Cohomology of Modules for Algebraic Groups and Exceptional Isogenies

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Thesis submitted for the degree of
Doctor of Philosophy

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December 2023

Acknowledgements

I consider myself extremely fortunate to have been supervised by David Stewart, and I would like to thank him for all his support and encouragement, as well as finding so much time for me and for all the time spent reading this thesis. I am also very grateful to him for introducing me to some wonderful mathematics, and for giving me the freedom to develop my own perspective on our projects. I must also thank him for countless illuminating discussions, unrelated to maths.

I would also like to express my full gratitude to my second supervisor, James Waldron, for his constant support and all of the time dedicated to me and this thesis. I also want to thank him for all our discussions related to all kinds of mathematics, from which I have learnt so much.

Both of their guidance has been invaluable to the completion of this document and I am incredibly lucky to have been their student.

I also need to thank my examiners, Martina Balagovic and Alison Parker, for suggesting improvements and helping me remove mistakes, as well as the very helpful conversations during the viva.

Thank you to the people in the Newcastle maths department and the people in my office. Firstly, I am remarkably grateful to Sam Mutter for being a wonderful friend, for all his help and support while writing this thesis, and for collaborating with me on a paper. Thank you to Devika and Aida for all the fun times spent together and all their moral support. I would also like to thank Francesca and Horacio for all of their helpful suggestions throughout my PhD.

Thank you to my amazing housemates, Will and Tasha, for being great friends and people.

A huge thank you to my friends, Andreea, Ina and Livia, whose support has been invaluable, and words fail to express how much they mean to me.

Finally, I must thank my family for always being there for me. In particular, my late mother who was always incredibly understanding and for supporting me no matter what. I also need to thank her for making me want me to always stay curious and keep learning. Thank you to my father, who has always encouraged my interest in mathematics and helping me out with things I didn't understand. I am grateful to my sister, Anca, for being the best sister anyone could ever wish for. This could not have been possible without them.

This thesis was produced under the financial support of the Romanian Government.

Abstract

For a simple, simply connected algebraic group G over an algebraically closed field k of characteristic $p > 0$, consider a a surjective endomorphism $\sigma : G \to G$ such that the fixed-point set $G(\sigma)$ is a Suzuki or Ree group. Further, write G_σ to denote the scheme-theoretic kernel of σ. Then, by utitlizing results of Jantzen and Bendel–Nakano–Pillen, we are able to compute the first cohomology for the Frobenius kernels with coefficients in the induced modules, $\mathrm{H}^1(G_\sigma, \mathrm{H}^0(\lambda))$, and extensions $\mathrm{Ext}^1_{G_\sigma}(L(\lambda), L(\mu))$ between the simple modules. When $G(\sigma)$ is a Ree group of type F_4, these results can be used to improve the known bounds for identifying extensions of simple modules in defining characteristic $\mathrm{Ext}^1_{G(\sigma)}(L(\lambda), L(\mu))$ with those for the algebraic group.

Contents

Chapter 1

Introduction

Let G be a simple, simply connected algebraic group over an algebraically closed field k of characteristic $p > 0$. Then, for a strict endomorphism $\sigma : G \to G$, the fixed point set of the points, $G(\sigma) := G(k)^\sigma$, is a finite group. Moreover, the scheme-theoretic kernel of σ is an infinitesimal subgroup of G and we denote it by G_σ. The study of cohomology of finite groups of Lie type has been of great interest throughout the years, as it encapsulates crucial information regarding the category of $kG(\sigma)$-modules.

The groundbreaking work of Cline, Parshall, Scott and van der Kallen (CPS75; CPSvdK77) relates rational cohomology to the cohomology of finite groups. Further work by Andersen (And87) then provides a general approach for Chevalley groups, with restrictions on the minimal bound on the characteristic p. However, since the cases of small values of p could not be tackled using this construction, a mixture of techniques arose, characterised by the fact that they relied on specific information concerning the groups and root systems. (See (Hum06, Chapter 12) for a literature review.)

In a series of papers (Sin92; Sin93; Sin94a; Sin94b), the 1-cohomology for the Suzuki-Ree groups was considered. In particular, in (Sin94a; Sin94b), Sin computed the 1-cohomology for the algebraic group of type F_4 in characteristic 2.

The authors Bendel, Nakano and Pillen have taken a different approach, in which extensions for the finite group are compared to extensions for the ambient algebraic group by analysing the structure of the infinite dimensional module $\mathrm{Ind}_{G(\sigma)}^{G} k$. See (BNP06) or (BNP$^+$15), for example.

We describe our results. For G with root system Φ, in cases $(\Phi, p) = (C_2, 2)$, $(G_2, 3)$ or $(F_4, 2)$, there exists a fixed purely inseparable isogeny $\tau : G \to G$ whose square is the Frobenius map F on G. Following (BT73, 3.3) we refer to a map $\sigma = \tau^r = F^{r/2}$ as an exceptional isogeny. This thesis has two main aims: first, to provide the explicit description of the 1-cohomology $\mathrm{H}^1(G_\sigma, V)$, where G_σ is the scheme-theoretic kernel of σ and V is an induced module or simple module; second, to apply these results in the case $\Phi = F_4$ to improve the known bounds for identifying extensions of simple modules for the Ree groups $G(\sigma)$ in defining characteristic with those for G.

The thesis develops the author's own work in (Rad22).

In Chapter 2 we fix some notation and remind the reader of certain facts regarding the structure of the Suzuki and Ree groups. Then the strict endomorphism σ is given by $\sigma = \tau^r = F^{r/2}$, for an odd positive integer r. In these cases, following (BT73, 3.3), we shall to refer to σ as an exceptional isogeny. Thus the fixed point set under σ becomes a Suzuki–Ree group and we denote the scheme-theoretic kernel G_σ by $G_{r/2}$. To differentiate it from the classical case, we call this infinitesimal subgroup of G an exotic or half Frobenius kernel.

In Section 3.1 we compute the 1-cohomology for the exotic Frobenius kernels with coefficients in the induced modules, $\mathrm{H}^1(G_{r/2}, \mathrm{H}^0(\lambda))$, for the Suzuki groups (Subsection 3.2), the Ree groups of type G_2 (Subsection 3.3) and of type F_4 (Subsection 3.4). Moreover, we calculate the extensions between simple modules for the classical and for the exotic Frobenius kernels.

Then in Chapter 4, we focus on the Ree groups of type F_4. We consider a certain truncation of $\mathrm{Ind}_{G(\sigma)}^{G} k$ and relate the finite group cohomology to the algebraic group cohomology. In Section 4.1, we precisely bound the weights in our truncated category (see Lemma 4.1.3), performing many spectral sequence computations involving half Frobenius kernels, instead of the classical ones. Thus, we ensure the sharpness of our bound on the size of the finite group using these methods. We observe, rather surprisingly, that the Ree groups of type F_4 exhibit very different behaviour compared to the other finite groups of Lie type (be it Chevalley, twisted or, indeed, Suzuki or Ree groups of type G_2).

In order to see this, first recall some of the terminology used in (BNP04), (BNP$^+$15) and (PSS13). Let $\mathcal{C}_t$ be the full subcategory of all finite-dimensional G-modules whose composition factors $L(\nu)$ have highest weights in the set $\pi_t = \{\nu \in X_+ : \langle \nu, \alpha_0^\vee \rangle < t\}$. The weight $\nu \in \pi_t$ is $(t-1)$-small.

Now, let σ denote the appropriate strict endomorphism, as discussed above. In Remark 4.1.4(a), we observe that in the case $G = F_4$, $p = 2$, $\sigma = F^{r/2}$ for r odd, the non-vanishing of $\mathrm{Ext}^1_{G_\sigma}(L(\lambda) \otimes V(\nu)^{(\sigma)}, L(\mu) \otimes \mathrm{H}^0(\nu))^{(-\sigma)}$ implies that the weights ν are $(h+4)$-small. This is in contrast to (BNP06, Lemma 5.2), (BNP$^+$15, Theorem 2.3.1) and an analogous argument for Suzuki-Ree groups, where for all (G, p, σ) aside from the case we consider, one has that the non-vanishing of $\mathrm{Ext}^1_{G_\sigma}(L(\lambda) \otimes V(\nu)^{(\sigma)}, L(\mu) \otimes \mathrm{H}^0(\nu))^{(-\sigma)}$ implies that the weights ν are $(h-1)$-small. This comes as a surprise, given the fact that similar methods were used.

In Section 4.2, we turn our attention to finite group extensions. We find that self-extensions between simple $kG(\sigma)$-modules vanish, provided $r \geq 13$. (Theorem 4.2.4). Finally, in Theorem 4.2.5, we conclude that, for $r \geq 13$, the Ext^1 group between simple $kG(\sigma)$-modules is isomorphic to the Ext^1 group between a specific pair of σ-restricted simple G-modules (which depends on the pair of $kG(\sigma)$-modules).

Chapter 2

Background

2.1 Categories

We begin with a brief discussion of notions in category theory necessary for our setup and following chapters. We direct the reader to (ML98) and (Kra22) for more details, with the aim to establish the nomenclature assumed in (Mil17, Appendix A) and (Jan03).

Definition 2.1.1. A *category* $\mathcal{C}$ is a triple $\mathcal{C} = (\mathrm{Ob}\,\mathcal{C}, \mathrm{Hom}\,\mathcal{C}, \circ)$ consisting of the following: the class of objects in $\mathcal{C}$, $\mathrm{Ob}\,\mathcal{C}$, the class of morphisms $\mathrm{Hom}\,\mathcal{C}$, which is the union of the sets $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and $\circ$ is a partial binary operation on $\mathrm{Hom}\,\mathcal{C}$. The triple $\mathcal{C}$ satisfies the following conditions:

(a) To each pair $A, B \in \mathrm{Ob}\mathcal{C}$, we associate the set of morphisms from A to B, denoted by $\mathrm{Hom}_{\mathcal{C}}(A, B)$, so that if $(A, B) \neq (C, D)$, then the intersection $\mathrm{Hom}_{\mathcal{C}}(A, B) \cap \mathrm{Hom}_{\mathcal{C}}(C, D) = \varnothing$.

(b) The operation $\circ$ is a well-defined associative unital composition. That is, for each $A, B, C \in \mathrm{Ob}\,\mathcal{C}$ the operation

$$\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C), \text{ given by } (\beta, \alpha) \mapsto \beta \circ \alpha$$

is well-defined and satisfies associativity and existence of the identity morphism.

The category $\mathcal{C}$ is said to be *small* if $\mathrm{Ob}\,\mathcal{C}$ is a set.

Definition 2.1.2. Let k be a field. A category $\mathcal{C}$ is said to be a *k-linear category* if for each pair of objects A, B in $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is equipped with a k-vector space structure such that the composition of morphisms in $\mathcal{C}$ is a k-bilinear map.

Definition 2.1.3. A *preadditive category* is a category in which all Hom-sets are abelian groups and composition of morphisms is bilinear. An *additive category* is a preadditive category that admits finite direct sums.

Definition 2.1.4. A category $\mathcal{C}$ is an *additive category* if the following conditions are satisfied.

(a) For any finite set of objects $A_1, A_2, \ldots, A_n$ in $\mathcal{C}$, there exists a direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_n$.

(b) For each pair of objects A, B in $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is equipped with an abelian group structure.

(c) For each triple A, B, C in $\mathcal{C}$, the composition

$$\circ : \mathrm{Hom}_{\mathcal{A}}(B, C) \times \mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{A}}(A, C)$$

is bilinear.

(d) There exists an object 0 in $\mathcal{C}$, called the zero object, such that 1_0 is the zero element of the abelian group $\mathrm{Hom}_{\mathcal{C}}(0, 0)$.

Example 2.1.5. (Module Categories) Let k be a field and A be a finite-dimensional associative k-algebra; we assume that A-modules of any k-algebra A are left A-modules unless otherwise specified.

(a) Left (respectively right) A-modules together with their module morphisms form a category, denoted by $\mathrm{Mod}\, A$ (respectively $\mathrm{Mod}\, A^{op}$). Moreover, the category of finitely generated left A-modules is denoted by $\mathrm{mod}\, A$ and the one of finitely generated right A-modules is denoted by $\mathrm{mod}\, A^{op}$.

(b) Let M be a left A-module and note that for any left A-module N, we have that the set of morphisms from M to N, denoted by $\mathrm{Hom}_A(M, N)$ is a k-vector space and $\mathrm{Mod}\, A$ is a k-linear category. It may be shown that $\mathrm{Hom}_A(M, -)$ is a *covariant* functor, and that $\mathrm{Hom}_A(-, M)$ is a *contravariant* functor.

Definition 2.1.6.

An *additive functor* between additive categories $\mathcal{A}$ and $\mathcal{B}$ is a functor $F : \mathcal{A} \to \mathcal{B}$ such that the maps

$$F\left(A, A'\right) : \mathrm{Hom}_{\mathcal{A}}\left(A, A'\right) \to \mathrm{Hom}_{\mathcal{B}}\left(F(A), F\left(A'\right)\right)$$

are group homomorphisms for all pairs of objects A and A' of $\mathcal{A}$. Additive categories can be thought of as a generalization of rings for which we can define the notions of ideals and quotient categories.

Definition 2.1.7. An *ideal (of morphisms)* $\mathcal{I}$ in $\mathcal{A}$ is a collection of subgroups $\mathcal{I}(A,B) \subseteq \operatorname{Hom}_{\mathcal{A}}(A,B)$, for every pair of objects A and B of $\mathcal{A}$, that is stable under composition. More specifically, this means that $b \circ f \circ a \in \mathcal{I}(A',B')$ for all $a \in \operatorname{Hom}_{\mathcal{A}}(A',A), b \in \operatorname{Hom}_{\mathcal{A}}(B,B')$ and $f \in \mathcal{I}(A,B)$.

Definition 2.1.8. Let $\mathcal{I}$ be an ideal of morphisms in $\mathcal{A}$. The quotient category $\mathcal{A}/\mathcal{I}$ has the same objects as $\mathcal{A}$ and Hom-sets

$$\operatorname{Hom}_{\mathcal{A}/\mathcal{I}}(A,B) := \operatorname{Hom}_{\mathcal{A}}(A,B)/\mathcal{I}(A,B).$$

Composition of morphisms in $\mathcal{A}/\mathcal{I}$ is induced by the composition law in $\mathcal{A}$. Note that the composition of morphisms in the quotient category is well-defined because $\mathcal{I}$ is stable under composition. It is straightforward to check that $\mathcal{A}/\mathcal{I}$ is an additive category and that the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}$ is additive.

Definition 2.1.9. A *Krull-Schmidt category* is an additive category where every object is isomorphic to a finite direct sum of objects having local endomorphism rings.

Note that Krull-Schmidt categories are well-behaved with respect to taking quotients by ideals of morphisms.

Lemma 2.1.10. Let $\mathcal{A}$ be a Krull-Schmidt category and let $\mathcal{I}$ be an ideal of morphisms in $\mathcal{A}$. Then $\mathcal{A}/\mathcal{I}$ is a Krull-Schmidt category.

Definition 2.1.11. Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $F, G : \mathcal{A} \to \mathcal{B}$ be functors; we may assume both are covariant without loss of generality. Let $\eta = \{\eta_A\}_{A \in \operatorname{Ob}\mathcal{A}}$ be a family of morphisms in $\mathcal{B}$ such that for each $A \in \operatorname{Ob}\mathcal{A}$, we have that $\eta_A \in \operatorname{Hom}_{\mathcal{B}}(F(A), G(A))$. We say that η is a *natural transformation* if for each $A, A' \in \operatorname{Ob}\mathcal{A}$ and each $\alpha \in \operatorname{Hom}_{\mathcal{A}}(A, A')$ the following diagram commutes

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(\alpha)"'] & G(A) \arrow[d, "G(\alpha)"] \\ F(A') \arrow[r, "\eta_{A'}"] & G(A') \end{tikzcd}$$

We say that η is a *natural equivalence* if in addition η_A is an isomorphism for each $A \in \operatorname{Ob}\mathcal{A}$.

A functor is said to be an *equivalence of categories* if it is fully faithful and essentially surjective. A sufficiently strong version of the axiom of global choice then implies the existence of a quasi-inverse to the functor. A natural transformation of functors is sometimes known as a *map of functors*.

Definition 2.1.12. Given categories $\mathcal{A}$ and $\mathcal{I}$, the functors $F : \mathcal{I} \to \mathcal{A}$ form the objects of the *functor category* $\mathcal{A}^{\mathcal{I}}$, and the morphisms in $\mathcal{A}^{\mathcal{I}}$ from F to G are the natural transformations $\eta : F \to G$.

Lastly, we present an example that appears in the following section.

Example 2.1.13. The *Yoneda embedding* is the functor $h : \mathcal{I} \to \mathrm{Set}^{\mathrm{op}}$ given by $h_i(j) = \mathrm{Hom}_{\mathcal{I}}(j, i)$, which is a fully faithful functor.

2.2 Algebraic Group Schemes

In this section we follow (Mil17) and (Jan03).

2.2.1 Algebraic schemes

First we establish some notational conventions from (Mil17). Let Alg_k^0 denote the category of *finitely generated k-algebras*. The objects of Alg_k^0 form a set, and so Alg_k^0 is small. We call the objects of Alg_k^0 small.

The inclusion functor $\mathrm{Alg}_k^0 \hookrightarrow \mathrm{Alg}_k$ is an equivalence of categories. Choosing a quasi-inverse amounts to choosing an ordered set of generators for each finitely generated k-algebra. Once a quasi-inverse has been chosen, every functor on Alg_k^0 has a well-defined extension to Alg_k.

We define a *k-functor* to be any functor $\mathrm{Alg}_k^0 \to \mathrm{Set}$. For example, if A is a k-algebra, then we get a k-functor $h^A : R \to \mathrm{Hom}(A, R)$.

Definition 2.2.1. For any k-algebra R, we can define a k-functor $\mathrm{Spm}_k R$ through $(\mathrm{Spm}_k R)(A) = \mathrm{Hom}_{k\text{-alg}}(R, A)$ for all A and

$$(\mathrm{Spm}_k R)(\varphi) : \mathrm{Hom}_{k\text{-alg}}(R, A) \to \mathrm{Hom}_{k\text{-alg}}\left(R, A'\right), \quad \alpha \mapsto \varphi \circ \varphi$$

for all homomorphisms $\varphi : A \to A'$. We say that $\mathrm{Spm}_k R$ is the *spectrum of* R.

Definition 2.2.2. Let k be a field and let R be any k-algebra. We define an affine scheme over k to be any k functor isomorphic to $\mathrm{Spm}_k R$.

Abstractly, we may define an affine scheme over k to be a representable functor from finitely generated commutative k-algebras to sets.

Definition 2.2.3. An affine scheme X over k is *algebraic* if the coordinate algebra $k[X]$ is isomorphic to a k-algebra of the form $k[T_1, \ldots, T_n]/I$, for some $n \in \mathbb{N}$ and some finitely generated ideal I of the polynomial ring $k[T_1, \ldots, T_n]$.

We say it is *reduced* if $k[X]$ contains no nilpotent elements apart from 0.

2.2.2 Algebraic Group Schemes

A *k-group functor* is a functor from the category of all k-algebras to the category of groups.

Any k-group functor may be regarded as a k-functor by composing it with the forgetful functor from Grp to Set.

Let G, H be two group functors. Denote by $\mathrm{Mor}(G, H)$ the set of natural transformations from G to H considered as k-functors. and by $\mathrm{Hom}(G, H)$ the set of all morphisms from G to H considered as k-group functors. So $\mathrm{Hom}(G, H)$ consists of all those $f \in \mathrm{Mor}(G, H)$ with $f(A)$ a group homomorphism for each k-algebra A.

These elements are called homomorphisms from G to H. Let $\mathrm{Aut}(G)$ be the group of all automorphisms of the k-group functor G.

An *affine k-group scheme* is a k-group functor that is an affine scheme over k when considered as a k-functor.

In this thesis we will use the term *algebraic group* only to refer to an affine k-group scheme of finite type over a field k. This means that it is a k-group functor. An algebraic k-group is a k-group scheme that is algebraic as an affine scheme. A k-group scheme is called *reduced* if it is so as an affine scheme.

2.2.3 Properties

Let A be a k-algebra and let G be a k-group functor.

A *subgroup functor* of G is a subfunctor H of G such that each $H(A)$ is a subgroup of $G(A)$. The intersection of subgroup functors is again a subgroup functor. The inverse image of a subgroup functor under a homomorphism is again one. A direct product of k-group functors is again a k-group functor.

A subgroup functor H of G is called *normal* if each $H(A)$ is a normal subgroup of $G(A)$. Again, normality is preserved under taking intersections and inverse images under homomorphisms. The *kernel* $\ker\varphi$ of a homomorphism $\varphi : G \to G'$ is always a normal subgroup functor.

2.3 Restricted Lie Algebras

Definition 2.3.1. A Lie algebra $\mathfrak{g}$ over k is a vector space equipped with a binary operation $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying $[x,x] = 0$ for all $x \in \mathfrak{g}$ and the Jacobi identity

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0, \quad \forall x,y,z \in \mathfrak{g}$$

Definition 2.3.2. Let $\mathfrak{g}$ be a finite-dimensional modular Lie algebra over k. Following [J], a map $[p] : \mathfrak{g} \to \mathfrak{g}, \quad g \mapsto g^{[p]}$ is called a *p-structure of* $\mathfrak{g}$ and we call $\mathfrak{g}$ a *restricted Lie algebra* if

(i) $\mathrm{ad}_{g[p]} = (\mathrm{ad}_g)^p$ for all $g \in \mathfrak{g}$;

(ii) $(\alpha g)^{[p]} = \alpha^p g^{[p]}$ for all $g \in \mathfrak{g}$ and $\alpha \in k$;

(iii) $(g+h)^{[p]} = g^{[p]} + h^{[p]} + \sum_{1\leq i\leq p-1} s_i(g,h)$, where the $s_i(g,h)$ can be obtained from $(\mathrm{ad}_{\lambda g+h})^{p-1}(g) = \sum_{1\leq i\leq p-1} i s_i(g,h)\lambda^{i-1}$,

In fact, conditions (*ii*) and (*iii*) are redundant due to the following result:

Theorem 2.3.3. (SF88, 2.2.3) Let $(e_j)_{j\in J}$ be a basis of a Lie algebra $\mathfrak{g}$ such that there exist $y_j \in \mathfrak{g}$ with $(\mathrm{ad}\, e_j)^p = \mathrm{ad}\, y_j$. Then there is a unique p-mapping $[p] : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that

$$e_j^{[p]} = y_j$$

for all $j \in J$.

Let A be an associative algebra over k. We call a *derivation* of A a k-linear map $D : A \to A$ such that, for all $f,g \in A$, we have $D(fg) = fD(g) + D(f)g$. We denote by $\mathrm{Der}_k(A)$ the space of all derivations of A. Note that $\mathrm{Der}_k(A)$ may be made into a Lie algebra using the Lie bracket $[D_1,D_2] = D_1 \circ D_2 - D_2 \circ D_1$.

Now suppose G is an algebraic k-group, with $k[G]$ its coordinate algebra. Given $g \in G$, we define $\lambda_g : k[G] \to k[G]$ by $(\lambda_g \cdot f)(x) = f(g^{-1}x)$, for all $f \in k[G]$ and $x \in G$.

Then, the *Lie algebra of* G is defined as

$$\mathrm{Lie}(G) := \{D \in \mathrm{Der}_k(k[G]) \mid D\lambda_g = \lambda_g D, \forall g \in G\}.$$

It follows that $\mathrm{Lie}(G)$ is a Lie algebra of the same dimension of G, and G acts on it via the *adjoint action*.

By (Jan04, A.2), for an algebraic k-group G, the Lie algebra $\mathrm{Lie}(G)$ has a natural structure as a restricted Lie algebra.

2.4 Reductive groups, Roots and Notation

We fix an algebraically closed field k of characteristic $p > 0$.

2.4.1 Reductive groups and Roots

We define a *torus* T to be an algebraic group over k that is isomorphic to a finite number of copies of $\mathbb{G}_m$. Since k is algebraically closed, all tori considered are *split*. We have that split tori are smooth, connected diagonalisable algebraic groups.

By (Mil17, 17.10), maximal tori in algebraic groups are all conjugate by an element of $G(k)$.

Since T is a torus, T is diagonalisable in every representation (cf. (Mil17, 12.12)); that is, every representation (M, r) of T admits a weight space decomposition:

$$M = \bigoplus_{\lambda \in X} M_\lambda.$$

The characters χ of T such that the eigenspaces $M_\lambda \neq 0$ are called *weights* of T on M and the non-zero eigenspaces are called *weight spaces*.

An algebraic group G over k is *unipotent* if every non-zero representation of G has a non-zero fixed vector. This is equivalent to saying its only irreducible representations are vector spaces equipped with a trivial action on G.

There is a maximal smooth connected normal unipotent subgroup of G called the *unipotent radical* of G, denoted by $R_u(G)$. Similarly, there is a maximal smooth connected normal solvable subgroup of G, denoted by $R(G)$.

An algebraic group G is *reductive* if it is a connected algebraic k-group whose unipotent radical is trivial, $R_u(G) = e$. G is *semisimple* if it is a connected algebraic k-group whose radical is trivial, $R(G) = e$.

An algebraic group G is *simple* if it is a connected non-trivial semisimple algebraic k-group with no proper normal subgroup.

Note that a simple algebraic group is reductive, since it has no non-trivial normal subgroup and, hence, has no normal unipotent subgroup (ie. $R_u(G) = e$).

A *Borel subgroup* B of G is a maximal smooth, connected solvable subgroup of G. By (Bor66, 4.2), $B = U \rtimes T$, where T is a maximal torus and U is its unipotent radical. Every weight $\lambda \in X \cong X(T)$ gives rise to a one-dimensional B-module k_λ, where T acts via λ and U acts trivially.

Let (G, T) be a split reductive group over k, and $\mathrm{Ad} : G \to GL_{\mathfrak{g}}$ the adjoint representation. Since T acts on $\mathrm{Lie}(G)$ and T is diaogalisable, we have a weight space decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in X(T)} \mathfrak{g}_\lambda,$$

where $\mathfrak{g}_0 = \mathfrak{g}^T$ and $\mathfrak{g}_\lambda$ denotes the subspace on which T acts through $\alpha \in X(T)$, a non-trivial character. Note that $\mathfrak{g}_0 = \mathrm{Lie}(T)$.

We call the characters α of T in the above decomposition the *roots* of (G, T) and they form a finite subset $\Phi(G, T)$ of $X(T)$, called the *root system* of G with respect to T. (cf. (Mil17, 21.a)) We use Bourbaki's conventions.

Let B be a Borel subgroup containing T a maximal torus and denote by $\Phi(B)$ the roots corresponding to B. We follow (Bor91, 13.18) and have

$$\mathrm{Lie}(B) = \mathfrak{b} = \mathfrak{b}^T \oplus \bigoplus_{\lambda \in \Phi(B)} \mathfrak{b}_\lambda,$$

where $\mathfrak{b}^T = \mathfrak{g}^T$, and if $\lambda \in \Phi(B)$, then $\mathfrak{b}_\alpha = \mathfrak{g}_\alpha$. Thus,

$$\mathrm{Lie}(B) = \mathfrak{g}^T \oplus \bigoplus_{\lambda \in \Phi(B)} \mathfrak{g}_\lambda,$$

and it follows that $\mathrm{Lie}(B)$ contains the roots associated to B. By (Bor91, Theorem 13.8(5a)), Φ is the disjoint union of $\Phi(B)$ and $-\Phi(B)$.

We conclude that $\mathrm{Lie}(B)$ contains exactly half of the roots, $\Phi(B)$, and the other half are their negatives. For the purpose of induction etc., it is convenient to assert that Lie(B) consists of the negative roots (cf. (Jan03, II, 1.7)).

Thus, we have $\Phi(G,T)^- := \Phi(B)$ and $\Phi(G,T)^+ := -\Phi(B)$ and $\Phi(G,T) = \Phi(G,T)^+ \sqcup \Phi(G,T)^-$, a decomposition of Φ into positive roots and negative roots, respectively.

By (Mil17, 21.41), such a decomposition corresponds bijectively to a set of simple roots Π and to a choice of a Borel subgroup containing the fixed maximal torus T.

An algebraic group is *simply-connected* if the character group of a maximal torus equals the full weight lattice (cf. (Hum75, 31.1)).

2.4.2 Notation

We fix notation. Now and for the remainder of this thesis, we fix an algebraically closed field k of characteristic $p > 0$ and denote by G a simply-connected simple algebraic group scheme over k.

We denote by T a maximal split torus in G and let Φ be the corresponding root system; let $\Pi = \{\alpha_1, ..., \alpha_n\}$ be the set of simple roots in the Bourbaki ordering (Bou82, Planches) and α_0 the maximal short root. Let B denote a Borel subgroup containing T, corresponding to the negative roots, and let U denote its unipotent radical. For our choice of σ, all these subgroups can be chosen to be σ-invariant.

Let $\langle \cdot, \cdot \rangle$ be the standard inner product on the Euclidean space $\mathbb{E} := \mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$. Then, let $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ be the coroot of $\alpha \in \Phi$ and let h be the Coxeter number of the root system.

We have the weight lattice $X(T) = X = \bigoplus \mathbb{Z}\omega_i$, for ω_i the fundamental dominant weights satisfying $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$, for α_j a simple root. Then $X_+ = \{\lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle \geq 0, \lambda \in \Pi\}$ is the cone of dominant weights.

Let W be the Weyl group of Φ, generated by the set of simple reflections $\{s_\beta \mid \beta \in \Pi\}$. For $\alpha \in \Phi$, $s_\alpha : \mathbb{E} \to \mathbb{E}$ is the orthogonal reflection in the hyperplane $H_\alpha \subset \mathbb{E}$ of vectors orthogonal to α. Write $\ell : W \to \mathbb{N}$ for the standard length function on W: for $w \in W$, $\ell(w)$ is the minimum number of simple reflections required to write w as a product of simple reflections. Moreover, note that W acts naturally on $X(T)$ via the dot action. (cf. (Jan03, II.1.5))

2.5 Strict Endomorphisms, Finite Groups of Lie Type and Frobenius Kernels

For the remainder of this thesis, we let G be a simple, simply-connected algebraic group over a field k of characteristic $p > 0$.

2.5.1 Strict Endomorphisms

Definition 2.5.1. (BNP$^+$15) We call a surjective endomorphism $\sigma : G \to G$ *strict* if the group $G(\sigma)$ of σ-fixed points is finite.

Definition 2.5.2. (GLS98, Definition 1.15.1) Let G be an algebraic group over k. Then

(a) $\mathrm{Aut}_0(G)$ is the group of all automorphisms of G as an algebraic group. Moreover, $\mathrm{Aut}(G)$ is the group of all automorphisms of G as an abstract group.

(b) $\mathrm{Aut}_1(G)$ is the set of all $\sigma \in \mathrm{Aut}(G)$ such that either σ or σ^{-1} is an endomorphism of G; note that $\mathrm{Aut}_1(G)$ is a group.

Theorem 2.5.3. ((GLS98, Theorem 1.15.6)) Let G be a simple algebraic group over k. Then, the following hold: if σ is an endomorphism of algebraic groups, then either $\sigma \in \mathrm{Aut}_0(G)$ or σ is a strict endomorphism.

An *isogeny* is a surjective morphism of algebraic groups with finite kernel.

Theorem 2.5.4. ((GLS98, Theorem 2.1.6)) Let G be a connected algebraic group over k and let σ be an endomorphism of G. Then there exist a maximal torus T and a Borel subgroup B of G, with B containing T, which are σ-invariant.

Now we fix some notation. Since G is simply-connected, we may also call it *universal.* Let U_α (or X_α) denote the 1-parameter unipotent subgroup of G corresponding to root α in Φ. Note that U_α is unipotent and that there exists an isomorphism as algebraic groups $x_\alpha : k \to U_\alpha$. Then, we have $tx_\alpha(c)t^{-1} = x_\alpha(\alpha(t)c)$, for all $t \in T$, $\alpha \in \Phi$ and $c \in k$.

Next, we discuss graph automorphisms of the Dynkin Diagram corresponding to Φ.

Suppose G is a simple algebraic group over k. Then, for any isometry ρ of the set of simple roots Π, there exists exactly one (since G is simply-connected) automorphism of G as an algebraic group, denoted by γ_ρ, such that $\gamma_\rho(x_\alpha(t)) = x_{\alpha^\rho}(t)$, for all $\alpha \in \pm\Pi$ and $t \in k$ (cf. (GLS98, 1.15.2)).

The non-trivial isometries are as follows. For root systems of type A_n, D_n and E_6, there exists a reflection, which acts in the following way: $\alpha_i \mapsto \alpha_{n+1-i}$ for A_n, as $\alpha_n - 1 \mapsto \alpha_n$ and $\alpha_i \mapsto \alpha_i$ otherwise for D_n for A_n and $\alpha_1 \mapsto \alpha_5$, $\alpha_2 \mapsto \alpha_4$ for E_6.

In addition, for D_4 there exists a *triality*, a rotation of order 3 sending $\alpha_2 \mapsto \alpha_2$, $\alpha_1 \mapsto \alpha_3$ and $\alpha_4 \mapsto \alpha_1$.

For all other choices of Φ, there exist only trivial graph automorphisms. Since G is simply-connected, the set of all automorphisms γ_ρ, denoted by Γ_0, is isomorphic to $\mathrm{Aut}(\pi)$. Note that in the cases A_n, D_n and E_6, we have $\mathrm{Aut}(\pi) \cong \mathbb{Z}_2$ and the order of ρ is 2.

Now, let us consider strict endomorphisms one may encounter. Let G be a simple algebraic group over k, with root system Φ and a set of simple roots Π. Then, with respect to some set of Chevalley generators of G, the following hold:

(a) For each power $q = p^r$ of p with $r \in \mathbb{Z}$, there is a unique strict endomorphism F^r such that $F^r(x_\alpha(t)) = x_\alpha(t^q)$, for all $\alpha \in \Pi$ and $t \in k$.

(b) Suppose $(\Phi, p) = (C_2, 2)$, $(F_4, 2)$, $(G_2, 3)$. In these cases, the root system is isomorphic to its dual root system, $\Phi \cong \Phi^\vee$ and the Dynkin diagram contains an edge of multiplicity p. That is, there exist long and short roots, for which the ratio of lengths squared equals p. Then there exists unique angle preserving and length changing bijection and a unique strict endomorphism, a special isogeny denoted $\tau = F^{1/2} : G \to G$, such that it maps $x_\alpha(c) \mapsto x_{\alpha^\vee}(c)$ when α is long and $x_\alpha(c) \mapsto x_{\alpha^\vee}(c^p)$ when α is short. Note that its square, the composite $\tau^2 = \tau \circ \tau : G \to G$ is just the standard Frobenius map relative to p, since G is generated by its root subgroups.

When τ is restricted to a map of maximal tori $T \to T'$, the comorphism $\tau^* : X(T') \to X(T)$ sends $\alpha^\vee \mapsto \alpha$ when α is long and sends $\alpha^\vee \mapsto p\alpha$ when α is short. Corresponding fundamental weights are mapped similarly.

We introduce some more notation, (cf. (Hum06, 5.3)). We denote the set of weights λ with $\langle \lambda, \alpha^\vee \rangle = 0$ for α short by $X(T)_L$. Similarly, we denote the set of weights λ with $\langle \lambda, \alpha^\vee \rangle = 0$ for α long by $X(T)_L$. A weight $\lambda \in X(T)_L$ is expressed as a $\mathbb{Z}$-linear combination of fundamental dominant weights corresponding to long simple roots and we say that it has *long support*. Similarly, we say $\lambda \in X(T)_S$ has *short support*.

Therefore, τ^* sends weights with short support to corresponding weights with long support. Furthermore, τ^* sends weights with long support to p times the corresponding weights with short support.

Then, by (GLS98, Theorem 1.15.7, Theorem 2.2.3), we obtain all of the possible forms of a strict endomorphism σ of G.

Theorem 2.5.5. Let G be a simple, simply-connected algebraic group over k and assume the notation above. Then a strict endomorphism $\sigma : G \to G$ has the following forms (up to conjugation by an inner automorphism of G):

(a) Let $\rho \in \mathrm{Aut}(\Pi)$ of order d and let $q = p^r$, for r a positive power of p. Then $\sigma = \gamma_\rho \circ F^r$.

(b) Suppose $(\Phi, p) = (C_2, 2)$, $(F_4, 2)$, $(G_2, 3)$. Then $\sigma = \tau \circ F^r = \tau^{2r+1}$, where $q = p^r$.

2.5.2 Finite Groups of Lie Type

Let G be a simple, simply-connected algebraic group over an algebraically closed field k of characteristic $p > 0$. Then, for a strict endomorphism $\sigma : G \to G$, the fixed point set of the points, $G(\sigma) := G(k)^\sigma$, is a *finite group of Lie type*. There are three cases to consider (see (GLS98, Chapter 2) and (BNP$^+$15)). Now we follow (BNP$^+$15) and direct the interested reader to their discussion for more details.

I. The Finite Chevalley Groups

For a positive integer r, let $q = p^r$. In these cases, there are only trivial graph automorphisms, so $\sigma = F^r$, the standard Frobenius map relative to q. Set $G\left(F^r\right) = G\left(\mathbb{F}_q\right)$, the group of F^r-fixed points. Note that The resulting finite group of fixed points coincides with the group of rational points $G\left(\mathbb{F}_q\right)$.

