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A comparison of a visual-spatial approach and a verbal approach to teaching early secondary school mathematics

PhD Thesis

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Declaration

I certify that all the material in this thesis which is not my own work has been identified and that no material is included which has been submitted for any other award or qualification.

Signed:

Date:

Preface

What is a number that a man may know it, and a man, that he may know a number?

(McCulloch, 1965, cited by Dehaene, 1992)

This quotation is used by Dehaene (1992) before he goes on to argue that psychology will be relevant to such a question, which has traditionally been considered to be more the concern of philosophy. Certainly if mathematics is to be understood as a part of human understanding, rather than as god-given knowledge, the discoveries and theories of cognitive psychology would appear to be worth considering. Furthermore, if the particular interest is in mathematics education, this will provide more necessity to involve psychological knowledge, since much has been built up around interest in learning and development. It would appear then that an investigation into learning mathematics must take into account characteristics of human thinking, so the subject areas of psychology, education and the philosophy of mathematics all need to be considered and their relationships examined.

This then is the foundational rationale to this research into the possibility of understanding mathematics through visual-spatial or verbal thinking. Although a particular influence on the author was classroom observation of children grasping mathematical ideas through visual representations and sometimes succeeding with visual tasks when they had failed equivalent problems involving words, this was all understood through the conception of mathematics, education and psychology as interdependent. For this reason, the introductory chapters (1-3) to this research aim to convey the wide range of ideas that are relevant to the investigation and to suggest some recurring themes. Only then does Chapter 4 describe the particular concerns and questions that the research hopes to address and the methodology chosen to achieve these aims. Chapter 5 details the results of the teaching experiment and then Chapter 6 attempts to relate these to the conceptions explored previously. An important intention was for the classroom research to be relevant to this broad understanding of mathematics and learning, as well as to the reality of teaching mathematics.

Abstract

Despite mathematicians valuing the ability to visualise a problem and psychologists finding positive correlations of visual-spatial ability with success in mathematics, many educationists remain unconvinced about the benefits of visualisation for mathematical understanding. This study compared a ‘visual’ to a ‘verbal’ teaching approach by teaching a range of early secondary school mathematics topics to two classes using one or other approach. The two classes were compared by considering their scores on a post-intervention test of mathematical competency, on which the verbally taught class scored significantly higher.

A major interest of the research was individual differences in underlying abilities or preferred learning styles, seen as underpinned by visual-spatial and verbal cognitive processes. A test was developed to measure participants’ general tendency to process information visually or verbally and the mathematics test results were also considered from the perspective of cognitive style. No interactions were found between teaching style and the learners’ preferred styles. The pupils identified as ‘visualisers’ did tend to perform more poorly on the mathematics test. However, further examination of the classroom performance and approaches taken to mathematics by these and other students led to doubt about the validity of the visualiser-verbaliser test used and indeed about the underlying constructs of visualiser and verbaliser cognitive styles.

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1. Mathematics

If the aim in teaching mathematics is that students achieve understanding, it is necessary to consider the nature of mathematical knowledge and understanding. The importance of tackling understanding and the nature of the learning process has been frequently emphasised. For example, Sierpinska and Lerman (1996) argue that because of the power of knowledge "an explicit engagement with the underlying epistemological assumptions of education, mathematics, teaching, learning and the child is an ethical requirement of the researcher and the teacher and others involved in education".

Yet the task of understanding a subject is presumably related to the nature of the subject, and perhaps this is particularly the case with mathematics where the required knowledge appears very different to other knowledge we hold about the world. Sfard (1991) takes this view, suggesting that difficulties with learning and teaching mathematics might be illuminated by considering the nature of mathematics and, in particular, by questioning what it is about mathematical entities that makes them hard to understand. Boaler (2002) argues that knowledge about mathematics as practiced by mathematicians could help to improve learning, since "prevailing dogma about what it means to know and be proficient in mathematics is extremely narrow in most countries". She suggests that the limited conception that many learners have of the nature of mathematics could be the root of many problems they experience with the subject.

There is a need then to consider the underlying nature of mathematics since this will be related to understanding in mathematics, perhaps giving insights into how such understanding can be achieved in the classroom.

1.1 The underlying nature of mathematics

To answer the question of what mathematics is, there would appear to be two obvious places to look for suggestions. These are philosophical inquiries into mathematics and the actual practice and beliefs of mathematicians. These are not mutually exclusive; Kitcher's (1984) theory about the nature of mathematical knowledge is based on the practice of mathematicians and any ideas about mathematics must be able to give reasonable explanations of what mathematicians are doing. In this vein, observing mathematicians may be informative and a number of researchers have interviewed mathematicians about their work (Burton, 1999; Sfard, 1994; Stylianou, 2002). However, it might be that mathematicians are not entirely aware of the underlying nature of mathematics through their practice and so they could hold beliefs about mathematics which are not entirely accurate. It would therefore be possible for there to be apparent contradictions between the philosophy and the practice of mathematics. This does in fact seem to be the case, and has been frequently remarked upon, with the theoretical ideas of philosophers and educationalists tending to be broadly constructivist, while mathematicians proceed in their work as if holding a realist position. It would appear that each position needs closer investigation before a way of accommodating both might be suggested.

1.11 Realism

A cheerful realism is presented by Gowers (2002) as the hallmark of mathematicians, "who either find it obvious that numbers exist or do not understand what is being asked". He further argues that they are right not to worry but should just get on with their mathematical work and it does indeed seem natural to assume that entities one works with have a straight-forward existence. Sierpinska and Lerman

(1996) make the point that mathematicians can avoid worrying about philosophy because at the technical level it is unimportant and holding different philosophical positions does not preclude communication.

However, just because an assumption allows one to get on with the work does not mean that it is an accurate assessment. Kitcher makes a related point when he argues (p61) that just because some mathematical truths seem self-evident does not mean that they actually are, as one cannot simply trust a feeling of transparency or direct knowledge. The philosophically casual mathematicians seem to be heading for a Platonic position in their conviction that mathematical entities have actual existence somewhere, but one that does not appear to depend on the physical world or on the actions of mathematicians. Kitcher describes the philosophical problems that result from simple Platonism and particularly from relying on the idea of apriori knowledge, which is independent of experience. It leads to explaining mathematical knowledge through "some mysterious intuition of abstract objects" (p91) and so makes the "sensible question" of why mathematics is useful in the real world "look like an unfathomable mystery" (p104/5).

Beyond the philosophical difficulties, Cobb et al (1992) highlight the main problem for a teacher with such a view of mathematics. This is that if certain concepts are assumed to be self-evident then the only teaching strategy is to be increasingly explicit, which they argue has been shown to fail. To continue to assume that concepts we now understand are somehow 'out there' leads, they complain, to continuing "to interpret the instructional problem as that of developing new and improved ways to express and transmit mathematical relationships that are self-evident to the expert". This tends to downgrade the activity and involvement of the learner and so risks ignoring a fundamental part of the learning process.

1.12 Constructivism

Since the activity of the learner is so obvious to those involved in teaching, educationalists have tended to adopt constructivist views of knowledge (Sfard, 1994). These have some philosophical underpinnings in the shape of Piaget's ideas about the growth of understanding but, as Cobb et al argue, they tend to be essentially vague ideas about constructing knowledge, which are held but not examined. As these authors go on to discuss, constructivism seems to suggest relativism about knowledge, when in fact the teacher has some very definite ideas about the outcomes required. The apparent hypocrisy of the teacher in claiming there is no absolute knowledge, then seemingly using their position to judge what is acceptable, leads Zevenbergen (1996) to claim that constructivism is a bourgeois justification for the influence of the powerful in deciding what counts as knowledge. Alternatively, though, the situation could just result from the nature of mathematical knowledge and the inadequacy of simple constructivism for explaining its accumulation. This is touched on by Zevenbergen's contention that the constructivism seen in mathematics education over-emphasises the individual, and other educationalists have indeed complained that constructivist explanations often ignore the social aspects of mathematics learning and the cultural side of the subject itself.

1.13 'Constructivist realism'

It would seem then that there is a need for an understanding of mathematics that can accommodate the entities mathematicians work with and the sense teachers have of shared concepts to be grasped, together with the individual mental construction of knowledge and understanding through engaging with mathematics. As Cobb et al put it, we need to sort out the conflict caused by the dualistic belief in both ‘maths in student’s head’ and ‘maths in the world’. As well as such a search for a solution being valid in its own right as an attempt at complete understanding, Cobb et al have shown that the lack of reconciliation is a problem for mathematicians and educators. Their solution is to see the culture of our society’s mathematical practices as the force that makes concepts seem self-evident, since it ensures that we have all made the relevant constructions and that they are similar enough for us to discuss them. If this seems too relativistic, potentially allowing mathematics to be anything agreed on by a culture, it is possible to use Kitcher’s arguments to ground the mathematical culture in the physical nature of the world. He claims that mathematical truths can all be traced back through the actions of the mathematical community to basic physical operations and so are constrained by the nature of the actual world. The physical constraints imposed by the world can be seen to include ourselves, as evolved animals in the world, and the resulting nature of our brains. Parallels can be drawn here with the explanation Johnson (1987) has for our understanding of the world, detailing the physical roots of many of our mental constructs, and describing his theory as ‘embodied’ realism. Similarly, Kitcher emphasises that his theory proposes a realist position, by claiming that mathematical truths are rooted in physical reality, but avoids the philosophical difficulties of self-evident truths. As has been noted, self-evident truths cause problems for educationalists too and this alternative version of realism also seems to accommodate the intuitions of teachers about people constructing knowledge. In some ways, teachers’ vaguely constructivist ideas seem to have more in common with Kitcher’s theory than with pure constructivism, in that he emphasises that mathematical operations are basically physical and not “private transactions in some inner medium”. This physical grounding seems similar to the intuitive feelings of mathematics teachers that pupils need solid items from which they can abstract mathematical understanding. Indeed, Presmeg (1992a) sees parallels between Johnson's 'body in the mind' and the 'internalization of bodily action' which is so central to the Piagetian ideas that have influenced educational practice.

1.2 Mathematical understanding

It would seem to be possible, then, to outline a broad conception of the nature of mathematics that accommodates the ideas and practice of mathematicians, on the one hand, and educationalists on the other, while avoiding philosophical contradictions and confusion. However this still leaves much about the detail of what can be considered to be mathematical understanding unresolved, although the theory of mathematics outlined above may be relevant in discussing some of these details.

1.21 Concrete and abstract

Dieudonné (1972) characterises mathematics as "a realm in which one worked only with abstractions, rather than the concrete reality of experience" (p100) and the abstract nature of mathematics does indeed seem vital to its power (see e.g. Gowers, 2003, p17-34). Furthermore, the idea that mathematics is founded on physical reality could be seen as supportive of the conception that the process of learning mathematics "begins with the concrete and 'ascends' (the metaphor is pertinent) to the abstract" (Noss et al, 1997). However, these authors dislike this idea of hierarchy and argue that actual mathematical reasoning goes back and forth from "formal to informal, analytic to perceptual, rigorous to intuitive".

Obviously just because the body of mathematical knowledge is rooted in the concrete does not mean that doing, or learning, mathematics need progress strictly from concrete to abstract. In fact there is plenty of evidence that it does not. Apart from the reports of practising mathematicians, mentioned by Noss et al, there are interpretations of learning situations which undermine this idea. Gravemeyer (1997) is one of many who point to a problem often experienced when using manipulatives in the classroom, arguing that "the mathematical concepts embodied in the didactical representations are only there for the experts who already have those concepts available to see". Taking a Kitcher-inspired view of mathematics suggests that this is overstating the case and that the concepts really are there. However, it does convey the difficulty of grasping them and argues against a simplistic progression within the individual learner from concrete objects to abstract structural understanding. Instead it seems likely that the beginnings of a concept have to be in a student's head for them to get anything from the manipulatives, but then the concrete procedures can help to develop the concepts, which in turn lead to further appreciation of the activities.

Similarly, it is sometimes assumed that because the concept of number is an abstraction from physical objects, the small child developing a number concept experiences this move from the concrete to the abstract. It is usually proposed that this is achieved through counting, which the child first learns as a routine activity, only later abstracting the commonalities of actual counting situations to reach numerical understanding. For example, Gray and Tall (1994) propose that the "process of counting [is] encapsulated as the concept of number". Yet this does not seem to explain how the child knows what to abstract, and so other researchers have appealed to an innate numerical tendency, or at least perceptual and cognitive biases, which support the early appreciation of small numbers. The extent to which these tendencies can be said to be concepts has led to researchers attempting to establish whether very young children, just beginning to count, appreciate certain principles about number (see Wynn, 1990, for a review of this argument). These investigations have only produced equivocal results, but it does seem reasonable to conclude that the individual child does not progress in a simple way from counting concrete objects to holding abstract number concepts (see Bruce & Threlfall, 2004 for a similar conclusion). Thus, although mathematics can be seen as progressing from concrete particulars to abstract generalities, we should not expect individual occasions of doing, or learning mathematics, to imitate this progression.

1.22 Process and concept

The tendency in the previous paragraph was to identify the process of counting with the concrete and number concepts with the abstract. This brings in a dichotomy even more popular in mathematics education than that between concrete and abstract entities, which is the distinction made between processes, or procedures, and concepts. The solution suggested to understanding how the concrete and the abstract relate was not to expect the general direction of mathematics to be mirrored by individuals so that their understanding can be classified as concrete or abstract, with the former neatly progressing to the latter. This has a parallel in the process-concept debate, with Sierpinska (1994) arguing that processes can not exist alone, but must have concepts to act on, although these might be rather weak or incomplete (p51). Elsewhere, we are reminded that proficiency with a process does not lead inevitably to the holding of the relevant concepts (Sfard, 1994; Gray and Tall, 1994).

However, the process-concept dichotomy is not merely a translation of the concrete-abstract distinction. In addition to being an ontological description of the mathematical entities, it can also function as a description of mathematical activity and understanding. A further important aspect is that the distinction is usually used to convey the levels of abstraction found in mathematics, where processes come to be seen as concepts, which are then used as processes to produce more abstract concepts. Sfard (1991) argues that this is possible because mathematical ideas can be understood as either processes or concepts (although she prefers the terms 'operational' and 'structural'), and in fact they are both, which might be what makes mathematics difficult. She is certainly pointing to a characteristic of mathematics that others have noticed and described in various ways. Davis (1984) considers that the way in which procedures are first carried out and only later can be scrutinised is like a verb becoming a noun. Dienes (1960) describes predicates becoming subjects, and argues that this 'taming' of predicates is the essence of mathematics. Wilder (1972) uses a slightly different linguistic metaphor when he talks about the transition of number words from adjective ('3 cats') to noun ('3'). Gray and Tall (1994) continue Sfard's idea that mathematical concepts are both concepts and process by coining the term 'procept' and arguing that appreciating a mathematical entity in this way is the key to success.

However, it must be noted that despite proclaimed attempts (Sfard, 1991) to value both sides of mathematical entities, there is still an implicit idea of hierarchy and progression, which reflects the more obvious direction of advance implied by the various analogies above. Sfard (1991) declares the structural and the operational "equal but different", yet then goes on to argue that mathematical development, both historically and of the individual, goes typically from operational to structural understanding. It is argued (Gray and Tall, 1994; Gray, 1991) that mathematical problems result when learners get stuck at the procedural level and fail to see entities as 'procepts'.

It would appear that there is a tendency to over-emphasise the conceptual side of mathematics, at the expense of processes, in much the same way that Noss et al (1997) believe that the abstract is overly elevated above the concrete. This produces similar contradictions and problems to those that they identified, but there would also seem to be further reasons not to overlook the processes of mathematics. A major one results from the understanding of the nature of mathematics developed previously, since the philosophical position of Kitcher, and others, depends fundamentally on the actions of mathematicians and the processes of doing mathematics. Kitcher draws attention to the fact that he emphasises processes over concepts, saying (p110) that he "replace[s] the notions of abstract mathematical objects, notions like that of a collect*ion*, with the notion of a kind of mathematical activity, collect*ing"*. Boaler (2002) suggests an educational implication of the tendency to forget about mathematical practice when she argues that it leads to the important mathematical action she calls "making connections" being overlooked in teaching materials. Also aware of the practical implications of investigating mathematical processes are the researchers who investigate the practice of mathematicians in the expectation that it will illuminate the nature of mathematics, but also that it will make suggestions for teaching (Burton, 1999; Stylianou, 2002).

1.23 Translation between ways of thinking

It would seem, then, that there are many reasons not to lose sight of the process part of mathematics and the place this has in complete understanding. If Sfard is correct in her assessment that it is the dual nature of mathematical entities that gives them their power, it is important that both aspects of mathematics are conveyed in the classroom. A practical example of this is Kieran's (1997) interpretation of the problems children often have with functions. She argues that these can arise from an exaggerated separation of 'algebra', in the form of unknowns and equations, from 'functions', which makes it difficult later to close the gap between the process side of functions (e.g. calculating according to a rule) and the conceptual side, involving graphical representation. Gray and Tall (1994) propose that mathematical symbolism particularly conveys the dual nature of mathematical entities, since the same notation is used to represent both a process and the product of the process. They argue that learners need to appreciate this ambiguity to be successful in mathematics.

It could be that the duality in the nature of mathematical entities is a particular case of a general aspect of mathematics. Sierpinska and Lerman (1996) quote Dieudonné (1992) as saying that a defining characteristic of mathematics is that it involves "different ways of speaking of a given concept" and translating between them. Other writers on mathematics have also drawn attention to this aspect of the subject, although with different distinctions in thought from the process-concept distinction explored above (e.g.Sierpinska, 1994). The alternative distinction usually made is between the visual, or geometric, side of mathematics and the verbal, or analytic, side. The two poles of this alternative dichotomy will now be considered.

1.3 The visual-verbal distinction in mathematics

1.31 The visual side of mathematics

It must be questioned what evidence there is for a visual side to mathematics. There are essentially two sorts of evidence; one comes from looking at people doing mathematics and considering the nature of the processes involved, while the other can be found by inquiring into the nature of mathematics, understood as a body of knowledge. Following Kitcher's theory of mathematics, the first of these sources is obviously important and could be seen as the essence of the other. However, there is also here a considerable overlap with psychology and theories of learning, which will be considered in more detail in Chapter 2. This will then be developed in Chapter 3 by the consideration of how mathematics and processes of visualisation can be understood and related. Here, therefore, the discussion will be limited to the visual aspects of mathematics that are implied by the history of mathematics and the body of knowledge accumulated, understood, as far as possible, in isolation from the on-going processes of mathematics.

The most explicit visual mathematics must be that involved in geometry, or shape and space, and the long history of this part of mathematics might suggest that it is a vital aspect. However, it could be concluded that such mathematics is only a primitive base, which true, analytic reasoning has gone beyond. This view is strongly opposed by Wilder (1972), who argues not only that Greek mathematicians achieved the first abstraction through geometry, which would be further developed analytically, but also that they "used geometry as a tool to do arithmetic and algebra". This point that visual techniques do not have to be tied to the specific and the concrete is frequently made by enthusiasts for the visual side of mathematics, often to encourage teachers to try to develop this side of their pupils. Examples may be given of successful visualising and individual images, and these will be considered further later, but other examples offered refer to the abstract ideas conveyed by visual techniques that are routinely employed. For instance, Arcavi (2003) talks about "conceptually rich images" that can convey lots of concepts, using as an example Cartesian co-ordinates. It may be concluded that not only mental images but also certain physical representations can convey abstract mathematical ideas.

It might be questioned why mathematics seems to have its roots in visual understanding, both historically and as a way of conveying much essential reasoning and knowledge. Kitcher traces all mathematics back to physical operations, which are mainly appreciated visually, so perhaps true mathematics is bound to have begun with geometry. For a more general explanation of why visual explanations work well for us as human beings, Johnson's (1987) theory of 'embodied reality' gives a prominent place to perception since this is such an important and fundamental way of learning about the world. The nature of human visual-spatial abilities will be further considered in Chapter 2 and at this stage it is enough to note that a rationale can be seen for them underpinning the human endeavour of mathematics. The extent of this influence will now be considered further.

As mathematics developed through history, visual aids were frequently important. They had practical uses, such as the use of the abacus, which can be seen as successful because of its ability, like Cartesian co-ordinates, to contain so many concepts. Sfard (1991) also mentions examples where visual representations, in the form of number lines and Argand diagrams, helped the further abstraction and development of mathematics, in these cases through assisting the understanding of negative and complex numbers. She conjectures that good visual representations might be similarly beneficial for individual learners, helping them to progress from an operational to a structural understanding. Similarly, Skemp (1987) proposes that visual thought is particularly appropriate to "integrate" ideas and convey "over all structure".

It has been pointed out that there is a visual side even to some of the aspects of mathematics usually considered verbal or analytic. For example, both Skemp (1987) and Davis (1984) discuss the visual-spatial side to mathematical symbolism, with Davis noting how the layout of mathematical reasoning can sometimes prompt the next stage in what he terms a "visually mediated sequence" (p35).

1.32 Language and mathematics

Having considered the influence of visual processes on mathematical understanding, it now seems appropriate to turn to the verbal, linguistic or analytic side of the subject. Obviously, these terms are not synonymous and this is one of the problems with reviewing the literature. Furthermore this will be a brief section because the linguistic side of mathematics is not the main focus of this research.

The most straightforward way that mathematics has a verbal side is in the need for communication (see Skemp, 1987, for discussion of the collective and social nature of verbal symbols) and this is particularly relevant for mathematics education. Thus various writers discuss the importance of using language carefully, with Kerslake (1991) stating: "Mathematics is hard for many people to learn; we do not make it any easier by using ill-defined words and by changing the interpretation of others without even a minimum acknowledgement". Sometimes words are used to convey metaphorical ideas and Nolder (1991) in the same anthology, considers some of the metaphors used in mathematics education. She notes that they stress particular aspects of an entity over others and will only work if the audience is familiar with the entity invoked in the comparison. Her detailed example concerns the likening of an equation to a balance, which will only succeed in conveying the characteristics of equations if learners are familiar with the analogous features of balances.

Such discussion of the communication of mathematics, though, sometimes makes assumptions about the nature of mathematical thinking and tends to identify effective communication with true understanding. For example, Orton (1999) notes with regret that "pupils often recognise pattern but lack the vocabulary to explain fully what they perceive" (p166). This seems a distinct over-extension of the idea of language as communication to seeing it as a necessary part of understanding, perhaps influenced by claims such as Vygotsky's (1986) that "real concepts are impossible without words". Yet the ability to step back, mentally, from the particular and understand the general is an ability valued by all mathematicians and probably this is what Orton desires for her pupils and seeing verbal description as the key. Whether such explicit, verbal understanding is necessarily so important in mathematical understanding, and in thinking generally, will be further considered in Chapters 2 and 3. It is enough here to note the sense some educationalists, in particular, have of its importance.

It is also important to be aware that other commentators see language as having a potentially damaging effect on mathematical understanding. Arnheim (1969, p.244) argues that "the function of language is essentially conservative and stabilising", useful for labelling and fixing concepts. It seems likely that this could be detrimental when dealing with the duality of mathematical entities and trying to engage with both their process and concept attributes. This would seem to be the view of Davis (1984), who states:

> We have observed students who placed their full reliance on natural language statements…What is required in the learning of mathematics is not the verbatim repeating of verbal statements, but the synthesis of appropriate mental frames to represent the concepts and procedures of mathematics (p202).

Aside from the idea that mathematical concepts do not reduce simply to words, which this suggests, there is the possibility that verbal expression may not be adequate for various elements of human thinking. This idea underpins the phenomenon of 'verbal over-shadowing', which will be considered further in Chapter 2. Here it is just worth noting that psychology researchers have found a number of tasks where performance is impaired if the participant attempts a verbal description. These tasks include a range of visual-spatial memory tasks (e.g. Schooler & Engster-Schooler, 1990), assessments relying on taste and smell and, most relevantly for this research, problem solving (Schooler et al, 1993). This last piece of work found that having participants describe their thinking as they solved problems interfered with insight.

There is then quite a range of views about the place of natural language in mathematics, from seeing it as primarily a communication tool, which might even cause problems for individual understanding, to seeing the verbal expression of understanding as the essence of that understanding. All these views, though, see the language as somewhat separate from the mathematics. In fact, Davis (1984) argues that human knowledge of all sorts, not just mathematics, is stored in a form which is "neither words nor pictures" (for a critique of this apparently reasonable position, see Kaufmann, 1996).

Yet some writers do appear to claim that mathematics is distinctly verbal. Discussing primary school numeracy, Anghileri's (1999) declares that "the transition from arranging 12 objects in four groups of three to the mathematical relationship between 12, 4 and 3 depends on verbalisation". Reacting to such assumptions, Dehaene (1992) argues against "the prevailing notion that human numerical abilities are deeply linked to language". Some writers seem to go further, arguing for verbal foundations to higher level mathematics through emphasising analytic thought, which tends to be identified with linguistic comprehension. For example, Stylianou (2002) follows Zazkis et al (1996) in distinguishing two sorts of mental processes involved in mathematical thought: visualisation and analysis. However, on closer inspection it is clear that their conception of analysis, by involving any sort of 'manipulation' of information, including mental images, should not be identified with verbal thought just because it is opposed to 'visualisation'. Other writers do want to draw closer parallels between analysis and language, though. Skemp (1987) attempts to compress all the possible dichotomies into a distinction between 'verbal-algebraic' kinds of symbol and thought, which are sequential and logical, and visual symbolism and thought, which integrates and synthesizes information.

However, it must be questioned whether this identification of logical thought with language, through the sequential nature of both, is actually legitimate. Although, as Skemp is not alone in pointing out, visual thought does not tend to be sequential, there seems no reason in principle why it cannot be. In fact, given Johnson's (1987) idea that a 'path' model, derived from experience of travelling, underpins much human logic and reasoning, it appears likely that visual representation might assist logical analysis. A practical instance of this occurring is reported by Bruer (1993), who describes how a computerised visual-spatial layout of the steps in a proof helped students to build up and understand the structures of reasoning required.

Finally, it should be noted that although Skemp (1987) distinguishes the verbal from the visual, he argues that both sorts of thinking are required by mathematics. This prompts a return to the idea that mathematical entities might have a peculiar duality about them; after all if they can convey both process and concept, why should they not be expressed both visually and verbally? In this regard it is worth remembering Wilder's (1972) contention that Greek mathematicians were able to approach aspects of what would now be called number theory through their abstract geometry, before Descartes reversed the direction of translation through his development of analytic geometry.

1.4 Mathematics in schools

It is sometimes questioned exactly how insights about the nature of mathematics should be applied to the teaching of the subject. Given Kitcher's theory of mathematics, it is clear that mathematics is essentially a practice, not a static body of knowledge. But it might be suggested that knowledge about mathematics has little relevance for education because ‘school mathematics’ is so different from ‘real mathematics’ that they amount to distinct practices. However, this would be to veer too much towards a radically constructivist position, when instead a Kitcher-inspired reality can be seen as providing an underlying ‘mathematics’ that results from all mathematical practice. Yet suggesting that all mathematics is related and linked still does not specify how ideas about mathematics might translate into teaching. As will become clear in this section, though, ideas that have developed about mathematics do suggest responses to particular challenges of learners and questions of teaching style and approach.

They also imply a general attitude to mathematics on the part of the teacher and suggest some aims for activity in the classroom. For example, the appreciation of the abstract character of mathematical entities suggests a general educational aim of developing abstract understanding in students. However, the previous discussion also warns against expecting abstraction to appear suddenly and miraculously, given the right concrete experiences. Instead, teachers should be recognising and developing the weak concepts that children hold. An example of such weak mathematical concepts, in the author’s own experience, is the appreciation that most secondary school students seem to have of numbers as somewhat abstract entities. They tend to see numbers as more than just parts of the counting sequence, being comfortable with the idea of different types of number such as square or odd.

The discussion of process and concept tended to conclude that both aspects are valuable, whether they are considered to refer to the nature of mathematical entities or to the understanding of mathematical activity and practice. However, classroom practice has tended to value one over the other. Hence the enthusiasm of Davis (1972) for doing things as the way into learning, in contrast to the then current “abstract ‘telling’”, of which he considered there to be too much. This has, of course, now given way to a perception that pupils are hampered because they cannot progress beyond the activities and procedures to grasp the mathematical concepts (Gray, 1991; Gray and Tall, 1994). An overall interpretation of this swinging of emphasis from one extreme to the other is given by Wing (1996). He argues that the ‘concept paradigm’ in twentieth century education and psychology “made whole communities of educational writers effectively blind” to the techniques and procedures that might also be necessary for success in mathematics, but that attempts to correct this anti-process bias when introducing young children to number has now produced “an equally unproductive obsession with counting”.

It can be seen, then, that the ideas discussed previously about the nature of mathematical entities and understanding can be related to mathematics education and, indeed, that they should be, given the conception of mathematics as linked through practice. However, as can be judged from the tendency of pedagogical theories to emphasise first one pole of a dichotomy and then the other, ideas about mathematics do not translate simply into classroom prescriptions.

1.41 The practice of mathematicians

What the philosophical emphasis on mathematical practice does suggest, clearly, though, is the relevance of mathematicians doing mathematics to the educational enterprise. This has been noted by a number of researchers who have interviewed mathematicians or observed them working on mathematical problems (Burton, 1999; Sfard, 1994; Stylianou, 2002). It seems worth asking what insights such work has provided for educationalists. Burton reports the importance that mathematicians attach to synthesis and making connections, recalling many of Skemp’s (1987) comments about synthesis and suggesting that such thinking is a valuable goal for teaching. However, these observations say very little about how such a goal should be achieved and so seem more to fall into the category of an insight into the nature of mathematics than a clear recommendation for teachers. Similarly, a main contention of Sfard (1991; 1994) is that complete understanding, taking in both the operational and the structural, is extremely hard to achieve, even for mathematicians. This would also appear to relate mostly to the appreciation of the nature of mathematics, although Sfard (1991) does argue for patience in attempting the transition from an operational to a structural conception. This does have relevance for teachers, and pupils, suggesting as it does that although students should aim for a complete understanding, they should be content at times with 'only' an instrumental understanding and not give up mathematics altogether. Since it seems likely that most teachers now tend to accept Skemp's (1976) distinction between 'relational' and 'instrumental' learning, and be inclined to put much higher value on the former, it is notable that Sfard defends the latter. This does suggest a slight change of emphasis in the classroom.

1.411 Visual methods

However, if the teacher or educationalist is looking for definite prescriptions for the classroom from the practice of mathematicians, it is the reports of visual techniques and thinking which tend most to give them. Many of the mathematicians interviewed by Burton and Sfard reported that they used visual images and diagrams, attaching some importance to these techniques in their work. Such findings, or the comments of particular mathematicians are often used by educationalists who are enthusiastic about developing pupils' visual thinking. It also fits in comfortably with much that is understood about the nature of mathematics, particularly if the attempt is made to root mathematics in our human, and very visual, appreciation of the physical world. However, there are still questions remaining about exactly how visual thinking and techniques should be used in the classroom. In particular, some mathematicians clearly attach more importance to them than do others and this suggests the general problem of individual differences, where visual ideas will be much more useful to some pupils than to others. This question of variation in existing abilities and preferred ways of thinking will be addressed in Chapters 2 and 3.

A problem that is more purely mathematical is the question of whether visual methods are appropriate for all areas and stages of school mathematics. There are clear reasons for using simple representations in the early years to help children develop a concept of number. This involves not just concrete items to count and compare, but also the abstract representations of a number line, and later '100 squares' and 'empty number lines'. However, it might be questioned whether visual thinking is appropriate to secondary school mathematics.

Those who promote visual methods are certain that it can be helpful, making suggestions for improving the understanding of functions, and then calculus, through work with graphical representations (e.g. Tall, 1996; Zimmerman and Cunningham, 1991). The idea that visual representations are not simply a concrete base from which to abstract mathematics is suggested by Sfard's (1991) observations. She proposed that the use of representations could help develop a student's understanding from operational to structural, as certain representations have helped in the historical development of mathematics. In these cases, the representation embodies the mathematical entity and, far from being a base to abstract from, provides a concept to tie down ideas that beforehand could only be defined by processes. The Argand diagram provided a conceptual reality for complex numbers, which before could only be thought of as the results of solving equations.

Certain representations might be useful because they convey both process and concept ideas and so encourage the development of ‘proceptual’ thinking. For example, number lines can be used in the processes of counting and arithmetic, but also embody many useful concepts such as the location of negative numbers and of fractions. Similarly, dividing and shading blocks to represent fractions is initially just a process, but having a picture, or mental image, of certain fractions should help them to be appreciated as conceptual entities.

If this all sounds too straightforward, Sfard (1991) does sound a note of caution when she points out that learners can sometimes come to identify a concept with a particular representation, which she describes as a “debased, quasi-structural approach” that is extremely limiting. This can be understood as a general appreciation of the problems many teachers and others have noticed with particular representations, where their use causes a lack of flexibility. This difficulty has a certain inevitability because visual representations will emphasise some aspects of a concept over others. For instance, the shading shapes and cutting up pizza approach to fractions suggests fractions as parts of wholes, but loses the sense of a fractional amount of time or distance on the way from zero to one, which may be conveyed by clocks or number lines. Similarly, Arnheim (1969) is cautious about number lines because they do not so clearly convey the ‘take-away’ sense of subtraction. In both these cases, the solutions would seem to be the use of plenty of different representations, encouraging flexibility and developing the idea that mathematical concepts are not tied to particular pictures, any more than they are purely the results of particular processes.

A more basic problem with the use of visual-spatial representation in schools that some researchers have noted is that of convincing learners that such methods are useful. It has been observed that children might be reluctant to use techniques involving apparatus or pictures which they perceive as childish (Munn, 1998; Gray & Pitta, 1996). Older learners often fail to make use of diagrams, which instead "take on a ritual character becoming mere appendages to problem solution rather than a part of its process" (Noss et al, 1999).

1.42 Identifying learners' mathematical problems

In theoretical outline there would seem to be some valuable ideas for a visual teaching approach, which fits in with the practice of mathematicians and with a coherent background theory of mathematics. It is possible to see that these ideas should be useful in the classroom, since they can be justified in the terms of the understanding developed of the nature of mathematics. However, it must be questioned just what problems learners have with mathematics and whether a visual approach is likely to be beneficial. As has been mentioned, one possibility is that visual methods are more useful to some learners than others and this will be addressed later. At this stage it is necessary to identify the general difficulties experienced with mathematics and question whether some of them might be alleviated by a more visual emphasis in the classroom.

Although the research was carried out some time ago, and there have been curriculum changes since then, the extensive Concepts in Secondary Mathematics and Science (CSMS) survey (Hart, 1981) of secondary school children in Britain remains a valuable resource. The hierarchies of understanding in various topics and, in particular, the common errors made by pupils, tend to concur with current experience of teaching secondary children. Major conclusions drawn by the researchers are "that mathematics is a very difficult subject for most children", although "all children make some progress". This would seem to suggest a more careful teaching of basic concepts and an awareness that it might be arduous for the child to develop a complete understanding that goes beyond procedure. Presumably with this sort of idea in mind, the researchers recommend more use of apparatus in the later primary and secondary years. This is an aspect of education that probably has changed in the last decade with more blocks and tiles in evidence in the secondary classroom, although these are perhaps more often used in presentations by the teacher than in 'hands-on' activities of the pupils. Returning to the CSMS research, this found diagrams were generally useful, although there is a general warning about visual representations contained in the finding about distance/time graphs. It is reported that "the essential relationship time/distance was lost in the visual aspects of the graph. Many children looked at the picture and described it in terms of going up or left rather than stating the meaning of the line segments". This is an instance of the obvious problem, noted by others (e.g. Arcavi, 2003) that learners may be misled by a visual representation if they mainly notice 'irrelevant' aspects, often the visually salient ones. Simple solutions are difficult to provide, but the answer would seem to be the use of a range of representations and awareness on the part of the teacher that this problem is likely. It should not be seen as a fundamental criticism of visual representations, since similar misunderstandings could clearly come about through the learner's incomplete understanding of a mathematical term or definition, or as the result of grasping the wrong aspect of a verbal metaphor (Nolder, 1991).

It has been suggested repeatedly that a general problem in mathematics results from learners failing to develop proper conceptual understanding and managing by knowing a lot of procedures. Skemp (1976) contrasts relational with instrumental understanding, and argues that teachers and learners need to have relational understanding as their goals for this to be achieved. More recently, Gray (1991) found low achievers across a wide age range (7 to 12 years) all similarly solving arithmetic problems by relying on counting procedures to a much greater extent than the higher achievers. He argues that the low achievers were put at a disadvantage by using these cumbersome procedures and that the root of their problem was that they had not developed a sense of numbers as conceptual entities, viewing them instead as parts of a process. Wing (1996) has made very similar observations about the early number skills of young children, arguing that counting is over-emphasised in infant schools. Gray (1991) urges a more explicit emphasis on known number facts and working with derived number facts, which has in fact made its way into the classroom through recent curriculum recommendations. For instance, the current National Curriculum (QCA, 1999) gives a high priority to working with number facts and deriving them in the programme of study for Key Stage 1. This might be successful through suggesting that numbers are individual entities, not just parts of the counting process. However, such ideas do not preclude the increased use of visual representations for numbers, as Wing (1996) suggests. These should similarly have the effect of conveying the sense of an entity, but might also be expected to convey a more rich conception. For example, a dot pattern suggests evenness or oddness and a link with geometry through the pattern of the dots.

