). Finally, in case (i) where $\alpha = \beta = 2$, we have $A'_{11} = A'_{12} = A'_{21} = A'_{22}$ with no further relations, such that $\operatorname{coker} = \langle A'_{11} \rangle \cong \mathbb{Z}$, and the result follows from Theorem 2.2.6. ■

2.4.5 Theorem (Order of identity in K_0 for unpointed tile systems) *Write $\kappa = \kappa(\alpha, \beta)$ to denote a complete bipartite graph for some $\alpha, \beta \geq 3$, and let $\mathcal{A} = \mathcal{A}(\Lambda'(\kappa))$ be the graph algebra induced by an unpointed tile system. Write $g := \gcd(\alpha - 2, \beta - 2)$. Then the order of the class of the identity $\mathrm{id}_{\mathcal{A}}$ in $K_0(\mathcal{A}(\Lambda'(\kappa)))$ is equal to g if g is odd, and $g/2$ if g is even.*

■ **Proof** Consider the notation used in the proof of Theorem 2.4.4. As with Theorem 2.2.10, we know that the order of $[\mathrm{id}_{\mathcal{A}}]$ in $K_0(\mathcal{A}(\kappa))$ is equal to the order of the sum of all tiles A'_{ij}; we write Σ for this sum.

We have that $A'_{pq} = \sum_{i \neq p} A'_{iq} = \sum_{j \neq q} A'_{pj}$, and so $\Sigma = (\alpha - 1)\Sigma = (\beta - 1)\Sigma$. From this, it follows that $g\Sigma = 0$. We also have $A'_{pq} = \sum_{i \neq p} \sum_{j \neq q} A'_{ij}$, so that

$$\Sigma = A'_{pq} + \sum_{i \neq p} A'_{iq} + \sum_{j \neq q} A'_{pj} + \sum_{i \neq p} \sum_{j \neq q} A'_{ij} = 4A'_{pq},$$

for any fixed p, q. But $2gA'_{pq} = 0$, and so if $g = 2h$ for some integer h, then $h\Sigma = 4hA'_{pq} = 0$, and we know that the order of Σ divides h. But there are no further relations in presentation (2.5) which further restrict the order of Σ, so we are done. ■

2.4.6 Proposition (Classification of graph algebras for unpointed tile systems) *Let $\alpha, \beta \geq 3$, let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph, and let $\Lambda'(\kappa)$ be the induced 2-rank graph. Then the isomorphism class of the universal $C^\star$-algebra $\mathcal{A}(\Lambda'(\kappa))$ is completely determined by its K-theory and the position of the class of the identity $\mathrm{id}_{\mathcal{A}}$ in $K_0(\mathcal{A}(\Lambda'(\kappa)))$.*

■ **Proof** The proof relies on identical results to those in §2.3. ■

§2.5 Pointed and unpointed $2t$-gon systems

2.5.1 In this section we suggest generalisations of the methods above for constructing $C^\star$-algebras associated to $2t$-gon systems, both for even and arbitrary $t \geq 1$.

When $t = 2$, we have an innate idea of what it means for two $2t$-gons to be "stackable:" functions we called *horizontal* and *vertical adjacency* in Definition 2.2.1. We extend this notion to all even $t \geq 2$ in as natural a way possible. The following directly generalises the definitions at the beginning of §2.1.

2.5.2 Definition (Pointed polygon system) Let G be a connected bipartite undirected graph on α white and β black vertices. Let U, V be sets with $|U| = 2t\alpha$, $|V| = 2t\beta$, and which are gifted with fixed-point-free involutions $u \mapsto \bar{u}$, $v \mapsto \bar{v}$ respectively. Construct using U and V the $2t$-hedron $P(G)$ from Theorem 2.1.3 which has G as its link at each vertex, and write $\mathcal{S}'(G) := \{A_e \mid e \in E(G)\}$ for the set of $2t$-gons which comprise $P(G)$. We call elements of $\mathcal{S}'_t(G)$ unpointed $2t$-gons, and denote them by $A_e = (x_1, y_1, \ldots, x_t, y_t)$.

Analogously to in Definition 2.1.5, we write $[x_1, y_1, \ldots, x_t, y_t]$ for a pointed $2t$-gon, that is, a $2t$-gon labelled anticlockwise and starting from a distinguished basepoint by the sequence $x_1, y_1, \ldots, x_t, y_t$, for some $x_i \in U$, $y_i \in V$. Write $\mathcal{S}_t = \mathcal{S}_t(G)$ for the set of $2t\alpha\beta$ pointed $2t$-gons. We call the datum $(G, U, V, \mathcal{S}_t)$ a $2t$-gon system. Similarly we call the datum $(G, U, V, \mathcal{S}'_t)$ an unpointed $2t$-gon system.

2.5.3 Consider the adjacency matrices M_1, M_2 from 2.2.1. We can view two pointed tiles (4-gons) $A = [x_1, y_1, x_2, y_2]$ and B as being horizontally adjacent if and only if, after reflecting A through an axis connecting the midpoints of x_1 and x_2, and then replacing x_1, x_2 by some $x'_1 \neq x_1$, $x'_2 \neq x_2$ respectively, we can obtain B.

Likewise, if and only if we can obtain B by reflecting A through an axis joining the midpoints of the y edges, and then changing the labels of those edges, do we say that A and B are vertically adjacent. This observation forms the basis of the definition of adjacency in general $2t$-gons.

2.5.4 Definition (U- and V-adjacency) Let t be an even integer, let $(G, U, V, \mathcal{S}_t)$ be a $2t$-gon system, and let $A = [x_1, y_1, \ldots, x_t, y_t] \in \mathcal{S}_t$ be a pointed $2t$-gon.

Reflect A through an axis joining the midpoints of sides labelled x_1 and $x_{(t/2)+1}$ to obtain a new pointed $2t$-gon $\left[\bar{x}_1, \bar{y}_t, \bar{x}_t, \bar{y}_{t-1}, \ldots, \bar{x}_2, \bar{y}_1\right]$. We say that a pointed $2t$-gon $B \in \mathcal{S}_t$ is V-adjacent to A if $B = \left[\bar{x}'_1, \bar{y}_t, \bar{x}'_t, \bar{y}_{t-1}, \ldots, \bar{x}'_2, \bar{y}_1\right]$, for some $x'_i \neq x_i$.

Similarly, reflect A such that $x_1 \mapsto \bar{x}_{(t/2)+1}$; we obtain a new pointed $2t$-gon

$$\left[\bar{x}_{(t/2)+1}, \bar{y}_{t/2}, \bar{x}_{t/2}, \ldots, \bar{y}_1, \bar{x}_1, \bar{y}_t, \bar{x}_t, \ldots, \bar{x}_{(t/2)+2}, \bar{y}_{(t/2)+1}\right]. \tag{2.6}$$

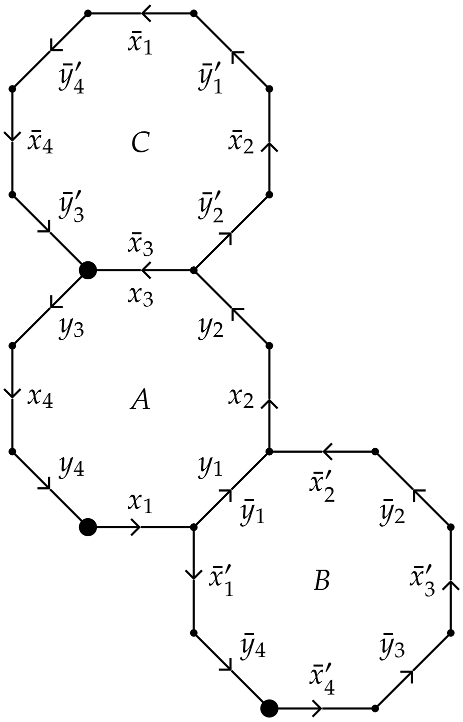


Figure 2.7: Consider pointed 8-gons $A = [x_1, y_1, \ldots, x_4, y_4]$, $B = [\bar{x}'_1, \bar{y}_4, \ldots, \bar{x}'_2, \bar{y}_1]$ and $C = [\bar{x}_3, \bar{y}'_2, \ldots, \bar{x}_4, \bar{y}'_3]$ in $\mathcal{S}_4$. We say that A and B are *V-adjacent*, and A and C are *U-adjacent*, and they can be arranged as shown. There is a unique octagon $D = [x'_3, y'_3, \ldots, x'_2, y'_2]$ which is both U-adjacent to B and V-adjacent to C.

We say that a pointed $2t$-gon $B \in \mathcal{S}_t$ is U-adjacent to A if B is of the form (2.6), but with all elements y_i replaced with some $y'_i \neq y_i$ (Figure 2.7).

We define the U- and V-adjacency matrices, M_U and M_V respectively, to be the $2t\alpha\beta \times 2t\alpha\beta$ matrices with AB-th entry 1 if A and B are U-adjacent (resp. V-adjacent), and 0 otherwise.

2.5.5 Proposition *Let t be an even integer, and $(\kappa, U, V, \mathcal{S}_t(\kappa))$ be a $2t$-gon system with adjacency matrices M_U, M_V. Then these matrices commute, and $(\kappa, U, V, \mathcal{S}_t(\kappa))$ has the UCE Property.*

Hence $(\kappa, U, V, \mathcal{S}_t(\kappa))$ induces a 2-rank graph $(\Lambda_t(\kappa), d)$.

■ **Proof** Consider the pointed $2t$-gon $A = \left[u_i^1, v_j^1, \ldots, u_i^t, v_j^t\right] \in \mathcal{S}_t(\kappa)$; those $2t$-gons corresponding to its reflections and rotations are treated similarly. Then a pointed $2t$-gon B is V-adjacent to A if and only if $B = \left[\bar{u}_k^1, \bar{v}_j^t, \ldots, \bar{u}_k^2, \bar{v}_j^1\right]$, for some $k \neq i$. Suppose B is such a $2t$-gon V-adjacent to A; then a pointed $2t$-gon D is U-adjacent to B if and only if

$$D = \left[u_k^{(t/2)+1}, v_l^{(t/2)+1}, \ldots, u_k^t, v_l^t, u_k^1, v_k^1, \ldots, u_k^{t/2}, v_l^{t/2}\right], \tag{2.7}$$

for some $l \neq j$. Likewise, C is U-adjacent to A if and only if

$$C = \left[\bar{u}_i^{(t/2)+1}, \bar{v}_l^{t/2}, \ldots, \bar{u}_i^1, \bar{v}_l^t, \ldots, \bar{u}_i^{(t/2)+2}, \bar{v}_l^{(t/2)+1}\right],$$

for some $l \neq j$. Clearly if C is such a $2t$-gon, then D is V-adjacent to C if and only if it is of the form (2.7). Exactly one such D exists in $\mathcal{S}_t(\kappa)$, so M_U and M_V commute. Then the $2t$-gon system $(\kappa, U, V, \mathcal{S}_t(\kappa))$ has the *UCE Property*, and the 2-rank graph structure follows from [KP00, §6] and the same arguments used in Proposition 2.2.4. ■

2.5.6 Recall the 2-rank graph $\Lambda(\kappa)$ induced from a tile system and its adjacency matrices M_1, M_2 in §2.1, and recall its associated universal $C^\star$-algebra $\mathcal{A}(\Lambda)$ from Definition 1.3.11. Similarly, we write $\Lambda_t(\kappa)$ for the 2-rank graph induced from the U- and V-adjacency matrices M_U and M_V, and observe that $\Lambda_t(\kappa)$ is row-finite, with finite vertex set and no sources. Hence from *Evans' Theorem* 2.2.6, we can deduce:

2.5.7 Theorem (K-theory for algebras of pointed $2t$-gon systems, t even) *Let $t \geq 2$ be an even integer, and for $\alpha, \beta \geq 2$, let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph on α white and β black vertices. Then, for $\epsilon = 0, 1$:*

$$K_\epsilon(\mathcal{A}(\Lambda_t(\kappa))) \cong (K_\epsilon(\mathcal{A}(\Lambda(\kappa))))^{t/2}.$$

■ **Proof** Fix t and assume without loss of generality that $\alpha \leq \beta$. Analogously to in the proof of Theorem 2.2.8, we denote the pointed $2t$-gons in $\mathcal{S}_t(\kappa)$ as follows:

- $(A_r)_{ij} := \left[u_i^r, v_j^r, \ldots, u_i^t, v_j^t, u_i^1, v_j^1, \ldots, u_i^{r-1}, v_j^{r-1}\right]$,
- $(B_r)_{ij} := \left[\bar{u}_i^r, \bar{v}_j^{r-1}, \ldots, \bar{u}_i^1, \bar{v}_j^t, \ldots, \bar{u}_i^{r+1}, \bar{v}_j^r\right]$,
- $(C_r)_{ij} := \left[\bar{u}_i^{(t/2)+r}, \bar{v}_j^{(t/2)+r-1}, \ldots, \bar{u}_i^1, \bar{v}_j^t, \ldots, \bar{u}_i^{(t/2)+r+1}, \bar{v}_j^{(t/2)+r}\right]$,
- $(D_r)_{ij} := \left[u_i^{(t/2)+r}, v_j^{(t/2)+r}, \ldots, u_i^t, v_j^t, u_i^1, v_j^1, \ldots, u_i^{(t/2)+r-1}, v_j^{(t/2)+r-1}\right]$,

for $i \in \{1, \ldots, \alpha\}$, $j \in \{1, \ldots, \beta\}$, $r \in \{1, \ldots, t/2\}$, and with addition in superscript indices defined modulo t. Note that each $S \in \mathcal{S}_t(\kappa)$ takes exactly one of the above forms. Then

$$\operatorname{coker}\left[\mathbf{1} - M_U^T, \mathbf{1} - M_V^T\right] = \left\langle (A_r)_{pq}, (B_r)_{pq}, (C_r)_{pq}, (D_r)_{pq} \;\middle|\; \begin{array}{l} (A_r)_{pq} = \sum_{i \neq p} (B_r)_{iq} = \sum_{j \neq q} (C_r)_{pj}, \\ (B_r)_{pq} = \sum_{i \neq p} (A_r)_{iq} = \sum_{j \neq q} (D_r)_{pj}, \\ (C_r)_{pq} = \sum_{i \neq p} (D_r)_{iq} = \sum_{j \neq q} (A_r)_{pj}, \\ (D_r)_{pq} = \sum_{i \neq p} (C_r)_{iq} = \sum_{j \neq q} (B_r)_{pj}, \\ \text{for } p \in \{1, \ldots, \alpha\}, q \in \{1, \ldots, \beta\}, \text{ and } r \in \{1, \ldots, t/2\} \end{array} \right\rangle.$$

But, comparing this to (2.2), we see this is precisely a presentation for the direct sum of $t/2$ copies of coker $\left[\mathbf{1} - M_1^T, \mathbf{1} - M_2^T\right]$ as in Theorem 2.2.8, and the result follows. ■

2.5.8 Theorem (Order of identity in K_0 for pointed polygon systems) *Let $\alpha, \beta \geq 3$, let $t \geq 2$ be even, let $\kappa = \kappa(\alpha, \beta)$ be a complete bipartite graph, and let $\mathcal{A} = \mathcal{A}(\Lambda_t(\kappa))$ be the graph algebra induced by a pointed $2t$-gon system. Then the order of the class of the identity $\mathrm{id}_{\mathcal{A}}$ in $K_0(\mathcal{A}(\Lambda_t(\kappa)))$ is equal to $g := \gcd(\alpha - 2, \beta - 2)$.*

Furthermore, the isomorphism class of $\mathcal{A}(\Lambda_t(\kappa))$ is completely determined by the K-groups in Theorem 2.5.7 and the order of $[\mathrm{id}_{\mathcal{A}}]$ in $K_0(\mathcal{A})$.

■ **Proof** The result follows from Theorems 2.2.10 and 2.5.7, and similar arguments to those considered in §2.3. ■

2.5.9 If we extend the concept of U- and V-adjacency from Definition 2.5.4 in the obvious way, we can obtain a generalisation of §2.4 for unpointed $2t$-gon systems of complete bipartite graphs.

Write $(\Lambda_t'(\kappa), d)$ for the 2-rank graph induced by these adjacency functions. We realise that the proof of Theorem 2.4.4 does not depend on the number of sides $2t$ of the polygons; hence nor do the K-groups associated to $\Lambda_t'(\kappa)$.

2.5.10 Theorem (K-theory for algebras of unpointed $2t$-gon systems) *For $\alpha, \beta \geq 2$, consider the complete bipartite graph on α white and β black vertices $\kappa = \kappa(\alpha, \beta)$. Then*

$$K_\epsilon(\mathcal{A}(\Lambda_t'(\kappa))) \cong K_\epsilon(\mathcal{A}(\Lambda'(\kappa))),$$

for $\epsilon = 0, 1$, and all $t \geq 1$. ■

2.5.11 Theorem (Order of identity in K_0 for unpointed polygon systems) *Let $t \geq 1$ be an arbitrary integer. For $\alpha, \beta \geq 3$, let $\kappa = \kappa(\alpha, \beta)$ be a complete bipartite graph, and let $\mathcal{A} = \mathcal{A}(\Lambda_t'(\kappa))$ be the graph algebra induced by an unpointed $2t$-gon system. Write $g := \gcd(\alpha - 2, \beta - 2)$. Then the order of the class of the identity $\mathrm{id}_{\mathcal{A}}$ in $K_0(\mathcal{A})$ is equal to g if g is odd, and $g/2$ if g is even.*

Furthermore, the isomorphism class of $\mathcal{A}(\Lambda_t'(\kappa))$ is completely determined by the K-groups in Theorem 2.5.10 and the order of $[\mathrm{id}_{\mathcal{A}}]$ in $K_0(\mathcal{A})$. ■

An alternative construction of a $2t$-gon system

2.5.12 Theorem 2.5.10 gives us a collection of K-groups for algebras corresponding to systems of $2t$-gons with an arbitrary even number of sides $2t$—in the pointed case however, Theorem 2.5.7 insists on $2t$ being divisible by four. This is a consequence of how we define adjacency in each instance: in the $2t$-hedron $P(\kappa)$, each face is adjacent to every other, and since the number of faces is not dependent on t, nor are the U- and V-adjacency matrices in an unpointed $2t$-gon system.

Adjacency in the pointed case is more difficult to define canonically. When $t = 2$ and we are dealing with tiles, there is an obvious pair of adjacency functions. We extended these in Definition 2.5.4, thinking of two $2t$-gons as adjacent if we can reflect one horizontally or vertically in order to obtain the form of the other. This works since horizontal and vertical reflections commute, and so the $2t$-gon system will have the *UCE Property*. If t is not even, then there are no two distinct reflections of $2t$-gons which commute and preserve the structure of pointed $2t$-gons. We must pick the same two reflections for both adjacency functions, else some combination of rotations and identity transformations. None of these options is a direct extension of our horizontal and vertical adjacency functions from 2.2.1, and so there is no natural choice.

We suggest that the following definitions of U- and V-adjacency for pointed $2t$-gons are the most intuitive for $t \geq 3$, based on the idea that adjacent $2t$-gons should have opposite orientations. They do not, however, generalise the tile systems from §2.1–2.4, themselves being the most natural constructions when $t = 2$. Hence, the previous constructions are given the spotlight in this section up until now.

The proof of Proposition 2.5.14 is almost identical to that of 2.5.5, together with Proposition 2.2.4. From this, along with Theorem 2.2.6, we can deduce Theorems 2.5.15 and 2.5.16.

2.5.13 Definition (Alternative U- and V-adjacency) Let $t \geq 1$ be a fixed arbitrary integer, let $(G, U, V, \mathcal{S}_t)$ be a $2t$-gon system, and let $A = [x_1, y_1, \ldots, x_t, y_t] \in \mathcal{S}_t$ be a pointed $2t$-gon.

A pointed $2t$-gon $B \in \mathcal{S}_t$ is V^*-adjacent to A if and only if $B = \left[\bar{x}'_1, \bar{y}_t, \ldots, \bar{x}'_2, \bar{y}_1\right]$ for some $x'_i \neq x_i$. Similarly, we say that a pointed $2t$-gon $C \in \mathcal{S}_t$ is U^*-adjacent to A if and only if $C = \left[\bar{x}_1, \bar{y}'_t, \ldots, \bar{x}_2, \bar{y}'_1\right]$, for some $y'_i \neq y_i$. We define the U^*- and V^*-adjacency matrices M^*_U and M^*_V respectively, as above.

2.5.14 Proposition *Let $(\kappa, U, V, \mathcal{S}_t(\kappa))$ be a $2t$-gon system with adjacency matrices M^*_U, M^*_V. Then $(\kappa, U, V, \mathcal{S}_t(\kappa))$ induces a 2-rank graph $\Lambda^*_t(\kappa)$, which is row-finite, with finite vertex set and no sources.* ■

■ **Proof** Consider the pointed $2t$-gon $A = \left[u^1_i, v^1_j, \ldots, u^t_i, v^t_j\right] \in \mathcal{S}_t(\kappa)$; those $2t$-gons corresponding to its reflections and rotations are treated similarly. Then a pointed $2t$-gon B is V^*-adjacent to A if and only if $B = \left[\bar{u}^1_k, \bar{v}^t_j, \ldots, \bar{u}^2_k, \bar{v}^1_j\right]$, for some $k \neq i$. Suppose B is such a $2t$-gon—then a pointed $2t$-gon D is U^*-adjacent to B if and only if $D = \left[u^1_k, v^1_l, \ldots, u^t_k, v^t_l\right]$, for some $l \neq j$.

Likewise, C is U^*-adjacent to A if and only if $C = \left[\bar{u}^1_i, \bar{v}^t_l, \ldots, \bar{u}^2_i, \bar{v}^1_l\right]$, for some $k \neq i$. Clearly if C is such a $2t$-gon, then D is the unique $2t$-gon which is both U^*-adjacent to B and V^*-adjacent to C. The induced 2-rank graph follows from [KP00, §6] and the same arguments used in Propositions 2.5.5 and 2.2.4. ■

2.5.15 Theorem (K-theory for algebras of pointed $2t$-gon systems, t arbitrary) *Let $t \geq 1$, and for $a, b \geq 0$, let $\kappa = \kappa(a+2, b+2)$ be the complete bipartite graph on $a+2$ white and $b+2$ black vertices. Without loss of generality, we assume that $a \leq b$. Then, for $\epsilon = 0, 1$:*

(i) If $a = b = 0$, then $K_\epsilon(\mathcal{A}(\Lambda_t^*(\kappa))) \cong \mathbb{Z}^{4t}$.

(ii) If $b \geq 1$ and a, b are coprime, then $K_\epsilon(\mathcal{A}(\Lambda_t^*(\kappa))) \cong \mathbb{Z}^{2t(a+1)(b+1)}$.

(iii) If $b \geq 1$ and a, b are not coprime, then

$$K_\epsilon(\mathcal{A}(\Lambda_t^*(\kappa))) \cong \mathbb{Z}^{2t(a+1)(b+1)} \oplus (\mathbb{Z}/g)^t,$$

where $g := \gcd(a, b)$.

■ **Proof** The proof unsurprisingly follows the same lines as those of Theorems 2.2.8, 2.4.4, and 2.5.7. Write $\alpha := a+2$, $\beta := b+2$, and let $\beta \geq 3$. We denote the pointed $2t$-gons in $\mathcal{S}_t(\kappa)$ as:

- $(A_r)_{ij} := \left[u_i^r, v_j^r, \ldots, u_i^t, v_j^t, u_i^1, v_j^1, \ldots, u_i^{r-1}, v_j^{r-1}\right]$,
- $(B_r)_{ij} := \left[\bar{u}_i^r, \bar{v}_j^{r-1}, \ldots, \bar{u}_i^1, \bar{v}_j^t, \ldots, \bar{u}_i^{r+1}, \bar{v}_j^r\right]$,

for $i \in \{1, \ldots, \alpha\}$, $j \in \{1, \ldots, \beta\}$, $r \in \{1, \ldots, t\}$, and with addition in superscript indices defined modulo t. Observe that each $S \in \mathcal{S}_t(\kappa)$ is either of the form $(A_r)_{ij}$ or $(B_r)_{ij}$. Then

$$\begin{aligned}\operatorname{coker}\left[\mathbf{1} - (M_U^*)^T, \mathbf{1} - (M_V^*)^T\right] = \Bigg\langle (A_r)_{pq}, (B_r)_{pq} \;\Bigg|\; & (A_r)_{pq} = \sum_{i \neq p} (B_r)_{iq} = \sum_{j \neq q} (B_r)_{pj}, \\ & (B_r)_{pq} = \sum_{i \neq p} (A_r)_{iq} = \sum_{j \neq q} (A_r)_{pj}, \\ & \text{for } i \in \{1, \ldots, \alpha\}, j \in \{1, \ldots, \beta\}, \text{ and } r \in \{1, \ldots, t\} \Bigg\rangle.\end{aligned}$$

As in the proof of 2.2.8, define $(J_r)_q := \sum_{i=1}^{\alpha} (A_r)_{iq}$, and $(I_r)_p := \sum_{j=1}^{\beta} (A_r)_{pj}$. Through a sequence of Tietze transformations, and using observations from previous proofs, we see that the above presentation for coker $=$ coker $\left[\mathbf{1} - (M_U^*)^T, \mathbf{1} - (M_V^*)^T\right]$ is equivalent to

$$\begin{aligned}\operatorname{coker} &= \Bigg\langle (A_r)_{pq} \;\Bigg|\; (A_r)_{pq} = \sum_{i \neq p} \sum_{k \neq i} (A_r)_{kq} = \sum_{j \neq q} \sum_{l \neq j} (A_r)_{pl}, \sum_{i \neq p} (A_r)_{iq} = \sum_{j \neq q} (A_r)_{pj} \Bigg\rangle \\ &= \Bigg\langle (A_r)_{pq} \;\Bigg|\; (\alpha - 2)(J_r)_q = (\beta - 2)(I_r)_p = 0, \sum_{i \neq p} (A_r)_{iq} = \sum_{j \neq q} (A_r)_{pj} \Bigg\rangle \\ &= \langle (A_r)_{pq} \mid (\alpha - 2)(J_r)_q = (\beta - 2)(I_r)_p = 0, (J_r)_q = (I_r)_p, \text{ for all } p, q \rangle.\end{aligned}$$

We can rewrite each $(A_r)_{i1}$ and $(A_r)_{1j}$ in terms of the other $(A_r)_{ij}$, the $(J_r)_q$, and the $(I_r)_p$, and hence remove them from the list of generators. Then, since $(J_r)_q = (I_r)_p$ for all $p \in \{1, \dots, \alpha\}$ and $q \in \{1, \dots, \beta\}$ we can remove all-but-one of these from the list of generators as well, leaving

$$\begin{aligned} \text{coker} = \langle (A_r)_{pq}, (J_r)_1 \mid (\alpha-2)(J_r)_1 = (\beta-2)(J_r)_1 = 0, \\ \text{for } p \in \{2, \dots, \alpha\}, q \in \{2, \dots, \beta\}, \text{ and } r \in \{1, \dots, t\} \rangle. \end{aligned} \tag{2.8}$$

We substitute $a = \alpha - 2$ and $b = \beta - 2$, and write $g := \gcd(a, b)$. Then (2.8) is a presentation for $\mathbb{Z}^{t(a+1)(b+1)} \oplus (\mathbb{Z}/g)^t$ if $g > 1$, and $\mathbb{Z}^{t(a+1)(b+1)}$ otherwise. If $\alpha = \beta = 2$, then (2.8) gives a presentation for $\mathbb{Z}^2$. Together with Theorem 2.2.6, this gives the desired result. ∎

2.5.16 Theorem (Order of identity in K_0 for alternative pointed polygon systems) *Let $t \geq 1$ be an arbitrary integer. For $\alpha, \beta \geq 3$, let $\kappa = \kappa(\alpha, \beta)$ be a complete bipartite graph, and write $\mathcal{A}$ to denote the graph algebra $\mathcal{A}(\Lambda_t^*(\kappa))$ induced by the adjacency matrices from 2.5.13. Then the order of the class of the identity $\mathrm{id}_{\mathcal{A}}$ in $K_0(\mathcal{A})$ is equal to $g := \gcd(\alpha - 2, \beta - 2)$.*

Furthermore, the isomorphism class of $\mathcal{A}$ is completely determined by the K-groups in Theorem 2.5.15 and the order of $[\mathrm{id}_{\mathcal{A}}]$ in $K_0(\mathcal{A})$. ∎

§2.6 The homology of $2t$-hedra

We round off this chapter with a brief discussion of the geometry of tile complexes and $2t$-hedra. Similarly to Theorem 2.5.10, the proof of 2.6.2 does not change with the number of sides $2t$ of the polygons, so Corollary 2.6.3 is immediate.

2.6.1 To each k-dimensional cell complex $\mathcal{M}$ can be associated a sequence $\{H_n(\mathcal{M})\}_{n\in\mathbb{N}}$ of Abelian groups, called the homology groups of $\mathcal{M}$. The nth homology group $H_n(\mathcal{M})$ provides a measure of the number of k-dimensional holes in the complex, as well as an idea of the "twistedness" of $\mathcal{M}$ (see [Hat02, §2.1]). For each $n \in \{0, \dots, k\}$, write $\mathcal{S}'_n$ to denote the set of n-dimensional cells in $\mathcal{M}$; we construct free Abelian groups $C_n \cong \mathbb{Z}\mathcal{S}'_n$ whose generators are indexed by the elements of $\mathcal{S}'_n$. Clearly $C_n = 0$ whenever $n > k$. Then there is a sequence of functions δ_n where $\delta_n \circ \delta_{n+1} = 0$ as follows:

$$\cdots \to 0 \longrightarrow C_k \xrightarrow{\delta_k} \cdots \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \longrightarrow 0. \tag{2.9}$$

This is called a chain complex, and the map $\delta_n : C_n \to C_{n-1}$ takes an n-cell to its *boundary*, a formal sum of its $(n-1)$-dimensional faces. We may choose how to order and orient the list of faces which make up the boundary of a cell (see, for example, Figure 3.1), so long as this decision is fixed beforehand. Thus $\delta_n : \mathbb{Z}\mathcal{S}'_n \to \mathbb{Z}\mathcal{S}'_{n-1}$ can be viewed as a $|\mathcal{S}'_n| \times |\mathcal{S}'_{n-1}|$ matrix with rows indexed by the n-cells of $\mathcal{M}$, columns by the $(n-1)$-cells, and with AB-th entry 1 if and only if B is a face of A (-1 if the cell $\bar{B}$ with the opposite orientation to B forms part of the boundary).

Some intricacies might arise if two faces in the boundary of a cell of dimension at least 2 are identical, but this can be avoided by subdividing the complex into smaller cells. Then $H_n := \ker(\delta_n)/\mathrm{im}(\delta_{n+1})$ is an Abelian group, and the torsion-free rank (see 2.2.5) of H_n counts the number of n-dimensional "holes" in $\mathcal{M}$.

2.6.2 Theorem (Homology of a tile complex) *Let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph on $\alpha \geq 2$ white and $\beta \geq 2$ black vertices, let $(\kappa, U, V, \mathcal{S}'(\kappa))$ be an unpointed tile system, and let $TC(\kappa)$ be its associated tile complex. Then the homology groups of $TC(\kappa)$ are given by*

$$H_n(TC(\kappa)) \cong \begin{cases} \mathbb{Z} & \textit{if } n = 0, \\ \mathbb{Z}^{\alpha+\beta-2} & \textit{if } n = 1, \\ \mathbb{Z}^{(\alpha-1)(\beta-1)} & \textit{if } n = 2, \\ 0 & \textit{if } n \geq 3. \end{cases}$$

■ **Proof** As $TC(\kappa)$ is a path-connected, 2-dimensional cell complex by construction, clearly $H_n(TC(\kappa)) \cong 0$ for $n = 0$ and $n \geq 3$.

The proof uses as its basis that of [NTV18, Prop. 3]. The boundary of each square in $TC(\kappa)$ is given by an element of $\mathcal{S}'(\kappa)$; write these elements as $(u_i^1, v_j^1, u_i^2, v_j^2)$. By construction, $TC(\kappa)$ has four vertices: each one the origin of all directed edges labelled u_i^1, v_j^1, u_i^2, and v_j^2 respectively. Each tile is homotopy equivalent to a point; pick tile $(u_1^1, v_1^1, u_1^2, v_1^2)$ and contract it, thereby identifying the four vertices. Call the resulting tile complex $TC_1(\kappa)$; this is a 2-dimensional cell complex whose edges are loops, and whose 2-cells comprise:

- $(\alpha - 1)(\beta - 1)$-many unpointed tiles $A'_{ij} = (u_i^1, v_j^1, u_i^2, v_j^2)$,
- $(\alpha - 1)$-many 2-gons X'_i with boundaries described analogously by (u_i^1, u_i^2),
- $(\beta - 1)$-many 2-gons Y'_j with boundaries described by (v_j^1, v_j^2),

for $i \in \{2, \ldots, \alpha\}$ and $j \in \{2, \ldots, \beta\}$. By construction, no two edges on the boundary of a square are identical, so we can consider the chain complex associated to $TC_1(\kappa)$ by 2.6.1:

$$\cdots \longrightarrow C_3 \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \longrightarrow 0.$$

Since $TC_1(\kappa)$ is 2-dimensional and has one vertex, this boils down to

$$0 \xrightarrow{0} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{0} 0,$$

and so $H_1(TC_1(\kappa)) \cong C_1/\mathrm{im}(\delta_2)$, and $H_2(TC_1(\kappa)) \cong \ker(\delta_2)$. We have $\delta_2(A'_{ij}) = u_i^1 + v_j^1 + u_i^2 + v_j^2$, $\delta_2(X'_i) = u_i^1 + u_i^2$, and $\delta_2(Y'_j) = v_j^1 + v_j^2$. Clearly $\ker(\delta_2)$ is generated by the set

$\{A'_{ij} - X'_i - Y'_j \mid 2 \leq i \leq \alpha, 2 \leq j \leq \beta\}$, such that $\ker(\delta_2) \cong \mathbb{Z}^{(\alpha-1)(\beta-1)}$.