II. The Twisted Steinberg Groups

Let $\rho \in \mathrm{Aut}(\Pi)$ be an isometry of order $d > 1$ and let γ_ρ be a nontrivial graph automorphism of G stabilizing B and T. That is, we are in the cases A_ℓ (with $\ell > 1$), D_ℓ (for $\ell \geq 4$), and E_6. For a positive integer r, set $\sigma = F^r \circ \gamma_\rho = \gamma_\rho \circ F^r : G \to G$. We say that σ is *twisted*. Then let $G(\sigma) := {}^d\Phi(q^2)$ be the finite group of σ-fixed points. Thus, $G(\sigma) = {}^2A_n\left(q^2\right), {}^2D_n\left(q^2\right), {}^3D_4\left(q^3\right)$, or ${}^2E_6\left(q^2\right)$. The group of fixed points is isomorphic to the group of rational points over $\mathbb{F}_q$ of a quasi-split but non-split group of the same type as G.

III. The Suzuki Groups and Ree Groups

Suppose $(\Phi, p) = (C_2, 2)$, $(F_4, 2)$, $(G_2, 3)$. Let $\tau := F^{1/2} : G \to G$ be a fixed purely inseparable isogeny satisfying $\left(F^{1/2}\right)^2 = F$. For an odd positive integer r, set $\sigma = F^{r/2} = \left(F^{1/2}\right)^r$. Thus, $G(\sigma) = {}^2C_2\left(2^{\frac{2m+1}{2}}\right), {}^2F_4\left(2^{\frac{2m+1}{2}}\right)$, or, ${}^2G_2\left(3^{\frac{2m+1}{2}}\right)$.

2.5.3 Frobenius Kernels

Let σ be a strict endomorphism. Then, by (GLS98, Theorem 2.1.2), it follows that $\ker(\sigma)$ is a finite subgroup of G.

In all the above cases, the group scheme-theoretic kernel G_σ of σ plays an important role. In case I, where $\sigma = F^r$, this kernel is commonly denoted G_r, and it is called the *r-th classical Frobenius kernel*. In case II, with $\sigma = F^r \circ \theta, \theta$ is an automorphism so that $G_\sigma = G_r$.

Classical Frobenius Kernels

We follow (Jan03, I.9). Let G be a k-group functor and $G^{(r)}$ is also a k-group functor, with $\sigma : G \to G^{(r)}$ a homomorphism of k-group functors.

The kernel, denoted G_r is a normal subgropup functor of G, known as the *r-th Frobenius kernel* of G. By (Jan03, I.9.4), here exists an ascending chain of normal subgroup functors of G

$$G_1 \subset G_2 \subset G_3 \subset \dots$$

Moreover, given H a subgroup functor of G, it follows that $H^{(r)}$ is a subgroup functor of $G^{(r)}$ and we have $H_r = H \cap G_r$ (cf. (Jan03, I.9.4(2))).

By (Jan03, Proposition 9.5), if G is a reduced k-group, then each F_G^r induces the isomorphism $G/G_r \cong G^{(r)}$. Finally, each Frobenius kernel G_r is an infinitesimal k-group (cf. (Jan03, I.9.6(1))).

In case III, we have $\sigma = F^{r/2} = \left(F^{1/2}\right)^r$, for r an odd positive integer. Denote the kernels G_σ by $G_{r/2}$ and we call them *exceptional/exotic Frobenius kernels*.

Exotic Frobenius Kernels

First, consider the exceptional isogeny $\tau : G \to G$. Let $G_\tau = \ker\tau$ and let $\mathfrak{g}_\tau = \mathrm{Lie(G_\tau)}$. By (Sin94b, 1.2), we have that G_τ is an infinitesimal group scheme of height one and, hence, its representation theory is equivalent to the representation theory of g_τ.

Moreover, the coordinate algebra of G_τ, $k[G_\tau]$, is the dual of the restricted enveloping algebra of the subalgebra of the Lie algebra of G generated by the short simple roots (cf. (BNP$^+$15, 2.2)).

By (CGP15, 7.16), we have that G_τ is normal in G, and G_τ is the unique minimal non-central normal k-subgroup scheme of G. We further discuss G_τ and its Lie algebra in Chapter 3, Section 3.1.

Now let $G_\sigma = G_{r/2}$, for an odd positive integer r. Using (Bor91, Proposition 17.9), it may be shown that we get an ascending chain of normal subgroup functors of G:

$$G_\tau \subset G_{\tau^2} = G_1 \subset G_{\tau^3} \subset G_{\tau^4} = G_2 \dots$$

Furthermore, each exotic Frobenius kernel $G_{r/2}$ is an infinitesimal k-group. For details, we direct the reader to (BT73, Section 3) and (Bor91, Section 17).

We have $G/G_{r/2} \cong G^{(r/2)}$, where $G^{(r/2)}$ has coordinate algebra $k[G]^{(r/2)}$ (cf. (BNP$^+$15, Remark 2.2.1(a))).

The Frobenius kernels G_r play a central role in the representation theory of G. These results are all available in cases I or II. In case III, for $G_{r/2}$, many of these results hold as well. We discuss various results concerning infinitesimal representation theory in Section 2.7.

2.6 Module Categories

Throughout this thesis, whenever we discuss modules over a group scheme or an algebra, they are assumed to be possibly infinite-dimensional, unless stated otherwise.

This section follows (Jan03, I.2). We denote the category of (finite-dimensional) modules over a k-group scheme H by $\mathrm{Mod}(H)$.

Definition 2.6.1 (G-Module)**.** A vector space V is said to be a *G-module* if it acts linearly on the associated vector group V_a; i.e. there is a natural transformation $\alpha : G \times V_a \to V_a$ such that for any k-algebra A, the map $G(A) \times V_a(A) \to V_a(A)$ is an action of the group

$G(A)$ on $V_a(A) = V \otimes_k A$ satisfying $g(a.v) = ag(v)$.

For G an algebraic group with a corresponding maximal split torus T, there exists an isomorphism between the weight lattice X and the character group $X(T)$ of T (that is, the group of k-group scheme homomorphisms from T to the multiplicative group scheme). Every T-module M admits a weight space decomposition, since T is a diagonalisable group scheme:

$$M = \bigoplus_{\lambda \in X} M_\lambda.$$

Recalling $B = U \rtimes T$, every weight $\lambda \in X(T)$ gives rise to a one-dimensional B-module k_λ, where T acts via λ and U acts trivially.

We refer to the objects of $\mathrm{Mod}(G)$ as *G-modules*; we write $\mathrm{Hom}_G(M,N)$ and $\mathrm{Ext}^i_G(M,N)$ for the space of homomorphisms and the Ext-groups between G-modules M and N, respectively. A G-module is *completely reducible* if it is isomorphic to a direct sum of simple G-modules, and the *socle* $\mathrm{soc}_G M$ is the largest completely reducible G-submodule of M. Moreover, given a simple G-module E, the sum of all simple G-submodules of M isomorphic to E is called the *E-isotypic component* of $\mathrm{soc}_G M$ and we denote it by $(\mathrm{soc}_G M)_E$.

The *radical* $\mathrm{rad}_G M$ is defined to be the smallest G-submodule of M such that $M/\mathrm{rad}_G M$ is completely reducible, and $\mathrm{hd}_G M := M/\mathrm{rad}_G M$ is known as the *head* of M. We say that a G-module is *uniserial* if it has a unique composition series. Recall that every G-module M has a weight space decomposition

$$M = \bigoplus_{\lambda \in X} M_\lambda;$$

$\lambda \in X$ is called a *weight* of M if $M_\lambda \neq 0$. The character of M is defined as the element

$$\mathrm{ch}\, M = \sum_{\lambda \in X} \dim (M_\lambda) \cdot e^\lambda$$

of the group ring $\mathbb{Z}[X]$. It has a basis consisting of formal exponentials $\{e^\lambda \mid \lambda \in X\}$, where $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$ for $\lambda, \mu \in X$.

Use W to denote $N_G(T)/C_G(T)$, which acts on $X(T)$; the standard action of W on X induces an action of W on $\mathbb{Z}[X(T)]$ by ring automorphisms. It turns out that the characters of all G-modules belong to the ring $\mathbb{Z}[X(T)]^W$ of W-fixed points in $\mathbb{Z}[X(T)]$. For G-modules M and N, the tensor product $M \otimes N$ (over k) has a canonical G-module structure, defined in the usual way. The dual space $M^* = \mathrm{Hom}_{\mathrm{k}}(M,k)$ of a G-module M also carries a natural G-module structure, defined again in the obvious way.

Taking duals is a contravariant autoequivalence of $\mathrm{Mod}(G)$, and we have

$$\operatorname{ch} M^* = \sum_{\lambda \in X} \dim(M_\lambda) \cdot e^{-\lambda}.$$

The natural evaluation map and coevaluation map

$$\mathrm{ev}_M : M \otimes M^* \longrightarrow k \quad \text{and} \quad \mathrm{coev}_M : k \longrightarrow M^* \otimes M$$

are homomorphisms of G-modules, where k denotes the trivial G-module. For G-modules N and N', there are natural isomorphisms $(M \otimes N)^* \cong N^* \otimes M^*$ and $\mathrm{Hom}_G(N \otimes M, N') \cong \mathrm{Hom}_G(N, N' \otimes M^*)$. There exists a second duality on $\mathrm{Mod}(G)$ called the *contravariant duality* and denoted by $M \mapsto M^\tau$ (cf. (Jan03, II.2.13).

On the level of characters, we have $\operatorname{ch} M = \operatorname{ch} M^\tau$, and we call a G-module M *contravariantly self-dual* if $M \cong M^\tau$. For G-modules M and N, there are natural isomorphisms $(M \otimes N)^\tau \cong N^\tau \otimes M^\tau$ and $\mathrm{Hom}_G(M, N) \cong \mathrm{Hom}_G(N^\tau, M^\tau)$.

Next we follow (Jan03, I.3, II.2) and consider some important G-modules. Let G be a k-group and H a subgroup functor of G. By (Jan03, I.3.1), every G-module M is viewed as an H-module in a natural way: for each k-algebra A, restrict the action of $G(A)$ to $H(A)$. The restriction functor obtained is exact.

According to (Jan03, I.3.2), we define the right adjoint functor Ind_H^G as follows:

$$\mathrm{Ind}_H^G = \{f \in \mathrm{Mor}(G, M_a) \mid f(gh) = h^{-1} f(g), \forall g \in G(A), h \in H(A), k\text{-algebra } A\}$$

We may also define the *induced module of* M from H to G by observing the natural $(G \times H)$-module structure on $M \otimes k[G]$. We let G act trivially on M and act by the left regular representation on $k[G]$; then, let H act as on M and by the right regular representation on $k[G]$ (cf. (Jan03, I.3.2)). Note that $(M \otimes k[G])^H$ is a G-submodule of $M \otimes k[G]$ and write $\mathrm{Ind}_H^G M := (M \otimes k[G])^H$.

Next, one may prove that Ind is the associated right adjoint functor to the restriction functor Res using the following important result, *Frobenius reciprocity.*

Proposition 2.6.2. (Frobenius Reciprocity, (Jan03, 3.4)) Let H be a flat subgroup scheme of G and M an H-module. Then, for any G-module N, there exists an isomorphism

$$\mathrm{Hom}_G(N, \mathrm{Ind}_B^G M) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G N, M).$$

Now, let $H = B$. To the *restriction functor*

$$\mathrm{Res}_B^G : \mathrm{Mod}(G) \longrightarrow \mathrm{Mod}(B)$$

is associated a *right adjoint induction functor*

$$\mathrm{Ind}_B^G : \mathrm{Mod}(B) \longrightarrow \mathrm{Mod}(G).$$

The induction of a simple B-module k_λ is non-zero if and only if λ is dominant. For $\lambda \in X^+$, we call

$$\mathrm{H}^0(\lambda) := \mathrm{Ind}_B^G(k_\lambda)$$

the *induced module of highest weight* λ. Since the weight spaces $\mathrm{H}^0(\lambda)_\mu$ are zero unless $\mu \leq \lambda$, and $\dim \mathrm{H}^0(\lambda)_\lambda = 1$, we can say that there is a highest weight. The characters $\chi(\lambda) := \mathrm{ch}\, \mathrm{H}^0(\lambda)$ of the induced modules are given by *Weyl's character formula*:

$$\chi(\lambda) = \frac{\sum_{w \in W} \det(w) \cdot e^{w(\lambda+\rho)}}{\sum_{w \in W} \det(w) \cdot e^{w\rho}},$$

and they form a basis of $\mathbb{Z}[X]^W$. The formula above can be used to define $\chi(\lambda) \in \mathbb{Z}[X]^W$ for any $\lambda \in X$ (and not just for dominant weights). It is readily verifiable that $\chi(w \cdot \lambda) = \det(w) \cdot \chi(\lambda)$ for all $\lambda \in X$ and $w \in W$, and that $\chi(\lambda) = 0$ if $\langle \lambda, \alpha^\vee \rangle = -1$ for some $\alpha \in \Pi$.

The induced module $\mathrm{H}^0(\lambda)$ has a unique simple submodule

$$L(\lambda) := \mathrm{soc}_G\, \mathrm{H}^0(\lambda).$$

We have that the simple G-modules $L(\lambda)$ with $\lambda \in X(T)_+$ form a set of representatives for the isomorphism classes of simple objects in $\mathrm{Mod}(G)$ (cf. (Jan03, II.2.3)). Every finite-dimensional G-module M has a finite composition series, and we write $[M : L(\lambda)]$ for the multiplicity of the simple module $L(\lambda)$ as a composition factor of M. Observe that $\mathrm{Mod}(G)$ is a Krull-Schmidt category, by the existence of finite composition series. For a G-module M and an indecomposable G-module N, we write $[M : N]_\oplus$ for the multiplicity of N in a Krull-Schmidt decomposition of M. The dual of a simple G-module is simple, and as $\mathrm{ch}\, L(\lambda)^* = \sum_{\mu \in X} \dim (L(\lambda)_\mu) \cdot e^{-\mu}$ and $-w_0\lambda$ is the unique dominant weight in the W-orbit of $-\lambda$, for all $\lambda \in X^+$, we have $L(\lambda)^* \cong L(-w_0\lambda)$.

The *Weyl module of highest weight* λ is

$$\mathrm{V}(\lambda) := \mathrm{H}(-w_0\lambda)^* \cong \mathrm{H}^0(\lambda)^\tau,$$

and has a unique maximal submodule $\mathrm{rad}_G\, \mathrm{V}(\lambda)$ and $\mathrm{V}(\lambda)/\,\mathrm{rad}_G\, \mathrm{V}(\lambda) \cong L(\lambda)$.

Remark 2.6.3 (Ext-Vanishing Property)**.** (Jan03, II.4.13) The Weyl modules and induced modules satisfy the following Ext-vanishing property:

$$\mathrm{Ext}^i_G(\mathrm{V}(\lambda), \mathrm{H}^0(\mu)) \cong \begin{cases} k & \text{if } i = 0 \text{ and } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

We can now prove the following well-known universal property:

Lemma 2.6.4. Let $\lambda \in X^+$and let M be a G-module with $\mathrm{hd}_G\, M \cong L(\lambda)$ and such that λ is maximal among the weights of M. Then there is a surjective homomorphism $\mathrm{V}(\lambda) \to M$.

Proof. Since λ is maximal among the weights of M, we have

$$\mathrm{Ext}^1_G\left(\mathrm{V}(\lambda), \mathrm{rad}_G\, M\right) \cong \mathrm{Ext}^1_G\left((\mathrm{rad}_G\, M)^*, \mathrm{H}^0(\lambda)\right) = 0,$$

by the Ext-Vanishing Property. We have a short exact sequence

$$0 \longrightarrow \mathrm{rad}_G\, M \longrightarrow M \longrightarrow L(\lambda) \longrightarrow 0$$

and, since the covariant functor $\mathrm{Hom}_G(V(\lambda), -)$ is left exact, we have the following short exact sequence

$$0 \to \mathrm{Hom}_G\left(\mathrm{V}(\lambda), \mathrm{rad}_G\, M\right) \to \mathrm{Hom}_G(\mathrm{V}(\lambda), M) \to \mathrm{Hom}_G(\mathrm{V}(\lambda), L(\lambda)) \to 0.$$

Therefore, there is a homomorphism $\varphi : \mathrm{V}(\lambda) \to M$ such that the composition of φ with the epimorphism $M \to L(\lambda)$ with kernel $\mathrm{rad}_G\, M$ is non-zero. Then, the image of φ is not contained in $\mathrm{rad}_G\, M$, the unique maximal submodule of M. We conclude that φ is surjective. □

The Linkage Principle

We follow (Jan03, II.6). The upshot is that the linkage principle describes the decomposition of $\mathrm{Mod}(G)$ into linkage classes that arise from a certain action of the affine Weyl group on X, which we shall define. The translation principle relates the different linkage classes via translation functors. Before recalling these results, we need to introduce some more notation, describing the alcove geometry with respect to the dot action.

Let $s_{\beta,r}$ for $\beta \in \Phi$ and $r \in \mathbb{Z}$ denote the *affine reflection* on $X(T)$ or $X_{\mathbb{R}} := X(T) \otimes_{\mathbb{Z}} \mathbb{R}$

and we have

$$s_{\beta,r}(\lambda) = \lambda - (\langle \lambda, \beta^\vee \rangle - r)\beta = s_\beta(\lambda) + r\beta,$$

for all λ. Then, the group generated by all such $s_{\beta,mp}$ for $m \in \mathbb{Z}$ is called the *affine Weyl group*, and it is denoted by W_p. It follows that $W_p \cong p\mathbb{Z}\Phi \rtimes W$, where $p\mathbb{Z}\Phi$ is the group acting by translations on $X(T)$ and W is the Weyl group (cf. (Jan03, II.6.1)).

We consider the *dot action*, $w.\lambda = w(\lambda + \rho) - \rho$, of W_p on $X(T)$ and $X_\mathbb{R}$.

The set of fixed points of a reflection $s = s_{\beta,m}$ with respect to the dot action is the affine hyperplane

$$H_s^p = H_{\beta,m}^p := \left\{ x \in X_\mathbb{R} \mid \langle x + \rho, \beta^\vee \rangle = pm \right\},$$

and the *p-alcoves* are the connected components of $X_\mathbb{R} \backslash \bigcup_{\beta,m} H_{\beta,m}^p$. A weight $\lambda \in X$ is called *p-singular* if it lies on at least one of the hyperplanes $H_{\beta,m}^p$, and *p-regular* if it lies in a p-alcove. Recall that we write $H_{\beta,m}$ for the hyperplane of fixed points of the affine reflection $s_{\beta,m}$ with respect to the standard action. We call

$$C_{\text{stand}} := p \cdot A_{\text{stand}} - \rho = \left\{ x \in X_\mathbb{R} \mid 0 < \langle x + \rho, \beta^\vee \rangle < p \text{ for all } \beta \in \Phi^+ \right\}$$

the *standard p-alcove*; its closure is a fundamental domain for the dot action of W_p on $X_\mathbb{R}$ (cf. (Jan03, II.6.2 (6))). By (Jan03, II.6.2), a p-alcove $C \subseteq X_\mathbb{R}$ is determined by a collection of integers $n_\beta(C)$, for $\beta \in \Phi^+$, such that

$$C = \left\{ x \in X_\mathbb{R} \mid n_\beta(C) \cdot p < \langle x + \rho, \beta^\vee \rangle < (n_\beta(C) + 1) \cdot p \text{ for all } \beta \in \Phi^+ \right\},$$

and we set $d(C) := \sum_\beta n_\beta(C)$. For all $\lambda \in X$ and $\beta \in \Phi^+$, we can choose $n_\beta(\lambda) \in \mathbb{Z}$ such that

$$n_\beta(\lambda) \cdot p \leq \langle \lambda + \rho, \beta^\vee \rangle < (n_\beta(\lambda) + 1) \cdot p,$$

and we set $d(\lambda) := \sum_\beta n_\beta(\lambda)$. The *linkage order* $\uparrow_p$ on X is the reflexive and transitive closure of the relation given by $\mu \uparrow_p \lambda$ if $\mu \leq \lambda$ and there exists a reflection $s \in W_p$ with $\lambda = s \cdot \mu$ (cf. (Jan03, II.6.4)).

Proposition 2.6.5 (The Strong Linkage Principle)**.** (Jan03, II.6.13) Let $\lambda \in X(T)$ such that $\langle \lambda + \rho, \beta^\vee \rangle \geq 0$ for all $\lambda \in \Phi^+$ and let $\mu \in X(T)^+$. If $L(\mu)$ is a composition factor of some $\mathrm{H}^i(w.\lambda)$ with $w \in W$ and $i \in \mathbb{N}$, then $\mu \uparrow_p \lambda$.

Corollary 2.6.6 (The Weak Linkage Principle)**.** (Jan03, II.6.17) If $\lambda, \mu \in X(T)^+$such that

$$\mathrm{Ext}_G^i(L(\lambda), L(\mu)) \neq 0$$

for some $i \geq 0$ then $\mu \in W_p \cdot \lambda$.

Therefore, we have the following result:

Proposition 2.6.7. (Jan03, II.6.20) If $\lambda, \mu \in X(T)^+$and $i \geq 0$ such that

$$\mathrm{Ext}^i_G(L(\lambda), \mathrm{H}^0(\mu)) \neq 0 \quad \text{or} \quad \mathrm{Ext}^i_G(\mathrm{H}^0(\lambda), \mathrm{H}^0(\mu)) \neq 0$$

then $\mu \uparrow_p \lambda$ and $i \leq d(\lambda) - d(\mu)$.

2.7 Infinitesimal Representation Theory

Recall that the group scheme G admits a strict endomorphism $\sigma : G \to G$ that fixes the Borel subgroup B and the maximal torus T. Moreover, recall the cases I-III discussed in Section 2.5.3. The Frobenius kernels $G_\sigma := \ker(\sigma)$ are infinitesimal subgroup schemes (cf. (Jan03, I.8.1)) and play an important role in the representation theory of G. In this section, we discuss some results concerning the representation theory of these subgroup schemes G_σ. Note that the results in cases I and II are available in (Jan03, II.3) and that many results hold in case III.

We follow (Jan03, II.3) and (BNP$^+$15, Remark 2.2.1). In cases I and II, we have $G_\sigma = G_r$. By (Jan03, I.9.10), given the strict endomorphism σ and a G-module M, let $M^{(r)}$ denote the twist of the module obtained by precomposing the action map with F^r. We may also define the untwist, $M^{(-r)}$, if G_r acts trivially on M.

Now suppose we are in case III, so let $r = 2s + 1$ be an odd positive integer and set $\sigma = \tau^r = F^{r/2}$. For a G-module M, let $M^{(\sigma)} = M^{(r/2)}$ denote the module obtained by making G act on M through σ. Moreover, if M is of the form $N^{(r/2)}$ for some G-module N, put $M^{(-\sigma)} = M^{(-r/2)} = N$.

Note the following useful results concerning the exotic Frobenius kernels. We have $G/G_{r/2} \cong G^{(r/2)}$, where $G^{(r/2)}$ has coordinate algebra $k[G]^{(r/2)}$. When G is identified with G, the $G/G_{r/2}$-module $\mathrm{H}^j(G_{r/2}, M)$ is identified with $\mathrm{H}^j(G_{r/2}, M)^{(-r/2)}$.

In cases I and II, G_r is a normal, infinitesimal, subgroup scheme of G and one may use various Lyndon-Hochschild-Serre spectral sequences. In case III, and $G_{r/2}$ is a normal subgroup scheme of G, so we may use Lyndon-Hochschild-Serre spectral sequences as well (cf. (BNP$^+$15, Remark 2.2.1(a))).

In all cases, let $\mathrm{Mod}\,(G_\sigma)$ denote the category of G_σ-modules, and $\mathrm{Hom}_{G_\sigma}(M, N)$ for the space of homomorphisms between G_σ-modules M and N. Since G_σ is a normal subgroup scheme of G, there is a natural restriction functor $\mathrm{Res}^G_{G_\sigma} : \mathrm{Mod}(G) \to \mathrm{Mod}\,(G_\sigma)$, and for every G-module M, the G_σ-fixed points M^{G_σ} form a G-submodule of M. Simlarly, for G-modules M, N, the isomorphisms $\mathrm{Hom}_{G_{r/2}}(M, N) \cong (N \otimes M^*)^{G_\sigma}$ yields a G-module structure on $\mathrm{Hom}_{G_{r/2}}(M, N)$.

Now, we must introduce the notion of σ-*restricted weights*; we denote by X_σ the set of all such weights. In cases I and II, the p^r-restricted weights are the dominant weights λ such that $\langle \lambda, \alpha^\vee \rangle < p^r$ for all $\alpha \in \Pi$. We also have that $\lambda \in X_+$ may be uniquely expressed as $\lambda = \lambda_0 + p^r \lambda_1$, with $\lambda_0 \in X_r$ and $\lambda_1 \in X_+$.

In case III, for $r = 2s + 1$ an odd positive integer, a dominant weight $\lambda \in X^+$ is called σ-*restricted* provided it satisfies the following conditions: $\langle \lambda, \alpha^\vee \rangle < p^{s+1}$, for α a short simple root, or $\langle \lambda, \alpha^\vee \rangle < p^s$, for α a long simple root.

Note that the irreducible $G(\sigma)$-modules are the restriction to $G(\sigma)$ of the irreducible G-modules $L(\sigma)$, for $\lambda \in X_\sigma$, a σ-restricted weight.

In addition, the set of σ-restricted weights, X_σ indexes the irreducible modules for the infinitesimal subgroup schemes G_σ; these irreducible modules are the restrictions to G_σ of the corresponding irreducible G-modules. That is, for $\lambda \in X_\sigma$, the modules $L(\lambda)$ form a set of representatives for the isomorphism classes of simple G_σ-modules.

Following (Jan03, II.3.16) and recalling the G-module structure on $\mathrm{Hom}_{G_\sigma}(M, N)$ in all cases I-III, we obtain that the G_σ-socle of a G-module M, denoted by $\mathrm{soc}_{\mathrm{G}_\sigma} M$, is a G-submodule of M, and there is an isomorphism of G-modules

$$\mathrm{soc}_{\mathrm{G}_\sigma} M \cong \bigoplus_{\lambda \in X_\sigma} L(\lambda) \otimes \mathrm{Hom}_{\mathrm{G}_\sigma}(L(\lambda), M).$$

In particular, if M is a simple G-module, then $\mathrm{soc}_{G_\sigma} M = M$. Hence any semisimple G-module is semisimple for G_σ. We apply these observations to a simple G-module, say $M = L(\mu)$ for some $\lambda \in X_+$. Observe that a composition factor of $\mathrm{Hom}_{\mathrm{G}_\sigma}(L(\lambda), L(\mu))$ must be of the form $L(\theta)^{(\sigma)}$ and follow the proof in (Jan03, II.3.16); we obtain the following important result:

Theorem 2.7.1. (Steinberg's tensor product theorem) Let $\sigma : G \to G$ denote a strict endomorphism and let $\sigma^* : X(T) \to X(T)$ denote the restriction to $X(T)$ of the comorphism σ^* of σ. Let $\lambda \in X^+$ and write $\lambda = \lambda_0 + \sigma\lambda_1$ with $\lambda_0 \in X_\sigma$ and $\lambda_1 \in X^+$. Then

$$L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{(\sigma)}.$$

We discuss the Steinberg Tensor Product Theorem in all of our cases, I-III. In case I, we have $\sigma = F^r$ and we may express $\lambda \in X_+$ uniquely as $\lambda = \lambda_0 + p^r \lambda_1$, for $\lambda_0 \in X_r$ and

$\lambda_1 \in X_+$. The Steinberg Tensor Product Theorem states that $L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{(r)}$.

In case II, we have $\sigma = F^r \circ \gamma$ and σ^* as described above. Then, we have $\lambda = \lambda_0 + \sigma^* \lambda_1$, for $\lambda_0 \in X_r$ and $\lambda_1 \in X_+$. Note that $\sigma^* = p^r \gamma$, where γ denotes the automorphism of X induced by the graph automorphism. Therefore, the Steinberg Tensor Product Theorem states that $L(\lambda) \cong L(\lambda_0) \otimes L(\gamma\lambda_1)^{(r)}$.

For the purposes of this thesis we need an extension of this theorem in case III. Suppose $\tau = F^{1/2}$ is a square root of the Frobenius morphism. Then we may similarly define the subsets of dominant weights $X_{1/2}$ and $X_{r/2}$ and for $\lambda \in X^+$ a unique decomposition $\lambda = \lambda_0 + (\tau^*)^r \lambda_1$, for $\lambda_0 \in X_{r/2}$ and $\lambda_1 \in X_+$. Then a refined version of the Steinberg Tensor Product Theorem gives $L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{(\tau^r)}$.

2.8 Cohomology

The main results of this thesis are statements about various cohomology groups of interest and in this section we provide an introduction to cohomology and different aspects of homological algebra. We direct the reader to (Wei94), (Kra22) for a more detailed discussion of these topics.

We begin with the definition of an abelian category (cf. (Wei94, Definition 1.2.2)).

Definition 2.8.1. An additive category $\mathcal{A}$ is *abelian* if for every morphism $\phi : X \to Y$ there exists a kernel and cokernel and if the canonical factorisation of ϕ induces an isomorphism $\tilde{\phi}$.

Let $\mathcal{A}$ be an additive category. A *cochain complex* $A = (A^\bullet, d_A^\bullet)$ in $\mathcal{A}$ is a sequence of objects $\left(A^i\right)_{i\in\mathbb{Z}}$ of $\mathcal{A}$ with morphisms $d_A^i \in \operatorname{Hom}_{\mathcal{A}}\left(A^i, A^{i+1}\right)$

$$\cdots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \longrightarrow \cdots$$

such that $d^i \circ d^{i-1} = 0$ for all $i \in \mathbb{Z}$. We think of a complex A as a graded object with A^i the term in cohomological degree i and d_A^i the i-th differential of A.

We denote by $\mathbf{C}(\mathcal{A})$ the category of complexes, where a morphism $\phi : X \to Y$ between complexes consists of morphisms $\phi^i : A^i \to B^i$ with $d_B^i \circ \phi^i = \phi^{i+1} \circ d_A^i$ for all $i \in \mathbb{Z}$.

For a complex A over $\mathcal{A}$, the condition that $d_A^i \circ d_A^{i-1} = 0$ means that the image of d_A^{i-1} is contained in the kernel of d_A^i, and we define the *i-th cohomology of A* as

$$H^i(A) := \ker\left(d_A^i\right) / \operatorname{im}\left(d_A^{i-1}\right).$$

A complex is called *exact in degree i* if its i-th cohomology is zero and *exact* if it is exact in all degrees. We also say *exact sequence* for a bounded exact complex and *short exact sequence* for an exact sequence with at most three non-zero terms. A chain map $f : A \to B$ induces homomorphisms

$$H^i(f) : H^i(A) \to H^i(B)$$

for all $i \in \mathbb{Z}$, and we call f a *quasi-isomorphism* if all $H^i(f)$ are isomorphisms.

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $F : \mathcal{A} \to \mathcal{B}$ be a covariant, additive functor. By (Wei94, Definition 1.6.6), we say that F is *left exact* (respectively *right exact*) if for every short exact sequence $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$, the sequence $0 \to F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C)$ (respectively $F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C) \to 0$) is exact.

Similarly if $F : \mathcal{A} \to \mathcal{B}$ is a contravariant functor, F is *left exact* (respectively *right exact*) if for every short exact sequence $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$, the sequence $0 \to F(C) \xrightarrow{F(\beta)} F(B) \xrightarrow{F(\alpha)} F(A)$ (respectively $F(C) \xrightarrow{F(\beta)} F(B) \xrightarrow{F(\alpha)} F(A) \to 0$) is exact.

A covariant or contravariant functor which is both left and right exact is called an *exact functor.*

Example 2.8.2. ((Wei94, Corollary 1.6.9)) Let $\mathcal{A}$ be an abelian category. Then for any object A in $\mathcal{A}$, the covariant functor $\operatorname{Hom}_{\mathcal{A}}(A, -)$ and the contravariant functor $\operatorname{Hom}_{\mathcal{A}}(-, A)$ are left exact.

An object I in $\mathcal{A}$ is *injective* if every admissible monomorphism $X \to Y$ induces a surjective map $\operatorname{Hom}_{\mathcal{A}}(Y, I) \to \operatorname{Hom}_{\mathcal{A}}(X, I)$ (cf. (Wei94, 2.3.4)). That is, I is *an injective object* if the contravariant functor $\operatorname{Hom}_{\mathcal{A}}(-, I)$ is exact. The full subcategory of injective objects in $\mathcal{A}$ is denoted by $\operatorname{Inj} \mathcal{A}$.

According to (Kra22, 2.1),the category $\mathcal{A}$ has *enough injective objects* if for every object $X \in \mathcal{A}$ there is an admissible monomorphism $X \to I$ such that I is injective.

An *injective presentation* of an object A in $\mathcal{A}$, is a short exact sequence in $\mathcal{A}$ of the form $0 \to A \to I^0 \to I^1$, where each I^i is an injective object in $\mathcal{A}$ (cf. (Kra22, 2.1.14)). Equivalently, we say $\mathcal{A}$ *has enough injectives* if for every object $A \in \mathcal{A}$ there is an injective presentation.

Let A be an object in an abelian category $\mathcal{A}$. By (Wei94, Definition 2.3.5), an *injective resolution* of A is a cochain complex I, with $I^i = 0$, for $i < 0$ and each object I^i is injective, and a map $A \to I^0$ such that the complex

$$0 \to A \to I^0 \to I^1 \to I^2 \to \dots$$

is exact. If $\mathcal{A}$ has enough injectives, then every object in $\mathcal{A}$ has an injective resolution (cf. (Wei94, Lemma 2.3.6)).

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between two abelian categories. If $\mathcal{A}$ has enough injectives, we can construct the right derived functors $\mathrm{R}^i F (i \geq 0)$ of F as follows. If A is an object of $\mathcal{A}$, choose an injective resolution $A \to I$ and define

$$\mathrm{R}^i F(A) = H^i(F(I)).$$

By (Wei94, 2.5.1), note that since $0 \to F(A) \to F\left(I^0\right) \to F\left(I^1\right)$ is exact, we always have $R^0 F(A) \cong F(A)$.

Next, we introduce Ext-functors:

Definition 2.8.3. Let $\mathcal{A}$ be an abelian category with enough injectives, with A an object in $\mathcal{A}$. For $i \geq 1$, we define $\mathrm{Ext}^i_{\mathcal{A}}(A, -) := \mathrm{R}^i \mathrm{Hom}_{\mathcal{A}}(A, -)$, that is the right derived functors of the covariant, left exact, additive functor $\mathrm{Hom}_{\mathcal{A}}(A, -)$, using injective resolutions.

Spectral Sequences

We recall some of the key facts about spectral sequences, for the unfamiliar reader. (See (McC01) or (Jan03) for an exhaustive treatment.) Let $\mathcal{C}$ be an abelian category; then, a spectral sequence (of cohomological type) consists of a family of bigraded objects $E_n = \bigoplus_{i,j \in \mathbb{Z}} E_n^{i,j}$ of $\mathcal{C}$ and differentials of bidegree $(n, -n+1)$, $d_n : E_n^{i,j} \to E_n^{i+n,j-n+1}$ and $d_n : E_n^{i-n,j+n-1} \to E_n^{i,j}$, which satisfy $d_n \circ d_n = 0$. The r-th stage of such an object is called its E_r-term (or E_r-page). We require

$$E_{n+1}^{i,j} \cong \mathrm{H}(E_n^{i,j}) \cong \frac{\ker(d_n : E_n^{i,j} \to E_n^{i+n,j-n+1})}{\mathrm{im}(d_n : E_n^{i-n,j+n-1} \to E_n^{i,j})}.$$

The collections $(E_n^{i,j})_{i,j}$ for fixed n are known as the *pages* of the spectral sequence, and we move to the next one by taking cohomology, using the isomorphism above.

A spectral sequence for which $E_n^{i,j} = 0$ whenever $i < 0$ or $j < 0$ is called a *first quadrant* spectral sequence.

For a given i and j, consider $E_n^{i,j}$, where $n > \max(i, j+1)$; then, the differentials d_n are trivial. Since $j + 1 - n < 0$, we have $E_n^{i+n,j-n+1} = 0$ and $d_n : E_n^{i,j} \to E_n^{i+n,j-n+1} = 0$ has $\ker d_n = E_n^{i,j}$. Then, $i - n < 0$ implies $E_n^{i-n,j+n-1} = 0$, so $d_n : E_n^{i-n,j+n-1} \to E_n^{i,j}$ has $\mathrm{im} d_n = 0$. Hence, $E_{n+1}^{i,j} \cong E_n^{i,j}$ and iterating this calculation yields $E_{n+k}^{i,j} = E_n^{i,j}$, for $k \geq 0$. We denote this value by $E_\infty^{i,j}$.

An n-th stage spectral sequence *converges* to groups H^m, ($E_n^{i,j} \Rightarrow \mathrm{H}^{i+j=m}$), if there is a filtration

$$H^{i+j} = F^0 H^{i+j} \subseteq F^1 H^{i+j} \subseteq \cdots \subseteq F^{i+j+1} H^{i+j} = 0,$$

such that $E_{\text{inf}}^{i,m-i} \cong F^i H^m / F^{i+1} H^m$. The spectral sequence has *stable values* $E_\infty^{i,j}$ (*abutments*). It follows that the stable values $E_\infty^{i,j}$ on the diagonal $i+j=m$ are the succesive quotients in the filtrations of H^m.