1.5 Summary

It is necessary to consider the nature of mathematics because any understanding of a subject will influence ideas about teaching it. Specifically, the theory of mathematics discussed above has particular implications for learning mathematics in that it emphasises the process of doing mathematics, as carried out by a range of people currently and as developed through its history. However, although the ideas discussed recognise the importance of constructing mathematical understanding, they also provide a basis for recognising a body of mathematical knowledge and understanding that goes beyond individual thinking. Such a conception allows the teacher to respect the effort of individual learners to construct knowledge, but provides a rationale for the inevitable guidance and planning on the part of the teacher.

In the light of these broad conclusions, it is interesting to consider the more detailed ideas that mathematicians and educationalists have advanced about teaching and learning mathematics. A range of dichotomies has been proposed to capture the way that mathematics appears to advance. These include distinguishing the concrete from the abstract and numerous distinctions made between processes and concepts. It has been argued that just because mathematics seems to develop from the concrete to the abstract and from process to concept, teachers should not expect the progress of individual students neatly to reflect this. However, the dual nature of mathematical entities should alert teachers to the general aim of developing learners' understanding of both aspects.

Finally, the ideas developed about mathematics can be used as a background for beginning to consider the place of visual thinking. Both the history of mathematics and the current practice of mathematicians suggest a place for visual representation and understanding. Certain problems of learning mathematics also appear likely to be minimised by the use of visual representation and the ideas suggested are open to understanding in the terms, such as the process-concept distinction, developed by various writers.

2. Cognition, learning and visual-spatial thinking.

Before attempting to understand mathematics visual-spatially, it is necessary to consider the various theories and ideas which have been advanced about this sort of thinking. In an everyday sense, visual-spatial thinking seems fairly straightforward. We have an idea of activities, such as jigsaw puzzles, that make demands on our visual-spatial skill and perhaps a feeling for the components of this, such as forming a visual image, comparing arrangements or shapes and noticing visual patterns. It is no surprise that psychologists have investigated performance on tasks designed to require visual-spatial strategies and can be more precise about what these strategies entail. However, this activity defined grasp of visual-spatial processing leaves out many important details and overall understanding. On the one hand, there is the tendency not to investigate precisely how observed visual-spatial behaviour is underpinned by psychological processes and the nature of these processes. On the other, is a failure to consider wider conceptions of visual-spatial thinking such as its development, relationship to language and verbal processes and how it fits into other general theories of psychological functioning. Such ideas about the place of visual-spatial thought and its foundations will be explored below.

2.1 Cognitive development

There are a number of perspectives from which to consider cognitive growth and so try to understand children's learning behaviour and the results of any attempt at teaching. It is also worth asking what place these ideas give to visual thinking. Piaget's theory, as he and his school developed it, sees learning as a process of constructing understanding. This starts from the repetition of basic movements, in the sensori-motor stage, but develops into the ability to represent information in an increasingly refined way. Such representation, either in words or images allows the child to reason beyond the immediate situation. First comes appreciation of changes to actual objects (concrete operations), then children progress to being able to reason about completely hypothetical entities and events (formal operations). The underlying theme here is an increasing abstraction away from the actual to the possible and similar ideas run through most conceptions of cognitive development (see Donaldson's 1978 discussion of 'de-centring'). This abstraction clearly makes possible more powerful thought and must be important for understanding the abstractions of mathematics (Chapter 1, section 1.21). Bruner gives even greater emphasis to this growing abstraction, arguing that a "benchmark of intellectual growth" is "increasing independence of response from the immediate environment" (1968, p.17-18). He proposes that it is the underlying representations that make this shift possible, with children first having 'enactive' representations, then 'iconic', which are linked to the world, and finally 'symbolic' representations that are completely abstract. Vygotsky ties his understanding of children's development to his observations about their ability to form concepts. He argues (1986) that early reasoning uses 'complexes', not concepts, because the child is not able to abstract and generalise a property away from its embodiment in a particular item.

The impetus for the increasing abstraction remains unclear. Piaget proposed a timetable to this development and, over the years, educationalists and psychologists have puzzled over the extent to which it can be speeded up. It has generally been found that there is a limit (see Adey & Shayer, 1993; Krutetskii, 1976, p.329-332, discusses this in relation to mathematics education) and suggestions have been made that this is based on mental processing capacity increasing due to maturation. Case (e.g. 1985) is among the neo-Piagetian thinkers who have interpreted Piaget's stages to reflect maturational changes in the brain. There have been various proposals that short-term memory, in particular, increases with age and it could be that limitations here hold back younger children or perhaps those struggling in any age group. However, it is by no means certain whether absolute capacity increases or just the ability to make efficient use of whatever is available. This latter possibility provided the impetus behind the development of ideas about metacognition, which will be considered further later, and Bruner alludes to the same idea when he notes that it is in the nature of abstraction to 'compact' or 'condense' information, which allows us to make better use of our brain's limited capacity. It can sensibly be argued (e.g. Halford, 1998) that development both in capacity and in knowledge handling are important and probably both change, contributing to progress.

These global theories of cognitive development give a general idea of the underlying transition, and additionally, in the case of Piaget, provide detailed information about changes in performance on specific tasks. However it is difficult to see how these ideas can be used to guide teaching beyond a fairly general recommendation to encourage children to construct understanding and generalise their knowledge. The work of Adey and Shayer (1993; 1990) confirms this view, since although they have developed a complete science course, this is essentially built on a general idea of developing abstraction and logical reasoning together with their own knowledge of teaching secondary science. Although they explain their aims, methods and observations in Piagetian terms, these could all be understood through somewhat different conceptions of learning and development.

However if theories of cognitive development do not prescribe mathematics teaching, it might still be worth considering what they have to say about visual-spatial thinking. Both Piaget and Bruner see visual thinking as a fairly primitive stage, although not the most basic sort of thought. Piaget proposes that during the early sensori-motor stage thoughts are all about actions and children appear to develop image-use somewhat later. He argues (Piaget & Inhelder, 1971) that this progression, which takes place during the child's second year when s/he is gaining language, is, similarly, a very early move towards symbolism. Like early language, images assist with the 'internalisation' that produces representational thought and they are therefore more than just remembered perceptions. However, this idea of images as a "system of intermediary agents made up of perceptual schemes" linking individual experience to shared concepts still demotes imagery to the position of being prior to, and more primitive than, language. Similarly, Bruner sees iconic thinking as a step in the progression from enactive to symbolic thinking, where iconic thought is still tied to actual examples in the real world. He mentions the idea of successful adult functioning encompassing all three, but this can never be independent of the underlying hierarchy, as is evident when he states that "intellectual development…run[s] the course of these three systems of representation until the human being is able to command all three". His conception of images will necessarily be rather concrete because of the part they play in his theory of increasing abstraction. Piaget acknowledges this emerging idea in his own thoughts, and a possible contradiction with it, when he discusses the visual-spatial abilities of mathematicians. His solution is that visual images about spatial relations are a special case where images can be more "fruitful" and symbolic, since they are more appropriate to the content. Yet, in his opinion, they are still "subordinated" to the "central operations core", which has to develop before children can make progress.

It can be seen then that there is a tendency within theories of cognitive development to view visual thinking as a primitive form that needs to be superseded. This is particularly the case in the ideas of Bruner, but it is more the result of his overall theory than of a detailed consideration of the nature of visual processing. Piaget, because he does investigate actual functioning, hints at an awareness of a more complex interpretation of visual thoughts and, interestingly, links this to spatial thought. As will become apparent, the nature of the relationship between the 'visual' and the 'spatial' is a continuing complication in this field but one which might prove illuminating.

2.2 Abilities and processes

A presumption of there being a link between visual and spatial thinking underlies work in the psychometric tradition, but it will be shown that the exact nature of the association is considered unclear. Furthermore such work in psychology often assumes a straightforward link between the visual-spatial processes used to solve particular problems and visual-spatial ability, which is assessed through these tasks. It is easy to find oneself assuming that a particular ability neatly results from, and so reflects the structure of, specific visual-spatial processes, resulting in a tendency to speak interchangeably of ability and process. However, as will be discussed, visual-spatial ability, whether measured with psychometric tests or estimated from success with particular everyday tasks, need not rely only on visual-spatial processes. Therefore a conclusion about the existence of specific visual-spatial cognitive processes does not, by necessity, lead to the validity of the idea of visual-spatial ability.

Historically, the psychometric standpoint developed as psychology began to move away from introspection and attempt an objective study of thinking. Building on the success of Binet's IQ test, there were various tests of general ability constructed, but these gave way to a realisation that overall success on any test requires a variety of potentially separable aptitudes. Different sorts of test can be written which require different sorts of abilities. A fairly obvious distinction can be made between tests that require verbal skills and those which are designed to be non-verbal, requiring reasoning about pictures and patterns. Of these, the ones most clearly dependent on visual thinking require the test-taker to imagine a visual stimulus and then transform this view, by mentally rotating or folding up the item or by changing the viewpoint. Results from these tests of visual-spatial ability correlate much more highly with each other than with tests depending on verbal skills. McGee (1979) in a review of the distinction of spatial ability concluded that "numerous factor analytical studies have yielded a spatial factor mathematically distinct from verbal ability". He points out that this result holds across males and females as well as across various ethnic, cultural and socio-economic groups.

However, such statistical findings could still leave doubt that the distinction of visual-spatial ability reflects something real in human thinking as opposed to an artefact of the way psychometric tests are produced and used. This is why Hunt (1994), in another review, notes the statistical evidence as one of three sorts of support for the validity of the concept. He also refers to experimental psychology's investigation of the dual task paradigm and to continuing neurological work. However, it might be argued that these research findings offer support more to the foundational idea of separate processes than to the elaborate concept of abilities.

Dual task experiments involve asking subjects to attempt two tasks at once and then considering how these interfere (see Baddeley, 1997, for a review). Baddeley et al (1975) carried out an experiment where subjects were asked to track with their eyes a moving rotor blade, which is a visually demanding task, while simultaneously carrying out another task that involved classification of elements of some information held in memory. In the verbal condition this was a sentence, the words of which needed to be classified (noun or non-noun) while in the visual condition this was the mental image of a block capital letter, the corners of which needed to be classified according to their position on the figure. Only the visual memory task significantly impaired the tracking, with similar results being found for comparisons of other concurrent tasks. These findings are understood by postulating two parallel systems within working memory, which are referred to as the visuo-spatial sketch pad (VSSP) and the articulatory loop (sometimes called the phonological loop). These provide short term storage for, respectively, visual and verbal information while tasks are being accomplished and are separate from executive, more general processing resources (Logie & Baddeley, 1990)

The related suggestion that two visual-spatial tasks are more difficult to do simultaneously because they both use the same parts of the brain, while a concurrent verbal task can make use of other parts, is supported by neurological research. This has developed from fairly crude EEG based research evidence linking the right hemisphere of the brain with visual-spatial processing and the left hemisphere with verbal processing (see Davidson & Ehrlichman, 1980, for discussion about such experiments) to more detailed observation of brain functioning (e.g. Johnsrude et al, 1999). These modern studies demonstrate the complexities involved in any task and seem to suggest quite a lot of individual variation in which brain regions are used

(Wendt & Risburg, 1994). Certain parietal regions are particularly involved in visual processing although the precise location of activity depends on the detail of the task (Alivisatos & Petrides, 1997). Some neurological work has suggested that localisation of brain activity might be linked to individual differences in visual-spatial and verbal abilities. Gevins and Smith (2000) report that their participants' hemispheric asymmetries in brain activity when doing a task were related to their relative cognitive strengths. However, the experiment did not establish whether the participants were using broadly visual-spatial or verbal processing on the task. Taken in total, though, all these observations of brain activity do suggest neurological foundations for the findings of the dual task experiments.

It has been argued (Kosslyn 1994) that these modern brain imaging techniques have largely resolved the philosophical argument about the functional reality of mental images, perhaps superseding the ingenious psychological experiments (Finke, 1980) which were already strongly suggesting that experience of imagery is more than just epiphenomenal. The argument arose because of a reawakened interest in mental imagery, which grew out of the information processing approach to cognition and which explicitly acknowledges that we can be more certain about processes than about abilities. Paivio (1971) argued that most information can be encoded either verbally or through images and the type of encoding that occurs can be influenced by a number of factors, including the nature of the information, demands of the task and also the encoding tendency of the subject. This last variable was expanded by Paivio, and others, into a theory of visual or verbal personal styles, which will be returned to later in this chapter. However, other researchers were more interested in visual imagery, as used by all subjects to solve certain visual-spatial problems. Shepard and Metzler (1971) showed that mentally rotating an image is, in some ways, remarkably similar to actually rotating an item: it takes longer to turn it further. Recent research (e.g. D'Angiulli, 2002; Mast & Kosslyn, 2002)has explored other ways in which imagining is like perceiving.

Yet there is a considerable problem in relating this visual processing to visual-spatial ability. The work of Gevins and Smith (2000), referred to above, appears to link neurological activity to cognitive abilities, but leaves out the linkage of recognised psychological processes. Although these can perhaps be inferred from the brain activity, this does not seem wholly satisfactory. The experiment suggests a neat connection from neurology through psychological process to psychometric ability but, taken alone, does not provide it, while other research demonstrates the difficulties involved. For instance, spatial test items may not always be solved by using mental imagery or anything that could really be called a spatial strategy. Even if all subjects taking a spatial test are using spatial processing this might be supplemented by other skills and these conceivably could account for differences in performance rather than any variety in spatial skill. Lohman and Kyllonen (1983), in their discussion of the various factors that affect the strategies used in a particular situation, comment that such variation in strategies is a particular problem for spatial tasks because "it is possible to construct verbal tasks where spatial strategies would be of little or no assistance (e.g. a simple vocabulary test). On the other hand it is extremely difficult to design spatial tasks that cannot be solved at least in part by some non-spatial strategy". The experiment of Roberts et al (1997) demonstrates another complication in relating task success to the performance of particular strategies, since their task seemed superficially to require a visual-spatial strategy but could actually be completed better using a non-visual strategy. They found that it was participants assessed as spatially able who realised this and so performed better on the task, through using the non-visual strategy.

A further problem is in relating the functional aspects of visual-spatial processing to the conscious experience of having an image. Marks (1999) points out that even some apparently image-reliant tasks, perhaps including the Shepard and Metzler rotation task, do not seem to require conscious mental imagery. In the original dual task experiment, Brooks (1967) found that all his subjects suffered interference that is suggestive of visual processing but only a minority reported experiencing "a clear image". Given this, it is not surprising that Richardson (1983) reports that spatial ability scores correlate poorly with the 'vividness of images' measures that try to measure the subjective experience and which are used by those interested in individual differences in visualising ability. Poltrok & Brown (1984) report similar findings while Di-vesta et al (1971) used factor analysis and found that the introspective measures loaded heavily on a 'social-desirability' factor that they identified, rather than on their 'imagery' factor, which emerged from various visual-spatial tests. Considering such difficulties with measurement Richardson (1977) proposes maintaining a strict distinction between a consciously experienced 'image' and other visual thought. However, it seems unlikely that such a distinction can always be maintained theoretically and it is extremely unlikely that it could be explained to research participants, whose reflections on their own processing Richardson considers to be valuable.

These difficulties of relating both the experience and the effects of visual imagery to spatial ability measurements might be helped by attempts to break down visual-spatial processing into components. There is repeated suggestion that the visual and the spatial might be fairly separate processes and, perhaps, abilities.

McGee (1979) argues that a review of the literature points to spatial ability being composed of “at least two spatial factors”. One of these appears to involve spatial visualisation whereas the other is less visual and more to do with orientation. The recent factor analysis of Burton and Fogarty (2003) demonstrates how a five factor model of spatial intelligence may allow for a resolution, with the single factor representing the visualisation aspect of spatial tasks essentially overlapping with an image quality factor derived from more purely visual tasks. They also try to say where the self report factor, and so the subjective experience, fits into this model but here their interpretation appears more arguable. Baddeley’s (1997) review of the dual task experiments that he and his colleagues conducted argues that the separation of visual and spatial processes explains some of their apparently conflicting experimental results. He points out that this fits in with the proposed separation of the visual from the spatial in the perceptual system (Kosslyn et al, 1990), which has also been referred to by other researchers (Farah et al, 1988; Postma & Dehaan, 1996) to underpin various proposals that, in general, spatial and visual cognitive processing may be separable. Extending these ideas, Knauff and Johnson-Laird (2002) argue from the premise that visual and spatial processing are distinctly different to the assertion that spatial processing is much more helpful than visual to logical reasoning.

Such ideas about the comparative utility of visual and spatial processing will be further explored when their relevance to mathematics is investigated (Chapter 3). However, the foregoing shows that it is one thing to distinguish visual-spatial processing from verbal, but quite another to analyse it conclusively. At the level of visual-spatial ability, as demonstrated by performance on certain tasks, this distinction becomes more difficult still since such a range of strategies may be exploited. As has been shown, there is some doubt about the legitimacy of linking visual processing with spatial, although, in practice, their distinctiveness when compared to verbal processing often makes this linking irresistible.

2.3 Other dichotomies

Having indicated the limitations of the visual-spatial/verbal distinction, at least when approached from a perspective of skills and abilities, it is worth looking at other dichotomies suggested by the psychological literature. It will be argued that links can be found between these systems and that they have relevance for the understanding of visual-spatial thinking.

2.31 Explicit and implicit processes

A major underlying theme is the distinction between explicit and implicit mental processes. This starts off with the straightforward recognition that we cannot possibly be consciously controlling all our brain processes or even simple activities, such as getting out of bed, would be impossibly complex. Studies of people with brain injuries (e.g.Humphreys & Riddoch, 1987) have shown the sorts of processes that are usually carried out without conscious control and have also indicated the difficulties of trying to compensate consciously for these processes when the relevant parts of the brain have been damaged.

Psychological research into so-called implicit processing has argued that it is not just the deploying and monitoring of physical subsystems, such as vision, which relies on non-conscious processing. There is evidence that more complicated cognitive processes continue unconsciously, with even judgements and decisions often being made without conscious control. Experiments with priming and masking have demonstrated the existence of processing below conscious awareness in everyday, but complex, learnt processes such as reading (Levy, 1993) and number knowledge (Dehaene et al, 1998). Similarly, a complex network of implicit knowledge supporting consciously controlled reasoning is suggested to lie behind the superior performance of experts across a range of domains (Ericsson & Smith, 1991). The possible complexity of such knowledge is suggested by psychological experiments into implicit learning.

Although many of these experiments rely on an apparently rather basic human ability to spot patterns in information, these patterns are often very complicated. Reber (1967) began this line of research by looking at participants' ability to learn an 'artificial grammar' of links between symbols. Through being given examples of sequences of symbols that followed the rules of the system, and of those which did not, subjects learnt the grammar to the extent that they could reliably classify new sequences as legitimate or not. They did not need explicitly to know the actual rules that generated the examples to be able to make use of the regularities they produced. This phenomenon was found with various adaptations of the artificial grammar method, including experiments where participants were able to distinguish examples from two different grammars, through "highly competent but inarticulate concept-identification" (Brooks, 1978) . Berry and Broadbent (1984) expanded this non-explicit rule learning by developing complicated artificial situations, described by computer programs following a number of inter-related rules, but manipulated by people through 'trial and error'. Participants rapidly learnt how to get the desired results from simulations of sugar production and of the responses of a fantasy person. It was found that generally they could succeed practically when they could not describe their procedures and other studies (Schooler et al, 1993; Reber 1993, p.47) have even suggested that attempting to verbalise an insight might actually interfere with it.

In these studies, attempting verbal explanations and descriptions was used to assess explicit knowledge but this link between verbal and explicit processing suggests the relevance of implicit and explicit processes for the understanding of visual-spatial processes. Although one is not synonomous with the other, verbal and explicit understanding are often linked, as they are above, while visual-spatial skills sometimes seem more implicit. ‘Verbal over-shadowing’ occurs when the non-verbal strategies that are more appropriate to certain tasks are used less effectively and inadequate or incomplete verbal descriptions are used instead. It should be noted that such use of verbal strategies seems to be partly consciously controlled, and therefore an example of explicit processing, but sometimes it seems to happen involuntarily. This appears to depend on the nature of the task and, for example, remembering easy to name pictures seems to lead to involuntary verbal labelling (Brandimonte & Gerbino, 1996).

Although, as this demonstrates, verbal knowledge should not be identified with explicit knowledge, there is a tendency, particularly in education, to do this. Some examples of this occurring in mathematics education were referred to previously (Chapter 1, section 1.32). Such ideas about explicit and verbal understanding are perhaps partly explained by interest in metacognition and, particularly, because of the way that this ideas has come to be understood. A large quantity of writing across the psychology and education literatures, over the last few decades, has appealed to the importance of metacognition. This is essentially the ability to be consciously aware of one’s own processing, reflect on it and control it. The idea originated in developmental psychology (see Flavell, 1976), where metacognitive skills were found to develop with age and correlate with success in particular tasks. This obviously builds on Piaget's observations that a child can first carry out sensori-motor actions and is only later able to represent them, most obviously in the form of speech but also in increasingly accurate and predictive images of the world (Piaget & Inhelder, 1971). Inhelder and Piaget (1958) saw representation as crucial for moving on to logical understanding in adolescence, where information needs to be organised and conclusions drawn. The tasks used to test developing logical reasoning require conscious and careful organising of information (e.g. to control variables or list possible combinations) and have links to ideas about metacognitive monitoring. Furthermore, although Piaget considered both image and speech based representations, others have tended to emphasise the verbal side of explicit, conscious understanding. For example, Vygotsky (1986) argues that the development in adolescence of mature reasoning requires language since 'real concepts are impossible without words'.

Yet even without the interpretation of metacognition as rather verbal, there would seem to be problems with over-emphasising this aspect of learning. It must be recognised that tasks such as the Piagetian combination task (Inhelder & Piaget, 1958)are of a particular sort, just as the implicit learning ones are and, similarly, repay a particular approach, but this time a more conscious one. By over-emphasising the importance of metacognition, educators risk ignoring the place that non-conscious processes have been shown to have in learning and, instead, concentrating only on processes that appear to be under conscious control. Furthermore, another symptom of the over-stating of metacognition is suggested by Adey and Shayer (1993), who complain that the term has become "over-used", being deployed to describe 'self-regulation' as well as 'self-knowledge'. They argue that self-regulation is not true metacognition and is better considered as a part of constructing understanding, where it is not given such overtones of explicit processing.

The direct relevance of being aware of the distinction between explicit and implicit processing for a study of visual-spatial ability is that while there is a tendency to identify explicit thought with verbal thought, many visual-spatial processes do not seem to be entirely consciously controlled. For example, mental images of past events just spring to mind and some people experience images of numberlines when asked to think about numbers (Seron et al, 1992). Even though, on the other hand, many unconscious processes appear to underlie language ability, actually expressing an idea in words is an extremely explicit process. However, we should be careful not to think that it is the only way to explicit understanding, even in circumstances where such learning is judged to be useful. The research of Chi et al (1994) found that encouraging and prompting language-based 'self-explanations' improved learners' understanding, but the researchers point out that they would expect benefits from "any form of constructive activity…even diagram drawing". Stylianou (2002) gives an indication of how such non-verbal, but explicit, elaboration could work with her study of mathematicians solving problems through drawing diagrams. The danger of inaccurately identifying verbal processes with explicit understanding is that, together with over-valuing verbal expression, it can lead to under-valuing other sorts of understanding. That this is a problem in education is suggested by Moseley's (2003) finding that Further Education teachers tended to rate 'working with visual patterns' as a relatively unimportant set of skills, with this being significantly more pronounced among teacher trainers. The inadequacy of this evaluation is conveyed by other research on implicit processing and verbal over-shadowing, which shows that non-verbal strategies, including visual ones, might often be more appropriate to a task.

To sum up the discussion so far, it has been argued that although mental processes can be considered to be broadly explicit or implicit, it is important not to emphasise one type of thinking over the other. This is particularly vital if, as sometimes seems to happen, verbal knowledge is identified with explicit processing and visual-spatial ability with implicit processing. The combination of this over-simplification with educationalists' tendency to over-value conscious control of thinking can lead to a very narrow view of what constitutes real understanding and one that does not seem particularly appropriate to mathematics.

2.32 Procedural and declarative knowledge

Another distinction, related to the explicit-implicit dichotomy, is that made between different types of knowledge. This distinction between declarative and procedural knowledge has its roots in epistemology but has been applied to the activity of learning, as understood by both psychology and education. The essential idea (Ryle, 1949) is that some knowledge involves 'knowing how' to do something, without necessarily being able to explain the actions involved; other knowledge involves 'knowing that' a certain fact is true, where this information can be readily described and communicated. Considered in this way, as primarily a dichotomy, the distinction is clearly very similar to that made between implicit and explicit processes. However, theorists using the procedural/declarative distinction more often try to relate one sort of knowledge to the other or attempt to explain how one sort of knowledge might develop into the other sort.

Hiebert and Lefevre (1986), in their discussion of this sort of reasoning, note the various divisions that have been made by different theorists and, particularly, the varying interpretations. Piaget, they point out, clearly thought that 'conceptual understanding' was a development from mere 'successful action', but Anderson (e.g. 1983) argued that experience allows 'declarative knowledge' to become automatic 'procedural knowledge', resulting in more efficient performance. It seems possible to think of examples for each of these directions of development and Hiebert and Lefevre conclude that since benefits can be seen for both directions, the important issue is that links exist between the two types of knowledge. They further argue that although distinguishing the two types of knowledge is "useful", it is not "exhaustive" as some knowledge seems to fall into both categories and some into neither. This is a conclusion to bear in mind when considering visual-spatial functioning, since the drawing of a picture or diagram can be seen as a demonstration of both sorts of knowledge. Many elements of the representation may be quite automatic, the subject having learnt how to give an impression of depth or to number axes, but the 'procedures' also relate to declarative knowledge of vanishing points or Cartesian coordinates. Another way of understanding this is in terms of processes and concepts since these ideas seem to have an existence beyond the subject's actions, and so can be seen to constitute concepts, but they depend on processes. As was discussed previously (Chapter 1), such involvement of both processes and concepts in mathematics is often argued for. Sierpinska (1994), in her discussion of mathematical understanding, comments that a process must have concepts to act on and Sfard (1991) also attempts to integrate the two ideas.

In conclusion then, psychological research has suggested various dichotomies in thought processes, which can be seen as variations on the theme of less conscious procedural cognition compared to more explicit knowledge about something. Although it has been shown that there is a tendency to interpret the distinction as a hierarchy, it has been argued that this is not reasonable. Various attempts have been made by both educationalists and psychologists to describe the ways that the two sorts of thinking interact and rely on each other. It has also been pointed out that distinguishing two types of thinking, learning or knowledge is sometimes inappropriate and this realisation can be linked to ideas, explored previously, about the nature of mathematical knowledge and understanding.

2.4 Styles

The realisation that different sorts of thinking may be appropriate to different situations, rather than necessarily superior or inferior can be related to another strand of psychological research: that of cognitive styles. From this perspective, individual differences in performance are due to people tending to use different styles, which are more or less appropriate to a particular situation, instead of possessing a particular level of ability. As Sternberg and Grigorenko (2001) point out in a recent review of this area, to justify the distinction of 'styles' from 'abilities', it is important that proposed styles do actually have tasks to which they are appropriate: "One become suspicious of the relation between a style and an ability when one of the two complementary styles always seems to be better", they comment. Similarly, Riding (2001) states that the "essential difference" between style and ability "is that performance on all tasks improves as ability increases, where as the effect of style on performance for an individual is either positive or negative depending on the nature of the task". Whether it is always possible or necessary so strongly to delineate between styles and abilities is an area that will be returned to. However, these propositions give a flavour of the assumptions embedded in the concepts of styles.

Cognitive styles cannot be ignored by this research because they have become a fairly common way of understanding the distinction between visual and verbal thinking. Cognitive styles are usually seen as preferred ways of thinking (Sternberg & Zhang, 2001; Riding, 2001), although sometimes a slightly different emphasis is conveyed by referring to ‘learning styles’. This term is often used to suggest a preference for particular external presentations rather than the predominance of certain internal processes. It could be argued that such preference for particular presentation style is only a part of the more wide-ranging construct of cognitive style. The differing interpretations conveyed by different writers leads to complaints about the inconsistent use of terms across the literature (Mayer & Massa, 2003). These writers go on to distinguish a preference for particular types of input from a tendency to use particular cognitive processes, but argue that both are elements of being “visual or verbal learners”. Since the psychological and neurological work reviewed previously (Section 2.2) suggests a basis for tendencies to process in a visual-spatial or verbal manner, the current research will tend to work with an ‘internal’ view of cognitive style. A major need in this area is to investigate how such tendencies do, in fact, relate to external factors, such as preference for using diagrams or success with a particular style of teaching.

It is argued that many descriptions of differing styles of thinking can be reduced to a verbal-imagery dimension (e.g. Riding & Rayner, 1998) and this is linked to the earlier interest of psychologists and others in differing tendencies to report and use mental images (e.g. Galton, 1880a). It is possible to see how an idea of visual and verbal styles could arise from the finding that there are different parts of the brain and different sorts of processes associated with the two sorts of thinking: using lots of one sort of process, rather than the other, could constitute a style. In this way, the concept of visual and verbal styles seems more justifiable than other conceptions of styles. This is the line of reasoning that Paivio (e.g. 1971) follows in moving from his theory of dual coding of information to classifying individuals according to differences in their preferred mode of encoding.

For there to be identifiable individual styles requires that the sort of processing used should depend on the person and not on the task, and this is suggested by the findings of MacLeod et al (1978). They found that in a simple task of making speeded logical judgments about the positioning of items, the participants were consistently using one of two possible methods. Measuring the time the subjects took to complete the components of the task revealed that some of them were translating the diagram into words and then comparing these with the given statement (as the experimenters had expected them to do), but others were translating the words into a mental diagram and then comparing this with the actual diagram. It seems legitimate to conclude that some of the participants tended to use a verbal style of processing whereas others tended to use a visual one. Research using other tasks has sometimes found evidence of such a tendency to use either a verbal or a visual-spatial style, relying on differing mental representations (Ford, 1995). This becomes an interesting perspective on learning if individuals turn out to be fairly consistent in their style of processing across all sorts of situation. Riding and Rayner (1998) argue that this is true of the visual-verbal style distinction, although even in Riding's conception there is a continuum with some individuals much more clearly 'visual' or 'verbal' than others. These people at either end of the continuum could be expected to be more consistent in their use of visual and verbal strategies. This will affect how and when visualisation is used by people doing mathematics, which will be considered in Chapter 3.

However, ignoring this problem for the moment, there is a clear relevance to education of the finding that some individuals tend to use one style to learn while others tend to use a very different one. A series of laboratory studies concerned with a different pair of cognitive styles, broadly 'holist' and 'serialist' (Pask, 1976) included an attempt to look at the effect of matching and mismatching teaching and learning styles (Pask & Scott, 1972). This found that most individuals were unable to adapt to the mismatched teaching style, continued to use their own style and did not perform as well as the participants who were taught using the style they preferred. Such a finding confirms the concerns that many teachers intuitively feel about differences between the ways they and some of their pupils think. As Leutner and Plas (1998) point out, new multimedia forms of instruction provide further impetus within education to understand preferences for a visual or verbal presentation. Furthermore, Riding and Douglas (1993) found that personal visual or verbal style can interact with presentation to make a difference to performance. They varied the mode of presentation of some teaching material (text and picture compared to text and text) and found that the additional picture significantly improved the performance of the participants they had classified as 'imagers', although it made no difference to the 'verbalisers'.

2.5 Summary

It has been argued that it is legitimate to talk about visual-spatial and verbal cognitive processes, although it is less certain how this distinction should be extended beyond cognitive processes to individuals: in particular, should we talk about 'abilities' or 'styles'? It has also been acknowledged that the main justification for distinguishing visual-spatial processing lies in its distinction from verbal processing rather than stemming from a definite understanding of the nature of visual-spatial processes. However, the distinction found between visual-spatial and verbal processing at both the psychological and the neurological level suggest a foundation for individual differences in general preference for, or proficiency with, visual or verbal processing.

Although there is plenty of disagreement in this area and often some of the necessary links between neurological activity, psychological processes and individual differences in performance are missing, there does seem to be some consensus around distinguishing the verbal from the visual-spatial at a number of levels. Dual task experiments and observations of brain activity appear to justify this distinction, rather than one between other sorts of processing and, mainly, do not suggest further division. Although visual and spatial processes might be somewhat dissociable, there does not seem to be an indication of continuing sub-dividing of visual, spatial or verbal processing. This suggests that considering broad visual-spatial or verbal styles, or abilities, is reasonable; these proficiencies or preferences can be seen as being applied to particular areas, rather than requiring the specification of many limited, subject-specific or context-specific skills.

Other dichotomies found in psychology, and which are sometimes linked to the visual/verbal distinction, have been explored and argued to be related to the philosophical distinction between ‘procedural’ and ‘declarative’ knowledge. The importance of both types of knowledge has been emphasised despite the tendency of some educationalists to value explicit over implicit understanding, as when the concept of metacognition is over-used.

3. Mathematics and visualisation

Having briefly considered ideas about the nature of mathematics (Chapter 1) and then, separately, the psychological background to ideas about human visual-spatial thinking (Chapter 2), it now seems appropriate to look at attempts to investigate the relationship of visual-spatial processing to mathematics. Any findings about the utility, or alternatively the superfluous nature, of visual-spatial thought during instances of doing mathematics have implications both for the understanding of mathematics and for what we mean by visual-spatial processes. Particularly important, though, are the implications of this relationship for mathematics teaching.

It should be noted that confusion sometimes results from the mixing up of visualisation, understood as a process which is practised on occasions, perhaps by everyone, and the identification of visualisers, who seem to experience and understand in a more visual way. Obviously these two conceptions do overlap, since a 'visualiser' would be expected to be using predominately visual processes. However, conflating the idea of a sort of process with an individual style introduces confusions and there is a tendency to end up arguing either for or against the general utility of visual-spatial thinking in mathematics. Therefore in what follows an attempt will be made to keep the two interpretations of visual-spatial mathematical understanding separate and to identify the implications that they have for each other in a precise way that does not conflate them.

3.1 Visualisation of mathematics

3.11 A visual-spatial sense of numbers

When dealing with numbers not everybody has the subjective experience of visual-spatial processing but a minority do describe such an experience. Galton (1880a) investigated the "tendency of certain persons to see numbers in definite and consistent arrangements or schemes" and found (1880b) that about 1 in 30 men and 1 in 15 women have number "forms", which are visual-spatial in nature and involve more than just a visual image of an isolated numeral. A more recent investigation (Seron et al, 1992), using this same definition, reported a somewhat higher proportion of the population, 14%, having this experience of numbers and concluded that consistencies in the reports strongly imply that such experience is genuine. Yet, however real and interesting the phenomenon, the people reporting a clear visual-spatial mental experience of numbers to Seron were still the minority and this finding fits in with other research into visual images in general. Brooks (1967) mentions that only a quarter of his subjects reported "a clear image" when carrying out his visual-spatial matrix task. The reason not to conclude that visual-spatial processing is just a strange, minority cognitive skill is also contained in this work, though, and in that which followed it (Baddeley, 1997; Baddeley et al, 1975), since the similar performance of all the subjects, in conditions where imagery is variously possible or difficult, implies that all subjects are using visual-spatial processes. Variations appear to occur in how distinctly they are conscious of these processes and it is interesting to consider why some people have a much stronger subjective experience of mental images than do others as well as the purpose of conscious mental imagery (Marks, 1999). However, the experimental evidence that all are using similar processes underpins the argument previously advanced (Chapter 2, section 2.2) that it is difficult to make rigid distinctions between conscious images and other visual thinking, as well as demonstrating that it is not possible just to dismiss specifically visual-spatial skills as a minority interest.

That such an idea, of generality underlying the variation in subjective experience, extends to the domain of numbers is suggested by Dehaene's 'SNARC' effect. This robust effect was discovered and has been investigated by Dehaene and his colleagues (e.g. Dehaene et al, 1993) as well as other researchers (e.g. Berch et al, 1999). The findings justify the identification of a Spatial-Numerical Association of Response Codes ('SNARC') and have direct relevance for the research into subjective experiences of number forms. Essentially, experimental participants, when asked to make speeded judgements about numbers (such as about size or parity) respond as though they do possess a mental left to right arrangement of the counting numbers. When they are requested to respond with either left or right hand, their responses are systematically faster to larger numbers when responding with the right hand, compared to responding with the left hand. Similarly, the responses to smaller numbers are relatively faster with the left hand. This would fit in with the finding of Hunter (1957) that even among people who claimed no number form, the vast majority (210 out of 250) reported vague spatial associations with numbers: "a feeling that numbers somehow recede from them". Such findings also support the use of a numberline in mathematical teaching, suggesting that it might be a very natural, and so easily accommodated, model for numbers.