Similarly, we have an Abelian group presentation for $H_1(TC_1(\kappa))$ as follows:

$$H_1(TC_1(\kappa)) \cong \Big\langle u_i^1, v_j^1, u_i^2, v_j^2 \;\Big|\; u_i^1 + v_j^1 + u_i^2 + v_j^2 = u_i^1 + u_i^2 = v_j^1 + v_j^2 = 0, \\ \text{for } 2 \leq i \leq \alpha, 2 \leq j \leq \beta \Big\rangle,$$

which, after substituting $u_i^2 = -u_i^1$ and $v_j^2 = -v_j^1$, gives

$$H_1(TC_1(\kappa)) \cong \big\langle u_i^1, v_j^1, \text{ for } 2 \leq i \leq \alpha, 2 \leq j \leq \beta \big\rangle.$$

This is a presentation for $\mathbb{Z}^{\alpha+\beta-2}$, and since $TC_1(\kappa)$ is homotopy equivalent to $TC(\kappa)$, we are done. ■

2.6.3 Corollary (Homology of a $2t$-hedron) *Let $(\kappa, U, V, \mathcal{S}'_t(\kappa))$ be an unpointed $2t$-gon system, and let $P(\kappa)$ be its associated $2t$-hedron. Then the homology groups of $P(\kappa)$ do not depend on t, that is:*

$$H_n(P(\kappa)) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}^{\alpha+\beta-2} & \text{if } n = 1, \\ \mathbb{Z}^{(\alpha-1)(\beta-1)} & \text{if } n = 2, \\ 0 & \text{if } n \geq 3. \end{cases}$$

■

Chapter 3

k-domino groups

In this chapter we use the basic tools introduced in Chapter 2 to construct an infinite family of k-rank graphs for *arbitrary* k.

Instead of forming sets of tiles, we look at sets of k-dimensional cubes, upon which we define k-many adjacency functions. We can glue together these cubes whenever they are adjacent (like dominoes), in a manner reminiscent of the *t-hedra* from Definition 2.1.2. We design our sets of cubes such that the k-dimensional cube complex $\mathcal{M}$ obtained as a result of this gluing has a particular link at each vertex—by doing this, we ensure that the adjacency matrices have a k-dimensional *UCE Property* (3.3.4, to be compared with 2.2.2) and induce a k-rank graph.

We define a *k-domino group* or *k-cube group* to be a group Γ whose generators and relations are induced by the squares in $\mathcal{M}$. A k-domino group acts freely and transitively on a product of k trees, a k-rank affine building, and the quotient of this action is $\mathcal{M}$. Through these correspondences we are able to weave between geometric and group-theoretical notions, eventually explaining what it means to have a *cube in the group* Γ.

Some facts about the K-theory of the $C^\star$-algebras associated to these higher-rank graphs were demonstrated in [MRV20], which this chapter cites as its principal source, and which was joint work with Aura-Cristiana Radu and Alina Vdovina. It was shown there that the K-groups together with the order of the class of the identity in K_0 determine the $C^\star$-algebras uniquely, up to isomorphism.

§3.1 Domino complexes

3.1.1 As in §2.1, we begin by constructing sets of squares with distinguished basepoints and orientations. Let $[a, b, c, d]$ denote the square whose boundary is labelled anticlockwise and starting from the base vertex by the sequence a, b, c, d, and give each side of the boundary a forward-directed arrow. We call such squares 2-dimensional dominoes, or sometimes just pointed squares.

Let $\{E_i \mid 1 \leq i \leq k\}$ be a family of alphabets: disjoint sets of respective *even* size m_i, each equipped with a fixed-point-free involution $x \mapsto \bar{x}$. For each $p, q \in \{1, \ldots, k\}$ with $p \neq q$, we write $F(p, q)$ to denote the set consisting of all those dominoes $[a_1, b_1, a_2, b_2]$, where $a_1, a_2 \in E_p$ and $b_1, b_2 \in E_q$. We identify the involution $e \mapsto \bar{e}$ with the reversion of the direction of an arrow.

3.1.2 **Definition (k-dimensional dominoes)** Let E_1, E_2, E_3 be alphabets of size at least 4, and consider three pointed squares $A_1 = [a_1, b_1, a_2, b_2] \in F(1, 2)$, $B_1 = [a', c_1, a_3, c_2] \in F(1, 3)$ and $C_1 = [b', c_3, b_3, c'] \in F(2, 3)$. If $a' = a_1$, then we can glue A_1 and B_1 together along their common side, in the manner of 2.1.2. If, in addition, $b' = \bar{b}_2$ and $c' = c_2$, then we can slot C_1 together with A_1 and B_1 to form half of a cube (as in Figure 3.1). We might be lucky enough to find three more pointed squares $A_2 \in F(1, 2)$, $B_2 \in F(1, 3)$ and $C_2 \in F(2, 3)$ which form the remaining three faces of the cube. If this is the case, then we write $S = [A_1, B_1, C_1, A_2, B_2, C_2]$ to denote the cube, regarded as oriented and with a basepoint, and call S a 3-dimensional domino.

The choice of basepoints and orientations of the 3-dimensional domino and its faces is arbitrary, but must remain fixed if we're considering a set of multiple dominoes.

If $E_1, \ldots, E_k$ are alphabets of size at least 4, then we can generalise the above definition to that of a k-dimensional domino: a pointed, oriented k-dimensional cube formed by gluing together a compatible set of pointed squares, 2^{k-2} from each set $F(p, q)$ for $p < q$.

3.1.3 We're going to define *adjacency* (3.1.16) for k-dimensional dominoes whenever they share a common face, similarly to in 2.2.1 and Figure 2.3 and as the terminology is designed to suggest. Then, we'll glue together adjacent dominoes to form k-dimensional analogues of the *t-hedra* from Definition 2.1.2, which we call *domino complexes* (3.1.10).

Given some alphabets $E_1, \ldots, E_k$, we aim to find a set of pointed squares $\mathcal{S}_2 \subseteq \bigsqcup_{p<q} F(p, q)$ which yield domino complexes with a *UCE Property* (3.3.4). Definition 3.1.2 doesn't tell us how to build such a set $\mathcal{S}_2$, so we look at the converse construction, beginning from our desired domino complex and seeing what its component squares look like. We describe conditions on a set of pointed squares which allow them to be compiled into a domino complex in 3.1.17.

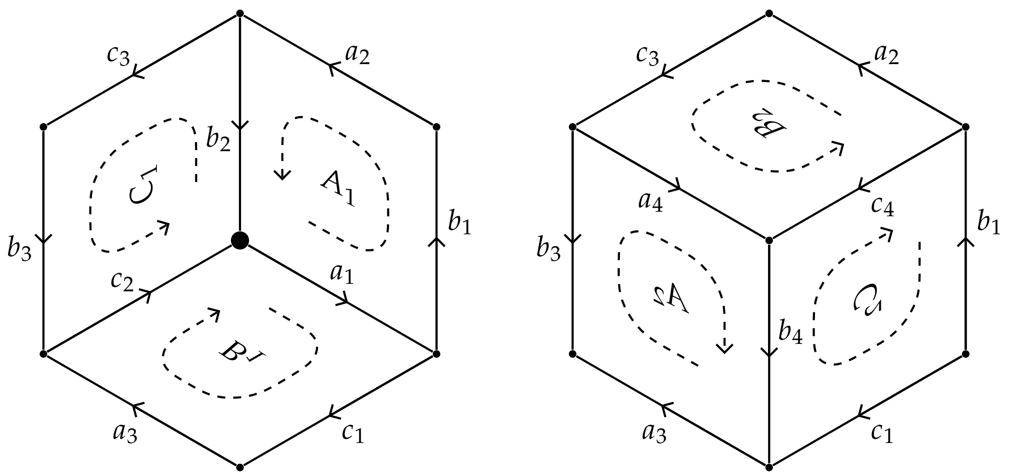


Figure 3.1: Depiction of the 3-dimensional domino $S = [A_1, B_1, C_1, A_2, B_2, C_2]$, which has faces $A_1 = [a_1, b_1, a_2, b_2]$, $B_1 = [a_1, c_1, a_3, c_2]$, $C_1 = \left[\bar{b}_2, c_3, b_3, c_2\right]$, $A_2 = \left[a_3, \bar{b}_3, a_4, b_4\right]$, $B_2 = [a_2, c_3, a_4, c_4]$, $C_2 = \left[\bar{b}_1, c_1, \bar{b}_4, c_4\right]$; the basepoint is represented by the large black dot.

We use this ordering and orientation of faces for our 3-dimensional examples, but as long as they are fixed beforehand, any other order and orientation are also valid.

Endowing a cube complex with an adjacency structure

3.1.4 For $n \geq 2$, we define $T(n)$ to be the regular tree of degree n. We may simply write T if the degree is not important.

Let $T(m_1), \ldots, T(m_k)$ be regular trees, and consider the product $T(m_1) \times \cdots \times T(m_k)$. This defines a k-dimensional cube complex (see 2.1.1) which we call Δ, and which is an *affine building* of rank k (see 3.2.8–3.2.9).

3.1.5 Definition (Clique complex) An n-dimensional simplex is the convex hull of $(n+1)$-many linearly-independent vertices $v_i \in \mathbb{R}^n$. Thus, a 3-dimensional simplex is a tetrahedron, a 2-dimensional simplex is a triangle, etc. The faces of a simplex are the lower-dimensional simplices defined by subsets of $\{v_i\}$. A (geometric) simplicial complex is a set of simplices which have been “glued together” along their faces, in the same manner as 2.1.1. Every face of a simplex in the complex is also in the complex, and the intersection of two simplices is a face of them both. The dimension of a simplicial complex is defined to be that of the highest-dimensional simplex it contains.

Let G be an undirected graph. A clique in G is a collection of vertices, any two of which are adjacent—that is, a collection of vertices which induces a complete subgraph. The clique complex of G is a simplicial complex where each clique with $(n + 1)$ vertices defines a simplex of dimension n. In other words, if there are three vertices each connected by an edge, then these defines a triangle in G; four mutually connected vertices always define a tetrahedron, and so on.

3.1.6 Definition (Adjacency structure) Let $\mathcal{M}$ be a k-dimensional cube complex with vertex set $\mathcal{S}_0$ and directed edges labelled from an alphabet E. The link $\mathrm{lk}_z(\mathcal{M})$ at a vertex $z \in \mathcal{S}_0$ is the $(k-1)$-dimensional cell complex obtained from the intersection of $\mathcal{M}$ with a small 2-sphere centred at z (compare with 2.1.2).

For each $z \in \mathcal{S}_0$, let $E(z)$ be the set of directed edges originating at z. Suppose that we can partition the edges $E = E_1 \sqcup \cdots \sqcup E_k$ in such a way that $\bar{e} \in E_i$ whenever $e \in E_i$. If each link $\mathrm{lk}_z(\mathcal{M})$ is the clique complex of the complete k-partite graph with vertices labelled by the elements of $E(z)$ and partition $E(z) = E(z)_1 \sqcup \cdots \sqcup E(z)_k$, then we say that the alphabets $E_1, \ldots, E_k$ form an adjacency structure for $\mathcal{M}$.

3.1.7 Proposition *Let $\mathcal{M}$ be a k-dimensional cube complex. The universal cover of $\mathcal{M}$ is a product of k trees $\tilde{\mathcal{M}} = T(m_1) \times \cdots \times T(m_k)$ if and only if the link at each vertex of $\mathcal{M}$ is the clique complex of a complete k-partite graph on vertex sets of size $|m_1|, \ldots, |m_k|$.*

■ **Proof** This proposition is a generalisation of [BW99, 10.2]. Observe that if the link $\mathrm{lk}_z(\mathcal{M})$ at a vertex z of $\mathcal{M}$ is such a clique complex, then $\mathrm{lk}_z(\mathcal{M})$ is a $(k-1)$-dimensional complex such that every cycle has length at least k. Hence $\mathrm{lk}_z(\mathcal{M})$ is CAT(1), and so by the *Gromov Link Condition* [Gro88, §4.2], $\mathcal{M}$ must be CAT(0). The result then follows from a relatively straightforward adaptation to [BW99, 4.3]. ■

3.1.8 So, a cube complex $\mathcal{M}$ admits an adjacency structure if and only if its universal cover is a product of trees. If this is the case, then the n-dimensional cells of $\mathcal{M}$ are n-cubes whose edges have labels from n of the alphabets $E_1, \ldots, E_k$, and where parallel edges have labels from the same alphabet. We'll see how this looks in three dimensions to begin with.

3.1.9 Definition (3-dimensional domino complex) Let $\mathcal{M}$ be a 3-dimensional cube complex with vertex set $\mathcal{S}_0$, directed edges labelled from some alphabet E, and adjacency structure E_1, E_2, E_3. Write $\mathcal{S}_2' = \mathcal{S}_2'(\mathcal{M})$ for the set of geometric squares of which $\mathcal{M}$ consists. The elements of $\mathcal{S}_2'$ can be written as ordered 4-tuples of their oriented edge labels (a_1, b_1, a_2, b_2) for $a_1, a_2 \in E_p$, $b_1, b_2 \in E_q$. We will always have $p \neq q$ here by the fact that the 1-skeletons of the links of $\mathcal{M}$ are tripartite graphs, meaning that adjacent vertices in the links cannot have labels from the same alphabet E_i.

We use square brackets if we wish to emphasise that a square in $\mathcal{M}$ is labelled according to some predetermined orientation and starting from some basepoint (compare with Definition 2.1.5). For each such square $S = [a_1, b_1, a_2, b_2]$, write

$$S_H := \left[\bar{a}_1, \bar{b}_2, \bar{a}_2, \bar{b}_1\right], \quad S_R := [a_2, b_2, a_1, b_1], \quad S_V := \left[\bar{a}_2, \bar{b}_1, \bar{a}_1, \bar{b}_2\right];$$

geometrically these can be interpreted as the pointed squares which lie in the same orbit of S under reflections in the a and/or b directions. We define the set

$$\mathcal{S}_2 := \{S = [a_1, b_1, a_2, b_2], S_H, S_R, S_V \mid (a_1, b_1, a_2, b_2) \in \mathcal{S}_2'\}.$$

We can identify the set $F(p,q)$ with $F(q,p)$ via the map $\varphi : [a_1, b_1, a_2, b_2] \mapsto [\bar{b}_2, \bar{a}_2, \bar{b}_1, \bar{a}_1]$. Thus we can say that $\mathcal{S}_2$ is the set of all pointed squares of the form $[a_1, b_1, a_2, b_2]$ with $a_1, a_2 \in E_p$ and $b_1, b_2 \in E_q$ for $p < q$.

Similarly, we write $\mathcal{S}'_3 = \mathcal{S}'_3(\mathcal{M})$ for the set of geometric cubes of which $\mathcal{M}$ consists, and we denote elements of $\mathcal{S}'_3$ by ordered 6-tuples of their faces $(A_1, B_1, C_1, A_2, B_2, C_2)$, where $A_1, A_2 \in F(p,q)$, $B_1, B_2 \in F(p,r)$, and $C_1, C_2 \in F(q,r)$. As above, we use square brackets to indicate that a cube is oriented and with a basepoint, and for each such cube $U = [A_1, B_1, C_1, A_2, B_2, C_2]$ we define

- $U_H := [(A_1)_H, (B_1)_H, (C_2)_H, (A_2)_H, (B_2)_H, (C_1)_H]$,
- $U_R := [(A_1)_R, B_2, C_2, (A_2)_R, B_1, C_1]$,
- $U_V := [(A_1)_V, (B_2)_H, (C_1)_H, (A_2)_V, (B_1)_H, (C_2)_H]$,
- $U_I := [(A_2)_H, (B_1)_V, (C_1)_V, (A_1)_H, (B_2)_V, (C_2)_V]$,
- $U_{HI} := [A_2, (B_1)_R, (C_2)_R, A_1, (B_2)_R, (C_1)_R]$,
- $U_{RI} := [(A_2)_V, (B_2)_V, (C_2)_V, (A_1)_V, (B_1)_V, (C_1)_V]$,
- $U_{VI} := [(A_2)_R, (B_2)_R, (C_1)_R, (A_1)_R, (B_1)_R, (C_2)_R]$.

These are the cubes $[X_1, \ldots, X_6]$ which belong to the same orbit as $[A_1, B_1, C_1, A_2, B_2, C_2]$ under action by the symmetry group of the cube, with the property that if $A_1 \in F(p,q)$, then $X_1 \in F(p,q)$. We write $\mathcal{S}_3$ for the set which consists of each $U \in \mathcal{S}'_3$ and all of the corresponding pointed cubes above, and refer to elements of $\mathcal{S}_3$ as dominoes or 3-dimensional dominoes, and this definition coincides with 3.1.2. The complex $\mathcal{M}$ is called a 3-dimensional domino complex. We write

$$F(p,q,r) := \{[A_1, B_1, C_1, A_2, B_2, C_2] \in \mathcal{S}_3 \mid A_i \in F(p,q), B_i \in F(p,r), \text{and } C_i \in F(q,r)\},$$

and identify $F(a,b,c)$ with $F(a,c,b)$ via the isomorphism

$$(A_1, B_1, C_1, A_2, B_2, C_2) \longmapsto (\varphi(A_1), (C_2)_H, (B_2)_H, \varphi(A_2), (C_1)_H, (B_1)_H).$$

Likewise we are able to identify $F(a,b,c)$ with each of the sets $F(\sigma(a,b,c))$, where σ is a permutation. This is all much more succinctly describable in pictures, so we urge the reader to consult Figure 3.2.

3.1.10 Definition (k-dimensional domino complex) Now let $\mathcal{M}$ be a k-dimensional cube complex with vertex set $\mathcal{S}_0$, directed edges labelled from some alphabet E, and adjacency structure $E_1, \ldots, E_k$. We define $\mathcal{S}_2$, $\mathcal{S}'_2$, $\mathcal{S}_3$ and $\mathcal{S}'_3$ as in 3.1.9, and for each $n \in \{4, \ldots, k\}$ we define the sets $\mathcal{S}'_n$ and $\mathcal{S}_n$ inductively as follows:

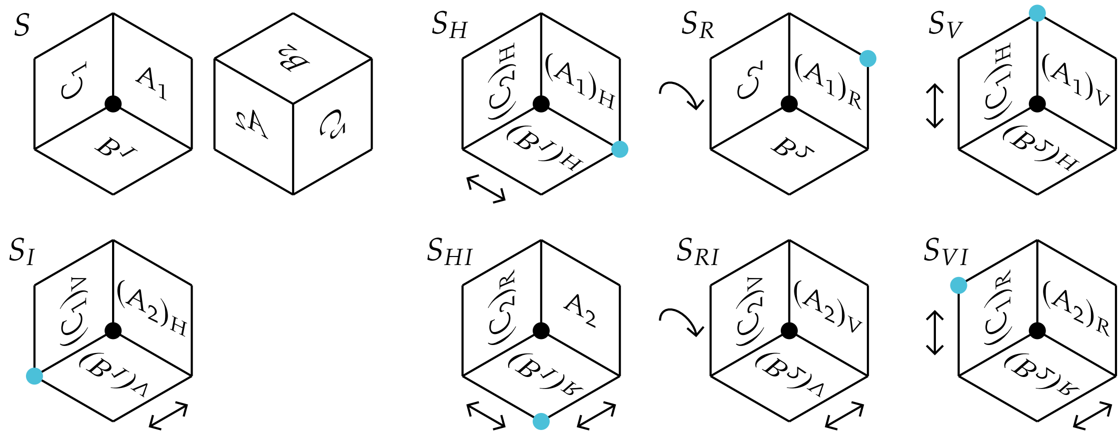


Figure 3.2: Let $U = [A_1, B_1, C_1, A_2, B_2, C_2] \in \mathcal{S}_3$ be a 3-dimensional domino. Then the seven cubes defined in 3.1.9 are also dominoes in $\mathcal{S}_3$. These are obtained by reflecting S according to the above arrows (a composition of two reflections might be denoted as a rotation by π). The symmetries keep the A_i, B_i and C_i faces "in the same positions" relative to the basepoint (in this figure the B_i faces always appear on the top and bottom of the dominoes, for example).

The black vertices represent the basepoints of the dominoes, while the blue vertices show where the basepoint of S maps to under the symmetry actions.

Write $\mathcal{S}'_n = \mathcal{S}'_n(\mathcal{M})$ for the set of geometric n-cubes of which $\mathcal{M}$ consists. We denote elements of $\mathcal{S}'_n$ by ordered $(2n)$-tuples of their *faces*, that is, the elements of $\mathcal{S}_{n-1}$ which are incident. We define the following sets inductively from 3.1.9:

$$F(p_1, \ldots, p_n) := \left\{ \left[A_1^1, \ldots, A_1^n, A_2^1, \ldots, A_2^n\right] \in \mathcal{S}_{n-1} \;\middle|\; A_1^i, A_2^i \in F\left(p_1, \ldots, \hat{p}_{n-i+1}, \ldots, p_n\right) \right\},$$

where the "hat" over $\hat{p}_{n-i+1}$ indicates to remove it from the list. As above, we can identify the sets $F(\sigma(p_1, \ldots, p_n))$ for each permutation σ. Then $\mathcal{S}_n$ is the set of all $(2n)$-tuples $(X_1, \ldots, X_{2n})$ which belong to the orbit of some $\left(A_1^1, \ldots, A_1^n, A_2^1, \ldots, A_2^n\right) \in \mathcal{S}'_n$ under the action of the group of those reflections of the n-cube with the property that if $A_1^1 \in F(p_1, \ldots, p_{n-1})$, then $X_1 \in F(p_1, \ldots, p_{n-1})$. These are the reflections of the n-cube through midplanes parallel to its faces.

We call the complex $\mathcal{M}$ a k-dimensional domino complex, and we call the elements of $\mathcal{S}_k$ dominoes or k-dimensional dominoes. They are pointed, oriented k-dimensional cubes whose k sets of parallel edges are labelled respectively from the alphabets $E_1, \ldots, E_k$, and with edges labelled from each E_i always lying in the same orientation relative to the basepoint (see Figure 3.2). This is compatible with 3.1.2.

Constructing a domino complex from suitable sets of pointed squares

3.1.11 In 3.1.8 we learned that the cells of a k-dimensional domino complex $\mathcal{M}$ are n-dimensional cubes with parallel edges labelled from the same alphabet E_i. In particular, the 2-cells can be regarded as equivalence classes of elements of $F(p,q)$ for some $p,q \in \{1,\ldots,k\}$ with $p < q$, under the equivalence relation from 2.1.5. By choosing our initial set of pointed squares carefully, we can ensure that they form a k-dimensional domino complex when glued together.

3.1.12 **Example (A suitable set of pointed squares)** Consider the alphabets $E_1 = \{a_1, a_2, \bar{a}_1, \bar{a}_2\}$, $E_2 = \{b_1, b_2, \bar{b}_1, \bar{b}_2\}$, $E_3 = \{c_1, c_2, \bar{c}_1, \bar{c}_2\}$, and let $\mathcal{T}_2'$ be the set of 12 geometric squares

$$\mathcal{T}_2' = \left\{(a_i, b_j, \bar{a}_i, \bar{b}_j), (a_i, c_j, \bar{a}_i, \bar{c}_j), (b_i, c_j, \bar{b}_i, \bar{c}_j) \;\middle|\; i,j \in \{1,2\}\right\}.$$

Using this, we construct the corresponding set of pointed squares

$$\mathcal{T}_2 = \left\{S, S_H, S_R, S_V \mid S = [x_i, y_j, x_i, y_j] \text{ and } (x_i, y_j, x_i, y_j) \in \mathcal{T}_2'\right\},$$

as described in Definition 3.1.9. We have designed this set of squares in such a way as to be able to glue them together into cubes; using Figure 3.1 as a template, we give the examples

- $U = \big[[a_1, b_1, \bar{a}_1, \bar{b}_1], [a_1, c_1, \bar{a}_1, \bar{c}_1], [b_1, c_1, \bar{b}_1, \bar{c}_1],$
  $[\bar{a}_1, b_1, a_1, \bar{b}_1], [\bar{a}_1, c_1, a_1, \bar{c}_1], [\bar{b}_1, c_1, b_1, \bar{c}_1]\big],$
- $V = \big[[a_1, b_1, \bar{a}_1, \bar{b}_1], [a_1, c_2, \bar{a}_1, \bar{c}_2], [b_1, c_2, \bar{b}_1, \bar{c}_2],$
  $[\bar{a}_1, b_1, a_1, \bar{b}_1], [\bar{a}_1, c_2, a_1, \bar{c}_2], [\bar{b}_1, c_2, b_1, \bar{c}_2]\big],$
- $W = \big[[\bar{a}_1, \bar{b}_1, a_1, b_1], [\bar{a}_1, c_1, a_1, \bar{c}_1], [\bar{b}_1, c_1, b_1, \bar{c}_1],$
  $[a_1, \bar{b}_1, \bar{a}_1, b_1], [a_1, c_1, \bar{a}_1, \bar{c}_1], [b_1, c_1, \bar{b}_1, \bar{c}_1]\big].$

After a bit of staring we can deduce that any cube built from this set of squares has to be of the form $[A, B, C, A_H, B_H, C_H]$ for some $A, B, C \in \mathcal{T}_2$. We write $\mathcal{T}_3$ for the set of all such 6-tuples, and call elements of $\mathcal{T}_3$ *dominoes*. With the notation of 3.1.9 we see that $W = U_R$; in other words, W and U define the same geometric cube, but different dominoes—we write $\mathcal{T}_3'$ for the set of these geometric cubes. Likewise, by comparing the first two *faces* (elements of the 6-tuples) of U and V, we immediately see that they define different dominoes and different geometric cubes. In fact, we just need to choose A and B, and then C and all of the other faces are forced into place. This quality will be true of all domino complexes (Theorem 3.1.18), and is hard to achieve by accident.

Now, we can form a cube complex from the set $\mathcal{T}_3'$ by identifying the faces of two cubes whenever they are the same (as long as this doesn't result in two edges with the same label pointing towards the same vertex, as in 3.1.16). This cube complex is a 3-dimensional domino complex. ♦

3.1.13 A combinatorial description of k-dimensional dominoes In 3.1.2 we defined dominoes as pointed and oriented k-dimensional cubes with parallel edges given labels from one of k alphabets. In the rest of this section and the next, we will require some notation for dealing with specific edges and faces of dominoes, but characterising them combinatorially becomes needlessly complicated very quickly. We therefore restrict ourselves to dominoes built from a set of pointed squares which has a *domino structure*.

3.1.14 Definition (Domino structure) Let $k \geq 2$, and let $E_1, \ldots, E_k$ be alphabets of respective even cardinalities $m_1, \ldots, m_k$, with each $m_i \geq 4$. For each distinct p and q, define the set $F(p,q) := E_p \times E_q \times E_p \times E_q$. We can uniquely represent each element $[a_1, b_1, a_2, b_2] \in \bigsqcup_{i<j} F(i,j)$ as a pointed square with the same labels, so it makes notational sense to use square brackets for elements of $\bigsqcup_{i<j} F(i,j)$, and to refer to them as pointed squares.

Suppose that $R \subseteq \bigsqcup_{i<j} F(i,j)$ is a subset of pointed squares with the following properties:

D1 For every $[a_1, b_1, a_2, b_2] \in R$, each of $\left[\bar{a}_1, \bar{b}_2, \bar{a}_2, \bar{b}_1\right]$, $[a_2, b_2, a_1, b_1]$, and $\left[\bar{a}_2, \bar{b}_1, \bar{a}_1, \bar{b}_2\right]$ is also in R_2, and all four such squares are distinct.

D2 Each of the projections of R onto the subproducts of the form $E_p \times E_q$ or $E_q \times E_p$, for all $p \neq q$, is bijective.

If $k = 2$, then conditions **D1** and **D2** are enough for R to define the *VH-datum* of [BM97; Wis96]. Under our terminology, we say that R is a set of pointed squares with a 2-domino structure.

From now on, we denote by $R(p,q)$ the set of pointed squares with labels from E_p and E_q, that is, $R(p,q) := R \cap F(p,q)$.

As a last illustration before presenting a definition for arbitrary k, suppose that $k = 3$, and consider two pointed squares $[a_1, b_1, a_2, b_2] \in R(p,q)$ and $[a_1, c_1, a_3, c_2] \in R(p,r)$, where $q \neq r$. Also suppose that we can find some *unique* elements $a_4 \in E_p$, $b_3, b_4 \in E_q$, $c_3, c_4 \in E_r$ such that $\left[\bar{b}_2, c_3, b_3, c_2\right]$, $\left[\bar{b}_1, c_1, \bar{b}_4, \bar{c}_4\right] \in R(q,r)$, $\left[a_2, c_3, a_4, \bar{c}_4\right] \in R(p,r)$, and $\left[a_3, \bar{b}_3, a_4, b_4\right] \in R(p,q)$.

We may suppose that the same unique pointed squares can be found if we are instead given $[a_1, b_1, a_2, b_2] \in R(p,q)$ and $\left[\bar{b}_2, c_3, b_3, c_2\right] \in R(q,r)$.

By interpreting these pointed squares purely geometrically (in the manner of Figure 3.1), we see that each of $a_1, \ldots, a_4, b_1, \ldots, b_4, c_1, \ldots, c_4$ labels the edges of a cube. In the manner above, we write $\mathcal{S}_3$ for the set of 6-tuples of elements of R which correspond to the faces of all such cubes, pointed and oriented according to some predetermined orientation, in the manner of 3.1.9.

Now fix $k \geq 3$ and suppose that R is a set of pointed squares with properties **D1** and **D2**; we construct sets $\mathcal{S}_n$ of n-dimensional dominoes which are uniquely determined from some initial set of n edges, one from each of n-many of the alphabets E_i.

Consider a subset $J \subseteq \{1, \ldots, k\}$ of cardinality $n \geq 2$, and fix n elements $u_j \in E_j$, one for each $j \in J$. Each pair (u_i, u_j) uniquely defines a pointed square $\left[u_i, u_j^i, \bar{u}_i^j, \bar{u}_j\right] \in R(i,j)$ by **D2**. Presume that the set R is designed in order that, whenever $L \subseteq J$ is a subset with $0 \leq |L| \leq (n-1)$, then we can find *unique* elements of the form $u_i^L \in E_i$ such that

D3 $\left[u_i^L, u_j^{L\cup\{i\}}, \bar{u}_i^{L\cup\{j\}}, \bar{u}_j^L\right] \in R(i,j)$, for all $i, j \in J \setminus L$ with $i < j$.

Then we say that R is a set of pointed squares with an n-domino structure. We write $\square(u_{j_1}, \ldots, u_{j_n})$ for the $2n$-tuple of elements of R which comprises each $\left[u_i, u_j^i, \bar{u}_i^j, \bar{u}_j\right]$ and the pointed squares above which they uniquely determine, listed according to some predetermined order. We write $\mathcal{S}_n$ for the set of all such $2n$-tuples $\square(u_{j_1}, \ldots, u_{j_n})$, since these can be regarded as n-dimensional dominoes whose basepoint is the vertex out of which u and the w_j are emitted (Figure 3.3).