Next, we discuss edge maps and trangressions (cf (Ben91, 3.2)). First, put $j=0$ and observe that each $E_n^{i,0}$ is a quotient of $E_2^{i,0}$ and that $E_{\text{inf}}^{i,0}$ is contained in H^i. Hence, there are maps

$$E_2^{i,0} \twoheadrightarrow E_3^{i,0} \twoheadrightarrow \cdots \twoheadrightarrow E_{\text{inf}}^{i,0} \hookrightarrow \mathrm{H}^i,$$

and the composite map $E_2^{i,0} \to \mathrm{H}^i$ is called the *horizontal edge map.*

Similarly, setting $i=0$, we obtain maps

$$\mathrm{H}^j \twoheadrightarrow E_\infty^{0,j} \hookrightarrow \ldots \hookrightarrow E_3^{0,j} \hookrightarrow E_2^{0,j},$$

where the composite map $\mathrm{H}^j \to E_2^{0,j}$ is called the *vertical edge map.* Furthermore, we call the differential $d_n : E_n^{0,n-1} \to E_n^{n,0}$ the *transgression.*

In this thesis, we make extensive use of second stage first quadrant spectral sequences. Set $n=2$ and we have the following important result (cf. (Bou07, Section 2, Exerc 15c), (Ben91, Proposition 3.2.10)).

Lemma 2.8.4. Let (E_n, d_n) be a second stage first quadrant spectral sequence of cohomological type converging to the graded vector space H^* and we have $F^{i+k}\,\mathrm{H}^i = 0$, for all $k > 0$. Then, $E_\infty^{0,0} = \mathrm{H}^0$ and we have a five-term exact sequence

$$0 \to E_2^{1,0} = E_\infty^{1,0} \to \mathrm{H}^1 \to E_2^{0,1} \to E_2^{2,0} \to \mathrm{H}^2\,.$$

Proof. Notice that $E_2^{1,0}$ has no maps into it or from it from the first quadrant. Thus, it injects to H^1; that is, we have $0 \to E^{1,0} \to \mathrm{H}^1 = E_\infty^1$.

Consider $E_3^{0,1} = \mathrm{H}(E_2^{0,1})$. We have

$$E_3^{0,1} = \frac{\ker(d_2 : E_2^{0,1} \to E_2^{2,0})}{\mathrm{im}(d_2 : E_2^{-2,2} \to E_2^{0,1})} = \ker(d_2 : E_2^{0,1} \to E_2^{2,0}).$$

Note that there is a differential $d_2 : E_2^{0,1} \to E_2^{2,0}$.

Similarly, there are no maps in or out of $E_3^{0,1}$ from the first quadrant, so H^1 is $E_2^{1,0} \oplus E_3^{0,1}$.

The latter, $E_3^{0,1}$, is the cohomology of the sequence $0 \to E_2^{0,1} \to E_2^{2,0}$. So that explains $0 \to E_2^{1,0} \to \mathrm{H}^1 \to E_2^{0,1} \to E_2^{2,0}$.

Finally the cokernel of the map $d_2 : E_2^{0,1} \to E_2^{2,0}$ is $E_3^{2,0}$ and this has no maps in or out, so it injects into H^2. This concludes the proof. $\square$

2.9 Inducing from the fixed point subgroup of a reductive group

Recall that σ is a strict endomorphism of G. In this section we follow (oGVAG12, Section 2) and adapt the results for all cases of G_σ.

We consider the exact induction functor $\mathcal{G}_\sigma(-) = \mathrm{Ind}_{G(\sigma)}^{G}(-)$ from the category of $kG(\sigma))$-modules to the category of G-modules. Set $\mathcal{G}_\sigma(-) = \mathrm{Ind}_{G(\sigma)}^{G}(-)$. Let $\iota : k \to \mathcal{G}_\sigma(k)$ be the homomorphism induced by Frobenius reciprocity from the identity map $\mathrm{id} : k \to k$, and set $N = \mathrm{coker}(\iota)$. Then there exists a short exact sequence of G-modules

$$0 \to k \xrightarrow{\iota} \mathcal{G}_\sigma(k) \to N \to 0. \tag{2.9.1}$$

Let M be a G-module. By the tensor identity (Jan03, I.3.6), $M \otimes \mathcal{G}_\sigma(k) \cong \mathcal{G}_\sigma(M)$. Then applying the exact functor $M \otimes -$ to the sequence above, one obtains the new short exact sequence

$$0 \to M \to \mathcal{G}_\sigma(M) \to M \otimes N \to 0, \tag{2.9.2}$$

and hence the associated long exact sequence in cohomology

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_G(k, M) \longrightarrow \mathrm{Hom}_G(k, \mathcal{G}_\sigma(M)) \longrightarrow \mathrm{Hom}_G(k, M \otimes N) \\ &\longrightarrow \mathrm{Ext}_G^1(k, M) \longrightarrow \mathrm{Ext}_G^1(k, \mathcal{G}_\sigma(M)) \longrightarrow \mathrm{Ext}_G^1(k, M \otimes N) \\ &\longrightarrow \mathrm{Ext}_G^2(k, M) \longrightarrow \mathrm{Ext}_G^2(k, \mathcal{G}_\sigma(M)) \longrightarrow \mathrm{Ext}_G^2(k, M \otimes N) \quad \longrightarrow \dots \end{aligned} \tag{2.9.3}$$

Since $G/G(\sigma)$ is affine, the induction functor is exact (cf. (Jan03, I.5.13)). Then, there exists an isomorphism for all $i \geq 0$, by generalized Frobenius reciprocity (Jan03, I.4.6),

$$\mathrm{Ext}_G^i(k, \mathcal{G}_\sigma(M)) \cong \mathrm{Ext}_{G(\sigma)}^i(k, M). \tag{2.9.4}$$

The isomorphism is realised by the composition of the maps

$$\mathrm{Ext}_G^i(k, \mathcal{G}_\sigma(M)) \xrightarrow{\varphi_1} \mathrm{Ext}_{G(\sigma)^i}(k, \mathcal{G}_\sigma(M)) \xrightarrow{\varphi_2} \mathrm{Ext}_{G(\sigma)}^i(k, M),$$

where φ_1 is the restriction map induced by the inclusion $G(\sigma) \subset G$, and φ_2 is induced by the evaluation homomorphism $\mathrm{ev} : \mathcal{G}_\sigma(M) \to M$, a homomorphism of $G(\sigma)$-modules. Since the map $M \to \mathcal{G}_\sigma(M)$ is induced by Frobenius reciprocity from the identity $\mathrm{id} : M \to M$, we have isomorphisms $\mathrm{Ext}^i_G(k, \mathcal{G}_\sigma(M)) \cong \mathrm{Ext}^i_{G(\sigma)}(k, M)$. Then, the maps in the long exact sequence, $\mathrm{Ext}^1_G(k, M) \to \mathrm{Ext}^1_G(k, \mathcal{G}_\sigma(M))$ become the cohomological restriction maps $\mathrm{Res} : \mathrm{Ext}^i_G(k, M) \to \mathrm{Ext}^i_{G(\sigma)}(k, M)$.

Therefore, we may rewrite the long exact sequence in the following way:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_G(k, M) \xrightarrow{\mathrm{Res}} \mathrm{Hom}_{G(\sigma)}(k, M) \longrightarrow \mathrm{Hom}_G(k, M \otimes N) \\ &\longrightarrow \mathrm{Ext}^1_G(k, M) \xrightarrow{\mathrm{Res}} \mathrm{Ext}^1_{G(\sigma)}(k, M) \longrightarrow \mathrm{Ext}^1_G(k, M \otimes N) \\ &\longrightarrow \mathrm{Ext}^2_G(k, M) \xrightarrow{\mathrm{Res}} \mathrm{Ext}^2_G(\sigma)(k, M) \longrightarrow \mathrm{Ext}^2_G(k, M \otimes N) \quad \longrightarrow \dots \end{aligned} \tag{2.9.5}$$

It follows from the long exact sequence (2.9.5) that whenever both $\mathrm{Ext}^{i-1}_G(k, M \otimes N) = 0$ and $\mathrm{Ext}^i_G(k, M \otimes N) = 0$, we have that the restriction maps $\mathrm{Res} : \mathrm{Ext}^i_G(k, M) \to \mathrm{Ext}^i_{G(\mathbb{F}_q)}(k, M)$ are isomorphisms.

Now, we follow (BNP$^+$15, 3.1). The coordinate algebra $k[G]$ is naturally a $G \times G$-module via the left and right regular actions, respectively. The action of $G \times G$ on $k[G]$ is given by $((x, \sigma(y)) * f)(g) = (x \cdot f \cdot \sigma(y)^{-1})(g) := f(\sigma(y)^{-1} g x)$.

Let $k[G]^{(1\times\sigma)} = k[G]_\sigma$ be the coordinate algebra viwewd as a G-module, where $x \in G$ acts as follows:

$$(x * f)(g) := (x \cdot f \cdot \sigma(x)^{-1})(g) = f(\sigma(x)^{-1} g x),$$

for $f \in k[G]$ and $g \in G$. Then, by (BNP$^+$15, 3.12), $\mathrm{Ind}^G_{G(\sigma)} k \cong k[G]_\sigma$, and it has a G-filtration with sections $\mathrm{H}^0(\lambda) \otimes \mathrm{H}^0(\lambda^*)^{(\sigma)}$, $\lambda \in X(T)^+$ appearing only once.

Therefore $\mathrm{Ext}^i_G(k, M \otimes N) = 0$ if

$$\mathrm{Ext}^i_G\left(k, M \otimes H^0(\mu) \otimes H^0(\mu^*)^{(\sigma)}\right) = 0 \quad \text{for all } 0 \neq \mu \in X(T)^+.$$

We expand upon this in Chapter 4.

Chapter 3

First Cohomology for the Kernels of Exceptional Isogenies

In this chapter we compute the 1-cohomology for the Frobenius kernels of the induced modules for the Suzuki and Ree groups and the extensions for the Frobenius kernels between simple modules for the Suzuki and Ree groups. Thus in this section G is a simply-connected algebraic group of type C_2 (3.2), G_2 (3.3) and F_4 (3.4).

We fix now and for the remainder of the thesis a positive odd integer $r = 2s + 1$, and define $\sigma = \tau^r$. Then, we take the kernel of σ, $G_\sigma = G_{(r/2)} = \ker \sigma : G \to G$, with a view to calculating invariants of G_σ.

3.1 Preliminaries

We adapt methods of Jantzen (Jan91) in order to compute the B_τ-cohomology. Then, based on an argument in (BNP04), we extend the results from B_τ to $B_{r/2}$; using an analogue of (Jan03, II.12.2(2)) for exotic Frobenius kernels, we obtain $\mathrm{H}^1(G_{r/2}, \mathrm{H}^0(\lambda))$. Moreover, we extend the G_1-cohomology results which were computed in (Sin94b) to calculate $\mathrm{Ext}^1_{G_s}(L(\lambda), L(\mu))$, for a positive integer s and $\lambda, \mu \in X_s(T)$ and for r an odd positive integer, $\mathrm{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu))$, for $\lambda, \mu \in X_{r/2}(T)$.

Since G is simply-connected, there exists a Chevalley basis for the Lie algebra $\mathfrak{g}_\mathbb{Z}$, which may be reduced modulo p to obtain the restricted Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. We write $\mathfrak{g} = \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} k$, where $\mathfrak{g}_\mathbb{Z} = \{X_\alpha, \alpha \in \Phi \mid H_\alpha = [X_\alpha, X_{-\alpha}], \alpha \in \Pi\}$. Hence, suppose that α, β are roots with $\alpha + \beta$ also a root, with the associated root vectors X_α, X_β and $X_{\alpha+\beta}$, respectively, in $\mathfrak{g}_{\mathbb{Z}}$. It follows that the commutator $[X_\alpha, X_\beta] = N_{\alpha\beta} X_{\alpha+\beta}$, for some integer $N_{\alpha\beta}$ (with possible values $0, \pm 1, \pm 2, \pm 3$).

Abusing notation, we shall also denote the element $X_\alpha \otimes 1$ of $\mathfrak{g}$ by X_α. Moreover, upon reduction modulo p, whenever we have α, β two short roots whose sum is a long root, the structure constant $N_{\alpha,\beta}$ will vanish.

Recall (Φ, p) is special, and therefore there exists a special isogeny τ, satisfying $\tau^2 = F$, the Frobenius map. This interacts with the root system in the following way. There is a subsystem of short roots denoted Φ_s. In case $(G, p) = (C_2, 2)$, $(G_2, 3)$ and $(F_4, 2)$ respectively, Φ_s is of type $A_1 \times A_1$, A_2 and D_4, respectively. Degeneracies in the commutator relations in our specific characteristics guarantee Lie subalgebras $\mathfrak{g}_s$ wih root system Φ_s which are generated by the root vectors corresponding to the elements of Φ_s, and maximal rank subgroups of G whose root system is Φ_s. The kernel of $d\tau$ is $\mathfrak{g}_s$, hence we write $\mathfrak{g}_\tau$ for this ideal. The kernel G_τ of τ is an infinitesimal group scheme of height one, whose representation theory is equivalent to the one of the restricted Lie algebra $\mathfrak{g}_\tau$. Since U is τ-stable, we get also a kernel U_τ, whose Lie algebra $\mathfrak{u}_\tau$ is the ideal in $\mathfrak{u}$ generated by negative short roots. We obtain an analogue of (Jan91, Lemma 2.1), noting that the proof follows analogously from Jantzen's proof:

Lemma 3.1.1. We have an isomorphism of B-modules

$$\mathrm{H}^1(U_\tau, k) \cong \mathrm{H}^1(\mathfrak{u}_\tau, k) \cong (\mathfrak{u}_\tau / [\mathfrak{u}_\tau, \mathfrak{u}_\tau])^*,$$

where $\mathfrak{u}_\tau = \mathrm{Lie}(U_\tau) = \langle X_\beta : \beta \in \Phi_s^- \rangle$.

Here, Φ_s^- denotes the set of the negative roots of Φ_s, the subsystem generated by the short roots.

Analogously to (Jan91, Prop 2.2) we have:

Lemma 3.1.2. Let β_i be a set of simple roots of Φ_s. Then,

$$\mathrm{H}^1(U_\tau, k) \cong \bigoplus_i k_{\beta_i}.$$

Proof. The subalgebra $[\mathfrak{u}_\tau, \mathfrak{u}_\tau]$ is spanned by all commutators $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha,\beta}$, for α, β negative short roots. Moreover, $N_{\alpha,\beta} \neq 0$ if and only if $\alpha + \beta$ is a short root. Using this, one checks $[\mathfrak{u}_\tau, \mathfrak{u}_\tau]$ is spanned by root vectors corresponding to non-simple roots. Thus $\mathfrak{u}_\tau / [\mathfrak{u}_\tau, \mathfrak{u}_\tau]$ has a basis with elements the classes of $X_{-\beta_i}$, being the weight vectors for T_τ for weights $-\beta_i$. The result follows by dualising. $\square$

Note that B_τ acts trivially on the weight module $k_{\tau(\lambda)} \cong k_\lambda^\tau$. Then we obtain:

Lemma 3.1.3. For $\lambda \in X_{r/2}$ and β_i simple roots of Φ_s, there exist the following isomorphisms

$$\begin{aligned} \mathrm{H}^1(B_\tau, \lambda) &\cong \left[\mathrm{H}^1(U_\tau, k) \otimes k_\lambda\right]^{T_\tau} \\ &\cong \left[\bigoplus_i k_{\beta_i+\lambda}\right]^{T_\tau}. \end{aligned}$$

Since any weight λ can be uniquely written as $\lambda = \lambda_0 + \tau(\lambda_1)$, for $\lambda_0 \in X_\tau(T)$ and $\lambda_1 \in X(T)$, we have $\mathrm{H}^1(B_\tau, \lambda) \cong \mathrm{H}^1(B_\tau, \lambda_0) \otimes \tau(\lambda_1)$. In particular, when λ is $r/2$-restricted, we have $\lambda = \lambda_0 + \tau(\lambda_1)$, for $\lambda_0 \in X_\tau(T)$ and $\lambda_1 \in X_s(T)$. Thus, it suffices to compute $\mathrm{H}^1(B_\tau, \lambda_0)$, for $\lambda_0 \in X_\tau(T)$.

Considered as a T-module, $\mathrm{H}^1(U_\tau, k) \otimes \lambda_0$ is the direct sum of certain $k_{\beta_i+\lambda_0}$, for β_i, as previously defined. Such a summand yields a non-zero contribution to $\mathrm{H}^1(B_\tau, \lambda_0)$ if and only if $\beta_i + \lambda_0 \in \tau X(T)$. Hence, the problem boils down to checking which of these weights belong to $\tau X(T)$.

Once we have established the appropriate $B_{r/2}$-cohomology, the next result yields the $G_{r/2}$-cohomology with coefficients in the induced modules.

Lemma 3.1.4. Let $\lambda \in X(T)_+$. Then

$$\mathrm{H}^1(G_{r/2}, \mathrm{H}^0(\lambda))^{(-r/2)} \cong \mathrm{Ind}_B^G(\mathrm{H}^1(B_{r/2}, \lambda)^{(-r/2)}). \tag{3.1.1}$$

Proof. By (BNP$^+$15, Remark 2.2.1, (2.2.3)), there exists a spectral sequence

$$E_2^{i,j} = R^i \operatorname{Ind}_B^G \mathrm{H}^j(B_{r/2}, \lambda)^{(-r/2)} \Rightarrow \mathrm{H}^{i+j=n}(G_{r/2}, \mathrm{H}^0(\lambda))^{(-r/2)},$$

for $\lambda \in X_+$ viewed as a one-dimensional B-module. First, note that the R^i are only defined for $i \geq 0$ and that the $\mathrm{H}^j(B_{r/2}, \lambda)$ are only defined for $j \geq 0$; this is a first quadrant spectral sequence and, hence, converges. It rise to the corresponding five-term exact sequence

$$0 \to E_2^{1,0} \to E_\infty^1 \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to E_\infty^2.$$

Following the programme in (Jan03, II.12.2), we firstly suppose that $\lambda \notin \tau^r X(T)$. Then $\mathrm{H}^0(B_{r/2}, \lambda) = 0$, forcing $E_2^{n,0} = 0$ and $E_\infty^1 \cong E_2^{0,1}$. Otherwise, $\lambda \in \tau^r X(T)$ and so we may write $\lambda = \tau^r \lambda'$ for some $\lambda' \in X(T)_+$. Then

$$E_2^{n,0} = R^n \operatorname{Ind}_B^G \mathrm{H}^0(B_{r/2}, \lambda)^{(-r/2)} \cong R^n \operatorname{Ind}_B^G \lambda' = 0,$$

for $n > 0$, by Kempf's vanishing theorem (cf. (Jan03, II.4.5)). Therefore, $E_\infty^1 \cong E_2^{0,1}$, as required. $\square$

By Kempf's vanishing theorem, $\mathrm{H}^0(\lambda) = \mathrm{Ind}_B^G \lambda$ is zero unless λ is dominant. For $\lambda \in X(T)_+$, one may use the preceding computations of $B_{r/2}$-cohomology to compute $\mathrm{H}^1\left(G_{r/2}, \mathrm{H}^0(\lambda)\right)$ thanks to the isomorphism in (3.1.1).

Moreover, one can make use of the G_1-cohomology with coefficients in simple modules, computed in (Sin94b, Proposition 2.3, 3.5, 4.11), to calculate $\mathrm{Ext}^1_{G_s}(L(\lambda), L(\mu))$, for a positive integer s and $\lambda, \mu \in X_s(T)$. Having established the G_s-cohomology, applying the Lyndon-Hochschild-Serre spectral sequence corresponding to $G_s \lhd G_{r/2}$ to compute $\mathrm{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu))$, for $\lambda, \mu \in X_{r/2}(T)$, completes the objectives set out for this section.

In the remaining sections we consider each case of (Φ, p) separately, and we compute the $B_{r/2}$-cohomology and $G_{r/2}$-cohomology explicitly.

3.2 C_2 in Characteristic 2

Let G be simply-connected of type C_2 over k of characteristic 2. Following (Bou82, Planche III), let $\Phi = \{\pm 2\epsilon_1, \pm 2\epsilon_2, \pm\epsilon_1 \pm \epsilon_2\}$ be the roots of a system of type C_2. Writing $\epsilon_1 = (1, 0)$ and $\epsilon_2 = (0, 1)$, a base of simple roots is $\Pi := \{\alpha_1, \alpha_2\}$, with $\alpha_1 = (1, -1)$ short, and $\alpha_2 = (0, 2)$ long; furthermore, the corresponding fundamental dominant weights are $\omega_1 = (1, 0)$, $\omega_2 = (1, 1)$. One checks that a set of simple roots of Φ_s is $\Pi_s := \{\alpha_1, \alpha_1 + \alpha_2\}$. We shall denote these simple roots by $\beta_1 = \alpha_1 = (1, -1)$, $\beta_2 = \alpha_1 + \alpha_2 = (1, 1)$. The special isogeny induces a $\mathbb{Z}$-linear map $\tau^* : X(T) \to X(T)$, under which $\omega_1 \mapsto \omega_2 \mapsto 2\omega_1$. From now on, we abuse notation, writing τ instead of τ^*. Thus, the τ-restricted weights are 0 and ω_1.

B_τ-Cohomology

Let $\lambda \in X_{r/2}$ be written as $\lambda = \lambda_0 + \tau(\lambda_1)$, for some $\lambda_1 \in X_s(T)$, such that $\mathrm{H}^1(B_\tau, \lambda) \cong \mathrm{H}^1(B_\tau, \lambda_0) \otimes \tau(\lambda_1)$. Thus, it suffices to compute $\mathrm{H}^1(B_\tau, \lambda_0)$, for $\lambda_0 \in X_\tau(T)$.

Theorem 3.2.1. Let $\lambda_0 \in X_\tau(T)$. Then

$$\mathrm{H}^1\left(B_\tau, \lambda_0\right) \cong \begin{cases} k^{(\tau)}_{\omega_2 - \omega_1} \oplus k^{(\tau)}_{\omega_1} & \text{if } \lambda_0 = k \\ 0 & \text{else.} \end{cases}$$

Proof. By Lemma 3.1.2, considered as a T-module, $\mathrm{H}^1(U_\tau, k) \otimes \lambda_0$ is the direct sum of certain $k_{\beta_i+\lambda_0}$, for $\beta_i \in \Pi_s$. By Lemma 3.1.3, such a summand yields a non-zero contribution to $\mathrm{H}^1(B_\tau, \lambda_0)$ if and only if $\beta_i + \lambda_0 \in \tau X(T)$. We now directly verify which of these weights belong to $\tau X(T)$.

First, suppose $\lambda_0 = 0$. Then, we have

$$\beta_1 + 0 = \alpha_1 = 2\omega_1 - \omega_2 = \tau(\omega_2 - \omega_1).$$

$$\beta_2 + 0 = \alpha_1 + \alpha_2 = \omega_2 = \tau(\omega_1).$$

Hence, $\mathrm{H}^1(B_\tau, k) \cong [\bigoplus_i k_{\beta_i+0}]^{T_\tau} \cong \left[k_{\tau(\omega_2-\omega_1)} \oplus k_{\tau\omega_1}\right]^{T_\tau} \cong k^{(\tau)}_{\omega_2-\omega_1} \oplus k^{(\tau)}_{\omega_1}$.

Now, suppose $\lambda_0 = \omega_1$ and we obtain

$$\beta_1 + \omega_1 = 3\omega_1 - \omega_2 \notin \tau X(T).$$

$$\beta_2 + \omega_1 = \omega_2 + \omega_1 \notin \tau X(T).$$

Then, $\mathrm{H}^1(B_\tau, \omega_1) \cong [\bigoplus_i k_{\beta_i+\omega_1}]^{T_\tau} = 0$, as neither belongs to $\tau X(T)$. □

$B_{r/2}$-Cohomology

In this subsection, we extend the calculations of the previous section in order to compute $\mathrm{H}^1(B_{r/2}, \lambda)$, for $\lambda \in X_{r/2}(T)$.

First, we note that in this case, the calculation of $\mathrm{H}^1(B_{r/2}, \lambda)$ requires, among other things, knowledge of the second B_s-cohomology with coefficients in a p^s-restricted weight; this was computed in (Wri11, Appendix C.2.6). For the reader's convenience, we list these cohomology groups here, with data extracted specifically for the underlying root system of G of type C_2.

Lemma 3.2.2. Assume the underlying root system of G is of type C_2. Let s be a positive integer, $p = 2$ with $\lambda' \in X_s(T)$ and $w \in W$. Then

$$\mathrm{H}^2(B_s, \lambda') \cong \begin{cases} \mathrm{H}^2(B_1, w \cdot 0 + 2\nu)^{(s)} & \text{if } \lambda' = 2^{s-1}(w \cdot 0 + 2\nu), \ell(w) = 0, 2 \\ \nu^{(s)} & \text{if } \lambda' = 2^s\nu + 2^l w \cdot 0, \ell(w) = 0, 2; \\ & \text{and } 0 \leq l < s-1 \\ \nu^{(s)} & \text{if } \lambda' = 2^s\nu - 2^l\alpha, \alpha \in \Pi, 0 \leq l \leq s-1; \\ & \text{and } l \neq s-1 \text{ if } \alpha = \alpha_2 \\ \nu^{(s)} & \text{if } \lambda' = 2^s\nu - 2^t\beta - 2^l\alpha, \alpha, \beta \in \Pi, 0 \leq l < t < s \\ \nu^{(s)} & \text{if } \lambda' = 2^s\nu - 2^l(\alpha_1 + \alpha_2), 0 \leq l < s-1 \\ M^{(s)} \otimes \nu^{(s)} & \text{if } \lambda' = 2^s\nu - 2^{s-1}\alpha_2 - 2^l\alpha, \alpha \in \Pi, 0 \leq l < s-1 \\ M^{(s)} \otimes \nu^{(s)} & \text{if } \lambda' = 2^s\nu - 2^{s-1}\alpha, \alpha \in \Pi \\ \mathrm{H}^1(B_{s-1}, M^{(-1)} \otimes \lambda_1) & \text{if } \lambda' = 2\lambda_1, \text{ for some } \lambda_1 \in X(T), s > 1 \\ \oplus\, \mathrm{H}^2(B_{s-1}, \lambda_1) & \\ 0 & \text{else.} \end{cases}$$

Here M denotes an indecomposable B-module with head k_{α_1} and socle k (cf. (Wri11, Appendix C.2.5)). Note that it is implicit in the statement of the lemma that $s \geq 1$ or $s \geq 2$, depending on the case.

If $r = 1$, we refer the reader to Theorem 3.2.1.

Theorem 3.2.3. Suppose $r = 2s + 1 > 1$ and let $\lambda \in X_{r/2}(T)$. Then, for $0 \leq i \leq s-2$, we have

$$\mathrm{H}^1\left(B_{r/2}, \lambda\right) \cong \begin{cases} k_{\omega_1}^{(r/2)} & \text{if } \lambda = (2^s - 1)\omega_2 = \tau^r\omega_1 - \beta_2 \\ k_{\omega_2}^{(r/2)} & \text{if } \lambda = (2^{s+1} - 2)\omega_1 + \omega_2 = \tau^r\omega_2 - \beta_1 \\ k_{\omega_1}^{(r/2)} & \text{if } \lambda = 2^s\omega_1 = \tau^r\omega_1 - \tau^{2s-1}\alpha_1 \\ M_{C_2}^{(r/2)} & \text{if } \lambda = 0 = \tau^r(\omega_2 - \omega_1) - \tau^{2s-1}\alpha_2 \\ k_{\omega_1}^{(r/2)} & \text{if } \lambda = (2^s - 2^{i+1})\omega_2 + 2^{i+1}\omega_1 = \tau^r\omega_1 - \tau^{2i+1}\alpha_1 \\ k_{\omega_2}^{(r/2)} & \text{if } \lambda = 2^{i+1}\omega_2 + (2^{s+1} - 2^{i+2})\omega_1 = \tau^r\omega_2 - \tau^{2i+1}\alpha_2 \\ 0 & \text{else.} \end{cases}$$

Here M_{C_2} denotes the 2-dimensional indecomposable B-module with head k_{ω_1} and socle $k_{\omega_2 - \omega_1}$ (cf. (BNP04, 2.2)).

We underline that the last two non-zero instances only occur when $s \geq 2$ (or $r \geq 5$).

Proof. The second equality in each case identifying two forms of λ is straightforward, recalling $\tau(\omega_1) = \omega_2$, $\beta_1 = (1, -1) = 2\omega_1 - \omega_2$ and $\beta_2 = (1, 1) = \omega_2$. For instance, suppose $\lambda = \tau^r\omega_1 - \beta_2$. Then, λ may be expressed as $\lambda = 2^s\tau\omega_1 - \omega_2 = (2^s - 1)\omega_2$. Hence we now prove that λ must be equal to one of the weights given by the first equality in each case. We consider the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(B_{r/2}/B_\tau, \mathrm{H}^j(B_\tau, \lambda)) \Rightarrow \mathrm{H}^{i+j}(B_{r/2}, \lambda)$$

and the corresponding five-term exact sequence

$$0 \to E_2^{1,0} \to E_\infty^1 \to E_2^{0,1} \to E_2^{2,0} \to E_\infty^2.$$

We will identify E_∞^1 with either $E_2^{0,1}$ or $E_2^{1,0}$ and compute all of the non-zero cases in this way. First, we fix some notation. Since $\lambda \in X_{r/2}(T)$, it may be uniquely expressed as $\lambda = \sum_{i=0}^{r-1} \tau^i\lambda_i$, where λ_i are τ-restricted weights. Then, we write $\lambda = \lambda_0 + \tau(\lambda')$, for $\lambda' = \sum_{j=1}^{r-1} \tau^{j-1}\lambda_j$. Suppose $E_2^{0,1} \neq 0$ and consider the $E_2^{0,1}$-term.
We have

$$\begin{aligned} E_2^{0,1} &= \mathrm{Hom}_{B_{r/2}/B_\tau}(k, \mathrm{H}^1(B_\tau, \lambda)) \\ &\cong \mathrm{Hom}_{B_{r/2}/B_\tau}(k, \mathrm{H}^1(B_\tau, \lambda_0) \otimes \tau(\lambda')). \end{aligned}$$

There is only one τ-restricted weight for which $\mathrm{H}^1(B_\tau, \lambda_0) \neq 0$, namely $\lambda_0 = 0$. In this case

$$\mathrm{H}^1(B_\tau, k) \cong k_{\omega_2-\omega_1}^{(\tau)} \oplus k_{\omega_1}^{(\tau)}.$$

Hence

$$\begin{aligned} E_2^{0,1} &= \mathrm{Hom}_{B_{r/2}/B_\tau}(k, (k_{\omega_1}^{(\tau)} \oplus k_{\omega_2-\omega_1}^{(\tau)}) \otimes \tau(\lambda')) \\ &\cong \mathrm{Hom}_{B_{(r-1)/2}}(k, (k_{\omega_1}^{(\tau)} \oplus k_{\omega_2-\omega_1}^{(\tau)}) \otimes k_{\lambda'}^{(\tau)}) \\ &\cong \mathrm{Hom}_{B_{(r-1)/2}}(k, k_{\omega_1+\lambda'}^{(\tau)} \oplus k_{\omega_2-\omega_1+\lambda'}^{(\tau)}). \end{aligned}$$

Now $\mathrm{Hom}_{B_{(r-1)/2}}(k, k_{\omega_1+\lambda'}^{(\tau)} \oplus k_{\omega_2-\omega_1+\lambda'}^{(\tau)})$ is non-zero if at least one of $\omega_1+\lambda'$ or $\omega_2-\omega_1+\lambda' \in \tau^{r-1}X(T)$. In fact, $\mathrm{Hom}_{B_{(r-1)/2}}(k, k_{\omega_1+\lambda'}^{(\tau)} \oplus k_{\omega_2-\omega_1+\lambda'}^{(\tau)})$ is at most one-dimensional: since $\omega_2 - 2\omega_1 \notin \tau^{r-1}X(T)$, at most one of $\omega_1 + \lambda'$ and $\omega_2 - \omega_1 + \lambda'$ is in $\tau^{r-1}X(T)$. We take each case in turn.

First, suppose $\omega_1 + \lambda' \in \tau^{r-1}X(T)$. As $p = 2$, we have $\lambda' = (a2^s - 1)\omega_1 + b2^s\omega_2 \in X_s$. It immediately follows $b = 0$ and $a = 1$, giving $\lambda = \lambda_0 + \tau(\lambda') = (2^s - 1)\omega_2$ and

$$E_2^{0,1} = \mathrm{Hom}_{B_{r/2}/B_\tau}(k, k_{\tau(\omega_1+\lambda')} \oplus k_{\tau(\omega_2-\omega_1+\lambda')}).$$

The first term in the target of the Hom is $k_{\omega_2+(2^s-1)\omega_2} = k_{2^s\omega_2}$. Thus $E^{0,1} \cong k_{2^s\omega_2} = (k_{\omega_1})^{(r/2)}$.

In the case $\omega_2 - \omega_1 + \lambda' \in \tau^{r-1}X(T)$, a similar argument leads us to conclude that $E_2^{0,1} = k_{\omega_2}^{(r/2)}$ and $\lambda = \omega_2 + (2^{s+1} - 2)\omega_1$. To conclude, for $\lambda \in X_{r/2}(T)$,

$$E_2^{0,1} \cong \begin{cases} k_{\omega_1}^{(r/2)} & \text{if } \lambda = (2^s - 1)\omega_2 \\ k_{\omega_2}^{(r/2)} & \text{if } \lambda = (2^{s+1} - 2)\omega_1 + \omega_2 \\ 0 & \text{else.} \end{cases}$$

Now suppose $E_2^{1,0} \neq 0$. We have

$$\begin{aligned} E_2^{1,0} &= \mathrm{H}^1(B_{r/2}/B_\tau, \mathrm{Hom}_{B_\tau}(k, \lambda)), \\ &= \mathrm{H}^1(B_{r/2}/B_\tau, \mathrm{Hom}_{B_\tau}(k, \lambda_0) \otimes \tau(\lambda')) \end{aligned}$$

so $\lambda_0 = 0$ and $\lambda = \tau(\lambda')$. Thus $E_2^{1,0} \cong \mathrm{H}^1(B_s, \lambda'^{(\tau)}) \cong \mathrm{H}^1(B_s, \lambda')^{(\tau)}$ for $\lambda = \tau\lambda'$. Notice that since $r - 1 = 2s > 0$, $B_{(r-1)/2} = B_s$ is a classical Frobenius kernel and $\mathrm{H}^1(B_s, \lambda')$ is the B_s-cohomology for $\lambda' \in X_s(T)$ computed in (BNP04, Theorem 2.7). We have

$$\mathrm{H}^1(B_s, \lambda') \cong \begin{cases} k_{\omega_1}^{(s)} & \text{if } \lambda' = 2^s\omega_1 - 2^{s-1}\alpha_1 \\ M_{C_2}^{(s)} & \text{if } \lambda' = 0 = 2^s(\omega_2 - \omega_1) - 2^{s-1}\alpha_2 \\ k_{\omega_j}^{(s)} & \text{if } \lambda' = 2^s\omega_\alpha - 2^i\alpha, \alpha \in \Pi, 0 \leq i \leq s-2 \\ 0 & \text{else.} \end{cases}$$

with M_{C_2} having the structure as claimed in the statement of the theorem. We note the implicit constraints on s in the different cases. Hence,

$$E_2^{1,0} \cong \mathrm{H}^1\left(B_s, \lambda'\right)^{(\tau)} \cong \begin{cases} k_{\omega_1}^{(r/2)} & \text{if } \lambda' = 2^s\omega_1 - 2^{s-1}\alpha_1 \\ M_{C_2}^{(r/2)} & \text{if } \lambda' = 0 = 2^s(\omega_2 - \omega_1) - 2^{s-1}\alpha_2 \\ k_{\omega_j}^{(r/2)} & \text{if } \lambda' = 2^s\omega_\alpha - 2^i\alpha, \alpha \in \Pi, 0 \leq i \leq s-2 \\ 0 & \text{else.} \end{cases}$$

One can recover λ from λ', recalling $\alpha_1 = 2\omega_1 - \omega_2$ and $\alpha_2 = 2\omega_2 - 2\omega_1$. For example if $\lambda' = 2^s\omega_1 - 2^{s-1}\alpha_1 = 2^{s-1}\omega_2$, then $\lambda = \tau\lambda' = 2^s\omega_1$. The other cases are similar.

In light of the above, observe that there is no choice of λ for which $E_2^{1,0}$ and $E_2^{0,1}$ are both non-zero. Hence if $E_2^{1,0} \neq 0$, then $E^{0,1_2} = 0$, implying that $E_\infty^1 \cong E_2^{1,0}$. Alternatively, suppose that $E_2^{0,1} \neq 0$, so $E_2^{1,0} = 0$. It remains to check whether the differential $d_2 : E_2^{0,1} \to E_2^{2,0}$ is the zero map. Assume $E_2^{2,0} \neq 0$ and we have

$$\begin{aligned} E_2^{2,0} &= \mathrm{H}^2(B_{r/2}/B_\tau, \mathrm{Hom}_{B_\tau}(k, \lambda)) \\ &\cong \mathrm{H}^2(B_{(r-1)/2}, \mathrm{Hom}_{B_\tau}(k,\lambda)^{(-\tau)})^{(\tau)} \\ &\cong \mathrm{H}^2(B_s, \lambda'^{(\tau)}) \cong \mathrm{H}^2(B_s, \lambda')^{(\tau)} \end{aligned}$$

for $\lambda = \tau\lambda'$. As before, $B_{(r-1)/2} = B_s$ is a classical Frobenius kernel and $\mathrm{H}^2(B_s, \lambda')$ is the second B_s-cohomology for $\lambda' \in X_s(T)$ computed in Lemma 3.2.2.