Furthermore, these finding are underpinned by various suggestions that the "number sense" (Barth et al, 2003) or "number module" (Butterworth, 1999) in the cognitive architecture works in a broadly visual-spatial way. Dehaene (1992) postulates a “magnitude representation”, that is separate from another two mental representations, one of which is based on verbal knowledge and the other on written numerals. This representation is used for tasks that require ‘quantification’ and approximation, as opposed to precise calculation, and automatic access to this representation underlies the SNARC effect. Barth et al (2003) argue that “numerosity representations” in adults are abstract entities, but they are “constructed from multiple perceptual cues”. Although these are not necessarily visual-spatial, much of the information we actually use to build up such understandings will tend to be visual. Supporting this, the experiments of Feigenson et al (2002) with infants show that they are very sensitive to visual information. These researchers argue that the familiar experimental results (e.g. Wynn, 1992) where infants apparently respond to number (‘subitizing’) actually result from infants noticing spatial extent properties, such as area or perimeter. Such image-based judgements could be the major part of the perceptual basis for the numerosity understanding, so that although it is not strictly, or purely, a visual-spatial representation, it is inextricably linked. This understanding provides a mechanism for the conviction of thinkers such as Arnheim (1969) that “counting is preceded by the perceptual grasp of groups” (p.211) and a rationale for the use of apparatus such as Cuisenaire rods in elementary classrooms.

However, many questions are left about the role played by visual-spatial processes in doing mathematics. Even the investigation into various aspects of the mental numberline leave open the functional aspect as Seron et al (1992) note when they conclude that “at present there exists no clear evidence about the role of number forms in calculation”. Leonard (1987) was not able to affect her subjects’ performance on multi-digit calculations through using visual interference, suggesting that they were not using visual-spatial processes. Such findings are not particularly surprising, given that research into children's calculation strategies (Gray, 1991) suggests that successful calculation relies heavily on learnt number facts, retrieved from memory, and that, therefore, practised calculation may not be a particularly visual or spatial process. However, Trbovich and Lefevre (2003) argue that the presentation of a multi-digit calculation affects the mental processes used. Their dual task experiments suggest that presenting a calculation horizontally leads to the use of the phonological loop in working memory while vertical presentation provokes the use of the visual-spatial sketchpad. In addition to this must be noted the calculation techniques of people who have mastered the Japanese abacus to the extent that they calculate using a mental abacus, manipulating a visual image of rows of beads (Hatano, 1997; Stigler, 1984).

However, there is rather more to mathematics, even at the school level, than calculations and it is worth asking how visual-spatial processes might be involved in other aspects. So far the consideration of general ideas about numbers have tended to suggest processes at the implicit level and, in moving to particular areas of mathematics and applications of number sense, there will also be a tendency to consider more explicit processes.

3.12 Using visual-spatial models in mathematics

Arnheim (1969) makes the case for the use of images in mathematics on the basis, which he argues throughout his book, that mental images are more than just concrete instances of perception. This allows him to claim that the images used in mathematics are better than mere views of actual items, with the visualised, unlike the drawn, square having perfect right angled corners and sides of exactly the same length. It is clear from this that he is urging an understanding of image that moves far beyond the sort of visual processes that are used in perception and some would argue that he is therefore including in his conception of visual thinking elements which are not really visual. However, others propose that such abstract ideas are in fact instances of visual thought. For example, Shepard (1978) declares that " so-called imageless thought may constitute just one end of a continuum of representational processes ranging from the most concrete and pictorial to the most abstract and conceptual." As will be seen, many researchers interested in visual mathematics have also proposed such ranges of visual processing (e.g. Presmeg, 1992b). In any case, the ideas Arnheim discusses clearly relate to a non-verbal appreciation of mathematics. This divergence from verbal understanding sets these ideas apart from those based on an assumption that mathematics is somehow language based (Chapter 1) and, as has been argued previously (Chapter 2), it is often hard to progress far beyond the conception of visual-spatial processes as those which are non-verbal.

The idea that mathematical understanding can be advanced by visual-spatial images is not just based on the conviction of enthusiasts for visual thought. Stylianou (2002) notes the use of diagrams, both in the historic development of mathematics and by individual mathematicians, before going on to study how practising mathematicians make use of diagrams when solving particular problems. Sfard (1991; 1994) conjectures that effective visual representations can be vital in helping individuals develop from an operational to a structural understanding of particular topics. She comments (1994) that "visual imagery is an integral component in the transition" and notes (1991) that this development mirrors the historical development of the subject where certain representations (e.g. the numberline for negative numbers; Argand diagrams for complex numbers) supported and advanced understanding. Examples of visual representations being helpful to individual learners have also been recorded. Many such instances occur when researchers investigate visualisers' approaches and these will be considered more fully later (Section 3.2). Of note though, is a small study (Edwards, 1998) which was not specifically interested in visual thought but demonstrates the utility of such thinking. In this research, only three out of ten 14-15 year olds, judging statements about odd or even numbers produced attempts at 'structural', generalised explanations. Of these students, two used a visual interpretation of the problem.

It seems, then, highly likely that pictures and diagrams should be useful in the maths classroom and textbook writers, for instance, certainly share this opinion, although their reliance on visual demonstrations varies. However, it must be questioned how we can decide what is useful to a learner and what might prove no more than a distraction. For example, Santos Bernard (1996) urges caution in using pictures in textbooks because children will tend to try to make use of even purely cosmetic illustrations. This would seem to be another instance of the general problem of the overly concrete interpretation of visual representations, considered by Arcavi (2003) and discussed previously (Chapter 1, section 1.42). Attempting to answer the question of how diagrams might be used profitably, there have been a small number of well controlled experiments where students were trained to use particular visual representation to help them with calculation problems (Lewis, 1989; Willis & Fuson, 1988). Lewis trained college students to use a numberline to represent the information contained in word problems. These were worded in a misleading manner and the numberline representation significantly improved performance compared to training in deconstructing the language. Although there is some suggestion by Lewis that using the numberline method also reduced errors on related arithmetic problems, this does not alter the fact that this is only a very narrow use of diagrams in mathematics.

Writers advocating a more general use of visual aids (e.g. Clements & Battista, 1991; Zimmerman & Cunningham, 1991) tend to base their assertions on their own experience and teaching practice, which does result in a lack of rigorous assessment of the methods. This is not to deny that there are some interesting ideas, which could be built into a general visual-spatial approach. For example, Waring (2000) proposes using 'picture-proofs' as a way into mathematical proof while Arcavi (1994) describes an entirely visually argued proof to show that imagery need not be crude and secondary to language. Chinn (1996) suggests that those struggling to learn multiplication tables should try to "get a picture" of the nature of multiplication, initially using coins, number strips and square to facilitate this. Wing (2001; 1996) argues that initial number work with small children should exploit the visual regularities and pattern of numbers, rather that over-emphasising counting.

A number of recent contributions to this identification of promising ideas discuss the use of computers and graphical calculators. Villarreal (2000) comments, "perhaps the computer has come to restore the value of the process of visualisation in mathematics education". Tall (1996) notes the use that is beginning to be made of such technology to enhance the understanding of functions, but argues that, at the time of writing, not all the possibilities of visualisation were being recognised. Elsewhere, he has been among educationalists calling for a more visual approach to learning calculus (Tall, 1991; Zimmerman, 1991). It must be noted that such proposals are not received uncritically in all quarters with, for example, Aspinwall et al (1997) arguing that a visual emphasis in calculus could lead to rather concrete images, which interfere with student attempts at abstraction. It is also worth noting that Dieudonné's (1992) assessment of the fundamental abstraction of mathematics includes the assertion that high level mathematics is distinctly non-visual, involving entities that “are not supported in any way by visual ‘pictures’” (p.2). However, here he is in apparent disagreement with other mathematicians (e.g. Gowers, 2002), perhaps through his overly simple pictorial interpretation of imagery, and in other places in his book he notes the importance of geometrical ideas and “’spatial’ language” (p.164).

In order to limit such arguments within education, based as they are on re-reading the literature and studying the occasional individual student, it seems important actually to test any curriculum proposals in the classroom. A programme that has been submitted to such testing, and which covers a sufficiently wide area of mathematical ideas to be interesting, is reported by Hershkowitz et al (1996). They describe the Argam visual skill programme, aimed at 3 to 8 year olds, which intends to develop “visual meta-processes”. The content of the programme units range from basic shape and space ideas to concepts like ratio and pattern, presented in a visual-spatial way with language kept to a minimum. The authors report that this programme appears to result in raised IQ score and increased ‘school readiness’. They further comment, however, that the programme appeared to work particularly well for “children who tended to be introverted or non-verbal” and this highlights an important concern with all these proposals for visual teaching: it is necessary to question whether these ideas will work for some students but not for others.

3.2 Visualisers doing mathematics

A cursory glance through the literature on cognitive abilities and styles (see Chapter 2) tends to provoke the conclusion that visual teaching will be more effective for some learners than for others since there are distinct individual differences in learners’ pre-existing thinking, whether this is understood as resulting from a range of abilities or differing styles. Without a particular style of teaching being adopted, correlational studies have found that high spatial ability generally predicts success in mathematics, as well as in science and technology (Smith, 1964). Evidence has been found for the importance of visual-spatial working memory to success in certain tests of mathematical competence (Reuhkala, 2001). Researchers have used differences in measurements of spatial ability between girls and boys to explain sex differences in mathematical reasoning on the basis that correlations between spatial and mathematical ability are causal in nature (Geary et al, 2000). Booth and Thomas (2000) suggest one way that this causal link could occur with their finding that, among mathematical under-achievers, those with higher visual-spatial ability were able to make better use of diagrams when solving mathematical problems. Pyke (2003) reports a similar finding.

However, the picture is considerably confused by the literature specifically relating to the mathematical performance of 'visualisers', those who tend to think in a more visual way. As will become evident, this work tends to consider strategies people use in mathematics, specifically, rather than any general tendency in their thought processes. This shifts the emphasis away from global cognitive style, or abilities, which not only changes the nature of the findings but also may be contributing to some apparent contradictions.

Over the years, many successful mathematicians have characterised themselves as visualisers and emphasised the importance to their work of their preferred way of thinking (see Stylianou, 2002, for a review). For example, Devlin (1994) states:

> Mathematicians may be able to express their thoughts using the language of algebra, but generally they do not think that way…every single one of us is able to manipulate mental pictures and shapes with ease.

However Krutetskii (1976) found that in Russian secondary schools, the students he classified as visualisers did not tend to be among the most successful performers in mathematics. Presmeg (1986) found a similar pattern of attainment among South African sixth formers who were studying mathematics. Lean and Clements (1981) classified participants according to their 'preferred mode of processing mathematical information'. This produced an 'analytic-visualiser' dimension that was weakly related to mathematical performance, with 'analytic' students tending to perform better than the visualisers.

The first thing to note about these findings is that the participants were all characterised as visualisers on the basis of their mathematical thinking styles, and the assessments do not say anything directly about general processing styles. Initially this seems perfectly legitimate, since it is mathematics with which we are concerned and it could be argued that an individual's style might vary according to the subject matter (although see Chapter 2, sections 2.2 and 2.5, for the rationale behind general processing tendencies and the agrument to support such a conception). However, this method of assessment introduces an important problem in that the association between those having visual images and poor mathematics performance could just be that struggling with mathematics leads to a resort to rather crude images. This seems likely given that struggling with any problem appears sometimes to encourage visual imagery and working at the limit of one's knowledge has been observed to cause images to arise involuntarily (Richardson, 1983, p.30). Such an interpretation is suggested by the study undertaken by Campbell et al (1995), which, unusually, did not identify visualisers on the basis of their mathematical processing style but through independent self reported vividness of visual imagery. They found that vivid visual imagery did not affect success on mathematical problems, which was instead related to general mathematical ability.

However, the more common findings of visualisers struggling with mathematics are still interesting for comprehending the relationship between mathematics and imagery. Yet it must be questioned whether they have much to say about characteristic individual thinking styles since, by considering the nature of visual images experienced as result of mathematical difficulties, they tacitly assume similarity between individuals. This conclusion is explicitly reached by Zazkis et al (1996) who remark that “there may be a more important question than that of classifying an individual”.

It must be noted, though, that researchers vary in their awareness of the problem of mathematical proficiency affecting reliance on visual imagery and the subtlety with which they use mathematical questions to gauge participants’ mathematical style. While Pitta (1998) used the same basic arithmetic questions for children with a wide range of achievement, Presmeg (1985) attempted some matching of question difficulty to participant ability. Krutetskii (1976) used a single battery of questions but his participants were all relatively high achievers. Taken together, this body of research still leads to an apparent contradiction when the weaker students, with their characteristic visualising tendencies, are compared to successful mathematical visualisers, such as practising mathematicians. Of course it is not known whether the mathematicians who consider themselves to be visualisers would be so assessed by the methods of Krutestski, Presmeg and Lean and Clements and this could be the solution to the apparent contradiction. However, assuming that they would be recognised as visualisers, the answer might lie in a closer consideration of the nature of the visualisation that subjects report. Presmeg (1992b) argues for the importance of differentiating different types of imagery used in mathematical reasoning and proposes a "continuum from specific to more general" images. The most abstract sort of imagery, "pattern imagery", she describes as "stripped of concrete details" and notes that it was only used repeatedly by one of the visualisers she studied. This student was the only visualiser to achieve an 'A' grade in 'A' Level mathematics.

Considering younger children, similar qualitative differences in the images described by high and low achievers have been reported (Pitta, 1998; Pitta & Gray, 1997; Gray et al, 2000). This research found that, when doing arithmetic, low achievers tended to experience images "that possess shape and, in many instances, colour" (Gray et al, 2000), which the high achievers did not. There is the methodological problem of all the subjects answering the same arithmetic questions so these were much harder for the low achievers. However, this criticism is tempered by the fact that the differences in imagery reported during calculations mirrored those found when the subjects were asked to provide descriptions of pictures, icons and verbally given concepts. In this way the tendency to visualise was considered more broadly and could not be arising simply as a result of difficulty with the subject matter of maths. Although this work could be hurriedly interpreted as suggesting that visualisers tend to struggle in mathematics, Pitta (1998) is careful to point out that "such labels ['visualiser' and 'non-visualiser'] do not provide an indicator of the level of numerical achievement of the children". To illustrate this she provides a case study of a successful Year 6 child who reported lots of visual images and seemed actively to use them when combining numbers with a result less than 20. Interestingly, these findings appear to reflect the result already noted that although, in general, images seem to be associated with weakness in mathematics there are notable exceptions of successful individuals whose visual imagery appears to help them.

The work of Pitta and of Presmeg can be seen to have important similarities. Their conclusions are very suggestive of the sort of visual thinking that might prove useful in mathematics, with the more useful images appearing to be at the spatial, rather than the visual, end of visual-spatial experience. Hegarty and Kozhevnikov (1999) reach similar conclusions having found that "pictorial images" were not associated with success on maths problems but "schemetic imagery" was. This is a way of explaining the apparent paradox of finding mathematically struggling visualisers while a few use images and succeed. However, it will be noted that the above discussion leads to a movement away from considering visualisers to contemplating aspects of visualisation. This produces, again, the question of whether it is necessary, or desirable, to talk about visualising in terms of individual style over and above understanding it as a process. One possible reason to consider individual visualisers is implied by Pitta's (1998) decision to include a case study of a student who could be characterised as such, which provides details of successful visual thinking. Similarly, it was by first identifying visualisers that Presmeg was able to discover "pattern imagery".

It must be questioned, though, what if anything such an approach can tell us about mathematical thinking in terms of cognitive style and beyond being merely a method of finding examples of process. A potential way of acknowledging differences in visual processing, while retaining an understanding based on cognitive styles, is proposed by Kozhevnikov et al (2002). They moved from the observation of different sorts of visual-spatial representation, some of which are more useful in mathematics (Hegarty & Kozhevnikov, 1999) to a proposal that there are two sorts of visualiser. One of these types is also high in spatial ability and, they argue, uses this ability to produce useful images in mathematics, while the other type is particularly low in spatial ability. It is students of this second type who are hampered in mathematics by their rather concrete visual images. Although the visualiser-verbaliser distinction was based only on mathematical processing (to fit in with previous research), the participants' spatial ability was measured using general psychometric tests. This reasoning makes sense of the contradictions that have threatened to undermine the whole idea of identifying visualisers and verbalisers, allowing there still to be a meaningful separation, at least in mathematics. It also fits in with the proposed separation of visual and spatial processes, discussed previously (Chapter 2, section 2.2) although more research needs to be done to see if the findings of dramatically differing spatial ability holds for all visualisers.

However, it should be noted that Kozhevnikov et al explain the mathematical difficulties of (some) visualisers in terms of problematic images, here understood as resulting from their spatial ability, rather than as a consequence of their 'cognitive balance' between visual and verbal processing. In contrast, Pitta (1998) notes that the mathematical high achiever who reports lots of images uses a general style, at least in arithmetic, which she sees as integrating the visual and the verbal. He is able to use visual images flexibly to support memory and they can be combined with verbal representations, such as internal speech. This description of the child's thinking, though, leads to questioning whether he is really a visualiser at all. In Krutetskii's terms it could be that he is actually a "harmonic" as opposed to "analytic" or "geometric" thinker. Children with such styles are, he found, much more common than either extreme and it seems possible that even the mathematicians who emphasise their visual thinking are also strong enough on the verbal side not to be true visualisers. This is similar to the view that Zazkis et al (1996) take, with their "visualise/analyse" model, arguing that "for most people both visual and analytic thinking may need to be present and integrated in order to construct rich understandings of mathematical concepts". Although their work uses a visual-analytic distinction instead of visual-verbal distinction, this seems a valid point and such an understanding also fits in with knowledge about the underlying visual-spatial and verbal processes, since it suggests efficient, parallel use of the two sorts of processing. However, it notably leaves open the possibility that there might be visualisers who do not integrate visual and verbal processes and perhaps do not have rich constructions of mathematical concepts, leading to the question of how these people might be found.

Obviously a lot depends on exactly how individuals are categorised. Some methods depend on detecting a bias towards processing one way or the other, rather than looking at absolute levels of particular sorts of processing, and, as Katz (1983) argues, this makes sense from a cognitive point of view. However, it could lead to apparent contradictions since everyday judgements tend to be based on absolute levels, as when it is noticed that a particular child often uses diagrams. Researchers identify visualisers using a variety of methods, some of which depend on opposing visual and verbal processing (e.g. Riding & Calvey, 1981; Paivio, 1971) while others consider absolute levels of visual processing, often through assessment of vividness of visual imagery (e.g. Marks, 1973). As has been discussed, researchers interested in mathematical processing tend to consider methods used by individuals attempting mathematical questions, leading to the difficulty of potentially confusing cause and effect in the relationship between imagery and mathematical success. A further problem is that it is difficult to determine whether these methods tend to identify ‘visualisers’ who are unbalanced in their approach, neglecting verbal methods, or instead those who use absolutely high levels of visualisation. This results in difficulties in accommodating findings from the various studies and in relating these to more general ideas about visualisers. Although a lot of information has been collected, it still appears very difficult both in mathematics and in general to answer clearly the question Katz (1983) posed: “What does it mean to be a high imager?”

3.3 Summary

Evidence from psychology and ideas about the nature of mathematics suggest a link between visual-spatial representations and mathematics. However, it is difficult to be precise about the nature of this and so understand the implications for mathematics education. Lesson ideas abound but few with rigorous demonstrations that they work and, if they do, how. Although, clearly, there will be individual differences in how students respond to particular styles of teaching, the problem of how to aid particular learners is not advanced by the confusion over the ‘visualiser’ style. Attempts to solve the contradiction of successful, visualiser mathematicians existing together with struggling visualiser students offer two sorts of explanation. One type emphasises the balance of cognitive styles, with success stemming from a flexible approach, integrating the visual and the verbal. More explanations, though, look to the nature of the visual images people experience and explain problems in mathematics by pointing to inadequacies of these images. Yet it is frequently unclear whether these are a cause or an effect of the mathematical difficulties.

4. Background to fieldwork

4.1 Introduction

4.11 General aims

As has been noted (Chapter 3), there is, in certain quarters, considerable enthusiasm for the use of visual-spatial representations in the mathematics classroom but little rigorous assessment of these ideas. It seems, therefore, that what is required is an experimental study comparing matched classes, with one taught using the methods of interest and the other covering the same content in a different way. Designing the alternative method to be a verbal approach allows for a direct comparison between two contrasting ways that learning is envisaged to take place. If, however, this research is to have clear implications for standard teaching, the intervention should not depart too far from the normal classroom situation. For example, it should make use of standard equipment, be taught to a class of normal size and cover a number of areas across the school mathematics curriculum. This last requirement also means that the results should be of more general interest to educators than is the case when a very narrow area of content is taught through a new approach. The only limit to the content in the present study was to avoid teaching 'shape and space' lessons. This is because it was felt that mathematics in this area is quite uncontroversially linked to visual spatial processes, where as the interest of this research is in the extent to which such processes underlie, or can be used to support, other mathematics such as number and reasoning work.

To ensure relevance to general educational experience, then, the approach and content of the visual and verbal lessons should not differ dramatically from standard classroom practice. In contrast, Presmeg (1985) argues that the conclusions of Suwarsono's (1982, cited by Presmeg) teaching experiment are limited by this weakness since his methods were not typical. The aim in this research is for each approach to draw on ideas suggested previously by teachers and educationalists, but with consistent underlying styles of teaching to help give them coherence.

On the other hand, the intention was to avoid the problem of the differing approaches being irrelevant additions to the teaching. Arnheim (1969, p.313) warns that, "It is not enough to pay lip service to the doctrine of visual aids" while Klein (2003) criticises 'learning styles' approaches where the teaching activity, intended to be in a particular style, "is irrelevant to the content being learned" (p.49). During this research it was intended that the visual and verbal approaches would not be mere additions but would instead be thought of as ways into mathematical understanding. The children would be encouraged to generate their own constructions in the particular modality, rather than just passively receive information presented in a certain form. Therefore the intention was, in the terms of Cronbach and Snow (1977) to capitalise on preferred thinking styles, rather than to compensate for them.

It has been shown that there are two alternative emphases in understanding the relationship of mathematics and visualisation. One concentrates on the process of visualisation, including the mental images produced and the use made of external representations, such as diagrams. The other is concerned with identifying 'visualisers' who experience and prefer to use such representation, either specifically in mathematics or as a more general feature of their cognitive processes. Therefore any practical research needs to take both these viewpoints into consideration and this research aims to do that. It intends to consider both visualisation processes, by investigating the results of visual lessons, and visualisers, through relating measurements of individuals' styles and abilities to outcomes. If the visual approach is generally valuable, perhaps because it provides an alternative, and possibly particularly appropriate, way of thinking about mathematics, it should lead to improvement in mathematics performance. This can be assessed by comparing the two classes. However, theorising based on the importance of individual differences, and the identification of visualisers, would suggest that improvements should not be looked for at the class level. Instead it is necessary to consider any change in the performance of individuals, relating their performance to their styles and abilities. A main objective of the research is essentially to set in competition the two broad explanations for any change in mathematical performance.

Presmeg (1985) reports that such a teaching experiment, looking for interactions of visual and verbal teaching and learning styles has been carried out previously (Suwarsono, 1982, cited by Presmeg). However, she criticises this and other ATI (aptitude-treatment interactions) studies for relying on individual ability measures, such as language and spatial test results, while considering styles of teaching. She argues for the necessity of following through theoretical arguments, which suggest matching teaching and leaning styles, rather than turning to assessments of individual cognitive abilities. Her own study, although it took a more qualitative approach to the interaction of teaching and learning, was careful to assess teachers and learners in terms of visual or verbal style, not ability. There is clearly a need for a more controlled teaching experiment, which is similarly consistent.

The interest in the outcomes for individuals with particular styles of thinking leads to the necessity of assessing these styles. Given that the major interest is with the visualiser-verbaliser dimension, the method of measuring this tendency is central to the research as a whole. It has been argued previously (Chapter 3) that the initially appealing idea of rating participants according to their styles of processing mathematics has limitations. Standard methods leave open the possibility that the visualisation found arises specifically as a reaction to, relatively difficult, mathematical problems. Overcoming this would mean tailoring the questions to the individual's mathematical competence, which would be methodologically difficult, if not impossible. Furthermore, even if this could be done, it still necessarily limits the impact of the research because it would only be informative about the cognitive styles individuals use when doing mathematics rather than adding to knowledge about global cognitive styles. Some conception of an individual's general abilities or style does seem to many psychologists to have theoretic value (Chapter 2, section 2.2), linking results from the factor analyses of psychometric tests, psychology experiments and suggestions from neuropsychology (Hunt, 1994). At the everyday level, although there are obviously differences between solving problems in mathematics and those in other domains, it is the same brain, with the same strengths and weaknesses, which does the solving. It does not seem sensible to suppose that there are no similarities or generalities in an individual's approaches to very different problems. Therefore this research will be centred on a characterisation of visualisers and verbalisers that is intended to reflect general cognitive tendencies, not just problem-solving preferences in mathematical situations. However, it is important not to take for granted the existence and utility of such a distinction and the characterisation will be compared with other global understandings of cognitive tendency, based on balances of abilities, and with the types of strategies observed in the mathematical processes of individuals.

4.12 Overview of method

In line with the above aims, the main study was designed to compare the outcomes of two matched classes, taught through contrasting methods, and also to consider the effects on individuals. Therefore a main requirement was for a tool to assess mathematics performance, which could be used pre-intervention to match the groups and post-intervention to consider improvement. In addition, there was the need for a range of instruments to assess individuals in terms of their cognitive strengths and weaknesses, as well as a method of rating them on a visualiser/verbaliser scale.

Furthermore, it was necessary to design a programme of lessons, using visual-spatial techniques, and a control programme covering the same content using a verbal approach. The intention was to test a broad, visual-spatial based teaching style against teaching with an emphasis on mathematical vocabulary and verbal explanation. The rationale behind the 'visual' teaching has previously been developed (Chapter 3, section 3.1), but it has also been indicated that, in contrast, many educators advocate a more verbal approach (Chapter1, section 1.32). Therefore, both styles of teaching can be justified and so could be expected to benefit the participants.

The decision was taken to work with Year 7 (11-12 years old) pupils for a number of reasons. Some of these are the result of considering internal, cognitive attributes of this group, while others relate to external, social factors. As many psychologists have discussed (Piaget, 1958; Vygotsky, 1986), children of this age are just beginning to demonstrate dramatically increased abstract thought with maturational change during adolescence perhaps underlying this (Chapter 2, section 2.1). Sierpinska (1994) argues that it is only at this stage in development, when the child is beginning to function conceptually and recognise inconsistencies, that ‘epistemological obstacles’ can be used to advance true mathematical understanding. Such ideas are reflected in the organisation of the school curriculum, with Year 7 students in Britain beginning a new ‘key stage’, which places greater emphasis on abstract thinking such as ‘generalising’ and ‘reasoning’ with opportunities provided to ‘transform’ and to ‘represent’ problems (Key Stage 3 National Strategy, p.15, DfEE, 2001).

Yet, as the CSMS (Hart, 1981) research demonstrates, this is also the point in mathematics education where children can really struggle. Adey and Shayer (1993) highlight similar concerns with secondary school science. Despite their own developing abilities and the widening opportunities provided by the school curriculum, many children do not make the anticipated progress towards abstract mathematical understanding. Therefore, this stage in education appears a promising point to try alternative methods or to compare certain approaches. There are reasons to consider that this is a potentially fruitful time, but also indications that this potential is often unfulfilled.

It was intended to test the utility of the visual approach in a normal school environment so a whole class was taught by one person (the researcher) with the lesson content ranging over many standard Year 7 areas, as they arose in the school’s scheme of work. The visual ideas were derived from various sources and suggestions (to be indicated in the lesson plans section), some of them having been tested by other researchers. The ‘verbal’ lessons covered the same content area, using the same questions and investigations, and, where appropriate, identical teaching materials. When these were visual in style they were translated into a verbal style. This meant that in general the verbal lessons followed from the visual lessons, which were designed first, and therefore some of them might have been limited by this. Both styles were introduced to the classes as being “a bit different” from their other maths lessons. The visual class were encouraged to “see numbers and explain things with diagrams and pictures” whereas the verbal class were told they should be “thinking harder about *how* we do maths and trying to explain things to each other”, resulting in verbal explanations.

4.13 Purpose of the pilot study

Preliminary work was carried out to test the particular methods of assessment, evaluate some lesson ideas and see whether the planned research, as a whole, seemed likely to produce illuminating results. The participants were a small class of Year 7 pupils attending a 9-13 middle school. Only lessons in the visual-spatial style were used and these were only a subset of the eventual programme of lessons. As a result, changes in mathematical performance were not expected and the quantitative pilot study outcomes will not be evaluated with those of the main study. However, more qualitative aspects such as how particular lessons were received will be considered where appropriate, since they may add to conclusions, especially as the pilot study involved a different school with a quite distinct ethos, perhaps due to the age range.

The main focus, at the pilot study stage, was on assessing the characteristics of the participants, using various methods and considering what other assessment tools were needed. In addition the visualiser/verbaliser scale that had been developed was used and the decision was made to continue to use it in this form. For this reason it is possible to combine scores on this scale collected in the preliminary study with those of the main study.

4.2 Method

4.21 Participants

The school involved in the main study was an 11-18 comprehensive school on the edge of a city (N.O.R. 1058). Its intake was quite mixed socially but its reputation was not particularly good and its exam results somewhat disappointing: 35% of pupils achieved five or more grade A to C GCSEs in 2001, compared to a national average that year of 50%. It, therefore, was not a sought after school and so tended to have a slightly disproportionate number of difficult children and an achievement range skewed towards the bottom of the range. It was also coping with adapting to a change of status, as it had recently become an 11-18 school, having previously been a 13-18 high school. Despite this, the mathematics teachers were motivated and enthusiastic. The staff comprised some respected senior teachers as well as some new recruits, including an ambitious Head of Department in his second year at the school.

The Year 7 children had been taught in mixed ability groups for their first half term. Then on the basis, mainly, of the 'Mathematics Competency Test' (Vernon, Miller and Izard, 1995) each pair of classes was rearranged to form a top group and a bottom group. Given the research aim of tackling student difficulties, it was two of these lower classes, containing children from roughly the lower achieving half of the school population, which participated in this research. These two classes were chosen because they had maths lessons at the same time on two days of the week so the two experimental classes could comprise half from each ordinary class.

The children were assigned to one or the other of the experimental groups through alphabetic lists of the children's names, split in the middle. It was hoped that this would ease organisational aspects as well as avoiding any definite sorting of the participants. However, the initial division produced an extremely uneven distribution of the girls and so some alterations were made. This was done so that the two existing classes were divided fairly equally between the experimental groups and so that the maths achievement profiles, derived from the MCT, in the two groups were comparable (see Table 4.1). The aim was for the range of scores and the number of children without scores to be similar. A one-way ANOVA revealed no difference between the mean MCT scores of the two groups.

Table 4.1 Descriptive statistics for MCT scores in the two intervention groups

Group	No MCT score	N	Range	Minimum	Maximum	Mean	Standard Deviation
Monday	2	21	17	2	19	14.10	4.37
Wednesday	3	20	18	2	20	13.80	5.07

Once assigned to the groups, the participants were taught in these groups for one lesson of 50 minutes per week during the ten weeks of the Spring Term. The 'visual' group had their lesson from 11:45 to 12:35 on Mondays and the 'verbal' group received theirs from 1:30 to 2:20 on Wednesdays. Although this meant that the times and days of the two lesson styles were not balanced, it was considered too disruptive and potentially confusing to change the classes over half way through the term.

4.22 Interviews with a subgroup

The decision was taken to interview a subset, taken from both classes, of the participants before and after the intervention. This was in an attempt to get more detailed qualitative information to relate to the quantitative data resulting from the various tests and measures. To this end, interviewees were asked some questions about their attitudes to mathematics, before the intervention, and afterwards, when some questions related specifically to the intervention lessons. They were also asked to work through some maths questions so that their tendencies to use visual or verbal approaches to mathematics could be assessed. This is similar to the technique used by other researchers (e.g. Krutetskii, 1976; Presmeg, 1985) in the field of visualisation and mathematics, but it was not possible, given time constraints, to carry out the procedure with all the participants. Similarly, it was desirable to assess mathematical understanding, as opposed to a simple rating of performance. Again this was only possible with the subgroup of participants, given that the tool developed was quite time consuming to use and demanded one-to-one attention.

During the pilot study these individual interviews produced some interesting opinions and ideas from the children and some very suggestive descriptions of their approaches to mathematical problems. For this reason, the interview procedure was unchanged for the main study. Unfortunately, though, the main study participants were much less forthcoming and less able to talk through their approaches to mathematics. However, it is felt that there is still plenty of qualitative data from the main study to consider because the full programme of lessons produced much more work and also more opportunity to interact with the children while they worked in the normal classroom surroundings.

4.23 Assessment

4.231 Assessment used

Mathematics

- MidYIS maths score: pre-intervention measure of maths achievement
- MCT score: pre and post intervention testing so change in score could be considered
- Question sort: grouping maths questions, as a measure of understanding, completed by the sample of participants (three from each teaching group) before and after the intervention
- Class work and question generation

Visual/verbal ability or tendency

- MidYIS vocabulary score: pre-intervention indication of verbal strength
- MidYIS non-verbal score: pre-intervention indication of non-verbal strength
- Spatial memory test: pre-intervention indication of spatial ability
- Recognition test: pre-intervention test that measured the tendency to encode information visually or verbally, expressed as a visual/verbal ratio
- Strategy choice: the sample interviewed worked through maths questions to see if their approaches indicated preferred thinking styles
- Preferred content: the interviewed participants were asked, pre-intervention, to identify areas of maths they enjoyed

Attitude

- Interview questions: pre-intervention questions asked about attitude to school maths; post-intervention questions related to the interventions
- Participation and behaviour during intervention lessons

4.24 Explanation of assessment tools

4.241 Assessment of all main study participants

4.2411 MidYIS test

This test is administered by the CEM Centre, Durham University and had been completed by virtually all the children, in October, near the beginning of Year 7. The MidYIS subtest scores considered were for 'Vocabulary', 'Non-verbal' (a score from a number of items ranging from recognising cross sections of solids to following the visual logic of patterns) and 'Maths'. There was also a score for 'Skills', but since this combined proof reading with matching sequences of numbers, letters and other symbols, it was difficult to know how to make use of it. CEM provides Cronbach's α scores of the reliability of these subtests, which are 0.90 for vocabulary, 0.89 for non-verbal and 0.93 for maths.

4.2412 Mathematics Competency Test (MCT)

Experience during the pilot study suggested the benefits of using an externally validated maths achievement test and, since the school had already administered the MCT to most of the participants at the beginning of the school year, it was decided to use this test. The test is designed to cover the areas of mathematics recognised by the

National Curriculum and uses a variety of styles of questions (see Chapter 5, Fig 5.3 for examples of MCT questions). The reliability measure given is an internal consistency of 0.94 for the whole test. The authors of the MCT consider that it is suitable for re-use so it was given to the participants immediately after the interventions by their usual class teachers. This meant that they all took the test at the same time with no warning or preparation.

4.2413 Question generation

An attempt was made to ask all the participants to generate questions with a particular answer, before and after the interventions. One aim was to look for inventiveness and mathematical fluency by considering the range of questions a child produced. Another was to consider any elements of visual or verbal presentation used by the participants, since this might provide another indication of preferred individual style or of assumptions being made about appropriate forms for mathematics.

4.2414 Class work

In both classes the participants worked on paper and all their work was collected.

4.2415 Spatial memory test

It was decided after the pilot study that some externally validated measures of visual-spatial ability should be sought to complement the recognition test measure of tendency to use a particular processing style. Although it was anticipated that the MidYIS scores would be useful, a test of spatial memory (from the Kaufman Battery, Kaufman & Kaufman, 1983) was also administered before the interventions to most of the participants. This involved remembering the positions on a grid of an increasing number of pictorial items. The individual subtests of the Kaufman Battery are said to have split-half reliability coefficients “typically in the 0.80s” (Kaufman & Kaufman, 1983).