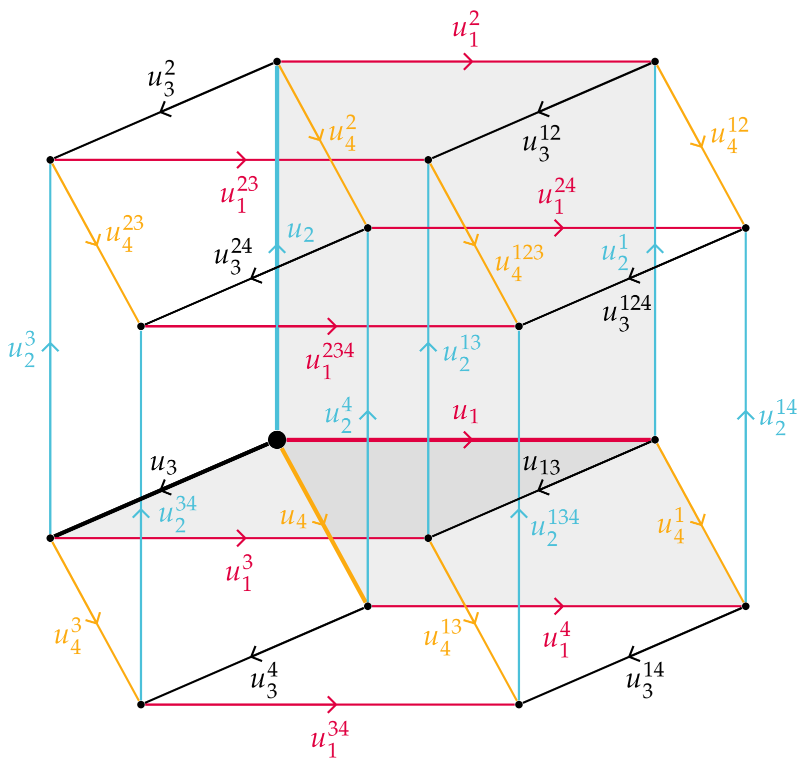


Figure 3.3: Above, we have illustrated the 4-dimensional domino $\square(u_1, u_2, u_3, u_4)$. If $E_1, \ldots, E_4$ are alphabets, and $R \subseteq \bigsqcup_{i<j} F(i,j)$ is a set of pointed squares with a 4-domino structure, then a 4-tuple $(u_1, u_2, u_3, u_4) \in E_1 \times \cdots \times E_4$ uniquely defines a 4-dimensional domino whose basepoint emits u and the w_j (**thick lines**). In this example, the initial elements first define three mutually adjacent pointed squares $\left[u_1, u_2^1, \bar{u}_1^2, \bar{u}_2\right]$, $\left[u_1, u_3^1, \bar{u}_1^3, \bar{u}_3\right]$, $\left[u_1, u_4^1, \bar{u}_1^4, \bar{u}_4\right]$, highlighted in grey. Then each of the remaining elements $u_i^L \in E_i$ are uniquely determined such that they label the edges of a 4-dimensional cube.

3.1.15 When R is a set of pointed squares with an n-domino structure, then it also has an m-domino structure for all $m \in \{2, \ldots, n\}$.

In particular, any set of $(n-m+1)$-many mutually incident m-dimensional dominoes in $\mathcal{S}_m$, each in a different set $R(p_1, \ldots, p_m)$, uniquely characterises an n-dimensional domino. We sometimes find it convenient to write $\square\{[u_p, u_j^p, \bar{u}_p^j, \bar{u}_j]\}_j$ to represent the n-dimensional domino uniquely characterised by $(n-1)$ pointed squares in $R(p, j)$, for some fixed $p \in J$ and $j \in J \setminus \{p\}$. In these instances we also find it convenient to assume that the fixed element $p = 1$; if it is not, we can of course rearrange the alphabets E_i so that the above constructions still make sense.

3.1.16 Definition (Adjacency of k-dimensional dominoes) Let $k \geq 2$, let $E_1, \ldots, E_k$ be alphabets, each with at least four elements, and let R be a set of pointed squares with labels from the sets E_i, and which has a k-domino structure. Consider two k-dimensional dominoes $U = \square(u_1, \ldots, u_k)$ and $V = \square(v_1, \ldots, v_k)$ in $\mathcal{S}_k$. We define adjacency matrices $M_1, \ldots, M_k$ to be square matrices with rows and columns indexed by $\mathcal{S}_k$, and with UV-th entry $M_i(U, V) := 1$ whenever

(i) $v_j^L = u_j^{L\cup\{i\}}$ for all $j \notin (L \cup \{i\})$,

(ii) $v_i^L \neq \bar{u}_i^L$,

for each subset $L \subseteq (\{1, \ldots, k\} \setminus \{i\})$ with $|L| \geq 0$. We set $M_i(U, V) := 0$ otherwise. For each $i \in \{1, \ldots, k\}$, we say that V is adjacent in the E_i direction, or E_i-adjacent, to U whenever $M_i(U, V) = 1$. Note that this definition of adjacency differs from those of §2.5.

3.1.17 Constructing a k-dimensional domino complex Let $k \geq 2$, let $E_1, \ldots, E_k$ be alphabets, each with at least four elements, and let R be a set of pointed squares with labels from the sets E_i, and which has a k-domino structure. Write $\mathcal{S}_k$ to denote the set of k-dimensional dominoes obtained via the domino structure, as constructed in 3.1.14, and let $M_1, \ldots, M_k$ be the corresponding adjacency matrices.

We can construct a cube complex $\mathcal{M}$ from the set of dominoes $\mathcal{S}_k$ by identifying the relevant edges from 3.1.16(i)–(ii) whenever two dominoes are adjacent, in the manner of 2.1.1

3.1.18 Theorem *Let R be a set of pointed squares with a k-domino structure, which gives rise to a set of dominoes $\mathcal{S}_k$. Consider the k-dimensional cube complex $\mathcal{M}$ obtained in 3.1.17 by identifying the $(k-1)$-dimensional faces of elements of $\mathcal{S}_k$ whenever they are adjacent. Then $\mathcal{M}$ is a k-dimensional domino complex, in the sense of 3.1.10.*

Conversely, any k-dimensional domino complex can be decomposed into a set of pointed squares which has a k-domino structure.

■ **Proof** Firstly, let $E_1, \ldots, E_k$ be alphabets of respective sizes m_i, and let $R \subseteq \bigsqcup_{i<j} F(i, j)$ be a set of pointed squares with a k-domino structure. By property **D2** of 3.1.14, the projection from $F(p, q)$ to each subproduct of the form $E_p \times E_q$ or $E_q \times E_p$ is a bijection. This means

that each k-dimensional domino constructed from R is E_i-adjacent to $(m_i - 1)$-many other dominoes, and gluing them together at their common face creates a vertex z whose link contains vertices indexed by E_i. By gluing adjacent dominoes together in each direction, we obtain a cube complex $\mathcal{M}$, and $\mathrm{lk}_z(\mathcal{M})$ has vertices indexed by $E = \bigsqcup_i E_i$, and after this procedure, z is the only vertex in $\mathcal{M}$.

Then in $\mathrm{lk}_z(\mathcal{M})$, there is an edge joining a to b for each pointed square $\square(a, b)$, where $a \in E_p$, $b \in E_q$. Furthermore, there is a solid triangle joining a, b, c whenever $\square(a, b, c) \in \mathcal{S}_3$, for $a \in E_p$, $b \in E_q$, $c \in E_r$, and more generally there is an $(n-1)$-dimensional simplex for each element of $\mathcal{S}_n$. Thus $\mathrm{lk}_z(\mathcal{M})$ is a clique complex of a k-partite graph with partition sets indexed by elements of $E_1, \dots, E_k$, and since z is the only vertex in $\mathcal{M}$, this implies that $\mathcal{M}$ is a k-dimensional domino complex.

Conversely, suppose that $\mathcal{M}$ is a k-dimensional domino complex with adjacency structure $E_1, \dots, E_k$, and consider the sets $\mathcal{S}_n(\mathcal{M})$ for each $n \in \{2, \dots, k\}$. Suppose for a contradiction that there is some n-tuple $(u_1, \dots, u_n) \in \prod_{i=1}^n E_i$ which *does not* uniquely define an n-dimensional domino via the construction in 3.1.14—this is to say that there are two distinct n-dimensional dominoes $U, U' \in \mathcal{S}_n$ whose basepoints emit vertices with labels $u_1, \dots, u_n$. Then there is some pair of pointed squares in $\mathcal{S}_2$ of the form $A = \left[a_1, b_1, a_2, b_2\right]$, $A' = [a_1, b_1, a_2', b_2']$, where $a_2 \neq a_2'$ are in E_p and $b_2 \neq b_2'$ are in E_q for $p < q$ (after potentially reordering the alphabets E_i).

Now, consider the link $\mathrm{lk}_z(\mathcal{M})$, where z is the vertex which receives edge b_1 and emits edges a_2 and a_2'. In order for the link z to be the clique complex of a complete k-partite graph, there must be some other pointed square $B = \left[a_3, b_3, a_4, \bar{b}_1\right] \in F(p, q)$ with $a_4 \neq \bar{a}_2$ and $a_4 \neq \bar{a}_2'$, such that $M_i(A, B) = M_i(A', B) = 1$. Since $|E_i| \geq 4$ for each i, we know that such elements exist. But then the link at z has two edges connecting a_1 to b_1, and so $\mathrm{lk}_z(\mathcal{M})$ cannot be a clique complex. So each n-tuple in $\prod_{i=1}^n E_i$ uniquely defines an n-dimensional domino in $\mathcal{M}$, and hence the set of 2-dimensional dominoes $\mathcal{S}_2(\mathcal{M})$ has an n-domino structure for each n. ■

§3.2 Identifying a domino complex with a domino group

3.2.1 Definition (k-domino group) Let $k \geq 2$, let $E_1, \dots, E_k$ be alphabets, and consider a set $R \subseteq \bigsqcup_{i<j} F(i, j)$ of pointed squares. We define a group

$$\Gamma := \langle E = E_1 \sqcup \cdots \sqcup E_k \mid x_1 y_1 x_2 y_2 = 1 \text{ whenever } [x_1, y_1, x_2, y_2] \in R \rangle.$$

If R has a k-domino structure, then we call Γ a k-domino group or a k-cube group, and we frequently use the notation $\Gamma = \langle E \mid R \rangle$. We call the alphabets $E_1, \dots, E_k$ the adjacency structure of the k-domino group Γ.

3.2.2 In the case where $k = 2$ and R satisfies only **D1** and **D2** of 3.1.14, the group $\Gamma = \langle E \mid R \rangle$ is called a BM-group, named after Burger and Mozes, and first developed by them and Wise in [Wis96; BM97; KR02].

In [KV19], an alternative condition was given on R, which is equivalent to properties **D1** and **D2**:

D1′ The product sets E_iE_j and E_jE_i are equal, contain no 2-torsion, and have cardinality $|E_iE_j| = |E_i| \cdot |E_j| = m_im_j$.

However, since there are multiple explicit constructions of domino complexes throughout this thesis, we find it more convenient to deal with properties **D1** and **D2**.

3.2.3 Lemma *Let $\Gamma = \langle E \mid R \rangle$ be a k-domino group with adjacency structure $E_1, \ldots, E_k$, and let $(\Gamma - p) = \langle (E - p) \mid (R - p) \rangle \subset \Gamma$ denote the subgroup generated by $E \setminus E_p$ and whose relations are obtained by removing all pointed squares which contain elements of E_p from R. Then $(\Gamma - p)$ is a $(k-1)$-domino group, with adjacency structure $E_1, \ldots, \hat{E}_p, \ldots, E_k$.*

By induction, we can form a $(k - m)$-domino subgroup by removing all elements of m-many alphabets $E_{p_1}, \ldots, E_{p_m}$ from the generating set E. We denote such a group by $(\Gamma - p_1 - \cdots - p_m)$.

■ **Proof** By disregarding all elements from some alphabet E_p, what remains are $(k-1)$-dimensional dominoes in $R(1, \ldots, \hat{p}, \ldots, k)$, and as it was remarked in 3.1.15 that a set of pointed squares R with an n-domino structure also has an m-domino structure for all $m < n$, we know that these dominoes are uniquely defined by some initial $(k-1)$-tuples of elements of $E \setminus E_p$. ■

3.2.4 Proposition *Let Γ be a k-domino group with adjacency structure $E_1, \ldots, E_k$, and consider*

$$(\Gamma - p_1 - \cdots - p_m) = \langle (E - p_1 - \cdots - p_m) \mid (R - p_1 - \cdots - p_m) \rangle,$$

the $(k - m)$-domino subgroup of Γ constructed in 3.2.3. Then

$$\Gamma = \Big(\big(((\Gamma - 1) *_{\langle (E-1)\cap(E-2) \rangle} (\Gamma - 2)\big) *_{\langle (E-3) \rangle} (\Gamma - 3)\big) *_{\langle (E-4) \rangle} \cdots \Big) *_{\langle (E-k) \rangle} (\Gamma - k),$$

where $*_{\mathcal{G}}$ denotes the amalgamated free product over a group $\mathcal{G}$.

■ **Proof** Firstly, write $G_2 := (\Gamma - 1) *_{\langle (E-1)\cap(E-2) \rangle} (\Gamma - 2)$, and then

$$G_{i+1} := G_i *_{\langle (E-i) \rangle} (\Gamma - i),$$

for all $i \in \{2, \ldots, k-1\}$. Then G_2 is the group generated by E, with relations $(R-1) \cup (R-2)$. At each step, we amalgamate over the free group generated by the intersection of E with $(E - i)$, which is $(E - i)$. Hence each G_i is generated by E, and has relation set given by $(R - 1) \cup \cdots \cup (R - i)$. But this set is equal to R, and so $G_k = \Gamma$. ■

3.2.5 It is important to note that the converse to 3.2.4 is not true—in general it is extremely rare to find a family of k-domino groups whose amalgamated product over the subgroups generated by their pairwise intersections forms a $(k+1)$-domino group. We present one method of [RSV19] for doing so in 4.1.6.

3.2.6 We may henceforth regard a k-domino group $\Gamma = \langle E \mid R \rangle$ geometrically as the corresponding domino complex with edges labelled by elements of Γ and pointed squares in R. If a clear distinction needs to be made, we might write $\mathcal{M}(\Gamma)$ for the geometric realisation of the domino complex. We often write $\mathcal{S}_n(\Gamma)$ for the set of n-dimensional dominoes arising from R, and sometimes describe elements of $\mathcal{S}_n(\Gamma)$ as cubes or dominoes in Γ.

Domino complexes as buildings

3.2.7 Jacques Tits first introduced *buildings* in order to understand the structure of semi-simple Lie groups over arbitrary fields. Since then, their intricate branching symmetries and ability to be characterised both geometrically and by way of their algebraic groups have enchanted mathematicians from fields as diverse as combinatorics and operator theory (see, for example [Vdo02; KS91], respectively).

Proposition 3.2.10 gives an important connection between the geometries of domino complexes and of buildings, and it is the branching property of buildings which ensures that we can make the classification of *domino algebras* in §3.5.

3.2.8 Definition (Tits building) A (Tits) building of rank k is a k-dimensional simplicial complex or cube complex Δ which can be expressed as the union of apartments, subcomplexes $\Sigma \subseteq \Delta$, which together have the following properties:

(i) Every cell (simplex or n-dimensional cube, for $n \in \{0, \dots, k\}$) in Δ is contained in a chamber, that is, a k-dimensional cell. Moreover, every cell in a subcomplex Σ is contained in a chamber in Σ.

(ii) Each $(k-1)$-dimensional cell in Δ is contained in exactly two chambers.

(iii) Given any two chambers $X, X' \in \Sigma$, we can find a sequence $X = X_0, X_1, \dots, X_s = X'$ of chambers such that X_i is adjacent to X_{i+1} for each i.

(iv) Given any two cells in Δ, there is some apartment which contains both of them.

(v) If Σ, Σ' are apartments which both contain two cells x, x', then there is an isomorphism $\Sigma \to \Sigma'$ which respects the structure of the complex, and which fixes the vertices of x and x'.

(vi) Each n-dimensional cell of Δ is contained in at least three chambers, for all $n < k$.

Property (vi) is that of thickness. Some authors reserve the term "building" for simplicial or cube complexes which satisfy (i)–(v), awarding the term thick building to those which also satisfy (vi).

All apartments Σ are isomorphic to one another. If each apartment is a tessellation of Euclidean space, then we say that the building Δ is affine (see [Bro89, VI]).

3.2.9 For our purposes, it is enough for chambers of a building to be k-dimensional cubes or simplices, but it should be noted that Tits originally defined buildings as strictly simplicial complexes (see [Bro89, IV]). In general, an apartment of a k-rank affine building can be regarded as a tessellation of k-dimensional Euclidean space by products of simplices, sometimes called polysimplices—then an n-dimensional cube is a product of n-many 1-dimensional simplices (edges). An affine building is said to be of type $\tilde{A}_n$ for $n \geq 1$, if its chambers are n-dimensional simplices. Thus a building whose chambers are 2-dimensional squares has type $\tilde{A}_1 \times \tilde{A}_1$, for example (see [BT72, §1]).

The product of k-many trees $\Delta := T(m_1) \times \cdots \times T(m_k)$ is an example of an type $\prod_{i=1}^{k} \tilde{A}_1$ building of rank k; if $m_i \geq 3$ for each i, then Δ is thick. The characterisation of k-domino groups in the following proposition was actually used as the definition in [Vdo21].

3.2.10 Proposition *A group Γ is a k-domino group if and only if it is a torsion-free $\prod_{i=1}^{k} \tilde{A}_1$ group, that is, one which acts freely and transitively on the set of vertices of the product Δ of k trees.*

■ **Proof** The proof that a torsion-free group of type $\prod_{i=1}^{k} \tilde{A}_1$ is a k-domino group follows the same argument as that of [KR02, 3.4]. By the same proof, the fact that a k-domino group Γ acts on a product of k-many trees follows from 3.1.7 and considerations in [BW99]. It is enough to note that elements of Γ correspond to paths in the 1-skeleton of Δ. Then to show that the action is free and transitive, the remainder of [KR02, 3.4] can be easily generalised from $k = 2$ to arbitrary k. ■

§3.3 The *UCE Property* in higher dimensions

In this section, we show that k-domino groups possess a k-dimensional *Unique Common Extension Property*, a cognate of 2.2.2. Before then, we take some time to familiarise ourselves with k-domino groups by inspecting some examples.

3.3.1 Example (A 3-domino group with 24 cubes) Consider the alphabets $E_1 = \{a_1, a_2, \bar{a}_1, \bar{a}_2\}$, $E_2 = \{b_1, b_2, b_3, \bar{b}_1, \bar{b}_2, \bar{b}_3\}$, $E_3 = \{c_1, c_2, c_3, c_4, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4\}$, and define the group $\Gamma = \Gamma_{\{3,5,7\}} := \langle E = E_1 \sqcup E_2 \sqcup E_3 \mid R\rangle$, as first seen in [RSV19, 3.17], where

$$\begin{aligned} R = \{&a_1b_1a_2b_2,\ a_1b_2a_2\bar{b}_1,\ a_1b_3\bar{a}_2b_1,\ a_1\bar{b}_3a_1\bar{b}_2,\ a_1\bar{b}_1\bar{a}_2b_3,\ a_2b_3a_2\bar{b}_2,\\ &a_1c_1\bar{a}_2\bar{c}_2,\ a_1c_2\bar{a}_1c_3,\ a_1c_3\bar{a}_2\bar{c}_4,\ a_1c_4a_1\bar{c}_1,\ a_1\bar{c}_4a_2c_2,\ a_1\bar{c}_3a_2c_1,\ a_2c_3a_2\bar{c}_2,\ a_2c_4\bar{a}_2c_1,\\ &c_1b_1c_3\bar{b}_3,\ c_1b_2c_4\bar{b}_2,\ c_1b_3\bar{c}_4b_2,\ c_1\bar{b}_3c_4b_3,\ c_1\bar{b}_2c_2b_1,\ c_1\bar{b}_1c_4\bar{b}_1,\\ &c_2b_2\bar{c}_3\bar{b}_3,\ c_2b_3c_4b_1,\ c_2\bar{b}_3c_3b_3,\ c_2\bar{b}_2c_3b_2,\ c_2\bar{b}_1c_3\bar{b}_1,\ c_3b_1c_4b_2\}. \end{aligned}$$

This is a 3-domino group with adjacency structure E_1, E_2, E_3. The set of relations R as displayed above is shorthand for the set of pointed squares

$$\mathcal{S}_2(\Gamma) = \{S = [x_1, y_1, x_2, y_2], S_H, S_R, S_V \mid x_1y_1x_2y_2 \in R\},$$

from which we construct the corresponding domino complex $\mathcal{M}(\Gamma)$ (see 3.2.6). Given each relation corresponding to a pointed square S, the relations corresponding to S_H, S_R and S_V are all implied, so we have omitted them.

Writing $|E_i| = m_i$, we compute that the domino complex $\mathcal{M}(\Gamma)$ comprises one vertex, $m_1 + m_2 + m_3 = 18$ directed edges (9 geometric edges), $m_1m_2 + m_1m_3 + m_2m_3 = 104$ pointed squares (26 geometric squares), and $m_1m_2m_3 = 192$ pointed cubes, or 3-dimensional dominoes (24 geometric cubes). ♦

3.3.2 Example (A 3-domino group with 27 cubes) Now consider three alphabets E_1, E_2 and E_3, each of cardinality 6, and the group $\Gamma = \Gamma'_{\{2,3,4\}}$ which was defined in [RSV19, 2.36] as $\Gamma := \langle E = E_1 \sqcup E_2 \sqcup E_3 \mid R\rangle$, where

$$\begin{aligned} R = &\\ \{&a_1b_1\bar{a}_2\bar{b}_3, a_1b_2a_3b_3, a_1b_3a_3b_2, a_1\bar{b}_1\bar{a}_3\bar{b}_1, a_1\bar{b}_2a_2\bar{b}_2, a_1\bar{b}_3\bar{a}_2b_1, a_2b_1\bar{a}_3b_2, a_2b_2\bar{a}_3b_1, a_2\bar{b}_3a_3\bar{b}_3,\\ &a_1c_1\bar{a}_2c_1, a_1c_2\bar{a}_1\bar{c}_2, a_1c_3a_3c_3, a_1\bar{c}_1a_1\bar{c}_3, a_2c_1a_2\bar{c}_2, a_2c_2\bar{a}_3c_2, a_2c_3\bar{a}_2\bar{c}_3, a_3c_1\bar{a}_3\bar{c}_1, a_3c_2a_3\bar{c}_3,\\ &b_1c_1\bar{b}_2\bar{c}_3, b_1c_2b_3c_3, b_1c_3b_3c_2, b_1\bar{c}_1\bar{b}_3\bar{c}_1, b_1\bar{c}_2b_2\bar{c}_2, b_1\bar{c}_3\bar{b}_2c_1, b_2c_1\bar{b}_3c_2, b_2c_2\bar{b}_3c_1, b_2\bar{c}_3b_3\bar{c}_3\}. \end{aligned}$$

This is a 3-domino group with adjacency structure $E_1 = \{a_i, \bar{a}_i\}$, $E_2 = \{b_i, \bar{b}_i\}$, $E_3 = \{c_i, \bar{c}_i\}$, and which acts freely and transitively on the product of three trees of valency six: $T(6) \times T(6) \times T(6)$, by Proposition 3.1.7. The corresponding cube complex $\mathcal{M}(\Gamma)$ has one vertex, 27 geometric squares labelled with the relators in R (giving rise to $27 \times 4 = 108$

pointed squares), and 27 geometric cubes ($27 \times 8 = 216$ 3-dimensional dominoes). We present the set of 27 geometric cubes, that is, elements of $\mathcal{S}_3'(\mathcal{M})$, in Figure 3.4. ♦

3.3.3 Lemma *Let Γ be a k-domino group with adjacency structure $E_1, \ldots, E_k$. Then each of the adjacency matrices $M_1, \ldots, M_k$ from 3.1.16 has entries in $\{0, 1\}$, and has at least three non-zero entries in each row.*

■ **Proof** Consider the k-dimensional domino $U = \square\{[u_1, u_j^1, \bar{u}_1^j, \bar{u}_j]\} \in \mathcal{S}_k(\Gamma)$, as constructed in 3.1.14. By property **D1** of k-cube groups and the fact that $|E_i| \geq 4$ for all i, we are able to find some k-dimensional domino $V = \square\{[\bar{v}_1, v_j, v_1^j, \bar{v}_j^1]\}$, where $\bar{v}_j = u_j^1$ and $v_1 \neq u_1$.

Property **D2** then implies that $v_1^j \neq u_1^j$ for all j. It follows that $M_1(A, B) = 1$, and a similar argument can be used for each $i \in \{2, \ldots, k\}$ to find a k-dimensional domino W_i with $M_i(U, W_i) = 1$. Hence, in each row in each of the matrices $M_1, \ldots, M_k$, there are at least three non-zero entries, and by definition these are $\{0, 1\}$-matrices. ■

3.3.4 Definition (UCE Property in k dimensions) Let $k \geq 3$, and let Γ be a k-domino group with adjacency structure $E_1, \ldots E_k$ and adjacency matrices $M_1, \ldots, M_k$. Let U, V_p, V_q, V_r be k-dimensional dominoes in $\mathcal{S}_k(\Gamma)$ such that $M_p(U, V_p) = M_q(U, V_q) = M_r(U, V_r) = 1$ for some distinct $p, q, r \in \{1, \ldots, k\}$. We say that the matrices M_i have the Unique Common Extension Property or UCE Property if we can find *unique* dominoes $W_{pq}, W_{pr}, W_{qr}, X \in \mathcal{S}_k(\Gamma)$ such that each of

$$M_p(V_q, W_{pq}),\ M_p(V_r, W_{pr}),\ M_q(V_p, W_{pq}),\ M_q(V_r, W_{pr}),\ M_r(V_p, W_{pr}),\ M_r(V_q, W_{qr}),$$

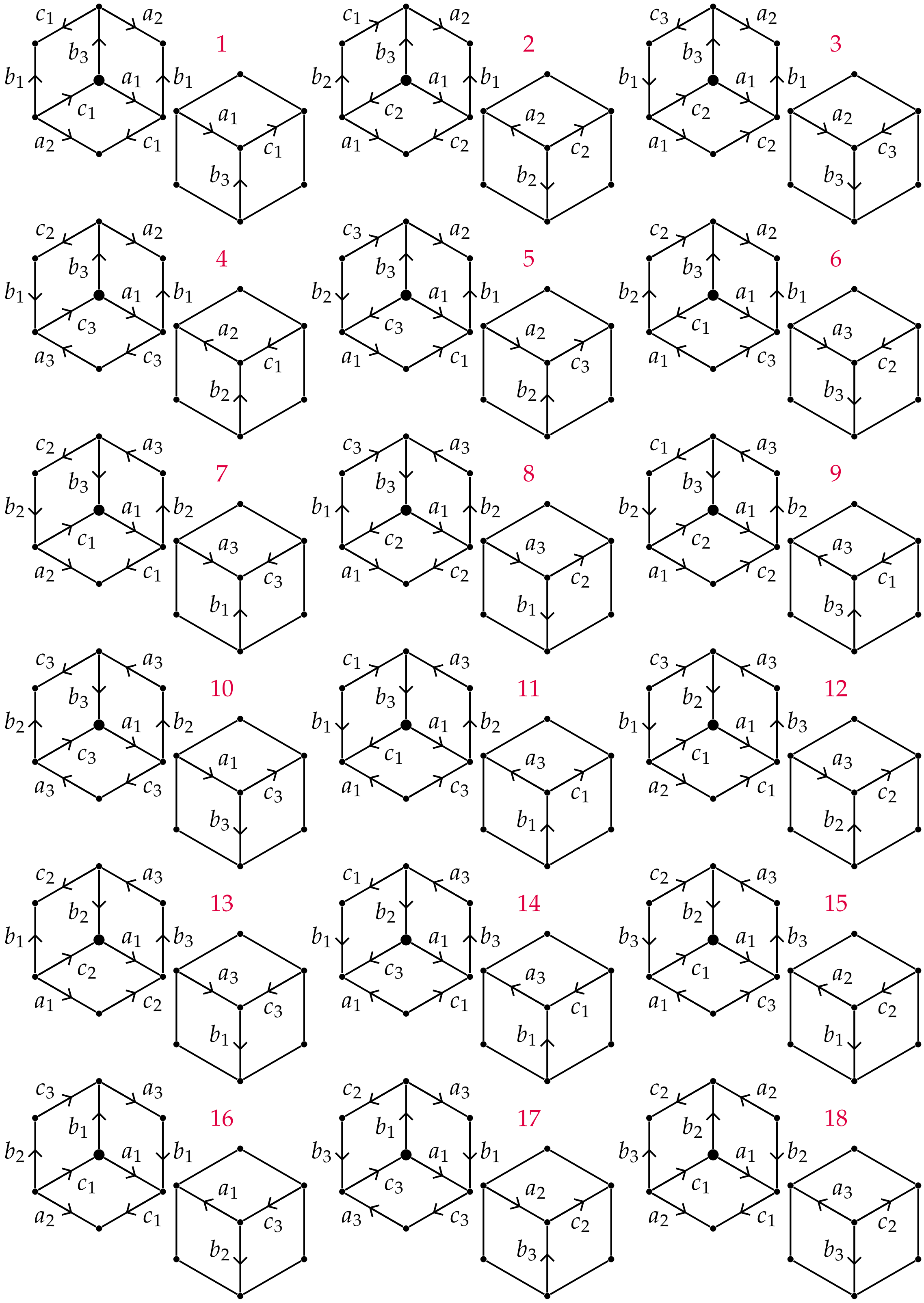
and each of

$$M_p(W_{qr}, X),\ M_q(W_{pr}, X),\ M_r(W_{pq}, X)$$

is equal to 1 (see Figure 3.5). In the case where $k = 2$, the definition reduces to that of 2.2.2: if $A, B, C \in \mathcal{S}_2$ are such that $M_1(A, B) = M_2(A, C) = 1$. Then M_1, M_2 have the *UCE Property* if there exists a unique pointed square $D \in \mathcal{S}_2$ such that $M_2(B, D) = M_1(C, D) = 1$.

3.3.5 The definition of the *Unique Common Extension Property* could be extended to deal with higher numbers of dominoes $V_1, \ldots, V_k \in \mathcal{S}_k$, with each V_i being E_i-adjacent to some initial domino U. By [RS99, 1.4], however, it turns out that given just three k-cubes V_p, V_q, V_r initially adjacent to U as above, having unique common extensions is enough to imply unique common extensions for any number of initial dominoes V_i.

3.3.6 Proposition *Let $\mathcal{M}$ be the k-dimensional cube complex with adjacency structure $E_1, \ldots, E_k$, as constructed in 3.1.17. Then its associated adjacency matrices $M_1, \ldots, M_k$ commute, and have the Unique Common Extension Property.*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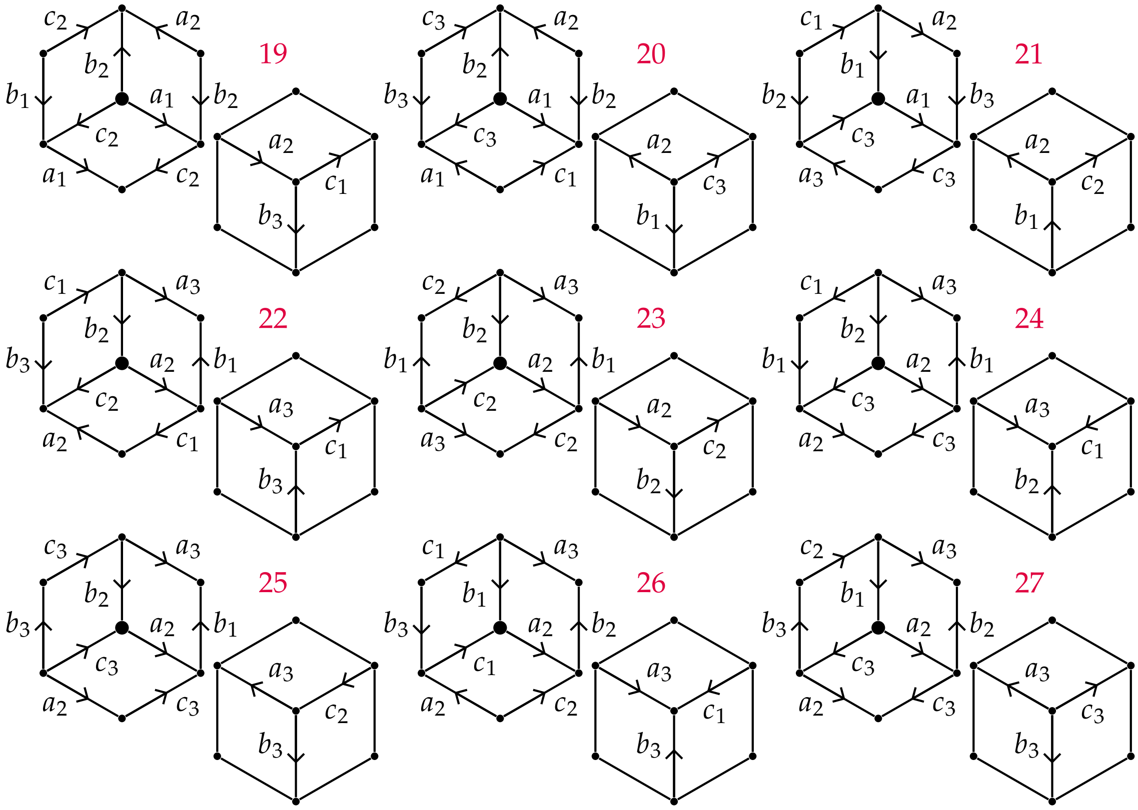


Figure 3.4: Continued from page 68. An illustration of the 27 geometric cubes in the group $\Gamma = \Gamma'_{\{2,3,4\}}$ from 3.3.2—each represents an element of $\mathcal{S}'_3(\Gamma)$, and by acting on this set by the group of reflections of the cube (see Figure 3.2), we obtain the 216 3-dimensional dominoes of $\mathcal{S}_3(\Gamma)$.