Since $E_2^{0,1} \neq 0$, $\lambda' = 2^s\omega_1 - \omega_1$ or $2^s\omega_2 + \omega_1 - \omega_2$. In each case the coefficient of ω_1 in λ' is odd, so λ' is not in the root lattice; however, since $E_2^{2,0} \neq 0$, we see from Lemma 3.2.2 that λ' is in the root lattice—a contradiction. It follows that the differential $d_2 : E_2^{0,1} \to E_2^{2,0}$ is the zero map. Therefore, if $E_2^{0,1} \neq 0$, we have $E_\infty^1 \cong E_2^{0,1}$. □

For a general $\lambda \in X(T)$, not necessarily lying in $X_{r/2}$, we proceed as in (BNP04, 2.8).

Corollary 3.2.4. Let $\lambda \in X(T)$ and $r = 2s+1 > 1$. Then $\mathrm{H}^1(B_{r/2}, \lambda) \neq 0$ if and only if $\lambda = \tau^r\omega - \tau^i\alpha$, for some weight $\omega \in X(T)$, and $\alpha \in \Pi$ with $0 \leq i \leq 2s-1$ or $\lambda = \tau^r\omega - \beta$, for some weight $\omega \in X(T)$, and $\beta \in \Pi_s$.

Proof. Suppose $\mathrm{H}^1(B_{r/2}, \lambda) \neq 0$. Then we may uniquely write $\lambda = \lambda_0 + \tau^r\lambda_1$, for $\lambda_0 \in X_{r/2}(T)$ and $\lambda_1 \in X(T)$. It follows that $\mathrm{H}^1(B_{r/2}, \lambda) \cong \mathrm{H}^1(B_{r/2}, \lambda_0) \otimes \tau^r\lambda_1$. Thus, by Theorem 3.2.3, $\mathrm{H}^1(B_{r/2}, \lambda_0) \neq 0$ if and only if $\lambda_0 = \tau^r\omega' - \tau^i\alpha$ for $\alpha \in \Pi$ and $0 \leq i \leq 2s-1$, or $\lambda_0 = \tau^r\omega' - \beta$ for $\beta \in \Pi_s$, with ω' the specific weight in the theorem. In the first case, we may then write $\lambda = \lambda_0 + \tau^r\lambda_1 = \tau^r\omega' - \tau^i\alpha + \tau^r\lambda_1 = \tau^r(\omega' + \lambda_1) - \tau^i\alpha = \tau^r\omega - \tau^i\alpha$. Secondly, we have $\lambda = \lambda_0 + \tau^r\lambda_1 = \tau^r\omega' - \beta + \tau^r\lambda_1 = \tau^r(\omega' + \lambda_1) - \beta = \tau^r\omega - \beta$. In both cases, we obtain the required form.

Conversely, suppose we are given any weight $\lambda = \tau^r\omega - \tau^i\alpha$, with $\alpha \in \Pi$ and $0 \leq i \leq 2s-1$ or $\lambda = \tau^r\omega - \beta$, for $\beta \in \Pi_s$. In either case, one can always express ω as $\omega = \omega' + \lambda_1$, for the required weight ω' in Theorem 3.2.3 and some weight $\lambda_1 \in X(T)$. Hence, $\mathrm{H}^1(B_{r/2}, \lambda) \neq 0$ for all such λ, as non-vanishing is independent of the choice of λ_1. □

Suppose $\mathrm{H}^1(B_{r/2}, \lambda) \neq 0$ and let (ζ, j) denote the appropriate pair, (α, i) or $(\beta, 1)$, as defined in the previous corollary. Now, given $\lambda = \tau^r\omega - \tau^j\zeta$, we may write $\lambda = \tau^r\omega' - \tau^j\zeta + \tau^r\lambda_1$, where ω' is chosen as per the list in Theorem 3.2.3, and so it follows that λ_1 is $\omega - \omega'$. Hence

$$\begin{aligned} \mathrm{H}^1(B_{r/2}, \lambda) &\cong \mathrm{H}^1(B_{r/2}, \lambda_0) \otimes k_{\lambda_1}^{(r/2)} \\ &\cong \mathrm{H}^1(B_{r/2}, \tau^r\omega' - \tau^j\zeta) \otimes k_{\omega-\omega'}^{(r/2)}. \end{aligned}$$

Direct verification, substituting the answers from Theorem 3.2.3, yields the following result

Theorem 3.2.5. Let $\lambda \in X(T)$. Then, for $0 \leq i \leq s-2$, we have

$$\mathrm{H}^1\left(B_{r/2}, \lambda\right) \cong \begin{cases} k_\omega^{(r/2)} & \text{if } \lambda = \tau^r\omega - \beta_i, \omega \in X(T), \beta_i \in \Pi_s \\ k_\omega^{(r/2)} & \text{if } \lambda = \tau^r\omega - \tau^{2s-1}\alpha_1, \omega \in X(T) \\ M_{C_2}^{(r/2)} \otimes k_{\omega+\omega_1-\omega_2}^{(r/2)} & \text{if } \lambda = \tau^r\omega - \tau^{2s-1}\alpha_2, \omega \in X(T) \\ k_\omega^{(r/2)} & \text{if } \lambda = \tau^r\omega - \tau^{2i+1}\alpha_j, \omega \in X(T), \alpha_j \in \Pi \\ 0 & \text{else.} \end{cases}$$

$G_{r/2}$-Cohomology of Induced Modules

By Kempf's vanishing theorem, $\mathrm{H}^0(\lambda) = \mathrm{Ind}_B^G \lambda$ is zero unless λ is dominant. For $\lambda \in X(T)_+$, one may use Theorem 3.2.1 and Theorem 3.2.3, respectively, to compute $\mathrm{H}^1(G_{r/2}, H^0(\lambda))$ with the aid of the isomorphism (3.1.1). Finally, we note that, by (BNP04, 3.1. Theorem (C)), $\mathrm{Ind}_B^G(M_{C_2}) = \mathrm{H}^0(\omega_1)$.

In case $r = 1$, we obtain (Sin94b, Lemma 2.1):

Theorem 3.2.6. Let $\lambda \in X_\tau(T)$. Then

$$\mathrm{H}^1\left(G_\tau, \mathrm{H}^0(\lambda)\right)^{(-\tau)} \cong \begin{cases} \mathrm{H}^0(\omega_1) \cong L(\omega_1) & \text{if } \lambda = 0 \\ 0 & \text{else.} \end{cases}$$

Otherwise, we have:

Theorem 3.2.7. Let $r = 2s+1 > 1$, $\lambda \in X_{r/2}(T)$ and $0 \leq i \leq s-2$. Then

$$\mathrm{H}^1(G_{r/2}, \mathrm{H}^0(\lambda))^{(-r/2)} \cong \begin{cases} \mathrm{H}^0(\omega_1) & \text{if } \lambda = (2^s-1)\omega_2 = \tau^r\omega_1 - \beta_2 \\ \mathrm{H}^0(\omega_2) & \text{if } \lambda = (2^{s+1}-2)\omega_1 + \omega_2 = \tau^r\omega_2 - \beta_1 \\ \mathrm{H}^0(\omega_1) & \text{if } \lambda = 2^s\omega_1 = \tau^r\omega_1 - \tau^{2s-1}\alpha_1 \\ \mathrm{H}^0(\omega_1) & \text{if } \lambda = 0 = \tau^r(\omega_2 - \omega_1) - \tau^{2s-1}\alpha_2 \\ \mathrm{H}^0(\omega_1) & \text{if } \lambda = (2^s - 2^{i+1})\omega_2 + 2^{i+1}\omega_1 = \tau^r\omega_1 - \tau^{2i+1}\alpha_1 \\ \mathrm{H}^0(\omega_2) & \text{if } \lambda = 2^{i+1}\omega_2 + (2^{s+1} - 2^{i+2})\omega_1 = \tau^r\omega_2 - \tau^{2i+1}\alpha_2 \\ 0 & \text{else.} \end{cases}$$

Next, one can make use of Theorem 3.2.5 to compute $\mathrm{H}^1(G_{r/2}, \mathrm{H}^0(\lambda))$ in terms of induced modules for all dominant weights λ, by applying the induction functor Ind_B^G. The only non-obvious case is dealt with in the remark below.

Remark 3.2.8. Let $\tau^r\omega - \tau^{2s-1}\alpha_2 \in X(T)_+$. Then $\langle \omega, \alpha_1^\vee \rangle \geq -1$ and $\langle \omega, \alpha_2^\vee \rangle \geq 1$. In this case, by (BNP04, Proposition 3.4 (B)(c)), $\mathrm{Ind}_B^G(M_{C_2} \otimes k_{\omega+\omega_1-\omega_2})$ has a filtration with factors satisfying the following short exact sequence

$$0 \to \mathrm{H}^0(\omega) \to \mathrm{Ind}_B^G(M_{C_2} \otimes k_{\omega+\omega_1-\omega_2}) \to \mathrm{H}^0(\omega + 2\omega_1 - \omega_2) \to 0.$$

(Observe that $\mathrm{H}^0(\omega+2\omega_1-\omega_2)$ is always present, but $\mathrm{H}^0(\omega)$ appears as a factor if $\langle \omega, \alpha_1^\vee \rangle \geq 0$.)

$G_{r/2}$ Extensions Between Simple Modules

In this subsection, we make use of the G_1-cohomology with coefficients in simple modules, computed in (Sin94b, Proposition 2.3), to calculate $\mathrm{Ext}^1_{G_s}(L(\lambda), L(\mu))^{(-s)}$, for a positive integer s and $\lambda, \mu \in X_s(T)$.

We begin with a result due to (HS53, Section 4).

Lemma 3.2.9. Let (E_n, d_n) be a second stage first quadrant spectral sequence of cohomological type converging to the graded vector space H^* and assume $E_2^{i,1} = 0$ for all $i \geq 0$. Then, we have a five-term exact sequence

$$0 \to E_3^{2,0} = E_\infty^{2,0} \to \mathrm{H}^2 \to E_3^{0,2} \to E_3^{3,0} \to \mathrm{H}^3\,.$$

Proof. Note that the $j = 1$ row vanishes; that is $E_2^{i,1} = 0$. Since every $E_r^{i,1}$ is a subquotient of $E_2^{i,1}$, then $E_r^{i,1} = 0$, for all r.

Note that, assuming $E_2^{i,1} = 0$, the five-term exact sequence discussed in Lemma 2.8.4 becomes an isomorphism $E_2^{1,0} \cong E_\infty^1 = \mathrm{H}^1$ and $E_2^{2,0} \hookrightarrow E_\infty^2 = \mathrm{H}^2$; $E_2^{2,0}$ injects in $E_\infty^2 = \mathrm{H}^2$, so we have $0 \to E_2^{2,0} \to E_\infty^2 = \mathrm{H}^2$. Since $E_2^{i,1} = 0$, we have $\mathrm{H}^2 = E_\infty^2 = E_\infty^{2,0} \oplus E_2^{0,2}$.

Consider $E_2^{0,2}$ and take the cohomology

$$\mathrm{H}(E_2^{0,2}) = \frac{\ker(\mathrm{d}_2 : \mathrm{E}_2^{0,2} \to \mathrm{E}_2^{2,1} = 0)}{\mathrm{im}(d_2 : E_2^{-2,3} \to E_2^{0,2})} = E_2^{0,2}.$$

Hence, $E_3^{0,2} \cong E_2^{0,2}$. There is also a differential $d_3 : E_3^{0,2} \to E_3^{3,0}$. Now, consider $\mathrm{H}(E_3^{0,2})$:

$$\mathrm{H}(E_4^{0,2}) = \frac{\ker(\mathrm{d}_3 : \mathrm{E}_3^{0,2} \to \mathrm{E}_3^{3,0})}{\mathrm{im}(d_3 : E_3^{-3,4} \to E_3^{0,2})} = \ker(d_3 : E_3^{0,2} \to E_3^{3,0}).$$

Therefore, $E_4^{0,2}$ is the cohomology of the sequence $0 \to E_3^{0,2} \to E_3^{3,0}$ and this explains

$$0 \to E_3^{2,0} \to E_\infty^2 = \mathrm{H}^2 \to E_3^{0,2} \xrightarrow{d_2} E_3^{3,0}.$$

Finally, the cokernel of the map $d_3 : E_3^{0,2} \to E_3^{3,0}$ is $E_2^{3,0}/\mathrm{im}(d_3 : E_3^{0,2} \to E_3^{3,0})$ and notice that it equals $E_4^{3,0}$. Observe that $E_4^{3,0}$ has no maps in or out of the first quadrant, so it injects into $E_\infty^3 = \mathrm{H}^3$. □

First, we underline that in this case, the calculation of $\mathrm{Ext}^1_{G_s}(L(\lambda), L(\mu))^{(-s)}$ requires knowledge of the following cohomology group.

Lemma 3.2.10. Let G be of type C_2 and $p = 2$. Then $\mathrm{H}^2(G_1, L(\omega_1)) = 0$.

Proof. We run the Lyndon-Hochschild-Serre spectral sequence corresponding to $G_\tau \lhd G_1$. The E_2-page is given by

$$E_2^{i,j} = \mathrm{H}^i(G_\tau, \mathrm{H}^j(G_\tau, L(\omega_1))^{(-\tau)})^{(\tau)}.$$

By Theorem 3.2.6, $\mathrm{H}^1(G_\tau, L(\omega_1)) = 0$, so $E_2^{i,1} = 0$. It follows that we obtain the following five-term exact sequence from Lemma 3.2.9

$$0 \to E_3^{2,0} \to E_\infty^2 \to E_3^{0,2} \to E_3^{3,0} \to E_\infty^3.$$

First, note that the $E_3^{2,0}$-term vanishes, since $\mathrm{Hom}_{G_\tau}(k, L(\omega_1)) = 0$. Moreover, $L(\omega_1)$ is an injective module for G_τ, so $\mathrm{H}^2(G_\tau, L(\omega_1)) = 0$. Therefore, $E_3^{0,2}$ also vanishes, so we conclude that $E_\infty^2 = \mathrm{H}^2(G_1, L(\omega_1)) = 0$. □

Theorem 3.2.11. Let s be a positive integer, $\lambda, \mu \in X_s(T)$ and $1 \leq i \leq s-1$. We write $\lambda = \lambda_0 + 2^{s-1}\lambda_1$, for $\lambda_0 = \sum_{i=0}^{s-2} p^i \lambda_{0,i} \in X_{s-1}$ and $\lambda_1 \in X_1$; we take a similar expression for μ. Then

$$\operatorname{Ext}^1_{G_s}(L(\lambda), L(\mu))^{(-s)} \cong \begin{cases} L(\omega_1) & \text{if } \lambda = \mu \text{ and } \lambda_1 = \mu_1 \in \{0, \omega_1\} \\ k & \text{if } \lambda - \mu = \pm 2^{s-1}\omega_2, (\lambda_{0,s-1}, \mu_{0,s-1}) = (0, \omega_2) \\ k & \text{if } \lambda - \mu = \pm 2^{i-1}\omega_2, (\lambda_{0,i-1}, \mu_{0,i-1}) = (0, \omega_2) \\ k & \text{if } \lambda - \mu = \pm 2^{i}\omega_1, (\lambda_{0,i}, \mu_{0,i}) \in \{(0, \omega_1), (\omega_2, \omega_1 + \omega_2)\} \\ & \qquad\qquad \text{and } \lambda_{0,i-1} = \mu_{0,i-1} \in \{0, \omega_1\} \\ 0 & \text{else.} \end{cases}$$

Note that it is implicit in the statement of the theorem that $s \geq 1$ or $s \geq 2$, depending on the case.

Proof. We proceed inductively. When $s = 1$, we refer the reader to (Sin94b, Proposition 2.3). Suppose $s > 1$ and consider the Lyndon-Hochschild-Serre spectral sequence corresponding to $G_{s-1} \lhd G_s$. The E_2-page is given by

$$E_2^{i,j} := \operatorname{Ext}^i_{G_1}(L(\lambda_1), \operatorname{Ext}^j_{G_{s-1}}(L(\lambda_0), L(\lambda_0))^{(-s+1)} \otimes L(\mu_1))^{(s-1)}.$$

First, consider the $E_2^{1,0}$-term. We have

$$E_2^{1,0} = \operatorname{Ext}^1_{G_1}(L(\lambda_1), \operatorname{Hom}_{G_{s-1}}(L(\lambda_0), L(\mu_0))^{(-s+1)} \otimes L(\mu_1))^{(s-1)}.$$

Note that $E_2^{1,0} \neq 0$ if and only if $\lambda_0 = \mu_0$, in which case we obtain

$$E_2^{1,0} = \mathrm{H}^1(G_1, L(\lambda_1))^{(s-1)} \cong \begin{cases} L(\omega_1)^{(s)} & \text{if } \lambda_1 = \mu_1 \in \{0, \omega_1\} \\ k & \text{if } (\lambda_1, \mu_1) = (0, \omega_2) \\ 0 & \text{else.} \end{cases}$$

(cf. (Sin94b, Proposition 2.3)). Therefore, recalling that $\lambda = \lambda_0 + 2^{s-1}\lambda_1$ and $\mu = \mu_0 + 2^{s-1}\mu_1$, we may conclude that for $\lambda, \mu \in X_s$,

$$E_2^{1,0} \cong \begin{cases} L(\omega_1)^{(s)} & \text{if } \lambda = \mu, \lambda_1 = \mu_1 \in \{0, \omega_1\} \\ k & \text{if } \lambda - \mu = \pm 2^{s-1}\omega_2, (\lambda_1, \mu_1) = (0, \omega_2) \\ 0 & \text{else.} \end{cases}$$

Now consider the $E_2^{0,1}$-term. We have

$$E_2^{0,1} = \mathrm{Hom}_{G_1}(L(\lambda_1), \mathrm{Ext}^1_{G_{s-1}}(L(\lambda_0), L(\mu_0))^{(-s+1)} \otimes L(\mu_1))^{(s-1)}.$$

We take each non-zero instance of $\mathrm{Ext}^1_{G_{s-1}}(L(\lambda_0), L(\mu_0))$ in turn. By the induction hypothesis, if $\lambda_0 = \mu_0$, with $\lambda_{0,s-2} = \mu_{0,s-2} \in \{0, \omega_1\}$, then we have

$$\begin{aligned} E_2^{0,1} &= \mathrm{Hom}_{G_1}(L(\lambda_1), L(\omega_1) \otimes L(\mu_1))^{(s-1)} \\ &\cong \mathrm{Hom}_{G_\tau}(L(\lambda_{1,1}), \mathrm{Hom}_{G_\tau}(L(\lambda_{1,0}), L(\mu_{1,0}) \otimes L(\omega_1))^{(-\tau)} \otimes L(\mu_{1,1}))^{(s-1/2)}, \end{aligned}$$

where $\lambda_1 = \lambda_{1,0} + \tau\lambda_{1,1}$ and μ_1 is expressed similarly. By (Sin94b, Proposition 2.2(c)(i)), $\mathrm{Hom}_{G_\tau}(L(\lambda_{1,0}), L(\mu_{1,0}) \otimes L(\omega_1)) \neq 0$ if and only if $(\lambda_{1,0}, \mu_{1,0}) = (0, \omega_1)$. Thus $E_2^{0,1} \neq 0$ if and only if $\lambda_{1,1} = \mu_{1,1}$; it follows that $E_2^{0,1} \cong k$ if $\lambda - \mu = \pm 2^{s-1}\omega_1$. The other cases follow similarly and we obtain, for $1 \leq i \leq s-1$

$$E_2^{0,1} \cong \begin{cases} k & \text{if } \lambda - \mu = \pm 2^{i-1}\omega_2, (\lambda_{0,i-1}, \mu_{0,i-1}) = (0, \omega_2) \\ k & \text{if } \lambda - \mu = \pm 2^{i}\omega_1, (\lambda_{0,i}, \mu_{0,i}) \in \{(0, \omega_1), (\omega_2, \omega_1 + \omega_2)\} \\ & \qquad\qquad\qquad \text{and } \lambda_{0,i-1} = \mu_{0,i-1} \in \{0, \omega_1\} \\ 0 & \text{else.} \end{cases}$$

Notice that there are no choices of λ and μ for which $E_2^{1,0}$ and $E_2^{0,1}$ are both non-zero. Hence if $E_2^{1,0} \neq 0$, then $E_2^{0,1} = 0$, implying that $E_\infty^1 \cong E_2^{1,0}$. Alternatively, suppose that $E_2^{0,1} \neq 0$, so $E_2^{1,0} = 0$. It remains to check whether the differential $d_2 : E_2^{0,1} \to E_2^{2,0}$ is the zero map. The $E_2^{2,0}$-term is

$$E_2^{2,0} = \mathrm{Ext}^2_{G_1}(L(\lambda_1), \mathrm{Hom}_{G_{s-1}}(L(\lambda_0), L(\mu_0))^{(-s+1)} \otimes L(\mu_1))^{(s-1)}.$$

We consider each choice of (λ, μ) for which $E_2^{0,1} \neq 0$. If $\lambda - \mu = \pm 2^{i-1}\omega_2$ or $\lambda - \mu = \pm 2^i \omega_1$, we obtain $\mathrm{Hom}_{G_{s-1}}(L(\lambda_0), L(\mu_0)) = 0$, so $E_2^{2,0} = 0$.

It remains to verify the case $\lambda - \mu = \pm 2^{s-1}\omega_1$. Then $E_2^{2,0} = \mathrm{H}^2(G_1, L(\omega_1))^{(s-1)}$, which vanishes by Lemma 3.2.10. It follows that $d_2 : E_2^{0,1} \to E_2^{2,0}$ is the zero map and we reach our conclusion. □

Next, making use of the previous theorem concerning the cohomology for classical Frobenius kernels, we compute $\mathrm{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu))$ for r an odd positive integer and $\lambda, \mu \in X_{r/2}$.

If $r = 1$, we refer the reader to (Sin94b, 2.1). Otherwise, we obtain

Theorem 3.2.12. Let s be a positive integer, $\lambda, \mu \in X_{r/2}(T)$ and $1 \leq i \leq s-1$. We write

$\lambda = \lambda_0 + 2^s\lambda_1$, for $\lambda_0 = \sum_{i=0}^{s-1} p^i\lambda_{0,i} \in X_s$ and $\lambda_1 \in X_\tau$; we take a similar expression for μ. Then

$$\operatorname{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu))^{(-r/2)} \cong \begin{cases} L(\omega_1) & \text{if } \lambda = \mu = \lambda_0 \in X_s \\ k & \text{if } \lambda - \mu = \pm 2^s\omega_1, \lambda_{0,s-1} = \mu_{0,s-1} \in \{0, \omega_1\} \\ k & \text{if } \lambda - \mu = \pm 2^{s-1}\omega_2, (\lambda_{0,s-1}, \mu_{0,s-1}) = (0, \omega_2) \\ k & \text{if } \lambda - \mu = \pm 2^{i-1}\omega_2, (\lambda_{0,i-1}, \mu_{0,i-1}) = (0, \omega_2) \\ k & \text{if } \lambda - \mu = \pm 2^i\omega_1, (\lambda_{0,i}, \mu_{0,i}) \in \{(0, \omega_1), (\omega_2, \omega_1 + \omega_2)\} \\ & \qquad\qquad \text{and } \lambda_{0,i-1} = \mu_{0,i-1} \in \{0, \omega_1\} \\ 0 & \text{else.} \end{cases}$$

Note that it is implicit in the statement of the theorem that $s \geq 1$ or $s \geq 2$, depending on the case.

Proof. Consider the Lyndon-Hochschild-Serre spectral sequence corresponding to $G_s \lhd G_{r/2}$. The E_2-page is given by

$$E_2^{i,j} := \operatorname{Ext}^i_{G_\tau}(L(\lambda_1), \operatorname{Ext}^j_{G_s}(L(\lambda_0), L(\mu_0))^{(-s)} \otimes L(\mu_1))^{(s)}.$$

First, consider the $E_2^{1,0}$-term. We have

$$E_2^{1,0} = \operatorname{Ext}^1_{G_\tau}(L(\lambda_1), \operatorname{Hom}_{G_s}(L(\lambda_0), L(\mu_0))^{(-s)} \otimes L(\mu_1))^{(s)}.$$

Note that $E_2^{1,0} \neq 0$ if and only if $\lambda_0 = \mu_0$, in which case we obtain

$$E_2^{1,0} = \operatorname{Ext}^1_{G_\tau}(L(\lambda_1), L(\mu_1))^{(s)}. \cong \begin{cases} L(\omega_1)^{(r/2)} & \text{if } \lambda_1 = \mu_1 \\ 0 & \text{else.} \end{cases}$$

(cf. Theorem 3.2.6 and (Sin94b, Lemma 2.1)). Thus, recalling $\lambda = \lambda_0 + 2^s\lambda_1$ and $\mu = \mu_0 + 2^s\mu_1$, we have $E_2^{1,0} = k$ if $\lambda = \mu = \lambda_0 \in X_s$ and vanishes otherwise. Next, consider the $E_2^{0,1}$-term:

$$E_2^{0,1} = \operatorname{Hom}_{G_\tau}(L(\lambda_1), \operatorname{Ext}^1_{G_s}(L(\lambda_0), L(\mu_0))^{(-s)} \otimes L(\mu_1))^{(s)}.$$

We take each non-zero instance of $\operatorname{Ext}^1_{G_s}(L(\lambda_0), L(\mu_0))^{(-s)}$ from Theorem 3.2.11 in turn. If $\lambda_0 = \mu_0$ and $\lambda_{0,s-1} = \mu_{0,s-1} \in \{0, \omega_1\}$, then

$$E_2^{0,1} = \operatorname{Hom}_{G_\tau}(L(\lambda_1), L(\mu_1) \otimes L(\omega_1))^{(s)},$$

which is non-zero if and only if $\lambda_1 - \mu_1 = \pm\omega_1$, by (Sin94b, Proposition 2.2(c)(i)); we obtain $E_2^{0,1} \cong k$ for $\lambda - \mu = \pm 2^s \omega_1$ and $\lambda_{0,s-1} = \mu_{0,s-1} \in \{0, \omega_1\}$. The other cases are similar. Moreover, we can recover λ and μ, recalling $\lambda = \lambda_0 + 2^s \lambda_1$ and $\mu = \mu_0 + 2^s \mu_1$. We get, for $1 \leq i \leq s-1$

$$E_2^{0,1} \cong \begin{cases} k & \text{if } \lambda - \mu = \pm 2^s \omega_1, \lambda_{0,s-1} = \mu_{0,s-1} \in \{0, \omega_1\} \\ k & \text{if } \lambda - \mu = \pm 2^{s-1} \omega_2, (\lambda_{0,s-1}, \mu_{0,s-1}) = (0, \omega_2) \\ k & \text{if } \lambda - \mu = \pm 2^{i-1} \omega_2, (\lambda_{0,i-1}, \mu_{0,i-1}) = (0, \omega_2) \\ k & \text{if } \lambda - \mu = \pm 2^{i} \omega_1, (\lambda_{0,i}, \mu_{0,i}) \in \{(0, \omega_1), (\omega_2, \omega_1 + \omega_2)\} \\ & \qquad \text{and } \lambda_{0,i-1} = \mu_{0,i-1} \in \{0, \omega_1\} \\ 0 & \text{else.} \end{cases}$$

Observe that there are no choices of λ and μ for which $E_2^{1,0}$ and $E_2^{0,1}$ are both non-zero. Hence if $E_2^{1,0} \neq 0$, then $E_2^{0,1} = 0$, implying that $E_\infty^1 \cong E_2^{1,0}$. Alternatively, suppose that $E_2^{0,1} \neq 0$, so $E_2^{1,0} = 0$. We must also investigate whether the differential $d_2 : E_2^{0,1} \to E_2^{2,0}$ is the zero map. The $E_2^{2,0}$-term is

$$E_2^{2,0} = \mathrm{Ext}^2_{G_\tau}(L(\lambda_1), \mathrm{Hom}_{G_s}(L(\lambda_0), L(\mu_0))^{(-s)} \otimes L(\mu_1))^{(s)}.$$

We consider each choice of (λ, μ) for which $E_2^{0,1} \neq 0$ in turn.

First, if $\lambda - \mu = \pm 2^s \omega_1, \lambda_{0,s-1} = \mu_{0,s-1} \in \{0, \omega_1\}$, it follows that

$$\begin{aligned} E_2^{2,0} &= \mathrm{Ext}^2_{G_\tau}(k, \mathrm{Hom}_{G_s}(L(\lambda_0), L(\mu_0))^{(-s)} \otimes L(\omega_1))^{(s)} \\ &\cong \mathrm{Ext}^2_{G_\tau}(k, L(\omega_1))^{(s)} = 0, \end{aligned}$$

since $L(\omega_1)$ is an injective module for G_τ. In all of the other cases, we have $\lambda_0 \neq \mu_0$, so $\mathrm{Hom}_{G_s}(L(\lambda_0), L(\mu_0)) = 0$, which forces $E_2^{2,0} = 0$. We conclude that d_2 is the zero map. Therefore, $E_2^{0,1} \neq 0$ implies $E_\infty^1 \cong E_2^{0,1}$. $\square$

3.3 G_2 in Characteristic 3

Let G be simply-connected of type G_2 over k of characteristic 3. Following (Bou82, Planche IX), let $\Phi = \{\pm(\epsilon_1 - \epsilon_2), \pm(\epsilon_1 - \epsilon_3), \pm(\epsilon_2 - \epsilon_3), \pm(2\epsilon_1 - \epsilon_2 - \epsilon_3), \pm(2\epsilon_2 - \epsilon_1 - \epsilon_3), \pm(2\epsilon_3 - \epsilon_1 - \epsilon_2)\}$ be the roots of a system of type G_2. Writing $\epsilon_1 = (1, 0, 0)$, $\epsilon_2 = (0, 1, 0)$ and $\epsilon_3 = (0, 0, 1)$, we may take a base of simple roots to be $\Pi := \{\alpha_1, \alpha_2\}$, with $\alpha_1 = (1, -1, 0)$ short, and $\alpha_2 = (-2, 1, 1)$ long; moreover, the corresponding fundamental dominant weights are $\omega_1 = (0, -1, 1)$ and $\omega_2 = (-1, -1, 2)$. We may check that a set of simple roots of Φ_s

is $\Pi_s := \{\alpha_1, \alpha_1 + \alpha_2\}$. We shall denote these simple roots by $\beta_1 = \alpha_1 = (1, -1, 0)$, $\beta_2 = \alpha_1 + \alpha_2 = (-1, 0, 1)$. The special isogeny induces a $\mathbb{Z}$-linear map $\tau^* : X(T) \to X(T)$, under which $\omega_1 \mapsto \omega_2 \mapsto 3\omega_1$. From this point onwards, we abuse notation, writing τ instead of τ^*. Thus, the τ-restricted weights are 0, ω_1 and $2\omega_1$.

B_τ-Cohomology

Let $\lambda \in X_{r/2}$ be expressed as $\lambda = \lambda_0 + \tau(\lambda_1)$, for $\lambda_0 \in X_\tau(T)$ and $\lambda_1 \in X_s(T)$, such that $\mathrm{H}^1(B_\tau, \lambda) \cong \mathrm{H}^1(B_\tau, \lambda_0) \otimes \tau(\lambda_1)$. Thus, it suffices to compute $\mathrm{H}^1(B_\tau, \lambda_0)$, for $\lambda_0 \in X_\tau(T)$.

Theorem 3.3.1. Let $\lambda_0 \in X_\tau(T)$. Then

$$\mathrm{H}^1(B_\tau, \lambda_0) \cong \begin{cases} k^{(\tau)}_{\omega_2 - \omega_1} \oplus k^{(\tau)}_{\omega_1} & \text{if } \lambda_0 = \omega_1 \\ 0 & \text{else.} \end{cases}$$

Proof. Once again, Lemma 3.1.2 tells us that, regarded as a T-module, $\mathrm{H}^1(U_\tau, k) \otimes \lambda_0$ is the direct sum of certain $k_{\beta_i + \lambda_0}$, for $\beta_i \in \Pi_s$, as previously defined. Such a summand yields a non-zero contribution to $\mathrm{H}^1(B_\tau, \lambda_0)$ if and only if $\beta_i + \lambda_0 \in \tau X(T)$, by Lemma 3.1.3. Hence, we need only check which of these weights belong to $\tau X(T)$.

First, suppose $\lambda_0 = 0$. It is readily checked that we have no non-zero contribution. We conclude that $\mathrm{H}^1(B_\tau, 0) = 0$.

Then, let $\lambda_0 = \omega_1$ and we have

$$\beta_1 + \omega_1 = 2\omega_1 - \omega_2 + \omega_1 = 3\omega_1 - \omega_2 = \tau(\omega_2 - \omega_1).$$

$$\beta_2 + \omega_1 = \omega_2 - \omega_1 + \omega_1 = \omega_2 = \tau(\omega_1).$$

Then, $\mathrm{H}^1(B_\tau, \omega_1) \cong [\bigoplus_i k_{\beta_i + \omega_1}]^{T_\tau} \cong \left[k_{\tau(\omega_2 - \omega_1)} \oplus k_{\tau\omega_1}\right]^{T_\tau} \cong k^{(\tau)}_{\omega_2 - \omega_1} \oplus k^{(\tau)}_{\omega_1}$.

Lastly, suppose $\lambda_0 = 2\omega_1$. We obtain

$$\beta_1 + 2\omega_1 = 2\omega_1 - \omega_2 + 2\omega_1 = 4\omega_1 - \omega_2 \notin \tau X(T).$$

$$\beta_2 + 2\omega_1 = \omega_2 - \omega_1 + 2\omega_1 = \omega_1 + \omega_2 \notin \tau X(T).$$

Then, $\mathrm{H}^1(B_\tau, 2\omega_1) \cong [\bigoplus_i k_{\beta_i + \omega_1}]^{T_\tau} = 0$, since none of them lie in $\tau X(T)$. □

$B_{r/2}$-Cohomology

In this subsection, we extend the results of the previous section to calculate $\mathrm{H}^1(B_{r/2}, \lambda)$, for $\lambda \in X_{r/2}(T)$.

First, when $r = 1$, we direct the reader to Theorem 3.3.1. Otherwise, we obtain

Theorem 3.3.2. Suppose $r = 2s + 1 > 1$ and let $\lambda \in X_{r/2}(T)$. Then, for $0 \leq i \leq s - 2$, we have

$$\mathrm{H}^1\left(B_{r/2}, \lambda\right) \cong \begin{cases} k_{\omega_1}^{(r/2)} & \text{if } \lambda = \omega_1 + (3^s - 1)\omega_2 = \tau^r\omega_1 - \beta_2 \\ k_{\omega_2}^{(r/2)} & \text{if } \lambda = (3^{s+1} - 2)\omega_1 + \omega_2 = \tau^r\omega_2 - \beta_1 \\ k_{\omega_1}^{(r/2)} & \text{if } \lambda = 3^s\omega_1 + 3^{s-1}\omega_2 = \tau^r\omega_1 - \tau^{2s-1}\alpha_1 \\ M_{G_2}^{(r/2)} & \text{if } \lambda = 3^s\omega_1 = \tau^r(\omega_2 - \omega_1) - \tau^{2s-1}\alpha_2 \\ k_{\omega_1}^{(r/2)} & \text{if } \lambda = (3^s - 3^i \cdot 2)\omega_2 + 3^{i+1}\omega_1 2 = \tau^r\omega_1 - \tau^{2i+1}\alpha_1 \\ k_{\omega_2}^{(r/2)} & \text{if } \lambda = 3^{i+1}\omega_2 + (3^{s+1} - 3^{i+1} \cdot 2)\omega_1 = \tau^r\omega_2 - \tau^{2i+1}\alpha_2 \\ 0 & \text{else.} \end{cases}$$

Here M_{G_2} denotes the 2-dimensional indecomposable B-module with head k_{ω_1} and socle $k_{\omega_2 - \omega_1}$ (cf. (BNP04, 2.2)). Moreover, the last two non-zero instances only occur for $s \geq 2$ (or $r \geq 5$).