As with all psychometric tests, there is concern about what ability the test is actually measuring, and in particular whether it is also testing verbal skill. Despite Lohman and Kyllonen’s (1983) warnings about the problems of finding non-verbal test items which cannot be attempted verbally, most researchers have concluded that the type of memory test used here can be said to measure non-verbal short term memory. Postma and Dehaen (1996) argue that although matching objects to positions seems to use verbal strategies, merely identifying the positions is a more purely non-verbal task. However they also mention that using a matrix, rather than an empty space, is more likely to lead to verbal strategies being helpful. When participants in the current research attempted the Kaufman spatial memory test there was some evidence of the occasional use of such verbal strategies, with muttered comments about “top-left” or “bottom line” being heard. However there was very little of this and most participants seemed to approach the task in a visual-spatial way, pointing out, for instance, the positions which fell into a memorable pattern, such as being arranged in a straight line, either vertical, horizontal or diagonal.

However, even if one can conclude that this test is assessing the capacity of the non-verbal part of short term memory (the visual-spatial scratch pad, as opposed to the articulatory loop, in Baddeley’s terms, see Chapter 2, section 2.2), there is still the question of whether it is a more visual or a more spatial test. Although it is described as a spatial test, recent research into visual-spatial working memory (e.g. Andrade et al., 2002; Hamilton et al, 2003) tends to use such tests to assess the visual elements of this memory. When spatial memory is of interest this is usually assessed using the recall of an ordered sequence of blocks, pointed out from an array. Clearly these two types of tasks are measuring slightly different elements of visual-spatial memory, but whether it is legitimate to categorise one as measuring visual span and the other as measuring spatial span seems more problematic. Interestingly, Reuhkala (2001), using these two sorts of task sometimes refers to them as visual and spatial memory tasks and sometimes as measuring, respectively, static and dynamic visual-spatial memory.

In conclusion, while it seems difficult to be precise about exactly what element of visual-spatial memory this test measures, it seems legitimate to state that it is not too contaminated by verbal strategies and is indeed assessing visual-spatial short term memory. This is, in turn, thought to be a vital component of visual-spatial ability so the test should add to information from the more general MidYIS test of non-verbal skills. Furthermore, it should complement the attempt to assess preferred cognitive style and perhaps suggest more about the relationship between preferred processing style and particular abilities.

4.2416 Recognition test

<u>Rationale</u>

For the reasons discussed previously (Chapters 2 and 3), it was intended that visual-verbal cognitive style should be assessed on the basis of general processing style. This makes the research more generally applicable and also avoids the problem of any apparent mathematical processing style being a possible result, not a cause, of an individual's difficulties with a set of maths problems. However, the existing imager and visualiser-verbaliser scales tend to be self report in nature. The fundamental problem with self-reporting is summed up by Kline (1998, p.158), who states:

> In summary, it is simply contrary to any reputable account of human psychology to imagine that much could be learned from simple questions. If it could, there would be no problems in understanding human behaviour and there would be no subject called psychology because there would be no need for it.

More specifically, self reports on mental images seem likely to suffer from differences between people in the criteria they use to rate their own images. Richardson (1977) reports that there is some evidence for imagery ratings reflecting general patterns of response, with ‘low imagers’ having higher criteria. Arnheim (1969, p.102) discusses the likelihood of someone failing to report an image because they did not consider that they had experienced one. Other researchers have reported that some introspective measures of imaging, when factor-analysed, tend to load heavily on social desirability (Di Vesta et al, 1971; Richardson, 1977).

Self reports of habitual ways of thinking, which give the opportunity to be either positively visual or verbal, such as Paivio’s IDQ (Paivio, 1971), seem less prone to these problems (Richardson, 1977). However there is considerable doubt about using such measures with children rather than adults. Presmeg (1985) found good correlations between a self-rating of visual-verbal tendency and her other measures for adult participants and older school children. For 14-15 year old children these correlations were much lower and she concluded that children of this age are too young to reflect accurately on their mental processes.

Since the current research was intending to investigate 11-12 year old children, it seemed unwise to use any sorts of self report or introspective measures. Therefore a method of assessing cognitive style through observing behaviour and without directing attention to this aim was sought. It has been argued (Leutner & Plas, 1998; Plas et al, 1998; Mayer & Massa, 2003) that a valid way to assess preferred learning style, avoiding problems of self report, is to ask participants to make a choice between visual or verbal presentations of information. However, learning style has been previously argued (Chapter2, section 2.4) to be only a part of cognitive style and a related view about the structure of the visualiser-verbaliser dimension is advanced by Mayer and Massa. More fundamental than a choice of presentation, which might occur for a range of reasons, would seem to be the cognitive processing that an individual tends to use. Justification for the distinction at this level between the visual and the verbal can be found in psychology and neurology (Chapter2, section 2.2). Therefore it was concluded that, since this research is interested in general processing tendency, what is needed is an exercise where information must be processed and then investigation of responses can establish whether the individual tends to encode the information visually or verbally.

The recognition test used was adapted from a procedure described by Richardson (1980) as a feasible method of indicating a person's coding preference, either visual or verbal, when remembering items. Since memory is so heavily involved in the theory of cognitive style, and there is plenty of evidence for separate processing of visual and verbal information in working memory (Chapter 2, section 2.2), predominant style of encoding in memory seems a reasonable way of considering coding preference. Riding's attempts to assess a person's position on a continuum between verbal and visual thinking styles (Riding and Rayner, 1998;

Riding and Calvey, 1981) also do so by considering the visual and verbal coding of remembered information. His tests compare speeds of responses to questions requiring the two codes and he points out that this means an individual's visual responses are compared to their own verbal responses, not to a normed response time, and so indicate a personal visual-verbal bias. However, Riding's otherwise carefully argued theoretical background makes one vital assumption about the nature of visual processing in that his visual coding questions ask about the colours of items , rather than any other visual aspect. This assumption that visual processing is essentially colourful seems unwarranted when it is considered, for example, that only some of those reporting distinct visual-spatial experiences of number (Galton, 1880; Seron et al, 1992. See Chapter 3, section 3.11) describe these as having colour. It seems quite likely that colour varies in importance for visual thinkers and that therefore the measurement of visual processing should not depend on it.

The recognition test used in this research does not have this disadvantage yet includes the benefit noted by Riding that an individual's level of verbal coding is compared to their own visual coding to generate a visual/verbal ratio which does not rely on absolute levels of response. The test relies on an effect found by an earlier researcher, and broadly supported by Richardson's work, that when people are asked to remember words and pictures, errors they make on a recognition test will reflect how they coded the original items. If they tend to encode verbally, they will make more false positive mistakes with words, having previously seen pictures of the items, than they will make with pictures, having previously seen the words. A more visual person would be expected to have the opposite ratio of visual to verbal false positive responses.

Method of testing

The recognition test involved twenty items to be remembered, of which half were pictures and half were words. These were chosen from a total of 30 pictures and their associated 30 words, with the choice and arrangement being random within the constraint that half the items were mathematics related and half were not. The items were chosen so that the words were easily read nouns, while the associated pictures were straightforward to recognise. The items were arranged in a random order in two columns on an A4 piece of paper, with the pictures drawn to fit into 3cm by 3cm spaces and the words printed in 18 point size (see Appendix A). The participants were given two minutes to study and remember the list of items, with the instructions that they should ask if they could not read any of the words but not ask about the pictures (since a verbal explanation from the researcher would have interfered with the participant's own encoding). The instructions, which were given to the participants verbally, were as follows:

> When I tell you, not before, I want you to turn the paper over. You'll have two minutes to look at the pictures and words and try to remember them. If you can't read any of the words, put your hand up and ask me. Please don't ask me about the pictures – just try to remember them. Does everyone understand?

A week after they had seen the list of items, the participants were shown the complete set of 60 words and pictures (see Appendix B), one at a time, and were asked to answer "yes" if they thought an item had been on the original list, "no" if they thought it was not. The items were increased in size (the pictures enlarged by a factor of 2; the words printed in 72 point size) and reproduced on cards that could be shown to the participants. They recorded their responses on a sheet numbered from 1 to 60.

From the response sheet it was possible to work out a discrimination score by subtracting the proportion of false positives from the proportion of correct positive responses. The number of responses made to the cross modal decoy items (the items in the opposite modality from the original twenty to-be-remembered items) was recorded and a visual/verbal ratio calculated from the number of 'visual errors' compared to the number of 'verbal errors'. The only difficulty with this method of calculation was when a participant had a zero score for either visual or verbal errors. If the verbal score is used as the divisor this means that no verbal errors make the ratio impossible to compute. Additionally, where no visual errors are made, the final ratio does not distinguish between someone making only a single verbal error and someone else making four of five such errors: both participants have a visual tendency ratio of zero. However, the alternative to a ratio of calculating differences between the two numbers of errors was unappealing because this would lose the essential idea that the test was trying to measure the participant's balance between visual and verbal thinking. Thus a difference score would assess someone with visual and verbal errors of 9 and 7, respectively, as similarly visual to someone with visual and verbal errors of 4 and 2, despite the fact that the second person has made twice as many visual as verbal errors. It was anticipated that the pilot study would provide guidance on the scoring, as well as on other aspects of the recognition test.

Piloting the recognition test

The recognition test was administered twice, at the beginning and end of the pilot study, with the two occasions being just over a month apart. In the pilot study school pupils were streamed in Years 7 and 8 for most lessons, according to broad academic criteria, and the deliberately small, low ability set formed the pilot study class. This setting had the advantage that if the organisational aspects of the recognition test could be managed by these children, then it should not present any problems for groups with a greater range of abilities and achievement.

The pilot study participants were able to follow the instructions and seemed able to read the words to be remembered. During the two minutes study session, one child checked that they had correctly read “reflection”, but nobody else asked about any words. The assumption that they could read the words is supported by a word recognition reading test (Appendix C), administered at the end of the pilot study, where all the participants were able to find all the to-be-remembered words from among similar words completely correctly.

For each participant, on each occasion of taking the test, a hit rate and a false positive rate were calculated. These proportional scores were used to give a basic measure, the discrimination score, Pr, of the individual’s ability to discriminate old from new items. There was no evidence of the problem noted by Richardson (1980) that the recognition rate for pictures tends to be much higher than the rate for words, with ceiling effects introducing complications. For pilot study participants, the correctly recognised pictures tended to be similar in number for each person to their number of correctly recognised words. This lack of a picture superiority effect could have been because of the fairly long time between presentation and test or could perhaps be the result of the style of pictures used. These were simple line drawings and diagrams, which perhaps do not provide the wealth of redundant information contained in a genuine picture.

There was a lot of variation between individuals, but the scores for each individual did not vary much between the two tests and there was a significant correlation between discrimination scores on the two tests (Pearson correlation coefficiant of 0.73, p=0.026, N=9). It seemed most likely that the false positive rate would change, given that by the second test all the items were somewhat familiar. Inspection did not suggest too much change, although there was evidence of a slight increase (from a mean of 0.3 to a mean of 0.4 for the nine pupils who experienced both tests). However this was not found by a t-test to be statistically significant (p=0.064).

A more far reaching problem was that it was considered possible that the scores of interest, the visual/verbal error ratios might just reflect some other aspect of answering style. For example, finding that the visual/verbal ratio was related to accurate memory or to any other aspect of performance would undermine the idea that the ratio was assessing processing style, since there is no reason why this should determine accuracy. So correlations were calculated between the ratios, the discrimination scores, the hit rates, the false positive rates and the rate of false recognition of the ‘decoy’ items (the ones remembered in one modality then seen in the other). This produced no significant correlations. There were few zero scores (only 3 out of the 46 error scores which resulted from the test and retest with the pilot study participants), so the potential problem of calculating visual/verbal ratios was judged not to be too serious in practice. The lack of discrimination between apparently extremely verbal participants, while not ideal, was felt to be acceptable because the main interest of the study was with the visual participants. Any visual participants who made no verbal errors present more of a problem, since their ratios cannot be calculated but this did not arise in the pilot study. Finally, it was questioned whether the 'decoys' were actually attracting more false positives than the other new items, which might be expected if a serious amount of cross-modality confusion had occurred. The decoy rate (mean 0.43) was compared to 0.5, which would be expected by chance, and was found not to differ significantly (p=0.11).

<u>Validity</u>

Since the recognition test was designed because it was judged that there was not another test appropriate to measuring general visual or verbal encoding tendency, this made validating the test difficult. As has been discussed, self report measures of habitual processing tendency have been successfully used with adults, but there is real doubt about whether children of this age can reflect accurately on their own thinking styles. Alternatively, any attempt to validate the test by comparing it to actual mathematical processing risks confusing struggles with a difficult question, which result in some reported images, with an immediate, successful use of a visual method. The main justification for this test must be that it can be justified theoretically and it can be argued (Kline, 2000) that this is a vital underpinning for any test if it is to avoid identifying supposed human traits which are actually only clustering of test items. Supporting this theoretical underpinning, pilot study analysis of all the scores resulting from the test did not suggest that the visual/verbal ratio was just reflecting some other aspect of answering style or general facility with the memory test. The implication is that the visual/verbal ratio really is measuring some other cognitive tendency, which theory suggests is individual coding preference.

The pilot study was of further use in giving general suggestions about whether the test was a valid measure of visual-verbal tendency since the participants' visual/verbal ratios could be compared informally with other observations of their behaviour and mathematical performance. Most of the participants were interviewed and they were forthcoming about the sorts of mathematics they liked and were able to give good descriptions of their thinking when working on maths problems. In addition, a pencil and paper test (Appendix D) was used before the pilot study to give an indication of the pupils' levels of knowledge of particular areas, which were going to be covered during the teaching. The items on this test could be divided into three groups according to their presentation, since they relied predominately on either words, a diagram or numerals and, for each participant, percentage accuracy could then be calculated for each type of question. In general, these revealed superiority on the diagram questions over questions involving words or numerals. However, there were some individuals who did not follow this pattern and others for whom it was exaggerated.

It should be noted that since the maths test was not designed to suggest visual or verbal style, it was inevitably an imperfect instrument. However it would be expected that individuals with a tendency to think more visually would show particular strength on questions that relied on diagrams, while those with a tendency to think more verbally would perform better on the word questions. Although there are difficulties in interpretation because virtually all the pupils did better on the diagram questions, differences can be observed between the patterns of accuracy for each pupil.

For three pupils, the plots of their percentage accuracy on the three types of maths test question appeared flatter than for the rest of the class. This is confused by the ceiling effect created by the diagram questions being answered much better by all the participants. However, for these three their diagram scores could have been higher, so they were not literally at the ceiling. This is quite a different pattern of results from the rest of the class and particularly different to that produced by the pupils identified as ‘visual’. After these results for these three pupils had been noticed, their visual/verbal ratios from the recognition test were considered and these were found to be generally lower than for the rest of the class (all below the sample median of 0.5). Two of the pupils, Sam and Cathy, were among the interviewees and both tended to answer mathematical questions by using counting strategies, rather than the imaging strategies reported by ‘visual’ pupils, and were occasionally misled by spurious number patterns.

When plots of percentage accuracy on the three types of question were examined to find participants whose patterns of responses suggested visual strength, four individuals were identified. For these children, the general pattern of superiority on diagram questions was exaggerated. Interviews with two of these pupils did suggest, strongly in the case of one of them, that they used visual methods to solve mathematics problems, although they also used other strategies. The third pupil’s interview suggested that he might prefer visual aspects and content within mathematics, but since his reading appeared weak it was thought that this could be more due to verbal weakness than to a preferred visual style. The fourth pupil was not interviewed.

Table 4.2 shows how having a visual/verbal ratio above the sample median of 0.5 co-occurs with the observed pattern on superiority on diagram items.

Table 4.2 Visual/verbal ratio and observed performance on test items

	visual/verbal ratio > 0.5	visual/verbal ratio ≤ 0.5	
Superior performance on diagram questions	3	1	4
Normal performance on diagram questions	0	6	6
	3	7	10

Although the small numbers involved suggest that it is unwise to attempt a Chi-square test, the cell values do appear quite different from that expected if the two variables were independent. However, this suggestion is lost if an alternative means of identifying visual strength on the maths test is used. If the difference is calculated between percentage accuracy on ‘diagram’ and on ‘word’ items and participants are divided according to whether their score differences are above or below the median difference of 20, Table 4.3 is the result.

Table 4.3 Visual/verbal ratio and performance on test items relative to median difference between word and diagram success

	visual/verbal ratio > 0.5	visual/verbal ratio ≤ 0.5	
Diagram accuracy – word accuracy > 20	2	3	5
Diagram accuracy – word accuracy < 20	1	4	5
	3	7	10

As will be observed, this table does not suggest a relationship between strength on diagram questions and the visual/verbal ratio.

Since the numbers are too small to allow a straightforward statistical analysis, it is not necessary to choose which method of categorisation to use to identify superior performance on the diagram questions, but the above tables do demonstrate the equivocal nature of some of the pilot study results. When information from the interviews and observations from the classroom were added to suggestions from the maths test, a pattern did seem to emerge, which included the visual/verbal ratio and could be seen as giving it validity. In particular it did seem possible to identify as verbalisers the three children with the lowest visual/verbal ratios, who were also less successful on diagram questions, more successful on word questions and in general tended to use verbal strategies. However, visualisers were more difficult to identify and this perhaps points to a difficulty in classifying people as decisively visual or verbal.

Despite these concerns, though, it seems reasonable to conclude that there did seem to be some coherence between the visual/verbal ratio, relative strength on diagram questions, reported preference for certain areas of mathematics and the use of visual strategies to answer mathematical problems. This is encouraging for the validity of the visual/verbal ratio, but a further important question, if it is to be used to indicate a person's cognitive style, is its reliability.

Reliability

Since the recognition test was carried out twice during the pilot study, it should be possible to calculate its test-retest reliability. However, there was some concern that, because the assessment used a recognition test, the second test would be very different in nature to the first one since on the second occasion all the test items will be somewhat familiar. It seemed most likely, therefore, that the false positive rate would change and there was evidence of a slight increase in this rate but this was not statistically significant. Supporting the idea that the test was approached in a similar way on both occasions the discrimination scores on the two tests correlate significantly.

Therefore, despite some initial concerns about using a test-retest measure of reliability this does actually seem to be a legitimate gauge of the test's reliability and the correlation should be considered between the visual/verbal ratios produced by the two occasions of testing. However, the next concern is whether it is appropriate to use a parametric correlation co-efficient on these ratios. Although the numbers of visual and verbal errors can be considered to be an interval scale, this seems less likely with the final ratios. The mathematical operation involved in their calculation has the effect of compressing the verbal end of the scale (to between 0 and 1) while stretching the visual end (comprising scores of 1 to 10). Furthermore, the visual/verbal ratios of zero do not distinguish between different verbal error rates. For these reasons, it seems more appropriate to consider the test-retest correlation using a non-parametric method. A Spearman's Rho correlation co-efficient was therefore calculated and found to be 0.724. This is a statistically significant and satisfactorily high correlation, but obviously the number of participants in the pilot study was very small (N=9).

Conclusions from piloting the recognition test

Together with the theoretical underpinning of the visual/verbal ratio, the pilot study observations suggested that the recognition test does measure visual-verbal thinking style. The test-retest correlation suggests that individual visual/verbal ratios are reasonably reliable measurements. For further quantitative assurance of validity and reliability, more participants were needed and these were provided by the main study.

Issues to be addressed by the main study

The main fieldwork provides a simple increase in sample size, when all the participants' data are combined, and also, by coming from a different school, the main study results allow a check that the pilot study pupils are not so unusual in some way that their results cannot be generalised.

Validity

As in the pilot study, some validity issues can be addressed by considering the relationship of various scores produced by the recognition test. In particular, a strong correlation between the visual/verbal ratio and the discrimination score would suggest that the ratio is failing to be a measure of style, uncontaminated by test proficiency or certain cognitive abilities. Additionally, the main study allows the comparison of the visual-verbal scores on several measures relating to visual-spatial and verbal abilities (spatial memory test, MidYIS non-verbal score and MidYIS vocabulary score). Previous research (Kozhevnikov et al, 2002; Hegarty & Kozhevnikov 1999) has generally found that visualiser-verbaliser measures do not correlate with tests of spatial ability, although there has been some suggestion of a negative correlation between spatial ability and tendency to process mathematical information visually (Lean & Clements, 1981). If the visual/verbal ratio is working as a measure of processing tendency' rather than ability in particular cognitive areas, it would be expected not to correlate highly with these other measures, although a non-significant tendency might be expected since ability and preferred style are unlikely to be entirely independent.

Reliability

As in the pilot study, the recognition test was given to the main study participants both before and after the intervention. Provided there are no reasons to suspect that the two samples of participants differ in important respects, it seems legitimate to combine the two sets of data and consider test-retest reliability for all the participants. Although the main study pupils experienced an interval between test and retest of approximately three months, compared with just over one month for the pilot study children, it is felt that these time intervals are comparable. In both cases it is much too long for active rehearsal of the material and both periods are considerably longer than the actual retention interval, of one week, used by the recognition test.

Limitations and extra investigation

Even with the increased sample size provide by the main study, a major limitation to confidence in the recognition test was that it had only been used on a fairly small sample. Furthermore, these children were all Year 7 pupils and they only represent a section of this year group since they were all taken from maths sets intended to contain the less able. Therefore it seemed sensible to administer the recognition test to some more children, including some older and more able students to see if the pattern of results, and particularly the distribution of visual/verbal ratios, was similar. This was done after the main study fieldwork was concluded, in the same school. The additional Year 7 pupils were 29 children, also from the lower half of the ability range. The older participants were 33 students nearing the end of Year 10 and taken from the two top sets, which represent the higher achieving 20-25% of the year group. They were therefore over three years older than the other participants and considerably more experienced and successful in secondary school mathematics.

In addition to administering the recognition test to these older pupils, it seemed advisable to use them in another attempt to validate the visual/verbal ratio by linking it to other indicators of visual or verbal style. Although the main study provided quantitative data relating stylistic tendency to particular cognitive abilities, there were problems with supporting qualitative data and, particularly, with relating the visual/verbal ratio to other suggestions of cognitive style. The interviews with the main study sample proved inconclusive (see Chapter 5, section 5.111 for more detailed description), while observation of pupils during the lessons did not suggest that the recognition test was clearly identifying pupils' thinking styles. However, it is always difficult to be sure exactly what strategies a person is employing and this is particularly the case in a classroom situation, where it is impossible to concentrate on any one child for long. Therefore interviews with people identified as visualisers or verbalisers by the recognition test seem advisable, where processing styles could be probed more carefully and in a number of ways.

Interviews with the older participants seemed likely to be more fruitful than those with the main study pupils, as these older individuals should be more able to reflect on their own abilities and stylistic tendencies. Since they were older, and also more capable with school mathematics, it was expected that they would be more confident and able to work through maths problems, which could reveal their approaches and methods. By specifically identifying visualisers and verbalisers to interview, it was hoped to make maximum use of each opportunity to carry out an interview.

The interviewees were six pupils taken from those with the highest visual/verbal ratios and another four pupils selected to match these in terms of mathematics performance and recognition test discrimination scores, but who were identified as verbalisers. These students were asked the questions relating to preferred areas of mathematics previously used (Section 4.242), before being asked to work through a number of problems, with pencil and paper provided. These questions included some from the bank of questions used with the Year 7 pupils (Appendix E) as well as some of those used in previous research (Appendix F). They were intended to be broadly appropriate for the age and ability of the interviewees and to allow a range of possible solution strategies, some more visual than others. At this stage, then, the participants' mathematical processing style had been probed indirectly by asking them about preferred mathematical topics and directly by asking them to demonstrate some problem solving.

Next it was decided to ask them to carry out three mental rotation tasks, which each involved mentally rotating a three dimensional configuration of blocks to find which of five pictured alternatives was correct (see Appendix G). The aim was to give an indication whether visual processing tendency is related to ability to visualise material and manipulate visual images. It was anticipated that this would add to any understanding based on the main study quantitative data about the relationship of preferred style to visual-spatial ability. Finally, the interviewees were asked to consider how they think, and in particular whether they tended to "think in pictures and diagrams, or in words". The intention was to investigate whether these people, who had been identified by the recognition test as visualisers or verbalisers, considered themselves to be visual or verbal thinkers. It was assumed that such self-report might have some validity for these older, and so more meta-cognitively aware, participants but it was recognised that there will always be doubts about any simple self report and the extent to which cognitive processes can be accurately introspected upon.

4.242 Additional assessment of subgroup of main study participants

4.2421 Maths question sorting

Questions were found or devised, and tested during the pilot study (Appendix H). Each needed one of a range of simple mathematical procedures to solve it but had surface features unrelated to the underlying mathematics. These questions were written on cards and the children seen individually were asked to sort them into groups according to whether they seemed to be "about the same sort of maths" and "needed the same sort of maths to answer them". The idea (based on the method of

Schoenfeld, 1985) was to see if the pupils tended to sort the questions according to superficial features or whether they could see through these to the mathematical operations underlying them, as this would suggest deeper understanding.

4.2422 Strategy choice

During the individual interviews, the interviewees were asked to try some mathematics questions (Appendix E), which were not given in any particular order. Some of these questions were chosen because they lent themselves more or less to either verbal or visual methods. Other questions were more open and it seemed interesting to see how individual children would approach them. This procedure was based on that used by Krutetskii (1976) to classify mathematically talented Russian children according to their tendency to use visual methods to solve a problem. Here, questions were chosen from the range available depending on the participant's success with previous questions, so that the problems were found by the child to be easy enough to attempt but not so easy that they could give an automatic response. The intention was to allow the children to talk through their approach to, and attempted solution of, individual questions then examine their responses to see if they suggested a preferred thinking style.

4.2423 Interviews of sample of participants

During the individual interviews, before the intervention, the children were asked the following questions to elicit broadly their attitudes to school mathematics. Some of the questions relate to their strengths and weaknesses, both to reveal more about their views of what constitutes mathematics and to examine any individual preferences. The mathematics mentioned might suggest an individual's preferred ways of thinking.

During the pilot study these questions had produced some interesting comments and opinions.

1) In general, do you enjoy maths lessons? Why?
2) Name some maths work you have enjoyed.
3) Name some maths work you haven't enjoyed.
4) What maths work do you find easy?
5) What maths work do you find hard?
6) Why do you think you have to do maths?
7) Imagine you're faced with a maths question that looks hard. What do you do to try to work it out? (Prompt: And then what? What if you get stuck in the middle?).

After the intervention lessons, the interviewees were asked the following questions to assess their response to the intervention.

Thinking about the lessons I took,

1) What do you remember doing?
2) What do you think might be useful in the future? Why?

4.3 Intervention Lessons

Introduction

As has been described above (Section 4.11), the lessons making up the interventions were designed to cover a range of mathematical content with the only limit to the content being to avoid teaching 'shape and space' lessons. This is because it was felt that mathematics in this area is quite uncontroversially linked to visual spatial processes, where as the interest of this research is in the extent to which such processes underlie, or can be used to support, other mathematics such as number and reasoning work. The visual lessons were as purely visual as possible in an attempt to avoid any problems of ‘verbal overshadowing’ (Chapter 1, section 1.32; Chapter 2, section 2.31).

All the lessons were planned to follow the school’s scheme of work, which had been organised to accommodate the Key Stage 3 National Strategy Framework (DfEE, 2001) for teaching mathematics. A particular effect of the Strategy was that the lessons were explicitly organised into three parts, with an ‘oral and mental starter’ forming an introduction to each lesson (these are fully described in the lesson plans) and some attempt at a ‘final plenary’ forming the conclusion. These are not always described in the lesson plans but some attempt was always made to round off the lessons by provoking the children to reflect on their work.

Lesson 1: Introduction to intervention and to data handling

Aim

It was intended to introduce the lessons as being distinct from the classes' usual mathematics but with relevance for them. The 'mental starter' was an occasion of question generation, which it was anticipated would reveal the breadth and style of the children's mathematical knowledge.

Introduction

An identical basic introduction was given to each intervention saying, "I'm going to be teaching you for this lesson every week this term. And what we're going to do should help you in your other maths lessons. We're going to think about maths a bit differently. And you're going to have to think for yourselves quite a lot. You're going to be working on paper and I won't be telling you exactly what to write and how. All I ask is that you put your name on each piece of paper."

Mental Starter

The activity was identical for the two classes and was introduced as follows: "First, here's something to get you thinking. I'm going to give you an answer to a maths question and I want you to make up some questions with this answer. What I'm looking for is as many different questions as possible, and make them as different as possible. Write your questions on your paper. Use pictures or diagrams if you like." The target answer given was '25'. Once pupils had finished writing, their questions were read out and shared. Following on from this activity, specific introductions to the lesson styles were given.

Introduction to the visual-spatial lessons

"Generally in these lessons, we're going to be trying to *see* numbers and explain things with diagrams and pictures. Now, in your other maths lessons this week, you're going to be looking at handling data. So I want us to have a think about that now." The class were asked for words from this content area, which were written on the board. They were then asked to draw diagrams to illustrate three of these concepts.

Introduction to the verbal lessons

"Generally in these lessons, we're going to be thinking harder about *how* we do maths and trying to explain things to each other. Now, in your other maths lessons this week, you're going to be looking at handling data. So I want us to have a think about that now." The class were asked for words from this content area, which were written on the board. They were then asked to write explanations for three of these concepts.

Lesson 2: Numbers and factors: Visual-spatial approach

Aim

The lesson was based on an idea underlying the use of Stern equipment as described by Wing (1996). Children are encouraged to build up conceptions of numbers based on how these numbers of items can be arranged rather than relying on counting. The arrangements are possible because of the mathematical properties of the number (e.g. the fact that 9 can be imagined as a 3 by 3 square is because it is the third square number). Counting only practises a single, well rehearsed, and by this age generally well understood, routine that does not reflect the particular qualities of individual numbers. The initial activities led to a visual presentation of factors.

Mental starter

The class was quickly shown two cards, each picturing six dots, and asked how many dots they could see. Given the short exposure, an accurate answer was only possible when the dots were arranged, as they were on one card, in two rows of three rather than scattered with no pattern, as they were on the other card (Figure 4.1). Attention was drawn to this.

Next other cards were shown of patterns of dots (Figure 4.2) and the pupils were asked to write down how many dots they had seen. This activity was taken from the Israeli Agam Project (reported by Hershowitz et al 1996) which is intended to develop the visual thinking of children. The exercise introduced the class to the idea of seeing "how many, without counting" by providing a task where this is necessary and where grouping strategies emerge.

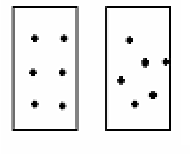


Fig 4.1 Cards featuring differing arrangements of six dots

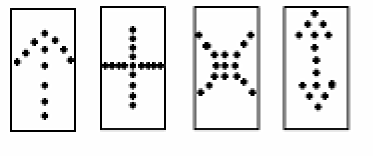


Fig 4.2 Cards featuring patterns of dots

How many without counting?

The class was asked to draw arrangements of dots to show the following numbers, trying out their arrangement on a neighbour and trying to think of alternatives:

9 (example)

12, 13, 20, 7

Factors

It was pointed out that rectangle patterns show factors and it was demonstrated how to use dot patterns to identify a number's factors. Pupils then tried to use this method to find all the factors of the following numbers:

14, 18, 21, 25, 19, 36.

They were asked to think about what it means if:

(i)A number of dots will not make a rectangle (primes)

(ii)A square can be made (square numbers)

Lesson 2: Numbers and factors: Verbal approach

Aim

It was intended to practise number facts and number bonds to increase arithmetic fluency. Also the lesson involved discussion of the meaning of the operations (e.g. multiply).

Mental starter

The number 6 was described in two ways as “the number after 5” and as “two 3s”. There was some discussion about which description makes identifying the number easier.

Next the class was asked to write down the numbers described as follows:

6 and two 3s;

four 4s and 1 more;

9 and four 2s;

3,2,1 twice and an extra 3.

These were descriptions of the cards used in the visual lesson.

How many without counting?

It was suggested that number facts and “breaking down” numbers help with calculations. Children were asked to give descriptions of the following numbers by breaking them into parts:

9 (example)

12, 13, 20, 7

Factors

Given that a good way of breaking a number down is by using its factors, a description, then definition, of the concept was elicited. Then pupils tried to find all the factors of the following numbers:

14, 18, 21, 25, 19, 36.

They were asked to think about

(i)How to know when all a number's factors have been found

(ii)When the number of factors is not even

Lesson 3: Calculations: Visual-spatial approach

Aim

The lesson was intended to promote the use of diagrams, in the context of helping to choose the correct operation to carry out. The empty number line was particularly emphasised as this has been found to be helpful by educators (e.g Verschaffel & De Corte, 1996) and researchers (Lewis, 1989; Willis & Fuson, 1988).

Mental starter

A paper number line, marked in units up to 23 was put up. The class was asked to "Think of a number that is:

(i) bigger than 18

(ii) smaller than 12

(iii) greater than 5.5

(iv) less than a half."

The answers were related to the number line and the class was asked to confirm which way one moves along the line for bigger numbers and which way for smaller numbers.

Choosing the right calculation

A copy of an exercise from the SMP B2 book (School Mathematics Project, 1985) was used (Fig 4.3) was used. The first question was discussed, then the children worked individually on the others. The second sheet (Fig 4.4), advocating 'thinking in pictures' was given out so the children could check their answers.

.

A Doing the right calculation

A calculator will do your calculations for you. But it will not tell you what calculation to do!

You have to think carefully about each problem, before you decide whether to **add, subtract, multiply** or **divide**.

A1 Read this problem carefully.

Stella came into the cellar and took away 79 bottles.
After she had gone, there were 28 bottles left in the cellar.
How many bottles were there before Stella came?

Which of these calculations gives the answer to the problem?

$79 + 28$ $\quad$ $79 - 28$ $\quad$ $28 - 79$

79×28 $\quad$ $79 \div 28$ $\quad$ $28 \div 79$

A2 Read each of these problems and then choose the calculation for it.

(a) The king shared out his farmland equally between his 7 daughters. Each daughter got 161 square miles.
How many square miles did the king share out?

$161 + 7$ $\quad$ $161 - 7$ $\quad$ $7 - 161$

161×7 $\quad$ $161 \div 7$ $\quad$ $7 \div 161$

(b) Baljit had 436 foreign stamps. His uncle gave him some more.
Then he had 712 altogether.
How many stamps did his uncle give him?

$436 + 712$ $\quad$ $436 - 712$ $\quad$ $712 - 436$

436×712 $\quad$ $436 \div 712$ $\quad$ $712 \div 436$

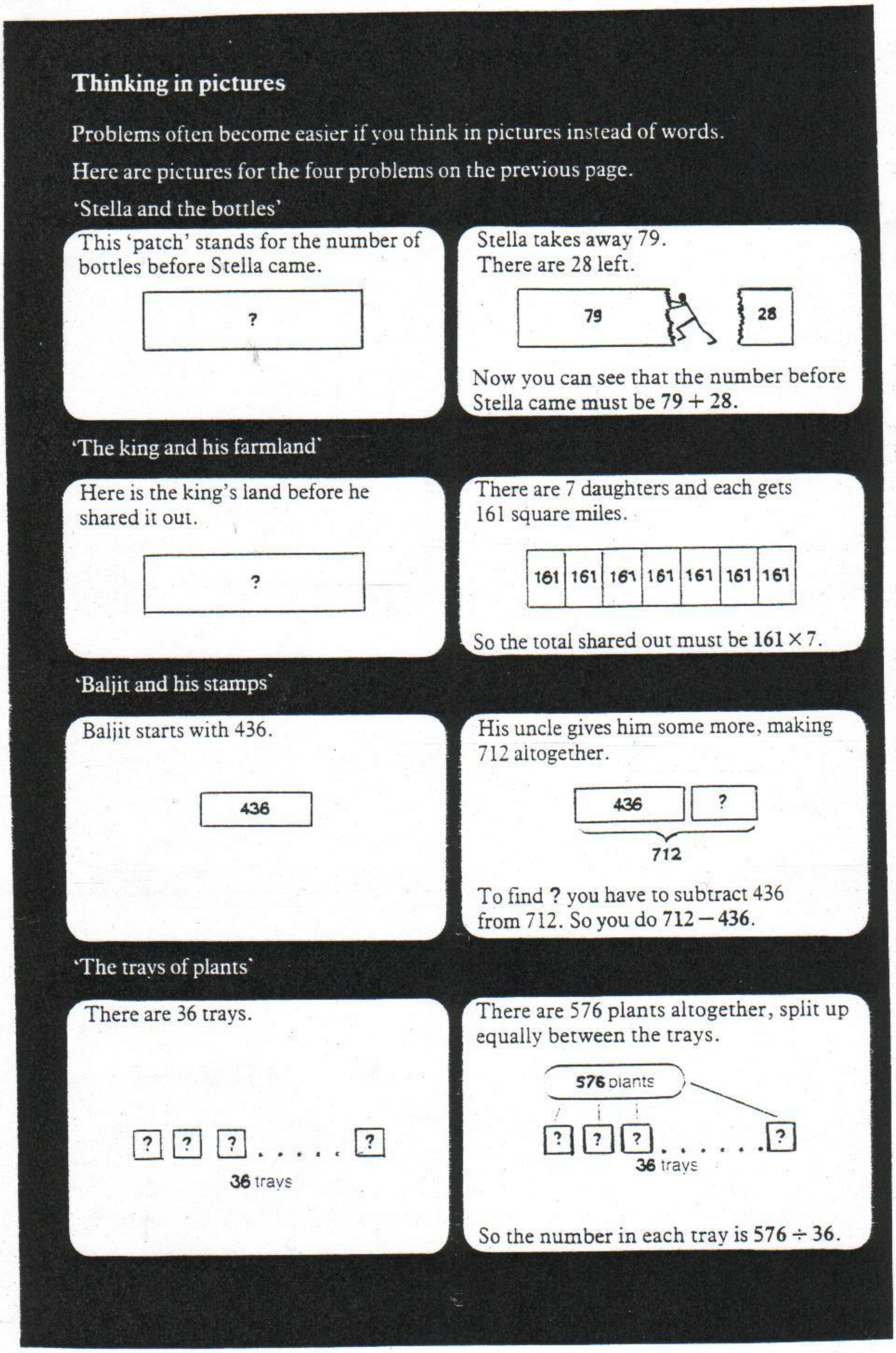
(c) In the greenhouse there were 36 trays of plants, with the same number of plants in each tray. Altogether there were 576 plants. How many plants were there in each tray?