Each triple $(a, b, c) \in E_1 \times E_2 \times E_3$ appears exactly once as the set of arrows emitted from a corner of one of the cubes above, for example the triple $(\bar{a}_2, b_3, \bar{c}_1)$ appears as the front-bottom corner of the first cube. Thus each such triple appears at the basepoint of exactly one 3-dimensional domino, as we expect from 3.1.14–3.1.15.

■ **Proof** In the case where $k = 2$, the result is proved in [KR02, 4.1].

Suppose, then, that $k \geq 3$, and let $U = \square(u_1, \ldots, u_k) \in \mathcal{S}_k$ be a k-dimensional domino. Pick distinct elements $p, q, r \in \{1, \ldots, k\}$, define three more k-dimensional dominoes $V_p := \square(v(p)_1, \ldots, v(p)_k)$, $V_q := \square(v(q)_1, \ldots, v(q)_k)$, $V_r := \square(v(r)_1, \ldots, v(r)_k)$ in $\mathcal{S}_k$, and suppose that $M_i(U, V_i) = 1$ for each $i \in \{p, q, r\}$. Then by 3.1.16, we have $v(i)^L_j = u^{L\cup\{i\}}_j$ and $v(i)^L_i \neq \bar{u}^L_i$ whenever $L \subseteq (\{1, \ldots, k\} \setminus \{i\})$ and $j \notin (L \cup \{i\})$.

By 3.3.3, we can find k-dimensional dominoes $W_{ij} = \square(w(ij)_1, \ldots, w(ij)_k) \in \mathcal{S}_k$ such that $M_j(V_i, W_{ij}) = 1$ whenever $j \in (\{p, q, r\} \setminus \{i\})$. Then W_{ij} is the unique k-dimensional domino with $w(ij)_l = u^{\{i,j\}}_l$ for all $l \in (\{1, \ldots, k\} \setminus \{i, j\})$, and $w(ij)_i = v(i)^j_i$ and $w(ij)_j = \bar{v}(j)^i_j$. In particular, we see that $M_j(V_i, W_{ij}) = 1$ if and only if $M_i(V_j, W_{ji}) = 1$, which is to say that $M_{ij} = M_{ji}$ for each i, j.

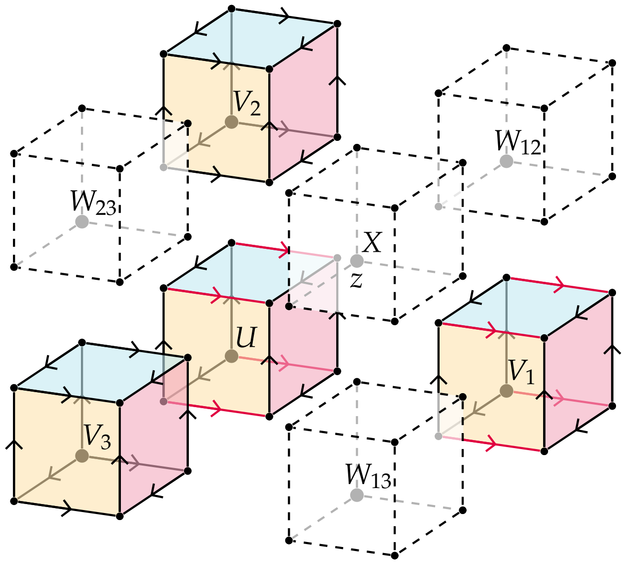


Figure 3.5: A visualisation of adjacency of k-dimensional dominoes when $k = 3$. Presume that edges perpendicular to the pink faces are labelled by elements of an alphabet E_1, those perpendicular to blue and yellow faces are labelled by E_2 and E_3, respectively. Domino V_1 is E_1-adjacent to U if the right face of V_1 aligns with the left face of U, and none of the pink edges of U is the inverse of the corresponding pink edge of V.

Likewise, V_2 and V_3 are respectively E_2- and E_3-adjacent to U if the correct faces match, and when they are glued together at their common face, no edge end up incident to its inverse. We generalise the notion of adjacency to k-dimensional dominoes by considering the similarity not just of face, but of $(k-1)$-dimensional sub-cubes (see 3.1.16).

The *UCE Property* from 3.3.4 would suggest, given k-dimensional dominoes U, V_1, V_2 and V_3 as above, the existence of *unique* dominoes W_{12}, W_{13}, W_{23} and X which are adjacent in a way that completes the $2\times2\times2$ configuration above. Each of those eight dominoes shares the vertex z.

Finally, consider a k-dimensional domino $X = \square(x_1 \ldots, x_k) \in \mathcal{S}_k$ such that $M_r(W_{pq}, X) = 1$; such a k-cube exists which satisfies 3.1.16 by Lemma 3.3.3. Then X is the unique k-dimensional domino defined by

- $x_i = u_i^{\{p,q,r\}}$ whenever $i \notin \{p, q, r\}$,
- $x_i = w(ij)_i^l$, for $i, j, l \in \{p, q, r\}$.

It is clear that X is the unique k-dimensional domino which also satisfies $M_q(W_{pr}, X) = M_p(W_{qr}, X) = 1$. ■

§3.4 The higher-rank graph induced by a domino group

3.4.1 Given a k-domino group Γ with adjacency structure, we can use the adjacency matrices $M_1, \ldots, M_k$ to construct a k-coloured directed graph G with vertex set $G^0 = \mathcal{S}_k(\Gamma)$ and an arrow of colour i from a domino V to another domino U whenever $M_i(U, V) = 1$.

3.4.2 The *UCE Property* is formulated slightly differently to the factorisation property of k-rank graphs (compare with 1.1.4, Figure 1.4, and the associativity property **C2** from 1.1.15). We claim that the *UCE Property* implies this associativity.

Indeed, by property **D2**, any two consecutive sides of a geometric square in the complex $\mathcal{M}(\Gamma)$ uniquely define the square. Then, n-many adjacent and mutually perpendicular edges uniquely determine a geometric n-cube (and such an n-cube exists, since the link at each vertex of $\mathcal{M}(\Gamma)$ is a clique complex of a complete k-partite graph). So, given k-dimensional dominoes U and X arranged as in Figure 3.5, then the remaining dominoes V_i, W_{ij} are determined by U and X, since each of them includes the vertex z common to U and X, and in each domino V_i, W_{ij} we can find k-many mutually perpendicular edges incident to z, which are also included in U or X.

This fact, together with 1.1.16 and 3.4.1, demonstrates that a k-domino group uniquely induces a k-rank graph, which we sometimes refer to as a k-domino graph. Since the set $\mathcal{S}_k(\Gamma)$ is finite, and the matrices M_i have non-empty rows by 3.3.3, we can write:

3.4.3 Theorem *Let Γ be a k-domino group with adjacency matrices $M_1, \ldots, M_k$. Then Γ induces a row-finite k-rank graph $(\Lambda(\Gamma), d)$ with no sources, vertex set $\Lambda(\Gamma)^{\mathbf{0}} := \mathcal{S}_k(\Gamma)$, and where for all vertices $U, V \in \mathcal{S}_k(\Gamma)$, there is a morphism $\lambda : V \to U$ of degree $d(\lambda) = \mathbf{e}_i$ precisely when $M_i(U, V) = 1$.* ■

Evans' spectral sequence

3.4.4 As we learnt in 1.3.3, we can associate a unital $C^\star$-algebra to any row-finite higher-rank graph with no sources; shortly we will do this for the k-domino graphs $\Lambda(\Gamma)$ above, and then we investigate their K-theory. Firstly, however, it behoves us to introduce *spectral sequences*: generalisations of exact sequences which are to be employed in the proofs of 3.4.7 and 3.4.11. We attempt to offer an overview of spectral sequences here, though their intricate nature deserves much more attention—we direct any unsatisfied readers to [McC00] for such a treatment.

3.4.5 Definition (Spectral sequence) Let C be an *Abelian* category (see [Mac78, I.8]). A spectral sequence (of homological type) is a family of objects and maps $\{(E^r, d^r)\}_{r \geq 1}$. For each positive integer r, E^r is an object *bigraded* by integers p and q, with

$$E^r := \bigoplus_{p,q \in \mathbb{Z}} E^r_{p,q}$$

for some $E^r_{p,q}$ in $\mathrm{Ob}(C)$. Each d^r is a map of degree $(-r, r-1)$ called a differential, with

$$d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}, \quad \text{and} \quad d^r : E^r_{p+r,q-r+1} \longrightarrow E^r_{p,q},$$

and which satisfies $d^r \circ d^r = 0$. The differentials d^r are not the same as the degree maps of higher-rank graphs. We insist that

$$E^{r+1}_{p,q} \cong H(E^r_{p,q}) := \frac{\ker\left(d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}\right)}{\operatorname{im}\left(d^r : E^r_{p+r,q-r+1} \longrightarrow E^r_{p,q}\right)}.$$

The collections $\left(E^r_{p,q}\right)$ for fixed r are known as the sheets of the spectral sequence. We "move to the next sheet" by taking the *homology* H, defined above. We call a spectral sequence bounded if the sequence of objects $E^r_{p,q}$ stabilises as $r \longrightarrow \infty$; we denote this limit by $E^\infty_{p,q}$, and call it the stable value.

We say that a bounded spectral sequence converges to a family of $\mathbb{Z}$-modules $\{\mathcal{K}_\epsilon\}_{\epsilon\in\mathbb{Z}}$ if there exists a finite *ascending filtration* of modules

$$0 = F_s(\mathcal{K}_\epsilon) \subseteq \cdots \subseteq F_{p-1}(\mathcal{K}_\epsilon) \subseteq F_p(\mathcal{K}_\epsilon) \subseteq F_{p+1}(\mathcal{K}_\epsilon) \subseteq \cdots \subseteq F_t(\mathcal{K}_\epsilon) = \mathcal{K}_\epsilon, \tag{3.1}$$

and an isomorphism

$$E^\infty_{p,q} \cong F_p(\mathcal{K}_{p+q})/F_{p-1}(\mathcal{K}_{p+q}), \tag{3.2}$$

for every pair (p, q).

Given a general chain complex $A := \cdots \to A_{i+1} \xrightarrow{\partial_{i+1}} A_i \xrightarrow{\partial_i} A_{i-1} \to \cdots$, we frequently write $H_i(A)$ to denote the ith homology $\ker(\partial_i)/\operatorname{im}(\partial_{i+1})$.

3.4.6 Lemma (Evans, 2008) *There exists a spectral sequence $\{(E^r, d^r)\}$ which converges to the family of $\mathbb{Z}$-modules $\{\mathcal{K}_\epsilon\}_{\epsilon\in\mathbb{Z}}$, where*

$$\mathcal{K}_\epsilon := \begin{cases} K_0(\mathcal{A}(\Lambda)) & \text{if } \epsilon \text{ is even,} \\ K_1(\mathcal{A}(\Lambda)) & \text{if } \epsilon \text{ is odd.} \end{cases}$$

with $E^\infty_{p,q} \cong E^5_{p,q} = 0$ whenever $p \in (\mathbb{Z} \setminus \{0, \ldots, 4\})$ or q is odd.

3.4.7 Theorem (Evans, 2008) *Define the sets*

$$N_l := \begin{cases} \{\boldsymbol{\mu} := (\mu_1, \ldots, \mu_l) \in \{1, \ldots, k\}^l \mid \mu_1 < \cdots < \mu_l\} & \text{if } l \in \{1, \ldots, k\}, \\ \{\,*\,\} & \text{if } l = 0, \\ \emptyset & \text{otherwise,} \end{cases}$$

and for $l \in \{1, \dots, k\}$ and $\boldsymbol{\mu} \in N_l$, define

$$\boldsymbol{\mu}^i := \begin{cases} (\mu_1, \dots, \hat{\mu}_i, \dots, \mu_l) \in N_{l-1} & \text{if } l > 1, \\ * & \text{if } l = 1. \end{cases}$$

Let Λ be a row-finite k-graph with no sources. Then there exists a spectral sequence $\{(E^r, d^r)\}$ which converges to $K_\epsilon(\mathcal{A}(\Lambda))$, with $E^\infty_{p,q} \cong E^{k+1}_{p,q}$, and

$$E^2_{p,q} \cong \begin{cases} H_p(\mathcal{D}_k) & \text{if } p \in \{0, 1, \dots, k\} \text{ and } q \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\mathcal{D}_k := \cdots \to (\mathcal{D}_k)_{p+1} \xrightarrow{\partial_{p+1}} (\mathcal{D}_k)_p \xrightarrow{\partial_p} (\mathcal{D}_k)_{p-1} \to \cdots$ is the chain complex with

$$(\mathcal{D}_k)_p := \begin{cases} \bigoplus_{\boldsymbol{\mu} \in N_p} \mathbb{Z}\Lambda^{\mathbf{0}} & \text{if } p \in \{0, 1, \dots, k\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbb{Z}\Lambda^{\mathbf{0}}$ represents the group of linear combinations of the elements of $\Lambda^{\mathbf{0}}$ with coefficients in $\mathbb{Z}$. The differentials $\partial_p : (\mathcal{D}_k)_p \to (\mathcal{D}_k)_{p-1}$ are defined by

$$\partial_p : \bigoplus_{\boldsymbol{\mu} \in N_p} m_{\boldsymbol{\mu}} \longmapsto \bigoplus_{\lambda \in N_{p-1}} \sum_{\boldsymbol{\mu} \in N_p} \sum_{i=1}^{p} (-1)^{i+1} \delta_{\lambda, \boldsymbol{\mu}^i} (\mathbf{1} - M^T_{\mu_i}) m_{\boldsymbol{\mu}},$$

for each $p \in \{1, \dots, k\}$, where $\delta_{\lambda, \boldsymbol{\mu}^i}$ is the Kronecker delta function, which takes value 1 when $\lambda = \boldsymbol{\mu}^i$, and 0 otherwise. ■

3.4.8 Theorem (Evans, 2008; 3.4.7 when $k = 3$) *Let Λ be a row-finite 3-graph with no sources, and with adjacency matrices M_1, M_2, M_3. Write $\mathbf{1}$ to denote the 3×3 identity matrix, and consider the chain complex $\mathcal{D}_3$:*

$$0 \longrightarrow \mathbb{Z}\Lambda^{\mathbf{0}} \xrightarrow{\partial_3} \bigoplus_{i=1}^{3} \mathbb{Z}\Lambda^{\mathbf{0}} \xrightarrow{\partial_2} \bigoplus_{i=1}^{3} \mathbb{Z}\Lambda^{\mathbf{0}} \xrightarrow{\partial_1} \mathbb{Z}\Lambda^{\mathbf{0}} \longrightarrow 0,$$

whose differentials $\partial_1, \partial_2, \partial_3$ are defined by the block matrices

$$\partial_1 := \begin{bmatrix} \mathbf{1} - M_1^T & \mathbf{1} - M_2^T & \mathbf{1} - M_3^T \end{bmatrix},$$
$$\partial_2 := \begin{bmatrix} M_2^T - \mathbf{1} & M_3^T - \mathbf{1} & 0 \\ \mathbf{1} - M_1^T & 0 & M_3^T - \mathbf{1} \\ 0 & \mathbf{1} - M_1^T & \mathbf{1} - M_2^T \end{bmatrix},$$
$$\partial_3 := \begin{bmatrix} \mathbf{1} - M_3^T \\ M_2^T - \mathbf{1} \\ \mathbf{1} - M_1^T \end{bmatrix}.$$

Then for some subgroups $G_0 \subseteq \operatorname{coker}(\partial_1)$ and $G_1 \subseteq \ker(\partial_3)$, there exists a short exact sequence

$$0 \longrightarrow \operatorname{coker}(\partial_1)/G_0 \longrightarrow K_0(\mathcal{A}(\Lambda)) \longrightarrow \ker(\partial_2)/\operatorname{im}(\partial_3) \longrightarrow 0,$$

and an isomorphism

$$K_1(\mathcal{A}(\Lambda)) \cong \ker(\partial_1)/\operatorname{im}(\partial_2) \oplus G_1,$$

where $\mathcal{A}(\Lambda)$ is the 3-rank graph $C^\star$-algebra associated to Λ, as defined in 1.3.11. ■

3.4.9 Corollary (Evans, 2008; $k = 3$) *In addition to the hypotheses of Theorem 3.4.8:*

(i) If ∂_1 is surjective, then:

(a) $K_0(\mathcal{A}(\Lambda)) \cong \ker(\partial_2)/\operatorname{im}(\partial_3)$.

(b) $K_1(\mathcal{A}(\Lambda)) \cong (\ker(\partial_1)/\operatorname{im}(\partial_2)) \oplus \ker(\partial_3)$.

(ii) If $\bigcap_i \ker\left(\mathbf{1} - M_i^T\right) = 0$, then there exists a short exact sequence

$$0 \longrightarrow \operatorname{coker}(\partial_1) \longrightarrow K_0(\mathcal{A}(\Lambda)) \longrightarrow \ker(\partial_2)/\operatorname{im}(\partial_3) \longrightarrow 0,$$

and an isomorphism

$$K_1(\mathcal{A}(\Lambda)) \cong \ker(\partial_1)/\operatorname{im}(\partial_2).$$

■

3.4.10 In [MRV20], versions of 3.4.8–3.4.9 were presented for the cases where $k = 4$ and $k = 5$. Although it is tempting to try to extend these results on k-graph $C^\star$-algebras to higher k, in actuality we get ever fuzzier information about the K-theory as k increases. This is because the groups $K_\epsilon(\mathcal{A}(\Lambda))$ lie at the heart of a nest of short exact sequences, surrounded by groups about which we only have partial information. Increasing the dimension k also increases the number of sequences, as can be seen in 3.4.11, to the point where above $k = 5$ we can deduce very little about the K-theory.

This being said, Corollaries 3.4.9 and 3.4.12 offer some conditions on the adjacency matrices of a higher-rank graph Λ which allow us to make sharper deductions about the K-theory. We present what is known about the K-theory of 4-rank graph algebras here.

3.4.11 Theorem (Mutter, Radu and Vdovina, 202-; 3.4.7 when $k = 4$) *Let Λ be a 4-graph which is row-finite and has no sources, and which has adjacency matrices $M_1, \ldots, M_4$. Write $\mathbf{1}$ to denote the 4×4 identity matrix, and consider the chain complex $\mathcal{D}_4$ defined by*

$$0 \longrightarrow \mathbb{Z}\Lambda^{\mathbf{0}} \xrightarrow{\partial_4} \bigoplus_{i=1}^{4} \mathbb{Z}\Lambda^{\mathbf{0}} \xrightarrow{\partial_3} \bigoplus_{i=1}^{6} \mathbb{Z}\Lambda^{\mathbf{0}} \xrightarrow{\partial_2} \bigoplus_{i=1}^{4} \mathbb{Z}\Lambda^{\mathbf{0}} \xrightarrow{\partial_1} \mathbb{Z}\Lambda^{\mathbf{0}} \longrightarrow 0,$$

whose differentials $\partial_1, \ldots, \partial_4$ are the group homomorphisms represented by block matrices

$$\partial_1 := \begin{bmatrix} \mathbf{1} - M_1^T & \mathbf{1} - M_2^T & \mathbf{1} - M_3^T & \mathbf{1} - M_4^T \end{bmatrix},$$

$$\partial_2 := \begin{bmatrix} M_2^T - \mathbf{1} & M_3^T - \mathbf{1} & M_4^T - \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} - M_1^T & 0 & 0 & M_3^T - \mathbf{1} & M_4^T - \mathbf{1} & 0 \\ 0 & \mathbf{1} - M_1^T & 0 & \mathbf{1} - M_2^T & 0 & M_4^T - \mathbf{1} \\ 0 & 0 & \mathbf{1} - M_1^T & 0 & \mathbf{1} - M_2^T & \mathbf{1} - M_3^T \end{bmatrix},$$

$$\partial_3 := \begin{bmatrix} \mathbf{1} - M_3^T & \mathbf{1} - M_4^T & 0 & 0 \\ M_2^T - \mathbf{1} & 0 & \mathbf{1} - M_4^T & 0 \\ 0 & M_2^T - \mathbf{1} & M_3^T - \mathbf{1} & 0 \\ \mathbf{1} - M_1^T & 0 & 0 & \mathbf{1} - M_4^T \\ 0 & \mathbf{1} - M_1^T & 0 & M_3^T - \mathbf{1} \\ 0 & 0 & \mathbf{1} - M_1^T & \mathbf{1} - M_2^T \end{bmatrix},$$

$$\partial_4 := \begin{bmatrix} M_4^T - \mathbf{1} \\ \mathbf{1} - M_3^T \\ M_2^T - \mathbf{1} \\ \mathbf{1} - M_1^T \end{bmatrix}.$$

Write $H_i(\mathcal{D}_4) := \ker(\partial_i)/\operatorname{im}(\partial_{i+1})$, and let F_2 be a factor in the ascending filtration (3.1) of the group $K_0(\mathcal{A}(\Lambda))$. Then, for some subgroups

$$G_0 \subseteq \operatorname{coker}(\partial_1), \quad G_1 \subseteq \ker(\partial_4), \quad G_2 \subseteq H_1(\mathcal{D}_4), \quad G_3 \subseteq H_3(\mathcal{D}_4),$$

there exist short exact sequences as follows:

(i) $0 \longrightarrow \operatorname{coker}(\partial_1)/G_0 \longrightarrow K_0(\mathcal{A}(\Lambda)) \longrightarrow \dfrac{K_0(\mathcal{A}(\Lambda))}{\operatorname{coker}(\partial_1)/G_0} \longrightarrow 0,$

(ii) $0 \longrightarrow \operatorname{coker}(\partial_1)/G_0 \longrightarrow F_2 \longrightarrow \dfrac{\ker(\partial_2)}{\operatorname{im}(\partial_3)} \longrightarrow 0,$

(iii) $0 \longrightarrow F_2 \longrightarrow K_0(\mathcal{A}(\Lambda)) \longrightarrow G_1 \longrightarrow 0,$

(iv) $0 \longrightarrow \dfrac{\ker(\partial_1)/\operatorname{im}(\partial_2)}{G_2} \longrightarrow K_1(\mathcal{A}(\Lambda)) \longrightarrow G_3 \longrightarrow 0,$

and sequence (iii) splits, such that $K_0(\mathcal{A}(\Lambda)) \cong F_2 \oplus G_1$.

■ **Proof** Write $\{(E^r, d^r)\}_{r\geq 1}$ to denote the spectral sequence of homological type from 3.4.7. We know from that theorem that $\{(E^r, d^r)\}$ is bounded, and that the stable value of $E^r_{p,q}$ is $E^\infty_{p,q} \cong E^{k+1}_{p,q} = E^5_{p,q}$. We may use the finite ascending filtration (3.1) and the isomorphism (3.2) for each family of $\mathbb{Z}$-modules $\{\mathcal{K}_\epsilon\}_{\epsilon\in\mathbb{Z}}$, since Lemma 3.4.6 tells us that the spectral sequence converges to $K_\epsilon(\mathcal{A}(\Lambda)) = K_{\epsilon \bmod 2}(\mathcal{A}(\Lambda))$.

We split the proof into cases based on the total degree $p+q$. Firstly, consider $K_0(\mathcal{A}(\Lambda))$, and write $K_0 = K_0(\mathcal{A}(\Lambda)) = K_{p+q}$, as in [Eva08, 3.3].

Case I: Fix the total degree, $p+q$, to be zero.

We must have $E^5_{p,q} = 0$ unless $p \in \{0,2,4\}$, since if p is odd and $p+q=0$, then q is also odd. Suppose, then, that $p \notin \{0,2,4\}$, so that $0 = E^5_{p,q} = F_p(\mathcal{K}_0)/F_{p-1}(\mathcal{K}_0)$, and hence $F_p(\mathcal{K}_0) \cong F_{p-1}(\mathcal{K}_0)$. We can deduce that, in our filtration, we have $F_1(\mathcal{K}_0) = F_0(\mathcal{K}_0)$, and $F_{i+1}(\mathcal{K}_0) = F_i(\mathcal{K}_0)$ for all $i \geq 2$.

By the same argument, it follows that $F_i(\mathcal{K}_0) = 0$ for all $i < 0$, and so the filtration bubbles down to

$$0 \subseteq F_0(\mathcal{K}_0) \subseteq F_2(\mathcal{K}_0) \subseteq \mathcal{K}_0.$$

Next, we consider the non-zero terms from the $\left(E^5_{p,q}\right)$ sheet. From (3.2), we have:

- $E^5_{0,0} \cong F_0(\mathcal{K}_0)$.
- $E^5_{2,-2} \cong F_2(\mathcal{K}_0)/F_1(\mathcal{K}_0) \cong F_2(\mathcal{K}_0)/F_0(\mathcal{K}_0)$.
- $E^5_{4,-4} \cong F_4(\mathcal{K}_0)/F_3(\mathcal{K}_0) \cong \mathcal{K}_0/F_2(\mathcal{K}_0)$.

We then obtain short exact sequences as follows:

(i′) $0 \longrightarrow E^5_{0,0} \longrightarrow \mathcal{K}_0 \longrightarrow \mathcal{K}_0/E^5_{0,0} \longrightarrow 0,$

(ii′) $0 \longrightarrow E^5_{0,0} \longrightarrow F_2(\mathcal{K}_0) \longrightarrow E^5_{2,-2} \longrightarrow 0,$

(iii′) $0 \longrightarrow F_2(\mathcal{K}_0) \longrightarrow \mathcal{K}_0 \longrightarrow E^5_{4,-4} \longrightarrow 0,$

which we use shortly to deduce the short exact sequences of the theorem. Before that, however, we turn our attention to $K_1(\mathcal{A}(\Lambda))$.

Case II: Now fix the total degree $p + q = 1$.

Note that, in order for $E^5_{p,q}$ to be non-zero, we must have $p \in \{0, \dots, 4\}$ and q even. But, the only pairs (p, q) of total degree 1 are $(1, 0)$ and $(3, -2)$. Thus, it follows analogously from [Eva08, 3.17] that there is a short exact sequence

$$0 \longrightarrow E^5_{1,0} \longrightarrow K_1(\mathcal{A}(\Lambda)) \longrightarrow E^5_{3,-2} \longrightarrow 0. \tag{3.3}$$

To complete the proof requires two more steps:

(a) For $K_1(\mathcal{A}(\Lambda))$, computation of the terms $E^5_{1,0}$ and $E^5_{3,-2}$ which appear in (3.3).

(b) For $K_0(\mathcal{A}(\Lambda))$, computation the terms $E^5_{0,0}$, $E^5_{2,-2}$, and $E^5_{4,-4}$.

Step (a): $E^5_{1,0}$ and $E^5_{3,-2}$.

Consider the differentials d^4, d^3, and d^2; since $E^4_{p,q} = 0$ whenever $p \in (\mathbb{Z} \setminus \{1, \dots, 4\})$, we necessarily have

$$d^4 : E^4_{p,q} \longrightarrow E^4_{p-4,q+3}, \qquad d^4 : E^4_{p+4,q-3} \longrightarrow E^4_{p,q},$$

for $p \in \{0, 4\}$. However, in either case this would imply that both of q and $q + 3$ or q and $q - 3$ are even: a contradiction. Hence d^4 must be the zero map.

Similarly, it follows that the only non-zero components of the d^3 differential are

$$d^3 : E^3_{3,q} \longrightarrow E_{0,q+2}, \qquad d^3 : E^3_{4,q} \longrightarrow E^3_{1,q+2},$$

whenever q is even. Furthermore, we can deduce that d^2 must also be the zero map, as in [Eva08, 3.16]. We therefore have the following isomorphisms:

$$E^5_{1,0} \cong H\big(E^4_{1,0}\big) = \frac{\ker\left(d^4 : E^4_{1,0} \to E^4_{-3,3}\right)}{\operatorname{im}\left(d^4 : E^4_{5,-3} \to E^4_{1,0}\right)} = E^4_{1,0},$$
$$E^4_{1,0} \cong H\big(E^3_{1,0}\big) = \frac{\ker\left(d^3 : E^3_{1,0} \to E^3_{-2,2}\right)}{\operatorname{im}\left(d^3 : E^3_{4,-2} \to E^3_{1,0}\right)} = E^3_{1,0}/\operatorname{im}\left(d^3 : E^3_{4,-2} \to E^3_{1,0}\right).$$

Now, let G_2 be a subgroup of $E^3_{1,0} = H_1(\mathcal{D}_4)$, namely $G_2 := \operatorname{im}\left(d^3 : E^3_{4,-2} \to E^3_{1,0}\right)$. Then

$$E^3_{1,0} \cong H\big(E^2_{1,0}\big) = E^2_{1,0} = H_1(\mathcal{D}_4),$$

so $E^5_{1,0} \cong (\ker(\partial_1)/\mathrm{im}(\partial_2))/G_2$, and it remains to compute $E^5_{3,-2}$. We have isomorphisms:

$$E^5_{3,-2} \cong H\big(E^4_{3,-2}\big) = \frac{\ker\big(d^4 : E^4_{3,-2} \to E^4_{-1,1}\big)}{\mathrm{im}\big(d^4 : E^4_{7,-5} \to E^4_{3,-2}\big)} = E^4_{3,-2},$$

$$E^4_{3,-2} \cong H\big(E^3_{3,-2}\big) = \frac{\ker\big(d^3 : E^3_{3,-2} \to E^3_{0,0}\big)}{\mathrm{im}\big(d^3 : E^3_{6,-4} \to E^3_{3,-2}\big)} = \ker\big(d^3_{3,-2}\big) \subseteq E^3_{3,-2}.$$

Combining the above, we find that $E^5_{3,-2} \cong \ker\big(d^3_{3,-2}\big) \subseteq E^3_{3,-2} = H_3(\mathcal{D}_4)$. Writing $G_3 := \ker\big(d^3_{3,-2}\big)$, which is a subgroup of $H_3(\mathcal{D}_4)$, we now have the short exact sequence (iv):

$$0 \longrightarrow H_1(\mathcal{D}_4)/G_2 \longrightarrow K_1(\mathcal{A}(\Lambda)) \longrightarrow G_3 \longrightarrow 0.$$

Step (b): $E^5_{0,0}$, $E^5_{2,-2}$, and $E^5_{4,-4}$.

Now consider $E^5_{0,0}$. We know that $E^5_{0,0} \cong H(E^4_{0,0}) = E^4_{0,0}$, since we have established that the differential d^4 is the zero map. We also know that

$$E^4_{0,0} \cong H\big(E^3_{0,0}\big) = \frac{\ker\big(d^3 : E^3_{0,0} \to E^3_{-3,0}\big)}{\mathrm{im}\big(d^3 : E^3_{3,-2} \to E^3_{0,0}\big)} = E^3_{0,0}/\mathrm{im}\big(d^3_{3,-2}\big).$$

Note that $E^3_{0,0} \cong H(E^2_{0,0}) = E^2_{0,0} = H_0(\mathcal{D}_4) = \mathrm{coker}(\partial_1)$, so that if we write $G_0 := \mathrm{im}(d^3_{3,-2})$, then we obtain $E^5_{0,0} = \mathrm{coker}(\partial_1)/G_0$. This, together with the sequence (i′) above, gives us the short exact sequence (i).