Proof. The second equality in each case identifying two forms of λ is readily verifiable, recalling $\tau(\omega_1) = \omega_2$, $\beta_1 = (1, -1, 0) = 2\omega_1 - \omega_2$ and $\beta_2 = -\omega_1 + \omega_2$. For instance, suppose $\lambda = \tau^r\omega_1 - \beta_2$. Then, λ may be expressed as $\lambda = 3^s\tau\omega_1 - (-\omega_1 + \omega_2) = \omega_1 + (3^s - 1)\omega_2$. Thus, we focus on proving that λ must be equal to one of the weights given by the first equality in each case. We consider the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(B_{r/2}/B_\tau, \mathrm{H}^j(B_\tau, \lambda)) \Rightarrow \mathrm{H}^{i+j}(B_{r/2}, \lambda)$$

and the corresponding five-term exact sequence

$$0 \to E_2^{1,0} \to E_\infty^1 \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to E_\infty^2.$$

As before, we will identify E_∞^1 with either $E_2^{0,1}$ or $E_2^{1,0}$ and we calculate all of the non-zero cases in this way. We begin by fixing some notation. Since $\lambda \in X_{r/2}(T)$, it has a unique τ-adic expansion and we write $\lambda = \sum_{i=0}^{r-1} \tau^i\lambda_i$, with λ_i τ-restricted weights. Then, $\lambda = \lambda_0 + \tau(\lambda')$, for $\lambda' = \sum_{j=1}^{r-1} \tau^{j-1}\lambda_j$. Suppose $E_2^{0,1} \neq 0$ and consider the $E_2^{0,1}$-term. We have

$$\begin{aligned} E_2^{0,1} &= \mathrm{Hom}_{B_{r/2}/B_\tau}(k, \mathrm{H}^1(B_\tau, \lambda)) \\ &\cong \mathrm{Hom}_{B_{r/2}/B_\tau}(k, \mathrm{H}^1(B_\tau, \lambda_0) \otimes \tau(\lambda')). \end{aligned}$$

There is only one τ-restricted weight for which $\mathrm{H}^1(B_\tau, \lambda_0) \neq 0$, namely $\lambda_0 = \omega_1$. In this case, we obtain

$$\mathrm{H}^1(B_\tau, \omega_1) \cong k_{\omega_2-\omega_1}^{(\tau)} \oplus k_{\omega_1}^{(\tau)}.$$

Hence

$$\begin{aligned} E_2^{0,1} &= \mathrm{Hom}_{B_{r/2}/B_\tau}(k, (k_{\omega_1}^{(\tau)} \oplus k_{\omega_2-\omega_1}^{(\tau)}) \otimes \tau(\lambda')) \\ &\cong \mathrm{Hom}_{B_{(r-1)/2}}(k, (k_{\omega_1}^{(\tau)} \oplus k_{\omega_2-\omega_1}^{(\tau)}) \otimes k_{\lambda'}^{(\tau)}) \\ &\cong \mathrm{Hom}_{B_{(r-1)/2}}(k, k_{\omega_1+\lambda'}^{(\tau)} \oplus k_{\omega_2-\omega_1+\lambda'}^{(\tau)}). \end{aligned}$$

Similarly to the proof of Theorem 3.2.3, $\mathrm{Hom}_{B_{(r-1)/2}}(k, k_{\omega_1+\lambda'}^{(\tau)} \oplus k_{\omega_2-\omega_1+\lambda'}^{(\tau)})$ is non-zero if at least one of $\omega_1 + \lambda'$ and $\omega_2 - \omega_1 + \lambda'$ belongs to $\tau^{r-1}X(T)$.

Moreover, $\mathrm{Hom}_{B_{(r-1)/2}}(k, k_{\omega_1+\lambda'}^{(\tau)} \oplus k_{\omega_2-\omega_1+\lambda'}^{(\tau)})$ is at most one-dimensional: since $\omega_2 - 2\omega_1 \notin \tau^{r-1}X(T)$, at most one of $\omega_1+\lambda'$ and $\omega_2-\omega_1+\lambda'$ lies in $\tau^{r-1}X(T)$. Thus, we consider both cases in turn to determine the possible values of λ and $E_2^{0,1}$. First, suppose $\omega_2 - \omega_1 + \lambda' \in \tau^{r-1}X(T)$. Since $p = 3$, we have $\lambda' = (a3^s + 1)\omega_1 + (b3^s - 1)\omega_2 \in X_s(T)$. It immediately follows that we must have $a = 0$, $b = 1$, in which case $\lambda' = \omega_1 + (3^s - 1)\omega_2$, giving $\lambda = (3^{s+1} - 2)\omega_1 + \omega_2$ and

$$E_2^{0,1} = \mathrm{Hom}_{B_{r/2}/B_\tau}(k, k_{\tau(\omega_1+\lambda')} \oplus k_{\tau(\omega_2-\omega_1+\lambda')}).$$

The second term in the target of the Hom is $k_{\tau(\omega_2-\omega_1+\omega_1+(3^s-1)\omega_2)} = k_{3^s\tau(\omega_2)}$. Thus $E_2^{0,1} \cong k_{3^s\tau(\omega_2)} = (k_{\omega_2})^{(r/2)}$.

In the case $\omega_1 + \lambda' \in \tau^{r-1}X(T)$, a similar argument leads us to conclude that $E_2^{0,1} = k_{\omega_1}^{(r/2)}$ for $\lambda = \omega_1 + (3^s - 1)\omega_2$.

To conclude, for $\lambda \in X_{r/2}(T)$,

$$E_2^{0,1} \cong \begin{cases} k_{\omega_1}^{(r/2)} & \text{if } \lambda = \omega_1 + (3^s - 1)\omega_2 \\ k_{\omega_2}^{(r/2)} & \text{if } \lambda = (3^{s+1} - 2)\omega_1 + \omega_2 \\ 0 & \text{else.} \end{cases}$$

Now suppose $E_2^{1,0} \neq 0$. We have

$$\begin{aligned} E^{1,0} &= \mathrm{H}^1(B_{r/2}/B_\tau, \mathrm{Hom}_{B_\tau}(k,\lambda)), \\ &= \mathrm{H}^1(B_{r/2}/B_\tau, \mathrm{Hom}_{B_\tau}(k,\lambda_0) \otimes \tau(\lambda')) \end{aligned}$$

so $\lambda_0 = 0$ and $\lambda = \tau(\lambda')$. Thus $E^{1,0} \cong \mathrm{H}^1(B_s, \lambda'^{(\tau)}) \cong \mathrm{H}^1(B_s,\lambda')^{(\tau)}$ for $\lambda = \tau\lambda'$. Notice that since $r-1 = 2s > 0$, $B_{(r-1)/2} = B_s$ is a classical Frobenius kernel and $\mathrm{H}^1(B_s,\lambda')$ is the B_s-cohomology for $\lambda' \in X_s(T)$ computed in (BNP04, Theorem 2.7). We have

$$\mathrm{H}^1(B_s,\lambda') \cong \begin{cases} k_{\omega_1}^{(s)} & \text{if } \lambda' = 3^{s-1}(\omega_1+\omega_2) \\ M_{G_2}^{(s)} & \text{if } \lambda' = 3^{s-1}\omega_2 \\ k_{\omega_j}^{(s)} & \text{if } \lambda' = 3^s\omega_j - 3^i\alpha_j, j \in \{1,2\}, 0 \le i \le s-2 \\ 0 & \text{else.} \end{cases}$$

where M_{G_2} has the structure as claimed in the statement of the theorem. We note the implicit constraints on s in the different cases. Thus,

$$E_2^{1,0} \cong \mathrm{H}^1(B_s,\lambda')^{(\tau)} \cong \begin{cases} k_{\omega_1}^{(r/2)} & \text{if } \lambda' = 3^{s-1}(\omega_1+\omega_2) \\ M_{G_2}^{(r/2)} & \text{if } \lambda' = 3^{s-1}\omega_2 \\ k_{\omega_j}^{(r/2)} & \text{if } \lambda' = 3^s\omega_j - 3^i\alpha_j, j \in \{1,2\}, 0 \le i \le s-2 \\ 0 & \text{else.} \end{cases}$$

We can recover λ from λ', recalling $\alpha_1 = 2\omega_1 - \omega_2$ and $\alpha_2 = -3\omega_1 + 2\omega_2$. For instance, if $\lambda' = 3^{s-1}(\omega_1+\omega_2)$, then $\lambda = \tau\lambda' = 3^s\omega_1 + 3^{s-1}\omega_2$. Note that the other cases follow similarly.

Finally, note that there is no choice of λ for which $E_2^{0,1}$ and $E_2^{1,0}$ are simultaneously non-zero. Hence, if $E_2^{1,0} \neq 0$, then $E_2^{0,1} = 0$ so $E_\infty^1 \cong E_2^{1,0}$. Alternatively, if $E_2^{0,1} \neq 0$, then $\lambda = \omega_1 + (3^s-1)\omega_2$ or $\lambda = (3^{s+1}-2)\omega_1 + \omega_2$, according to the earlier discussion. Note that in either case, $\lambda \notin \tau X(T)$, pushing $\mathrm{Hom}_{B_\tau}(k,\lambda) = 0$. Hence $E_2^{1,0} = E_2^{2,0} = 0$, meaning that $E_\infty^1 \cong E_2^{0,1}$. $\square$

Now, for completeness, for a general $\lambda \in X(T)$, not necessarily lying in $X_{r/2}$, we proceed as in (BNP04, 2.8). First, we make the following observation and we note that the proof is identical to the proof of Corollary 3.2.4.

Corollary 3.3.3. Let $\lambda \in X(T)$. Then $\mathrm{H}^1(B_{r/2},\lambda) \neq 0$ if and only if $\lambda = \tau^r\omega - \tau^i\alpha$, for some weight $\omega \in X(T)$, and $\alpha \in \Pi$ with $0 \le i \le 2s-1$ or $\lambda = \tau^r\omega - \beta$, for some weight

$\omega \in X(T)$, and $\beta \in \Pi_s$.

Now, we denote by (ζ, j) the pair (α, i) or $(\beta, 1)$, respectively, as defined in the previous corollary. Now, we write $\lambda = \tau^r\omega' - \tau^j\zeta + \tau^r\lambda_1$, for a given $\lambda = \tau^r\omega - \tau^j\zeta$. Supposing the $B_{r/2}$-cohomology does not vanish on λ, then ω' is as given in Theorem 3.3.2 and $\lambda_1 \in X(T)$. Then, set $\lambda_1 = \omega - \omega'$ and we obtain

$$\begin{aligned} \mathrm{H}^1(B_{r/2}, \lambda) &\cong \mathrm{H}^1(B_{r/2}, \lambda_0) \otimes k_{\lambda_1}^{(r/2)} \\ &\cong \mathrm{H}^1(B_{r/2}, \tau^r\omega' - \tau^j\zeta) \otimes k_{\omega-\omega'}^{(r/2)}. \end{aligned}$$

One then substitutes the results from Theorem 3.3.2. We omit the details for brevity and obtain

Theorem 3.3.4. Let $\lambda \in X(T)$ and $0 \leq i \leq s-2$. Then

$$\mathrm{H}^1\left(B_{r/2}, \lambda\right) \cong \begin{cases} k_\omega^{(r/2)} & \text{if } \lambda = \tau^r\omega - \tau^{2s-1}\alpha_1, \omega \in X(T) \\ M_{G_2}^{(r/2)} \otimes k_{\omega+\omega_1-\omega_2}^{(r/2)} & \text{if } \lambda = \tau^r\omega - \tau^{2s-1}\alpha_2, \omega \in X(T) \\ k_\omega^{(r/2)} & \text{if } \lambda = \tau^r\omega - \tau^{2i+1}\alpha_j, \omega \in X(T), \alpha_j \in \Pi \\ 0 & \text{else.} \end{cases}$$

$G_{r/2}$-Cohomology of Induced Modules

Using Kempf's vanishing theorem, Theorem 3.3.1, Theorem 3.3.2 and (3.1.1), we compute $\mathrm{H}^1(G_{r/2}, \mathrm{H}^0(\lambda))$ for $\lambda \in X_{r/2}$. Furthermore, we note that, by (BNP04, 3.1, Theorem (B)), $\mathrm{Ind}_B^G(M_{G_2}) = \mathrm{H}^0(\omega_1)$.

In case $r = 1$, we obtain (Sin94b, Lemma 3.2):

Theorem 3.3.5. Let $\lambda \in X_\tau(T)$. Then

$$\mathrm{H}^1\left(G_\tau, \mathrm{H}^0(\lambda)\right)^{(-\tau)} \cong \begin{cases} \mathrm{H}^0(\omega_1) & \text{if } \lambda = \omega_1 \\ 0 & \text{else.} \end{cases}$$

Now, assume $r > 1$.

Theorem 3.3.6. Let $\lambda \in X_{r/2}(T)$ and $0 \leq i \leq s-2$. Then

$$\mathrm{H}^1(G_{r/2}, \mathrm{H}^0(\lambda))^{(-r/2)} \cong \begin{cases} \mathrm{H}^0(\omega_1) & \text{if } \lambda = \omega_1 + (3^s - 1)\omega_2 = \tau^r\omega_1 - \beta_2 \\ \mathrm{H}^0(\omega_2) & \text{if } \lambda = (3^{s+1} - 2)\omega_1 + \omega_2 = \tau^r\omega_2 - \beta_1 \\ \mathrm{H}^0(\omega_1) & \text{if } \lambda = 3^s\omega_1 + 3^{s-1}\omega_2 = \tau^r\omega_1 - \tau^{2s-1}\alpha_1 \\ \mathrm{H}^0(\omega_1) & \text{if } \lambda = 3^s\omega_1 = \tau^r(\omega_2 - \omega_1) - \tau^{2s-1}\alpha_2 \\ \mathrm{H}^0(\omega_1) & \text{if } \lambda = (3^{s+1} - 3^i \cdot 2)\omega_2 + 3^{i+1}\omega_1 = \tau^r\omega_1 - \tau^{2i+1}\alpha_1 \\ \mathrm{H}^0(\omega_2) & \text{if } \lambda = 3^{i+1}\omega_2 + (3^{s+1} - 3^{i+1} \cdot 2)\omega_1 = \tau^r\omega_2 - \tau^{2i+1}\alpha_2 \\ 0 & \text{else.} \end{cases}$$

Lastly, based on Theorem 3.3.4, one may calculate $\mathrm{H}^1(G_{r/2}, \mathrm{H}^0(\lambda))$ in terms of induced modules for all dominant weights λ, by applying the induction functor Ind_B^G. We handle the only non-obvious case in the following remark.

Remark 3.3.7. Let $\tau^r\omega - \tau^{2s-1}\alpha_2 \in X(T)_+$. Then $\langle \omega, \alpha_1^\vee \rangle \geq -1$ and $\langle \omega, \alpha_2^\vee \rangle \geq 1$. In this case, by (BNP04, Proposition 3.4 (A)), we note that

(i) if $\langle \omega, \alpha_1^\vee \rangle \geq 0$, then $\mathrm{Ind}_B^G(M_{G_2} \otimes k_{\omega+\omega_1-\omega_2})$ has a filtration with factors satisfying the following short exact sequence

$$0 \to \mathrm{H}^0(\omega) \to \mathrm{Ind}_B^G(M_{C_2} \otimes k_{\omega+\omega_1-\omega_2}) \to \mathrm{H}^0(\omega + 2\omega_1 - \omega_2) \to 0.$$

(ii) if $\langle \omega, \alpha_1^\vee \rangle = -1$, then $\mathrm{Ind}_B^G(M_{G_2} \otimes k_{\omega+\omega_1-\omega_2}) \cong \mathrm{H}^0(\omega + 2\omega_1 - \omega_2)$.

$G_{r/2}$ Extensions Between Simple Modules

In this subsection, we make use of the G_1-cohomology with coefficients in simple modules, computed in (Sin94b, Proposition 3.5), to calculate $\mathrm{Ext}^1_{G_s}(L(\lambda), L(\mu))^{(-s)}$, for a positive integer s and $\lambda, \mu \in X_s(T)$.

Theorem 3.3.8. Let s be a positive integer, with $\lambda = \sum_{i=0}^{s-1} p^i\lambda_i$ and $\mu = \sum_{i=0}^{s-1} p^i\mu_i \in X_s(T)$. Put $d := \min\{i | \lambda_i \neq \mu_i\}$. Then write $\lambda = \lambda' + p^d\lambda''$, for $\lambda' = \sum_{i=0}^{d-1} p^i\lambda_i \in X_d$ and $\lambda'' = \sum_{i=d}^{s-1} p^{i-d}\lambda_i \in X_{s-d}$; we take a similar expression for μ. We may express any digit $\lambda_i = \lambda_{i,0} + \tau\lambda_{i,1}$, with $\lambda_{i,j} \in X_\tau$. Let $n_1n_2 := n_1\omega_1 + n_2\omega_2 \in X_1$. Then, we denote by

$$A := \{(00, 11), (01, 10), (01, 11), (01, 12), (02, 11), (02, 12)\} \text{ and}$$
$$B := \{(00, 01), (10, 11), (20, 21)\},$$

(a) Suppose $d = s-1$. Then we have

$$\operatorname{Ext}^1_{G_s}(L(\lambda), L(\mu))^{(-s)} \cong \begin{cases} k & \text{if } (\lambda_{s-1}, \mu_{s-1}) \in A \\ L(\omega_1) & \text{if } (\lambda_{s-1}, \mu_{s-1}) \in B \\ 0 & \text{else.} \end{cases}$$

(b) Suppose $d \neq s-1$. Then we have

$$\operatorname{Ext}^1_{G_s}(L(\lambda), L(\mu))^{(-s)} \cong \begin{cases} k & \text{if } (\lambda_d, \mu_d) \in A \text{ and } \lambda'' = \mu'' \\ k & \text{if } (\lambda_d, \mu_d) \in B \\ & (\lambda''_0, \mu''_0) \in \{(2\omega_1, \omega_1), (\omega_1, \omega_1), (2\omega_1, 2\omega_1)\} \\ & \text{and } \lambda''_1 = \mu''_1 \\ 0 & \text{else.} \end{cases}$$

Note that it is implicit in the statement of the theorem that $s \geq 1$ or $s \geq 2$, depending on the case.

Proof. We apply the Lyndon-Hochschild-Serre spectral sequence corresponding to $G_d \lhd G_s$ and we have

$$E_2^{i,j} := \operatorname{Ext}^i_{G_{s-d}}(L(\lambda''), \operatorname{Ext}^j_{G_d}(L(\lambda'), L(\mu'))^{(-d)} \otimes L(\mu''))^{(d)}.$$

By definition, $\lambda' = \mu'$, so $E_2^{0,1} = 0$, which forces $E^1_\infty \cong E_2^{1,0} = \operatorname{Ext}^1_{G_{s-d}}(L(\lambda''), L(\mu''))^{(d)}$.

(a) First, suppose $s - d = 1$. Then $E^1_\infty \cong E_2^{1,0} = \operatorname{Ext}^1_{G_1}(L(\lambda_{s-1}), L(\mu_{s-1}))^{(s-1)}$, which was computed in (Sin94b, Table III). We obtain

$$E_2^{1,0} \cong \begin{cases} k & \text{if } (\lambda_{s-1}, \mu_{s-1}) \in A \\ L(\omega_1)^{(s)} & \text{if } (\lambda_{s-1}, \mu_{s-1}) \in B \\ 0 & \text{else.} \end{cases}$$

(b) Now suppose $s-d \neq 1$, so we may apply the Lyndon-Hochscild-Serre spectral sequence corresponding to $G_1 \lhd G_{s-d}$ to $M = \operatorname{Ext}^1_{G_{s-d}}(L(\lambda''), L(\mu''))$. Write $\lambda'' = \lambda_d + p\lambda'''$, for $\lambda_d \in X_1$ and $\lambda''' \in X_{s-d-1}$ and we express μ'' similarly. The E_2-page is given by

$$E_2^{i,j} := \operatorname{Ext}^i_{G_{s-d-1}}(L(\lambda'''), \operatorname{Ext}^j_{G_1}(L(\lambda_d), L(\mu_d))^{(-1)} \otimes L(\mu'''))^{(1)}.$$

Since, by definition, $\lambda_d \neq \mu_d$, it follows that $E_2^{i,0} = 0$ for $i > 0$, so we obtain $M \cong E_2^{0,1} =$

$\mathrm{Hom}_{G_{s-d-1}}(L(\lambda'''), \mathrm{Ext}^1_{G_1}(L(\lambda_d), L(\mu_d))^{(-1)} \otimes L(\mu'''))^{(1)}$.

Now put $W := \mathrm{Ext}^1_{G_1}(L(\lambda_d), L(\mu_d))^{(-1)}$. Then, by (Sin94b, Table III), depending on the pair (λ_d, μ_d), $W \in \{k, L(\omega_1)\}$. We consider each value of W in turn.

Case I: $W = k$, which occurs when $(\lambda_d, \mu_d) \in A$

In this case we have

$$M \cong \mathrm{Hom}_{G_{s-d-1}}(L(\lambda'''), L(\mu'''))^{(1)} \cong k,$$

if and only if $\lambda''' = \mu'''$ and vanishes otherwise.

Case II: $W = L(\omega_1)$, which occurs when $(\lambda_d, \mu_d) \in B$

In this case, we obtain

$$\begin{aligned} M &\cong \mathrm{Hom}_{G_{s-d-1}}(L(\lambda'''), L(\mu''') \otimes L(\omega_1))^{(1)} \\ &\cong \mathrm{Hom}_{G_{s-d-3/2}}(L(\lambda'''_1), \mathrm{Hom}_{G_\tau}(L(\lambda'''_0), L(\mu'''_0) \otimes L(\omega_1))^{(-\tau)} \otimes L(\mu'''_1))^{(3/2)}, \end{aligned}$$

for $\lambda''' = \lambda'''_0 + \tau\lambda'''_1$, $\lambda'''_0 = \lambda_{d+1,0} \in X_\tau$ and $\lambda'''_1 \in X_{s-d-3/2}$ and μ''' expressed similarly. Thus, $M \neq 0$ if and only if $\lambda'''_1 = \mu'''_1$ and, by (Sin94b, Lemma 3.3), we have

$$M \cong \begin{cases} k & \text{if } (\lambda_d, \mu_d) \in B, (\lambda_{d+1,0}, \mu_{d+1,0}) = (2\omega_1, \omega_1) \\ k & \text{if } (\lambda_d, \mu_d) \in B, \lambda_{d+1,0} = \mu_{d+1,0} \in \{\omega_1, 2\omega_1\} \\ 0 & \text{else.} \end{cases}$$

□

Next, we compute $\mathrm{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu))^{(-r/2)}$ for r an odd positive integer and $\lambda, \mu \in X_{r/2}$, making use of the previous theorem concerning the cohomology for classical Frobenius kernels.

If $r = 1$, we refer the reader to (Sin94b, Lemma 3.2). Otherwise we obtain

Theorem 3.3.9. Suppose $r = 2s + 1 > 1$ and let $\lambda, \mu \in X_{r/2}(T)$. Write $\lambda = \lambda_0 + \tau\lambda'$, for $\lambda_0 \in X_\tau$ and $\lambda' = \sum_{i=0}^{2s-1} \tau^i \lambda_{1,i} \in X_s$. Moreover, we write $\lambda_{i+1} := \lambda_{1,2i} + \tau\lambda_{1,2i+1} \in X_1$ for $i \geq 0$ and we take similar expressions for μ. Put $d := \min\{i \geq 1 | \lambda_i \neq \mu_i\}$. Recall A, B from Theorem 3.3.8.

(a) Suppose $\lambda_0 = \mu_0$. Then we have

$$\mathrm{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu))^{(-r/2)} \cong \begin{cases} k & \text{if } (\lambda_s, \mu_s) \in A \\ L(\omega_1) & \text{if } (\lambda_s, \mu_s) \in B \\ k & \text{if } (\lambda_{d+1}, \mu_{d+1}) \in A \\ k \oplus k & \text{if } (\lambda_{d+1}, \mu_{d+1}) \in B \\ k & \text{if } (\lambda_{d+1}, \mu_{d+1}) \in B; \\ & (\lambda_{1,2d+2}, \mu_{1,2d+2}) \in \{(2\omega_1, \omega_1), (\omega_1, \omega_1), (2\omega_1, 2\omega_1)\} \\ 0 & \text{else.} \end{cases}$$

(b) Suppose $\lambda_0 \neq \mu_0$. Then we have

$$\mathrm{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu))^{(-r/2)} \cong \begin{cases} k & \text{if } \lambda - \mu = \pm\omega_1 \pm \tau\omega_1 \\ k & \text{if } \lambda - \mu = \pm\omega_1 \\ & \quad \text{and } \lambda_{1,0} = \mu_{1,0} \in \{2\omega_1, \omega_1\} \\ 0 & \text{else.} \end{cases}$$

Proof. We consider the Lyndon-Hochschild-Serre spectral sequence corresponding to $G_\tau \lhd G_{r/2}$. The E_2-page is given by

$$E_2^{i,j} := \mathrm{Ext}^i_{G_s}(L(\lambda'), \mathrm{Ext}^j_{G_\tau}(L(\lambda_0), L(\mu_0))^{(-\tau)} \otimes L(\mu'))^{(\tau)}.$$

First, suppose $\lambda_0 = \mu_0$. Then $E_2^{0,1} = 0$, so $E^1_\infty \cong E_2^{1,0}$ and $E_2^{1,0} := \mathrm{Ext}^1_{G_s}(L(\lambda'), L(\mu'))^{(\tau)}$, which was computed in Theorem 3.3.8.

Now suppose $\lambda_0 \neq \mu_0$. Then $E_2^{i,0} = 0$ for $i > 0$, so $E^1_\infty \cong E_2^{0,1}$ and we obtain

$$E_2^{0,1} := \mathrm{Hom}_{G_s}(L(\lambda'), \mathrm{Ext}^1_{G_\tau}(L(\lambda_0), L(\mu_0))^{(-\tau)} \otimes L(\mu'))^{(\tau)}.$$

By (Sin94b, Lemma 3.2), this vanishes unless $(\lambda_0, \mu_0) = (\omega_1, 0)$. In this case we obtain

$$\begin{aligned} E^1_\infty &\cong \mathrm{Hom}_{G_s}(L(\lambda'), L(\mu') \otimes L(\omega_1))^{(\tau)} \\ &\cong \mathrm{Hom}_{G_{s-1/2}}(L(\lambda''), \mathrm{Hom}_{G_\tau}(L(\lambda_{1,0}), L(\mu_{1,0}) \otimes L(\omega_1))^{(-\tau)} \otimes L(\mu''))^{(1)}, \end{aligned}$$

where $\tau\lambda'' = \lambda - (\lambda_0 + \tau\lambda_{1,0})$ and μ'' is expressed similarly. Then, $E_2^{0,1} \neq 0$ if and only if

$\lambda'' = \mu''$, and using (Sin94b, Lemma 3.3), we obtain

$$E^1_\infty \cong \begin{cases} k & \text{if } (\lambda_{1,0}, \mu_{1,0}) = (2\omega_1, \omega_1) \\ k & \text{if } \lambda_{1,0} = \mu_{1,0} \in \{\omega_1, 2\omega_1\} \\ 0 & \text{else.} \end{cases} \qquad \square$$

3.4 F_4 in Characteristic 2

Let G be simply-connected of type F_4 over k of characteristic 2. Following (Bou82, Planche VIII), let $\Phi = \{\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j, \frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\}$ be the roots of a system of type F_4. Writing $\epsilon_1 = (1,0,0,0)$, $\epsilon_2 = (0,1,0,0)$, $\epsilon_3 = (0,0,1,0)$ and $\epsilon_4 = (0,0,0,1)$, a base of simple roots is $\Pi := \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ with $\alpha_1 = (0,1,-1,0)$, $\alpha_2 = (0,0,1,-1)$, $\alpha_3 = (0,0,0,1)$ and $\alpha_4 = \frac{1}{2}(1,-1,-1,-1)$; furthermore, the corresponding fundamental dominant weights are $\omega_1 = (1,1,0,0)$, $\omega_2 = (2,1,1,0)$, $\omega_3 = \frac{1}{2}(3,1,1,1)$ and $\omega_4 = (1,0,0,0)$. Then one can check that a set of simple roots of Φ_s is $\Pi_s := \{\alpha_3, \alpha_4, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$, with α_4 being the central node in the Dynkin Diagram. We shall denote these simple roots by $\beta_1 = \alpha_3 = (0,0,0,1)$, $\beta_2 = \alpha_4 = \frac{1}{2}(1,-1,-1,-1)$, $\beta_3 = \alpha_2 + \alpha_3 = (0,0,1,0)$ and $\beta_4 = \alpha_1 + \alpha_2 + \alpha_3 = (0,1,0,0)$. The special isogeny induces a $\mathbb{Z}$-linear map τ^* as before, under which $\omega_4 \mapsto \omega_1 \mapsto 2\omega_4$ and $\omega_3 \mapsto \omega_2 \mapsto 2\omega_3$. We henceforth abuse notation, writing τ instead of τ^*. Consequently, the τ-restricted weights are 0, ω_3, ω_4 and $\omega_3 + \omega_4$.

B_τ-Cohomology

For a given $\lambda \in X_{r/2}$, we write $\lambda = \lambda_0 + \tau(\lambda_1)$, for $\lambda_0 \in X_\tau(T)$ and $\lambda_1 \in X_s(T)$, such that $\mathrm{H}^1(B_\tau, \lambda) \cong \mathrm{H}^1(B_\tau, \lambda_0) \otimes \tau(\lambda_1)$. Thus, it suffices to compute $\mathrm{H}^1(B_\tau, \lambda_0)$, for $\lambda_0 \in X_\tau(T)$.

Theorem 3.4.1. Let $\lambda_0 \in X_\tau(T)$. Then

$$\mathrm{H}^1(B_\tau, \lambda_0) \cong \begin{cases} k^{(\tau)}_{\omega_4} \oplus k^{(\tau)}_{\omega_2 - \omega_3} \oplus k^{(\tau)}_{\omega_3 - \omega_4} & \text{if } \lambda_0 = \omega_4 \\ k^{(\tau)}_{\omega_1} & \text{if } \lambda_0 = \omega_3 \\ 0 & \text{else.} \end{cases}$$

Proof. Much like in the other cases, regarded as a T-module, $\mathrm{H}^1(U_\tau, k) \otimes \lambda_0$ is the direct sum of certain $k_{\beta_i + \lambda_0}$, for $\beta_i \in \Pi_s$. Given the fact that such a summand yields a non-zero contribution to $\mathrm{H}^1(B_\tau, \lambda_0)$ if and only if $\beta_i + \lambda_0 \in \tau X(T)$, we now inspect which of these weights belong to $\tau X(T)$.

To begin with, suppose $\lambda_0 = 0$. It is readily verified that we have no non-zero contribution. Therefore, $\mathrm{H}^1(B_\tau, k) = 0$.

Then, suppose $\lambda_0 = \omega_4$. We have

$$\beta_1 + \omega_4 = (1,0,0,1) = -\omega_2 + 2\omega_3 = \tau(\omega_2 - \omega_3).$$

$$\beta_2 + \omega_4 = \frac{1}{2}(3,-1,-1,-1) = -\omega_3 + 3\omega_4 \notin \tau X(T).$$

$$\beta_3 + \omega_4 = (1,0,1,0) = -\omega_1 + \omega_2 = \tau(\omega_3 - \omega_4).$$

$$\beta_4 + \omega_4 = (1,1,0,0) = \omega_1 = \tau(\omega_4).$$

Hence,

$$\begin{aligned}\mathrm{H}^1(B_\tau, \omega_4) &\cong \left[\bigoplus_i k_{\beta_i+\omega_1}\right]^{T_\tau} \cong \left[k_{\tau(\omega_4)} \oplus k_{\tau(\omega_2-\omega_3)} \oplus k_{\tau(\omega_3-\omega_4)}\right]^{T_\tau} \\ &\cong k^{(\tau)}_{\omega_4} \oplus k^{(\tau)}_{\omega_2-\omega_3} \oplus k^{(\tau)}_{\omega_3-\omega_4}.\end{aligned}$$

Now let $\lambda_0 = \omega_3$ and we obtain

$$\beta_1 + \omega_3 = \frac{1}{2}(3,1,1,3) = -\omega_2 + 3\omega_3 - \omega_4 \notin \tau X(T).$$

$$\beta_2 + \omega_3 = (2,0,0,0) = 2\omega_4 = \tau(\omega_1).$$

$$\beta_3 + \omega_3 = \frac{1}{2}(3,1,3,1) = -\omega_1 + \omega_2 + \omega_3 - \omega_4 \notin \tau X(T).$$

$$\beta_4 + \omega_3 = \frac{1}{2}(3,3,1,1) = \omega_1 + \omega_3 - \omega_4 \notin \tau X(T).$$

Then, $\mathrm{H}^1(B_\tau, \omega_3) \cong k^{(\tau)}_{\omega_1}$.

Finally, for $\lambda_0 = \omega_3 + \omega_4$, we get

$$\beta_1 + \omega_3 + \omega_4 = \frac{1}{2}(5,1,1,3) = -\omega_2 + 3\omega_3 \notin \tau X(T).$$

$$\beta_2 + \omega_3 + \omega_4 = (3,0,0,0) = 3\omega_4 \notin \tau X(T).$$

$$\beta_3 + \omega_3 + \omega_4 = \frac{1}{2}(5,1,3,1) = -\omega_1 + \omega_2 + \omega_3 \notin \tau X(T).$$

$$\beta_4 + \omega_3 + \omega_4 = \frac{1}{2}(5,3,1,1) = \omega_1 + \omega_3 \notin \tau X(T).$$

Then, $\mathrm{H}^1(B_\tau, \omega_3 + \omega_4) = 0$. □

$B_{r/2}$-Cohomology

In this subsection, we extend the results of the previous section to calculate $\mathrm{H}^1(B_{r/2}, \lambda)$, for $\lambda \in X_{r/2}(T)$.

If $r = 1$, we direct the reader to Theorem 3.4.1.

Theorem 3.4.2. Suppose $r = 2s + 1 > 1$ and let $\lambda \in X_{r/2}(T)$. Then, for $0 \leq i \leq s - 2$, we have

$$\mathrm{H}^1\left(B_{r/2}, \lambda\right) \cong \begin{cases} k_{\omega_1}^{(r/2)} & \text{if } \lambda = \omega_3 + 2(2^s - 1)\omega_4 = \tau^r\omega_1 - \beta_2 \\ k_{\omega_2}^{(r/2)} & \text{if } \lambda = \omega_4 + \omega_2 + 2(2^s - 1)\omega_3 = \tau^r\omega_2 - \beta_1 \\ k_{\omega_3}^{(r/2)} & \text{if } \lambda = \omega_4 + \omega_1 + (2^s - 1)\omega_2 = \tau^r\omega_3 - \beta_3 \\ k_{\omega_4}^{(r/2)} & \text{if } \lambda = \omega_4 + (2^s - 1)\omega_1 = \tau^r\omega_4 - \beta_4 \\ k_{\omega_1}^{(r/2)} & \text{if } \lambda = 2^s\omega_3 = \tau^r\omega_1 - \tau^{2s-1}\alpha_1 \\ k_{\omega_3}^{(r/2)} & \text{if } \lambda = 2^s\omega_3 + 2^{s-1}\omega_1 = \tau^r\omega_3 - \tau^{2s-1}\alpha_3 \\ k_{\omega_4}^{(r/2)} & \text{if } \lambda = 2^{s-1}\omega_2 = \tau^r\omega_4 - \tau^{2s-1}\alpha_4 \\ M_{F_4}^{(r/2)} & \text{if } \lambda = 2^s\omega_4 = \tau^r(\omega_2 - \omega_3) - \tau^{2s-1}\alpha_2 \\ k_{\omega_1}^{(r/2)} & \text{if } \lambda = (2^{s+1} - 2^{i+2})\omega_4 + 2^{i+1}\omega_3 = \tau^r\omega_1 - \tau^{2i+1}\alpha_1 \\ k_{\omega_2}^{(r/2)} & \text{if } \lambda = 2^{i+1}\omega_2 + (2^{s+1} - 2^{i+2})\omega_3 + 2^{i+1}\omega_4 = \tau^r\omega_2 - \tau^{2i+1}\alpha_2 \\ k_{\omega_3}^{(r/2)} & \text{if } \lambda = 2^i\omega_1 + (2^s - 2^{i+1})\omega_2 + 2^{i+1}\omega_3 = \tau^r\omega_2 - \tau^{2i+1}\alpha_2 \\ k_{\omega_4}^{(r/2)} & \text{if } \lambda = (2^s - 2^{i+1})\omega_1 + 2^i\omega_2 = \tau^r\omega_4 - \tau^{2i+1}\alpha_4 \\ 0 & \text{else.} \end{cases}$$

Here M_{F_4} denotes the 3-dimensional indecomposable B-module with the following factors: head k_{ω_4}, $k_{\omega_3-\omega_4}$ and socle $k_{\omega_2-\omega_3}$ (cf. (BNP04, 2.2)). We underline that the last four non-zero instances only occur when $s \geq 2$ (or $r \geq 5$).

Proof. The second equality in each case identifying two forms of λ follows immediately, recalling $\tau(\omega_4) = \omega_1$ and $\tau(\omega_3) = \omega_2$, $\beta_1 = -\omega_2 + 2\omega_3 - \omega_4$, $\beta_2 = -\omega_3 + 2\omega_4$, $\beta_3 = -\omega_1 + \omega_2 - \omega_4$ and $\beta_4 = \omega_1 - \omega_4$. For instance, suppose $\lambda = \tau^r\omega_1 - \beta_2$. Then, λ may be expressed as $\lambda = 2^s\tau\omega_1 - (-\omega_3 + 2\omega_4) = \omega_3 + 2(2^s - 1)\omega_4$.We thus show that λ must be equal to one of the weights given by the first equality in each case. We consider the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(B_{r/2}/B_\tau, \mathrm{H}^j(B_\tau, \lambda)) \Rightarrow \mathrm{H}^{i+j}(B_{r/2}, \lambda)$$

and the corresponding five-term exact sequence

$$0 \to E_2^{1,0} \to E_\infty^1 \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to E_\infty^2.$$

Much like in the previous subsections, we shall identify E_∞^1 with either $E_2^{0,1}$ or $E_2^{1,0}$, in order to determine all of the non-zero cases. We must first fix some notation. Since $\lambda \in X_{r/2}(T)$, we may uniquely write $\lambda = \sum_{i=0}^{r-1} \tau^i \lambda_i$, where λ_i are τ-restricted. Then, $\lambda = \lambda_0 + \tau(\lambda')$, for $\lambda' = \sum_{j=1}^{r-1} \tau^{j-1}\lambda_j$. Suppose $E_2^{0,1} \neq 0$ and we have We have

$$\begin{aligned} E_2^{0,1} &= \mathrm{Hom}_{B_{r/2}/B_\tau}(k, \mathrm{H}^1(B_\tau, \lambda)) \\ &\cong \mathrm{Hom}_{B_{r/2}/B_\tau}(k, \mathrm{H}^1(B_\tau, \lambda_0) \otimes \tau(\lambda')). \end{aligned}$$

There are two τ-restricted weights for which $\mathrm{H}^1(B_\tau, \lambda_0) \neq 0$, namely ω_4 and ω_3, and we consider each case in turn.