$36 + 576$ $\quad$ $36 - 576$ $\quad$ $576 - 36$

36×576 $\quad$ $36 \div 576$ $\quad$ $576 \div 36$

Find out if your answers to questions A1 and A2 are right or wrong before you continue.

Fig 4.3 An exercise on choosing the right calculation



Thinking in pictures

Problems often become easier if you think in pictures instead of words.

Here are pictures for the four problems on the previous page.

'Stella and the bottles'

This 'patch' stands for the number of bottles before Stella came.

?

Stella takes away 79. There are 28 left.

79 28

Now you can see that the number before Stella came must be **79 + 28.**

'The king and his farmland'

Here is the king's land before he shared it out.

?

There are 7 daughters and each gets 161 square miles.

161	161	161	161	161	161	161

So the total shared out must be **161** × 7.

'Baljit and his stamps'

Baljit starts with 436.

436

His uncle gives him some more, making 712 altogether.

436 ?

712

To find **?** you have to subtract 436 from 712. So you do **712 − 436.**

'The trays of plants'

There are 36 trays.

? ? ? . . . . . ?

36 trays

There are 576 plants altogether, split up equally between the trays.

576 plants

? ? ? . . . . . ?

36 trays

So the number in each tray is **576 ÷ 36.**

Fig 4.4 Textbook description of 'thinking in pictures'

Inconsistent questions

It was then argued that diagrams can be helpful and that number lines are particularly good. A sheet of questions (Fig 4.5) where the wording is ‘inconsistent’ with the mathematical operation required (as Lewis, 1989) was used. The first question was demonstrated on the board using an empty number line on which to represent the information.

For each question draw a numberline, or other diagram, to show which calculation needs to be done. Then do the correct calculation and answer the question.

1) David earns £26 a week less than Stephen. David's pay is £172. How much is Stephen's?

2) Cornish pasties from the Town Bakery cost 14p more than those from Jo's Pie Shop. At Jo's Pie Shop Cornish pasties cost 47p each. How much are they at the Town Bakery?

3) Jade has 16 more felt pens than Lisa. Jade has 40 felt pens. How many has Lisa got?

4) On a computer game, Laura's score is 750 points less than Ben's. Laura's score is 980 points. What is Ben's score?

5) Danielle is 17cm shorter than her brother. He is 151cm tall. How tall is Danielle?

6) James has saved £65 more than Emma. James has saved £251. How much has Emma saved?

7) Michael has sold 26 more raffle tickets than Andrew. Andrew has sold 15 raffle tickets. How many has Michael sold?

8) On a computer game, Louise scores twice as many points as Sarah. Louise's score is 1450 points. What is Sarah's score?

9) Mike, John and David share out a packet of sweets equally between them. Each of them gets 18 sweets. How many were in the packet?

Fig 4.5 Work sheet of inconsistently worded questions

Lesson 3: Calculations: Verbal approach

Aim

A key to choosing the correct operation in simple addition and subtraction is recognising whether the answer will need to be bigger or smaller than the starting point. Therefore the lesson attempted to emphasise the concept of number size and the words used to convey numerosity information.

Mental starter

The class was asked to "Think of a number that is:

(i) bigger than 18

(ii) smaller than 12

(iii) greater than 5.5

(iv) less than a half."

They were then asked for other words meaning 'bigger' and 'smaller'.

Choosing the right calculation

A copy of an exercise from the SMP B2 book was used (Fig 4.3) was used. The first question was discussed, then the children worked individually on the others. It was suggested that underlining words in the questions might help to make sense of them.

Inconsistent questions

The sheet of questions (Fig 4.5) where the wording is 'inconsistent' with the mathematical operation required (as Lewis, 1989) was used. The written instructions were altered from those used in the visual lesson to the following:

"For each question, underline the important words and think which calculation needs to be done. Then do the correct calculation and answer the question."

Lesson 4: Number patterns: Visual-spatial approach

Aim

It was intended to consider odd and even numbers as a particular case of multiples. Dot pattern representation was again shown and this could be used in the later investigation to construct a pictorial proof (see Waring, 2000, for discussion of pictorial proof).

Mental starter

Dot patterns were drawn for some even numbers (Fig 4.6) and the class was asked what sort of numbers they represented. Such use of dot patterns to convey the nature of odd and even numbers is frequently mentioned by teachers and educationalists. For example, Frobisher (1999) discusses the use of such representation while Davis (1972) specifically recommends the associated "proof by visualisation" to explain why the sum of two odd numbers is even. The introduction to even numbers was developed by asking how they knew they were even, what odd numbers would look like and also by pointing out that even numbers are multiples of 2 (relating this to the dot pattern). Dot patterns were then drawn for multiples of 3, 5 and 6 and named by the class as the relevant multiples. It was demonstrated that multiples can also be shown on a number line as the result of equal jumps.

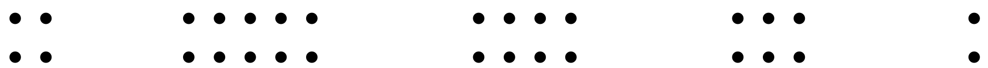


Fig 4.6 Dot patterns for some even numbers

Investigation

Worksheets of the Odds and Evens investigation (Fig 4.7) were used. It was emphasised that it is not enough just to notice patterns in maths; you also have to be able to show why they occur so you know you've found a real pattern.

Adding Numbers Investigation

Choose some numbers and fill in the spaces in these sums.
Don't forget to work out the answers!

☐ + ☐ = ☐
even number + even number

☐ + ☐ = ☐
even number + odd number

☐ + ☐ = ☐
odd number + even number

☐ + ☐ = ☐
odd number + odd number

Choose different numbers and make up some more sums for each of these groups

even number + even number

1)

2)

3)

4)

odd number + odd number

1)

2)

3)

4)

even number + odd number

1)

2)

3)

4)

odd number + even number

1)

2)

3)

4)

Now answer these questions on a different piece of paper:
1) What do you notice about the answers to your sums? (Hint: Which are odd, which are even?)
2) Can you explain (in pictures and words) what you have noticed?

Fig 4.7 Worksheet for odds and evens investigation

Lesson 4: Number patterns: Verbal approach

Aim

It was intended to consider odd and even numbers as a particular case of multiples. Ideas of 'explaining why' were used, in the context of an investigation, to introduce preliminary ideas about proof (Waring, 2000).

Mental starter

Some even numbers were written on the board (4, 10, 8, 6, 2, 24, 16, 44, so in arabic numerals and not in numerical order) and the class was asked what sort of numbers they were. This was developed by asking how they knew and whether they could describe or explain what an even number is. The definition was extended to include the fact that even numbers are multiples of 2 and a definition of odd numbers was requested. Multiples of 3, 5 and 6 were written up and named by the class as the relevant multiples. This was illustrated by writing the appropriate multiplication calculation under each number.

Investigation

Worksheets of the Odds and Evens investigation (Fig 4.7) were used. It was emphasised that it is not enough just to notice patterns in maths; you also have to be able to explain why they occur so you know you've found a real pattern.

Lesson 5: Patterns and sequences: Visual-spatial approach

Aim

This lesson intended to start from square numbers, since they are types of number where a visual-spatial link is particularly clear, but then extend the idea of visual pattern to number sequences.

Mental starter

The class was reminded that multiples can be represented by rectangles of dots and the possibility was discussed of the dots making a square. The class was asked whether all numbers could produce a square, leading to the question of which ones will. The square numbers were then represented by an ordered sequence of dot patterns (Fig 4.8). It was mentioned that this was a sequence and the class was asked to answer the following:

(i) Find the 7th square number

(ii) Find the 13th square number

(iii) Is 81 a square number? If it is, what is its position in the sequence?

(iv) Is 196 a square number? If it is, what is its position in the sequence?

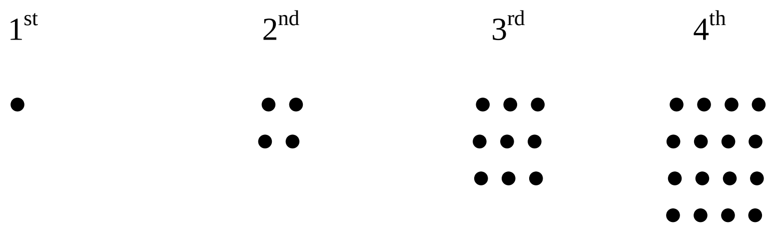


Fig 4.8 Dot patterns representing the first four square numbers

Sequences

Using the sequence 1, 4, 7, 10…it was noted that some visual representations might be no help (Fig 4.9), but that using the numberline helps to see the rule for finding the next number. It was demonstrated that this makes it possible to construct an informative diagram that conveys the rule for the sequence (Fig 4.10) and this could help to find a number deep within the sequence. The following exercise was then given:

For each sequence

a) Find the next two numbers in the sequence

b) Find the rule

c) Draw diagrams to show that your rule is correct and why it works.

2, 8, 14, 20…

21, 17, 13, 9…

3, 6, 9, 12…

1, 2, 4, 8…

23, 25, 27, 29…

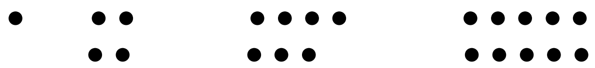


Fig 4.9 Unhelpful visual representation

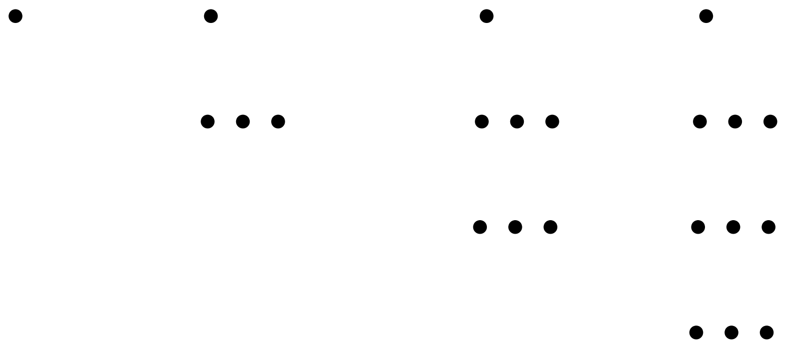


Fig 4.10 More helpful visual representation

Lesson 5: Patterns and sequences: Verbal approach

Aim

This lesson was intended to be a fairly straightforward exercise with sequences, but emphasising clarity of explanation

Mental starter

The class was reminded about multiples by being asked to explain the concept and some examples were written down. Square numbers were introduced as a special case, within the multiples, of "a number times by itself". The square numbers were then given as an ordered sequence of multiplications and their results (Fig 4.11). It was mentioned that this was a sequence and the class was asked to answer the following:

(i) Find the 7th square number

(ii) Find the 13th square number

(iii) Is 81 a square number? If it is, what is its position in the sequence?

(iv) Is 196 a square number? If it is, what is its position in the sequence?

1^{st}	2^{nd}	3^{rd}	4^{th}
1×1	2×2	3×3	4×4
1	4	9	16

Fig 4.11 Number patterns representing the first four square numbers

Sequences

The sequence 1, 4, 7, 10… was given and the class was asked for the next two numbers. The rule was elicited for the next number, but it was discussed that we really need a rule for any number in the sequence. The following exercise was then given:

For each sequence

a) Find the next two numbers in the sequence

b) Find the rule for the next number

c) (Extra!) Can you find a way of predicting any number in the sequence?

2, 8, 14, 20…

21, 17, 13, 9…

3, 6, 9, 12…

1, 2, 4, 8…

23, 25, 27, 29…

Lesson 6: Functions: Visual-spatial approach

Aim

The intention was to begin to consider functions through the idea of operating on numbers, then to introduce a system of icons to convey the rule, which would bridge the gap between numbers and algebraic symbolism.

Mental starter

Pairs of inputs and outputs were written on the board (Fig 4.12). Pupils, working individually, were required to fill the gaps.

2 → 7	1 → 2 ½
10 →	5 ½ →
→ 8	6 →
5 → 10	→ 13
4 → 8	5 → 10
15 → 30	7 → 16
7 →	2 → 1
→ 11	10 →
	→ 19

Fig 4.12 Inputs and outputs exercise

Rules, icons and mapping diagrams

Pupils were helped to write the function rules for the above pairs using icons (e.g. * → * + 5; # → ##). Davis (1972) proposes a similar use of iconic symbols to convey numerical rules. A mapping diagram was also drawn for the first function, to provide another way of visualising the relationship. It was also demonstrated that it is possible to begin with the rule, choose inputs and then find their outputs. The class then worked through an exercise, using the iconic representation (Fig 4.13).

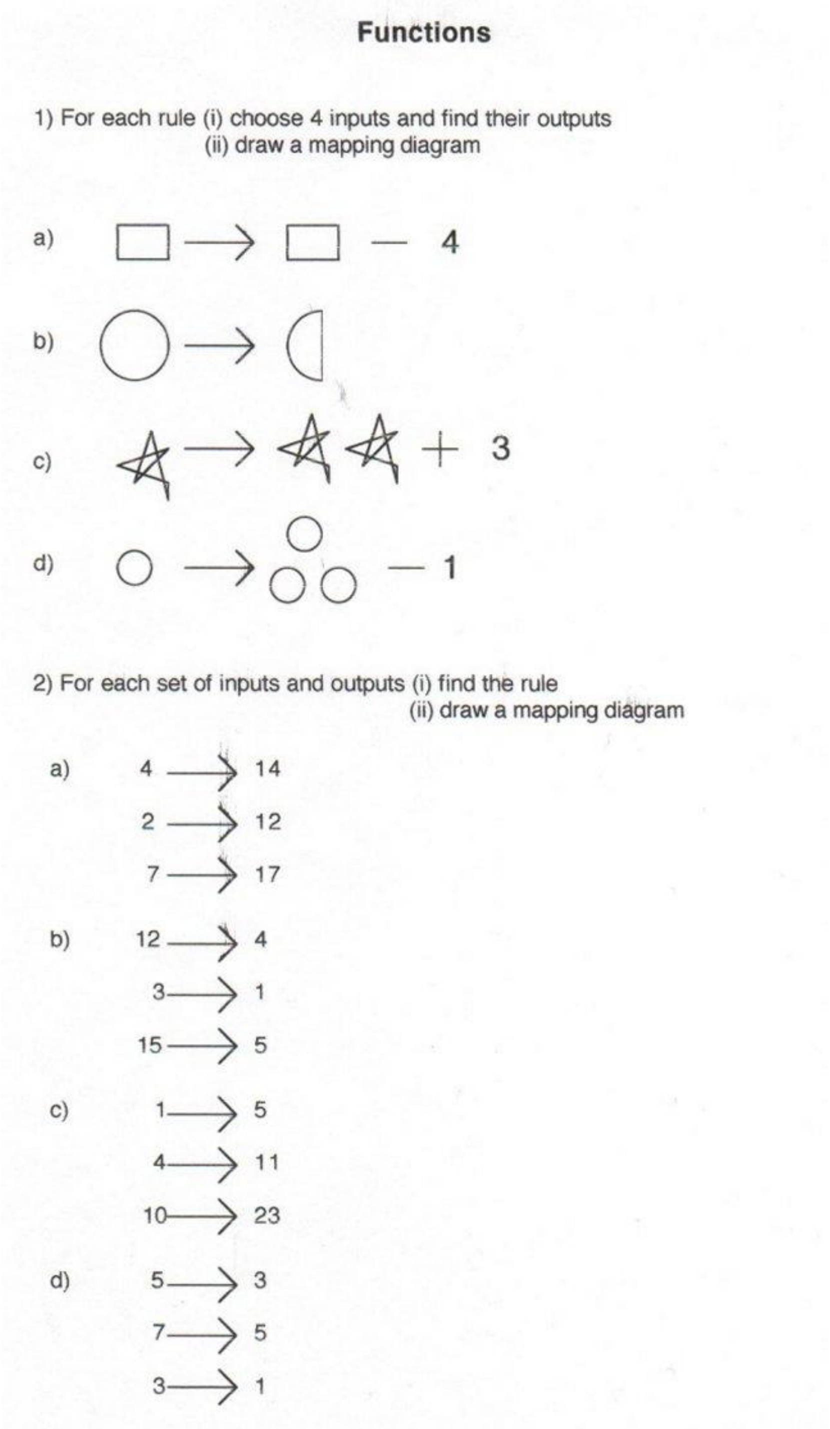


Fig 4.13 Functions exercise using iconic approach

Lesson 6: Functions: Verbal approach

Aim

The intention was to begin to consider functions through the idea of operating on numbers, and to emphasise the importance of framing function rules clearly. The use of correct vocabulary was instrumental in this.

Mental starter

Pairs of inputs and outputs were written on the board (Fig 4.12). Pupils, working individually, were required to fill the gaps.

Rules and mapping diagrams

The rules for the above functions (Fig 4.12) were elicited and pupils were helped to write them clearly, using the words ‘input’ and ‘output’. A mapping diagram was also drawn for the first function, to provide another way of understanding the relationship. It was also demonstrated that it is possible to begin with the rule, choose inputs and then find their outputs. The class then worked through an exercise, using written instructions as function rules (Fig 4.14).

Functions

1)For each rule (i) choose 4 inputs and find their outputs
(ii) draw a mapping diagram

a) Take away 4 from the input

b) The output is half the input

c) Double the input and then add 3

d) Multiply the input by 3, then take 1 away

2) For each set of inputs and outputs (i) find the rule
(ii) draw a mapping diagram

a) $4 \rightarrow 14$
$2 \rightarrow 12$
$7 \rightarrow 17$

b) $12 \rightarrow 4$
$3 \rightarrow 1$
$15 \rightarrow 5$

c) $1 \rightarrow 5$
$4 \rightarrow 11$
$10 \rightarrow 23$

d) $5 \rightarrow 3$
$7 \rightarrow 5$
$3 \rightarrow 1$

Fig 4.14 Functions exercise using verbal approach

Lesson 7: Pyramids Investigation: Visual-spatial approach

Aim

This lesson used an investigation where the final proof is possible using very basic algebra. It was anticipated that some pupils might use the iconic approach of the previous week.

Mental starter

This was intended, primarily, to remind the class about the representation used previously. They were given a function rule, expressed in this way (▲ → ▲ – 5) and asked to provide outputs to some inputs (10, 17, 8, 4). Pupils were then asked for functions that will "make numbers bigger" leading to various suggestions of $x \rightarrow kx$ (where k is an integer and $k \geq 2$), expressed iconically.

Investigation

A pyramid was drawn and three numbers (provided by the class) entered into the bottom squares. Using the rule that two adjacent numbers must be summed to produce the number for the square above them, the squares were filled (see Fig 4.15). The class was asked to consider whether the total would be the same if the order of the original three numbers was changed, and then a different total was demonstrated. The question raised was how to get the biggest total for any given set of numbers. The pupils then worked individually to answer these two questions:

1) Try other numbers. How do you arrange them to get the biggest total?
2) Prove your answer will *always* work.

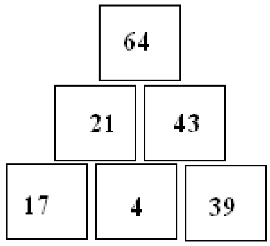


Fig 4.15 Example of pyramid used for pyramid investigation

Lesson 7: Pyramids Investigation: Verbal Approach

Aim

This lesson used the same investigation as the visual-spatial approach, but was working towards a good explanation, which would hinge on the idea that "doubling makes bigger".

Mental starter

This was intended, primarily, to remind the class about function rules . They were given a function rule, expressed in words, and asked to provide outputs to some inputs (10, 17, 8, 4). Pupils were then asked for functions that will "make numbers bigger" leading to various suggestions of $x \rightarrow kx$ (where k is an integer and $k \geq 2$), such as "double" and "treble".

Investigation

A pyramid was drawn and three numbers (provided by the class) entered into the bottom squares. Using the rule that two adjacent numbers must be summed to produce the number for the square above them, the squares were filled (see Fig 4.15). The class was asked to consider whether the total would be the same if the order of the original three numbers was changed, and then a different total was demonstrated. The question raised was how to get the biggest total for any given set of numbers. The pupils then worked individually to answer these two questions:

1) Try other numbers. How do you arrange them to get the biggest total?
2) Prove your answer will *always* work.

Lesson 8: Fractions, decimals and percentages: Visual-spatial approach

Aim

The intention was to suggest the variety of visual illustrations which can be used to express a particular fraction. Although the shaded boxes understanding of fractions can support misconceptions about addition of fractions (Silver, 1986), it was anticipated that using a variety of shapes and sizes would reduce this tendency while making explicit the equivalence of fractions, decimals and percentages.

Mental starter

On the board was drawn a range of diagrams (Fig 4.16) and class members were invited to come and join with a line any that were “equivalent” or “the same”. Discussion was deliberately avoided; any incorrect lines were simply noted as such and erased. Pupils were encouraged to go beyond merely producing pairs and the drawing of lines continued until no more were possible.

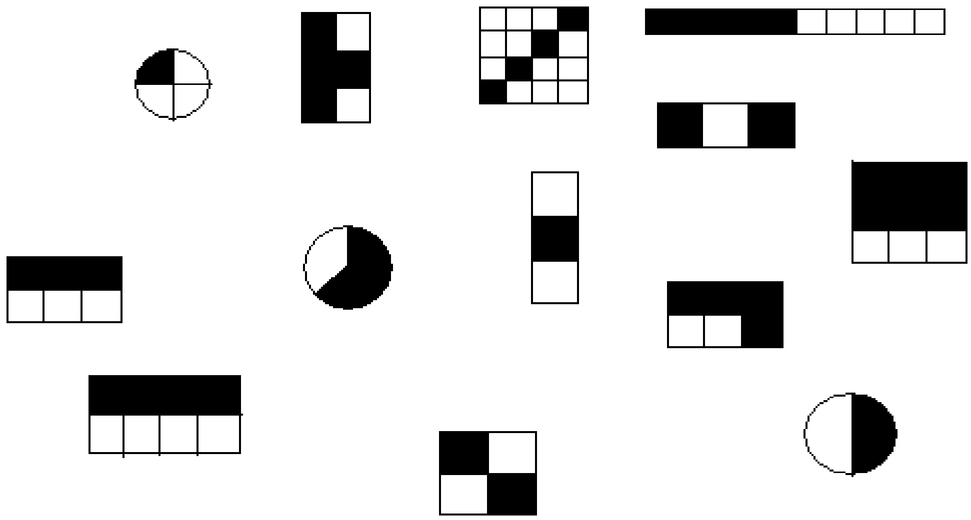


Fig 4.16 Diagrams of fractions

Introduction to main lesson

Returning to the number line, the class was asked what is “between the numbers”, producing the idea of ‘parts of a whole’ with fractions, decimals and percentages as particular cases.

Shading parts

A brief introduction reminding pupils how to shade diagrams was given. This mentioned coping with an inappropriate number of squares to be shaded and managing decimals by remembering that 0.1 = 1/10, 0.01 = 1/100, etc. Then the class worked on the shading parts sheet (Fig 4.17). After percentages were defined as “out of 100” the percentage sheet (Fig 4.18) was also completed.

Equivalence

As pupils finished the sheets, extra wholes were given to shade and there was discussion with individuals, partly provoked by the mental starter activity, about equivalence.

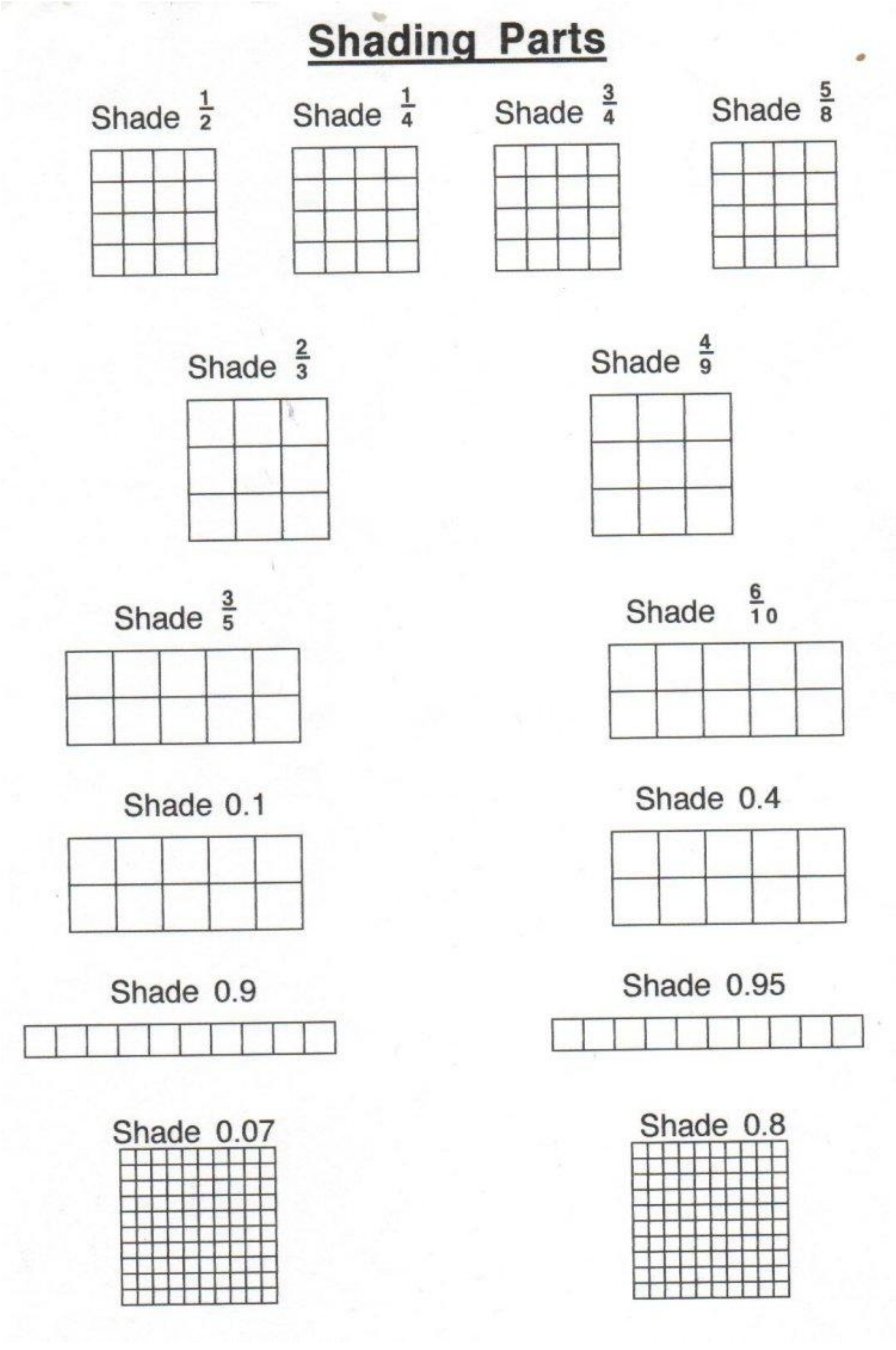


Fig 4.17 Shading parts worksheet

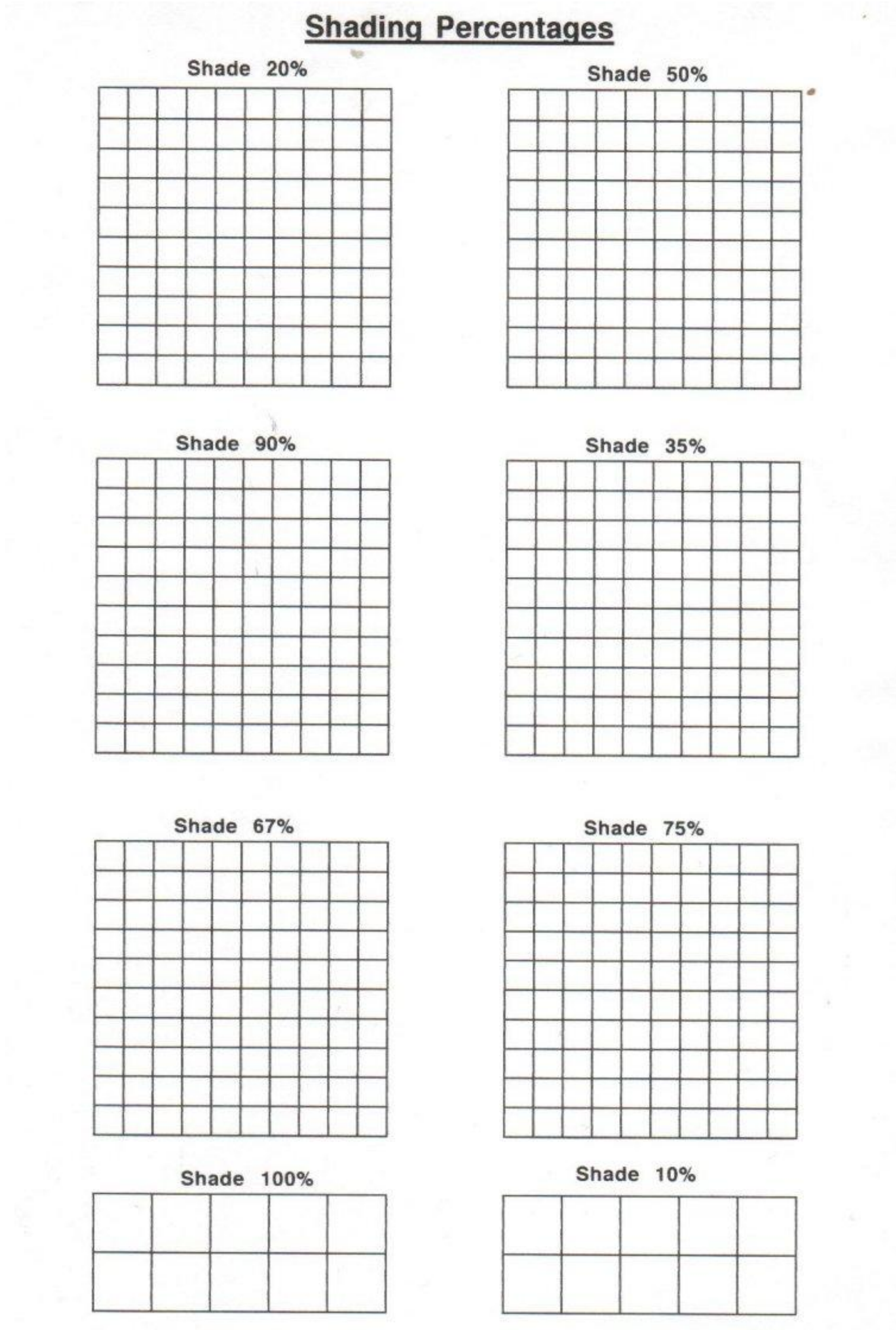


Fig 4.18 Shading percentages worksheet

Lesson 8: Fractions, decimals and percentages: Verbal Approach

Aim

The aim was to approach the idea of equivalence by clearly defining fractions, decimals and percentages as all being ‘parts of a whole’. This would be supported by calculations that would suggest the equivalence of specific quantities.

Mental starter

On the board was a range of fractions, expressed using numerical notation or words and, for the simple case of a half, using decimal and percentage notation (Fig 4.19). These were chosen and arranged so as to be identical to those used in the visual lesson. Pupils were invited to come and join with a line any that were “equivalent” or “the same”.

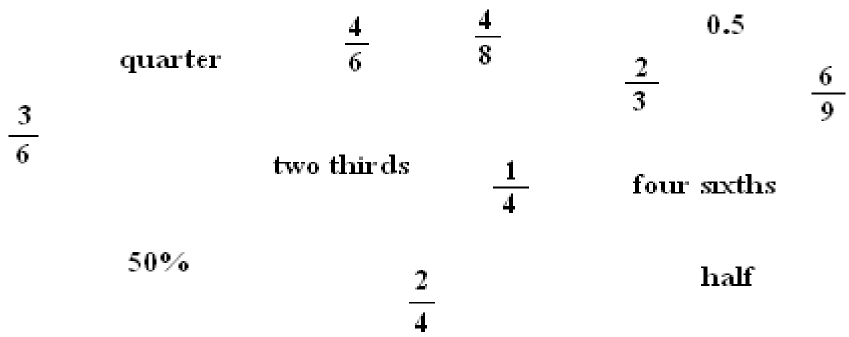


Fig 4.19 Non-visual expressions of fractions

Introduction to main lesson

The concept of 'parts of a whole' was used to elicit the ideas of fractions, decimals and percentages from the class. Fractions were then considered, in particular noting that ¾ means "three out of four equal parts", with the equality of the parts being emphasised. The numerator and denominator were defined.

Finding parts

The items on the sheets completed in the visual lesson had been translated into calculations of parts of given quantities (Fig 4.20). This sheet was then worked on, with the children being instructed to reflect on their own calculations with the aim of answering the following questions:

(i) *How* do you find a fraction of an amount?

(ii) *How* do you find a percentage of an amount?

(iii) Can you explain the links between fractions, percentages and decimals? (Why is 0.8×10 the same as 8/10 of 10 and 20% of £50 the same as 1/5 of £50?)

They were then asked to design a poster explaining about decimals, fractions and percentages using their answers to these questions.

Finding Parts

Find $\frac{1}{2}$ of 16

$\frac{1}{4}$ of 16

$\frac{3}{4}$ of 16

$\frac{5}{8}$ of 16

$\frac{2}{3}$ of 9

$\frac{4}{9}$ of 9

$\frac{3}{5}$ of 10

$\frac{6}{10}$ of 10

67% of 100

75% of 100

10% of 10

100% of 10

Work out 0.1×10

0.4×10

0.9×10

0.95×10

0.07×100

0.8×100

Fig 4.20 Finding parts worksheet

Lesson 9: Ratio: Visual-spatial approach

Aim

The intention was to introduce ratio based on the idea of an endlessly repeating pattern, expressed by coloured blocks or shaded squares. It was hoped that the pupils could then use shading to solve more involved word problems.

Mental starter

This revised the idea of sequences and provided a foundation to ratio, through practising doubling, trebling and finding other multiples. The first three terms of some simple sequences were given (Fig 4.21) and pupils were asked to provide the next two terms for each one.

4	3	7	4	5	2	1
8	6	14	8	10	6	4
12	9	21	16	20	18	16

Fig 4.21 Initial terms of some sequences

Introduction to ratio

Seven blocks were shown and the ratio of yellow to red was described as “one to six”. This was also written as 1:6. The number of blocks was doubled and then trebled, and the class was asked how many of each colour would be needed. It was emphasised that the ratio of yellow to red would still be 1:6. The ratio of red to yellow was also requested. Then the blocks were used to illustrate ratios of 2:3 and 3:2.

Shading practice

Pupils were instructed to shade lines of squared paper so that the ratio of shaded to unshaded squares conformed to each of the following ratios: 1:4, 1:12, 2:3, 3:1 and 5:4. Shading for the first ratio was demonstrated.

Problems

A sheet of word problems (Fig 4.22) was attempted, with pupils being encouraged to use drawings or diagrams to help them.

Some ratio questions

1) In a country dance there are 7 boys and 6 girls in every line. 42 boys take part in the dance.
How many girls take part?

2) In a class, the ratio of girls to boys is 3 to 2. There are 8 boys.
How many girls are there?

3) In a zoo there are 40 male monkeys and 10 females.
What is the ratio of males to females? Give your answer in the simplest way.

4) There are 3 chocolate biscuits in every 5 biscuits in a box. There are 30 biscuits in the box.
How many of them are chocolate biscuits?

5) Blagdon School is having an outing. The children vote on whether to go to the ice rink or to the cinema. They vote 2 to 1 in favour of the ice rink. 300 children voted for the ice rink.
How many voted for the cinema?