Turning our attention to $E^5_{2,-2}$ and $E^5_{4,-4}$ now, we know that $E^5_{2,-2} \cong H(E^4_{2,-2}) = E^4_{2,-2}$, by virtue of d^4 being the zero map. We also have an isomorphism

$$E^4_{2,-2} \cong H\big(E^3_{2,-2}\big) = \frac{\ker\big(d^3 : E^3_{2,-2} \to E^3_{-1,0}\big)}{\mathrm{im}\big(d^3 : E^3_{5,-4} \to E^3_{2,-2}\big)} = E^3_{2,-2} \cong H\big(E^2_{2,-2}\big) = H_2(\mathcal{D}_4),$$

from which we deduce that $E^5_{2,-2} \cong H_2(\mathcal{D}_4)$. Together with (ii′) and the above, this gives us sequence (ii). We also know that $E^5_{4,-4} \cong H(E^4_{4,-4}) = E^5_{4,-4}$, and

$$E^4_{4,-4} \cong H\big(E^3_{4,-4}\big) = \frac{\ker\big(d^3 : E^3_{4,-4} \to E^3_{1,-2}\big)}{\mathrm{im}\big(d^3 : E^3_{7,-6} \to E^3_{4,-4}\big)} = \ker\big(d^3_{4,-4}\big) \subseteq E^3_{4,-4},$$

and so $E^3_{4,-4} \cong H\big(E^2_{4,-4}\big) = E^2_{4,-4} = H_4(\mathcal{D}_4)$. By writing $G_1 := \ker(d^3_{4,-4})$, and putting this together with (iii′), we obtain the sequence (iii). Finally, we know that $H_4(\mathcal{D}_4)$ is a free Abelian group, and since subgroups of such groups are also free Abelian, it follows that G_1 is free Abelian, and hence that sequence (iii) splits. ∎

3.4.12 Corollary (Mutter, Radu and Vdovina, 202-; $k = 4$) *In addition to the hypotheses of 3.4.11:*

(i) If ∂_1 is surjective, then there exists an isomorphism $F_2 \cong \ker(\partial_2)/\mathrm{im}(\partial_3)$, and the short exact sequences 3.4.11(i)–(iv) reduce to an isomorphism and an exact sequence:

(a) $K_0(\mathcal{A}(\Lambda)) \cong \dfrac{\ker(\partial_2)}{\mathrm{im}(\partial_3)} \oplus G_1$,

(b) $0 \longrightarrow \dfrac{\ker(\partial_1)/\mathrm{im}(\partial_2)}{G_2} \longrightarrow K_1(\mathcal{A}(\Lambda)) \longrightarrow \ker(\partial_3)/\mathrm{im}(\partial_4) \longrightarrow 0.$

(ii) If the intersection $\bigcap_i \ker\left(\mathbf{1} - M_i^T\right) = 0$, then $K_0(\mathcal{A}(\Lambda)) \cong F_2$, and the sequences reduce to:

(a) $0 \longrightarrow \mathrm{coker}(\partial_1)/G_0 \longrightarrow K_0(\mathcal{A}(\Lambda)) \longrightarrow \ker(\partial_3)/\mathrm{im}(\partial_2) \longrightarrow 0,$

(b) $0 \longrightarrow \ker(\partial_1)/\mathrm{im}(\partial_2) \longrightarrow K_1(\mathcal{A}(\Lambda)) \longrightarrow G_3 \longrightarrow 0.$

■ **Proof** To demonstrate (i), we suppose that ∂_1 is surjective, such that $\mathrm{coker}(\partial_1) = 0$ and $F_2(K_0) \cong \ker(\partial_2)/\mathrm{im}(\partial_3)$. Then the split exact sequence (iii) from 3.4.11 gives us (i)(a).

Then, we also know that $0 = \mathrm{coker}(\partial_1) = H_0(\mathcal{D}_4) \cong E^3_{0,0}$, and so $d^3 : E^3_{3,-2} \to E^3_{0,0}$ must be the zero map. Hence $\ker(d^3_{3,-2}) = E^3_{3,-2} \cong H_3(\mathcal{D}_4)$, and we obtain (i)(b) by using the sequence 3.4.11(iv).

Now, in order to show (ii), we suppose that $\bigcap_i \left(\mathbf{1} - M_i^T\right) = 0$. Then $\ker(\partial_4) = 0$, and hence $G_1 = 0$ and $K_0 \cong F_2$. This gives us (ii)(a).

Finally, from 3.4.11(iv) we obtain the new sequence

$$0 \longrightarrow \frac{\ker(\partial_1)/\mathrm{im}(\partial_2)}{G_2} \longrightarrow K_1 \longrightarrow G_3 \longrightarrow 0,$$

where G_2 and G_3 are the groups defined by

$$\begin{aligned} G_2 &:= \mathrm{im}\left(d^3_{4,-2} : E^3_{4,-2} \longrightarrow E^3_{1,0} = H_1(\mathcal{D}_4)\right), \\ G_3 &:= \ker\left(d^3_{3,-2}\right) \subseteq \ker(\partial_3)/\mathrm{im}(\partial_4). \end{aligned}$$

However, we also know that $E^3_{4,-2} \cong H(E^2_{4,-2}) \cong E^2_{4,-2} \cong H_4(\mathcal{D}_4) = \ker(\partial_4)$. But because $\ker(\partial_4) = 0$, it follows that the differential $\partial^3_{4,-2}$ has domain 0, and is hence the zero map. Therefore $G_2 = 0$, and the result follows. ■

§3.5 Properties of domino graph algebras

3.5.1 Let Γ be a k-domino group with adjacency matrices $M_1, \ldots, M_k$; by 3.4.3, there is a k-rank graph $\Lambda = \Lambda(\Gamma)$ which has vertices indexed by the set of k-dimensional dominoes $\mathcal{S}_k(\Gamma)$, and a morphism between two dominoes whenever they are adjacent. We call k-rank graphs which arise in this way *k-domino graphs*, and we call the universal $C^\star$-algebra of a k-domino graph a k-domino graph algebra.

In this section we investigate some desirable properties of domino graph algebras, and eventually show that they fall under the same classification as the *tile systems* of Chapter 2, meaning that they satisfy the criteria of Theorem 2.3.11.

3.5.2 Lemma *Let Γ be a k-domino group with adjacency structure $E_1, \ldots, E_k$. Then the induced k-rank graph $\Lambda(\Gamma)$ satisfies the Aperiodicity Condition of 2.3.1.*

■ **Proof** The proof is almost identical to that of 2.3.3, and follows as a result of the observations in [RS99, §2]. Since $|E_i| \geq 4$ for each i, there are always at least two k-dimensional dominoes which are E_i-adjacent to some domino U. Hence, we can always exit some cycle by diverting our path down another edge in (the 1-skeleton of) $\Lambda(\Gamma)$ at any vertex U.

Likewise, given a k-dimensional domino V, there are always at least three dominoes to which V is E_i-adjacent. Thus we may always find an infinite aperiodic path, and $\Lambda(\Gamma)$ satisfies the *Aperiodicity Condition*. ■

3.5.3 Lemma *Let Γ be a k-domino group, and let $\Lambda(\Gamma)$ be its induced k-rank graph. Then $\Lambda(\Gamma)$ is strongly connected, that is, for any two vertices $U, V \in \Lambda(\Gamma)^{\mathbf{0}}$, there is a path linking U to V.*

■ **Proof** We give a geometric proof involving the domino complex $\mathcal{M}(\Gamma)$, although we point out that this can also be proved in the manner of [KR02, 4.2].

Let Γ be a k-domino group with adjacency structure $E_1, \ldots, E_k$, write $|E_i| = m_i$ for each i, and consider a k-dimensional domino U in the domino complex $\mathcal{M}(\Gamma)$. Let U_H be the domino obtained by reflecting S through the edges labelled by elements of E_1, leaving the basepoint and orientation the same as in S (in the manner of Figure 3.2).

Firstly, we show that there is a sequence of k-dimensional dominoes $U = V_0, \ldots, V_n = U_H$ such that $M_1(V_j, V_{j+1}) = 1$ for all j, that is, so that each domino is E_1-adjacent to the next.

Each k-dimensional domino X in $\mathcal{M}(\Gamma)$ contains two $(k-1)$-faces ($(k-1)$-dimensional subdominoes) labelled by elements of $E_2, \ldots, E_k$. Since the dominoes have a predetermined orientation, we can label these faces X^L and X^R, such that $M_1(X, Y) = 1$ if and only if $Y^L = X^R$ and $Y \neq X_H$. We may therefore assign to each domino X the pair (X^L, X^R) such that, in any sequence (V_j) of adjacent dominoes, $V_{j+1}^L = V_j^R$, and $V_{j+1} \neq (V_j)_H$, for all j.

Observe that each $(k-1)$-dimensional domino appears as X^L (resp. X^R) for some $X \in \mathcal{S}_k$ precisely m_1 times, and that, by assumption, $m_1 \geq 4$. Write $A_0 := U^R$, and let V_1 be a k-dimensional domino which is E_1-adjacent to U; such a domino exists by the above observation. If $V_1^R = A_0$, then $M_1(V_1, U_H) = 1$ and we are done.

Assume then that $V_1^R = A_1 \neq A_0$, and let V_2 be E_1-adjacent to V_1. If $V_2^R = A_0$, then $M_1(V_2, U_H) = 1$, and if $V_2^R = A_1$, then $M_1(V_2, (V_1)_H) = 1$, and $M_1((V_1)_H, U_H) = 1$. In both cases, we have constructed a sequence of adjacent dominoes linking U to U_H.

If $V_q^R = V_p^R$ for any $p < q$, then we obtain the sequence we desire. But also, by the fact that each $(k-1)$-dimensional domino appears as X^R for some $X \in \mathcal{S}_k$ an even number of times, there must be some $q > p$ for which $V_q^R = V_p^R$. Hence such a sequence exists, and there is a path connecting the vertices labelled U and U_H in $\Lambda(\Gamma)$.

In the same manner, we may show that there is a sequence of adjacent dominoes connecting each $U \in \mathcal{S}_k$ to each of its symmetries, being the ones which belong to the same orbit as U under the action of the group of reflections of the k-dimensional cube.

Now, we construct the set $\mathcal{P}$ of all k-dimensional dominoes which can be reached by a sequence of adjacent dominoes (in any sequence of directions) from an initial domino U. Certainly U_H is in $\mathcal{P}$, by the above. Moreover, by virtue of Proposition 3.1.7, $\mathcal{P}$ contains $(m_1 - 1)$-many more *distinct* dominoes which are E_1-adjacent to U, to total m_1 distinct dominoes. Each of these dominoes is E_2-adjacent to m_2 dominoes by the same argument. These are distinct from each other by the uniqueness property of **D3** from 3.1.14.

We may proceed inductively to find that $\mathcal{P}$ must contain at least $\prod_{i=1}^{k} m_i$ distinct k-dimensional dominoes, but this is precisely $|\mathcal{S}_k| = |\Lambda(\Gamma)^{\mathbf{0}}|$. Hence, given any two dominoes U, V, there is a sequence of adjacent dominoes from U to V. Equivalently, given any vertex labelled by U in $\Lambda(\Gamma)^{\mathbf{0}}$, there is a path from U to every other vertex. ■

3.5.4 Theorem (Classification of domino graph algebras) *Let Γ be a k-domino group with adjacency structure $E_1, \ldots, E_k$, and let $(\Lambda(\Gamma), d)$ be the induced k-graph. Then $\mathcal{A}(\Gamma) := \mathcal{A}(\Lambda(\Gamma))$ is separable, nuclear, purely infinite, simple, and satisfies the Universal Coefficient Theorem. Hence $\mathcal{A}(\Gamma)$ is completely determined by its K-theory and the class of $\mathrm{id}_{\mathcal{A}(\Gamma)}$ in K_0, up to isomorphism.*

■ **Proof** From 3.5.3 and 2.3.7, it follows that $\mathcal{A}(\Gamma)$ is simple. Also by 3.5.3, together with the fact that $|E_i| \geq 4$ for all i, it follows that for every $U \in \Lambda(\Gamma)^{\mathbf{0}}$ we can find $\lambda, \mu \in \Lambda(\Gamma)$ such that $d(\mu) \neq \mathbf{0}$, $r(\lambda) = U$, and $s(\lambda) = r(\mu) = s(\mu)$. Hence from Theorem 2.3.8 it follows that $\mathcal{A}(\Gamma)$ is purely infinite.

From Theorem 3.4.3 we know that $\Lambda(\Gamma)$ is a row-finite k-graph with no sources, and in 2.3.6 it is shown that such a k-graph has a corresponding $C^\star$-algebra which is separable, nuclear, unital, and satisfies the *Universal Coefficient Theorem*, thereby putting us in a situation where we can apply the *Kirchberg–Phillips Classification* (Theorem 2.3.11). ■

3.5.5 Proposition (Order of identity in K_0 for domino graph algebras) *Fix $k \geq 2$ and let Γ be a k-domino group with adjacency structure $E_1, \ldots, E_k$, where $|E_i| = m_i$ for each i, and define $\rho := \gcd\{(m_i/2) - 1\}_i$. Then the order of the class of $\mathrm{id}_{\mathcal{A}}$ in $K_0(\mathcal{A}(\Gamma))$ divides ρ.*

■ **Proof** This proposition can be proved via a simple generalisation of the method used in [KR02, 5.4]. ■

3.5.6 Conjecture *Let Γ be a k-cube group with adjacency structure $E_1, \dots, E_k$, where $|E_i| = m_i$, and define $\rho := \gcd\{(m_i/2) - 1\}_i$. Factorise ρ as $2^q r$, where r is an odd number: if ρ is odd then write $q = 0$. Then the order of the class of $\mathrm{id}_{\mathcal{A}}$ in $K_0(\mathcal{A}(\Gamma))$ is at most ρ, and is:*

(i) Equal to ρ if ρ is odd,

(ii) Divisible by $\rho/(2^q)$ whenever $q \in \{1, \dots, k-1\}$,

(iii) Divisible by $\rho/(2^{k-1})$ whenever $q \geq (k-1)$.

3.5.7 If *Matui's HK-Conjecture* (see 4.2 and [FKPS19; Mat16]) were to be confirmed in the case of domino graph algebras, then we would be able to refine the statement of Proposition 3.5.5 to that of Conjecture 3.5.6 by means of the following argument:

Given a higher-rank graph Λ, Proposition 1.3.13 tells us that the sum of all elements of $\mathcal{A}(\Lambda)$ of the form t_v, where $v \in \Lambda^{\mathbf{0}}$, is an identity for $\mathcal{A}(\Lambda)$. In particular, for a k-domino group Γ, the sum $\sum_{U \in \mathcal{S}(\Gamma)} t_U$ is an identity operator in $\mathcal{A}(\Gamma)$. Now, recall the map

$$\partial_1 : \mathbb{Z}\Lambda(\Gamma)^{\mathbf{0}} \longrightarrow \bigoplus_{i=1}^{k} \mathbb{Z}\Lambda(\Gamma)^{\mathbf{0}}$$

defined in 3.4.7 by the matrix $\left[\mathbf{1} - M_1^T, \dots, \mathbf{1} - M_k^T\right]$. The *Covariance Relation* of [KR02, §5] generalises to k-graphs, and so from [RS01] and *Matui's Conjecture* it would follow that the map

$$\varphi : \operatorname{coker}(\partial_1) = \left\langle U \in \mathcal{S}_k \;\middle|\; \sum_{V \in \mathcal{S}_k} M_i(U, V) \cdot U \right\rangle \longrightarrow K_0(\mathcal{A}(\Gamma)),$$

which takes a k-dimensional domino U to its class $[U]$ in K_0 is injective. But each column of M_i has exactly $(m_i - 1)$ ones, the rest of the entries being zero, and so $\Sigma = (m_i - 1)\Sigma$ for each $i \in \{1, \dots, k\}$, where $\Sigma := \sum_{U \in \mathcal{S}_k}$. Since $\sum_{U \in \mathcal{S}_k} t_U$ is an identity in $\mathcal{A}(\Gamma)$, the class $[\mathrm{id}_{\mathcal{A}}] \in K_0$ is the image of Σ under φ. By the above, we also know that $(m_i - 2)\Sigma$ is zero for each i.

Write $2\rho = \gcd\{m_i - 2\}_i$, and define the map $\psi : \operatorname{coker}(\partial_1) \to \mathbb{Z}/2\rho$ by $\psi(S) := 1 \bmod 2\rho$, as in the proof of [KR02, 5.4]. Now,

$$\prod_i (m_i - 2) = \left(\prod_i m_i\right) - 2^k \bmod 2\rho,$$

and since $(m_i - 2) = 0 \bmod 2\rho$, this means that $\psi(\Sigma) = 2^k \bmod 2\rho$, and hence that $\rho \cdot \psi(\Sigma) = 0 \bmod 2\rho$. If ρ is odd, then $\psi(\Sigma)$ has order ρ in $\mathbb{Z}/2\rho$. If ρ is even, then $\rho = 2^q r$ for some odd number r, and $\rho \cdot \psi(\Sigma) = 2^{k+q} r \bmod (2^{q+1} r)$. Thus the order of Σ in $\operatorname{coker}(\partial_1)$ is divisible by ρ in the former case, and by $\max\{\rho/(2^q), \rho/(2^{k-1})\}$ in the latter.

Chapter 4

The geometry and K-theory of domino graph algebras

We now provide some explicit examples of k-rank graph algebras based on the models introduced in Chapter 3, and we compute aspects of their K-theory. As we mentioned in 3.1.12, construction of a set of pointed squares which has a k-domino structure is very hard without the right tools, but the *Rungtanapirom–Stix–Vdovina Algorithm* in 4.1.6 provides a bountiful source.

In each case, we attempt to determine the K-theory of the associated k-rank graph algebra (in a manner reminiscent of Chapter 2), and calculate the *cellular homology* (see 2.6.1) of the k-dimensional domino complexes (à la §2.6). We discuss to what extent this is possible, and observe in 4.4.5 some common themes in the calculations which might lead to a more complete theory. We note in §4.2 that a more thorough classification of domino graph algebras could be possible, pending *Matui's HK-Conjecture* (4.2.2).

One of our aims in this chapter and in the author's research beyond this thesis is to expand upon the theorem of Raeburn–Szymański that every free Abelian group arises as $K_1(\mathcal{A})$ of some 1-graph algebra $\mathcal{A}$; there is no such restriction on K_1 of a higher-rank graph algebra. We display in §4.1 a number of 3-rank graph algebras which between them engender a wide variety of K-groups. The K-theory of certain domino graph algebra relies, at least in part, on the greatest common divisor of the sizes of the alphabets (refer to §4.5). It would be our ultimate aim to have a procedural method for constructing a higher-rank graph algebra with any desired K-groups, and we provide suggestions for further research to that end, and highlight potential limitations, throughout the chapter.

§4.1 3-domino group computations

4.1.1 Example (3.1.12 revisited) Consider the direct product of three free groups, each with two generators, defined as follows:

$$\mathbb{F}_2^3 := \langle a_1, a_2, b_1, b_2, c_1, c_2 \mid [a_i, b_j], [a_i, c_j], [b_i, c_j], \text{ for all } i, j \in \{1, 2\}\rangle,$$

where $[x, y]$ denotes the *commutator* $xy\bar{x}\bar{y}$. This is a 3-domino group with adjacency structure $E_1 = \{a_i, \bar{a}_i\}$, $E_2 = \{b_i, \bar{b}_i\}$, $E_3 = \{c_i, \bar{c}_i\}$, which we first came across in 3.1.12.

We construct the chain complex $\mathcal{D}_3$ from 3.4.8 using the three adjacency matrices M_1, M_2, M_3; the domino complex $\mathcal{M}(\Gamma)$ comprises eight geometric cubes (64 3-dimensional dominoes), so the adjacency matrices will be of dimension 64×64. Writing $\mathbf{1}$ to denote the 3×3 identity matrix, we obtain the chain complex:

$$0 \longrightarrow \mathbb{Z}^{64} \xrightarrow{\partial_3} (\mathbb{Z}^{64} \oplus \mathbb{Z}^{64} \oplus \mathbb{Z}^{64}) \xrightarrow{\partial_2} (\mathbb{Z}^{64} \oplus \mathbb{Z}^{64} \oplus \mathbb{Z}^{64}) \xrightarrow{\partial_1} \mathbb{Z}^{64} \longrightarrow 0, \qquad (4.1)$$

where the differentials ∂_1, ∂_2, ∂_3 are defined via the adjacency matrices, as in 3.4.8. Using the algebraic software package `MAGMA` (developed by [BCP97]), we compute the relevant cokernels, kernels, and images of the differentials. We find that $\operatorname{coker}(\partial_1) \cong \ker(\partial_3) \cong \mathbb{Z}^8$, and $\ker(\partial_2)/\operatorname{im}(\partial_3) \cong \ker(\partial_1)/\operatorname{im}(\partial_2) \cong \mathbb{Z}^{24}$. Then there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}^8/G_0 \longrightarrow K_0(\mathcal{A}(\Gamma)) \longrightarrow \mathbb{Z}^{24} \longrightarrow 0,$$

and an isomorphism $K_1(\mathcal{A}(\Gamma)) \cong \mathbb{Z}^{24} \oplus G_1$, where G_0 and G_1 are subgroups of $\mathbb{Z}^8$.

Since all of the groups from (4.1) are free Abelian, then G_0 and G_1 must also be free Abelian, and so are K_0 and K_1. Hence, using arguments of Spielberg from [Spi91] and outlined in [Rob00, §§1, 7], it can be shown that $\mathcal{A}(\mathbb{F}_2^3) \cong \mathcal{A}(\Lambda(\mathbb{F}_2^2)) \otimes \mathcal{A}(\mathbb{F}_2)$ (compare with [KR02, 6.1]). We can then use the *Künneth Theorem* for tensor products (1.2.18) and 2.2.6 to see that

$$\begin{aligned} K_0(\mathcal{A}(\mathbb{F}_2^3)) &\cong K_0(\mathcal{A}(\mathbb{F}_2^2)) \otimes K_0(\mathcal{A}(\mathbb{F}_2)) \oplus K_1(\mathcal{A}(\mathbb{F}_2^2)) \otimes K_1(\mathcal{A}(\mathbb{F}_2)) \\ &\cong \mathbb{Z}^8 \otimes \mathbb{Z}^2 \oplus \mathbb{Z}^8 \otimes \mathbb{Z}^2 \\ &\cong \mathbb{Z}^{32}, \end{aligned}$$

and similarly that

$$K_1(\mathcal{A}(\mathbb{F}_2^3)) \cong (\mathbb{Z}^8 \otimes \mathbb{Z}^2 \oplus \mathbb{Z}^8 \otimes \mathbb{Z}^2) \cong \mathbb{Z}^{32}.$$

From this we are able to deduce that $G_0 = 0$, and $G_1 \cong \mathbb{Z}^8$. This complies with *Matui's HK-Conjecture*, which says that $K_1(\mathcal{A}(\mathbb{F}_2^3)) \cong H_1(\mathcal{D}_3) \oplus H_3(\mathcal{D}_3)$ (see 4.2.2). ♦

4.1.2 In the example above (and 4.3.1), the groups $\operatorname{coker}(\partial_1)$ have been torsion-free, allowing for the deployment of the *Künneth Theorem* 1.2.18. This will not be the case for general

k-domino groups, not even for products of free groups with more than two generators. Indeed, it is notoriously hard to find higher-rank graph algebras whose K-theory we can determine, and which are not obvious products of lower-rank algebras.

One other clue we can follow is the next useful proposition from [Eva08, 4.1]:

4.1.3 Proposition (Evans, 2008) *Fix $k \geq 2$, and let Λ be a k-rank graph which is row-finite, has no sources, and has a finite number of vertices. Then the torsion-free rank (see 2.2.5) of $K_0(\mathcal{A}(\Lambda))$ is equal to that of $K_1(\mathcal{A}(\Lambda))$.* ■

4.1.4 Example (Product of three free groups of order 3) Now consider the product $\Gamma = \mathbb{F}_3^3$ of three free groups, each with three generators: this is a 3-cube group whose corresponding cube complex has as universal cover $T(6) \times T(6) \times T(6)$. We again construct the chain complex $\mathcal{D}_3$ from 3.4.8 using the three 216×216 adjacency matrices, to find that:

- $\operatorname{coker}(\partial_1) \cong \mathbb{Z}^{27} \oplus (\mathbb{Z}/2)^{37}$,
- $\ker(\partial_1)/\operatorname{im}(\partial_2) \cong \mathbb{Z}^{81} \oplus (\mathbb{Z}/2)^{74}$,
- $\ker(\partial_2)/\operatorname{im}(\partial_3) \cong \mathbb{Z}^{81} \oplus (\mathbb{Z}/2)^{37}$,
- $\ker(\partial_3) \cong \mathbb{Z}^{27}$.

Hence we have a short exact sequence

$$0 \longrightarrow \frac{\mathbb{Z}^{27} \oplus (\mathbb{Z}/2)^{37}}{G_0} \longrightarrow K_0(\mathcal{A}(\Gamma)) \longrightarrow \mathbb{Z}^{81} \oplus (\mathbb{Z}/2)^{37} \longrightarrow 0, \tag{4.2}$$

for some $G_0 \subseteq \mathbb{Z}^{27} \oplus (\mathbb{Z}/2)^{37}$, and an isomorphism $K_1(\mathcal{A}(\Gamma)) \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2)^{74}$, where $r \in \{81, \ldots, 108\}$. We deduce from (4.2) and the fact from 4.1.3 that K_0 and K_1 must have the same rank r.

Write A, B, C for the adjacency structure of Γ. By 3.2.3, the three subgroups of Γ isomorphic to $\mathbb{F}_3^2$, obtained by removing one of A, B or C from the generating set, are each 2-domino groups (or *BM-groups*). The 3-domino group Γ is a free product with amalgamation of these three groups (by 3.2.4). The K-theory of their induced k-rank graph algebras is given by

$$K_0\big(\mathcal{A}\big(\Lambda\big(\mathbb{F}_3^2\big)\big)\big) \cong K_1\big(\mathcal{A}\big(\Lambda\big(\mathbb{F}_3^2\big)\big)\big) \cong \mathbb{Z}^{18} \oplus (\mathbb{Z}/2)^7.$$

Compare this to the K-theory of the k-rank graph algebra induced by Γ, calculated above. There is no immediately obvious structure inherited by the K-theory of $\mathcal{A}(\Gamma)$ from the K-theory of the $C^\star$-algebras induced by its 2-domino subgroups. In 4.5.3, we investigate how higher-rank domino groups can be obtained from those of lower rank by taking the direct product with a free group, and we discuss how this impacts the K-theory of the domino graph algebras. ♦

4.1.5 A direct product Γ of k free groups, each of which has at least two generators, is always a k-domino group. This follows from 3.2.10, since Γ acts freely and transitively on its Cayley graph which is the 1-skeleton of a product of trees. The groups from 3.3.1 and 3.3.2 don't arise in this way, however. Rather, they are constructed by means of an algorithm first developed in [RSV19, 2.8], and which we have implemented using `Python` in 5.1.2–5.1.3. The algorithm unfolds as follows:

4.1.6 Rungtanapirom–Stix–Vdovina k-domino group Algorithm Let q be a prime number, or a power of a prime number, and consider the field $\mathbb{K} = \mathbb{K}_{q^2}$ with q^2 elements. Let δ be a generator of $\mathbb{K} \setminus \{0\}$ as multiplicative group, and let $i, j \in A := \mathbb{Z}/(q^2-1)$ be such that $i \neq j \bmod (q-1)$. There is a unique element $x_{ij} \in A$ which satisfies $\delta^{x_{ij}} = 1 + \delta^{j-i}$. There is another unique element $y_{ij} := x_{ij} + i - j$ for each x_{ij}, such that $\delta^{y_{ij}} = 1 + \delta^{i-j}$.

Now, define $s, t \in A$ by $s(i,j) := i - x_{ij}(q-1)$ and $t(i,j) := j - y_{ij}(q-1)$. Consider a set X indexed by elements of k-many cosets of $A/(q-1)$, that is,

$$X := \{a_i \mid i \in \mathbb{Z}/(q^2-1), \text{ and } i = j \bmod (q-1) \text{ for some } j \in J\},$$

where $J \subseteq \{0, \ldots, q-2\}$ is a set of size k. Then, define the group $\Gamma_{J,\delta}$ by

$$\Gamma_{J,\delta} := \langle a_i \in X \mid a_{i+(q^2-1)/2} a_i = 1 \text{ and } a_i a_j a_{t(i,j)} a_{s(i,j)} = 1, \text{ for all } i, j \rangle, \tag{4.3}$$

whenever q is odd, and

$$\Gamma_{J,\delta} := \langle a_i \in X \mid a_i a_i = 1 \text{ and } a_i a_j a_{t(i,j)} a_{s(i,j)} = 1, \text{ for all } i, j \rangle \tag{4.4}$$

if q is even. Then $\Gamma_{J,\delta}$ is a k-domino group, with adjacency structure given by the k-many sets of the form $\{a_i \mid i = j \bmod (q-1)\}$, one for each $j \in J$.

4.1.7 Using the Smith normal form to compute cellular homology To any $m \times n$ matrix M with integer entries, we can associate a unique diagonal matrix Smith(M) (not necessarily square), whose diagonal entries $\alpha_1, \ldots, \alpha_s$ are called the elementary divisors of M. Write rk(M) to denote the *rank* of the matrix M; different from the *torsion-free rank* of an Abelian group, the rank of a matrix is the dimension of the vector space spanned by its columns. Then $\mathrm{rk}(M) \leq \min(m,n)$ and the elementary divisors are non-negative integers which satisfy $\alpha_i \mid \alpha_{i+1}$ for each $i < \mathrm{rk}(M)$, and $\alpha_i = 0$ otherwise. The cokernel of Smith(M) is isomorphic to the cokernel of M.

Now, presume that $\mathcal{M}$ is a k-dimensional cell complex with the property that the boundary of each n-dimensional cell with $n \geq 2$ contains no $(n-1)$-dimensional face more than once. Then from (2.9) we have the chain complex

$$\cdots \longrightarrow 0 \xrightarrow{\delta_{k+1}} C_k \xrightarrow{\delta_k} \cdots \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \longrightarrow 0,$$

and the homology groups are given by $H_n(\mathcal{M}) := \ker(\delta_n)/\mathrm{im}(\delta_{n+1})$. The boundary maps δ_n are $|C_{n-1}| \times |C_n|$ matrices. Then there are isomorphisms

$$H_n(\mathcal{M}) \cong \mathbb{Z}^{|C_n| - \mathrm{rk}(\delta_n) - \mathrm{rk}(\delta_{n+1})} \oplus \bigoplus_{i=1}^{\mathrm{rk}(\delta_{n+1})} \mathbb{Z}/\alpha_i,$$

where α_i are the non-zero elementary divisors of δ_{n+1} (see, for example, [DHSW03] for details). Observe that, for a k-dimensional complex, H_k will be torsion-free, since δ_{k+1} is the zero map.

4.1.8 Example (3.3.1 revisited) Recall the group $\Gamma = \Gamma_{\{3,5,7\}}$ from 3.3.1. This example of a 3-domino group can be constructed via the *RSV Algorithm* (4.1.6), and we compute the groups from 3.4.8 in order to glean some insight into the 3-rank graph algebra $\mathcal{A}(\Gamma)$.

To obtain $\mathrm{coker}(\partial_1)$, it suffices to list the elementary divisors of ∂_1, since the cokernel of a linear map is isomorphic to the cokernel of its Smith normal form. By using `MAGMA`, we learn that $\mathrm{Smith}(\partial_1)$ is a 192×576 diagonal matrix with entries

$$\underbrace{1, \ldots, 1}_{\text{182 times}}, 4, 4, 12, \underbrace{0, \ldots, 0}_{\text{7 times}}.$$

Then $\mathrm{im}(\mathrm{Smith}(\partial_1)) \subset \mathbb{Z}^{192}$ is a subspace isomorphic to $\mathbb{Z}^{182} \oplus (4\mathbb{Z})^2 \oplus 12\mathbb{Z}$, and hence

$$\begin{aligned} \mathrm{coker}(\partial_1) &= \mathbb{Z}^{192}/\mathrm{im}(\mathrm{Smith}(\partial_1)) \\ &\cong \mathbb{Z}^7 \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/12) \cong \mathbb{Z}^7 \oplus (\mathbb{Z}/3) \oplus (\mathbb{Z}/4)^3. \end{aligned}$$

Similarly, we compute $\ker(\partial_3) \cong \mathbb{Z}^7$. Then, by writing the maps as homomorphisms (see lines 270–288 in 5.1.6), we can ask `MAGMA` to compute:

- $\ker(\partial_1)/\mathrm{im}(\partial_2) \cong \mathbb{Z}^{21} \oplus (\mathbb{Z}/2)^6 \oplus (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/4)^4$,
- $\ker(\partial_2)/\mathrm{im}(\partial_2) \cong \mathbb{Z}^{21} \oplus (\mathbb{Z}/3) \oplus (\mathbb{Z}/4)^3$.

Then the short exact sequence from 3.4.8 is

$$0 \longrightarrow \frac{\mathbb{Z}^7 \oplus (\mathbb{Z}/3) \oplus (\mathbb{Z}/4)^3}{G_0} \longrightarrow K_0(\mathcal{A}(\Gamma)) \longrightarrow \mathbb{Z}^{21} \oplus (\mathbb{Z}/3) \oplus (\mathbb{Z}/4)^3 \longrightarrow 0,$$

and there is also an isomorphism

$$K_1(\mathcal{A}(\Gamma)) \cong \mathbb{Z}^{21} \oplus (\mathbb{Z}/2)^6 \oplus (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/4)^4 \oplus G_1,$$

for some subgroups $G_0 \subseteq \mathbb{Z}^7 \oplus (\mathbb{Z}/3) \oplus (\mathbb{Z}/4)^3$ and $G_1 \subseteq \mathbb{Z}^7$. From this and 4.1.3, we can deduce that the torsion-free part of K_0 is isomorphic to $\mathbb{Z}^r$, and that $K_1 \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2)^6 \oplus (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/4)^4$, for some $r \in \{21, \ldots, 28\}$.