First, suppose $\lambda_0 = \omega_4$ and we have $\mathrm{H}^1(B_\tau, \omega_4) = k_{\omega_4}^{(\tau)} \oplus k_{\omega_2-\omega_3}^{(\tau)} \oplus k_{\omega_3-\omega_4}^{(\tau)}$. Hence

$$\begin{aligned} E_2^{0,1} &= \mathrm{Hom}_{B_{r/2}/B_\tau}(k, (k_{\omega_4}^{(\tau)} \oplus k_{\omega_2-\omega_3}^{(\tau)} \oplus k_{\omega_3-\omega_4}^{(\tau)}) \otimes \tau(\lambda')) \\ &\cong \mathrm{Hom}_{B_{(r-1)/2}}(k, (k_{\omega_4}^{(\tau)} \oplus k_{\omega_2-\omega_3}^{(\tau)} \oplus k_{\omega_3-\omega_4}^{(\tau)}) \otimes k_{\lambda'}^{(\tau)}) \\ &\cong \mathrm{Hom}_{B_{(r-1)/2}}(k, k_{\omega_4+\lambda'}^{(\tau)} \oplus k_{\omega_2-\omega_3+\lambda'}^{(\tau)} \oplus k_{\omega_3-\omega_4+\lambda'}^{(\tau)}). \end{aligned}$$

Notice that $\mathrm{Hom}_{B_{(r-1)/2}}(k, k_{\omega_4+\lambda'}^{(\tau)} \oplus k_{\omega_2-\omega_3+\lambda'}^{(\tau)} \oplus k_{\omega_3-\omega_4+\lambda'}^{(\tau)})$ is either zero or one-dimensional: at most one of $\omega_4 + \lambda'$, $\omega_2 - \omega_3 + \lambda'$ or $\omega_3 - \omega_4 + \lambda' \in \tau^{r-1}X(T)$.

First, suppose $\omega_4 + \lambda' \in \tau^{r-1}X(T)$. As $p = 2$, we obtain $\lambda' = a2^s\omega_1 + b2^s\omega_2 + c2^s\omega_3 + (d2^s - 1)\omega_4 \in X_s(T)$. It follows that we must have $a = b = c = 0$ and $d = 1$, pushing $\lambda = \lambda_0 + \tau(\lambda') = \omega_4 + (2^s - 1)\omega_1$ and

$$E_2^{0,1} = \mathrm{Hom}_{B_{r/2}/B_\tau}(k, k_{\tau(\omega_4+\lambda')} \oplus k_{\tau(\omega_2-\omega_3+\lambda')} \oplus k_{\tau(\omega_3-\omega_4+\lambda')}).$$

The first term in the target of the Hom is $k_{\omega_1+(2^s-1)\omega_1} = k_{2^s\omega_1}$. Thus $E_2^{0,1} \cong k_{2^s\omega_1} = k_{\omega_4}^{(r/2)}$.

Now assume $\omega_2 - \omega_3 + \lambda' \in \tau^{r-1}X(T)$ and a similar argument leads us to conclude that $E_2^{0,1} = k_{\omega_2}^{(r/2)}$ for $\lambda = \omega_2 + 2(2^s - 1)\omega_3 + \omega_4$.

Lastly, suppose $\omega_3 - \omega_4 + \lambda' \in \tau^{r-1}X(T)$, and we obtain $E_2^{0,1} = k_{\omega_3}^{(r/2)}$ for $\lambda = \omega_1 + (2^s - 1)\omega_2 + \omega_4$.

Analogously, the case where $\lambda_0 = \omega_3$ leads to $E_2^{0,1} \cong k_{\omega_1}^{(r/2)}$, when $\lambda = \omega_3 + 2(2^s - 1)\omega_4$.

Overall, we conclude that for $\lambda \in X_{r/2}(T)$,

$$E_2^{0,1} \cong \begin{cases} k_{\omega_1}^{(r/2)} & \text{if } \lambda = \omega_3 + 2(2^s - 1)\omega_4 \\ k_{\omega_2}^{(r/2)} & \text{if } \lambda = \omega_4 + \omega_2 + 2(2^s - 1)\omega_3 \\ k_{\omega_3}^{(r/2)} & \text{if } \lambda = \omega_4 + \omega_1 + (2^s - 1)\omega_2 \\ k_{\omega_4}^{(r/2)} & \text{if } \lambda = \omega_4 + (2^s - 1)\omega_1 \\ 0 & \text{else.} \end{cases}$$

Now suppose $E_2^{1,0} \neq 0$. We have

$$\begin{aligned} E_2^{1,0} &= \mathrm{H}^1(B_{r/2}/B_\tau, \mathrm{Hom}_{B_\tau}(k, \lambda)), \\ &= \mathrm{H}^1(B_{r/2}/B_\tau, \mathrm{Hom}_{B_\tau}(k, \lambda_0) \otimes \tau(\lambda')) \end{aligned}$$

so $\lambda_0 = 0$ and $\lambda = \tau(\lambda')$. Thus $E_2^{1,0} \cong \mathrm{H}^1(B_s, \lambda'^{(\tau)}) \cong \mathrm{H}^1(B_s, \lambda')^{(\tau)}$ for $\lambda = \tau\lambda'$. Notice that since $r - 1 = 2s > 0$, $B_{(r-1)/2} = B_s$ is a classical Frobenius kernel and $\mathrm{H}^1(B_s, \lambda')$ is the B_s-cohomology for $\lambda' \in X_s(T)$ computed in (BNP04, Theorem 2.7). We have

$$\mathrm{H}^1(B_s, \lambda') \cong \begin{cases} k_{\omega_j}^{(s)} & \text{if } \lambda' = 2^s\omega_j - 2^{s-1}\alpha_j, j \in \{1, 3, 4\} \\ M_{F_4}^{(s)} & \text{if } \lambda' = 2^{s-1}\omega_1 \\ k_{\omega_\alpha}^{(s)} & \text{if } \lambda' = 2^s\omega_\alpha - 2^i\alpha, \alpha \in \Pi, 0 \leq i \leq s - 2 \\ 0 & \text{else.} \end{cases}$$

with M_{F_4} having the structure as claimed in the statement of the theorem. We note the implicit constraints on s in the different cases. Thus,

$$E_2^{1,0} \cong \mathrm{H}^1(B_s, \lambda')^{(\tau)} \cong \begin{cases} k_{\omega_j}^{(r/2)} & \text{if } \lambda' = 2^s\omega_j - 2^{s-1}\alpha_j, j \in \{1, 3, 4\} \\ M_{F_4}^{(r/2)} & \text{if } \lambda' = 2^{s-1}\omega_1 \\ k_{\omega_\alpha}^{(r/2)} & \text{if } \lambda' = 2^s\omega_\alpha - 2^i\alpha, \alpha \in \Pi, 0 \leq i \leq s - 2 \\ 0 & \text{else.} \end{cases}$$

Lastly, one may recover λ from λ', recalling $\alpha_1 = 2\omega_1 - \omega_2, \alpha_2 = -\omega_1 + 2\omega_2 - 2\omega_3$, $\alpha_3 = -\omega_2 + 2\omega_3 - \omega_4$ and $\alpha_4 = -\omega_3 + 2\omega_4$.

For example, when $\lambda' = 2^s\omega_1 - 2^{s-1}\alpha_1$, then $\lambda = \tau\lambda' = 2^s\omega_4 - 2^{s-1}\tau(2\omega_1 - \omega_2) = 2^s\omega_3$. The other cases follow similarly.

By the discussion above, notice that there is no λ for which $E_2^{0,1}$ and $E_2^{1,0}$ are both non-zero. Thus, if $E_2^{0,1} = 0$, then $E_\infty^1 \cong E_2^{1,0}$. Alternatively, if $E_2^{0,1} \neq 0$, then λ must be one of the following: either $\lambda = \omega_3 + 2(2^s-1)\omega_4$, $\lambda = \omega_4 + \omega_2 + 2(2^s-1)\omega_3$, $\lambda = \omega_4 + \omega_1 + (2^s-1)\omega_2$ or $\lambda = \omega_4 + (2^s-1)\omega_1$. Clearly, in all of these cases, $\lambda \notin \tau X(T)$, thus forcing $\mathrm{Hom}_{B_\tau}(k, \lambda) = 0$. Hence $E_2^{1,0} = E_2^{2,0} = 0$, implying that $E_\infty^1 \cong E_2^{0,1}$. $\square$

For a general $\lambda \in X(T)$, not necessarily lying in $X_{r/2}$, we proceed as in (BNP04, 2.8). First, we make the following observation, whose proof is identical to the proof of Corollary 3.2.4:

Corollary 3.4.3. Let $\lambda \in X(T)$. Then $\mathrm{H}^1(B_{r/2}, \lambda) \neq 0$ if and only if $\lambda = \tau^r\omega - \tau^i\alpha$, for some weight $\omega \in X(T)$, and $\alpha \in \Pi$ with $0 \leq i \leq 2s-1$ or $\lambda = \tau^r\omega - \beta$, for some weight $\omega \in X(T)$, and $\beta \in \Pi_s$.

Like in the previous cases, let (ζ, j) denote the appropriate pair, (α, i) or $(\beta, 1)$, defined in the previous corollary. Given $\lambda = \tau^r\omega - \tau^j\zeta$, we may write $\lambda = \tau^r\omega' - \tau^j\zeta + \tau^r\lambda_1$. The non-vanishing of $\mathrm{H}^1(B_{r/2}, \lambda)$ is solely dependent on the choice of λ_0, so ω' is as given in Theorem 3.4.2 for some $\lambda_1 \in X(T)$. Then, set $\lambda_1 = \omega - \omega'$ and we get $\mathrm{H}^1(B_{r/2}, \lambda) \cong \mathrm{H}^1(B_{r/2}, \tau^r\omega' - \tau^j\zeta) \otimes k_{\omega-\omega'}^{(r/2)}$.

Substituting the results from Theorem 3.4.2 leads to the the following result

Theorem 3.4.4. Let $\lambda \in X(T)$. and $0 \leq i \leq s-2$. Then

$$\mathrm{H}^1\left(B_{r/2}, \lambda\right) \cong \begin{cases} k_\omega^{(r/2)} & \text{if } \lambda = \tau^r\omega - \beta, \omega \in X(T), \beta \in \Pi_s \\ k_\omega^{(r/2)} & \text{if } \lambda = \tau^r\omega - \tau^{2s-1}\alpha_j, \omega \in X(T), \\ & \quad \alpha_j \in \Pi, j \in \{1,3,4\} \\ M_{F_4}^{(r/2)} \otimes k_{\omega+\omega_3-\omega_2}^{(r/2)} & \text{if } \lambda = \tau^r\omega - \tau^{2s-1}\alpha_2, \omega \in X(T) \\ k_\omega^{(r/2)} & \text{if } \lambda = \tau^r\omega - \tau^{2i+1}\alpha_j, \omega \in X(T), \alpha_j \in \Pi \\ 0 & \text{else.} \end{cases}$$

$G_{r/2}$-Cohomology of Induced Modules

Using Kempf's vanishing theorem, Theorem 3.4.1, Theorem 3.4.2 and (3.1.1), we compute $\mathrm{H}^1(G_{r/2}, \mathrm{H}^0(\lambda))$ for $\lambda \in X_{r/2}$. Finally, we note that, by (BNP04, 3.1, Theorem (C)), $\mathrm{Ind}_B^G(M_{F_4}) = \mathrm{H}^0(\omega_4)$.

In case $r = 1$, we obtain (Sin94b, Lemma 4.5):

Theorem 3.4.5. Let $\lambda \in X_\tau(T)$. Then

$$\mathrm{H}^1(G_\tau, \mathrm{H}^0(\lambda))^{(-\tau)} \cong \begin{cases} \mathrm{H}^0(\omega_4) & \text{if } \lambda = \omega_4 \\ \mathrm{H}^0(\omega_1) & \text{if } \lambda = \omega_3 \\ 0 & \text{else.} \end{cases}$$

Now, let $r > 1$.

Theorem 3.4.6. Let $\lambda \in X_{r/2}(T)$ and $0 \leq i \leq s-2$. Then

$$\mathrm{H}^1(G_{r/2}, \mathrm{H}^0(\lambda))^{(-r/2)} \cong \begin{cases} \mathrm{H}^0(\omega_1) & \text{if } \lambda = \omega_3 + 2(2^s-1)\omega_4 = \tau^r\omega_1 - \beta_2 \\ \mathrm{H}^0(\omega_2) & \text{if } \lambda = \omega_4 + \omega_2 + 2(2^s-1)\omega_3 = \tau^r\omega_2 - \beta_1 \\ \mathrm{H}^0(\omega_3) & \text{if } \lambda = \omega_4 + \omega_1 + (2^s-1)\omega_2 = \tau^r\omega_3 - \beta_3 \\ \mathrm{H}^0(\omega_4) & \text{if } \lambda = \omega_4 + (2^s-1)\omega_1 = \tau^r\omega_4 - \beta_4 \\ \mathrm{H}^0(\omega_1) & \text{if } \lambda = 2^s\omega_3 = \tau^r\omega_1 - \tau^{2s-1}\alpha_1 \\ \mathrm{H}^0(\omega_3) & \text{if } \lambda = 2^s\omega_3 + 2^{s-1}\omega_1 = \tau^r\omega_3 - \tau^{2s-1}\alpha_3 \\ \mathrm{H}^0(\omega_4) & \text{if } \lambda = 2^{s-1}\omega_2 = \tau^r\omega_4 - \tau^{2s-1}\alpha_4 \\ \mathrm{H}^0(\omega_4) & \text{if } \lambda = 2^s\omega_4 = \tau^r(\omega_2 - \omega_3) - \tau^{2s-1}\alpha_2 \\ \mathrm{H}^0(\omega_1) & \text{if } \lambda = (2^{s+1} - 2^{i+2})\omega_4 + 2^{i+1}\omega_3 = \tau^r\omega_1 - \tau^{2i+1}\alpha_1 \\ \mathrm{H}^0(\omega_2) & \text{if } \lambda = 2^{i+1}\omega_2 + (2^{s+1} - 2^{i+2})\omega_3 + 2^{i+1}\omega_4 \\ & \qquad = \tau^r\omega_2 - \tau^{2i+1}\alpha_2 \\ \mathrm{H}^0(\omega_3) & \text{if } \lambda = 2^i\omega_1 + (2^s - 2^{i+1})\omega_2 + 2^{i+1}\omega_3 \\ & \qquad = \tau^r\omega_3 - \tau^{2i+1}\alpha_3 \\ \mathrm{H}^0(\omega_4) & \text{if } \lambda = (2^s - 2^{i+1})\omega_1 + 2^i\omega_2 = \tau^r\omega_4 - \tau^{2i+1}\alpha_4, \\ 0 & \text{else.} \end{cases}$$

One can use Theorem 3.4.4 to determine $\mathrm{H}^1(G_{r/2}, \mathrm{H}^0(\lambda))$ in terms of induced modules for all dominant weights λ, by applying the induction functor Ind_B^G. The remark below deals with the only non-obvious case.

Remark 3.4.7. Let $\tau^r\omega - \tau^{2s-1}\alpha_2 \in X(T)_+$. Then $\langle \omega, \alpha_1^\vee \rangle \geq 0$, $\langle \omega, \alpha_2^\vee \rangle \geq 1$, $\langle \omega, \alpha_3^\vee \rangle \geq -1$ and $\langle \omega, \alpha_4^\vee \rangle \geq 0$.

By (BNP04, Proposition 3.4 (B)(d)), $\mathrm{Ind}_B^G(M_{F_4} \otimes k_{\omega+\omega_3-\omega_2})$ has a filtration with the following factors, from top to bottom: $\mathrm{H}^0(\omega+\omega_3+\omega_4-\omega_2)$, $\mathrm{H}^0(\omega+2\omega_3-\omega_4-\omega_2)$ and $\mathrm{H}^0(\omega)$. Furthermore, observe that:

(i) $\mathrm{H}^0(\omega+\omega_3+\omega_4-\omega_2)$ is always present.

(ii) $\mathrm{H}^0(\omega+2\omega_3-\omega_4-\omega_2)$ appears as a factor if $\langle\omega,\alpha_4^\vee\rangle\geq 1$ and does not if $\langle\omega,\alpha_4^\vee\rangle=0$.

(iii) $\mathrm{H}^0(\omega)$ is present if $\langle\omega,\alpha_3^\vee\rangle\geq 0$ and is not present if $\langle\omega,\alpha_3^\vee\rangle=-1$.

$G_{r/2}$ Extensions Between Simple Modules

In this subsection, we make use of the G_1-extensions between simple modules, computed in (Sin94b, Proposition 4.11), to calculate $\mathrm{Ext}^1_{G_s}(L(\lambda),L(\mu))^{(-s)}$, for a positive integer s and $\lambda,\mu\in X_s(T)$.

Theorem 3.4.8. Let s be a positive integer, with $\lambda=\sum_{i=0}^{s-1}p^i\lambda_i$ and $\mu=\sum_{i=0}^{s-1}p^i\mu_i\in X_s(T)$. Put $d:=\min\{i|\lambda_i\neq\mu_i\}$. Then write $\lambda=\lambda'+p^d\lambda''$, for $\lambda'=\sum_{i=0}^{d-1}p^i\lambda_i\in X_d$ and $\lambda''=\sum_{i=d}^{s-1}p^{i-d}\lambda_i\in X_{s-d}$; we take a similar expression for μ. We may express any digit $\lambda_i=\lambda_{i,0}+\tau\lambda_{i,1}$, with $\lambda_{i,j}\in X_\tau$. We assume some notation from (Sin94b, Proposition 4.11), for brevity. Let the symbol $a\in\{0,3,4,\bar\rho\}$ correspond to some τ-restricted weight (eg. $\bar\rho$ stands for $\omega_3+\omega_4$). Then, the symbols ab denote $\alpha+\tau\beta$, the appropriate p-restricted weight, where a and b correspond to α and β, respectively. (For instance, the symbol $0\bar\rho$ stands for $0+\tau(\omega_3+\omega_4)=\omega_1+\omega_2$.). Then, we denote by

$$\begin{aligned}
A:=\{&(00,44),(04,03),(04,40),(04,44),(03,43),(03,4\bar\rho),(0\bar\rho,4\bar\rho),\\
&(40,30),(44,43),(44,34),(43,33),(4\bar\rho,3\bar\rho),(34,33),(\bar\rho4,\bar\rho3)\}\text{ and}\\
B:=\{&(0\bar\rho,4\bar\rho)\},\\
C:=\{&(00,04),(40,44),(30,34),(\bar\rho0,\bar\rho4)\},\\
D:=\{&(00,30),(04,34),(03,33),(0\bar\rho,3\bar\rho)\},\\
E:=\{&(00,03),(40,43),(30,33),(\bar\rho0,\bar\rho3)\}
\end{aligned}$$

(a) Suppose $d = s - 1$. Then we have

$$\mathrm{Ext}^1_{G_s}(L(\lambda), L(\mu))^{(-s)} \cong \begin{cases} k & \text{if } (\lambda_{s-1}, \mu_{s-1}) \in A \\ k \oplus k & \text{if } (\lambda_{s-1}, \mu_{s-1}) \in B \\ L(\omega_4) & \text{if } (\lambda_{s-1}, \mu_{s-1}) \in C \\ k \oplus L(\omega_4) & \text{if } (\lambda_{s-1}, \mu_{s-1}) \in D \\ k \oplus L(\omega_1) & \text{if } (\lambda_{s-1}, \mu_{s-1}) \in E \\ 0 & \text{else.} \end{cases}$$

(b) Suppose $d \neq s - 1$. Then we have

$$\mathrm{Ext}^1_{G_s}(L(\lambda), L(\mu))^{(-s)} \cong \begin{cases} k & \text{if } (\lambda_d, \mu_d) \in A \text{ and } \lambda'' = \mu'' \\ k \oplus k & \text{if } (\lambda_d, \mu_d) \in B \text{ and } \lambda'' = \mu'' \\ k & \text{if } (\lambda_d, \mu_d) \in C \\ & \text{and } (\lambda_{d+1,0}, \mu_{d+1,0}) \in \{(0,4), (3,\bar{\rho}), (4,4), (3,3)\} \\ k \oplus k & \text{if } (\lambda_d, \mu_d) \in C, \lambda_{d+1,0} = \mu_{d+1,0} = \bar{\rho} \\ k \oplus k \oplus k & \text{if } (\lambda_d, \mu_d) \in D, \lambda_{d+1,0} = \mu_{d+1,0} = \bar{\rho} \\ k \oplus k & \text{if } (\lambda_d, \mu_d) \in D, \lambda_{d+1,0} = \mu_{d+1,0} \in \{\omega_4, \omega_3\} \\ k & \text{if } (\lambda_d, \mu_d) \in D, \lambda_{d+1,0} = \mu_{d+1,0} = 0 \\ k & \text{if } (\lambda_d, \mu_d) \in D, (\lambda_{d+1,0}, \mu_{d+1,0}) \in \{(0,4), (3,\bar{\rho})\} \\ k \oplus k \oplus k & \text{if } (\lambda_d, \mu_d) \in E, \lambda_{d+1,1} = \mu_{d+1,1} = \bar{\rho} \\ k \oplus k & \text{if } (\lambda_d, \mu_d) \in E, \lambda_{d+1,1} = \mu_{d+1,1} \in \{\omega_4, \omega_3\} \\ k & \text{if } (\lambda_d, \mu_d) \in E, \lambda_{d+1,1} = \mu_{d+1,1} \in \{\omega_4, \omega_3\} \\ k & \text{if } (\lambda_d, \mu_d) \in E, (\lambda_{d+1,1}, \mu_{d+1,1}) \in \{(0,4), (3,\bar{\rho})\} \\ k & \text{if } (\lambda_d, \mu_d) \in E, \lambda_{d+1,1} = \mu_{d+1,1} = 0 \\ 0 & \text{else.} \end{cases}$$

Note that it is implicit in the statement of the theorem that $s \geq 1$ or $s \geq 2$, depending on the case.

Proof. We apply the Lyndon-Hochschild-Serre spectral sequence corresponding to $G_d \triangleleft G_s$ and we have

$$E_2^{i,j} := \mathrm{Ext}^i_{G_{s-d}}(L(\lambda''), \mathrm{Ext}^j_{G_d}(L(\lambda'), L(\mu'))^{(-d)} \otimes L(\mu''))^{(d)}.$$

By definition, $\lambda' = \mu'$, so $E_2^{0,1} = 0$, which forces $E^1_\infty \cong E_2^{1,0} = \mathrm{Ext}^1_{G_{s-d}}(L(\lambda''), L(\mu''))^{(d)}$.

(a) First, suppose $s-d=1$. Then $E_\infty^1 \cong E_2^{1,0} = \operatorname{Ext}^1_{G_1}(L(\lambda_{s-1}), L(\mu_{s-1}))^{(s-1)}$, which was computed in (Sin94b, Table VI). We obtain

$$E^{1,0} \cong \begin{cases} k & \text{if } (\lambda_{s-1}, \mu_{s-1}) \in A \\ k \oplus k & \text{if } (\lambda_{s-1}, \mu_{s-1}) \in B \\ L(\omega_4)^{(s)} & \text{if } (\lambda_{s-1}, \mu_{s-1}) \in C \\ k \oplus L(\omega_4)^{(s)} & \text{if } (\lambda_{s-1}, \mu_{s-1}) \in D \\ k \oplus L(\omega_1)^{(s)} & \text{if } (\lambda_{s-1}, \mu_{s-1}) \in E \\ 0 & \text{else.} \end{cases}$$

(b) Now suppose $s-d \neq 1$, so we may apply the Lyndon-Hochschild-Serre spectral sequence corresponding to $G_1 \lhd G_{s-d}$ to $M = \operatorname{Ext}^1_{G_{s-d}}(L(\lambda''), L(\mu''))$. Write $\lambda'' = \lambda_d + p\lambda'''$, for $\lambda_d \in X_1$ and $\lambda''' \in X_{s-d-1}$ and we express μ'' similarly. The E_2-page is given by

$$E_2^{i,j} := \operatorname{Ext}^i_{G_{s-d-1}}(L(\lambda'''), \operatorname{Ext}^j_{G_1}(L(\lambda_d), L(\mu_d))^{(-1)} \otimes L(\mu'''))^{(1)}.$$

Since, by definition, $\lambda_d \neq \mu_d$, it follows that $E_2^{i,0} = 0$ for $i > 0$, so we obtain

$$M \cong E_2^{0,1} \cong \operatorname{Hom}_{G_{s-d-1}}(L(\lambda'''), \operatorname{Ext}^1_{G_1}(L(\lambda_d), L(\mu_d))^{(-1)} \otimes L(\mu'''))^{(1)}.$$

Put $W := \operatorname{Ext}^1_{G_1}(L(\lambda_d), L(\mu_d))^{(-1)}$. Then, by (Sin94b, Table VI), depending on the pair (λ_d, μ_d), $W \in \{k, k \oplus k, k \oplus L(\omega_1), k \oplus L(\omega_4), L(\omega_4)\}$. We consider each value of W in turn.

Case I: $W = k$, which occurs when $(\lambda_d, \mu_d) \in A$

In this case we have

$$M \cong \operatorname{Hom}_{G_{s-d-1}}(L(\lambda'''), L(\mu'''))^{(1)} \cong k,$$

if and only if $\lambda''' = \mu'''$ and vanishes otherwise.

Case II: $W = k \oplus k$, which occurs when $(\lambda_d, \mu_d) \in B$

Then, it follows that

$$M \cong \bigoplus_2 \operatorname{Hom}_{G_{s-d-1}}(L(\lambda'''), L(\mu'''))^{(1)} \cong k \oplus k,$$

if and only if $\lambda''' = \mu'''$ and vanishes otherwise.

Case III: $W = L(\omega_4)$, which occurs when $(\lambda_d, \mu_d) \in C$

In this case, we obtain

$$\begin{aligned} M &\cong \mathrm{Hom}_{G_{s-d-1}}(L(\lambda'''), L(\mu''') \otimes L(\omega_4))^{(1)} \\ &\cong \mathrm{Hom}_{G_{s-d-3/2}}(L(\lambda_1'''), \mathrm{Hom}_{G_\tau}(L(\lambda_0'''), L(\mu_0''') \otimes L(\omega_4))^{(-\tau)} \otimes L(\mu_1'''))^{(3/2)}, \end{aligned}$$

for $\lambda''' = \lambda_0''' + \tau\lambda_1'''$, $\lambda_0''' = \lambda_{d+1,0} \in X_\tau$ and $\lambda_1''' \in X_{s-d-3/2}$ and μ''' expressed similarly. Thus, $M \neq 0$ if and we have

$$M \cong \begin{cases} k & \text{if } (\lambda_d, \mu_d) \in C, (\lambda_{d+1,0}, \mu_{d+1,0}) \in \{(0,4), (3,\bar{\rho}), (4,4), (3,3)\} \\ k \oplus k & \text{if } (\lambda_d, \mu_d) \in C, \lambda_{d+1,0} = \mu_{d+1,0} = \bar{\rho} \\ 0 & \text{else.} \end{cases}$$

Case IV: $W = k \oplus L(\omega_4)$**, which occurs when** $(\lambda_d, \mu_d) \in D$

It follows that

$$\begin{aligned} M \cong &\mathrm{Hom}_{G_{s-d-1}}(L(\lambda'''), L(\mu'''))^{(1)} \oplus \\ &\mathrm{Hom}_{G_{s-d-1}}(L(\lambda'''), L(\mu''') \otimes L(\omega_4))^{(1)}. \end{aligned}$$

Using Case III above, a similar calculation yields

$$M \cong \begin{cases} k \oplus k \oplus k & \text{if } (\lambda_d, \mu_d) \in D, \lambda_{d+1,0} = \mu_{d+1,0} = \bar{\rho} \\ k \oplus k & \text{if } (\lambda_d, \mu_d) \in D, \lambda_{d+1,0} = \mu_{d+1,0} \in \{\omega_4, \omega_3\} \\ k & \text{if } (\lambda_d, \mu_d) \in D, \lambda_{d+1,0} = \mu_{d+1,0} = 0 \\ k & \text{if } (\lambda_d, \mu_d) \in D, (\lambda_{d+1,0}, \mu_{d+1,0}) \in \{(0,4), (3,\bar{\rho})\} \\ 0 & \text{else.} \end{cases}$$

Case V: $W = k \oplus L(\omega_1)$**, which occurs when** $(\lambda_d, \mu_d) \in E$

It follows that

$$M \cong \mathrm{Hom}_{G_{s-d-1}}(L(\lambda'''), L(\mu'''))^{(1)} \oplus \mathrm{Hom}_{G_{s-d-1}}(L(\lambda'''), L(\mu''') \otimes L(\omega_1))^{(1)}.$$

We have

$$\begin{aligned} M_1 := &\mathrm{Hom}_{G_{s-d-1}}(L(\lambda'''), L(\mu''') \otimes L(\omega_1))^{(1)} \\ &\cong \mathrm{Hom}_{G_{s-d-3/2}}(L(\lambda_1'''), \mathrm{Hom}_{G_\tau}(L(\lambda_0'''), L(\mu_0'''))^{(-\tau)} \otimes L(\omega_4) \otimes L(\mu_1'''))^{(3/2)}. \end{aligned}$$

This in non-zero if and only if $\lambda_0''' = \mu_0'''$, in which case we obtain

$$M_1 = \mathrm{Hom}_{G_{s-d-3/2}}(L(\lambda_1'''), L(\mu_1''') \otimes L(\omega_4))^{(3/2)}.$$

Using (Sin94b, Table IV), we obtain

$$M_1 \cong \begin{cases} k \oplus k & \text{if } (\lambda_d, \mu_d) \in E, \lambda_{d+1,1} = \mu_{d+1,1} = \bar{\rho} \\ k & \text{if } (\lambda_d, \mu_d) \in E, (\lambda_{d+1,1}, \mu_{d+1,1}) \in \{(0,4), (3, \bar{\rho}), (4,4), (3,3)\} \\ 0 & \text{else.} \end{cases}$$

Lastly, we recover the answer recalling $M = M_1 \oplus \mathrm{Hom}_{G_{s-d-3/2}}(L(\lambda_1'''), L(\mu_1'''))^{(3/2)}$. $\square$

Next, with the aid of the previous theorem concerning the cohomology for classical Frobenius kernels, we compute $\mathrm{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu))$ for r an odd positive integer and $\lambda, \mu \in X_{r/2}$.

If $r = 1$, we refer the reader to (Sin94b, Lemma 4.5(a),4.6,4.9). First, we underline that in this case, the calculation of $\mathrm{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu))$ relies on the following remark.

Remark 3.4.9. In the literature, there is a discrepancy between results and we discuss it for the benefit of the reader.

(a) In (BNP+15, Remark 2.3.2(b)), the authors claim that, by (Sin94b, Lemma 4.6), $\mathrm{H}^1(G_\tau, L(\omega_3))^{(\tau^{-1})} \cong k \oplus L(2\omega_4)$.

(b) However, by (Sin94b, Lemma 4.6), we have $\mathrm{H}^1(G_\tau, L(\omega_3)) \cong k \oplus L(2\omega_4)$, so that $\mathrm{H}^1(G_\tau, L(\omega_3))^{(\tau^{-1})} \cong k \oplus L(\omega_1)$. This is consistent with its proof; moreover, it is easily verifiable that it is consistent with the computation of the G_1-extensions. (cf. (Sin94b, Lemma 4.11))

If $r > 1$, we get:

Theorem 3.4.10. Suppose $r = 2s + 1 > 1$ and let $\lambda, \mu \in X_{r/2}(T)$. Write $\lambda = \lambda_0 + \tau\lambda'$, for $\lambda_0 \in X_\tau$ and $\lambda' = \sum_{i=0}^{2s-1} \tau^i \lambda_{1,i} \in X_s$. Let $\lambda_{i+1} := \lambda_{1,2i} + \tau\lambda_{1,2i+1} \in X_1$, and we take similar expressions for μ. Put $d := \min\{i \geq 1 | \lambda_i \neq \mu_i\}$. Recall A, B, C, D, E from Theorem 3.4.8.

(a) Suppose $\lambda_0 = \mu_0$. Then we have

$$\operatorname{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu))^{(-r/2)} \cong \begin{cases} k & \text{if } (\lambda_s, \mu_s) \in A \\ k \oplus k & \text{if } (\lambda_s, \mu_s) \in B \\ L(\omega_4) & \text{if } (\lambda_s, \mu_s) \in C \\ k \oplus L(\omega_4) & \text{if } (\lambda_s, \mu_s) \in D \\ k \oplus L(\omega_1) & \text{if } (\lambda_s, \mu_s) \in E \\ k & \text{if } (\lambda_{d+1}, \mu_{d+1}) \in A \\ k \oplus k & \text{if } (\lambda_{d+1}, \mu_{d+1}) \in B \\ k & \text{if } (\lambda_{d+1}, \mu_{d+1}) \in C \\ & \text{and } (\lambda_{1,2d+2}, \mu_{1,2d+2}) \in \{(0,4), (3,\bar{\rho}), (4,4), (3,3)\} \\ k \oplus k & \text{if } (\lambda_{d+1}, \mu_{d+1}) \in C, \lambda_{1,2d+2} = \mu_{1,2d+2} = \bar{\rho} \\ k \oplus k \oplus k & \text{if } (\lambda_{d+1}, \mu_{d+1}) \in D, \lambda_{1,2d+2} = \mu_{1,2d+2} = \bar{\rho} \\ k \oplus k & \text{if } (\lambda_{d+1}, \mu_{d+1}) \in D, \lambda_{1,2d+2} = \mu_{1,2d+2} \in \{\omega_4, \omega_3\} \\ k & \text{if } (\lambda_{d+1}, \mu_{d+1}) \in D, \lambda_{1,2d+2} = \mu_{1,2d+2} = 0 \\ k & \text{if } (\lambda_{d+1}, \mu_{d+1}) \in D \\ & \text{and } (\lambda_{1,2d+2}, \mu_{1,2d+2}) \in \{(0,4), (3,\bar{\rho})\} \\ k \oplus k \oplus k & \text{if } (\lambda_{d+1}, \mu_{d+1}) \in E, \lambda_{1,2d+3} = \mu_{1,2d+3} = \bar{\rho} \\ k \oplus k & \text{if } (\lambda_{d+1}, \mu_{d+1}) \in E, \lambda_{1,2d+3} = \mu_{1,2d+3} \in \{\omega_4, \omega_3\} \\ k & \text{if } (\lambda_{d+1}, \mu_{d+1}) \in E, \lambda_{1,2d+3} = \mu_{1,2d+3} \in \{\omega_4, \omega_3\} \\ k & \text{if } (\lambda_{d+1}, \mu_{d+1}) \in E \\ & \text{and } (\lambda_{1,2d+3}, \mu_{1,2d+3}) \in \{(0,4), (3,\bar{\rho})\} \\ k & \text{if } (\lambda_{d+1}, \mu_{d+1}) \in E, \lambda_{1,2d+3} = \mu_{1,2d+3} = 0 \\ 0 & \text{else.} \end{cases}$$

(b) Suppose $\lambda_0 \neq \mu_0$. Then we have

$$\operatorname{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu))^{(-r/2)} \cong \begin{cases} k & \text{if } (\lambda_0, \mu_0) = (0, \omega_4) \\ & \quad \text{and } (\lambda_{1,0}, \mu_{1,0}) \in \{(0,4), (3,\bar{\rho}), (4,4), (3,3)\} \\ k \oplus k & \text{if } (\lambda_0, \mu_0) = (0, \omega_4) \text{ and } \lambda_{1,0} = \mu_{1,0} = \bar{\rho} \\ k \oplus k \oplus k & \text{if } (\lambda_0, \mu_0) = (0, \omega_3) \text{ and } \lambda_{1,1} = \mu_{1,1} = \bar{\rho} \\ k \oplus k & \text{if } (\lambda_0, \mu_0) = (0, \omega_3) \\ & \quad \text{and } \lambda_{1,1} = \mu_{1,1} \in \{\omega_4, \omega_3\} \\ k & \text{if } (\lambda_0, \mu_0) = (0, \omega_3) \\ & \quad \text{and } (\lambda_{1,1}, \mu_{1,1}) \in \{(0,4), (3,\bar{\rho})\} \\ k & \text{if } (\lambda_0, \mu_0) = (0, \omega_3) \\ & \quad \text{and } \lambda_{1,1} = \mu_{1,1} = 0 \\ k & \text{if } (\lambda_0, \mu_0) = (\omega_4, \omega_3) \\ 0 & \text{else.} \end{cases}$$

Proof. We consider the Lyndon-Hochschild-Serre spectral sequence corresponding to $G_\tau \lhd G_{r/2}$. The E_2-page is given by

$$E_2^{i,j} := \operatorname{Ext}^i_{G_s}(L(\lambda'), \operatorname{Ext}^j_{G_\tau}(L(\lambda_0), L(\mu_0))^{(-\tau)} \otimes L(\mu'))^{(\tau)}.$$

First, suppose $\lambda_0 = \mu_0$. Then $E_2^{0,1} = 0$, so $E^1_\infty \cong E_2^{1,0}$ and we have

$$E_2^{1,0} := \operatorname{Ext}^1_{G_s}(L(\lambda'), L(\mu'))^{(\tau)},$$

which was computed in Theorem 3.4.8.