6) Some children voted between a safari park and a zoo for a school visit. The result was 10 : 3 in favour of the safari park.
How many children voted in favour of the zoo?

7) You can make one "Peach Surprise" out of 2 peaches and 3 scoops of ice cream. You have 26 peaches.
a) How many "Peach Surprises" can you make?
b) How many scoops of ice cream will you need to make them?

8) The ratio of sugar to flour in a cake is 3 : 8. I have 150g of sugar.
How much flour do I need?

9) In a cafe, coffee is made from two types of beans, from Java and Colombia, in the ratio 2 : 3.
How much of each type of bean will be needed to make 500 grams of coffee?

10) A girl spent her savings of £40 on books and clothes in the ratio 1 : 3.
How much did she spend on clothes?

Make up some questions of your own that involve ratio...and answer them!

Fig 4.22 Worksheet of ratio problems

Lesson 9: Ratio: Verbal Approach

Aim

It was intended to introduce ratio based on the idea of diluting orange squash, since this would provide a comprehensible application but it is not one that relies on visual pattern. A calculation method based on this model was developed, which was intended to aid pupils when solving more involved problems

Mental starter

This revised the idea of sequences and provided a foundation to ratio, through practising doubling, trebling and finding other multiples. The first three terms of some simple sequences were given (Fig 4.41) and pupils were asked to provide the next two terms for each one.

Introduction to ratio

It was stated that in mixing up some orange squash it is necessary to use squash to water in the ratio of "one to six", with this being explained as the need for six cups of water for every one cup of squash. The ratio was also written as 1:6. The class was then asked how much water would be needed to achieve the same strength if two cups or three cups of squash were used. They wwere then asked what the orange would taste like if squash to water was used in the ratio of 6:1.

Calculation practice

It was demonstrated how a table layout could be used to generate various amounts of drink, all with the same strength, for any given ratio (Fig 4.23). Pupils were then asked to use this method to produce various quantities of drink with strengths governed by the following ratios: 1:4, 1:12, 2:3, 3:1 and 5:4.

squash	water
1	4
2	8
3	
6	

Fig 4.23 Demonstration of table layout for ratio problems

Problems

A sheet of word problems (Fig 4.22) was attempted, with pupils being encouraged to use the table layout to help them.

Lesson 10: Staircases investigation: Visual-spatial approach

Aim

An investigation was given where a visual insight (that the 'staircases' were simply reorganisations of progressively bigger squares) would allow pupils to see that an observed numerical pattern (the sequence of square numbers) must continue. It was hoped that the earlier use of dot patterns would make the insight more likely.

Mental starter

The first three 'staircases' (Fig 4.24) were copied from the board and the pupils were instructed to "draw the next three staircases".

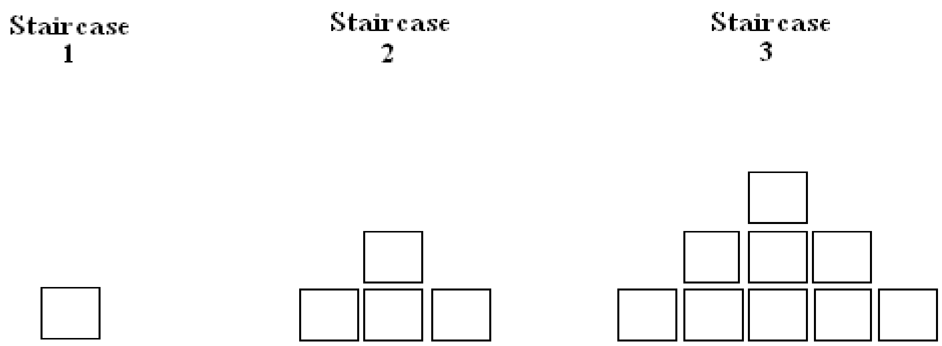


Fig 4.24 First three staircases for staircases investigation

Investigation

Instructions on the board led the class through the stages of the investigation:

1) Copy and complete the table:

Staircase height	1	2	3	4	5	6
Squares needed		4				

2) Predict how many squares you will need for staircases 7 and 8. Check by drawing
3) Can you make up a rule to say how many squares you need for any staircase?
4) Can you explain *why* your rule works?

Children worked independently before a class discussion and a demonstration of the way the staircases can be seen as squares.

Question generation

As a final activity, the challenge to produce maths questions with a particular answer (in this case, 16) was repeated. As in the first lesson, different sorts of question were requested and pupils' ideas were shared.

Lesson 10: Staircases investigation: Verbal Approach

Aim

The same investigation was given as was used in the visual lesson, but it was anticipated that the visual insight would be less likely to occur and the approach emphasised "answering the question of why" the number pattern occurs.

Mental starter

The first three 'staircases' (Fig 4.24) were copied from the board and the pupils were instructed to "draw the next three staircases".

Investigation

Instructions on the board led the class through the stages of the investigation:

1) Copy and complete the table:

Staircase height	1	2	3	4	5	6
Squares needed		4				

2) Predict how many squares you will need for staircases 7 and 8. Check by drawing
3) Can you make up a rule to say how many squares you need for any staircase?
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Question generation

As a final activity, the challenge to produce maths questions with a particular answer (in this case, 16) was repeated. As in the first lesson, different sorts of question were requested and pupils' ideas were shared.

5. Results

5.1 Comparing the two interventions

5.11 The reception of the intervention lessons

5.111 General impressions

During the main fieldwork, the two sets of lessons had apparently similar receptions. In both cases, there was initial curiosity about a different teacher, which declined at a fairly similar rate under the two approaches. In both cases there were some pupils who consistently applied themselves and others who it was hard to interest, so that although the material covered and work produced varied widely across the students, this was not related in some simple way to the approach. When preparing and teaching the lessons, it was felt that some mathematical topics worked particularly well with the visual approach, but that others seemed to lend themselves more to a verbal approach. This will be covered in more detail below.

In the interviews, after the interventions, the subsets of participants were questioned about the lessons and again there was not a consistent distinction between the responses of the children depending on the approach. The main study children often mentioned the recognition test as the activity they remembered, presumably because this was the experience which departed most from their usual school lessons. Both the main study and pilot study participants also showed a tendency to judge this as the most ‘useful’ thing they had done. Apart from the interest in the recognition test, there was no consistency in what was remembered. The pilot study pupils (who all experienced visual lessons) recalled a range of subject matter and all made reference to the idea of looking at numbers as visual groupings or “without counting”.

They linked this with familiar ideas such as “times-tables” (because “that’s the same as grouping”) and “the difference between odd and even”. In contrast, the main study visual lessons did not provoke such encouraging reports, but the main study interviewees were generally much less forthcoming. Children from both intervention classes claimed not to be able to remember the topics covered or gave one word answers, such as “fractions” or “graphs”, which did not refer to the teaching approach. Only one participant, from the verbal group, made any such reference when she answered that she remembered “sequences and explaining how you did it”.

The reasons for different reactions from the pilot compared to the main study participants are unclear, but probably lie in differences in the cultures of the two schools. The differing class sizes for the intervention lessons (15 in the pilot study, 23 in the main study) did not seem to be the deciding factor, since the different response styles were evident before the interventions, when the interviewees in the pilot study were much more willing to talk than were those in the main study.

The schools themselves were quite different, with the pilot study taking place in a middle school, which the participants had been attending for nearly three years, while the main study participants were only beginning their first year at a much larger, and perhaps more intimidating, secondary school. This presumably partly accounts for the pilot study pupils seeming much more confident and happy to talk. In addition, it is worth noting that the middle school was a semi-rural school with an impressive reputation, which it could be imagined would affect the confidence of the pupils and their expectation that an outsider would be interested in their opinions. Whatever the explanation, it seems unlikely that the different interview responses to the pilot and main study lessons reflected any reliable distinction between the two approaches. The main study interviewees in general seemed unimpressed, whichever teaching approach they had experienced. The fact that the visual lessons did get a much better response from the pilot study interviewees might suggest some sort of interaction between intervention approach and general school style, but without pilot study verbal lessons to compare with, this can not be concluded. Furthermore, the distinction between the reception of the lessons did not seem to extend much beyond the interview responses, with the work produced and classroom activity being broadly similar.

5.112 Visual lessons

A striking feature of the visual lessons, from the teacher’s point of view was how well some of the visual ideas worked as introductions to particular topics. For the lesson on calculation, the mental starter of using a number line to support the idea of ‘bigger’ and ’smaller’ provided a focus for the class. Answers were given enthusiastically and the children were clearly comfortable relating the idea of number size to position on the line. Similarly, the visual introduction to ratio, using coloured blocks, seemed to convey, fairly effortlessly, a number of mathematical ideas. Repeated patterns of blocks were built up and labelled with appropriate ratios without the need for lengthy explanations. The visual supports also functioned simply as visual aids, providing something for the children to look at and so making it easier to hold their attention.

However, the difficulty came in encouraging the children to make use of visual methods when solving mathematical problems. During the ratio lessons, the pupils carried out the shading squares exercise and this was extremely helpful in clearing up misunderstandings about the nature of ratio. Yet they were very reluctant to use shaded squares to help them solve the later word problems (which had low numbers so this method could have been useful). Only one child made any attempt to use shaded squares. He shaded squares to support his abstraction to listing the numbers involved, with each increase according to the ratio, until he reached the target number. Much more common than this, though, was to abandon completely the suggested approach, and with it the concept of ratio. Even Gavin, a child who seemed generally to benefit from a visual approach, insisted on answering all the word problems through calculation. He got almost all of them wrong because he was simply taking the difference between the numbers in the ratio and adding it to or subtracting it from the given number, rather than appreciating the nature of ratio and so the need to work with multiplication and division. It would seem, then, that the visual demonstrations of ratio had not given him a transferable sense of the nature of ratio, so that he soon fell back on numerical tricks, based on superficial features of the numbers involved in the problems. He behaved similarly with the word problems given during the calculations lesson, which were intended to be assisted by the use of a number line. When questioned, he claimed that he did not need to use diagrams to help him, clearly seeing the diagrams as a support that he had outgrown. Similar views were expressed by some of the pilot study class when presented with these problems and there does seem to be a general problem of persuading older children that visual methods are not baby methods.

However, even when pupils attempt to use visual methods in their own work they may have difficulty. Although they can appreciate a visual demonstration, where the abstraction from situation to diagram is done for them, they often have problems constructing visual descriptions for themselves, where they have to do the abstraction. This was clearly seen in many of the attempts to use number lines for word problems. Faced with the information in a question, many participants could not see how to translate this into positions on a number line, although they could appreciate the idea once it was done for them. When they attempted it for themselves the most common error was putting the difference between the two quantities in the question as a position on the number line. Furthermore, when the pupils were finding the diagram construction difficult they were inclined simply to abandon it and get on to the ‘real’ problem of doing a calculation. This tendency to see diagrams as “mere appendages” (Noss et al, 1997) has been noted by other researchers and in this case probably indicates a problem with the style of question.

A quite different sort of activity was provided during the lessons by the use of three investigations. In two of these (‘Odds and Evens’ and ‘Staircases’) it was anticipated that a visual understanding would make it possible for pupils to progress beyond pattern-spotting to an appreciation of why particular effects occur, and so head towards proper proof. The Pyramids investigation was intended to provide a use for the iconic formulae, which had been introduced the previous week, and which should help pupils to see the general pattern beyond the results for particular numbers.

In all three cases, there were some individual successes. The dot patterns approach to numbers had been introduced in a previous lesson, and was then applied to even numbers and multiples of other numbers as the mental starter to that lesson. One child, Gavin, was able to apply this to the problem of the investigation and gave a good, general explanation of how odd and even numbers combine by sketching dot patterns. At the teacher’s suggestion, he drew these more carefully so they could function as an explanation and, interestingly, took the decision to include a written description of what happens (Fig 5.1).

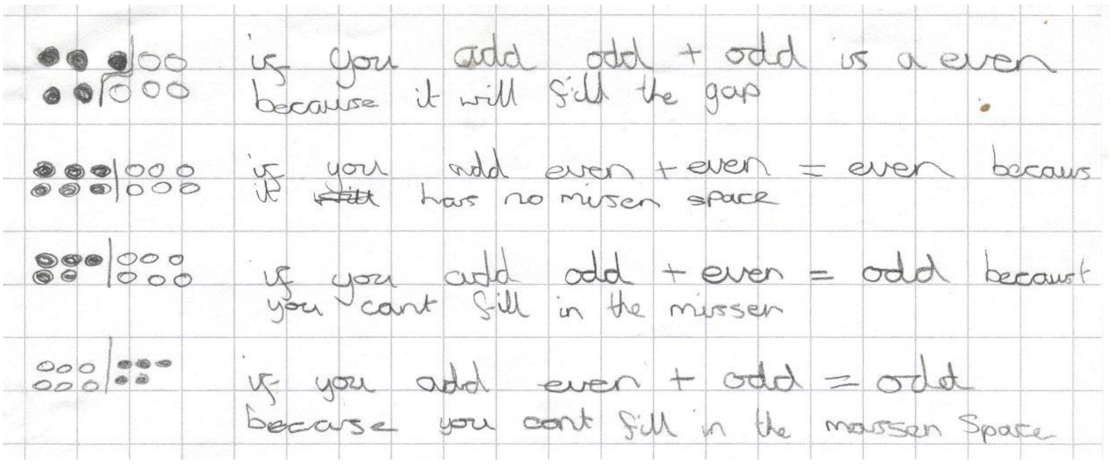


Fig 5.1 Gavin's odds and evens picture proof

A less impressive instance of a visual approach helping an individual child was seen during the Pyramids investigation. Kate had concluded that to make the largest total it is necessary to put the biggest starting number in the central position, but she could not say why this was the case. At the teacher's suggestion, she substituted iconic symbols for numbers, worked up the pyramid and realised that the top number results from a doubling of the central starter number added to the other two starter numbers. In conversation, she seemed to appreciate the significance of this discovery, facilitated by the iconic symbols, but when her work was handed in, no reference was made to it. By being disappointed by this, the researcher may just be falling into the trap of not believing that understanding has occurred unless it is described in words. However, no other methods, such as underlining or arrows, were used to draw attention to the relevant working. This leaves one wondering whether Kate's apparent insight was very transient, and perhaps therefore not of any long term use to her mathematical understanding.

In contrast to the reaction of individuals to the chance to explain results with visual support, the visual background did prove helpful to the teacher in providing explanations and tentative proofs to the class as whole. This was particularly striking at the end of the Staircases investigation, when the teacher was able to show why the numbers of blocks turned out to be square numbers by rearranging some staircases into squares, pointing out that this would work with staircases of any size. In the verbal lessons, it was not possible to produce such a neat demonstration of the reasons for the results the children had found for this investigation or for the Odds and Evens investigation. However, this way that a visual approach could be beneficial when used by the teacher shows again the difference between being able to appreciate a visual abstraction, when it is provided, and trying construct one for yourself.

This does seem to be a general conclusion that can be drawn from the reception of the visual lessons. Even if a teacher can overcome the tendencies for diagrams to be seen as awkward additions or as childish supports, there may still be difficulties where children have to construct representations. Although they might be able to understand the abstractions involved when they are done for them, they seem to experience great difficulty in carrying out such abstractions. This is similar to the problem teachers experience in many areas of mathematics when abstraction is required. For example, children are often able to plug numbers into formulae but struggle to write a formula based on a given relationship since this requires them actually to construct the abstract representation, rather than merely working with an existing abstract entity. It might be because visual representations work so well as immediate demonstrations of mathematical relationships and ideas that there is a tendency to overlook the rather high level thought that will be involved in constructing them and be surprised that pupils are not better able to adapt the methods to their own use. However, where they are able to make this leap, the visual representations can prove extremely beneficial, as Gavin's Odds and Evens proof shows. Even where they only make use of visual methods as fairly routine exercises, as in shading squares for ratio or shading parts for fractions, decimals and percentages, this can be less dramatically beneficial in that it ingrains important principles and may also provide opportunities for the teacher to spot, and correct, misunderstandings.

5.113 Verbal lessons

The verbal lessons were more challenging for the teacher to start well, since the mental starters often depended more on explanation and discussion by the class and this was sometimes unfocused or did not include all the pupils. An exception to this difficulty with the verbal starters was that used for the lesson on fractions, decimals and percentages. In this case, a focus was provided, since the children were asked to link expressions that were equivalent, just as happened in the visual lesson, but for the verbal lesson the fractions were expressed by numerical notation or words. The activity seemed successful in both forms, but arguably the verbal presentation made the point more strongly that fractions, decimals and percentages are similar in all being parts of wholes. Such a conclusion is suggested by the fact that during interview one of the pupils from the verbal class stated that he remembered that "we did fractions, which were equivalent to decimals". It is worth noting that he used the word 'equivalent' and perhaps this suggests a benefit of explicitly using correct vocabulary in that it makes it easier for children to reflect, clearly and coherently, on what they have learned.

Another lesson that involved a lot of vocabulary was the one on number bonds and factors, and in this case the teacher was able to coax the pupils into an interesting discussion about what certain mathematical terms really mean by asking them to explain terms such as ‘times’ and ‘multiply’. In this case it did seem worthwhile to draw out concepts that the pupils undoubtedly had, but rarely reflected upon, as a way of deepening their understanding. However, it must be questioned whether such reflection is always helpful and, furthermore, recognised that sometimes such ‘discussions’ end up merely rehearsing definitions. This seemed to be the case when the lesson came to factors and definitions were provided but they did not appear to add anything to the concept of a factor, in the way that the visual lesson activity of drawing rectangular dot patterns might have done.

Two lessons where verbal descriptions of mathematical situations seemed to be produced very naturally were the lessons on functions and sequences. In dealing with sequences, the pupils found it very obvious to produce a verbal rule explaining how to generate the next number. They were much more successful at this than the visual class pupils were at trying to draw diagrams showing how the sequence continued. Similarly, when considering functions, the verbal class responded well to requests to write or speak rules, and they seemed to appreciate the need for clarity and the avoidance of ambiguity, which was helped by using correct vocabulary. In contrast, although the iconic formulae introduced in the visual lessons were well understood when given as rules to use to generate outputs, the pupils found it more difficult to use this representation to convey a function rule, given the inputs and outputs. Despite being asked for this representation, some of them could only give the rule in words. However, it should be noted that the iconic diagrams are a distinct, further abstracting, step away from the specific numbers towards describing the general relationship algebraically. Although they might not come as naturally as verbal rules, they might still be useful as a step towards algebra, and could perhaps be beneficially combined with attention to verbal descriptions of rules and regularities.

This could be seen as an instance of the general problem that, without further abstraction, the careful descriptions encouraged by the verbal lessons might not progress to anything mathematically useful. Such a difficulty was seen in the investigation lessons, particularly during the Staircase investigation, which just fizzled out as no verbal explanation was really possible. Similarly, while working on the Pyramid investigation, many of the pupils produced clearly written observations about the pattern but they could not progress to an explanation of why this happened or even appreciate why this was needed. This was despite the mental starter requesting functions that ‘make numbers bigger’, which resulted in function rules such as ‘double’ and ‘treble’, so that the pupils had the vocabulary, and hopefully the concepts, to solve the explanation problem.

In contrast to difficulties with these two investigations, the Odds and Evens investigation was much more successful. Unlike the visually taught pupils, those in the verbal class had not had an appropriate representation for odd and even numbers suggested to them. Instead, earlier in the lesson, there had been discussion about the definition of even numbers as multiples of two, which had involved pupil comments about how one can tell if a number is even by looking at its final digit. This seemed to lay the foundations for a spontaneous whole-class attempt at the end of the lesson to come up with a numerical proof, case by case, considering the unit digit. Members of the class argued that only unit combinations had to be investigated as a rule for them would hold for all numbers, since the digit determines whether a number is even. This reasoning from the definition of even numbers is much less elegant than the visual proof, demonstrated by the teacher to the visual class, but it involved many more children, who appeared to be grasping the aim of attempting to prove something.

5.114 General conclusion

During both sets of lessons there were successful parts, where large proportions of the classes offered suggestions or examples and seemed to be fully engaged with the subject matter. Although it was often easier initially to get pupils' attention with a visual approach that gave them something to look at, the verbal approach, in asking for descriptions and definitions, often provoked more tangible involvement of the children.

In both sets of lessons, the children generally accepted the styles of teaching and appeared to be trying to use these styles to learn. The members of the visual class attempted diagrams, while the members of the verbal class tried to give verbal descriptions and explanations. It therefore seems reasonable to consider the intervention lessons generally to have encouraged the two styles of learning. However, it is not possible to be certain how each child is thinking through every instant of the lesson. The potential for a child to be using a different style from the one suggested is illustrated by the beginning of an attempt at a visual explanation of the Odds and Evens investigation produced by a child in the verbally taught class (Fig 5.2).

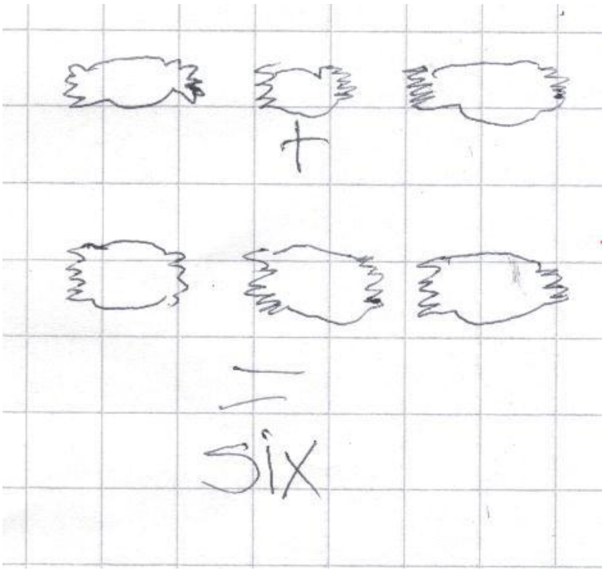


Fig 5.2 Picture produced in verbal lesson

In general, both teaching approaches allowed individual progress to be made by pupils, but in differing ways. Although the verbal approach provided the teacher and a pupil with a shared vocabulary, it was the visual approach which seemed to produce the occasional sudden insights, presumably because creating a diagram is more an act of abstraction than simply describing a situation. However, the construction by pupils of visual representations was found to be difficult and this added to their reluctance to use them. This was despite the ease with which pupils could understand a concept when it was conveyed visually, which contrasts with the often vague and imperfect discussion necessary to introduce the same idea verbally. Interestingly, the visual representations of mathematics did not seem to result in overly concrete conceptions, as various researchers have warned may happen (e.g. Krutetskii, 1976, p.326). Perhaps this was because the representations used were designed to be abstract and to encourage thinking beyond the immediate situation, even if this did make them difficult for the children to apply.

5.12 The effects of the intervention lessons

5.121 Mathematics understanding

The questions generated by the children in both teaching groups were similarly lacking in range, making very little use of words or pictures but relying on numerals and mainly simple operations (e.g. addition and subtraction) on those numerals. Of the questions produced by the Monday class, 94% relied on numerals with only 4% and 2% involving, respectively, words and pictures. In the Wednesday group, the situation was similar with 93% of questions based on numerals, 6% on words and 1% on pictures. After the interventions no improvement was observed in either group: if anything, the later questions were less varied as the children were presumably less interested in the task on the second occasion. Again very few questions making use of words or pictures were produced by pupils in either class

The test of understanding given to the sample interviewed suggested some interesting tendencies in terms of which ideas from the lessons the children were able to make use of after the intervention. They did use ideas such as 'sequences' and 'odds and evens', both implicitly and explicitly, to group the questions, which they generally did not do before the intervention. However, this was complicated by the fact that simply repeating the exercise seemed to make participants less prone to being misled by surface features of the questions. The two children who only attempted the activity after the intervention did a lot of sorting according to such features (this involved the implicit use of time in one case and "days and weeks" as an explicit label in the other), while this decreased among the children sorting for the second time. No coherent distinction could be made between the performances of members of the two teaching groups and , given the concerns about a practice effect, it is difficult to draw any conclusions about changes in individual understanding due to the intervention teaching from the question sorting.

Distinctions between the groups in terms of the quality of the work produced in class have been suggested above, in the reporting of the reception of the lessons, where it was concluded that there was no simple advantage to either intervention. In both groups there was considerable variation between individuals and variation from lesson to lesson in the output of many individuals. While some conclusions can be advanced about the classroom experience of a visual approach compared to a verbal one, these do not translate into an appreciable advantage for either method in terms of quantity or quality of work produced.

5.122 Mathematics performance

It was similarly difficult to reach any conclusions about relative mathematical performance from the qualitative indicators already considered. However, a major intention of this research was to be guided by quantitative evidence and it is therefore necessary now to turn to performance on the Mathematics Competency Test (MCT).

5.1221 Initial considerations

Pre-intervention group similarity

The children were assigned to the two groups in a broadly random way, but with an attempt being made to balance the distributions of MCT scores in the two groups (see Chapter 4, section 4.21). The other scores from the qualitative tests were not available and were not considered. However, when these were examined after the teaching interventions, it was found that none of the group means differed significantly. These statistics are all included in Table 5.1, together with the MCT scores.

Table 5.1 Pre-intervention statistics for the original groups

	Intervention group	N	Mean	Std. Deviation
MidYIS vocabulary	Monday	22	89.45	9.34
	Wednesday	23	95.35	14.25
MidYIS non-verbal	Monday	22	99.18	10.19
	Wednesday	22	95.36	13.08
MidYIS maths	Monday	22	91.91	8.58
	Wednesday	23	95.00	12.74
MidYIS skills	Monday	22	90.09	9.00
	Wednesday	22	95.09	16.42
Spatial memory test	Monday	16	11.69	3.03
	Wednesday	16	11.13	2.16
Visual/verbal ratio	Monday	18	1.03	0.82
	Wednesday	18	1.30	1.75
Pre-intervention MCT	Monday	21	14.10	4.37
	Wednesday	20	13.80	5.07

Absentees

The decision was taken to remove from the analysis of change in maths performance those individuals who had been absent for five or more of the intervention lessons. It was felt that any change in their performance could not be sensibly attributed to a series of lessons when they had attended no more than half of them. This amounted to three individuals from the Monday (visual) group and four from the Wednesday (verbal) group. In what follows any analysis considering changes over the intervention period will ignore these seven, although their scores will be used to generate any correlations between pre-intervention measures. The modified statistics are shown in Tables 5.2 and 5.3. As with the original groups, it was found that the mean scores on the various measures did not differ significantly between the groups. Although the standard deviations of the Wednesday group's scores tend to be larger, Leverne's test for equality of variances did not find this difference to be significant for any of the measures.

Table 5.2 Pre-intervention statistics for the groups with absentees removed

	Intervention group	N	Mean	Std. Deviation
MidYIS vocabulary	Monday	19	90.05	9.72
	Wednesday	19	95.63	14.82
MidYIS non-verbal	Monday	19	98.89	10.19
	Wednesday	18	98.83	11.22
MidYIS maths	Monday	19	92.95	8.22
	Wednesday	19	97.37	12.92
MidYIS skills	Monday	19	91.21	8.75
	Wednesday	18	96.94	16.82
Spatial memory test	Monday	15	11.67	3.13
	Wednesday	14	11.57	2.21
Visual/verbal ratio	Monday	16	1.04	.87
	Wednesday	18	1.30	1.75

Table 5.3 MCT scores before and after the intervention

	Intervention group	N	Mean	Std. Deviation	Sig.(2-tailed)
MCT-pre	Monday	19	13.84	4.50	
intervention	Wednesday	17	14.65	4.78	0.606
MCT-post	Monday	17	14.88	4.30	
intervention	Wednesday	19	19.32	7.27	0.035

5.1222 Maths Competency Test improvement

Inspection of the distributions of the MCT scores suggested that they were not Normally distributed. However, neither the Kolmogorov-Smirnov or Shapiro-Wilk test of Normality concluded that the divergence from Normality was statistically significant (Appendix I). After the intervention lessons most of the participants had improved MCT scores but this appeared to be more pronounced among the Wednesday group, who had received verbal style lessons (see Table 5.3). Since there was a good correlation between pre and post intervention scores (Pearson correlation of 0.669, significant at the 1% level), a regression was completed. This used the pre-intervention MCT score for each participant to predict their post-intervention MCT score, with the resulting standardised residuals used as a measure of improvement, referred to as MCT gain. Table 5.4 shows how these compare for the two teaching groups.

Table 5.4 MCT gain in the two intervention groups

	Intervention group	N	Mean	Std. Deviation	Sig.(2-tailed)
MCT gain	Monday	17	-0.34	0.79	
	Wednesday	17	0.34	1.06	0.043

There is a significant difference between the two teaching groups in their post-intervention performance and in improvement on the MCT ($p<0.05$). Therefore, a particular teaching approach benefited a whole class, although it was the verbal style that achieved this.

Effect Size

Since it was concluded that the MCT scores and the MCT gain were Normally distributed, it seemed reasonable to calculate an effect size for the difference between the MCT improvement in the two classes. This was calculated, using a pooled estimate for the standard deviation, for both the post-intervention MCT scores and the standardised residuals. In both cases, the resulting effect sizes were corrected using the approximation of Hedges and Olkin, 1985. This produces effect sizes of 0.7 (correct to 1 decimal place) for both measures of difference: the post intervention raw scores and MCT gain. Therefore the statistically significant difference found between the mathematical performances of the two teaching groups does appear to indicate a fairly sizable effect due to the verbal style of teaching.

The nature of the MCT

It seemed worth considering whether the verbal group's superiority on the post-intervention MCT covered all the questions or was limited to a certain style of question. In particular, some of the questions made quite heavy demands on literacy skills. Therefore the items on the test were classified, according to literacy demands, into three types (see Table 5.5 and Fig 5.3).

Table 5.5 Classification according to literacy demands, used on the MCT questions

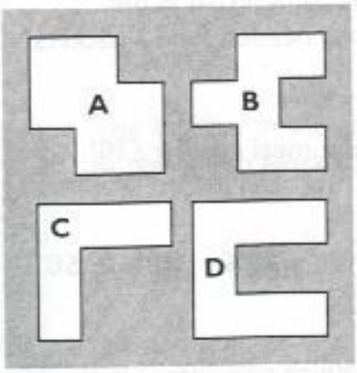
Type	Question style	Item numbers	Total on test
1	Numerals only or reading of everyday words	4,8,12,13,14,15,16,18,21,22,27,28, 31,32,33,36,38,44	18
2	A mathematical or numerical definition is needed	1,5,7,9,10,11,17,20,25,30,43,46	12
3	Heavy literacy demands involving instructions or definitions embedded in several sentences	2,3,6,19,23,24,26,29,34,35,37,39 40,41,42,45	16

Type 1

4 **7.6 9.3 6.7 8.9**

From this list, write the number which has the smallest value. ________

33



Which two shapes are the same size? ____ and ____

Type 2

7 **24 33 19 26**

Which number in this list is a multiple of 3 and also less than 27? ________

30 Write ¾ as a percentage. ________

Type 3

2 A bus with 27 passengers stopped to let 5 off and to collect 7 more. How many passengers are now on the bus? ________

Timetable

Lilydale	1118	2217	2251
Ringwood	1134	2237	2307
Burnley	1208	2311	2341
Museum	1226	2335	0005

34 Write the departure time for the train from Ringwood which arrives at Museum just after midnight.

____

Fig 5.3 Examples of MCT questions of the three types

An independent assessor who sorted the questions according to this classification system produced results with an agreement level of 76% (see Appendix J for this alternative sorting). When assessor agreement is corrected for chance (Bakeman & Gottman, 1986) the resulting Cohen's kappa is 0.64, which indicates an agreement significantly greater than chance. The performances of members of the two teaching groups on these fractions of the MCT were then compared (Table 5.6). Using the alternative sorting of the independent assessor produced substantially the same results with the same pattern of statistical significance (Appendix J).

Table 5.6 Performance on the different types of MCT questions, sorted according to literacy demands, within the intervention groups

	Intervention group	N	Mean	Std. Deviation	Sig. (2-tailed)
Type 1	Monday	17	6.24	2.36	
questions	Wednesday	19	7.89	3.49	0.108
Type 2	Monday	17	5.29	1.65	
questions	Wednesday	19	6.37	2.22	0.112
Type 3	Monday	17	3.35	1.73	
questions	Wednesday	19	5.05	2.50	0.025

As can be seen, there is no significant difference between the teaching groups on the first two types of question but there is a significant difference in scores on the questions with heavy literacy demands. It is on these that the verbal group's scores are significantly higher than the visual group's.

The MCT questions were then re-classified according to whether they made demands on visual-spatial skills (see Table 5.7)

Table 5.7 Classification according to visual-spatial demands, used on the MCT questions

Type	Question style	Item numbers	Total on test
1	Diagram or picture presentation	5,6,10,12,15,19,21,28,33,37,39,46	12
2	Question does not rely on understanding a diagram	1,2,3,4,7,8,9,11,13,14,16,17,18,20 22,23,24,25,26,27,29,30,31,32,34 35,36,38,40,41,42,43,44,45	34

The teaching groups' scores were then compared according to this classification (Table 5.8) and the significant difference is found for the questions making little demand on visual-spatial skills.

Table 5.8 Performance on the different types of MCT questions, sorted according to visual-spatial demands, within the intervention groups

	Intervention group	N	Mean	Std. Deviation	Sig. (2-tailed)
Type 1 questions	Monday	17	4.06	1.68	
	Wednesday	19	5.21	2.28	0.096
Type 2 questions	Monday	17	10.82	3.15	
	Wednesday	19	14.11	5.31	0.030^

(^It was found by applying Levene's test for equality of variances that equal variances could not be assumed and t was calculated accordingly.)

Profiles of MCT improvement in the two classes

Although on the pre-intervention MCT the profiles of scores within the two teaching groups were similar, this was not the case with the post-intervention scores or the improvement scores. Figure 5.4 shows box plots of MCT gain.

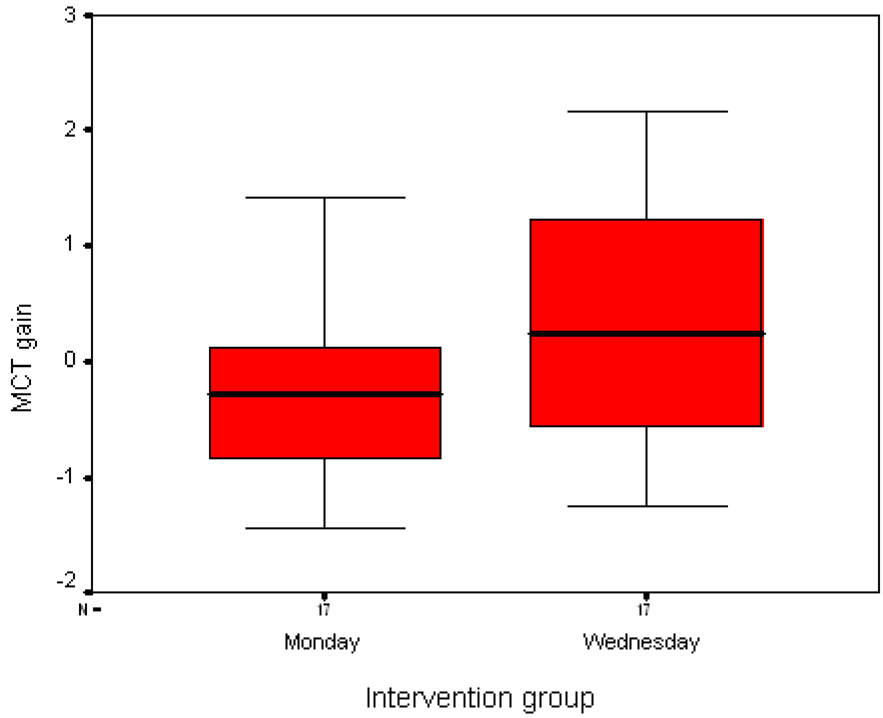


Fig 5.4 Distributions of MCT gain in the intervention groups

From the plots it can be seen that although the Wednesday group has generally improved, there is a lot of variation and the children at the lower end of the spectrum have done very little better than those in the other group. The extent of the range in the Monday group is mainly due to the score of one participant whose improvement on the MCT was far above that of the rest of the group (this was Gavin, whose work was noted previously and included as Fig 5.1). Clearly there is more to explaining these results than simply stating that the verbal lessons caused a general improvement in maths performance, as measured by the MCT. Comparing the class profiles suggests some individuals benefited much more from the verbal lessons than others and at least one child apparently gained from the visual lessons. Therefore it is necessary to look at individual differences in initial skills and achievement and their influence on outcome.

Furthermore, the breakdown of MCT performance into success on different styles of question suggests that the teaching interventions were affecting elements of maths performance, rather than general maths achievement. The next stage of analysis appears to be to question how individuals with differing abilities, and perhaps cognitive styles, performed on the different question styles within the MCT and find what differences, if any, can be put down to the teaching approaches.