We also calculate the cellular homology of the domino complex $\mathcal{M}(\Gamma)$ through 4.1.7 and 4.1.9 to be as follows:

$$H_n(\mathcal{M}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/4)^2 & \text{if } n = 1, \\ (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3) \oplus (\mathbb{Z}/4) & \text{if } n = 2, \\ \mathbb{Z}^7 & \text{if } n = 3, \\ 0 & \text{if } n \geq 4. \end{cases}$$

♦

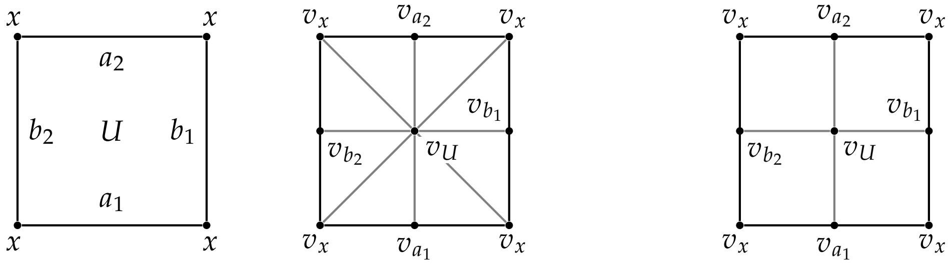
4.1.9 In its present state, the cube complex $\mathcal{M}(\Gamma_{\{3,5,7\}})$ from 4.1.8 does not satisfy the conditions of 2.6.1 since, for example, it contains the geometric square $(a_1, c_2, \bar{a}_1, c_3)$. This would be sent to $a_1 + c_2 - a_1 + c_3 = c_2 + c_3$ by the map δ_2, which does not accurately describe its boundary. This situation is likely to arise given a general domino complex $\mathcal{M}$—to overcome it we may consider the *barycentric subdivision* of $\mathcal{M}$. Traditionally, this is a simplicial complex formed by taking $\mathcal{M}$, adding a vertex v_X to the centre of each cell $X \in \mathcal{M}$ (identifying $v_x = x$ if x is a vertex in $\mathcal{M}$), and drawing an edge between two vertices v_X and v_Y whenever Y is a lower-dimensional cell contained in X, unless an edge already exists (Figure 4.1a).

For our purposes, however, it is sufficient to include an edge between v_X and v_Y only when the dimension of Y is precisely one less than the dimension of X. In this way, the subdivided complex remains a cube complex, which we denote by $\mathcal{M}'$ (Figure 4.1b). The barycentric subdivision of $\mathcal{M}'$ is *homotopy equivalent* to $\mathcal{M}$, meaning in particular that its cellular homology groups are unchanged. Furthermore, it is clear that $\mathcal{M}'$ contains no loops, nor cells which have repeated faces on their boundary; we can therefore apply 2.6.1 and 4.1.7 on $\mathcal{M}'$ to find the homology of any domino complex $\mathcal{M}$.

4.1.10 Example (3.3.2 revisited) Recall now the group $\Gamma = \Gamma'_{\{2,3,4\}}$ from 3.3.2; it has nine generators, which together label 108 2-dimensional dominoes, and 216 3-dimensional dominoes (depicted in Figure 3.4).

As in Example 4.1.8, we use `MAGMA` to compute the relevant kernels and cokernels from 3.4.8, culminating with:

- $\operatorname{coker}(\partial_1) \cong \mathbb{Z}^9 \oplus (\mathbb{Z}/2) \oplus (\mathbb{Z}/4) \oplus (\mathbb{Z}/5)^2 \oplus (\mathbb{Z}/16)$,
- $\ker(\partial_3) \cong \mathbb{Z}^9$,
- $\ker(\partial_1)/\operatorname{im}(\partial_2) \cong \mathbb{Z}^{27} \oplus (\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/8)^2$,
- $\ker(\partial_2)/\operatorname{im}(\partial_3) \cong \mathbb{Z}^{27} \oplus (\mathbb{Z}/2) \oplus (\mathbb{Z}/4) \oplus (\mathbb{Z}/5)^2 \oplus (\mathbb{Z}/16)$.



(a) On the left is a 2-cell U which might occur in a domino complex $\mathcal{M}$. On the right is U as it would appear after classical barycentric subdivision.

(b) Here is the image of U in $\mathcal{M}'$. Vertices v_X and v_Y are joined by an edge in the barycentric subdivision only when $\dim Y = \dim X - 1$.

Figure 4.1: Diagram showing the two notions of *barycentric subdivision* of a geometric square U from 4.1.9. In (a), U is subdivided until it becomes a simplicial complex, whereas in (b), it is divided into a cube complex; the definition corresponding to (b) is the one we use. The barycentric subdivision $\mathcal{M}'$ of a domino complex $\mathcal{M}$ has no loops, even if $\mathcal{M}$ does.

Then we obtain the short exact sequence

$$0 \longrightarrow \frac{\mathbb{Z}^9 \oplus \mathcal{G}}{G_0} \longrightarrow K_0(\mathcal{A}(\Gamma)) \longrightarrow \mathbb{Z}^{27} \oplus \mathcal{G} \longrightarrow 0,$$

where $\mathcal{G} = (\mathbb{Z}/2) \oplus (\mathbb{Z}/4) \oplus (\mathbb{Z}/5)^2 \oplus (\mathbb{Z}/16)$, and there is an isomorphism

$$K_1(\mathcal{A}(\Gamma)) \cong \mathbb{Z}^{27} \oplus (\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/8)^2 \oplus G_1,$$

for some groups $G_0 \subseteq \mathbb{Z}^9 \oplus \mathcal{G}$ and $G_1 \subseteq \mathbb{Z}^9$. The torsion-free rank of K_0 is r, and $K_1 \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/8)^2$, for some $r \in \{27, \ldots, 36\}$. The K_1 group in particular is distinct from those of Examples 4.1.4 and 4.1.8, so we may use Theorem 3.5.4 to conclude that the 3-rank graph $C^\star$-algebras induced by each of the domino complexes are non-isomorphic.

We now calculate the cellular homology of the domino complex $\mathcal{M}(\Gamma)$, via the barycentric subdivision $\mathcal{M}'$. We find that:

$$H_n(\mathcal{M}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ (\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/5)^2 & \text{if } n = 1, \\ (\mathbb{Z}/4) \oplus (\mathbb{Z}/5)^2 \oplus (\mathbb{Z}/8) & \text{if } n = 2, \\ \mathbb{Z}^9 & \text{if } n = 3, \\ 0 & \text{if } n \geq 4. \end{cases}$$

♦

4.1.11 Example (A 3-domino group arising from 4.1.6 with $q = 7$) Let us now perform the *RSV Algorithm* (4.1.6) with input parameter $q = 7$. As a set, we may regard the finite field $\mathbb{K} = \mathbb{K}_{49}$ with 49 elements as the set of polynomials $\{\alpha + \beta z \mid \alpha, \beta \in \mathbb{Z}_7\}$. We use the `Python` package `finitefield` from [Kun14] along with the algorithm given in 5.1.2 to find that $\delta = 2+z \in \mathbb{K}$ is a primitive element of $\mathbb{K}$, that is, a generator of $\mathbb{K} \setminus \{0\}$ as a multiplicative group. Then a subset of integers $J \subseteq \{0, \ldots, 5\}$ of size k defines a k-domino group $\Gamma_{J,\delta}$ by 4.1.6—in this example we choose $J = \{0, 1, 5\}$ such that $\Gamma = \Gamma_{\{0,1,5\},\delta}$ is a 3-domino group. Our implementation of the algorithm in `Python` outputs the following list of lists of elements of $\mathbb{Z}/48$:

```
[[0,1,42,1], [0,5,18,5], [0,7,6,7], [0,11,42,23], [0,13,0,43], [0,17,6,29],
[0,19,36,25], [0,23,42,11], [0,25,36,19], [0,29,6,17], [0,31,12,37],
[0,35,30,35], [0,37,12,31], [0,41,0,47], [1,5,31,11], [1,6,19,6], [1,11,31,5],
[1,17,25,41], [1,18,7,30], [1,23,19,29], [1,29,19,23], [1,30,7,18],
[1,35,1,47], [1,36,31,36], [5,6,47,6], [5,7,41,7], [5,12,11,12], [5,13,29,37],
[5,19,17,19], [5,36,17,42], [5,42,17,36], [6,13,12,13], [6,23,12,35],
[6,35,12,23], [6,37,18,43], [6,41,36,41], [6,43,18,37], [7,11,37,17],
[7,17,37,11], [7,23,31,47], [7,42,37,42], [11,13,47,13], [11,18,17,18],
[11,19,35,43], [12,19,18,19], [12,47,42,47], [13,17,43,23], [13,23,43,17]]
```

For each integer i that appears above, we let a_i be a generator for Γ, and each 4-tuple (i, j, t, s) defines a relation $a_i a_j a_t a_s = 1$, as in (4.3). Finally, we identify $\bar{a}_i$ with $a_{i+24 \bmod 48}$, so that Γ can be expressed as a group with 12 generators and 48 relations:

$$\Gamma_{\{0,1,5\},\delta} = \left\langle a_0, a_6, a_{12}, a_{18}, a_1, a_7, a_{13}, a_{19}, a_5, a_{11}, a_{17}, a_{23} \;\middle|\; a_0 a_1 \bar{a}_{18} a_1,\ a_0 a_5 a_{18} a_5, \ldots \right\rangle.$$

We can then feed these relations into Programs 5.1.4 and 5.2.2, which output:

- $\operatorname{coker}(\partial_1) \cong \mathbb{Z}^{28} \oplus (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/4) \oplus (\mathbb{Z}/7)^2 \oplus (\mathbb{Z}/9)$,
- $\ker(\partial_3) \cong \mathbb{Z}^{28}$,
- $\ker(\partial_1)/\operatorname{im}(\partial_2) \cong \mathbb{Z}^{84} \oplus (\mathbb{Z}/2)^8 \oplus (\mathbb{Z}/3)^8$,
- $\ker(\partial_2)/\operatorname{im}(\partial_3) \cong \mathbb{Z}^{84} \oplus (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/4) \oplus (\mathbb{Z}/7)^2 \oplus (\mathbb{Z}/9)$,

and

$$H_n(\mathcal{M}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ (\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/7)^2 & \text{if } n = 1, \\ (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3) \oplus (\mathbb{Z}/7)^2 \oplus (\mathbb{Z}/9) & \text{if } n = 2, \\ \mathbb{Z}^{28} & \text{if } n = 3, \\ 0 & \text{if } n \geq 4. \end{cases}$$

Then $K_0(\mathcal{A}(\Gamma))$ satisfies

$$0 \longrightarrow \frac{\mathbb{Z}^{27} \oplus \mathcal{G}}{G_0} \longrightarrow K_0(\mathcal{A}(\Gamma)) \longrightarrow \mathbb{Z}^{84} \oplus \mathcal{G} \longrightarrow 0,$$

where $\mathcal{G}$ is the finite part of $\operatorname{coker}(\partial_1)$, and $K_1(\mathcal{A}(\Gamma))$ satisfies

$$K_1(\mathcal{A}(\Gamma)) \cong \mathbb{Z}^{84} \oplus (\mathbb{Z}/2)^8 \oplus (\mathbb{Z}/3)^8 \oplus G_1,$$

for some groups $G_0 \subseteq \mathbb{Z}^{27} \oplus \mathcal{G}$ and $G_1 \subseteq \mathbb{Z}^{27}$. Then, similarly to in Examples 4.1.4, 4.1.8 and 4.1.10, we can deduce that $K_0 \cong \mathbb{Z}^r \oplus \mathcal{G}'$ and $K_1 \cong \mathbb{Z}^r \oplus (\mathbb{Z}/6)^8$, for some $r \in \{84, \ldots, 111\}$, and some finite Abelian group $\mathcal{G}'$. ♦

4.1.12 Even given an initial parameter q and a primitive element δ of $\mathbb{K}_{q^2}$, it should be noted that the choice of subset $J \subseteq \{0, \ldots, q-2\}$ still affects the group produced by the *RSV Algorithm*. For example, with an identical set-up to 4.1.11, picking $J = \{2, 4, 5\}$ leads to a group $\Gamma_{\{2,4,5\},\delta}$ whose associated domino complex has identical cellular homology to that of $\Gamma_{\{0,1,5\},\delta}$, but whose domino graph algebra is different. Namely, the torsion part of $K_1\big(\mathcal{A}\big(\Gamma_{\{2,4,5\},\delta}\big)\big)$is isomorphic to $(\mathbb{Z}/6)^8$, and the group denoted by $\mathcal{G}$ in 4.1.11 is instead $(\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/7)^2 \oplus (\mathbb{Z}/9)$.

4.1.13 For each of the 3-domino groups Γ above, we computed two things: the exact sequences of Evans' from 3.4.8, and the cellular homology of the complex $\mathcal{M}(\Gamma)$. These calculations require the manipulation of large matrices and homomorphisms, for which we used computer algebra software `MAGMA`. We wrote algorithms in `Python3` which compute the matrices in a manner which `MAGMA` can recognise: a technique called *metaprogramming*. Most of these codes have been reproduced in Chapter 5, and any which have not are available from the author upon request.

§4.2 The *HK-Conjecture*

4.2.1 Kumjian and Pask in [KP00] constructed a *groupoid* $\mathcal{G}_\Lambda$ (that is, a category in which every morphism is invertible) for each higher-rank graph (Λ, d). To each groupoid $\mathcal{G}$ can be associated a (reduced) $C^\star$-algebra $C^\star_r(\mathcal{G})$ by a method of [Ren80], and in [KP00, 3.5] it was also shown that $\mathcal{A}(\Lambda) \cong C^\star_r(\mathcal{G}_\Lambda)$.

Then, in [FKPS19] it was shown that the homology of the chain complex $\mathcal{D}_k$ from 3.4.7 is equivalent to the *homology of the groupoid* $\mathcal{G}_\Lambda$ (see, for example, [Mat11; LSV20, §7]). When (Λ, d) is a 1- or 2-rank graph, then the isomorphism (4.5) below holds. Furthermore, we know from [FKPS19, 7.9] that any 3-rank graph which satisfies the criteria of Corollary 3.4.9(ii) is also one for which the *HK-Conjecture* is verified.

4.2.2 Matui's HK-Conjecture *Let $\mathcal{G}$ be a groupoid which satisfies the conditions of [Mat16, 3.5]. Then there is an isomorphism*

$$K_\epsilon(C^\star_r(\mathcal{G})) \cong \bigoplus_{p=0}^{\infty} H_{2p+\epsilon}(\mathcal{G}).$$

4.2.3 With the considerations in 4.2.1, we can rephrase 4.2.2 in the context of higher-rank graphs: if (Λ, d) is a k-rank graph such that $\mathcal{G}_\Lambda$ is a groupoid which satisfies the same conditions of [Mat16, 3.5], then

$$K_\epsilon(\mathcal{A}(\Lambda)) \cong \bigoplus_{p=0}^{\infty} H_{2p+\epsilon}(\mathcal{D}_k), \tag{4.5}$$

where $\mathcal{D}_k$ is the chain complex from 2.2.6.

4.2.4 It is not known precisely under which conditions Conjecture 4.2.2 holds, but it is known not to be true in general, as counterexamples have been found (for example, by Scarparo in [Sca20]).

Were the conjecture verified in the case of 3-domino graph algebras $\mathcal{A}(\Gamma)$, then we would have an isomorphism $K_1(\mathcal{A}(\Gamma)) \cong H_1(\mathcal{D}_3) \oplus H_3(\mathcal{D}_3)$. In particular, this would mean that the torsion-free ranks r of $K_\epsilon(\mathcal{A}(\Gamma))$ would be maximal in their respective ranges, which were established in each of the examples in §4.1. We would then have a precise characterisation of K_1 in each case.

§4.3 Higher dimensions and limitations

Using the *RSV Algorithm*, we can construct sets of relations for k-domino groups. As we discussed in 3.4.10, however, the quality of information we are able to deduce from *Evans' Theorem* about k-domino graph algebras diminishes sharply as k increases.

4.3.1 Example (Product of k-many free groups of order 2) Now fix $k \geq 2$ and consider the group $\Gamma = \mathbb{F}_2^k$: the product of k-many free groups of order two. This can be presented with generating set $E = \bigsqcup_{p=1}^k E_p$ where $E_p = \{x_p^1, x_p^2\}$, and relation set

$$R = \left\{ x_p^i x_q^j \bar{x}_p^i \bar{x}_q^j \;\middle|\; p, q \in \{1, \ldots, k\} \text{ with } p \neq q, \text{ and } i, j \in \{1, 2\} \right\}.$$

It is a k-domino group, and k-dimensional dominoes in Γ will look like k-dimensional cubes where parallel edges all have the same label and orientation—as such, there will be $4^k = 2^{2k}$ elements of $\mathcal{S}_k(\Gamma)$. By iterating the arguments employed in 4.1.1, we find that $K_0(\mathcal{A}(\Gamma)) \cong K_1(\mathcal{A}(\Gamma)) \cong \mathbb{Z}^{2k-1}$.

By using the algorithm in 5.1.4, we can write down the adjacency matrices $M_1, \ldots, M_k$ which correspond to the set $\mathcal{S}_k(\Gamma)$. When $k = 4$, for example, these matrices will have dimension 256×256. Then the construction from 3.4.11 begets sequences:

$$\text{(i)} \quad 0 \longrightarrow \mathbb{Z}^{16}/G_0 \longrightarrow K_0(\mathcal{A}(\mathbb{F}_2^4)) \longrightarrow \frac{K_0(\mathcal{A}(\mathbb{F}_2^4))}{\mathbb{Z}^{16}/G_0} \longrightarrow 0,$$

$$\text{(ii)} \quad 0 \longrightarrow \mathbb{Z}^{16}/G_0 \longrightarrow F_2 \longrightarrow \mathbb{Z}^{96} \longrightarrow 0,$$

$$\text{(iii)} \quad 0 \longrightarrow F_2 \longrightarrow K_0(\mathcal{A}(\mathbb{F}_2^4)) \longrightarrow G_1 \longrightarrow 0,$$

$$\text{(iv)} \quad 0 \longrightarrow \mathbb{Z}^{64}/G_2 \longrightarrow K_1(\mathcal{A}(\mathbb{F}_2^4)) \longrightarrow G_3 \longrightarrow 0,$$

for some subgroups $G_0, G_1 \subseteq \mathbb{Z}^{16}$ and $G_2, G_3 \subseteq \mathbb{Z}^{64}$. Since $K_0 \cong K_1 \cong \mathbb{Z}^{128}$ and each group in the above sequences is free Abelian, we can deduce that $G_2 \cong 0$, $G_3 \cong \mathbb{Z}^{64}$, and $G_0 \oplus G_1 \cong \mathbb{Z}^{16}$. ♦

4.3.2 Example (A 4-domino group) We can perform the *RSV Algorithm* to construct a k-domino group for arbitrary $k \geq 2$. For this example, consider $k = 4$ and $q = 7$, so that we can recycle the primitive element $\delta = 2 + z \in \mathbb{K}_{49}$ from 4.1.11. Then choosing a subset $J \subset \{0, \ldots, 5\}$ of size four defines a 4-domino group $\Gamma_{J,\delta}$. We appoint $J = \{1, 2, 3, 4\}$ this time. The algorithm outputs a group $\Gamma = \Gamma_{\{1,2,3,4\},\delta} = \langle X \mid R \rangle$ with generators

$$X = \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3\},$$

and 54 relations. We have renamed the generators in order to better emphasise the adjacency structure. Then Γ has relations:

$a_1b_1\bar{a}_3b_1,\ a_1\bar{b}_1\bar{a}_2b_3,\ a_1b_2a_2b_2,\ a_1\bar{b}_2a_3\bar{b}_3,\ a_1b_3\bar{a}_2\bar{b}_1,\ a_1\bar{b}_3a_3\bar{b}_2,\ a_2b_3a_3b_3,\ a_3b_1\bar{a}_2b_2,\ a_3b_2\bar{a}_2b_1$
$a_1c_1\bar{a}_2c_1,\ a_1\bar{c}_1a_1\bar{c}_3,\ a_1c_2\bar{a}_1\bar{c}_2,\ a_1c_3a_3c_3,\ a_2c_1a_2\bar{c}_2,\ a_2c_2\bar{a}_3c_2,\ a_2c_3\bar{a}_2\bar{c}_3,\ a_3\bar{c}_1\bar{a}_3c_1,\ a_3c_2a_3\bar{c}_3$
$a_1d_1\bar{a}_3d_3,\ a_1\bar{d}_1a_2d_2,\ a_1d_2a_2\bar{d}_1,\ a_1\bar{d}_2a_1\bar{d}_3,\ a_1d_3\bar{a}_3d_1,\ a_2d_1a_2\bar{d}_3,\ a_2\bar{d}_2a_3d_3,\ a_2d_3a_3\bar{d}_2,\ a_3d_1a_3d_2$
$b_1c_1\bar{b}_3c_1,\ b_1\bar{c}_1\bar{b}_2c_3,\ b_1c_2b_2c_2,\ b_1\bar{c}_2b_3\bar{c}_3,\ b_1c_3\bar{b}_2\bar{c}_1,\ b_1\bar{c}_3b_3\bar{c}_2,\ b_2c_3b_3c_3,\ b_3c_1\bar{b}_2c_2,\ b_3c_2\bar{b}_2c_1,$
$b_1d_1\bar{b}_2d_1,\ b_1\bar{d}_1b_1\bar{d}_3,\ b_1d_2\bar{b}_1\bar{d}_2,\ b_1d_3b_3d_3,\ b_2d_1b_2\bar{d}_2,\ b_2d_2\bar{b}_3d_2,\ b_2d_3\bar{b}_2\bar{d}_3,\ b_3\bar{d}_1\bar{b}_3d_1,\ b_3d_2b_3\bar{d}_3$
$c_1d_1\bar{c}_3d_1,\ c_1\bar{d}_1\bar{c}_2d_3,\ c_1d_2c_2d_2,\ c_1\bar{d}_2c_3\bar{d}_3,\ c_1d_3\bar{c}_2\bar{d}_1,\ c_1\bar{d}_3c_3\bar{d}_2,\ c_2d_3c_3d_3,\ c_3d_1\bar{c}_2d_2,\ c_3d_2\bar{c}_2d_1.$

4.3.3 In Example 4.3.2, Γ is a 4-domino group whose corresponding adjacency matrices are of size $6^4 \times 6^4$. The version of `MAGMA` we have been using has a limit on the dimensions of the input, which these matrices exceed. To overcome this, it might be possible to instead consider the cellular homology of the associated domino complex—it is much less expensive to perform computations on the boundary matrices δ_n of 2.6.1 than it is to work with the homologies of 2.2.6, and we now turn to discuss the potential connections between these two invariants.

§4.4 The geometry of k-domino complexes

4.4.1 At this point, let us re-examine the examples of 3-domino groups in §4.1, namely 4.1.10–4.1.11. In each instance, we used the algorithms from Chapter 5 to compute the differential maps of Theorem 3.4.8 and the cellular homology groups of the corresponding 3-domino complex. We notice that in every example the groups $H_3(\mathcal{M}(\Gamma))$ and $\ker \partial_3$ are isomorphic, infinite Abelian groups with the same torsion-free rank as that of $\operatorname{coker} \partial_1$. Indeed, a theorem of [Rob05] establishes that $H_2(\mathcal{M}(\Gamma)) \cong \ker \partial_2$ whenever Γ provides a 2-domino group. While Theorem 4.4.3 is a k-dimensional generalisation of this result, we firstly consider the case where $k = 3$, which allows us to make use of diagrams and the terminology of the symmetry of cubes for clarity.

4.4.2 Theorem (Third homology of a 3-domino complex) *Let Γ be a 3-domino group with adjacency structure E_1, E_2, E_3, let M_1, M_2, M_3 be the adjacency matrices from 3.1.16, and let $\mathcal{M}$ be the associated 3-dimensional domino complex formed by gluing together adjacent dominoes. Recall the map $\partial_1 : (\mathbb{Z}\mathcal{S}_3)^3 \to \mathbb{Z}\mathcal{S}_3$ from 3.4.7, and consider its matrix transpose $\partial_1^T = [\mathbf{1} - M_1, \mathbf{1} - M_2, \mathbf{1} - M_3]^T$. Then $H_3(\mathcal{M}) \cong \ker\left(\partial_1^T\right)$.*

■ **Proof** Write $\mathcal{S}_2$ for the set of all pointed squares, and $\mathcal{S}_3$ for the set of all 3-dimensional dominoes (pointed and oriented cubes) in Γ. Write $F = [x_1, y_1, x_2, y_2]$ to denote a pointed square in $\mathcal{S}_2$, and write

$$X = [A_1(X), B_1(X), C_1(X), A_2(X), B_2(X), C_2(X)]$$

to denote a 3-dimensional domino in $\mathcal{S}_3$. Assume that M_1, M_2, M_3 are the matrices describing adjacency in directions parallel to reflection in the H, V and I directions, respectively, according to Figure 3.2. Then $M_1(X, Y) = 1$ only if $C_1(Y) = C_2(X)_H$, $M_2(X, Y) = 1$ only if $B_1(Y) = B_2(X)_H$, and $M_3(X, Y) = 1$ only if $A_1(Y) = A_2(X)_H$, and as long as no edges cancel out after identifying adjacent dominoes—that is, 3.1.16(i)–(ii) are satisfied. Define three functions $T_i : \mathbb{Z}\mathcal{S}_3 \to \mathbb{Z}\mathcal{S}_3$ by:

$$\begin{aligned} T_1(X) &:= \sum_{M_1(X,Y)=1} Y = -X_H + \sum_{C_2(X)_H = C_1(Y)} Y, \\ T_2(X) &:= \sum_{M_2(X,Y)=1} Y = -X_V + \sum_{B_2(X)_H = B_1(Y)} Y, \\ T_3(X) &:= \sum_{M_3(X,Y)=1} Y = -X_I + \sum_{A_2(X)_H = A_1(Y)} Y. \end{aligned} \tag{4.6}$$

Now define a family of functions $\psi_n : \mathcal{S}_n' \to \mathcal{S}_n$ for $n \in \{1, 2, 3\}$ which selects a representative element from each coset of $\mathcal{S}_n' \cong \mathcal{S}_n/(\mathbb{Z}/2)^n$, that is, a function which takes a geometric n-cell of $\mathcal{M}$ and gives it a basepoint and orientation, turning it into an n-dimensional domino. Denote these maps by $\psi_n : X' \mapsto X$, and write $\mathcal{S}_n^+ := \psi_n\left(\mathcal{S}_n'\right)$. In the instance where $n = 1$, the set $\mathcal{S}_1^+ \subset (E_1 \sqcup E_2 \sqcup E_3)$ can be written as $\mathcal{S}_1^+ := \{x \mid x, \bar{x} \in (E_1 \sqcup E_2 \sqcup E_3)$ define the same geometric edge$\}$. Then define $\varphi : \mathbb{Z}\mathcal{S}_3^+ \to \mathbb{Z}\mathcal{S}_3$ by:

$$\varphi(X) := X - X_H - X_V - X_I + X_{HV} + X_{HI} + X_{VI} - X_{HVI},$$

where $X = \psi_3(X')$. We have

$$\begin{aligned}
(\mathbf{1} - T_1)(X) &= X_H + X - \sum_{C_2(X)_H = C_1(Y)} Y,\\
(\mathbf{1} - T_1)(X_H) &= X + X_H - \sum_{C_2(X_H)_H = C_1(Y)} Y,\\
(\mathbf{1} - T_1)(X_V) &= X_{HV} + X_V - \sum_{C_2(X_V)_H = C_1(Y)} Y,
\end{aligned}$$

and so on, such that

$$\begin{aligned}
(\mathbf{1} - T_1) \circ \varphi(X) = & - \sum_{C_2(X)_H = C_1(Y)} Y + \sum_{C_2(X_H)_H = C_1(Y)} Y\\
& + \sum_{C_2(X_V)_H = C_1(Y)} Y + \sum_{C_2(X_I)_H = C_1(Y)} Y\\
& - \sum_{C_2(X_{HV})_H = C_1(Y)} Y - \sum_{C_2(X_{HI})_H = C_1(Y)} Y\\
& - \sum_{C_2(X_{VI})_H = C_1(Y)} Y + \sum_{C_2(X_{HVI})_H = C_1(Y)} Y,
\end{aligned}$$

for each $X \in \mathcal{S}_3^+$. The formulas are similar for $(\mathbf{1} - T_2) \circ \varphi(X)$ and $(\mathbf{1} - T_3) \circ \varphi(X)$. Now, $\mathcal{S}_2$ can be expressed as the disjoint union

$$\mathcal{S}_2 = \mathcal{S}_2^+ \sqcup \left(\mathcal{S}_2^+\right)_H \sqcup \left(\mathcal{S}_2^+\right)_V \sqcup \left(\mathcal{S}_2^+\right)_{HV}, \tag{4.7}$$

where $\left(\mathcal{S}_2^+\right)_i := \left\{F_i \mid F \in \mathcal{S}_2^+\right\}$. Consider the function $\epsilon_2 : \mathbb{Z}\mathcal{S}_2 \to \mathbb{Z}\mathcal{S}_2^+$ defined by

$$\epsilon_2(A) = \begin{cases} F & \text{if } F \in \mathcal{S}_2^+,\\ -F_H & \text{if } F \in \left(\mathcal{S}_2^+\right)_H,\\ -F_V & \text{if } F \in \left(\mathcal{S}_2^+\right)_V,\\ F_{HV} & \text{if } F \in \left(\mathcal{S}_2^+\right)_{HV}. \end{cases}$$

The elements of each set $\mathcal{S}_n^+$ are in one-to-one correspondence with the n-dimensional cells in $\mathcal{M}$, and we define a boundary map $\delta_2 : \mathbb{Z}\mathcal{S}_3^+ \to \mathbb{Z}\mathcal{S}_2^+$ by

$$\delta_2(X) := \epsilon_2(A_1(X) + B_1(X) + C_1(X) - A_2(X) - B_2(X) - C_2(X)),$$

such that the cellular homology group $H_3(\mathcal{M}) \cong \ker \partial_2$. Now we define another function $\varphi' := \mathbb{Z}\mathcal{S}_2 \to \mathbb{Z}\mathcal{S}_3 \oplus \mathbb{Z}\mathcal{S}_3 \oplus \mathbb{Z}\mathcal{S}_3$ by

$$\varphi'(F) := \begin{cases} 0 \oplus 0 \oplus L(C,F) & \text{if } F \in F(2,3), \\ 0 \oplus L(B,F) \oplus 0 & \text{if } F \in F(1,3), \\ L(A,F) \oplus 0 \oplus 0 & \text{if } F \in F(1,2), \end{cases}$$

where

$$L(t,F) := -\sum_{t_1(Y)=F} Y + \sum_{t_1(Y)=F_H} Y + \sum_{t_1(Y)=F_V} Y - \sum_{t_1(Y)=F_{HV}} Y, \tag{4.8}$$

and $F(p,q)$ is the set of all pointed squares in $\mathcal{S}_2$ with labels from alphabets E_p and E_q. Notice that $\varphi'(F) = \varphi' \circ \epsilon_2(F)$ for all $F \in \mathcal{S}_2$. Then we have the following diagram:

$$\begin{tikzcd} \mathbb{Z}\mathcal{S}_3^+ \arrow[r, "\varphi"] \arrow[d, "\delta_2"'] & \mathbb{Z}\mathcal{S}_3 \arrow[d, "(\mathbf{1}-T) := \begin{bmatrix} \mathbf{1}-T_1 \\ \mathbf{1}-T_2 \\ \mathbf{1}-T_3 \end{bmatrix}"] \\ \mathbb{Z}\mathcal{S}_2^+ \arrow[r, "\varphi'"'] & (\mathbb{Z}\mathcal{S}_3)^3 \end{tikzcd} \tag{4.9}$$

which we claim is commutative. To see why, let $X \in \mathcal{S}_3$, and observe that from (4.8) we get

$$\begin{aligned} L(C, C_1(X) - C_2(X)) &= \sum_{C_1(Y)=C_1(X)} Y - \sum_{C_1(Y)=C_1(X)_H} Y \\ &\quad - \sum_{C_1(Y)=C_1(X)_V} Y + \sum_{C_1(Y)=C_1(X)_{HV}} Y \\ &\quad + \sum_{C_1(Y)=C_2(X)} Y - \sum_{C_1(Y)=C_2(X)_H} Y \\ &\quad - \sum_{C_1(Y)=C_2(X)_V} Y + \sum_{C_1(Y)=C_2(X)_{HV}} Y. \end{aligned}$$

But also

$$\begin{aligned} &C_2(X) = C_2(X), \quad C_2(X)_H = C_2(X_V), \quad C_2(X)_V = C_2(X_I), \quad C_2(X)_{HV} = C_2(X_{VI}), \\ &C_1(X) = C_2(X_{VH}), \quad C_1(X)_H = C_2(X_H), \quad C_1(X)_V = C_2(X_{HVI}), \quad C_1(X)_{HV} = C_2(X_{HI}), \end{aligned}$$

so it follows that $L(C, C_1(X) - C_2(X)) = (\mathbf{1} - T_1) \circ \varphi(X)$ for each $X \in \mathcal{S}_3^+$. Similarly, it can be shown that $L(B, B_1(X) - B_2(X) = (\mathbf{1} - T_2) \circ \varphi(X)$ and $L(A, A_1(X) - A_2(X) = (\mathbf{1} - T_3) \circ \varphi(X)$. Hence $(\mathbf{1} - T) \circ \varphi = \varphi'(A_1(X) + B_1(X) + C_1(X) - A_2(X) - B_2(X) - C_2(X))$, which is to say that the diagram in (4.9) commutes.