Now suppose $\lambda_0 \neq \mu_0$. Then $E_2^{i,0} = 0$ for $i > 0$, so $E^1_\infty \cong E_2^{0,1}$ and we obtain

$$E_2^{0,1} := \operatorname{Hom}_{G_s}(L(\lambda'), \operatorname{Ext}^1_{G_\tau}(L(\lambda_0), L(\mu_0))^{(-\tau)} \otimes L(\mu'))^{(\tau)}.$$

By (Sin94b, 4.5(a), 4.6, 4.9), this vanishes unless $(\lambda_0, \mu_0) \in \{(0, \omega_4), (0, \omega_3), (\omega_4, \omega_3)\}$. We consider each case in turn.

Case I: $(\lambda_0, \mu_0) = (0, \omega_4)$

In this case we obtain

$$\begin{aligned} E^1_\infty &\cong \operatorname{Hom}_{G_s}(L(\lambda'), L(\mu') \otimes L(\omega_4))^{(\tau)} \\ &\cong \operatorname{Hom}_{G_{s-1/2}}(L(\lambda''), \operatorname{Hom}_{G_\tau}(L(\lambda_{1,0}), L(\mu_{1,0}) \otimes L(\omega_4))^{(-\tau)} \otimes L(\mu''))^{(1)}, \end{aligned}$$

where $\tau\lambda'' = \lambda - (\lambda_0 + \tau\lambda_{1,0})$ and μ'' is expressed similarly. Then, by (Sin94b, Table IV), we obtain for $\lambda'' = \mu''$

$$E_\infty^1 \cong \begin{cases} k & \text{if } (\lambda_{1,0}, \mu_{1,0}) \in \{(0,4), (3,\tilde{\rho}), (4,4), (3,3)\} \\ k \oplus k & \text{if } \lambda_{1,0} = \mu_{1,0} = \tilde{\rho} \\ 0 & \text{else.} \end{cases}$$

Case II: $(\lambda_0, \mu_0) = (0, \omega_3)$ It follows that

$$E_2^{0,1} \cong \mathrm{Hom}_{G_s}(L(\lambda'), L(\mu'))^{(\tau)} \oplus \mathrm{Hom}_{G_s}(L(\lambda'), L(\mu') \otimes L(\omega_1))^{(\tau)}.$$

We turn our attention to $M := \mathrm{Hom}_{G_s}(L(\lambda'), L(\mu') \otimes L(\omega_1))^{(\tau)}$ and we have

$$M \cong \mathrm{Hom}_{G_{s-1/2}}(L(\lambda''), \mathrm{Hom}_{G_\tau}(L(\lambda_{1,0}), L(\mu_{1,0}))^{(-\tau)} \otimes L(\omega_4) \otimes L(\mu))^{(\tau)},$$

where $\tau\lambda'' = \lambda - (\lambda_0 + \tau\lambda_{1,0})$ and μ'' is expressed similarly. This is non-zero if and only if $\lambda_{1,0} = \mu_{1,0}$, in which case

$$\begin{aligned} M &\cong \mathrm{Hom}_{G_{s-1/2}}(L(\lambda''), L(\mu'') \otimes L(\omega_4))^{(1)} \\ &\cong \mathrm{Hom}_{G_{s-1}}(L(\lambda'''), \mathrm{Hom}_{G_\tau}(L(\lambda_{1,1}), L(\mu_{1,1}) \otimes L(\omega_4))^{(-\tau)} \otimes L(\mu'''))^{(3/2)}, \end{aligned}$$

where $\tau\lambda''' = \lambda - (\lambda_0 + \tau\lambda_{1,0} + 2\lambda_{1,1})$ and μ''' is expressed similarly. Then, by (Sin94b, Table IV), we obtain for $\lambda''' = \mu'''$

$$M \cong \begin{cases} k & \text{if } (\lambda_{1,1}, \mu_{1,1}) \in \{(0,4), (3,\tilde{\rho}), (4,4), (3,3)\} \\ k \oplus k & \text{if } \lambda_{1,1} = \mu_{1,1} = \tilde{\rho} \\ 0 & \text{else.} \end{cases}$$

Then, recalling $E_2^{0,1} = M \oplus \mathrm{Hom}_{G_s}(L(\lambda'), L(\mu'))^{(\tau)}$, we obtain

$$E_2^{0,1} \cong \begin{cases} k \oplus k \oplus k & \text{if } \lambda_{1,1} = \mu_{1,1} = \tilde{\rho} \\ k \oplus k & \text{if } \lambda_{1,1} = \mu_{1,1} \in \{\omega_3, \omega_4\} \\ k & \text{if } (\lambda_{1,1}, \mu_{1,1}) \in \{(0,4), (3,\tilde{\rho})\} \\ k & \text{if } \lambda_{1,1} = \mu_{1,1} = 0 \\ 0 & \text{else.} \end{cases}$$

Case III: $(\lambda_0, \mu_0) = (\omega_4, \omega_3)$

It follows that $E_2^{0,1} \cong \mathrm{Hom}_{G_s}(L(\lambda'), L(\mu'))^{(\tau)} \cong k$ if and only if $\lambda' = \mu'$ and vanishes otherwise. $\square$

Corollary 3.4.11. Suppose $r = 2s+1 > 1$ and let $\lambda \in X_{r/2}(T)$, $0 \leq i \leq s-1$ and $0 \leq j \leq s-2$. Then

$$\mathrm{H}^1(G_{r/2}, L(\lambda))^{(-r/2)} \cong \begin{cases} L(\omega_4) & \text{if } \lambda = 2^s\omega_4 \\ k \oplus L(\omega_1) & \text{if } \lambda = 2^s\omega_3 \\ k \oplus L(\omega_4) & \text{if } \lambda = 2^{s-1}\omega_2 \\ k & \text{if } \lambda = 2^i(\omega_1 + 2\omega_4) \\ k & \text{if } \lambda = 2^i\omega_2 \\ k & \text{if } \lambda = 2^i(\omega_1 + \omega_4) \\ k & \text{if } \lambda = 2^i(\omega_2 + \omega_3) \\ k & \text{if } \lambda = 2^i(\omega_2 + \omega_3 + 2\omega_4) \\ k & \text{if } \lambda = 2^j(\omega_2 + 2\omega_1) \\ 0 & \text{else.} \end{cases}$$

Chapter 4

Bounding Cohomology for the Ree Groups of Type F_4

In this section we turn our attention to the extensions between simple modules for the Ree groups of type F_4, for which we aim to prove results using the (BNP06) approach.

To begin with, we briefly discuss the motivation behind the (BNP06) framework. Their method relies on the use of a certain truncated category of G-modules. In such a category, the weights of the G-modules have a suitable upper bound, and it is a highest weight category (see (CPS88, Definition 3.1) for a definition). We refer the reader to (BNP01, 4.2 and 4.5), or (Don86, S. 1), for a more general treatment of truncated categories.

In Subsection 4.1, we provide precise definitions and results on which we base our construction.

4.1 Filtering $\operatorname{Ind}_{G(\sigma)}^{G} k$

We begin by fixing some notation and introducing some terminology. For the trivial module k, set $\mathcal{G}(k) := \operatorname{Ind}_{G(\sigma)}^{G} k$; it is an infinite-dimensional module since the coset space $G/G(\sigma)$ is affine. Then for any finite set of dominant weights $\pi \subseteq X(T)_+$, we define $\mathcal{G}_\pi(k)$ to be the maximal G-submodule of $\mathcal{G}(k)$ having composition factors with weights in π.

Now, observe the following result from (BNP$^+$15) concerning the structure of $\mathcal{G}(k)$.

Theorem 4.1.1. ((BNP$^+$15, Prop 3.1.2)) The G-module $\mathcal{G}(k)$ has a filtration with factors of the form $\mathrm{H}^0(\nu)\otimes\mathrm{H}^0(\nu^*)^{(\sigma)}$, one for each $\nu \in X(T)_+$ and occurring in an order compatible with the dominance order on X_+.

Since $G/G(\sigma)$ is affine, the induction functor is exact (cf. (Jan03, I.5.13)). Then, by generalised Frobenius reciprocity (cf. (Jan03, I.4.6)), there exists an isomorphism for each $n \geq 0$ and any two G-modules V, W:

$$\mathrm{Ext}^n_{G(\sigma)}(V, W) \cong \mathrm{Ext}^n_G(V, W \otimes \mathcal{G}(k)). \tag{4.1.1}$$

In view of Theorem 4.1.1, in order to apply (4.1.1) and study $\mathrm{Ext}^1_{G(\sigma)}(L(\lambda), L(\mu))$ for $\lambda, \mu \in X_\sigma$, we must investigate the Ext-groups

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu) \otimes \mathrm{H}^0(\nu) \otimes \mathrm{H}^0(\nu^*)^{(r/2)}) \cong \mathrm{Ext}^1_G(L(\lambda) \otimes V(\nu)^{(r/2)}, L(\mu) \otimes \mathrm{H}^0(\nu)),$$

for all $\nu \neq 0$, $\nu \in X_+$. First, we provide a way to identify homomorphisms over $G_{r/2}$ with homomorphisms over G, under a certain condition. This holds for the Suzuki groups and the Ree groups.

Lemma 4.1.2. Let $r \in \mathbb{N}$ and set $s = \lfloor r/2 \rfloor$. Let $\lambda, \mu \in X_{r/2}$ and $\nu \in X_+$. We have:

(a) If $\langle \nu, \alpha_0^\vee \rangle < p^s$, then the G-module $\mathrm{Hom}_{G_{r/2}}(L(\lambda), L(\mu) \otimes \mathrm{H}^0(\nu))$ has trivial G-structure, meaning that it is isomorphic to $\mathrm{Hom}_G(L(\lambda), L(\mu) \otimes \mathrm{H}^0(\nu))$.

(b) If $\tau^r\theta$ is a weight of $\mathrm{Hom}_{G_{r/2}}(L(\lambda), L(\mu) \otimes \mathrm{H}^0(\nu))$, then $\langle \tau^r\theta, \alpha_0^\vee \rangle \leq \langle \nu, \alpha_0^\vee \rangle$.

Proof. (a) This is (BNP06, Proposition 3.1) when r is even. When r is odd, we use the same argument. Without loss of generality, we may assume $\langle \mu, \alpha_0^\vee \rangle \leq \langle \lambda, \alpha_0^\vee \rangle$. Since all G-composition factors of $\mathrm{Hom}_{G_{r/2}}(L(\lambda), L(\mu) \otimes \mathrm{H}^0(\nu))$ are $G_{r/2}$-trivial, they must be of the form $L(\theta)^{(r/2)}$, for some $\theta \in X(T)$. Let $L(\theta)^{(r/2)}$ be such a factor and then a weight of $L(\mu) \otimes \mathrm{H}^0(\nu)$ will be $\lambda + \tau^r\theta$; we obtain

$$\left\langle \lambda + \tau^r\theta, \alpha_0^\vee \right\rangle \leq \left\langle \mu + \nu, \alpha_0^\vee \right\rangle \leq \left\langle \lambda + \nu, \alpha_0^\vee \right\rangle$$

(with the last inequality following from the assumption). Thus

$$p^s \left\langle \theta, \alpha_0^\vee \right\rangle \leq p^s \left\langle \tau\theta, \alpha_0^\vee \right\rangle \leq \left\langle \nu, \alpha_0^\vee \right\rangle < p^s,$$

(with the last inequality following from the hypothesis), pushing $\theta = 0$, and thus proving the claim.

Part (b) follows immediately from the proof of part (a). $\square$

From this point onwards, unless stated otherwise, we let G be of type F_4 and $p = 2$. Next we prove a result in the flavour of (BNP06, Lemma 5.2).

Lemma 4.1.3. Let $\lambda, \mu \in X_{r/2}(T)$ and $\nu \in X(T)_+$. Assume further that $2^s > 4$. If $\mathrm{Ext}^1_G(L(\lambda) \otimes V(\nu)^{(r/2)}, L(\mu) \otimes \mathrm{H}^0(\nu)) \neq 0$, then $\langle \nu, \alpha_0^\vee \rangle < 17 = h+5$. Furthermore, except for possibly one dominant weight, namely $\nu = 8\omega_4$, the non-vanishing implies $\langle \nu, \alpha_0^\vee \rangle < 16$.

Proof. Consider the Lyndon-Hochschild-Serre spectral sequence

$$\begin{aligned} E_2^{i,j} &= \mathrm{Ext}^i_{G/G_{r/2}}(V(\nu)^{(r/2)}, \mathrm{Ext}^j_{G_{r/2}}(L(\lambda), L(\mu) \otimes \mathrm{H}^0(\nu))) \\ &\Rightarrow \mathrm{Ext}^{i+j}_G(L(\lambda) \otimes V(\nu)^{(r/2)}, L(\mu) \otimes \mathrm{H}^0(\nu)). \end{aligned}$$

Consider the $E_2^{i,0}$-term:

$$E_2^{i,0} = \mathrm{Ext}^i_{G/G_{r/2}}(V(\nu)^{(r/2)}, \mathrm{Hom}_{G_{r/2}}(L(\lambda), L(\mu) \otimes \mathrm{H}^0(\nu)).$$

It follows from Lemma 4.1.2 (b) that any weight θ of $\mathrm{Hom}_{G_{r/2}}(L(\lambda), L(\mu) \otimes \mathrm{H}^0(\nu))^{(-r/2)}$ satisfies $\langle \theta, \alpha_0^\vee \rangle \leq \frac{1}{p^s} \langle \nu, \alpha_0^\vee \rangle < \langle \nu, \alpha_0^\vee \rangle$. Since $V(\nu)$ is projective in the category of modules with weights β so that $\langle \beta, \alpha_0^\vee \rangle < \langle \nu, \alpha_0^\vee \rangle$, we may conclude that the $E_2^{i,0}$ terms vanish. Therefore

$$E^1_\infty \cong E_2^{0,1} \cong \mathrm{Hom}_{G/G_{r/2}}(V(\nu)^{(r/2)}, \mathrm{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu) \otimes \mathrm{H}^0(\nu))).$$

Let $\tau^r\gamma$ be a weight of a composition factor of $\mathrm{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu) \otimes \mathrm{H}^0(\nu))$. We claim that

$$\left\langle \tau^r\gamma, \alpha_0^\vee \right\rangle \leq \left\langle \lambda + \mu + \nu, \alpha_0^\vee \right\rangle + 2^s. \tag{4.1.2}$$

In order to show this, first consider $\mathrm{H}^1(G_{r/2}, L(\lambda) \otimes L(\mu) \otimes \mathrm{H}^0(\nu))$. Let $L(\sigma_0) \otimes L(\sigma_1)^{(r/2)}$ be a composition factor of $L(\lambda) \otimes L(\mu) \otimes \mathrm{H}^0(\nu)$, for some $\sigma_0 \in X_{r/2}$ and $\sigma_1 \in X_+$. Hence, in order to bound the weights of $\mathrm{H}^1(G_{r/2}, L(\lambda) \otimes L(\mu) \otimes \mathrm{H}^0(\nu))$, we must evaluate the weights of $\mathrm{H}^1(G_{r/2}, L(\sigma_0)) \otimes L(\sigma_1)^{(r/2)}$.

Observe that $\mathrm{H}^1(G_{r/2}, L(\sigma_0))^{(-r/2)}$ for $\sigma_0 \in X_{r/2}$ was computed in Theorem 3.4.11. Let $\tau^r\theta$ denote a weight of $\mathrm{H}^1(G_{r/2}, L(\sigma_0))$. We claim that it must satisfy

$$\left\langle \tau^r\theta, \alpha_0^\vee \right\rangle \leq \left\langle \sigma_0, \alpha_0^\vee \right\rangle + 2^s. \tag{4.1.3}$$

We consider each non-zero instance in the theorem in turn. We present the explicit computation of the case $\sigma_0 = 2^s\omega_3$, for which $\mathrm{H}^1(G_{r/2}, L(\sigma_0))^{(-r/2)} \cong k \oplus L(\omega_1)$. Since

$0 \leq \omega_1$, we may assume $\theta = \omega_1$. We obtain

$$\left\langle \tau^r \omega_1, \alpha_0^\vee \right\rangle = 2^s \cdot 4 \leq 2^s \cdot 3 + 2^s.$$

Similar calculations for all of the other choices of (σ_0, θ) lead us to conclude that the inequality (4.1.3) holds and this proves the claim.

Thus, if $\tau^r\gamma$ is a weight of $\mathrm{H}^1(G_{r/2}, L(\lambda) \otimes L(\mu) \otimes \mathrm{H}^0(\nu))$, we have $\langle \tau^r\gamma, \alpha_0^\vee \rangle \leq \langle \tau^r\theta, \alpha_0^\vee \rangle + \langle \tau^r\sigma_1, \alpha_0^\vee \rangle$, for $L(\sigma_0) \otimes L(\sigma_1)^{(r/2)}$ a composition factor of $L(\lambda) \otimes L(\mu) \otimes \mathrm{H}^0(\nu)$ and θ a weight of $\mathrm{H}^1(G_{r/2}, L(\sigma_0))^{(-r/2)}$. Using (4.1.3), we obtain

$$\begin{aligned} \left\langle \tau^r\gamma, \alpha_0^\vee \right\rangle &\leq \left\langle \tau^r\theta, \alpha_0^\vee \right\rangle + \left\langle \tau^r\sigma_1, \alpha_0^\vee \right\rangle \leq \left\langle \sigma_0, \alpha_0^\vee \right\rangle + 2^s + \left\langle \tau^r\sigma_1, \alpha_0^\vee \right\rangle \\ &\leq \left\langle \lambda + \mu + \nu, \alpha_0^\vee \right\rangle + 2^s. \end{aligned}$$

This verifies (4.1.2).

Consider the short exact sequence

$$0 \to L(\mu) \to \mathrm{St}_{r/2} \otimes L\left((2^s - 1)(\omega_1 + \omega_2) + (2^{s+1} - 1)(\omega_3 + \omega_4) + w_0\mu\right) \to R \to 0.$$

Using the long exact sequence of cohomology, along with the fact that $\mathrm{St}_{r/2}$ is injective as a $G_{r/2}$-module, one obtains a surjection

$$\mathrm{Hom}_{G_{r/2}}(L(\lambda), R \otimes \mathrm{H}^0(\nu)) \twoheadrightarrow \mathrm{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu) \otimes \mathrm{H}^0(\nu)).$$

Hence, any weight $\tau^r\gamma$ of $\mathrm{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu) \otimes \mathrm{H}^0(\nu))$ also satisfies

$$\begin{aligned} \left\langle \tau^r\gamma, \alpha_0^\vee \right\rangle \leq 2(2^s - 1) \left\langle \tau(\omega_3 + \omega_4), \alpha_0^\vee \right\rangle + 2(2^{s+1} - 1) \left\langle \omega_3 + \omega_4, \alpha_0^\vee \right\rangle \\ - \left\langle \lambda, \alpha_0^\vee \right\rangle - \left\langle \mu, \alpha_0^\vee \right\rangle + \left\langle \nu, \alpha_0^\vee \right\rangle. \end{aligned} \tag{4.1.4}$$

Adding (4.1.2) and (4.1.4) and dividing by two yields

$$\begin{aligned} \left\langle \tau^r\gamma, \alpha_0^\vee \right\rangle &\leq (2^s - 1) \left\langle \tau(\omega_3 + \omega_4), \alpha_0^\vee \right\rangle + (2^{s+1} - 1) \left\langle \omega_3 + \omega_4, \alpha_0^\vee \right\rangle + \left\langle \nu, \alpha_0^\vee \right\rangle + 2^{s-1}. \\ \left\langle \tau^r\gamma, \alpha_0^\vee \right\rangle &\leq (2^s - 1) \cdot 6 + (2^{s+1} - 1) \cdot 5 + 2^{s-1} + \left\langle \nu, \alpha_0^\vee \right\rangle. \end{aligned} \tag{4.1.5}$$

Since we assume $E_2^{0,1} \neq 0$, we may assume $\tau^r\nu$ is a weight of $\mathrm{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu) \otimes \mathrm{H}^0(\nu))$. Therefore, put $\gamma = \nu$ to get

$$\left\langle \tau^r\nu, \alpha_0^\vee \right\rangle \leq (2^s - 1) \cdot 6 + (2^{s+1} - 1) \cdot 5 + 2^{s-1} + \left\langle \nu, \alpha_0^\vee \right\rangle. \tag{4.1.6}$$

Then, we have

$$\langle \tau^r \nu, \alpha_0^\vee \rangle - \langle \nu, \alpha_0^\vee \rangle \leq (2^s - 1) \cdot 6 + (2^{s+1} - 1) \cdot 5 + 2^{s-1}. \tag{4.1.7}$$

Therefore, to finish the proof, we must investigate the link between $\langle \nu, \alpha_0^\vee \rangle$ and $\langle \tau^r \nu, \alpha_0^\vee \rangle$.

Since $\nu \in X(T)_+$, we may write $\nu = a\omega_1 + b\omega_2 + c\omega_3 + d\omega_4$, for some non-negative integers a, b, c, d. Then, $\tau\nu = 2a\omega_4 + 2b\omega_3 + c\omega_2 + d\omega_1$.

Furthermore, recalling $\langle \omega_4, \alpha_0^\vee \rangle = 2, \langle \omega_3, \alpha_0^\vee \rangle = 3, \langle \omega_2, \alpha_0^\vee \rangle = 4, \langle \omega_1, \alpha_0^\vee \rangle = 2$, we have $\langle \tau\nu, \alpha_0^\vee \rangle = \langle \nu, \alpha_0^\vee \rangle + 2a + 2b + c$. Since $\langle \tau\nu, \alpha_0^\vee \rangle \geq \langle \nu, \alpha_0^\vee \rangle$, inequality (4.1.7) yields

$$\langle \tau^r \nu, \alpha_0^\vee \rangle - \langle \tau\nu, \alpha_0^\vee \rangle \leq \langle \tau^r \nu, \alpha_0^\vee \rangle - \langle \nu, \alpha_0^\vee \rangle \leq (2^s - 1) \cdot 6 + (2^{s+1} - 1) \cdot 5 + 2^{s-1},$$

giving

$$\langle \tau\nu, \alpha_0^\vee \rangle \leq 6 + \frac{2^{s+1} - 1}{2^s - 1} \cdot 5 + \frac{2^{s-1}}{2^s - 1}. \tag{4.1.8}$$

Notice that, if $s \geq 3$, $\langle \tau\nu, \alpha_0^\vee \rangle < 18$ and if $s \geq 4$, $\langle \tau\nu, \alpha_0^\vee \rangle < 17$. Recall that $\langle \tau\nu, \alpha_0^\vee \rangle = \langle \nu, \alpha_0^\vee \rangle + 2a + 2b + c$, with $a, b, c \geq 0$. First, they are equal only when $a = b = c = 0$ and thus we get $\langle \nu, \alpha_0^\vee \rangle = 2d < 18$. Since d is a non-negative integer, we must have $d \leq 8$, in which case $\langle \nu, \alpha_0^\vee \rangle \leq 16$ (with equality only for $d = 8$ and $\nu = 8\omega_4$).

It remains to investigate the case $\langle \tau\nu, \alpha_0^\vee \rangle \neq \langle \nu, \alpha_0^\vee \rangle$, for which $2a + 2b + c > 0$. It is readily verifiable that $2a + 2b + c \geq 2$ implies $\langle \nu, \alpha_0^\vee \rangle < 16$. Otherwise, $2a + 2b + c = 1$ and it immediately follows that $c = 1$ and $a = b = 0$. Therefore $\nu = \omega_3 + d\omega_4$, with $\langle \nu, \alpha_0^\vee \rangle = 3 + 2d < 17$. This inequality forces $d \leq 6$, in which case, $\langle \nu, \alpha_0^\vee \rangle \leq 15 < 16$, as claimed. $\square$

Remark 4.1.4. (a) Suppose $(\Phi, p) = (F_4, 2)$ and $\sigma = \tau^r$, for $r = 2s + 1$. Then, Lemma 4.1.3 shows that for $\lambda, \mu \in X_\sigma$ and

$$\mathrm{Ext}^1_G(L(\lambda) \otimes V(\nu)^{(r/2)}, L(\mu) \otimes \mathrm{H}^0(\nu)) \neq 0,$$

then $\langle \nu, \alpha_0^\vee \rangle \leq h + 4$.

(b) Let $\sigma : G \to G$ denote the appropriate strict endomorphism so that $G(\sigma)$ is a finite group of Lie type and G_σ the associated scheme-theoretic kernel. Then by (BNP06, Lemma 5.2), (BNP+15, Theorem 2.3.1) and a similar argument for Suzuki-Ree groups, for all (G, p, σ) aside from the case where $G = F_4$, $p = 2$ and σ is an exceptional isogeny, $\mathrm{Ext}^1_G(L(\lambda) \otimes V(\nu)^{(\sigma)}, L(\mu) \otimes \mathrm{H}^0(\nu)) \neq 0$ implies that ν is $(h-1)$-small. This implies that the situation in case (a) differs significantly from all of the situations discussed in (b).

By Lemma 4.1.3, we know that $\mathrm{Ext}^1_G(L(\lambda)\otimes V(\nu)^{(r/2)}, L(\mu)\otimes \mathrm{H}^0(\nu)) \neq 0$ implies $\langle \nu, \alpha_0^\vee\rangle < 17$. Thus, let us define $\Gamma \subseteq X_+$ to be the following set of dominant weights:

$$\Gamma = \{\nu \in X(T)_+ \mid \langle \nu, \alpha_0^\vee\rangle < 17\},$$

and let $\mathcal{G}_\Gamma(k)$ be the finite-dimensional truncated submodule of $\mathcal{G}(k)$ with composition factors with highest weights in Γ.

We obtain for $\lambda, \mu \in X_\sigma$,

$$\mathrm{Ext}^1_{G(\sigma)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\lambda), L(\mu)\otimes \mathcal{G}_\Gamma(k)). \tag{4.1.9}$$

4.2 Finite Group Extensions

Next, we make use of (4.1.9) and Theorem 4.1.1 to deduce some information concerning $\mathrm{Ext}^1_{G(\sigma)}(L(\lambda), L(\mu))$, under some conditions on the size of the finite group, ${}^2F_4(2^{2s+1})$ – the conditions will therefore be imposed on the value of s and hence $r = 2s+1$.

First, by (BNP06, (5.3.1)), we have for W a G-module with a filtration $0 = W_0 \subset W_1 \subset W_2 \subset \ldots \subset W_l = W$, for all G-modules V,

$$\dim \mathrm{Ext}^1_G(V, W) \leq \sum_{n=1}^{l} \dim \mathrm{Ext}^1_G(V, W_n/W_{n-1}). \tag{4.2.1}$$

We begin with an auxiliary result.

Lemma 4.2.1. Let $t = 2k+1$ be a positive integer, $\lambda, \mu \in X_{t/2}$. Then

(a) $\mathrm{Ext}^1_{G_{t/2}}(L(\lambda), L(\mu)\otimes L(\omega_4))^{(-t/2)}$ has weights that are 6-small.

(b) $\mathrm{Ext}^1_{G_{t/2}}(L(\lambda), L(\mu)\otimes L(\omega_1))^{(-t/2)}$ has weights that are 12-small.

In particular, $L(8\omega_4)$ cannot be a composition factor of $\mathrm{Ext}^1_{G_{t/2}}(L(\lambda), L(\mu)\otimes L(\gamma))^{(-t/2)}$, for $\gamma \in \{\omega_1, \omega_4\}$.

Proof. (a) First, consider the case $\gamma = \omega_4$. We apply the Lyndon-Hochschild-Serre spectral sequence corresponding to $G_\tau \lhd G_{t/2}$ and obtain

$$E_2^{i,j} := \mathrm{Ext}^i_{G_k}(L(\lambda_1), \mathrm{Ext}^j_{G_\tau}(L(\lambda_0), L(\mu_0)\otimes L(\omega_4))^{(-\tau)} \otimes L(\mu_1))^{(\tau)},$$

with the associated five-term exact sequence.

Then, for each pair (λ_0, μ_0), we consider the $E_2^{0,1}$ and $E_2^{1,0}$ terms. We have

$$\begin{aligned} E_2^{1,0} &= \mathrm{Ext}^1_{G_k}(L(\lambda_1), \mathrm{Hom}_{G_\tau}(L(\lambda_0), L(\mu_0) \otimes L(\omega_4))^{(-\tau)} \otimes L(\mu_1))^{(\tau)} \\ E_2^{0,1} &= \mathrm{Hom}_{G_k}(L(\lambda_1), \mathrm{Ext}^1_{G_\tau}(L(\lambda_0), L(\mu_0) \otimes L(\omega_4))^{(-\tau)} \otimes L(\mu_1))^{(\tau)}. \end{aligned}$$

In particular, we need to consider the weights of $\mathrm{Hom}_{G_\tau}(L(\lambda_0), L(\mu_0) \otimes L(\omega_4))^{(-\tau)}$ (found in the $E_2^{1,0}$ term) and $\mathrm{Ext}^1_{G_\tau}(L(\lambda_0), L(\mu_0) \otimes L(\omega_4))^{(-\tau)}$ (found in the $E^{0,1}$ term) and note the cases where there is a potential overlap.

First, notice that the G_τ-multiplicities $[L(\mu_0) \otimes L(\omega_4) : L(\lambda_0)]_{G_\tau}$ (from (Sin94a, Table 2)) are smaller than the dimension of the smallest nontrivial G-module, so

$$\mathrm{Hom}_{G_\tau}(L(\lambda_0), L(\mu_0) \otimes L(\omega_4))^{(-\tau)} \cong \mathrm{Hom}_{G_\tau}(L(\lambda_0), L(\mu_0) \otimes L(\omega_4)).$$

This was computed in the proof of (Sin94b, Lemma 4.7) and (Sin94b, Table IV) and we reproduce it for the reader's convenience.

$\mathrm{Hom}_{G_\tau}(L(\lambda_0), L(\mu_0) \otimes L(\omega_4))^{(-\tau)}$	0	ω_4	ω_3	$\omega_3 + \omega_4$
0	0	k	0	0
ω_4	k	k	0	0
ω_3	0	0	k	k
$\omega_3 + \omega_4$	0	0	k	$k \oplus k$

Table 4.1: (cf. (Sin94b, Table IV))

Then, notice that any weight of $\mathrm{Ext}^1_{G_\tau}(L(\lambda_0), L(\mu_0) \otimes L(\omega_4))^{(-\tau)}$, for $\lambda_0, \mu_0 \in X_\tau$ must be no higher than any weight of $\mathrm{H}^1(G_\tau, L(\theta))^{(-\tau)}$, for $L(\theta)$ a composition factor of $L(\lambda_0) \otimes L(\mu_0) \otimes L(\omega_4)$.

Note that when at least one of the weights λ_0, μ_0 is zero, the answer is given by (Sin94b, Lemma 4.5 (a)). When one of the weights λ_0, μ_0 is $\omega_3 + \omega_4$, then $\mathrm{Ext}^1_{G_\tau}(L(\lambda_0), L(\mu_0) \otimes L(\omega_4))^{(-\tau)} = 0$, as $L(\omega_3 + \omega_4)$ is an injective module for G_τ. Thus, suppose $\lambda_0, \mu_0 \notin \{0, \omega_3 + \omega_4\}$.

First, let $\lambda_0 = \mu_0 = \omega_4$. By inspection of (Sin94a, Table 2), $L(\omega_4) \otimes L(\omega_4) \otimes L(\omega_4)$ has composition factors $L(\theta)$, with $\theta \in \{0, \omega_4, \omega_1, \omega_3, 2\omega_4, \omega_1+\omega_4, \omega_2, \omega_3+\omega_4\}$. Now, we need to consider each choice of θ in turn and bound the possible weights of $\mathrm{H}^1(G_\tau, L(\theta))^{(-\tau)}$.

For $\theta \in X_\tau$, $\mathrm{H}^1(G_\tau, L(\theta))^{(-\tau)}$ was computed in (Sin94b, Lemma 4.5) and its weights are 2-

small. Then, for $\theta \in \{\omega_1, 2\omega_4, \omega_2\}$, since $\theta \in \tau X(T)$, it follows that $\mathrm{H}^1(G_\tau, L(\theta))^{(-\tau)} = 0$. Lastly, when $\theta = \omega_1 + \omega_4$, we obtain $\mathrm{H}^1(G_\tau, L(\theta))^{(-\tau)} \cong \mathrm{H}^1(G_\tau, L(\omega_4))^{(-\tau)} \otimes L(\omega_4) \cong L(\omega_4) \otimes L(\omega_4)$, which has weights that are 4-small. We conclude that $\mathrm{Ext}^1_{G_\tau}(L(\omega_4), L(\omega_4) \otimes L(\omega_4))^{(-\tau)}$ must have weights that are 4-small.

Then, suppose $(\lambda_0, \mu_0) \in (\omega_3, \omega_4)$. Again, by inspection of (Sin94a, Table 2), $L(\omega_3) \otimes L(\omega_4) \otimes L(\omega_4)$ has composition factors $L(\theta)$, with $\theta \in \{0,\ \omega_4,\ \omega_1,\ \omega_3,\ 2\omega_4,\ \omega_1 + \omega_4,\ \omega_2,\ \omega_3 + \omega_4,\ 3\omega_4,\ 2\omega_1,\ \omega_1 + \omega_3,\ \omega_1 + 2\omega_4,\ \omega_2 + \omega_4,\ 2\omega_3,\ \omega_3 + 2\omega_4\}$. Now, we need to consider each choice of θ in turn and bound the possible weights of $\mathrm{H}^1(G_\tau, L(\theta))^{(-\tau)}$.

Suppose $\theta = \omega_2 + \omega_4$ and we obtain $\mathrm{H}^1(G_\tau, L(\theta))^{(-\tau)} \cong \mathrm{H}^1(G_\tau, L(\omega_4))^{(-\tau)} \otimes L(\omega_3) \cong L(\omega_4) \otimes L(\omega_3)$, which has weights that are 5-small. All of the other cases follow analogously and yield weights that are at most 4-small. Then $\mathrm{Ext}^1_{G_\tau}(L(\omega_3), L(\omega_4) \otimes L(\omega_4))^{(-\tau)}$ must have weights that are 5-small.

Lastly, let $\lambda_0 = \mu_0 = \omega_3$. Again, by inspection of (Sin94a, Table 2), $L(\omega_3) \otimes L(\omega_4) \otimes L(\omega_4)$ has composition factors $L(\theta)$, with $\theta \in \{0,\ \omega_4,\ \omega_1,\ \omega_3,\ 2\omega_4,\ \omega_1 + \omega_4,\ \omega_2,\ \omega_3 + \omega_4,\ 3\omega_4,\ 2\omega_1,\ \omega_1 + \omega_3,\ \omega_1 + 2\omega_4,\ \omega_2 + \omega_4,\ 2\omega_3,\ \omega_3 + 2\omega_4,\ \omega_1 + \omega_2,\ \omega_1 + \omega_3 + \omega_4,\ 2\omega_1 + \omega_4,\ \omega_2 + \omega_3,\ 4\omega_4,\ 2\omega_3 + \omega_4,\ \omega_1 + 3\omega_4\}$. A similar calculation shows that $\mathrm{H}^1(G_\tau, L(\theta))^{(-\tau)}$ has weights that are 6-small. Therefore, $\mathrm{Ext}^1_{G_\tau}(L(\omega_3), L(\omega_3) \otimes L(\omega_4))^{(-\tau)}$ must have weights that are 6-small.

We record the potential weights found above in the following table, noting that $\{4, 5, 6\}$ denote the fact that the weights of $\mathrm{Ext}^1_{G_\tau}(L(\lambda_0), L(\mu_0) \otimes L(\omega_4))^{(-\tau)}$ are 4-small, 5-small or 6-small, respectively. Now, we need to compute the $E_2^{0,1}$ and $E_2^{i,0}$-terms in the spectral

$\mathrm{Ext}^1_{G_\tau}(L(\lambda_0), L(\mu_0) \otimes L(\omega_4))^{(-\tau)}$	k	$L(\omega_4)$	$L(\omega_3)$	$L(\omega_3 + \omega_4)$
k	$L(\omega_4)$	0	k	0
$L(\omega_4)$	0	4	5	0
$L(\omega_3)$	k	5	6	0
$L(\omega_3 + \omega_4)$	0	0	0	0

Table 4.2

sequence. First, we turn our attention to the cases where $\mathrm{Hom}_{G_\tau}(L(\lambda_0), L(\mu_0) \otimes L(\omega_4))^{(-\tau)}$ (found in the $E_2^{i,0}$ term) and $\mathrm{Ext}^1_{G_\tau}(L(\lambda_0), L(\mu_0) \otimes L(\omega_4))^{(-\tau)}$ (found in the $E_2^{0,1}$ term) are not both non-zero.