5.2 The influence of individuals on outcome

If the focus of interest is on looking at how individual variation might have interacted with the lesson style, it is necessary to look more carefully at the individuals who comprised the two groups. It has already been established that there were no pre-existing statistically significant differences between the raw scores of the pupils in the two classes on a number of measures. However, it seems worth asking whether there were more subtle differences in the abilities and styles of the pupils in the groups, perhaps in the way these correlate, or which could be causing interactions between individual variables and the teaching style.

5.21 The measurement of visual/verbal tendency

A key measure of individual variation used by this research was the visual/verbal ratio calculated from the results of the recognition test. Since this was not an externally validated measure, there are many concerns about its legitimacy. Some of these were addressed by the pilot study work, but the larger sample involved in the main study allows further consideration of validity and reliability issues, which will be reported here. The additional interviews with some of the sample of older participants were particularly focused on elucidating the nature of visual and verbal style found by the test and this will be considered in more detail later.

The raw data of numbers of visual and verbal errors made by the main study and pilot study participants are included as Appendix K together with an alternative subtractive visual tendency score. This alternative scoring system results in similar correlations, both within the recognition test and with the other assessment measures, so it was not pursued and subsequent analysis only involves the visual/verbal ratio.

5.211 Distribution of scores

Figure 5.5 shows the distribution of visual/verbal ratios for the main study participants. The distribution is somewhat skewed, but this reflects how the ratio is calculated, with all the ‘verbal’ scores compressed to between 0 and 1. Similar distributions were obtained for the pilot study sample, the additional Year 7 pupils and for the Year 10 pupils (Appendix L). Table 5.9 shows descriptive statistics for the four samples. Since the visual/verbal ratio is not an interval measure, the most meaningful measures of average and spread are the median and inter quartile range, respectively. However, means and standard deviations are also shown to provide more of a picture of the data.

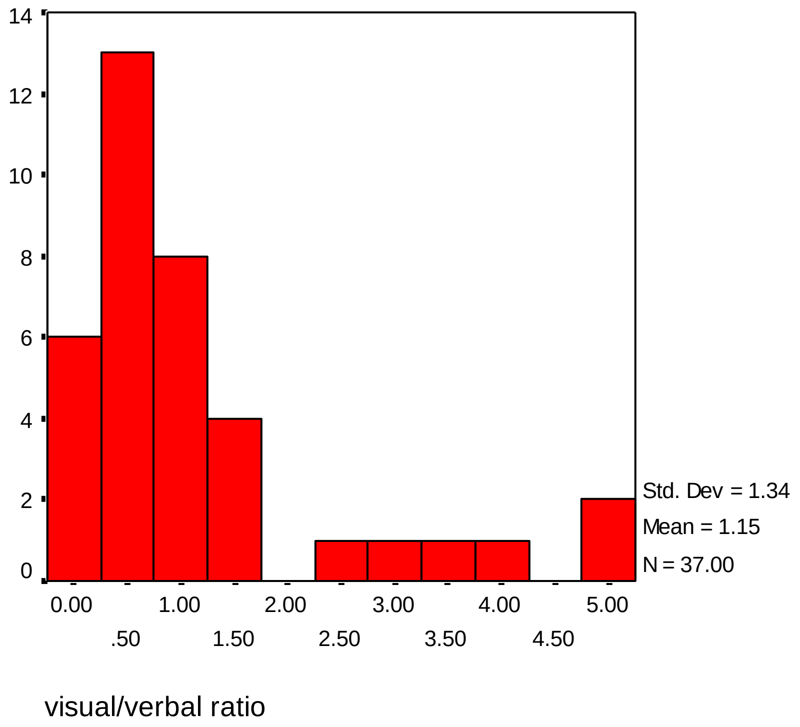


Fig 5.5 Distribution of visual/verbal ratios for main study participants

Table 5.9 Descriptive statistics for the visual/verbal ratio in the various samples of participants

Sample	Main study	Pilot study	Additional Year7	Year 10
N	37	11	29	33
Mean	1.15	0.59	0.78	0.73
Median	0.67	0.5	0.4	0.5
Std. Deviation	1.34	0.53	0.99	0.73
IQR	1.16	0.42	0.85	0.88

As can be seen, the descriptive statistics are broadly similar for all the samples, although the main study data did include somewhat more high visual scores, which is reflected in the higher values of average and range. The distributions of scores are not dramatically dissimilar, however, and, in particular, it is worth noting that the scores for the older pupils do not generally differ from those of the younger pupils. Thus there is no evidence that the test is functioning differently when used for people of different ages.

5.212 Internal indicators of validity

The pilot study compared a number of scores generated by the recognition test with the visual/verbal ratio and concluded that there was no systematic relationship between any of them and the ratio. The most important measure to compare with the ratio is the discrimination score, Pr, since a significant correlation would suggest that the ratio of visual to verbal errors simply reflects a differential ability to succeed on the memory test, through appropriate strategy choice, rather than relating to habitual style of information processing. In light of this concern, the discrimination scores were correlated with the visual/verbal ratios for all the samples and for each administration of the recognition test where this was repeated.

Although it might be strictly correct to use non-parametric measures of correlation for any correlations involving the visual/verbal ratio, it was found that for this correlation, as for many others, the non-parametric correlation co-efficients were extremely similar to the parametric correlation co-efficients. Since it would prove rather confusing to be using a mixture of co-efficients for correlations between main study assessment measurements, the decision was taken to use only parametric correlations. The exception to this is the case of the test-retest reliability of the visual/verbal ratio, where since it is only the visual/verbal ratio that is being considered and because this is so clearly not an interval scale, a non-parametric correlation co-efficient will continue to be calculated and reported.

The Pearson correlations of the visual/verbal ratio and the discrimination score for each administration of the recognition test are shown in Table 5.10.

Table 5.10 Correlation of visual/verbal ratio and discrimination scores for the various samples of participants

Sample	N	Correlaton: Pr and visual/verbal ratio
Main study 1st test	37	0.029
Main study 2nd test	32	-0.364*
Pilot study 1st test	11	-0.330
Pilot study 2nd test	12	-0.354
Additional Year 7	29	-0.646**
Year 10	33	-0.086

* Significant at 5% level

** Significant at 1% level

As can be seen, over all the samples, there is a tendency for the visual/verbal ratio to correlate negatively with the discrimination score, although this only reaches statistical significance for two samples: the additional sample of Year 7 pupils and the second administration of the test during the main study. How much importance can be attached to this finding will be considered later.

5.213 Reliability

Test-retest reliability

Consideration of various internal aspects of the recognition test, such as the distribution of visual/verbal ratios and their correlation with other scores, tends to suggest that the test functioned similarly for the pilot study and main study participants. Therefore it seems legitimate to consider their data together where needed. When the data were combined to calculate a test-retest measure of reliability the result was a Spearman's Rho correlation co-efficient of 0.478 (N=36, p=0.03). Although this is a statistically significant correlation, it is not as high as test-retest correlations are expected to be (see e.g. Kline, 2000) and does suggest that the measurements of visual tendency might not be reliable. It was questioned whether a difficulty might be that the visual/verbal ratios are based on only a relatively small proportion of the recognition test responses (20 out of 60), and furthermore that the scores of some participants will be based on a very few responses, if they did not make many errors. The visual/verbal ratio of these participants would be particularly unreliable. Therefore the decision was taken to calculate test-retest reliability for the subset of the sample whose error scores were all at least one, and so exclude those who were making so few errors that they actually scored zero on one or more of the error scores. This produced a test-retest correlation co-efficient of 0.568 (N=30, p=0.01), which is somewhat higher, but still leaves some doubts about the reliability of the visual-verbal measure.

Item analysis

Item analysis was conducted on the ten visual decoy items and on the ten verbal decoy items, using the main study and pilot study responses from the initial recognition tests. This produced Guttman Split Half Reliability co-efficients of 0.49 and 0.56 for the verbal and visual items respectively. Cronbach's α scores, to two decimal places, were also 0.49 and 0.56. These might be considered rather low, but there is some disagreement about the necessity of extremely high internal reliability (see Kline, 2000, p.31), especially where the items are intended to measure a broad tendency. However, it seems wise to conclude that the recognition test measure of visual-verbal tendency reflects a number of factors and tendencies, and is not a uni-dimensional measure of processing style. That this cannot be reduced to a distinction between the mathematical items and the everyday items on the test was revealed by the item analysis. This showed that the items that correlated most strongly were a mixture of these two sorts of items.

5.214 External indicators of validity

Table 5.11 shows how the visual/verbal ratio correlates with the other measures which relate to individual style or ability, in the visual-verbal dimension, for the main study participants. Shown are Pearson correlation coefficients and the number of subjects each correlation is based on

Table 5.11 Correlations between visual and verbal measures

	Visual /verbal ratio	Spatial memory test	MidYIS non-verbal	MidYIS vocab-ulary	MidYIS vocab and nonverbal difference
Visual/verbal	1	.095	-.092	-.121	.056
ratio	37	28	36	37	36
Spatial memory		1	.503**	.012	.448*
test		32	31	32	31
MidYIS			1	.236	.626**
non-verbal			44	44	44
MidYIS				1	-.609**
vocabulary				45	44
MidYIS vocab					1
and nonverbal					
difference					44

** Correlation is significant at the 0.01 level (2-tailed).

* Correlation is significant at the 0.05 level (2-tailed).

As can be seen, there are no statistically significant correlations between the ratio and the other measures. This suggests that the visual/verbal ratio is succeeding in its aim of measuring something other than particular cognitive ability (given the relationships with the spatial memory, MidYIS non-verbal and MidYIS vocabulary scores) or even relative cognitive ability (given the relationship with the difference between MidYIS vocabulary and non-verbal scores). Further consideration of exactly what it might be measuring will take place later, involving other parts of the research.

5.215 Conclusions on the use of the visual/verbal ratio

Taken as a whole, the examination of the data produced by the recognition test suggests that there are no irredeemable flaws in the instrument, in that it seems to be measuring a similar tendency in a number of different samples and this could be a tendency to process information visually. There are no clear reasons definitely to reject this assertion, although a more detailed examination of the data, together with observations from the main study and extra study interviews may shed further light on this. Although there are doubts about the reliability of the measure, given the test-retest correlation, the measure does not seem entirely appropriate to a test-retest paradigm; since it is a recognition test using the same items on the two occasions (see Chapter 4, section 4.2416 for discussion of this). Therefore, although caution should perhaps be exercised when assessing a participant's visual or verbal tendency, based on this measure, it seems reasonable to continue to consider it to be indicative.

5.22 Individual differences and mathematics performance

An initial overview of the data from the perspective of individual differences can be gained by looking at the correlations between the various measures in the two teaching groups. Tables 5.12 and 5.13 show Pearson correlations and the number of subjects each correlation is based on.

Table 5.12: Correlations of assessment measures: Monday (visual) group

	MidYIS vocab-ulary	MidYIS non-verbal	MidYIS maths	MidYIS skills	Spatial memory	Visual/ verbal ratio	MCT gain
MidYIS vocabulary	1	-.040	.129	.320	-.273	-.089	.217
	19	19	19	19	15	16	16
MidYIS non-verbal		1	.291	-.013	.403	-.112	-.022
		19	19	19	15	16	16
MidYIS maths			1	.399	-.056	.413	-.240
			19	19	15	16	16
MidYIS skills				1	-.457	.098	-.247
				19	15	16	16
Spatial memory					1	.397	.264
					15	14	14
Visual/verbal ratio						1	-.493
						16	14
MCT gain							1
							17

Table 5.13: Correlations of assessment measures: Wednesday (verbal) group

	MidYIS vocab-ulary	MidYIS non-verbal	MidYIS maths	MidYIS skills	Spatial memory	Visual/ verbal ratio	MCT gain
MidYIS vocabulary	1	.523*	.776**	.669**	.387	-.159	.542*
	19	18	19	18	14	18	17
MidYIS non-verbal		1	.630**	.462	.532	-.147	.571*
		18	18	18	13	17	16
MidYIS maths			1	.651**	.539	.007	.459
			19	18	14	18	17
MidYIS skills				1	-.013	-.248	.441
				18	13	17	16
Spatial memory					1	-.144	.439
					14	13	13
Visual/verbal ratio						1	-.572*
						18	17
MCT gain							1
							17

** Correlation is significant at the 0.01 level (2-tailed).

* Correlation is significant at the 0.05 level (2-tailed).

From the correlation tables, it is striking that the measure which does not correlate with all the others is the measure of visual tendency derived from the recognition test. This is consistent with previous research findings that verbaliser-verbaliser measures do not correlate with tests of spatial ability (Kozhevnikov et al, 2002; Hegarty &

Kozhevnikov 1999). Furthermore, this measure is negatively correlated with the standardised residual measure of improvement on the MCT: the more ‘visual’ children tended to fail to improve their MCT score. This relationship is more apparent, and significant, in the Wednesday group than it is in the Monday group. This finding will be considered in more detail later.

However in terms of questioning how the characteristics of the individual pupils combined with the interventional approaches, there are two issues of interest here. Firstly, there is the question of the differences from group to group in correlations between pre-intervention measures, which might suggest pre-existing differences between the groups that could explain the differential change in MCT performance. Secondly, it is necessary to consider the possibility of predicting the MCT gain in the two groups from the pre-intervention indicators of visual or verbal strength, since any differences between the groups could suggest interactions between pupil style and teaching approach.

5.221 The individuals in the two teaching groups

The overall patterns of correlations do differ between the two teaching groups, with the Wednesday group’s scores on the pre-intervention measures showing a much stronger tendency to correlate positively with each other. This is particularly pronounced for the MidYIS measures. The chance probability of finding five significant results when fifteen are considered (Sakoda et al, 1954) is very low ($p<0.01$), so it is legitimate to consider these correlations to have some meaning. However, the difference compared to the Monday group is probably due mainly to the differences between the two groups in variance on all the measures. The variances in all the scores are considerably higher in the data of the Wednesday group and this will make the scores more prone to higher correlation coefficients. Although it would be possible to compare each correlation across the two groups, the low numbers involved make it unwise to do so. Therefore it should be concluded that the apparent differences between the two groups are probably just the result of the differences in variances.

5.222 Interaction between pupils and teaching

An initial way to look for evidence of interactions is to consider the correlations of the standardised residuals with the pre-intervention indicators of abilities and styles, inquiring whether these differ according to the teaching group. Significantly different correlations from group to group between a pre-intervention measure and the standardised residual might suggest that possession of that quality has a differential effect depending on the teaching style. Obviously it would be expected, given the theoretical background of this research, that the indicators of visual or verbal strength or processing tendency should be most likely to interact with the style of teaching.

Considering the correlations, however, leads to the finding that there are no statistically significant differences between the two groups in their correlations between particular measures and the standardised residuals. This would suggest that there might not be any interactions between the children's individual abilities and styles and the teaching approach. However, it is difficult to use the various measures of cognitive strength to look for interactions with lesson style because they tend to correlate positively with each other. This is particularly pronounced for the Wednesday group, where it is possible to predict success by considering any of the MidYIS sub-scores and it is difficult to identify any particular skills or thinking styles that predispose a child to benefit from the verbal lessons.

Since a distinct focus of interest for this research is the possibility of finding interactions between the tendencies of pupil and teacher to use a visual or verbal approach, further analysis will be conducted to check the evidence for any interactions.

Analysis of variance

To establish whether there were any interactions between the styles of the children and the teaching approaches a series of two-way ANOVAs was conducted. The measures were identified that related to visual or verbal ability or thinking style. These were:

Visual/verbal ratio from the recognition test

Spatial memory test score

MidYIS non-verbal score

MidYIS vocabulary score

Difference between MidYIS non-verbal and vocabulary scores.

For each measure, the participants were classified as 'high' or 'low' depending on whether their scores were above or below the mean of all the participants' scores on that measure (the median was used for the visual/verbal ratio). Two-way ANOVAS were then carried out, considering the effects of each measure together with the intervention group on the MCT gain. No significant interactions were found between the teaching group and any of the visual-verbal indicators.

However, because of concerns about the different question styles on the MCT, the analysis was repeated using, first, the scores on the questions with heavy literacy demands and then the scores on the visually presented questions as the dependent variable. Again, there were no significant interactions.

5.223 Predicting MCT gain for individuals

Although there was no evidence of systematic interactions of lesson and pupil style, the fact still remains that in both classes some children's mathematics performance improved much more than others did, and it is worth questioning whether these successes and failures can be predicted from the pre-intervention measures. In particular, it will be recalled that the visual/verbal ratio measure of processing style correlated significantly and negatively with mathematics improvement. It must be noted, though, that considering all the pupils' scores together means that these analyses cease to distinguish between pupils in terms of teaching intervention so the individual outcomes must then be seen as a product of all their teaching experiences, both inside and outside the intervention lessons. The relevant correlations, for the non-absentee participants of both groups, are in Table 5.14. This shows Pearson correlations and the number of subjects each correlation is based on.

Table 5.14: Correlations of assessment measures: Participants from both intervention groups

	MidYIS vocab-ulary	MidYIS non-verbal	MidYIS maths	MidYIS skills	Spatial memory	Visual / verbal ratio	MCT gain
MidYIS vocabulary	1	.282	.604**	.593**	.056	-.123	.478**
	38	37	38	37	29	34	33
MidYIS non-verbal		1	.437**	.286	.433*	-.127	.354*
		37	37	37	28	33	32
MidYIS maths			1	.576**	.233	.108	.271
			38	37	29	34	33
MidYIS skills				1	-.169	-.159	.277
				37	28	33	32
Spatial memory					1	.071	.268
					29	27	27
Visual/verbal ratio						1	-.499**
						34	31
MCT gain							1
							34

** Correlation is significant at the 0.01 level (2-tailed).

* Correlation is significant at the 0.05 level (2-tailed).

Predicting maths improvement by combining MidYIS sub scores

As a further measure of visual-spatial strength compared to verbal ability, it was decided to calculate a difference score for each participant by subtracting their vocabulary score from their non-verbal score. Also, since these measures correlated notably with improvement on the MCT (see Table 5.14), a combined score was calculated by adding the two sub scores for each participant. The correlations of these scores with MCT gain are shown in Table 5.15. As can be seen, the difference score does not correlate with improvement, whereas the combined score correlates highly and positively. The addition of scores on the other measures used was found not to improve this correlation.

Table 5.15 Correlations of combined MidYIS scores with MCT gain

	MidYIS vocab and non-verbal difference	MidYIS vocab and non-verbal combined	MCT gain
MidYIS vocab and non-verbal difference	1	-.160	-0.170
	37	37	32
MidYIS vocab and non-verbal combined		1	0.521**
		37	32
MCT gain			1
			34

** Correlation is significant at the 0.01 level (2-tailed).

Inspection of the scatter diagram of this combined score plotted against MCT gain (Fig 5.6) suggests the relationship is linear, implying that the MidYIS combined score is similarly important at all levels of MCT performance.

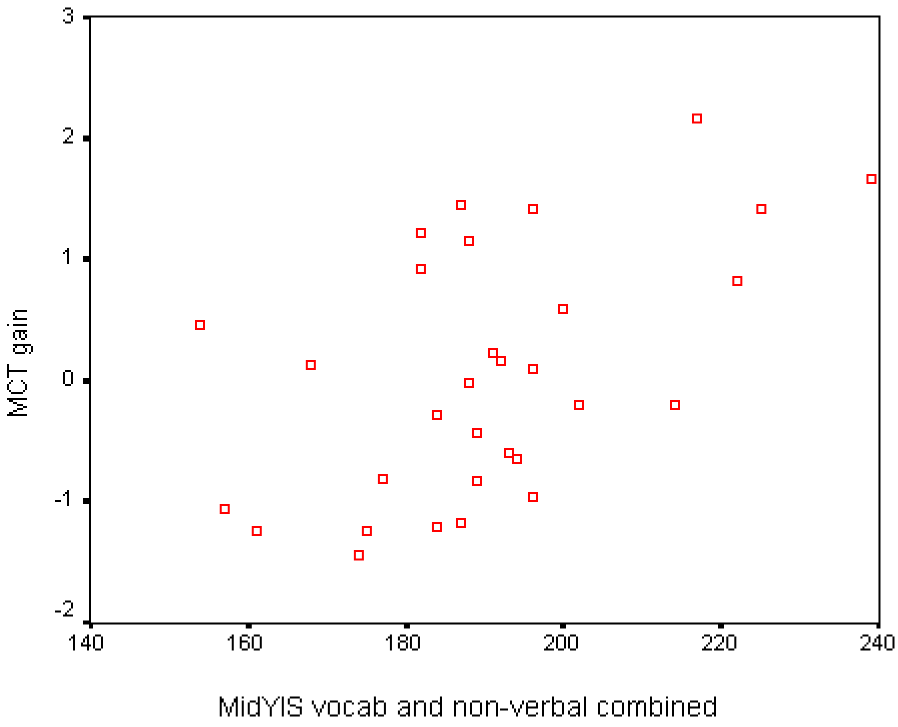


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Fig 5.6 Scatter plot of MidYIS vocab and non-verbal combined score with MCT gain

5.3 Visual and verbal cognitive styles

Since the analysis so far has found no evidence of an interaction between teacher and pupil styles, however this is measured, it must be questioned whether the visual-verbal distinction is worth making at all. Suggestions that it has validity in terms of the relationship between the teaching approach and the style of assessment have arisen in the finding that the verbally taught class significantly out-performed the visual class on the MCT questions with heavy literacy demands. However, it is also possible to infer that the visual-verbal distinction might be illuminatingly applied to individuals from the finding that visual tendency in the main study pupils correlated negatively with mathematics improvement. To make sense of this finding, though, requires a more systematic examination of the relevant main study results and further exploration of the nature of the visual/verbal ratio though consideration of styles and behaviour observed in participants in the pilot, main and extra studies.

5.31 Visual tendency and MCT improvement

Table 5.16 shows the correlations of the main study measures which relate to individual style or ability in the visual-verbal dimension. Shown are the correlations between these measures and the standardised residual measure of improvement in mathematical performance, MCT gain, for the non-absentee participants.

Table 5.16 Correlatons of visual and verbal measures with MCT gain

	Visual/ verbal ratio	Spatial memory test	MidYIS non-verbal	MidYIS vocab-ulary	MidYIS vocab and nonverbal difference	MCT gain
Visual/verbal	1	.071	-.127	-.123	.039	-.499**
ratio	34	27	33	34	33	31
Spatial memory		1	.433*	.056	.366	.268
test		29	28	29	28	27
MidYIS			1	.282	.514**	.354*
non-verbal			37	37	37	32
MidYIS				1	-.678**	.478**
vocabulary				38	37	33
MidYIS vocab					1	-.170
and non-verbal						
difference					37	32
MCT gain						1
						34

** Correlation is significant at the 0.01 level (2-tailed).

* Correlation is significant at the 0.05 level (2-tailed).

Of these measures, only the visual/verbal ratio can be assumed to be a measure of style, while the other seem more to measure ability in a particular cognitive area. This will be quite narrow in the case of spatial memory and much broader in the case of the MidYIS non-verbal score. The MidYIS difference score can be seen as a measure of relative cognitive strength, since it subtracts the score on the vocabulary test from that on the non-verbal test, and it therefore might be more related to preferred style than are the raw ability scores. However, there is a tendency for these scores to correlate positively with those for spatial memory (this is statistically significant when the scores of all the initial participants are considered, as can be seen from Table 5.11). This suggests that the difference score is more likely to be measuring, partly, proficiency in various spatial strategies and skills. Neither of these measures correlates with the visual/verbal ratio, implying that this is not measuring ability in some aspect of visual-spatial processing but may be assessing a tendency to use a processing style, which is fairly independent of success with that style.

Considering Table 5.16, it can be seen that both the MidYIS scores have significant correlations with MCT gain. This relationship of the MidYIS sub-scores to improved performance on the MCT has been previously noted and suggests that the skills needed to succeed on these differing tests may have important areas of overlap. The spatial memory test correlates much more modestly, suggesting that this underlying, basic aspect of visual-spatial ability is not particularly important for mathematical performance. However, the most interesting result is the finding that the visual/verbal ratio correlates significantly and negatively with mathematical improvement. Inspection of the scatterplot (Fig 5.7) reveals the nature of the relationship.

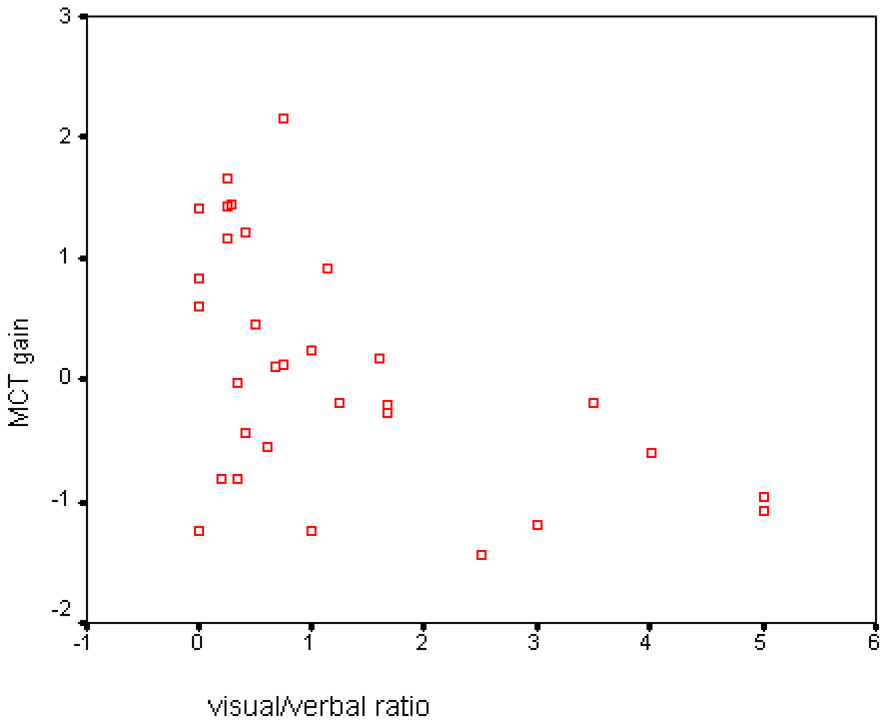

Fig 5.7 Scatter plot of visual/verbal ratio and MCT gain

Clearly there is not a linear relationship but a marked tendency for the more 'visual' participants to do particularly badly while the maths performance of the more 'verbal' children varies across the whole range.

5.32 Being a visualiser

5.321 Quantitative results

Given that the visual/verbal ratio had not been standardised on a large sample, it is difficult to be certain about what score to take as indicating a 'visualiser'. However, the scatter diagram above shows a cluster of participants with relatively much higher visual/verbal ratios. A beginning to the investigation of the nature of the visual tendency identified by the recognition test can be made by considering these six participants. Their scores on a number of measures are contained in Table 5.17.

Table 5.17 Scores on a number of measures of the Year 7 visualisers identified through the visual/verbal ratio

Participant: Group	Visual/verbal ratio	Pr	MCT: Pre-intervention	MCT: Post-intervention	Spatial memory	MidYIS non-verbal	MidYIS vocab
A: Wed	4	0.35	17	17	10	100	93
B: Wed	3.5	0.3	17	19		109	105
C: Wed	5	0.18	11	9	11	74	83
D: Wed	5	0.6	20	18	12	105	91
E: Mon	2.5	0.18	14	10	12	87	87
F: Mon	3	0.75	19	16	16	102	85

These scores confirm that many of the relationships of the other measures to the visual/verbal ratio extend to these extreme cases. As was found in both the pilot and main study data sets, there is no evidence of a correlation between visual tendency and success on the recognition test: these six children have a very wide range of discrimination scores. Similarly, their MidYIS non-verbal scores range from below to above average, although their vocabulary scores are (with one exception) uniformly low, which does not concur with the general finding of no relationship between vocabulary sores and the visual/verbal ratio. The spatial memory scores are fairly evenly distributed around the mean, of 11.4, found for the main study participants, which is in line with the finding from all the participants of no correlation between spatial memory and visual tendency. Interestingly, only one score diverges very far from this mean, providing no evidence for the suggestion (Kozhevnikov et al, 2002) that visualisers tend to be either high or low on spatial measures.

As Table 5.17 indicates, the main study visualisers mainly scored lower on the post-intervention MCT than on the test taken previously, suggesting that they really had not gained anything from the intervention lessons (nor from their other lessons), despite two of them experiencing a visual approach. Such general difficulty leads to considering the possibility that the visual/verbal ratio might just be identifying a fairly general problem with higher level cognitive processes; either a long term problem or perhaps a falling behind in cognitive development, which could potentially right itself. In either case, it becomes necessary to question whether this recognition test finds visualisers in a group of people who can be considered to be more able or cognitively advanced. As has been noted, however, the test does find visualisers in an older and more able sample, so it is not the case that the supposed visualisers are simply all less able. There remains the possibility, though, that there might be a number of ways of achieving a high visual/verbal ratio and generally poor cognitive functioning might be one of them. This will be further considered.

5.322 Qualitative results

If the visualisers found by the recognition test are not just low achievers, however, this still does not answer questions about what does characterise their thought and, particularly, their approach to mathematics. Here the qualitative data collected through interview should be suggestive. As has been mentioned (Section 5.111), the main study pupils were not very forthcoming when interviewed and, in addition the interviewees did not include any of the participants later identified by the recognition test as particularly visual thinkers. As has been reported (Chapter 4, section 4.2416) in relation to the validity of the recognition test, the interviews during the pilot study can be interpreted as implying some links between preferences for mathematics topics, strategies on maths questions and superiority on certain types of question (specifically word-based questions compared to those based on diagrams). However, it seems unwise to place too much weight on these suggestions for several reasons. One problem is that they are far from unarguable, with the results probably bearing a number of interpretations. This equivocation is partly caused by the way that this interviewing took place, at the beginning of the research when the questions that needed answering were still quite ill-defined and when the main intention was to test the materials and interview questions, rather than systematically to study the results. Furthermore, as has been noted, there are concerns about the quality and accuracy of reflection and the self-awareness possible with children of this age. For these reasons, the targeted interviews with older pupils were carried out after the main study fieldwork was complete and therefore now it is appropriate to turn to the results of these interviews.

Table 5.18 contains the observations made in the interviews with the Year 10 students identified by the recognition test as visualisers. It will be recalled that unlike the visualisers found among the main study Year 7 pupils, these interviewees were all reasonably successful in mathematics and in other school subjects. This is demonstrated by their inclusion in the top sets and also by scores in a recent test, consisting of past GCSE questions, which were found to be unrelated to thinking style (see Appendix M).

Table 5.18 Observations about the Year 10 visualisers identified through the visual/verbal ratio

'X' indicates a failure to use a technique or to solve an item correctly.

'/' indicates the use of a technique or successful solution of a test item.

'//' indicates more pronounced use of a technique or rapid, successful solution of an item.

Pupil		G	H	I	J	K	L
Rec. Test	Visual errors	6	1	3	2	1	5
	Verbal errors	5	0	2	1	0	4
	V/V ratio	1.2	2	1.5	2	2	1.25
Maths	Likes	Circle formulae; Pythagoras	Trig.	Graphs	Circles; Area and perimeter; Percentages	Trig; Algebra	Addition; "Easy stuff"
	Dislikes	Trig.	Algebra: Simultaneo-us equations	Algebra	Algebra	Long division/ multiply	Algebra; Fractions
Strategies	Diagrams	X	X	X	//	//	X
	Mental imagery	X	/	/	X	X	/
	Words & numerals	//	/	X	//	/	/

Pupil		G	H	I	J	K	L
Visual-spatial test	Item 1	//	//	//	X	/	/
	Item 2	/	//	//	/	/	/
	Item 3	/	/	//	/	X	X
	Comments	Imagined item turning; No words	Mentally rotated item	Mental image of item	Considered features and parts	Tried to imagine moveme nt, but "it was hard"	Considere d features and parts
Self report		Visual thinker	Visual thinker	Visual thinker	Visual thinker	"I use diagrams ."	"Both?"

As can be seen from the above table, there is considerable variation among these visualisers in their use of visual methods to answer questions and in their preferences for different areas of school mathematics. Even when the visualisers made use of visual methods, these were not always helpful. Student K solved Question S (A man planted a tree at each of the two ends of a straight, 25 metres long path. He then planted a tree every 5 metres along the path (along one side only). How many trees were planted along the path altogether?) through the careful use of a diagram. However Student H described her approach as follows: "[I] imagined the line and put one there and one there [indicates ends] and then imagined the others", having incorrectly answered “5”. Here the reliance on an incorrect visual image suggests a tendency towards visual thought, but perhaps not a beneficial one.

It is worth noting that two of the participants used diagrams but not mental images to solve the maths problems, while another three used mental images but no diagrams. Two of these imagers did seem more proficient than the diagram users on the visual-spatial mental manipulation and reported mental images of the whole item being rotated. The diagram-users reported more laborious attempts to imagine the movement of parts of the items and feature-by-feature assessment. This suggests that for all four of them habitual methods reflect ability to hold and control mental images, with the students who find such processes more difficult tending to use diagrams while the ones who experience clear mental images use such images instead of diagrams. However another participant who appeared very capable of the mental manipulations (Pupil G) did not make any use of any sorts of visual methods to answer the maths problems (which he did not answer very successfully).

Amongst the variation found between these visualisers, the one part of the interviews which did seem to correlate well with the visual/verbal ratio was the self report, since all the participants, apart from one, identified themselves as visualisers. However, before this can be taken seriously, it is necessary to see how apparent verbalisers answered the same inquiry about thinking style.

5.33 Being a verbaliser

As other researchers have noted (Presmeg, 1985), a limitation of much work concerned with teaching and learning styles is that it has concentrated on the characteristics of visual thought and shown less interest in the verbalisers. Although this research is similarly somewhat biased some apparent verbal processing tendencies were noted during the pilot study and someYear 10 verbalisers were identified and interviewed.

5.331 Interviews with Year 10 verbalisers

The results of these interviews are shown in Table 5.19.

Table 5.19 Observations about the Year 10 verbalisers identified through the visual/verbal ratio

'X' indicates a failure to use a technique or to solve an item correctly.

'O' indicates an unsuccessful, rejected attempt to use a technique.

'/' indicates the use of a technique or successful solution of a test item.

'//' indicates more pronounced use of a technique or rapid, successful solution of an item.

Pupil		N	O	P	Q
Rec. test	Visual errors	1	2	0	1
	Verbal errors	3	5	2	2
	V/V ratio	0.33	0.4	0	0.5
Maths	Likes	Probability: Triangles; Trig; Circles	Addition; Algebra	Algebra	Multiplication; Division
	Dislikes	Algebra: Fractions	Fractions	Trig; Ratio	Trig; Algebra; Equations

Pupil		N	O	P	Q
Strategies	Diagrams	O	X	O	//
	Mental imagery	X	X	/	/
	Words and numerals	/	//	/	X
Visual-spatial test	Item 1	/	X	/	
	Item 2	//	X	/	
	Item 3	//	X	/	
	Comments	Saw blocks as faces and tried to rotate	Tried to imagine item turning	Considered features and parts	
Self-report		Visual; Likes to draw diagrams	Visual: Has mental images	Thinks in words	Thinks in words

As was the case with the Year 10 visualisers, there is a lot of variation between these students in their mathematical preferences and in the strategies they used to solve the problems given during the interviews. A major aspect to note is that the self reports, perhaps because of the activities that preceded them, do not prove to be entirely dependable. Although the visualisers mostly considered themselves to think visually, so did two of the four verbalisers.

To further confuse matters, it was the two students who identified themselves as thinking “mostly in words” who showed much more evidence of using visual techniques, both mental images and diagrams, to solve the maths problems. One of these students, Student Q, was the only pupil interviewed who produced a completely visual solution (Fig 5.7) to the question, “Only four teams took part in a football competition. Each team played against each of the other teams once. How many matches were there in the competition?” The possibility of this solution is noted by Presmeg (1985), who used this question as part of her assessment of the mathematical processing styles of her participants. Here student Q successfully used the diagram to gain understanding of the problem and then to correct his initial mistake in assuming two matches between each pair of teams (indicated in Fig 5.8 by pairs of lines, which were drawn first, but single lines drawn later). This seems strikingly successful visual thinking for a student who is presumed to be a verbaliser.