At this point, we demonstrate that the homomorphisms φ and φ' are injective. Firstly, the function $\varphi : \mathbb{Z}\mathcal{S}_3^+ \to \mathbb{Z}\mathcal{S}_3$ takes as its argument some 3-dimensional domino $X_\bullet \in \mathcal{S}_3^+$, but

$\mathcal{S}_3^+$ contains exactly one representative from the orbit of each domino $X \in \mathcal{S}_3$. Since $\varphi(X_\bullet)$ is the sum of elements from the same orbit as $X_\bullet$, this means that φ is injective.

To show that φ' is an injection when restricted to $\mathbb{Z}\mathcal{S}_2^+$, define $\xi : (\mathbb{Z}\mathcal{S}_3)^3 \to \mathbb{Z}\mathcal{S}_2^+$ by $\xi(X, Y, Z) := \epsilon_2(C_2(X) + B_2(Y) + A_2(Z))$, such that $\xi \circ \varphi'(F) = nF$ for some $n \in \mathbb{Z}$. By applying ϵ_2 we ensure that $n \neq 0$ for all $F \in \mathcal{S}_2^+$, and so $\xi \circ \varphi' : \mathbb{Z}\mathcal{S}_2^+ \to \mathbb{Z}\mathcal{S}_2^+$ is injective; it follows then that φ' is also injective.

Now, we know that diagram (4.9) commutes and that φ, φ' are injective. Therefore, if $\varphi(X) \in \ker(\mathbf{1} - T)$, then $\varphi' \circ \delta_2(X) = 0$, and so $\delta_2(X) = 0$. In other words, X is a domino which, as a geometric cube, would lie in $H_3(\mathcal{M})$. Conversely, if $X' \in H_3(\mathcal{M})$, then $\delta_2 \circ \varphi' \circ \psi_3(X') = (\mathbf{1} - T) \circ \varphi(X') = 0$, and so $\varphi(X') \in \ker(\mathbf{1} - T)$. Hence there is an isomorphism $H_3(\mathcal{M}) \cong \varphi\left(\mathbb{Z}\mathcal{S}_3^+\right) \cap \ker(\mathbf{1} - T)$ given by the restriction of φ to $H_3(\mathcal{M})$. It is straightforward to check that $\ker(\mathbf{1} - T) \cong \ker\left(\partial_1^T\right)$, so that $H_3(\mathcal{M}) \cong \varphi\left(\mathbb{Z}\mathcal{S}_3^+\right) \cap \ker\left(\partial_1^T\right)$.

Finally, it follows from [Rob05, 4.4] that $\varphi\left(\mathbb{Z}\mathcal{S}_3^+\right) \cap \ker(\mathbf{1} - T) = \ker(\mathbf{1} - T)$, and so we can conclude that $H_3(\mathcal{M}) \cong \ker\left(\partial_1^T\right)$. ■

4.4.3 Theorem (kth homology of a k-domino complex) *Consider a k-domino group Γ with adjacency structure $E_1, \ldots, E_k$. Let $M_1, \ldots, M_k$ be the adjacency matrices from 3.1.16, and let $\mathcal{M}$ be the associated k-dimensional domino complex. Recall the map $\partial_1 : (\mathbb{Z}\mathcal{S}_k)^k \to \mathbb{Z}\mathcal{S}_k$ from 3.4.7, and consider its matrix transpose $\partial_1^T = [\mathbf{1} - M_1, \cdots, \mathbf{1} - M_k]^T$. Then $H_k(\mathcal{M}) \cong \ker\left(\partial_1^T\right)$.*

■ **Proof** The arguments are identical to those in the proof of Theorem 4.4.2, with the following modifications:

As usual, we write $\mathcal{S}_n$ for the set of all n-dimensional dominoes (pointed and oriented n-cubes) in Γ, for each $n \in \{1, \ldots, k\}$. Then we define k-many functions $T_i : \mathbb{Z}\mathcal{S}_k \to \mathbb{Z}\mathcal{S}_k$ analogously to (4.6).

Similarly, define a family of functions $\psi_n : \mathcal{S}'_n \to \mathcal{S}_n$ for $n \in \{1, \ldots, k\}$ which selects a representative element from each coset of $\mathcal{S}'_n \cong \mathcal{S}_n/(\mathbb{Z}/2)^n$. Denote these maps by $\psi_n : X' \mapsto X$, and write $\mathcal{S}_n^+ := \psi_n(\mathcal{S}'_n)$. Then define $\varphi(X) := \sum_j (-1)^j X_j$, where $X_j \in \mathcal{S}_k$ lies in the same orbit as X under the action of the reflection group, and can be obtained from X (in the sense of Figure 3.2) via a minimum of j-many reflections.

Proceed as in the proof of Theorem 4.4.2, writing expressions for each $(\mathbf{1} - T_i) \circ \varphi(X)$. Notice that $\mathcal{S}_{k-1}$ can be partitioned into 2^{k-1}-many sets in the manner of (4.7)—we use the notation $\left(\mathcal{S}_{k-1}^+\right)_\alpha := \left\{F_\alpha \mid F \in \mathcal{S}_{k-1}^+\right\}$, where α represents a sequence of reflections.

Define a map $\epsilon_{k-1} : \mathbb{Z}\mathcal{S}_{k-1} \to \mathbb{Z}\mathcal{S}_{k-1}^+$ which returns $(-1)^j F_j$ whenever $F_j \in \left(\mathcal{S}_{k-1}^+\right)_j$ can be obtained from F through j-many distinct reflections. The elements of each set $\mathcal{S}_n^+$ are in one-to-one correspondence with the n-dimensional cells in $\mathcal{M}$, and we define a boundary map $\delta_{k-1} : \mathbb{Z}\mathcal{S}_k^+ \to \mathbb{Z}\mathcal{S}_{k-1}^+$ such that $\delta_{k-1}(X)$ is the sum of the $(k-1)$-dimensional faces of X, prefixing by (-1) exactly one face from each set of the form $F(1, \ldots, \hat{n}, \ldots, k)$, before passing this sum through the function ϵ_{k-1}.

We define the function $\varphi' := \mathbb{Z}\mathcal{S}_k \to (\mathbb{Z}\mathcal{S}_k)^k$ analogously to its namesake in Theorem 4.4.2, so that

$$\begin{tikzcd} \mathbb{Z}\mathcal{S}_k^+ \arrow[r, "\varphi"] \arrow[d, "\delta_{k-1}"'] & \mathbb{Z}\mathcal{S}_k \arrow[d, "(\mathbf{1}-T) := \begin{bmatrix} \mathbf{1}-T_1 \\ \vdots \\ \mathbf{1}-T_k \end{bmatrix}"] \\ \mathbb{Z}\mathcal{S}_{k-1}^+ \arrow[r, "\varphi'"'] & (\mathbb{Z}\mathcal{S}_k)^k \end{tikzcd} \tag{4.10}$$

commutes. It is straightforward to show that the functions φ, φ' are injective, and furthermore that $\ker(\mathbf{1}-T) \cong \ker\left(\partial_1^T\right)$. Hence $H_k(\mathcal{M}) \cong \varphi\left(\mathbb{Z}\mathcal{S}_k^+\right) \cap \ker\left(\partial_1^T\right)$.

Together with the fact from [Rob05, 4.4] that $\varphi(\mathbb{Z}\mathcal{S}_k^+) \cap \ker(\mathbf{1}-T) = \ker(\mathbf{1}-T)$ for each k, this proves the result. ■

4.4.4 Proposition *Let Γ be a k-domino group with adjacency structure $E_1, \dots, E_k$ and adjacency matrices $M_1, \dots, M_k$. Consider the maps ∂_1 and ∂_k from 3.4.7. Then $H_k(\mathcal{M}(\Gamma)) \cong \operatorname{coker}(\partial_1) \cong \ker(\partial_k) \oplus \mathcal{G}$ for some finite Abelian group $\mathcal{G}$.*

■ **Proof** Firstly, as elementary row operations on matrices do not affect their kernels, then we can successively multiply rows of ∂_k by (-1) and rearrange them to find that

$$\ker(\partial_k) = \ker \begin{bmatrix} \mathbf{1} - M_k^T \\ M_{k-1}^T - \mathbf{1} \\ \vdots \\ (-1)^k\left(M_1^T - \mathbf{1}\right) \end{bmatrix} \cong \ker \begin{bmatrix} \mathbf{1} - M_k^T \\ \vdots \\ \mathbf{1} - M_1^T \end{bmatrix} \cong \ker \begin{bmatrix} \mathbf{1} - M_1^T \\ \vdots \\ \mathbf{1} - M_k^T \end{bmatrix}. \tag{4.11}$$

We claim that the row space (the vector space of linear combinations of row vectors) of M_i^T is isomorphic to that of M_i for each $i \in \{1, \dots, k\}$. To see why, let $X \in \mathcal{S}_k(\Gamma)$ be an arbitrary k-dimensional domino in Γ—then $M_i(X, Y) = 1$ if and only if Y is E_i-adjacent to X. For a domino $S \in \mathcal{S}_k$, let $S_i \in \mathcal{S}_k$ denote the domino obtained by reflecting S in the direction parallel to the edges labelled from alphabet E_i (for example, the dominoes named S_H, S_V, and S_I in Figure 3.2). Then $M_i(X, Y) = 1$ if and only if $M_i(Y_i, X_i) = 1$, if and only if $M_i^T(X_i, Y_i) = 1$. Then there is an isomorphism from the row space of M_i to the row space of M_i^T given by the map taking each generator S of $\mathbb{Z}\mathcal{S}_k$ to S_i. In particular, this means that $\ker(\mathbf{1} - M_i) \cong \ker\left(\mathbf{1} - M_i^T\right)$ for each i, and together with (4.11), that $\ker(\partial_k) \cong \ker\left(\partial_1^T\right)$.

Finally, it is well-known that the kernel of a matrix map between vector spaces must have the same rank as the cokernel of its transpose, and so we are done. ■

4.4.5 Observe that the cellular homology of a k-dimensional domino complex $\mathcal{M}(\Gamma)$ is closely related to the K-theory of its induced k-rank graph algebra $\mathcal{A}(\Gamma)$. Apart from the fact that $H_k(\mathcal{M})$ is isomorphic to the infinite part of $\operatorname{coker}(\partial_1)$ by Theorem 4.4.3, it is not clear exactly how. Looking at the case where $k = 3$, we see that the orders of the cyclic subgroups of $H_n(\mathcal{M})$ have the same prime factors as those of the homologies $\ker(\partial_1)/\mathrm{im}(\partial_2)$ in each of the examples from §4.1, but we do not know their precise relationship. Furthermore, it seems as though

$$\ker(\partial_2)/\mathrm{im}(\partial_3) \cong \mathrm{coker}(\partial_1) \oplus \mathbb{Z}^{2\,\mathrm{rk}(\mathrm{coker}(\partial_1))},$$

meaning that the torsion-free rank r of $K_0(\mathcal{A}(\Gamma))$ and $K_1(\mathcal{A}(\Gamma))$ lies somewhere in the range $r \in \{3\,\mathrm{rk}(\mathrm{coker}(\partial_1)), \ldots, 4\,\mathrm{rk}(\mathrm{coker}(\partial_1))\}$. An affirmation of *Matui's Conjecture* (4.2.2) in the case of domino graph algebras would suggest that $r = 4\,\mathrm{rk}(\mathrm{coker}(\partial_1))$.

4.4.6 To round off this section, we present a formulation of the relationship between the *Euler characteristic* (see [Hat02, §2.2]) and the homology of a cell complex, which allows us to find the torsion-free rank of a homology group whenever we know the ranks of the others.

4.4.7 Proposition (Euler characteristic in terms of homology) *Let Γ be a k-domino group with adjacency structure $E_1, \ldots, E_k$ of respective cardinalities $m_1, \ldots, m_k$, and consider the domino complex $\mathcal{M} = \mathcal{M}(\Gamma)$. Then*

$$\sum_{n=1}^{k} (-1)^n |\mathcal{S}'_n(\Gamma)| = \sum_{n=1}^{k} (-1)^n \,\mathrm{rk}(H_n(\mathcal{M})).$$

■ **Proof** Consider the barycentric subdivision $\mathcal{M}'$ of $\mathcal{M}$ from 4.1.9: by 4.1.7, the torsion-free rank of $H_n(\mathcal{M}')$ is equal to that of $|C'_n| - \mathrm{rk}(\delta_n) - \mathrm{rk}(\delta_{n+1})$, where $\mathrm{rk}(\delta)$ is the dimension of the vector space spanned by the columns of the matrix δ, and $|C'_n|$ is the number of geometric n-cubes in $\mathcal{M}'$. By alternating addition and subtraction of these equations, we obtain the result. ■

§4.5 Products of free groups

We established in 4.1.5 that a direct product of k-many free groups, each of order at least two, defines a k-domino group. We explore some of their properties as k-domino groups.

4.5.1 Theorem *Let $\Gamma = \mathbb{F}_{\alpha_1} \times \cdots \times \mathbb{F}_{\alpha_k}$, and consider the associated domino complex $\mathcal{M} = \mathcal{M}(\Gamma)$. Then*

$$H_n(\mathcal{M}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}^{\sigma(n)} & \text{if } n \in \{1, \ldots, k\}, \\ 0 & \text{otherwise,} \end{cases}$$

where, for each $n \in \{1, \ldots, k\}$, we define

$$\sigma(n) := \sum_{\substack{1 \le t_i \le k, \\ t_i \text{ distinct}}} \alpha_{t_1} \cdots \alpha_{t_n}.$$

■ **Proof** Consider the chain complex (2.9):

$$0 \longrightarrow C_k \longrightarrow \cdots \longrightarrow C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0,$$

from 2.6.1, where C_n is the free Abelian group whose generators are indexed by the n-dimensional geometric cubes in $\mathcal{M}$. Then $C_n \cong \mathbb{Z}^{\sigma(n)}$ for each $n \in \{1, \ldots, k\}$.

The boundary maps δ_n take n-cubes in $\mathcal{M}$ to the formal sum of their $(n-1)$-faces. Since Γ is a product of free groups, the 2-cubes (geometric squares) in $\mathcal{M}$ are all of the form $A = (x, y, \bar{x}, \bar{y})$, for $x \in \mathbb{F}_\alpha$, $y \in \mathbb{F}_\beta$. This means that $\delta_2(A) = x + y - x - y = 0$ for all $A \in C_2$. Inductively, we can see that the opposite $(n-1)$-faces of each n-cube in $\mathcal{M}$ are the same geometric $(n-1)$-cube but with opposite orientations, leading to different signs in the formal sum. Hence each map δ_n for $n \geq 2$ is the zero map.

Thus $H_n \cong \ker(\delta_n)/\operatorname{im}(\delta_{n+1}) = C_n/\{0\} = C_n$ for each $n \geq 2$. Furthermore, $\mathcal{M}$ has one vertex—call this v—so $\delta_1(x) = v - v = 0$ for all edges $x \in \mathcal{M}$, and hence $H_1 \cong C_1$. Finally, since $\mathcal{M}$ is connected and k-dimensional, we have $H_0 \cong \mathbb{Z}$ and $H_n \cong 0$ for all $n > k$. ■

Building 3-domino groups from 2-domino groups

4.5.2 Using Proposition 3.2.4, we can build k-domino groups whose generators and relations are induced by those of k-many $(k-1)$-domino groups. A simple way to do this is to take a 2-domino group Γ' and let $\Gamma = \Gamma' \times \mathbb{F}_\gamma$, where $\mathbb{F}_\gamma$ is the free group on γ generators.

In effect, we are turning 2-dimensional dominoes into "square prisms," as depicted in Figure 4.2.

4.5.3 Theorem *Let Γ' be a 2-domino group with adjacency structure E_1, E_2 and adjacency matrices M_1', M_2'. Consider the map $\partial_{\Gamma'} : \mathbb{Z}\mathcal{S}_2(\Gamma') \oplus \mathbb{Z}\mathcal{S}_2(\Gamma') \longrightarrow \mathbb{Z}\mathcal{S}_2(\Gamma')$ from 2.2.6 and 3.4.7 defined by the block matrix $\partial_{\Gamma'} := \left[\mathbf{1} - (M_1')^T, \mathbf{1} - (M_2')^T\right]$.*

Let $C' := \operatorname{coker}(\partial_{\Gamma'})$, which by the Fundamental Theorem of finitely-generated Abelian groups (2.2.5), we can write as $\bigoplus_i \mathbb{Z}/t_i$ for some non-negative integers i, where $\mathbb{Z}/0 := \mathbb{Z}$. Let $\Gamma = \Gamma' \times \mathbb{F}_\gamma$, for some $\gamma \geq 2$, and regard Γ as a 3-domino group with adjacency structure E_1, E_2, F, where $F = \{c_1, \bar{c}_1 \ldots, c_\gamma, \bar{c}_\gamma\}$ generates $\mathbb{F}_\gamma$, and write M_1, M_2, M_3 to denote the respective adjacency matrices.

Consider ∂_1, the differential map from 3.4.8 corresponding to Γ. Then

$$\operatorname{coker}(\partial_1) = (C')^\gamma \oplus \bigoplus_i \mathbb{Z}/\gcd(t_i, \gamma - 1),$$

where $\gcd(0, x) := x$.

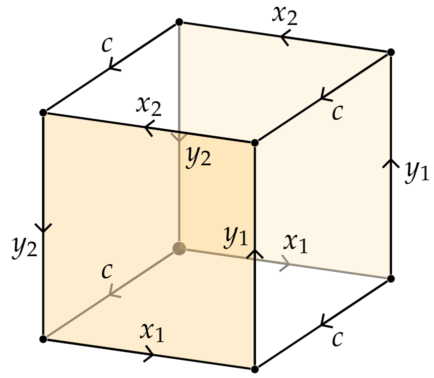


Figure 4.2: Consider a $(k-1)$-domino group Γ' with adjacency structure $E_1, \ldots, E_{k-1}$, and another alphabet $F = \{c_1, \bar{c}_1, \ldots, c_\gamma, \bar{c}_\gamma\}$, where $\gamma \geq 2$. Let L be the set of pointed squares $\{[x, c, \bar{x}, \bar{c}] \mid x \in E_1 \sqcup \ldots \sqcup E_{k-1}$ and $c \in F\}$. Then $\mathcal{S}_2(\Gamma') \cup L$ is a set of pointed squares with a k-domino structure, and $\Gamma = \Gamma' \times \mathbb{F}_F$ is a k-domino group.

Pictorially, to obtain a k-dimensional domino in Γ, we duplicate a $(k-1)$-dimensional domino from Γ', and join the matching vertices of the two copies with edges labelled by an element of F.

■ **Proof** Firstly, we have

$$\operatorname{coker}(\partial_{\Gamma'}) = \left\langle A \in \mathcal{S}_2(\Gamma') \;\middle|\; A = \sum_{B \in \mathcal{S}_2(\Gamma')} M_1'(A, B) \cdot B = \sum_{B \in \mathcal{S}_2(\Gamma')} M_2'(A, B) \cdot B \right\rangle,$$

and

$$\operatorname{coker}(\partial_1) = \left\langle X \in \mathcal{S}_3(\Gamma) \;\middle|\; X = \sum_{Y \in \mathcal{S}_3(\Gamma)} M_i(X, Y) \cdot Y, \text{ for } i \in \{1, 2, 3\} \right\rangle.$$

By construction, $|\mathcal{S}_3(\Gamma)| = 2\gamma \cdot |\mathcal{S}_2(\Gamma')|$, since each pointed square $A = [a, b, \bar{a}, \bar{b}] \in \mathcal{S}_2(\Gamma')$ gives rise to 2γ-many 3-dimensional dominoes of the form $\square([a, b, \bar{a}, \bar{b}], [a, c_i, \bar{a}, \bar{c}_i])$, using the notation of 3.1.15. Then, potentially after reordering the elements of the alphabets E_1 and E_2, we see that M_1 and M_2 are $(2\gamma \cdot |\mathcal{S}_2(\Gamma')|) \times (2\gamma \cdot |\mathcal{S}_2(\Gamma')|)$ block matrices with 2γ copies of M_1' (respectively M_2') down the diagonal, and $\mathbf{0}$ elsewhere.

Thus, to find $\operatorname{coker}(\partial_1)$, we can take 2γ copies of the generators and relations of $\operatorname{coker}(\partial_{\Gamma'})$, each indexed by an element of F. But these are not all of the relations; we have yet to consider those induced by M_3. Indeed, for each domino $X \in \mathcal{S}_3(\Gamma)$, we have to add the relation

$$X = X + \sum_{X \neq Y \in \mathcal{S}_3(\Gamma)} (Y + Y_I),$$

to $\operatorname{coker}(\partial_1)$, where Y_I is obtained by reflecting Y in the F direction, as in Figure 3.2. By summing all of these, we have $(\gamma - 1)(X + X_I) = 0$, for all $X \in \mathcal{S}_3(\Gamma)$. We partition the set $\mathcal{S}_3(\Gamma)$ into two sets $\mathcal{S}_3^+$, $\mathcal{S}_3^-$ of equal size, where $\mathcal{S}_3^- := \{X_I^+ \mid X^+ \in \mathcal{S}_3^+\}$. We may then replace the set $\mathcal{S}_3(\Gamma)$ of generators of coker(∂_1) with the set $\mathcal{S}_3^+ \cup \{(X^+ + X_I^+) \mid X^+ \in \mathcal{S}_3^+\}$, and we note that each of the relations involving X_I^+ will also satisfy $(\gamma-1)(X^+ + X_I^+) = 0$.

Hence, instead of 2γ copies of Γ', we have γ copies of Γ' corresponding to the elements of $\mathcal{S}_3^+$, and then γ-many generators of the form $Z = (X^+ + X_I^+)$, which satisfy relations $t_i Z = 0$ and $(\gamma-1)Z = 0$, that is, $\gcd(t_i, \gamma-1)Z = 0$. ■

4.5.4 Whenever $\Gamma = \Gamma' \times \mathbb{F}_\gamma$ is one of these 3-domino groups constructed from *prisms*, 4.5.3 and 4.4.4 give us a formula for the highest cellular homology group $H_3(\mathcal{M}(\Gamma))$ in terms of the adjacency matrices of the 2-domino group Γ'. Theorem 4.5.5 presents similar formulas for the remaining cellular homology groups of such a complex.

4.5.5 Theorem *Let Γ' be a 2-domino group with corresponding domino complex $\mathcal{M}' = \mathcal{M}(\Gamma')$, and let $\Gamma = \Gamma' \times \mathbb{F}_\gamma$ be the 3-domino group described in 4.5.3, which has corresponding domino complex $\mathcal{M} = \mathcal{M}(\Gamma)$. Then $H_1(\mathcal{M}) = H_1(\mathcal{M}') \oplus \mathbb{Z}^\gamma$ and $H_2(\mathcal{M})$ is equal to the direct sum of γ copies of $H_1(\mathcal{M}')$, plus one copy of $H_2(\mathcal{M}')$.*

■ **Proof** Firstly, consider the chain complexes from 2.6.1 corresponding to $\mathcal{M}$ and $\mathcal{M}'$:

$$0 \to C_3 \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \to 0, \qquad 0 \to C_2' \xrightarrow{\delta_2'} C_1' \xrightarrow{\delta_1'} C_0' \to 0;$$

the groups C_n and C_n' are free Abelian groups whose generators are indexed by the n-dimensional cells of Γ and Γ', respectively.

Denote by $\mathcal{G}^{\mathrm{ab}}$ the *Abelianisation* of a group $\mathcal{G}$; conventionally we write "+" to represent the group operation of an Abelian group, such that

$$\Gamma^{\mathrm{ab}} = \langle x \in C_1 \mid \delta_2(A) \text{ whenever } A \in C_2, \text{ and } x + x' = x' + x \text{ for all } x, x' \in C_1 \rangle.$$

Since $\mathcal{M}$ only has one vertex, each edge x in C_1 is a loop, and so $\ker(\delta_1) = C_1$. Then $H_1(\mathcal{M}) = \ker(\delta_1)/\mathrm{im}(\delta_2) \cong C_1/\mathrm{im}(\delta_2)$. But this is exactly $\cong \Gamma^{\mathrm{ab}}$, since $\mathcal{M}$ having only one vertex means that $\Gamma = \pi_1(\mathcal{M})$, the *fundamental group* of the complex $\mathcal{M}$. The same is true for $H_1(\mathcal{M}')$. Writing E_1, E_2 for the adjacency structure of Γ, and E_1, E_2, F for that of Γ, we have $\Gamma' = \langle E_1, E_2 \mid R \rangle$, where

$$R := \{x_1 y_1 x_2 y_2 \mid [x_1, y_1, x_2, y_2] \in \mathcal{S}_2(\Gamma')\},$$

and so

$$\Gamma = \langle E_1, E_2, F \mid R \cup \{[a, c_i], [b, c_i] \mid a \in E_1, b \in E_2, c_i \in F\} \rangle,$$

which is isomorphic to $\Gamma' \oplus \mathbb{Z}^\gamma$. Hence $H_1(\mathcal{M}) \cong H_1(\mathcal{M}') \oplus \mathbb{Z}^\gamma$.

Now, we aim to find $H_2(\mathcal{M}) = \ker(\delta_2)/\mathrm{im}(\delta_3)$. Each geometric 3-cube $X' \in \mathcal{S}_3'(\Gamma)$ is mapped under δ_3 to a sum of the form

$$\begin{aligned} &A + (a_1, c_i, \bar{a}_1, \bar{c}_i) + (b_1, c_i, \bar{b}_1, \bar{c}_i) + (a_2, c_i, \bar{a}_2, \bar{c}_i) + (b_2, c_i, \bar{b}_2, \bar{c}_i) - A \\ &= (a_1, c_i, \bar{a}_1, \bar{c}_i) + (b_1, c_i, \bar{b}_1, \bar{c}_i) + (a_2, c_i, \bar{a}_2, \bar{c}_i) + (b_2, c_i, \bar{b}_2, \bar{c}_i), \end{aligned}$$

where $A = (x, c_i, \bar{x}, \bar{c}_i) \in \mathcal{S}_2'(\Gamma')$, and $c_i' = \{c_i, \bar{c}_i\}$ is a pair of mutually inverse elements of F (a geometric edge in $\mathcal{M}$). Elements of $\mathrm{im}(\delta_3)$ can be therefore indexed by such pairs (A, c_i'); clearly $\mathrm{im}(\delta_3)$ is isomorphic to the direct sum of γ copies of $\mathrm{im}(\delta_2')$. Now, any geometric square in $\mathcal{M}$ which has some edge from F must be in $\ker(\delta_2)$, since it will be of the form $S = (x, c_i, \bar{x}, \bar{c}_i)$, so that $\delta_2(S) = x + c_i - x - c_i = 0$; write $L \subset \mathcal{S}_2'(\Gamma)$ to denote the set of all such geometric squares. Furthermore, if $X \in \ker(\delta_2')$, then $X \in \ker(\delta_2)$, so $\ker(\delta_2) \supseteq \ker(\delta_2') \cup L$; the reverse inclusion is clear, so these two sets are actually equal. Then

$$H_2(\mathcal{M}) \cong \frac{\ker(\partial_2)}{\mathrm{im}(\partial_3)} \cong \frac{\ker(\partial_2)}{(\mathrm{im}(\partial_2'))^\gamma} \cong \frac{\ker(\partial_2') \cup L}{(\mathrm{im}(\partial_2'))^\gamma}.$$

We calculate that $L/(\mathrm{im}(\partial_2'))^\gamma$ is isomorphic to

$$\frac{\{(x, c_i, \bar{x}, \bar{c}_i) \mid x \in E_1 \sqcup E_2, c_i \in F\}}{\{(x_1, c_i, \bar{x}_1, \bar{c}_i) + (y_1, c_i, \bar{y}_1, \bar{c}_i) + (x_2, c_i, \bar{x}_2, \bar{c}_i) + (y_2, c_i, \bar{y}_2, \bar{c}_i) \mid (x_1, y_1, x_2, y_2) \in \mathcal{S}_2'(\Gamma')\}},$$

which is isomorphic to the direct sum of γ copies of $H_1(\mathcal{M}')$. It remains to compute $\ker(\partial_2')/(\mathrm{im}(\partial_2'))^\gamma$, but since no elements of the denominator appear in the numerator, this is just C_2', which is $H_2(\mathcal{M}')$. Hence $H_2(\mathcal{M}) \cong H_1(\mathcal{M}')^\gamma \oplus H_2(\mathcal{M}')$. ∎

4.5.6 Example (Product of three free groups) Consider Theorems 4.5.3 and 4.5.5 when applied to the product of three free groups $\Gamma = \mathbb{F}_\alpha \times \mathbb{F}_\beta \times \mathbb{F}_\gamma$—here, $\Gamma' = \mathbb{F}_\alpha \times \mathbb{F}_\beta$ can be considered as the corresponding 2-domino group. From [KR02, 6.1], we find that

$$\mathrm{coker}(\partial_{\Gamma'}) \cong \mathbb{Z}^{\alpha\beta} \oplus (\mathbb{Z}/(\alpha-1))^\beta \oplus (\mathbb{Z}/(\beta-1))^\alpha \oplus (\mathbb{Z}/g),$$

where $g := \gcd(\alpha-1, \beta-1)$. Then from 4.5.3, we get

$$\begin{aligned} \mathrm{coker}(\partial_1) \cong \mathbb{Z}^{\alpha\beta\gamma} &\oplus (\mathbb{Z}/(\alpha-1))^{\beta\gamma} \oplus (\mathbb{Z}/(\beta-1))^{\alpha\gamma} \oplus (\mathbb{Z}/(\gamma-1))^{\alpha\beta} \\ &\oplus (\mathbb{Z}/g(\alpha,\beta))^\gamma \oplus (\mathbb{Z}/g(\alpha,\gamma))^\beta \oplus (\mathbb{Z}/g(\beta,\gamma))^\alpha \oplus (\mathbb{Z}/g(\alpha,\beta,\gamma)), \end{aligned}$$

where $g(x_1, \ldots, x_s) := \gcd(x_1 - 1, \ldots, x_s - 1)$. Setting $\alpha = \beta = \gamma = 3$, we recover the group $\mathrm{coker}(\partial_1) \cong \mathbb{Z}^{27} \oplus (\mathbb{Z}/2)^{37}$ which we computed in Example 4.1.4.

Likewise, we showed in 4.5.1 that $H_1(\mathcal{M}(\Gamma')) \cong \mathbb{Z}^{\alpha+\beta}$, and $H_2(\mathcal{M}(\Gamma')) \cong \mathbb{Z}^{\alpha\beta}$. Then, writing $\mathcal{M} = \mathcal{M}(\Gamma)$ and referring to Theorem 4.5.5, we must have $H_1(\mathcal{M}) \cong (\mathbb{Z}^{\alpha+\beta}) \oplus \mathbb{Z}^\gamma$, and $H_2(\mathcal{M}) \cong (\mathbb{Z}^{\alpha+\beta})^\gamma \oplus \mathbb{Z}^{\alpha\beta}$. By Proposition 4.4.4 and the above, we also know that $H_3(\mathcal{M}) \cong$

$\mathbb{Z}^{\alpha\beta\gamma}$, so that:

$$H_n(\mathcal{M}(\Gamma)) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}^{\alpha+\beta+\gamma} & \text{if } n = 1, \\ \mathbb{Z}^{\alpha\beta+\alpha\gamma+\beta\gamma} & \text{if } n = 2, \\ \mathbb{Z}^{\alpha\beta\gamma} & \text{if } n = 3, \\ 0 & \text{if } n \geq 4, \end{cases}$$

as expected. ♦

§4.6 Future directions

4.6.1 Recall from 1.3.7–1.3.8 Raeburn's and Szymański's observations that no $C^\star$-algebra whose K_1 group has torsion can be represented as a graph algebra, and conversely, that any two finitely-generated Abelian groups, one of which having no torsion, are the K-groups of some graph algebra. In this section, we have presented a new class of higher-rank graph $C^\star$-algebras, and have deduced a number of facts about their K-theory, including the finite (torsion) part of their K_1 groups, which is often non-trivial. A compelling question would be, then, what conditions are there on the K-groups of higher-rank graph $C^\star$-algebras? Moreover, is there a constructive method for finding a higher-rank graph algebra with some predetermined K-groups?

As we have discussed in 3.4.10 and §4.3, determining the K-theory of higher-rank graph algebras is no walk in the park, but the new models of higher-rank graphs we present in this thesis indicate some links between the structure of the algebras and the geometry of the associated cell complexes. Since it is often easier to work with the complexes than with the algebras, it would be fruitful to pin down the exact relationship between the geometric and algebraic invariants.