Suppose $\lambda_0 = \mu_0 = 0$. We obtain

$$E_2^{i,0} = \mathrm{Ext}^i_{G_k}(L(\lambda_1), \mathrm{Hom}_{G_\tau}(k, L(\omega_4))^{(-\tau)} \otimes L(\mu_1))^{(\tau)} = 0,$$

and then

$$E_\infty^1 \cong E_2^{0,1} = \mathrm{Hom}_{G_k}(L(\lambda_1), \mathrm{Ext}^1_{G_\tau}(k, L(\omega_4))^{(-\tau)} \otimes L(\mu_1))^{(\tau)}$$
$$\cong \mathrm{Hom}_{G_k}(L(\lambda_1), L(\omega_4) \otimes L(\mu_1))^{(\tau)}.$$

Note that any weight $2^k\gamma$ of $\mathrm{Hom}_{G_k}(L(\lambda_1), L(\omega_4) \otimes L(\mu_1))$ must satisfy $\langle \lambda_1 + 2^k\gamma, \alpha_0^\vee\rangle \leq \langle \mu_1 + \omega_4, \alpha_0^\vee\rangle$. Without loss of generality, we may assume $\langle \lambda_1, \alpha_0^\vee\rangle \leq \langle \mu_1, \alpha_0^\vee\rangle$ and we get $\langle \gamma, \alpha_0^\vee\rangle \leq 2^{1-k}$. Thus, since $k \geq 1$, we necessarily have $\langle \gamma, \alpha_0^\vee\rangle = 0$. Then $E_\infty^1 \cong E_2^{0,1}$, which has trivial G-structure.

Similarly, in the case $(\lambda_0, \mu_0) = (0, \omega_3)$, we have $E_\infty^1 \cong E_2^{0,1} = \mathrm{Hom}_{G_k}(L(\lambda_1), L(\mu_1)) = k$ if $\lambda_1 = \mu_1$ and vanishes otherwise.

An analogous calculation for $(\lambda_0, \mu_0) = (\omega_3, \omega_4)$ yields $E_\infty^1 \cong E_2^{0,1} = \mathrm{Hom}_{G_k}(L(\lambda_1), L(\mu_1) \otimes L(\theta))^{(\tau)}k$, where $\langle \theta, \alpha_0^\vee\rangle \leq 5$. Then, any weight γ of $\mathrm{Hom}_{G_k}(L(\lambda_1), L(\theta) \otimes L(\mu_1))^{(-k)}$ must satisfy $\langle \gamma, \alpha_0^\vee\rangle \leq \frac{5}{2^k} \leq 2$.

It remains to consider the cases $\lambda_0 = \mu_0 = \omega_4$ and $\lambda_0 = \mu_0 = \omega_3$. Suppose $\lambda_0 = \mu_0 = \lambda_4$. By Table 4.1, we have $\mathrm{Hom}_{G_\tau}(L(\omega_4), L(\omega_4) \otimes L(\omega_4))^{(-\tau)} = k$ and, by Table 4.2, $\mathrm{Ext}^1_{G_\tau}(L(\omega_4), L(\omega_4) \otimes L(\omega_4))^{(-\tau)}$ has weights that are 4-small. Thus, we have $E_2^{i,0} = \mathrm{Ext}^i_{G_k}(L(\lambda_1), L(\mu_1))^{(\tau)}$ and $E^{0,1} = \mathrm{Hom}_{G_k}(L(\lambda_1), L(\mu_1) \otimes L(\theta))^{(\tau)}$, with $\langle \theta, \alpha_0^\vee\rangle \leq 4$.

Note that any weight of E_∞^1 is no higher than any weight of $E_2^{1,0}$ and $E_2^{0,1}$. By Theorem 3.4.8, any weight of $\mathrm{Ext}^1_{G_k}(L(\lambda_1), L(\mu_1))^{(-k)}$ is 2-small. A similar argument as the ones above shows that $\mathrm{Hom}_{G_k}(L(\lambda_1), L(\mu_1) \otimes L(\theta))$ has weights that are 2-small.

Finally, a similar calculation in the case $\lambda_0 = \mu_0 = \omega_3$ shows that the weights of $\mathrm{Ext}^1_{G_{t/2}}(L(\lambda), L(\mu) \otimes L(\omega_4))^{(-t/2)}$ must be 3-small.

(b) Now consider $\mathrm{Ext}^1_{G_{t/2}}(L(\lambda), L(\mu) \otimes L(\omega_1))^{(-t/2)}$. We run the Lydon-Hochschild-Serre spectral sequence corresponding to $G_\tau \lhd G_{r/2}$. The E_2-page is given by

$$E_2^{i,j} := \mathrm{Ext}^i_{G_k}(L(\lambda_1), \mathrm{Ext}^j_{G_\tau}(L(\lambda_0), L(\mu_0))^{(-\tau)} \otimes L(\omega_4) \otimes L(\mu_1))^{(\tau)},$$

First, supposing $\lambda_0 = \mu_0$, we obtain

$$E_2^{0,1} = \mathrm{Hom}_{G_k}(L(\lambda_1), \mathrm{Ext}^1_{G_\tau}(L(\lambda_0), L(\lambda_0))^{(-\tau)} \otimes L(\omega_4) \otimes L(\mu_1))^{(\tau)} = 0,$$

since $\mathrm{Ext}^1_{G_\tau}(L(\lambda_0), L(\lambda_0))^{(-\tau)} = 0$. Then,

$$E_\infty^1 \cong E_2^{1,0} = \mathrm{Ext}^1_{G_k}(L(\lambda_1), L(\omega_4) \otimes L(\mu_1))^{(\tau)}.$$

Since $L(\omega_4) \cong \mathrm{H}^0(\omega_4)$, for our purposes, it suffices to invoke (BNP06, (5.2.4)) to conclude that $E_2^{1,0}$ has weights that are 12-small.

Then, suppose $\lambda_0 \neq \mu_0$. It immediately follows that $E_2^{i,0} = 0$ for $i > 0$. Thus,

$$E_\infty^1 \cong E_2^{0,1} = \mathrm{Hom}_{G_k}(L(\lambda_1), \mathrm{Ext}^1_{G_\tau}(L(\lambda_0), L(\mu_0))^{(-\tau)} \otimes L(\omega_4) \otimes L(\mu_1))^{(\tau)}.$$

Note that $E^{0,1} \neq 0$ if and only if $\mathrm{Ext}^1_{G_\tau}(L(\lambda_0), L(\mu_0))^{(-\tau)} \neq 0$. Thus, it remains to consider the cases $(\lambda_0, \mu_0) \in \{(0, \omega_4), (0, \omega_3), (\omega_4, \omega_3)\}$.

Let $(\lambda_0, \mu_0) = (0, \omega_4)$. Then $E_\infty^1 \cong E_2^{0,1} = \mathrm{Hom}_{G_k}(L(\lambda_1) \otimes L(\omega_4), L(\omega_4) \otimes L(\mu_1))^{(\tau)}$, which has trivial G-structure.

When $(\lambda_0, \mu_0) = (0, \omega_3)$, we obtain

$$E_\infty^1 \cong E_2^{0,1} = \mathrm{Hom}_{G_k}(L(\lambda_1), L(\mu_1))^{(\tau)} \oplus \mathrm{Hom}_{G_k}(L(\lambda_1), L(\mu_1) \otimes L(\omega_1))^{(\tau)},$$

which has trivial G-structure.

Lastly, $(\lambda_0, \mu_0) = (\omega_4, \omega_3)$ gives $E_\infty^1 \cong E_2^{0,1} = \mathrm{Hom}_{G_k}(L(\lambda_1), L(\mu_1))^{(\tau)} = k$ if $\lambda_1 = \mu_1$ and zero otherwise. Thus, our claim follows. $\square$

Proposition 4.2.2. Let $s \geq 6$, such that $r \geq 13$. Let $\lambda, \mu \in X_{r/2}$ and $\Gamma' = \Gamma \setminus \{0\}$. Then, the following hold:

(a) We have

$$\dim \mathrm{Ext}^1_{G(\sigma)}(L(\lambda), L(\mu)) \leq \dim \mathrm{Ext}^1_G(L(\lambda), L(\mu)) + \dim R,$$

where

$$\begin{aligned} R &\cong \bigoplus_{\nu \in \Gamma'} \mathrm{Ext}^1_G(L(\lambda) \otimes V(\nu)^{(r/2)}, L(\mu) \otimes \mathrm{H}^0(\nu)) \\ &\cong \bigoplus_{\nu \in \Gamma'} \mathrm{Hom}_{G/G_{r/2}}(V(\nu)^{(r/2)}, \mathrm{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu) \otimes \mathrm{H}^0(\nu))). \end{aligned}$$

(b) Let $\frac{7}{2} \leq \frac{t}{2} \leq \frac{2s-7}{2}$. Set $\lambda = \lambda_0 + \tau^t \lambda_1$ and $\mu = \mu_0 + \tau^t \mu_1$ with $\lambda_0, \mu_0 \in X_{t/2}$ and $\lambda_1, \mu_1 \in X_{\frac{r-t}{2}}$. Then we may re-identify R as

$$\begin{aligned} R &\cong \bigoplus_{\nu \in \Gamma'} \mathrm{Ext}^1_G(L\left(\lambda_1\right) \otimes V(\nu)^{(\frac{r-t}{2})}, L\left(\mu_1\right)) \otimes \mathrm{Hom}_G(L(\lambda_0), L(\mu_0) \otimes \mathrm{H}^0(\nu)) \\ &\cong \bigoplus_{\nu \in \Gamma'} \mathrm{Hom}_G(V(\nu)^{(\frac{r-t}{2})}, \mathrm{Ext}^1_{G_{\frac{r-t}{2}}}(L(\lambda_1), L(\mu_1))) \otimes \mathrm{Hom}_G(L(\lambda_0), L(\mu_0) \otimes \mathrm{H}^0(\nu)). \end{aligned}$$

Proof. (a) Note that by the previous discussion and Theorem 4.1.1, $\mathcal{G}_\Gamma(k)$ has a filtration with factors of the form $\mathrm{H}^0(\nu) \otimes \mathrm{H}^0(\nu^*)^{(r/2)}$, exactly one for each $\nu \in \Gamma$.

Now, by (4.1.9) and (4.2.1), we obtain

$$\begin{aligned}\dim \operatorname{Ext}^1_{G(\sigma)}(L(\lambda), L(\mu)) &= \dim \operatorname{Ext}^1_G(L(\lambda), L(\mu)\otimes \mathcal{G}_\Gamma(k))\\ &\leq \sum_{\nu\in\Gamma} \dim \operatorname{Ext}^1_G(L(\lambda)\otimes V(\nu)^{(r/2)}, L(\mu)\otimes \mathrm{H}^0(\nu))\\ &= \dim \operatorname{Ext}^1_G(L(\lambda), L(\mu)) + \\ &\qquad \sum_{\nu\in\Gamma'} \dim \operatorname{Ext}^1_G(L(\lambda)\otimes V(\nu)^{(r/2)}, L(\mu)\otimes \mathrm{H}^0(\nu)).\end{aligned}$$

The first isomorphism is an immediate consequence of (4.1.9) and the properties of $\mathcal{G}_{\Gamma'}(k)$. For the other isomorphism, note that since $2^s \geq 2^5 > 17$, we may apply Lemma 4.1.2 (a) to conclude that $\operatorname{Hom}_{G_{r/2}}(L(\lambda), L(\mu)\otimes \mathrm{H}^0(\nu))$ has trivial G-structure.

Now, let $M := \operatorname{Ext}^1_G(L(\lambda)\otimes V(\nu)^{(r/2)}, L(\mu)\otimes \mathrm{H}^0(\nu))$ and we run the Lyndon-Hochschild-Serre spectral sequence corresponding to $G_{r/2}\lhd G$. First, we investigate the $E_2^{i,0}$-term and we get

$$\begin{aligned}E_2^{i,0} &\cong \operatorname{Ext}^i_{G/G_{r/2}}(V(\nu)^{(r/2)}, \operatorname{Hom}_{G_{r/2}}(L(\lambda), L(\mu)\otimes \mathrm{H}^0(\nu)))\\ &\cong \operatorname{Ext}^i_G(V(\nu), k)\otimes \operatorname{Hom}_{G_{r/2}}(L(\lambda), L(\mu)\otimes \mathrm{H}^0(\nu)).\end{aligned}$$

By (Jan03, II.4.13), $\operatorname{Ext}^i_G(V(\nu), k) = 0$ for $i > 0$, so we conclude that the $E_2^{i,0}$-terms all vanish. Hence $M \cong E_2^{0,1}$, giving

$$R \cong \bigoplus_{\nu\in\Gamma'} \operatorname{Hom}_{G/G_{r/2}}(V(\nu)^{(r/2)}, \operatorname{Ext}^1_{G_{r/2}}(L(\lambda), L(\mu)\otimes \mathrm{H}^0(\nu))),$$

the desired result.

For (b), let λ and μ be expressed as suggested. We apply the Lyndon-Hochschild-Serre spectral sequence corresponding to $G_{t/2}\lhd G$ to the terms in the first expression for R in part (a). The E_2-page is given by

$$E_2^{i,j} := \operatorname{Ext}^i_{G/G_t}(L(\lambda_1)^{(t/2)}\otimes V(\nu)^{(r/2)}, \operatorname{Ext}^j_{G_{t/2}}(L(\lambda_0), L(\mu_0)\otimes \mathrm{H}^0(\nu))\otimes L(\mu_1)^{(t/2)}).$$

First we consider the $E_2^{0,1}$-term.

$$\begin{aligned}E_2^{0,1} &\cong \operatorname{Hom}_{G/G_{t/2}}(L(\lambda_1)^{(t/2)}\otimes V(\nu)^{(r/2)}, \operatorname{Ext}^1_{G_{t/2}}(L(\lambda_0), L(\mu_0)\otimes \mathrm{H}^0(\nu))\otimes L(\mu_1)^{(t/2)})\\ &\cong \operatorname{Hom}_G(L(\lambda_1)\otimes V(\nu)^{(\frac{r-t}{2})}, \operatorname{Ext}^1_{G_{t/2}}(L(\lambda_0), L(\mu_0)\otimes \mathrm{H}^0(\nu))^{(-t/2)}\otimes L(\mu_1)).\end{aligned}$$

We consider the cases $t = 2k$ even and $t = 2k+1$ odd in turn. We begin with $t = 2k$.

We have

$$E_2^{0,1} \cong \mathrm{Hom}_G(L(\lambda_1) \otimes V(\nu)^{(s-k+1/2)}, \mathrm{Ext}^1_{G_k}(L(\lambda_0), L(\mu_0) \otimes \mathrm{H}^0(\nu))^{(-k)} \otimes L(\mu_1)).$$

By (BNP06, (5.2.4)), any weight γ of $\mathrm{Ext}^1_{G_k}(L(\lambda_0), L(\mu_0) \otimes \mathrm{H}^0(\nu))^{(-k)}$ satisfies $\langle \gamma, \alpha_0^\vee \rangle \leq \frac{2^k-1}{2^k}(h-1) + \frac{\langle \nu, \alpha_0^\vee \rangle}{2^k} + \frac{3}{4} < h = 12$. Assume without loss of generality that $\langle \mu_1, \alpha_0^\vee \rangle \leq \langle \lambda_1, \alpha_0^\vee \rangle$. Therefore, $E_2^{0,1}$ vanishes unless $\langle \tau^{r-t}\nu, \alpha_0^\vee \rangle \leq \langle \gamma, \alpha_0^\vee \rangle$. We obtain $\langle \tau^{r-t}\nu, \alpha_0^\vee \rangle = 2^{s-k}\langle \tau\nu, \alpha_0^\vee \rangle \leq \langle \gamma, \alpha_0^\vee \rangle < 12$. Assuming $\nu \neq 0$, we have $\langle \tau\nu, \alpha_0^\vee \rangle \geq 2$, so $E_2^{0,1} = 0$, since $s - k \geq 3$. Thus, we have $E_\infty^1 \cong E_2^{1,0}$.

Then, suppose $t = 2k+1$. We have

$$E_2^{0,1} \cong \mathrm{Hom}_G(L(\lambda_1) \otimes V(\nu)^{(s-k)}, \mathrm{Ext}^1_{G_{t/2}}(L(\lambda_0), L(\mu_0) \otimes \mathrm{H}^0(\nu))^{(-t/2)} \otimes L(\mu_1)). \quad (4.2.2)$$

By (4.1.8), any weight γ of $\mathrm{Ext}^1_{G_{t/2}}(L(\lambda_0), L(\mu_0) \otimes \mathrm{H}^0(\nu))^{(-t/2)}$ satisfies $\langle \gamma, \alpha_0^\vee \rangle \leq 6 + \frac{2^{k+1}-1}{2^k-1} \cdot 5 + \frac{2^{k-1}}{2^k-1} \leq 16$. Assume without loss of generality that $\langle \mu_1, \alpha_0^\vee \rangle \leq \langle \lambda_1, \alpha_0^\vee \rangle$. Therefore, $E_2^{0,1}$ vanishes unless $2^{s-k}\langle \nu, \alpha_0^\vee \rangle \leq \langle \gamma, \alpha_0^\vee \rangle \leq 16$. We obtain $\langle \nu, \alpha_0^\vee \rangle \leq \frac{\langle \gamma, \alpha_0^\vee \rangle}{2^{s-k}} \leq 2$. Assuming $\nu \neq 0$, we have $\langle \tau\nu, \alpha_0^\vee \rangle \geq 2$, so $E_2^{0,1} = 0$, unless $\gamma = 8\omega_4$ and $\nu \in \{\omega_1, \omega_4\}$.

By Lemma 4.2.1, $L(8\omega_4)$ is not a composition factor of

$$\mathrm{Ext}^1_{G_{t/2}}(L(\lambda_0), L(\mu_0) \otimes \mathrm{H}^0(\nu))^{(-t/2)},$$

for $\nu \in \{\omega_1, \omega_4\}$, meaning that in (4.2.2), $E_2^{0,1} = 0$. Otherwise, if $\langle \nu, \alpha_0^\vee \rangle \geq 3$, $E_2^{0,1}$ also vanishes. Thus $E_\infty^1 \cong E_2^{1,0}$.

It remains to compute the $E_2^{1,0}$-term. We have

$$E_2^{1,0} \cong \mathrm{Ext}^1_{G/G_{t/2}}(L(\lambda_1)^{(t/2)} \otimes V(\nu)^{(r/2)}, \mathrm{Hom}_{G_{t/2}}(L(\lambda_0), L(\mu_0) \otimes \mathrm{H}^0(\nu)) \otimes L(\mu_1)^{(t/2)}).$$

By Lemma 4.1.2 (b), any weight γ of $\mathrm{Hom}_{G_{t/2}}(L(\lambda_0), L(\mu_0) \otimes \mathrm{H}^0(\nu))^{(-t/2)}$ satisfies $\langle \tau^t\gamma, \alpha_0^\vee \rangle \leq \langle \nu, \alpha_0^\vee \rangle \leq 16$.

In either case, $t = 2k$ or $t = 2k+1$, γ can only be non-trivial when $\nu = 8\omega_4$; then $\gamma \in \{0, \omega_1, \omega_4\}$ for $t = 2k$ or $\gamma \in \{0, \omega_4\}$ for $t = 2k+1$. Therefore, suppose $\nu = 8\omega_4$ and we apply the Lyndon-Hochschild-Serre spectral sequence corresponding to $G_{\frac{r-t}{2}} \lhd G$ to $\mathrm{Ext}^1_G(L(\lambda_1) \otimes V(8\omega_4)^{(\frac{r-t}{2})}, L(\gamma) \otimes L(\mu_1))$. The E_2-page is given by

$$E_2^{i,j} := \mathrm{Ext}^i_G(V(8\omega_4), \mathrm{Ext}^j_{G_{\frac{r-t}{2}}}(L(\lambda_1), L(\gamma) \otimes L(\mu_1))).$$

Note that since $s - k \geq 3$, $\mathrm{Hom}_{G_{\frac{r-t}{2}}}(L(\lambda_1), L(\gamma) \otimes L(\mu_1))$ has trivial G-structure, so the

$E_2^{i,0}$-term vanishes. Thus, we get $E_\infty^1 \cong E_2^{0,1} = \mathrm{Hom}_G(V(8\omega_4), \mathrm{Ext}^1_{G_{\frac{r-t}{2}}}(L(\lambda_1), L(\gamma) \otimes L(\mu_1)))$. By Lemma 4.2.1, this also vanishes.

Therefore, we may assume $\langle \nu, \alpha_0^\vee \rangle \leq 15$. Thus, any weight of $\mathrm{Hom}_{G_{t/2}}(L(\lambda_0), L(\mu_0) \otimes \mathrm{H}^0(\nu))$ must satisfy $\langle \tau^t\gamma, \alpha_0^\vee \rangle \leq \langle \nu, \alpha_0^\vee \rangle \leq 15$. It follows that in either case, $\langle \gamma, \alpha_0^\vee \rangle \leq \frac{\langle \nu, \alpha_0^\vee \rangle}{2} \leq \frac{15}{8} < 2$, so necessarily $\gamma = 0$.

We may conclude that

$$\begin{aligned} E_2^{1,0} &\cong \mathrm{Ext}^1_{G/G_{t/2}}(L(\lambda_1)^{(t/2)} \otimes V(\nu)^{(r/2)}, L(\mu_1)^{(t/2)}) \otimes \mathrm{Hom}_{G_{t/2}}(L(\lambda_0), L(\mu_0) \otimes \mathrm{H}^0(\nu)) \\ &\cong \mathrm{Ext}^1_G(L(\lambda_1) \otimes V(\nu)^{(\frac{r-t}{2})}, L(\mu_1)) \otimes \mathrm{Hom}_G(L(\lambda_0), L(\mu_0) \otimes \mathrm{H}^0(\nu)). \end{aligned}$$

This is the first re-identification. Now, consider $\mathrm{Ext}^1_G(L(\lambda_1) \otimes V(\nu)^{(\frac{r-t}{2})}, L(\mu_1))$ and we apply the Lyndon-Hochschild-Serre spectral sequence corresponding to $G_{\frac{r-t}{2}} \lhd G$. First, consider the $E_2^{i,0}$-term, for $i > 0$. We obtain

$$E_2^{i,0} \cong \mathrm{Ext}^i_{G/G_{\frac{r-t}{2}}}(V(\nu)^{(\frac{r-t}{2})}, \mathrm{Hom}_{G_{\frac{r-t}{2}}}(L(\lambda_1), L(\mu_1))).$$

Then, there are two possibilities – either $\lambda_1 = \mu_1$ or not. If they are not equal, it follows that $\mathrm{Hom}_{G_{\frac{r-t}{2}}}(L(\lambda_1), L(\mu_1))$ automatically vanishes, so $E_2^{1,0} = E_2^{2,0} = 0$. If they are equal, $\mathrm{Hom}_{G_{\frac{r-t}{2}}}(L(\lambda_1), L(\lambda_1))$ has trivial G-structure, and, once again $E_2^{1,0}$ and $E_2^{2,0}$ vanish, as $\mathrm{Ext}^i_G(V(\nu), k) = 0$, $i > 0$ (cf. (Jan03, II.4.13)). We may now conclude that

$$\mathrm{Ext}^1_G(L(\lambda_1) \otimes V(\nu)^{(\frac{r-t}{2})}, L(\mu_1)) \cong \mathrm{Hom}_{G/G_{\frac{r-t}{2}}}(V(\nu)^{(\frac{r-t}{2})}, \mathrm{Ext}^1_{G_{\frac{r-t}{2}}}(L(\lambda_1), L(\mu_1))),$$

and this completes the proof. □

Corollary 4.2.3. With the hypothesis of the previous proposition, there exists an isomorphism $\mathrm{Ext}^1_{G(\sigma)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\lambda), L(\mu))$ if either of the following hold:

(i) $\mathrm{Ext}^1_{G_{\frac{r-t}{2}}}(L(\lambda_1), L(\mu_1)) = 0$

(ii) $\mathrm{Hom}_G(L(\lambda_0), L(\mu_0) \otimes \mathrm{H}^0(\nu)) = 0$, for all $\nu \in \Gamma'$.

Next, we provide an analogue of (BNP06, Theorem 5.4) showing that generically, for the Ree groups of type F_4, self-extensions between simple modules vanish.

Theorem 4.2.4. Let $r = 2s + 1$ be odd with $s \geq 6$. Then

$$\mathrm{Ext}^1_{G(\sigma)}(L(\lambda), L(\lambda)) = 0,$$

for all $\lambda \in X_\sigma$.

Proof. We know that self-extensions for classical Frobenius kernels vanish, as G is not of type C_n (cf. (Jan03, II.12.9)); hence $\operatorname{Ext}^1_{G_s}(L(\lambda), L(\lambda)) = 0$ for any $\lambda \in X_s$.

We aim to extend this result by replacing s with $r/2$. When $r = 1$ the result follows from (Sin94b, 1.7(1)(2),4.5). Suppose $r \neq 1$ and let $\lambda = \lambda_0 + \tau^{r-1}\lambda_1 = \lambda_0 + 2^s\lambda_1$ with $\lambda_1 \in X_\tau$. We apply the Lyndon-Hochschild-Serre spectral sequence corresponding to $G_s \lhd G_{r/2}$. The E_2-page is given by

$$E_2^{i,j} = \operatorname{Ext}^i_{G_{r/2}/G_s}(L(\lambda_1)^{(s)}, \operatorname{Ext}^j_{G_s}(L(\lambda_0), L(\lambda_0)) \otimes L(\lambda_1)^{(s)}).$$

First consider the $E_2^{1,0}$-term:

$$\begin{aligned} E_2^{1,0} &= \operatorname{Ext}^1_{G_\tau}(L(\lambda_1), \operatorname{Hom}_{G_s}(L(\lambda_0), L(\lambda_0))^{(-s)} \otimes L(\lambda_1)) \\ &\cong \operatorname{Ext}^1_{G_\tau}(L(\lambda_1), L(\lambda_1))^{(s)} = 0, \end{aligned}$$

by the discussion above. Now, we turn our attention to the $E_2^{0,1}$-term, which is isomorphic to

$$\operatorname{Hom}_{G_\tau}(L(\lambda_1), \operatorname{Ext}^1_{G_s}(L(\lambda_0), L(\lambda_0))^{(-s)} \otimes L(\lambda_1))^{(s)} = 0,$$

by (Jan03, II.12.9). Therefore, $\operatorname{Ext}^1_{G_{r/2}}(L(\lambda), L(\lambda)) = 0$, for any $\lambda \in X_{r/2}$.

Having evaluated the self-extensions for $G_{r/2}$, we now express λ as $\lambda = \lambda_0 + \tau^t\lambda_1$, with $\lambda_0 \in X_{t/2}$ and $\lambda_1 \in X_{\frac{r-t}{2}}$. Then, since $s \geq 6$ and $\operatorname{Ext}^1_{G_{\frac{r-t}{2}}}(L(\lambda_1), L(\lambda_1)) = 0$, we may apply Corollary 4.2.3(i) and the claim follows. $\square$

Finally, the following theorem relates extensions between simple $kG(\sigma)$-modules and extensions between simple G-modules.

Theorem 4.2.5. Assume $r = 2s + 1$ with $s \geq 6$. Given $\lambda, \mu \in X_\sigma$, let

$$\begin{aligned} \lambda &= \sum_{i=0}^{r-1} \tau^i \lambda_{i/2} \\ &= \lambda_0 + \tau\lambda_{1/2} + 2\lambda_1 + \tau^3\lambda_{3/2} + \cdots + 2^s\lambda_{(r-1)/2} \end{aligned}$$

be the τ-adic expansion of λ, and take a similar expression for μ. Then there exists an integer $0 \leq n < r$ such that

$$\operatorname{Ext}^1_{G(\sigma)}(L(\lambda), L(\mu)) \cong \operatorname{Ext}^1_G(L(\tilde{\lambda}), L(\tilde{\mu}))$$

where
$$\tilde{\lambda} = \sum_{i=0}^{n-1} \tau^i \lambda_{\frac{i+r-n}{2}} + \sum_{i=n}^{r-1} \tau^i \lambda_{\frac{i-n}{2}}.$$

Proof. We express λ and $\tilde{\lambda}$ in this way, motivated by the fact that $V^{(r/2)} \cong_{G(\sigma)} V$ for any $G(\sigma)$-module V. Hence, applying Steinberg's Tensor Product Theorem leads to the isomorphism $L(\tilde{\lambda}) \cong_{G(\sigma)} L(\lambda)^{(n/2)}$.

By (Sin94a, 2.1(c)) there is an injection $\mathrm{Ext}^1_G(L(\tilde{\lambda}), L(\tilde{\mu})) \hookrightarrow \mathrm{Ext}^1_{G(\sigma)}(L(\tilde{\lambda}), L(\tilde{\mu}))$ and since τ is an automorphism of $G(\sigma)$, we have $\mathrm{Ext}^1_{G(\sigma)}(L(\tilde{\lambda}), L(\tilde{\mu})) \cong \mathrm{Ext}^1_{G(\sigma)}(L(\lambda), L(\mu))$. Thus it suffices to show (by dimensions) that there is also an injection

$$\mathrm{Ext}^1_{G(\sigma)}(L(\tilde{\lambda}), L(\tilde{\mu})) \hookrightarrow \mathrm{Ext}^1_G(L(\tilde{\lambda}), L(\tilde{\mu})).$$

First, suppose $\lambda = \mu$ and the claim follows from Theorem 4.2.4 with $n = 0$. Now assume $\lambda \neq \mu$. Then there exists $0 \leq i \leq r$ such that $\lambda_{i/2} \neq \mu_{i/2}$. Due to the discussion above, we may choose the integer n such that the differing digits in the τ-adic expansion of $\tilde{\lambda}$ and $\tilde{\mu}$ are in a certain position, namely $\tilde{\lambda}_{\frac{2s-8}{2}} \neq \tilde{\mu}_{\frac{2s-8}{2}}$. Thus, put $n = 2s - 8 - i$ if $i \leq 2s - 8$ and $n = r + 2s - 8 - i$ if $i \geq 2s - 8$. Therefore, we write $\tilde{\lambda} = \lambda' + \tau^{2s-8}\lambda'' + \tau^{2s-7}\lambda'''$ with $\lambda' \in X_{s-4}$, $\lambda'' = \tilde{\lambda}_{\frac{2s-8}{2}}$ and $\lambda''' \in X_4$, and take a similar expression for μ.

Then, we apply Proposition 4.2.2(b) with $\frac{t}{2} = \frac{2s-7}{2}$. Thus

$$\dim \mathrm{Ext}^1_{G(\sigma)}(L(\tilde{\lambda}), L(\tilde{\mu})) \leq \dim \mathrm{Ext}^1_G(L(\tilde{\lambda}), L(\tilde{\mu})) + \dim R,$$

where R is isomorphic to

$$\bigoplus_{\nu \in \Gamma'} \mathrm{Ext}^1_G(L(\lambda''') \otimes V(\nu)^{(4)}, L(\mu''')) \otimes \mathrm{Hom}_G(L(\lambda' + 2^{s-4}\lambda''), L(\mu' + 2^{s-4}\mu'') \otimes \mathrm{H}^0(\nu)).$$

We turn our attention to $\mathrm{Hom}_G(L(\lambda' + 2^{s-4}\lambda''), L(\mu' + 2^{s-4}\mu'') \otimes \mathrm{H}^0(\nu))$. We have

$$\begin{aligned} &\mathrm{Hom}_G(L(\lambda') \otimes L(\lambda'')^{(s-4)}, L(\mu') \otimes L(\mu'')^{(s-4)} \otimes \mathrm{H}^0(\nu)) \\ &\quad \cong \mathrm{Hom}_{G/G_{s-4}}(L(\lambda'')^{(s-4)}, \mathrm{Hom}_{G_{s-4}}(L(\lambda'), L(\mu') \otimes \mathrm{H}^0(\nu)) \otimes L(\mu'')^{(s-4)}). \end{aligned}$$

Consider $2^{s-4}\theta$ a weight of $\mathrm{Hom}_{G_{s-4}}(L(\lambda'), L(\mu') \otimes \mathrm{H}^0(\nu))$ and by Lemma 4.1.2 (b), it follows that $\langle 2^{s-4}\theta, \alpha_0^\vee \rangle \leq \langle \nu, \alpha_0^\vee \rangle$. Thus $\langle \theta, \alpha_0^\vee \rangle \leq \frac{\langle \nu, \alpha_0^\vee \rangle}{2^{s-4}} \leq \frac{16}{2^2} \leq 4$. However, note that the proof of Lemma 4.1.3 implies $\langle \nu, \alpha_0^\vee \rangle \leq 15$. Therefore, $\langle \theta, \alpha_0^\vee \rangle \leq \frac{15}{2^2}$, so $\langle \theta, \alpha_0^\vee \rangle \leq 3$. Thus, $\theta \in \{0, \omega_1, \omega_3, \omega_4\}$.

First, suppose $\mathrm{Hom}_{G_{s-4}}(L(\lambda'), L(\mu') \otimes \mathrm{H}^0(\nu))$ has trivial G-structure, and we may write

$$\begin{aligned}
&\mathrm{Hom}_G(L(\lambda') \otimes L(\lambda'')^{(s-4)}, L(\mu') \otimes L(\mu'')^{(s-4)} \otimes \mathrm{H}^0(\nu)) \\
&\quad \cong \mathrm{Hom}_{G/G_{s-4}}(L(\lambda'')^{(s-4)}, \mathrm{Hom}_{G_{s-4}}(L(\lambda'), L(\mu') \otimes \mathrm{H}^0(\nu)) \otimes L(\mu'')^{(s-4)}) \\
&\quad \cong \mathrm{Hom}_G(L(\lambda''), L(\mu'')) \otimes \mathrm{Hom}_G(L(\lambda'), L(\mu') \otimes \mathrm{H}^0(\nu)).
\end{aligned}$$

Since $\lambda'' = \tilde{\lambda}_{\frac{2s-8}{2}} \neq \tilde{\mu}_{\frac{2s-8}{2}} = \mu''$, all of the corresponding summands of R vanish.

It remains to consider the case in which $2^{s-4}\theta$ is a potential non-zero weight of $\mathrm{Hom}_{G_{s-4}}(L(\lambda'), L(\mu') \otimes \mathrm{H}^0(\nu))$. We obtain

$$\begin{aligned}
&\mathrm{Hom}_G(L(\lambda') \otimes L(\lambda'')^{(s-4)}, L(\mu') \otimes L(\mu'')^{(s-4)} \otimes \mathrm{H}^0(\nu)) \\
&\quad \cong \mathrm{Hom}_{G/G_{s-4}}(L(\lambda'')^{(s-4)}, \mathrm{Hom}_{G_{s-4}}(L(\lambda'), L(\mu') \otimes \mathrm{H}^0(\nu)) \otimes L(\mu'')^{(s-4)}) \\
&\quad \cong \mathrm{Hom}_G(L(\lambda''), L(\theta) \otimes L(\mu'')).
\end{aligned}$$

Moreover, we know that $\lambda'' = \tilde{\lambda}_{\frac{2s-8}{2}} \neq \tilde{\mu}_{\frac{2s-8}{2}} = \mu'' \in X_\tau$. Then, careful consideration using (Sin94b, Table V) shows that there exist λ'', μ'' such that $\mathrm{Hom}_G(L(\lambda''), L(\theta) \otimes L(\mu'')) \neq 0$, for $\theta \in \{0, \omega_1, \omega_3, \omega_4\}$.

Thus, since we cannot yet conclude that the summand of R corresponding to the aforementioned cases vanishes, we must turn our attention to $\mathrm{Ext}^1_G(L(\lambda''') \otimes V(\nu)^{(4)}, L(\mu'''))$, for $\lambda''', \mu''' \in X_4$. We run the Lyndon-Hochschild-Serre spectral sequence corresponding to $G_4 \lhd G$. First, consider the $E_2^{i,0}$-term for $i > 0$:

$$E_2^{i,0} := \mathrm{Ext}^1_{G/G_4}(V(\nu)^{(4)}, \mathrm{Hom}_{G_4}(L(\lambda'''), L(\mu'''))).$$

Since $\mathrm{Hom}_{G_4}(L(\lambda'''), L(\mu'''))$ is either zero or has trivial G-structure, it follows that $E_2^{1,0} = E_2^{2,0} = 0$, so we have $E_\infty^1 \cong E_2^{0,1}$. Thus

$$\begin{aligned}
&\mathrm{Ext}^1_G(L(\lambda''') \otimes V(\nu)^{(4)}, L(\mu''')) \\
&\qquad \cong \mathrm{Hom}_G(V(\nu), \mathrm{Ext}^1_{G_4}(L(\lambda'''), L(\mu'''))^{(-4)}).
\end{aligned}$$

Then, notice that by Lemma 3.4.8, any weight ζ of $\mathrm{Ext}^1_{G_4}(L(\lambda'''), L(\mu'''))^{(-4)}$ must satisfy $\langle \zeta, \alpha_0^\vee \rangle \leq 2$. Recall that we needed to consider the cases for which $\frac{\langle \nu, \alpha_0^\vee \rangle}{2^{s-4}} \neq 0$, so when $\langle \nu, \alpha_0^\vee \rangle \geq 2 \cdot 2^2 = 8$. Hence, note that $\mathrm{Hom}_G(V(\nu), \mathrm{Ext}^1_{G_4}(L(\lambda'''), L(\mu'''))^{(-4)}) = 0$, so we conclude that $\mathrm{Ext}^1_G(L(\lambda''') \otimes V(\nu)^{(4)}, L(\mu''')) = 0$. Thus, all of the summands vanish, giving $R = 0$ and the claim follows. □

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