We have suggested one potential way to simplify computations in §4.5, by showing how increasing the dimension of a k-domino group in a predictable way affects the associated graph algebra and complex, allowing for a wider variety of higher-rank graph algebras to be studied, while maintaining familiarity with their K-theory. Alternatively, one could work from the top down—now that there is a solid source of examples of k-rank graphs for arbitrary k (namely, 4.1.6), it would be interesting to look at their relationship with their lower-rank components (in the manner of 3.2.4).

Chapter 5

Appendix: Programs for generating domino groups

In this appendix, we exhibit some of the programs we used to generate the domino groups and the information about them, including the relevant cokernels and kernels from 3.4.8, and the cellular homology groups of the domino complexes.

§5.1 K-theory computations for domino groups

5.1.1 Recall the *RSV Algorithm* from 4.1.6, which takes as input a prime power q and a list of k-many integers between 0 and $(q-2)$, and outputs a set of pointed squares which has a k-domino structure. We implement this using `Python3`, using the package `finitefield` of [Kun14]. The *RSV Algorithm* requires arithmetic be done in a field of order q^2, and this package generates finite fields of arbitrary prime-power order. In a slight deviation from the description in 4.1.6, our realisation of the algorithm using `finitefield` takes as input a set J of k-many indices between 0 and $(p-2)$ (not $(q-2)$) which will label the alphabets of the k-domino group. We present the first part of the algorithm below, with an example input of $q = 5$, and $J = \{1, 2, 3\}$.

5.1.2 Finding a primitive element of a finite field

```
import functools
from finitefield import FiniteField
from finitefield import generateIrreduciblePolynomial
#FiniteField takes two arguments, p, n, and generates the field of order p^n
p = 5
n = 1
q = p^n
```

```
# label is a list of k-many integers from 0..p-2.
# It is used to generate the alphabets for the k-domino group.
label = [1,2,3]
dimension = len(label)
F = FiniteField(p, 2*n)
def change_base(i, p):
  """Assumes i, p integers.
  Returns a list with digits of i in base p (in reverse order)."""
  digits = []
  while i:
    digits.append(int(i % p))
    i //= p
  return digits
@functools.lru_cache(maxsize=None)
def field_elts(q, p, n):
  """Finds a generator d for the multiplicative group of F.
  Returns an ordered list of elements of F,
  the ith element is the ith power of d."""
  def find_generator_q2():
    generator_found = False
    for i in range(p, s):
      first = F(change_base(i, p))
      ith = first
      elt_list = [first]
      while ith != 1 and len(elt_list) < s:
        ith = ith*first
        elt_list.append(ith)
        if len(elt_list) == s:
          generator_found = True
          elt_list.insert(0,elt_list.pop())
          break
        else: continue
      if generator_found == True:
        return elt_list
    if generator_found == False:
      print("No generator found")
  return find_generator_q2()
multiplicative_group = field_elts(q, p, n)
d = multiplicative_group[1]
print("The field F = F_{} has primitive element d = {}.".format(q**2, d))
```

5.1.3 With the above parameters, the algorithm outputs:

```
The field F = F_25 has primitive element d = 3 + 1 x^1 ∈ F_{5^2}
```

which we can then use to build the relations of the k-domino group $\Gamma_{J,\delta}$. We omit the Python implementation of the rest of the *RSV Algorithm*, as it is fairly routine. Now, the output of the algorithm is a list of lists, each of length 4, representing the pointed squares in $\Gamma_{J,\delta}$. We might choose to relabel some of the individual elements for clarity (for example, in 5.1.4 we have chosen to represent inverse elements of $\Gamma_{J,\delta}$ by a change of sign).

Below is the algorithm we used to generate the 3-dimensional dominoes, given a set of pointed tiles with 3-domino structure. We have a k-dimensional version, which works by similar principles.

5.1.4 Program for building 3-dimensional dominoes from a list of squares

```python
import time
# The prime power q used to build the relations of Gamma.
q = 5
# Each generator of Gamma is in one of the alphabets {i mod a}, for a in label.
label = [1,2,3]
# The RSV Algorithm (4.1.6) will give us a list of relations: unpt_sqs.
# These example relations come from a product of three free groups F2^3.
unpt_sqs = [
  [1, 2, -1, -2], [1, 6, -1, -6], [5, 2, -5, -2], [5, 6, -5, -6],
  [3, 1, -3, -1], [3, 5, -3, -5], [7, 1, -7, -1], [7, 5, -7, -5],
  [3, 2, -3, -2], [3, 6, -3, -6], [7, 2, -7, -2], [7, 6, -7, -6]
]
start_time = time.time()
print('Here is the list of relations (squares):', unpt_sqs)

# In which set of the form F(a,b...) are the squares/cubes X?
def edge_labels(X):
  """Assumes X either a list of 4 (square) or of 6 (cube).
  Returns a list of 2 or 3, a sublist of label"""
  if len(X) == 4:
    return [abs(X[0]) % (q-1), abs(X[1]) % (q-1)]
  elif len(X) == 6:
    return [abs(X[0][0]) % (q-1), abs(X[0][1]) % (q-1), abs(X[1][0]) % (q-1)]
# e.g. the first row of elements in unpt_sqs have edge_labels = [1,2].
def sqs_type_x(inpt, x):
  """Assumes inpt is a list of squares or cubes, x is a list of 2 or 3 int.
  Returns all elements of inpt with the same edge_labels as x."""
```

```
    outpt = []
    for X in inpt:
        if edge_labels(X) == x:
            outpt.append(X)
    return outpt

# The following functions assume A is an element of unpt_sqs,
# treated as a pointed square (an ordered list).
# They return a pointed square (list of 4) corresponding to a symmetry of A.
def Sh(A):
    return [-A[0], -A[3], -A[2], -A[1]]
def Sv(A):
    return [-A[2], -A[1], -A[0], -A[3]]
def Sr(A):
    return[A[2], A[3], A[0], A[1]]
# sqs is a list of all pointed squares. It has length 4 times that of unpt_sqs.
sqs = []
for A in unpt_sqs:
    sqs.extend([A, Sh(A), Sv(A), Sr(A)])
# Finally, all_sqs contains each square in sqs twice:
# once of type F(a,b) and once of type F(b,a).
# This makes it easier to find squares which are adjacent.
all_sqs = []
for A in sqs:
    all_sqs += [A, [A[1], A[2], A[3], A[0]]]

def cubes_based_at(A):
    """Assumes A is a square (list of 4) from all_sqs.
    Returns a list of cubes (lists of 6) whose first element is A."""
    cube_list = []
    adj = []
    sqs_same_type = []
    sqs_other_type = []
    # Firstly, decide which of all_sqs are adjacent to A, and add them to adj.
    for B in all_sqs:
        if set(edge_labels(B)) != set(edge_labels(A)):
            sqs_other_type.append(B)
            if edge_labels(B)[1] == edge_labels(B)[0] and B[3] == -A[0]:
                adj.append(B)
        else:
```

```
      sqs_same_type.append(B)
  # Now, build a cube uniquely defined by A and B from adj.
  for B in adj:
    cube = [A,B,0,0,0,0]
    for C in sqs_other_type:
      if C[0] == -A[3] and C[3] == -B[0]:
        cube[2] = C
        for D in sqs_same_type:
          if D[0] == -B[1] and D[1] == -C[2]:
            cube[3] = D
        for D in sqs_other_type:
          if D[0] == -C[1] and D[1] == -A[2] and D[3] == -cube[3][2]:
            cube[4] = D
          if D[0] == -A[1] and D[1] == -B[2] and D[2] == -cube[3][3]:
            cube[5] = D
    if 0 in cube:
      print("Error: no valid square found to complete cube:", cube)
    else:
      cube_list.append(cube)
  return cube_list

def cubes_with_base_in(squareset):
  """Assumes squareset is a list of squares.
  Returns a list of cubes whose first elements are elements of squareset."""
  cubes_list = []
  for A in squareset:
    cubes_list = cubes_list + cubes_based_at(A)
  return cubes_list
# all_cubes is a list of all 3-dimensional cubes in all orientations.
all_cubes = cubes_with_base_in(all_sqs)

# Many of the cubes in all_cubes will be duplicates, or will be
# symmetries of one another. We sort this out now.
# The following functions are 3-d analogues of the symmetry functions Sx.
def Ch(U):
  return [Sh(U[0]), Sv(U[1]), Sh(U[5]), Sh(U[3]), Sv(U[4]), Sh(U[2])]
def Cv(U):
  return [Sv(U[0]), Sh(U[4]), Sh(U[2]), Sv(U[3]), Sh(U[1]), Sh(U[5])]
def Cr(U):
  return [Sr(U[0]), Sr(U[4]), U[5], Sr(U[3]), Sr(U[1]), U[2]]
```

```
def Ci(U):
  return [Sh(U[3]), Sh(U[1]), Sv(U[2]), Sh(U[0]), Sh(U[4]), Sv(U[5])]
def Chi(U):
  return [U[3], Sr(U[1]), Sr(U[5]), U[0], Sr(U[4]), Sr(U[2])]
def Cvi(U):
  return [Sr(U[3]), U[4], Sr(U[2]), Sr(U[0]), U[1], Sr(U[5])]
def Cri(U):
  return [Sv(U[3]), Sv(U[4]), Sv(U[5]), Sv(U[0]), Sv(U[1]), Sv(U[2])]
# cube_symmetries takes a cube and returns a list of its eight symmetries.
def cube_symmetries(U):
  return [U, Ch(U), Cv(U), Cr(U), Ci(U), Chi(U), Cvi(U), Cri(U)]
# eliminate_dupl removes duplicates and symmetries from a set of cubes.
def eliminate_dupl(cubeset):
  cube_syms = []
  small_cubeset = []
  for U in cubeset:
    if U in cube_syms:
      continue
    else:
      small_cubeset.append(U)
      cube_syms.extend(cube_symmetries(U))
  return small_cubeset
# pt_cubes the list of all 3-d dominoes.
# unpt_cubes is the list of all 3-d dominoes modulo symmetries.
pt_cubes = []
unpt_cubes = []
for U in eliminate_dupl(
  cubes_with_base_in(
    sqs_type_x(all_sqs,[label[0], label[1]])
  )
):
  unpt_cubes.append(U)
  pt_cubes.extend(cube_symmetries(U))
print(
"Those relations give rise to", len(unpt_cubes), "geometric cubes, and",
len(pt_cubes), "dominoes."
)
print("Here is the list of cubes formed from those relations:")
for U in unpt_cubes:
  print(U)
```

```
print("****************")
end_time = time.time()
total_time = end_time - start_time
print("This took", round(total_time, 3), "seconds.")
```

5.1.5 The above algorithm, upon input of the relations of a product of three free groups, each on two generators, outputs the following:

```
Here is the list of relations (squares): [[1,2,-1,-2], [1,6,-1,-6],
[5,2,-5,-2], [5,6,-5,-6], [3,1,-3,-1], [3,5,-3,-5], [7,1,-7,-1], [7,5,-7,-5],
[3,2,-3,-2], [3,6,-3,-6], [7,2,-7,-2], [7,6,-7,-6]]
Those relations give rise to 8 geometric cubes, and 64 dominoes.
Here is the list of cubes formed from those relations:
[[1,2,-1,-2], [3,1,-3,-1], [2,3,-2,-3], [-1,2,1,-2], [-3,1,3,-1], [-2,3,2,-3]]
[[1,2,-1,-2], [7,1,-7,-1], [2,7,-2,-7], [-1,2,1,-2], [-7,1,7,-1], [-2,7,2,-7]]
[[1,6,-1,-6], [3,1,-3,-1], [6,3,-6,-3], [-1,6,1,-6], [-3,1,3,-1], [-6,3,6,-3]]
[[1,6,-1,-6], [7,1,-7,-1], [6,7,-6,-7], [-1,6,1,-6], [-7,1,7,-1], [-6,7,6,-7]]
[[5,2,-5,-2], [3,5,-3,-5], [2,3,-2,-3], [-5,2,5,-2], [-3,5,3,-5], [-2,3,2,-3]]
[[5,2,-5,-2], [7,5,-7,-5], [2,7,-2,-7], [-5,2,5,-2], [-7,5,7,-5], [-2,7,2,-7]]
[[5,6,-5,-6], [3,5,-3,-5], [6,3,-6,-3], [-5,6,5,-6], [-3,5,3,-5], [-6,3,6,-3]]
[[5,6,-5,-6], [7,5,-7,-5], [6,7,-6,-7], [-5,6,5,-6], [-7,5,7,-5], [-6,7,6,-7]]
****************
This took 0.168 seconds.
```

These are the dominoes which Example 3.1.12 leads us to expect. We then design a program which determines when two dominoes are adjacent, and which outputs three adjacency matrices.

```
import math
import numpy as np
n = len(pt_cubes)
# The functions fi take two dominoes from pt_cubes as arguments,
# and return 1 if they are adjacent in the i direction, and 0 else.
def f1(U, V):
  if U[1] == Sh(V[4]) and U[0][1] != -V[0][1]:
    adjacency = 1
  else: adjacency = 0
  return adjacency
def f2(U, V):
  if U[2] == Sh(V[5]) and U[0][0] != -V[0][0]:
    adjacency = 1
  else: adjacency = 0
```

```
  return adjacency
def f3(U, V):
  if U[0] == Sh(V[3]) and U[1][0] != -V[1][0]:
    adjacency = 1
  else: adjacency = 0
  return adjacency
# Now we construct the adjacency matrices from 3.1.16.
M = [[], [], []]
def f(U, V):
  return [f1(U,V), f2(U,V), f3(U,V)]
for U in pt_cubes:
  new_row = [[], [], []]
  for i in range(3):
    for B in pt_cubes:
      new_row[i].append(f(A, B)[i])
    M[i].append(new_row[i])
M = [np.array(i) for i in M]
MT = [np.transpose(i) for i in M]
print("Matrices done")
# I and O are the n x n identity and zero matrices.
I = np.eye(n, dtype=int)
O = np.zeros((n,n), dtype=int)
# dn are the matrices representing the differentials in 3.4.8.
d1 = np.block([I-MT[0], I-MT[1], I-MT[2]])
d2 = np.block([[MT[1]-I,MT[2]-I,O], [I-MT[0],O,MT[2]-I], [O,I-MT[0],I-MT[1]]])
d3 = np.block([[I-MT[2]], [MT[1]-I], [I-MT[0]]])
```

5.1.6 Finally, we write a short script which converts the above matrices into a format which MAGMA can interpret.

```
def group_rels(d, y):
  """Assumes d is one of the block matrices di, y is a variable (string).
  Returns a string of relations."""
  rels = ""
  num_rows = len(d)
  num_cols = len(d[0])
  for j in range(num_cols):
    for i in range(num_rows):
      if d[i][j] == 0:
        pass
      else:
```

```
            rels += str(d[i][j]) + "*" + y + "[" + str(i+1) + "] + "
        rels = rels[0:len(rels)-3] + ", "
    rels = rels[0:len(rels)-2]
    return rels
def print_homs():
    """Takes no input. Prints relations readable by MAGMA and returns None."""
    txtfile = "3_cubes_q_{}_label_{}.txt".format(q, label)
    f = open(txtfile, "a")
    print(
        'A<[a]> := FreeAbelianGroup(',len(d3[0]),');\n',
        'B<[b]> := FreeAbelianGroup(',len(d2[0]),');\n',
        'C<[c]> := FreeAbelianGroup(',len(d1[0]),');\n',
        'D<[d]> := FreeAbelianGroup(',len(d3[0]),');', file=f
    )
    print(
        'd1 := hom< C -> D | ',group_rels(d1,'d'),'>;\n',
        'K1 := Kernel(d1);\n',
        'd2 := hom< B -> K1 | ',group_rels(d2,'c'),'>;\n',
        'K2 := Kernel(d2);\n',
        'd3 := hom< A -> K2 | ',group_rels(d3,'b'),'>;\n',
        'K3 := Kernel(d3);', file=f
    )
    f.close()
```

Running the function `print_homs()` outputs something which looks like

```
A<[a]> := FreeAbelianGroup( 64 );
B<[b]> := FreeAbelianGroup( 192 );
C<[c]> := FreeAbelianGroup( 192 );
D<[d]> := FreeAbelianGroup( 64 );
d1 := hom< C -> D |  -1*d[17] + -1*d[19], -1*d[18] + -1*d[20], -1*d[17] ...
K1 := Kernel(d1);
d2 := hom< B -> K1 |  1*c[33] + 1*c[34] + -1*c[81] + -1*c[83], 1*c[33] ...
K2 := Kernel(d2);
d3 := hom< A -> K2 |  -1*b[9] + -1*b[13] + 1*b[97] + 1*b[98] + -1*b[145] ...
K3 := Kernel(d3);
```

which `MAGMA` interprets as a sequence of homomorphisms; this is precisely the chain complex from 3.4.8. We can suffix this by the `MAGMA` functions

```
> D / Image(d1); // Returns coker(d1)
> K1 / Image(d2); // Returns ker(d1) / im(d2)
```

```
> K2 / Image(d3); // Returns ker(d2) / im(d3)
> K3; // Returns ker(d3)
```

in order to retrieve the desired kernels, cokernels and homologies.

5.1.7 It is easy to extend Algorithm 5.1.4 in order to build 4-domino groups with a function along the line of "`hypercubes_based_at(C)`," analogous to the one on line 102. Indeed, since the edges of a k-dimensional domino can be enumerated systematically (as in 3.1.14 and Figure 3.3), and can be encoded as an ordered $2k$-tuple of $(k-1)$-dominoes, it is also straightforward to extrapolate this to a program which constructs k-dimensional dominoes from a set of squares which has a k-domino structure.

This being said, the computation software `MAGMA` which we rely on to provide the relevant groups has a limit on the size of data input. Even the smallest example of a 4-domino group obtained by Algorithm 4.1.6, Example 4.3.2, has adjacency matrices which are too large for the standard installation of `MAGMA` to parse. Since the information about the K-theory of k-rank graph algebras via Theorem 2.2.6 gets more and more nebulous as k increases, computation of the (co)kernels and images of the differentials becomes less pertinent as well.

§5.2 Computing the homology of a domino complex

5.2.1 To compute the cellular homology groups of a domino complex $\mathcal{M}$, we ought firstly to consider the barycentric subdivision $\mathcal{M}'$ (by 4.1.7 and 4.1.9). The following algorithm is designed to run after 5.1.4, using the list of geometric cubes `unpt_cubes` obtained at line 182 as an input. With the list of squares obtained as relations of $\mathbb{F}_2^3$ we've been using as an example so far, these are the cubes displayed at line 204.

5.2.2 Program to compute the cellular homology of a 3-domino complex We retain the same preamble as 5.1.4, and use the same values for `q`, `label`, and `unpt_sqs`. Note that `MAGMA` dictates that matrices act on the right, so the matrices we construct below are transpose to the ones in 4.1.7.

```
    import sys
    np.set_printoptions(threshold=sys.maxsize)
305 # Input a name for the group.
    name = "F_2^3"
    # gens is a list of generators for the 3-domino group.
    # sizes returns a list of the number of generators in each alphabet,
    # that is, a list [m1 / 2, m2 / 2, m3 / 2].
310 gens = list(set([abs(i) for sq in unpt_sqs for i in sq]))
    sizes = [len([i for i in gens if i % (q-1) == a]) for a in label]
```

```
# bn is the number of cells of dimension n in the cube complex.
b0 = 1
b1 = sizes[0] + sizes[1] + sizes[2]
b2 = sizes[0]*sizes[1] + sizes[0]*sizes[2] + sizes[1]*sizes[2]
b3 = sizes[0]*sizes[1]*sizes[2]
# cn is the number of n-dimensional cells in the barycentric subdivision.
c0 = b0 + b1 + b2 + b3
c1 = 2*b1 + 4*b2 + 6*b3
c2 = 4*b2 + 12*b3
c3 = 8*b3
if len(gens) != b1 or len(unpt_sqs) != b2 or len(unpt_cubes) != b3:
  raise ValueError("Something wrong with sizes of the labelling sets.")
# Recall the functions Sh, Sv, Sr from line 82.
def all_syms(A):
  """Assumes A is a square from unpt_sqs (a list of length 4).
  Returns a list of length 8, the eight symmetries (incl. rot by pi) of A."""
  A2 = [A[1], A[2], A[3], A[0]]
  return [A, A2, Sh(A), Sh(A2), Sv(A), Sv(A2), Sr(A), Sr(A2)]
```

With this set-up, we can now compute the boundary matrices ∂_i of the chain complex

$$0 \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0,$$

where C_n = cn is a free Abelian group with generators indexed by the cells of dimension n in the complex $\mathcal{M}'$. Then each ∂_n is a $|C_n| \times |C_{n-1}|$ matrix, with rows indexed by n-dimensional cells, and columns indexed by $(n-1)$-dimensional cells. In the barycentric subdivision $\mathcal{M}'$, each vertex is labelled by some cell from $\mathcal{M}$: below these are represented by lists c0_b0, c0_b1, c0_b2, c0_b3 of vertices. Then there are $\left(\sum_{n=1}^{3} 2n|B_n|\right)$-many edges in $\mathcal{M}'$, represented by lists c1_b1, c1_b2, c1_b3.

```
# d1 is a c1 x c0 matrix.
# It is the map which sends edges to their start and end vertices.
# Every row contains exactly one -1 and one +1.
d1 = np.zeros((c1, c0)).astype(int)
# The columns are labelled by vertices indexed by c0_b0, c0_b1, c0_b2, c0_b3.
# The rows are labelled by edges of the form a0, a1 for a in b1,
# by A0, A1, A2, A3 for A in b2, and by U0,..., U5 for U in b3.
c1_b1 = [[2*a, 2*a + 1] for a in range(b1)]
last_c1_b1 = c1_b1[-1][-1]
c1_b2 = [[last_c1_b1 + 1 + 4*A + i for i in range(4)] for A in range(b2)]
last_c1_b2 = s1_sqr[-1][-1]
```

```
c1_b3 = [[last_c1_b2 + 1 + 6*U + i for i in range(6)] for U in range(b3)]
# Firstly, we map edges a0, a1 in c1_b1 to x in c0_b0 and a in c0_b1.
for edg in c1_b1:
  d1[edg[0]][0], d1[edg[1]][0] = -1, 1
  d1[edg[0]][c1_b1.index(edg) + 1], d1[edg[1]][c1_b1.index(edg) + 1] = 1, -1
# Next, map the Ai in c1_b2 to A in c0_b2 and some a in c0_b1.
for edg in c1_b2:
  for i in range(4):
    d1[edg[i]][b0 + b1 + c1_b2.index(edg)] = 1
    for gen in gens:
      if abs(unpt_sqs[c1_b2.index(edg)][i]) == gen:
        d1[edg[i]][1 + gens.index(gen)] = -1
        break
# Finally, map the Ui in c1_b3 to U in c0_b3 and some A in c0_b2.
for edg in c1_b3:
  for i in range(6):
    d1[edg[i]][b0 + b1 + b2 + c1_b3.index(edg)] = 1
    for sq in unpt_sqs:
      if unpt_cubes[c1_b3.index(edg)][i] in all_syms(sq):
        d1[edg[i]][1 + b1 + unpt_sqs.index(sq)] = -1
```

Similarly, the rows of the $|C_2| \times |C_1|$ matrix ∂_2 are indexed by the 2-cells of $\mathcal{M}'$; there are four of these for each geometric square in $\mathcal{M}$, and 12 for each geometric cube. The columns of ∂_2 are indexed in the same way as the rows of ∂_1. The map ∂_2 sends a square to its four boundary edges, so each row of ∂_2 will have exactly four non-zero entries.

```
d2 = np.zeros((c2, c1)).astype(int)
c2_b2 = [[4*A + i for i in range(4)] for A in range(b2)]
last_c2_b2 = c2_b2[-1][-1]
c2_b3 = [[last_c2_b2 + 1 + 12*U + i for i in range(12)] for U in range(b3)]
# Firstly, we map the four small squares formed by subdividing each A in b2,
# A20, A21, A22, A23, to their boundaries.
# Subsquare A21 is adjacent to two edges A0, A1 in the interior of A,
# and two edges u, v on the boundary of A.
def edg_exterior(edg):
  """Assumes edg is an element of a pair c1_b1.
  Returns, based on the sign of edg, its index in the pair."""
  sgn = np.sign(edg)
  if sgn == 1: return 0
  elif sgn == -1: return 1
  else: raise ValueError("Something is wrong: one of the labels is 0")
```

```
for sq in c2_b2:
  for i in range(4):
    k = (i - 1) % 4
    d2[sq[i]][last_c1_b1+1+sq[i]], d2[sq[i]][last_c1_b1+1+sq[0]+k] = 1, -1
    for gen in gens:
      u = unpt_sqs[c2_b2.index(sq)][i]
      v = unpt_sqs[c2_b2.index(sq)][k]
      if gen == abs(u):
        d2[sq[i]][c1_b1[gens.index(gen)][edg_exterior(u)]] = np.sign(u)
      if gen == abs(v):
        d2[sq[i]][c1_b1[gens.index(gen)][1 - edg_exterior(v)]] = np.sign(v)
# We deal with the 12 small squares formed by subdividing each cube in b3.
# Each square is adjacent to two edges on the interior of U,
# and two edges u, v on the surface of U.
# If we walk in a cycle around the outside of a cube,
# here are the faces we will visit (depending on the direction).
cycles = [[1,5,4,2], [2,3,5,0], [0,4,3,1]]
# This function tells us the interior labels.
def sq_interior(sq):
  q = int(np.floor(sq/4))
  r = sq % 4
  # The sgn here tells us whether the matrix entry will be +1 or -1.
  return [
    {"edg": cycles[q][r], "sgn": +1},
    {"edg": cycles[q][r-1], "sgn": -1}
  ]
# This function tells us the exterior labels.
def sq_exterior_no_symmetry(sq):
  interior = sq_interior(sq)
  r = sq % 4
  edg_numbers = [[0,3], [1,2], [2,3], [1,0]]
  return [[interior[i], edg_numbers[r][i]] for i in range(2)]
def sym_number(face):
  """Assumes face is a list of len 4.
  Returns the sq from unpt_sqs which is the same geometric square,
  and the index of all_syms(sq) which is the same pointed square."""
  for sq in unpt_sqs:
    syms = all_syms(sq)
    if face in syms:
      return {"sq_number": unpt_sqs.index(sq), "sym": syms.index(face)}
```

```
def sq_exterior(cube, position):
  """Assumes cube is a cube (list of len 6),
  and position is an interior subsquare (int from 0--11).
  Returns the labels of the interior subsquare."""
  A = sq_exterior_no_symmetry(position)
  n = [sym_number(face)["sym"] for face in cube]
  # p asks which cycle from cycles are we looking at: 0, 1 or 2?
  p = int(np.floor(position/4))
  r = position % 4
  adj_faces = [cycles[p][r], cycles[p][r-1]]
  change = [n[f] for f in adj_faces]
  for i in [0,1]:
  # If one of the relevant faces of the cube is from the following set of
  # parallel faces, then flipping the face horizontally changes the edge
  # labels of the interior subsquare by +2.
    if adj_faces[i] in [cycles[p][1], cycles[p][3]] and change[i] >= 4:
      change[i] += 2
    A[i][1] = (A[i][1] + change[i]) % 4
  return A

for sq in c2_b3:
  for i in range(12):
    for j in sq_interior(i):
      d2[sq[i]][last_c1_b2 + 1 + 6*c2_b3.index(sq) + j["edg"]] = j["sgn"]
    cube = unpt_cubes[c2_b3.index(sq)]
    ext_labels = sq_exterior(cube, i)
    for j in ext_labels:
      sq_index = sym_number(cube[j[0]["edg"]])["sq_number"]
      d2[sqr[i]][last_c1_b1 + 1 + 4*sq_index + abs(j[1])] = j[0]["sgn"]
```

Finally, we build the differential map ∂_3 as a $|C_3| \times |C_2|$ matrix, with columns indexed in the same way as the rows of ∂_2, and with eight rows for each cube in $\mathcal{M}$ corresponding to the eight subcubes obtained after barycentric subdivision.

```
d3 = np.zeros((c3, c2)).astype(int)
c3_b3 = [[8*c + i for i in range(8)] for c in range(b3)]
# Each cube is adjacent to six faces on the inside of U,
# and three faces on the outside of U, for U in b3.
def cub_interior(cub):
  r = cub % 8
  face_numbers = [
```

```
    [[0,−1], [4,−1], [8,−1]],
    [[1,−1], [7,−1], [8,+1]],
    [[2,−1], [7,+1], [9,+1]],
    [[3,−1], [4,+1], [9,−1]],
    [[0,+1], [5,−1], [11,−1]],
    [[1,+1], [6,−1], [11,+1]],
    [[2,+1], [6,+1], [10,+1]],
    [[3,+1], [5,+1], [10,−1]]
  ]
  # The sgn here tells us whether the matrix entry will be +1 or -1.
  return [{"face": i[0], "sgn": i[1]} for i in face_numbers[r]]
# This function tells us the exterior labels of an element of c3_b3.
def cub_exterior_no_symmetry(cub):
  """Assumes cub is an index of c3_b3.
  Returns int 0--5, the face number, and int 0--2, the subface number."""
  r = cub % 8
  face_numbers = [
    [[0,0], [1,0], [2,0]],
    [[0,1], [1,3], [5,1]],
    [[0,2], [4,2], [5,0]],
    [[0,3], [4,1], [2,1]],
    [[3,1], [1,1], [2,3]],
    [[3,0], [1,2], [5,2]],
    [[3,3], [4,3], [5,3]],
    [[3,2], [4,0], [2,2]]
  ]
  return face_numbers[r]
# Recall the functions Ch, Cv, Cr, Ci, Chi, Cvi, Cri,
# and cube_symmetries from line 149.
def cube_sym_number(cube):
  """Assumes cube is an element of pt_cubes (list of len 6).
  Gives an index of cube_symmetries(cube) corresponding to
  the orientation of cube w.r.t. the standard orientation.
  Returns the cube number, and then the symmetry number."""
  for unpt_cube in unpt_cubes:
    syms = cube_symmetries(unpt_cube)
    if cube in syms:
      return {
        "cube_number": unpt_cubes.index(unpt_cube), "sym": syms.index(cube)
      }
```

```
    else: return None
def subface_labels(face):
  n = sym_number(face)["sym"]
  labels = [
    {"sq_number": sym_number(face)["sq_number"], "subsquare": i, "sgn": +1}
    for i in range(4)
  ]
  if n < 4:
    labels = [labels[(i+n) % 4] for i in range(4)]
  if n >= 4:
    labels = [labels[(n-i+1) % 4] for i in range(4)]
    for label in labels: label["sgn"] = -1
  return labels
# This function makes adjustments to the sign of exterior labels,
# based on the sym_number of the cube, analogously to that on line 416.
def cub_exterior(cube, position):
  """Assumes cube is an element of pt_cubes,
  position is a subcube 0--7 for which we want the exterior labels.
  Returns a list of three signed subsquares which form the exterior faces."""
  X = cub_exterior_no_symmetry(position)
  new_labels = [subface_labels(face) for face in cube]
  return [new_labels[X[i][0]][X[i][1]] for i in range(3)]

for cub in c3_b3:
  for i in range(8):
    for j in cub_interior(i):
      d3[cub[i]][last_c2_b2 + 1 + 12*c3_b3.index(cub) + j["face"]] = j["sgn"]
    cube = unpt_cubes[c3_b3.index(cub)]
    for j in cub_exterior(cube, i):
      d3[cub[i]][4*j["sq_number"] + j["subsquare"]] = j["sgn"]
```

Now we have the differential matrices, we print them in a format which can be parsed by `MAGMA`.

```
# Check the differential matrices form an exact sequence.
d21 = np.matmul(d2, d1)
d32 = np.matmul(d3, d2)
print("Do the dn form an exact sequence?", np.all(d32 == 0))
# Print code readable by MAGMA.
def flatten(matr):
  return [j for i in matr for j in i]
```

```
def print_matrices():
  txtfile = "bdry_matrices_{}.txt".format(name)
  f = open(txtfile, "a")
  print("The geometric squares are",unpt_sqs, file=f)
  print("The geometric cubes are",unpt_cubes, file=f)
  print(
    "height_d1:={}; width_d1:={}; height_d2:={}; height_d3:={};".format(
      len(d1), len(d1[0]), len(d2), len(d3)
    ), file=f
  )
  print(
    "d1 := Matrix(IntegerRing(),{},{},{});".format(
      len(d1), len(d1[0]), flatten(d1)
    ), file=f
  )
  print(
    "d2 := Matrix(IntegerRing(),{},{},{});".format(
      len(d2), len(d2[0]), flatten(d2)
    ), file=f
  )
  print(
    "d3 := Matrix(IntegerRing(),{},{},{});".format(
      len(d3), len(d3[0]), flatten(d3)
    ), file=f
  )
  f.close()
  print("done")
print_matrices()
```

Finally, in `MAGMA` we can run the functions `Rank(dn)` and `ElementaryDivisors(dn)` which provide the ranks of the matrices δ_n and the entries in Smith(δ_n), the data needed to calculate the cellular homology groups in 4.1.7.

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