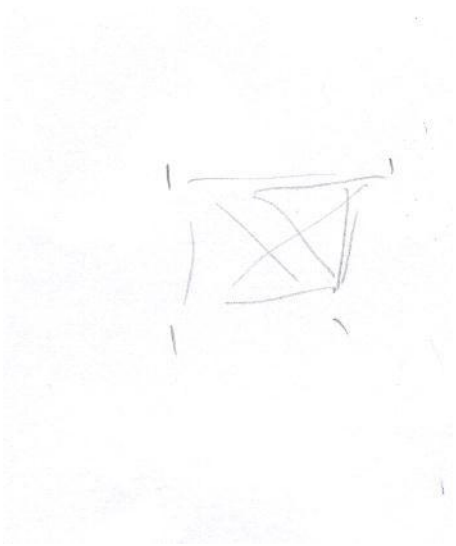

Fig 5.8 Diagram produced by Year 10 verbaliser

When these students did use non-visual methods there was some evidence of the tendency noted during the pilot study (Chapter 4, section 4.2416) for students to be misled by number patterns and facts. Question S (Section 5.322) frequently provoked such mistakes by the Year 10 students. However both visualisers and verbalisers gave the incorrect answer of "5" and justified this through non-visual numerical reasoning. Student I, a visualiser, said, "5, isn't it? [I] just divided 25 by 5" while Student O, a verbaliser, explained "5, like 25 metres and he's done it every 5 metres and there's 5 in 25."

5.332 Main study verbalisers

Returning to the scatter diagram of MCT gain and the visual/verbal ratio (Fig 5.6), it is possible to identify verbalisers who achieve across the range of MCT gain. However, an interesting comparison can be made between the six visualisers who performed poorly on the post intervention MCT and six participants whose MCT gain was similarly low but whose visual/verbal ratios identify them as verbalisers. Their scores on a number of measures are shown in Table 5.20 and can be compared with the scores of the Year 7 visualisers in Table 5.17.

Table 5.20 Scores on a number of measures of the Year 7 verbalisers identified through the visual/verbal ratio

Participant: Group	Visual/ verbal ratio	Pr	MCT: Pre-intervention	MCT: Post-intervention	Spatial memory	MidYIS non-verbal	MidYIS vocab
R: Wed	0.4	0.35	13	14	11	102	87
S: Wed	0.6	0.3	2	3	7		71
T: Wed	0	0.23	13	10		78	83
U: Mon	0.2	0.5	16	15	6	94	83
V: Mon	0.33	0.03	14	17	8	85	103
W: Mon	0.33	0.43	14	13	12	98	91

As with the visualisers, there is no suggestion of a link between intervention group and MCT success. These failing verbalisers were spread between the two teaching groups and neither teaching approach seems to have benefited them. A comparison of scores on the other measures reveals a tendency for the verbalisers to have lower scores on all the measures. While the visualisers' scores are distributed around the means for each measure, these verbalisers tend to have scores below these means. This extends to the MCT scores where, although the MCT gains of the visualisers are similarly low, the test scores on which these are based tend to be higher. It would appear then that the low achieving verbalisers do differ from the visualisers, who, although they failed to raise their MCT scores, have scores on other measures which suggest the potential for achievement.

These generally unsuccessful Year 7 verbalisers provide a contrast to the more successful Year 10 verbalisers and, of course, among the main study participants there were children with low visual/verbal ratios and large MCT gains (see Fig 5.6). Therefore, even among the main study participants, being a verbaliser is not associated with any particular level of achievement. This is different from the Year 7 visualisers, all of whom performed relatively poorly on the post intervention MCT.

6. Discussion

The central aim is to consider how the research results can be understood, given the previously discussed background issues, and how they shed further light on these issues. It will be necessary to consider both the quantitative results and the experiences of teaching and learning during the interventions, which are conveyed by the participants' work and their interactions with the teacher.

Specifically, it is intended, as previously, to discuss the idea of visualisers separately from the more general issue of visualisation. Even if, as will be argued, the utility of the cognitive style hypothesis is in doubt, this still leaves plenty that can be observed about visualisation, understood both as an individual way of understanding and as an aspect of teaching and learning mathematics.

6.1 Experimental outcomes

Initially, it seems wise to note the main outcomes of the research and to consider whether they appear to be genuine and legitimate, before looking at the questions they raise.

6.11 The success of the verbal approach

6.111 Pre-intervention group similarities

The central finding is the success of the verbal lessons in raising the participants' MCT performance. As is argued in the results section (Chapter 5, sections 5.1221 & 5.221), although the two teaching groups were not identical, there is no reason to conclude that pre-existing differences caused, in any simple way, the later diversion of scores. The means of the pre-intervention MCT scores are particularly close, with these scores being similarly distributed. There are no statistically significant differences between the group means for any of the other measures used, further implying that the groups are comparable. However, as has been noted (Chapter 5, section 5.1221), there is a consistent tendency for the Wednesday (verbally taught) group's scores to be more widely distributed on most of these measures. The variances for this group are higher on all the measures, apart from the spatial memory test (and there is only a slight difference on the MidYIS non-verbal test). Although it was not found that any of these variances differ significantly, this perhaps begins to suggest a subtle, but consistent, difference in the abilities of the children within the two groups, with the Wednesday group tending to contain children at either ends of the various ranges. What, if any, effect this difference is likely to have had on the outcome remains to be considered.

The suggestion of such a tendency leads to the question of whether it is, in general, the same children at the tops and bottoms of the various ranges, so making correlations between the tests more pronounced in this group. This might be important because 'good all-rounder' children would perhaps be expected to benefit more from any approach and it might be that the presence of more of them in the Wednesday group explains the outcome, especially given the widely distributed post-intervention MCT scores in this group. Considering the correlations of all the pre-intervention scores suggests that the tendency for the scores to correlate is generally more pronounced in the verbally taught group, but this is likely to result from the generally higher variances on many of the scores in this group. When teaching the two groups, the researcher was not aware of a difference in the general make-up of the classes. The numbers involved make it unwise to compare correlations directly so these observations are relied upon in drawing the conclusion that there would seem to be no reason to argue that the two groups differed significantly in their proportions of generally able and generally less able children.

Therefore this central finding is unlikely to be purely an anomaly, resulting from the pre-existing differences in the make-up of the groups. The only difference between the groups that stands up to scrutiny is the observation that the Wednesday group pupils tend to have more wide-ranging scores on the measures used, although actual differences in variances are non-significant. The most this suggests is slightly more variation in abilities in the verbally taught group group, although this was not reflected in the distributions of the pre-intervention MCT scores. There is not evidence to suggest that there was a particularly large proportion of more generally able children in the group and it is anyway difficult to specify what effect this could be expected to have on the outcome.

6.112 The nature of the success

Any suggestion that the apparent success of the verbal approach might be entirely explained by pre-existing differences between the groups is considerably undermined by the fact that this group did not improve in all areas of the MCT or produce particularly impressive classwork. The fact that the gain over the visually taught group was only seen in answering word-based MCT questions is difficult to explain through proposing a generally more able group, while it does strongly suggest the influence of the intervention teaching. However, this also serves to show the limits of the teaching approach, with the benefits not extending to more general improvements in mathematics performance or to much suggestion of improved understanding.

Of course, examples can be found where children in the Wednesday group did show genuine understanding, and sometimes the verbal approach did seem particularly to support and encourage this. As has been mentioned previously (Chapter 5, section 5.113), there was a class discussion about the meaning of some basic arithmetic terms, which seemed particularly illuminating. As well as allowing some children to deepen their understanding of certain terms, this discussion probably also supported the later work on functions and sequences, where the class members were successful in finding and describing mathematical rules in words. It appeared that producing explanations did help the children concerned to clarify and organise ideas and this could be a way that a verbal approach could enhance learning. Such a mechanism is supported by theories of constructing understanding and, especially, by ideas about 'self-explanations' (Chi & Bassok, 1989; Chi et al, 1994). However, as Chi and Bassok point out, it is important that such explanations are insightful and, generally, of high quality. Merely producing verbal explanations, if these only repeat information given, does not tend to improve performance. This problem was illustrated on many occasions during the research, such as, for example, when definitions of factors were produced in a routine manner. Also, during a different lesson, a child who liked to try to explain in words attempted to write down her conclusions about the Pyramids investigation. She ended up with only a lengthy description of adding numbers together, which completely ignores the central question of why putting the largest number in the middle is important:

> "The highest number should be in the middle bottom block so when you add the numbers on either side the next 2 blocks above will be 2 higher numbers and when you add them together the number in the top block will be a higher number"

In addition to not helping to produce genuine understanding, an attempted explanation such as this could actually be hindering the process since the child feels that they have written something and stops. Although Chi et al did find that in general prompting self explanations does help understanding, this later work used a piece of written exposition, rather different from the mechanics problems used by Chi and Bassok and also quite different from the sort of insight required by the Pyramids investigation. It would seem that for more routine understanding, where ideas just need to be put together or perhaps integrated with previous knowledge, the exact nature of any self-explanation is less important. Therefore this conception of the role of explanation could underpin at least some of the apparent understanding seen in the verbally taught class.

However, such ideas lead to two obvious questions. Who is benefiting from the explanations and how deep is the understanding produced? The second question indicates a return to the problem that began this section, of determining the extent of the mathematical improvement indicated by the MCT scores. This is clearly an over-arching concern that will be addressed throughout the discussion, so the question of who benefits will be tackled first.

The self-explanation framework obviously implies that any attempted explanations are mainly of benefit to the individual who makes them, even if this is done during a class discussion. Of course, involving various members of the group, and prompts by the teacher, makes it likely that more ideas are produced and so may be integrated. Furthermore, this combining of ideas might help weaker pupils by providing concepts that they are then able to put together. However, none of this seems likely to provide much benefit to the children who do not contribute to the discussion and, despite the teacher’s best efforts, there will always be some who do not contribute. It is worrying that such discussions could allow children with uncertainties to fall increasingly far behind, as they do not feel able to join in and then do not reap the rewards of trying to organise their thoughts. Of course there are presumably other ways to learn and individual discussion with the teacher during the lesson will also occur, as it did during the research. However, it remains that an important part of the learning process is not available in full to some of the class. It will be remembered that the MCT gain in the verbally taught class was only evident for some of the children and the distribution of the scores suggests that others had been left behind. The use of class discussion to encourage and support self-explanations, generally by the more capable children, could be one reason for this pattern of results.

It must now be questioned how useful these self-explanations were for the children who made them. As has been mentioned, the ideas produced during discussions varied from repeating of definitions to struggling with underlying mathematical ideas. Taking as evidence subsequent written work and later comments, the discussions sometimes appeared to be enhancing understanding but on other occasions did not. It might be anticipated that MCT performance would reflect understanding but it is difficult to know how much understanding is demonstrated by a certain level of performance on a test such as the MCT. Obviously, some mathematical understanding is required and if two people score extremely differently on the test, it seems safe to conclude that one has more understanding than the other, but at the level of small differences, it is more difficult to interpret the results. When the post-intervention test results were broken down according to question style, the verbally taught group's significant superiority was found to be due to their performance on the more word dependent questions. This suggests that their general mathematical understanding had not been enhanced by their attempts at explanations and any improved understanding did not extend to questions posed through diagrams or pictures. Furthermore, the Wednesday group did not perform particularly well on questions using ‘numerals only or reading of everyday words’ or on those where ‘a mathematical or numerical definition is needed’. So the problem was not that the verbal lessons had somehow neglected diagrams and not integrated them into the understanding developed. These lessons, despite the discussions of vocabulary and mathematical terms, also seem to have failed to improve the children’s ability to manage basic mathematical and numerical ideas. They were statistically no more or less likely than the visually taught class to be able to answer questions such as “Which number in this list is a multiple of 3 and also less than 27?” The questions that the Wednesday group scored more highly on, which made ‘heavy literacy demands involving instructions or definitions embedded in several sentences’ often involved only very basic numeracy skills.

So, although in the classroom, it sometimes appeared that the understandings voiced and developed were quite sophisticated, this does not seem to have generally improved the children’s understanding. Of course, individual explanations might have been beneficial but it does not seem in general that these were able to support consistent, and transferable, improvements in understanding, which could assist in answering all sorts of maths test questions. Yet the high proportion of verbal descriptions and explanations in the teaching materials, teacher’s remarks and encouraged in the children’s responses did seem to have improved the participants’ proficiency with words. The MCT scores imply that the Wednesday class members were more able to disentangle the numerical content from a lengthy description or ‘realistic’ setting. Arguably, this does point to improved understanding, but of a rather more narrow and specific sort than was initially envisaged or is usually associated with the idea of self-explanations.

6.113 The characteristics of the intervention lessons

If, as has been argued, it can be concluded that the two classes were comparable, if not identical, then particular observations and effects can be said to be due to the two interventions, and the differences between them. The question is then whether the two approaches were comparable. This will inevitably be difficult to answer, depending as it does upon what terms the approaches are compared and the theoretical assumptions of any criteria. However, any comparison would presumably require both sets of lessons to cover the same content, relate similarly to other mathematics lessons and to offer similar opportunities for developing understanding. In addition, each approach should address recommendations of enthusiasts for that approach and try to fulfil the specific needs created by the approach.

As has been described (Chapter 4, sections 4.12 & 4.3), the intervention lessons were designed to be as similar as possible, given the differing approaches. They made use of either identical teaching material, questions and investigations with appropriately altered instructions or exercises that were designed to be equivalent, though encouraging differing styles of thought. Examples of the latter include the visual class drawing dot patterns of numbers while the verbal class worked on number bonds (Lesson 2), and the visual class shading squares in a given ratio while the verbal class computed sequences of numbers according to a ratio (Lesson 9). A more complete comparison of the totality of the lessons can be made by studying the lesson plans (Chapter 4, section 4.3). As was noted (Chapter 4, section 4.12), one potential concern with the intervention lessons is that the visual lessons were always designed first and the verbal lesson planning then adapted the material. However, since this seems likely to have disadvantaged the verbal lessons, over the visual ones, such concerns can be ignored given the finding that the verbal lessons improved MCT performance.

An alternative concern, given the experimental design, is that because the verbal lessons were always delivered second, and made use of much of the material from the visual lessons, the teaching might have been generally better. Having already taught the visual lessons, the teacher in the verbal lessons was more aware of immediate difficulties and misunderstandings that the children were likely to have with the content or with certain material and was therefore in a better position to minimise them. However, if this is the reason for the differences in MCT gain, the advantage to the verbal class would be expected to extend beyond a certain type of MCT question, which it does not.

Neither approach was intended to be particularly innovative, with the aim being to reflect the generality of mathematics lessons and the constraints teachers work within. As has been noted (Chapter 4, section 4.11), previous research has sometimes been criticised for a failure to compare approaches which are typical of mathematics teaching methods (Presmeg, 1985). It was intended here that the two approaches would use activities that are typical of secondary mathematics teaching, simply bringing together ideas based on a visual or verbal approach and giving them coherence. Also, the lessons were designed to fit in with the other mathematics lessons the participants were receiving, covering content as it arose in the school's scheme of work. Considering the lesson plans leads to the conclusion that these aims were met and the children's mathematics teachers did not appear to think that the interventions disrupted their coverage of the term's work

Of course, this similarity to the bulk of school mathematics might be considered a limitation of the interventions, given that secondary school mathematics is often criticised by pupils and educators. As was noted previously (Chapter 5, section 5.11), there were occasions when the traditional question and answer format of some of the exercises interfered with mathematical thinking by suggesting an emphasis on answers rather than on understanding. This was arguably more of a problem in the visual lessons, where the diagrams were sometimes perceived as inessential diversions (Chapter 5, section 5.112), but also, during the verbal lessons, the desire to describe and define could be seen as undermining more sophisticated concept-building (Chapter 5, section 5.113). However, these problems did not occur with all the teaching materials and activities, since an attempt was made to use a variety of activities, and it seems more useful to draw attention to the failings of generally used techniques than to difficulties experienced with very innovative, and possibly idiosyncratic, methods.

In many ways, then, the two series of lessons do appear comparable, but if it is to be a fair test between them, as visual and verbal approaches, they both need to be following the recommendations of research and theory about such approaches. Unfortunately, even then there is the problem of establishing whether the children were actually trying to make use of visual or verbal ways of thinking, but this will be returned to. The starting point must be to establish that the lessons, as planned, were reasonable attempts at the two approaches.

Some of the requirements for the visual approach have been discussed previously (Chapter 1, section 1.411; Chapter 3, section 3.12) and some of these lead to equivalent aims for the verbal lessons, although with different ways to fulfil them. A major concern that is often voiced specifically about visual approaches to mathematics is that they may be too concrete and particular, so undermining an understanding of the abstraction and generality of mathematics. Although enthusiasts for a visual approach argue that this is not a necessity of visual representations (e.g. Arnheim, 1969) and examples have been collected of abstract images (Presmeg, 1992b; Krutetskii,1976) it is still a problem for a teaching approach to be confident of encouraging such ideas, rather than the irrelevant, and perhaps confusing, pictures in the head described by Pitta and Gray (1997). A key point is that the representations provided by the teacher, both as demonstrations and as methods for the pupils to use, must be at the abstract end of the spectrum. This suggests avoiding pictorial representations in favour of those that make their abstract nature clear by not having superficial similarities to that which they represent. An example of this contained in the intervention lessons is the encouragement to use an empty number line to represent quantities, which could alternatively been represented by more literal drawings (e.g. a picture of two people of differing sizes for a problem about height). As well as suggesting the general desirability of abstraction, this also conveys the related point about the power in mathematics of generalising, since the empty number line diagram can be used for such a range of problems.

Another way of conveying the importance of the general through diagrams is by taking care to offer a number of representations. This point tends to be made in any discussion of visual representations in mathematics learning (e.g. Arcavi, 2003). During the intervention lessons, this was attempted at a number of levels. In general, if the number system was represented, different representations were used, such as dot patterns and number lines. For example, during the lesson on number patterns (Chapter 4, section 4.3: Lesson 4, visual-spatial approach), dot patterns were used to introduce the idea of even numbers and multiples of other numbers, but it was demonstrated that multiples can also be shown as resulting from equal length jumps along a number line. Additionally, whenever particular types of representation were used in a lesson, it was emphasised that individual examples could be drawn differently while meaning the same thing. This was the point of the activity of drawing dot patterns for various numbers (Chapter 4, section 4.3: Lesson 2, visual-spatial approach), while the lesson on fractions (Chapter 4, section 4.3: Lesson 8, visual-spatial approach) used shaded boxes of differing shapes and sizes.

Unfortunately, providing and encouraging the construction of, a range of representations does not ensure that the concepts developed by the students are suitably general and abstract, but it is usually agreed that it helps and the visual intervention lessons did address this requirement. Another aspect of good mathematical understanding that teachers can aim for, but learners might still fail to achieve, is the holding of conceptual, rather than procedural, ideas about mathematical entities. There is less of a consensus about how this might be achieved, and no specific recommendations for visual teaching. However, as has been discussed (Chapter 1, section 1.411), the use of visual representations seems likely to encourage the idea of entities such as numbers having an existence beyond being the result of a procedure. It has been noted (Sfard, 1991) that this has historically been a beneficial effect of diagrams introduced into mathematics. Furthermore, if the vital part of understanding is, as Gray and Tall (1994) argue, to conceive of mathematical entities, ‘proceptually’, as both process and concept (see Chapter 1, section 1.22), then many visual representations seem particularly appropriate since they can encompass both actions and objects. During the intervention lessons, there was evidence of visual representations working in this way. For example, the concept of ratio was approached both as a visible entity, through holding up coloured cubes, and as a process with particular characteristics, through shading squares. The children seemed to see the cubes conceptually, since they were comfortable identifying a particular arrangement with a ratio, but the shading exercise allowed them to see the ratio as part of an on-going process. The shading activity brought to light some misconceptions about ratio, in that some children first attempted to shade squares as though for a non-equivalent fraction (so 1:12 was interpreted as 1/12). Through the process of shading squares they began to understand more about the nature of ratio, so the procedure seemed to expand their initial concept of ratio and this new conception combined process with concept.

The concerns about mathematical understanding raised above are clearly relevant for the verbal lessons. It is sometimes assumed that the application of language to an observation is inevitably an abstraction, since language is symbolic. However, it can be argued that language actually serves the cause of the particular, since "a verbal name is a fixed label" (Arnheim, 1969, p244), while there is also a concern that words may be used without underlying understanding (see e.g. Piaget's opinion on early counting, Piaget, 1952, p.29) and therefore no abstraction can be said to be occurring. During the intervention lessons, the aim was to avoid empty words by encouraging a questioning attitude and asking for further explanations. Mathematical ideas about generality were conveyed by such activities as flexible combining of numbers to make different totals (Chapter 4, section 4.3: Lesson 2, verbal approach). However, this can only really be seen as evidence of abstraction if the assumption is made that the children's ideas about number were based on concrete experiences with particular items, which number words allow them to abstract from. Yet, as has been argued previously (Chapter 1, section 1.21), such a conception of individual mathematical development has many problems, although it might be adequate description of how the body of mathematical knowledge has historically developed. With individual learners, the teacher is presented with the problem that numbers are the abstraction that needs to be grasped, but the use of number words gives little clue whether this has happened. So the children answering the inconsistently worded questions (Chapter 4, section 4.3: Lesson 3) might be abstracting number relationships from the problems, just as the visually taught children could demonstrate through using an empty number line. Alternatively, the verbally taught children might be only spotting and manipulating digits in an effort to get an answer. Similarly, the verbal rules offered during the functions lesson(Chapter 4, section 4.3: Lesson 6, verbal approach), could be instances of abstracting information from a collection of paired numbers, or be seen as attempts to limit a mathematical relationship to a rather abbreviated idea of what ‘double’ means. The temptation is either to see the abstraction as inevitable, given the use of language, or to consider the use of words as a poor substitute for real abstract understanding. Piaget argued that it is possible to discover the actual state of a child’s understanding, through cunning experiments and careful questioning (e.g.Piaget & Inhelder, 1956, p.212). However, in a busy mathematics classroom, such interviews are not possible and the teacher has to depend on guesses and suggestions in assessing the level of abstraction that the child is achieving.

A distinct indication is likely to be conveyed by how the learner uses language in mathematical discussions. In particular, a ‘proceptual’ understanding (Gray &Tall, 1994) could be expected to be indicated through flexible descriptions both in terms of processes (“times by itself”) and concepts (“square number”). While modelling such language use was not an explicit aim of the verbal lessons, it seems likely that this was one effect of all the discussions and attempted explanations. For example, the function rules were elicited as descriptions of procedures to carry out on some numbers, but then were described more conceptually as “function rules”. On other occasions terms that seemed to have become rather empty through over-use (e.g. "times") were reconnected to the processes through questioning by the teacher about meaning. The intention was to expand the learner's concept and it could be see as conveying the 'dual nature' of mathematical entities (Sfard, 1991). Unfortunately, though, this could be interpreted as undermining a learner's abstraction by tying a term back down to a procedure. It depends on what level of abstraction is assumed to be in existence and, as has been argued above, this is a difficult judgement to make on the basis of language use.

A major consideration for both sets of lessons is the issue of metacognition and the extent to which the two approaches support and encourage self-monitoring. As has been previously discussed (Chapter 2, sections 2.1 & 2.31), metacognitive awareness is assumed to be an important link between children's underlying abilities and knowledge, and their successful completion of a task. Although some have argued that the term is being over-used (Adey & Shayer, 1993), it is still a useful concept, which can explain why children sometimes fail to make good use of skills and strategies that they appear to have at their disposal. Therefore, any teaching approach should be based on an awareness of the issue and attempt to support metacognition. Since this seems to suggest a certain level of self-awareness and reflection, the verbal lessons seem more obviously to fulfil this need. It has already been concluded that the idea of self-explanations can be used fruitfully to interpret the interactions in the verbal lessons, and this concept is linked to self-monitoring. Chi and Bassok (1989) found that the successful students who produced a higher quality, and quantity, of self-explanations were also more accurate in their self-monitoring. This is not really surprising, given that the conception of self-explanations involves a fairly conscious, effortful construction of meaning, involving questioning and elaborating. During the verbal lessons, such an approach was, in effect, modelled by the teacher during the class discussions when various mathematical ideas were linked and questions were asked of the class to clarify understanding. Furthermore, the emphasis during the series of lessons, and particularly evident when investigations were used, was on explanation. The children were asked, “Why does that happen?” and “Can you explain?” This was intended to encourage reflection and looking back at the work to try to see a reason. The format of the investigations themselves was also an encouragement to elements of self-monitoring, such as planning and organising, since the presentation of the problem was more open than was the case with the traditional questions. Interestingly, the children found even this level of self-organisation quite hard. In both classes, there was initial confusion over the Pyramid investigation because the children could not see how to begin testing the possible arrangements of numbers in a systematic way.

Apart from this element of self-monitoring in the investigations, it might appear that the visual lessons gave less support to such behaviour. Certainly this was a concern during the designing of the lessons, although, as has been argued (Chapter 2, section 2.31), there is not absolute certainty that the very explicit sorts of self-awareness, more obviously prompted by the verbal teaching, are the only or best ways to think. However, there is an implicit sense of self-explanations and self-awareness about many of the activities used in the visual lessons while other exercises aimed to provoke a non-verbal, but quite explicit, sort of understanding. The representations that the children were encouraged to construct can be seen as self-explanations of a non-verbal sort as they pin down certain concepts so they can be examined and integrated with other knowledge. Such a purpose was conveyed to the class through the use of diagrams, such as the numbered number line, by the teacher to communicate and clarify ideas. Also on occasions, such as when they drew dot patterns for numbers, the pupils were explicitly directed to compare their drawings with others, so encouraging them to see visual representation as a way of holding ideas so they can be examined. As has been noted elsewhere (Chapter 5, section 5.112) however, the difficulty was in persuading the children to make use of diagrams. Although they would generally follow instructions to shade squares or draw dot patterns, they showed their lack of conviction about these methods in the way they were reluctant to use them to tackle further questions or problems. Although dot patterns would have provided a visual proof to the Odds and Evens investigation, only two children in the class tried to use them (one succeeding with the proof) and, similarly, during the ratio lessons, the square-shading method was swiftly ditched by all but one member of the class when it came to solving ratio problems.

It is tempting to see this as analogous to the difficulty in the verbal class of persuading the children to reflect on explanation. Although the verbal lessons did encourage self-reflection in a number of quite explicit ways, there is still no guarantee that the responses of the children were any more than superficial gestures. As has been mentioned, the class discussions only sometimes appeared to provoke real reflection and the children were generally unable to see why the investigations needed genuine explanations, rather than mere descriptions. Therefore, although the methods were different, both teaching approaches attempted to support and encourage the development of self-reflection, but problems were encountered in achieving the genuine engagement of the children with this sort of thinking.

The idea that metacognition was in fact encouraged more effectively by the verbal lessons and that this explains the superior MCT gain in the verbal class is undermined by the finding that this gain does not extend to all types of MCT question.

Furthermore, there was no evidence from the interviews with a subgroup of participants or from classwork that understanding had been improved by the verbal approach, as might be expected if metacognition had been particularly developed.

6.114 Summary

To summarise the above discussion on the effect of the verbal teaching approach, it is possible to conclude that the main finding, of this approach improving certain aspects of mathematical performance, stands up to scrutiny. The two intervention groups were not judged to have been so different in make-up that this is likely to have caused the result. Furthermore, a comparison of the two sets of intervention lessons allows the tentative conclusion that it is a legitimate comparison to make. Considering the sometime diverging ways that the two approaches attempted to convey mathematical understanding underpins the idea of reasonable comparison, but also suggests more about how the two sorts of teaching might work. In this consideration, it was important to remember, and try to explain, the finding that the verbal lessons only improved performance on the MCT questions particularly dependent on literacy skills.

6.12 The failure of the visualisers

The other main, notable, outcome of this research is the finding that across both teaching approaches the children identified as visualisers tended to fail to improve their MCT score. This idea of visualisers tending to struggle mathematically is broadly in line with other findings and the consensus in the literature, which is that visualisers have difficulty with mathematics. However, it should be noted that such a general disadvantage is not supported here since the MidYIS maths score, in common with the other MidYIS indicators, did not correlate with the visual/verbal ratio. There is, as has been previously discussed (Chapter 3, section 3.2; Chapter 4, sections 4.11 & 4.2416), a problem of how visualisers are identified and, particularly, whether this identification is separate from rating their mathematics performance. In this research the aim was to identify visualisers using a method separate from mathematics to avoid the difficulty of distinguishing a struggling student resorting to visual images from a habitual user of visual methods, which may or may not cause mathematical difficulties. However, this method of identifying visualisers has problems of its own and these need to be considered, together with the other findings from this research about the overall performance, and other behaviour, of these visualisers.

6.121 Validity of the visualiser-verbaliser scale used in this research

Just as the legitimacy of drawing conclusions from this research about visual and verbal teaching depends on the experimental details of the fieldwork, so any ideas about visualisers depend on the validity of the visual/verbal ratio. As has been described (Chapter 4, section 4.11), the aim was to find a means of measuring habitual tendency to think in a more visual or more verbal way. The decision to consider a global visual/verbal tendency was based on the conclusion (Chapter 2, section 2.5) that proposing the existence of such a concept is a reasonable response to previous research on dual coding (Paivio, 1971), distinct cognitive processes (Baddeley, 1997) and to some of the work on cognitive styles (Riding & Rayner, 1998). Also the very real problem of limiting the concept to mathematical processing is that this confuses the issue of any difficulties visualisers seem to have with mathematics as these could be either a cause or an effect of their visual processing (see Chapter 3, section 3.2).

Some of the more well-used tests of global visual tendency rely on participants' judgements about their own conscious experience of visual imagery leading to many problems of comparing judgements, but also presupposing that vivid conscious experience is the essence of being a visualiser. Given that many more people respond in experimental situations as though they are processing visually (Brooks, 1967; Dehaene et al 1993) than report a conscious experience of imagery, this identification of visualising and conscious experience seems problematic. Even the more reliable self report measures (see Richardson, 1977 for a review), based on people's assessments of their own tendencies to use visual or verbal methods in everyday life, encounter a more slight form of this problem. In the present research it seemed particularly unwise to use any sort of self-report measure given the age of the participants (see Chapter 4, section 4.2416).

The only visualiser-verbaliser scale that seemed at all appropriate to the needs of this research was the scale developed by Riding and colleagues (e.g. Riding and Rayner, 1998), since this aims to be a scale measuring general processing style that does not rely on self assessment. This visualiser-verbaliser scale is a ratio of the time a participant takes to respond to questions where the compared attribute of the two items is visual, compared to the time taken with a verbal attribute, that of membership of a concept. However, the scale is based on the assumption that a particularly salient visual attribute of an item is its colour. This does not seem an entirely justified assumption and seems likely to presuppose that the resulting visual style is rather literal and concrete. This seemed likely to be a particular problem when mathematical performance is investigated as so many writers consider that it is the more abstract or spatial end of the spectrum of visual representations that is, beneficially, involved in mathematical thinking (Presmeg, 1992b; Gray & Pitta, 1996). The decision was then taken to develop a new method of assessing visual tendency, based, like Riding's scale, on an opaque rest with the visual/verbal score resulting from a comparison of the visual and verbal processing presumed to underlie performance. This took the form of a memory test with pictures and words to be remembered, with mistakes made on a later recognition test assumed to reflect the participants' predominant, visual or verbal, style of encoding. Although occasional attempts have been made at using such a method of assessing cognitive style and it is considered feasible (Richardson, 1980), no replicable test has been developed. Therefore it is not possible to compare the results of this test with an earlier, or similar, version to establish its concurrent validity. Furthermore, the concerns that other methods of measuring visual tendency may be measuring different tendencies preclude a comparison of the research participants' scores on a number of visualiser or imager measures. However, to support the validity of the visual/verbal ratio measure, which is suggested by its theoretical background, its relationship to other measures used in the research, and the reliability of the scores, some additional work was carried out. This involved interviewing a sample of older children and attempting to link the visual/verbal tendency assessment derived from the recognition test to other indications of cognitive tendency, such as preferences for certain areas of mathematics, methods used on maths problems, achievement on a test of visual-spatial manipulation and their general assessments of their own habitual thinking styles.

The central concern to emerge from the main fieldwork was that the visual/verbal ratio did not seem to be suitably reliable. Although the test-retest correlation found during the pilot study was judged to be adequate, the correlation resulting from the main study and pilot study scores is rather low (Chapter 5, section 5.213). Item analysis scores also fell short of what is usually considered desirable on psychometric tests (Kline, 2000). However, there is some disagreement about the necessity of high internal consistency, particularly if the aim is to measure a broad construct, such as visualising tendency is assumed to be (Kline, 2000, p.31).

It would appear then that the visual tendency measure is not perfectly reliable but the question then arises over whether it is, imperfectly, measuring visual/verbal cognitive style or, in fact, some other ability or tendency. Evidence that the ratio might be measuring something else comes from the finding that there was a tendency for the visual/verbal ratio to correlate negatively with discrimination scores. This suggests that the ratio might just be another indication of a failure to do well on the memory test, which was presented to the participants as the aim of the exercise. A mechanism for this lack of achievement could be a failure to make use of appropriate strategies and engage in general self-monitoring behaviour. In particular, an obvious strategy to use when faced with the task of remembering pictures and words would be to elaborate the items, most obviously by naming the pictures. Brandimonte and Gerbino (1996) report that the expectation of a recall test tends to provoke verbal labelling of pictures, while the expectation of a recognition test encourages the use of visual memory. Although these participants were not told what form the memory test would take and so might vary in their expectations, informal comments suggested that many of them were expecting a test of recall. Certainly verbal labelling facilitates various sorts of rehearsal and self-testing, which some of the participants were observed to engage in. Such a strategy could be expected to improve the final discrimination score, but would also tend to lead to ‘verbal’ type errors and so a verbaliser score on the visual/verbal ratio. This interpretation means that the visual/verbal ratio is not so much measuring cognitive style, as the ability to make sensible use of cognitive skills and strategies, given what is known about the task. Such an explanation fits in with the contention that a more realistic assessment of human cognition rejects cognitive styles and concentrates on individual ability to make optimal use of all sorts of processes and skills (e.g. Roberts & Newton, 2001;

Klein, 2003). It also draws attention to the problems produced by a test that depends on both unconscious encoding tendencies and on the style of consciously used strategies. Although such a test could be expected to reflect visualiser and verbaliser styles more completely, it also produces more concerns about validity.

However if this were the whole story of the visualiser-verbaliser measure, it would perhaps be expected that the measure would correlate more consistently with the discrimination score, reaching statistical significance in more than two of the samples (Chapter 5, section 5.212). The finding that most undermines this idea, though, is the discovery of visualisers in the Year 10 group of older, and more successful, students. The distribution of visual/verbal ratio scores in this group was judged similar to the distributions of scores in the other samples, of younger and less mathematically able children (Chapter 5, section 5.211). Furthermore, in this group, the discrimination scores did not correlate with the visual tendency scores. This could be due to the students being more generally capable, and meta-cognitively aware, so they were all fairly equally likely to engage in the strategies mentioned above as well as other, perhaps visual, ones. In that case, realising the utility of attempting to use a strategy ceases to be the deciding factor in outcome level of achievement, which presumably is then determined by a range of factors. Meanwhile, a more general engagement with a variety of strategies by all the participants decreases the tendency of the visual/verbal ratio to reflect general ability. This argument suggests that the group of visualisers indicated by the test are more likely to be genuine visualisers, rather than a mixture of visualisers and those who could not work out how to succeed on the test. Indeed, as has been noted, if all the supposed visualisers were merely ineffectual test-takers, it would be expected that none would be found among the relatively successful Year 10 pupils.

It might be imagined that interviewing these students who were identified as visualisers would reveal whether they were actually visualisers, but this then means returning to the problem of defining a visualiser. However, the interviews were, in fact, quite revealing in that they suggest the considerable variation present among these visualisers in their preferences, tendencies and styles. Also, since a comparable sample of verbalisers were interviewed, it was possible to compare the strategies actually used on maths problems and the self-reports of habitual processing. There was a lot more over-lap than might be expected, especially over the general self-rating. In general, though, it seems possible to perceive a core tendency towards visual processing among the visualisers, compared to the verbalisers, and together with the observations above about strategy deployment, this suggests some validity for the visual/verbal ratio. However, the variety in the visual styles found among these pupils despite them being of a similar age and mathematical ability, does bolster a general idea that the characteristics of a visualiser might be very broad indeed.

6.122 Understanding the findings for the main study visualisers

Although there might be some remaining concerns about the visual/verbal ratio, these will be returned to and the whole concept of visualisers and verbalisers further explored. At this stage there does not seem to be sufficient reason to dismiss the visual/verbal ratio as a measure and it is worth considering the results of the main study research that concern the children identified as visualisers.

The main finding was of a significant negative correlation between the visual/verbal ratio and MCT gain. In fact, it was found that the visual/verbal ratio is a better predictor of MCT gain than any other single measure, although the MidYIS vocabulary and non-verbal scores, when combined do correlate somewhat more strongly. This raw result could be taken to indicate, in line with previous research (e.g. Lean & Clements, 1981), that visualisers do indeed struggle with mathematics. However, a major problem with such an interpretation is that the visual/verbal ratio does not correlate with the MidYIS maths score, which would be expected if the visualisers had a general problem with mathematics. It would seem then that this result indicates a tendency towards a limited failure of those pupils identified as visualisers to thrive through the intervention lessons. This result holds statistically significantly across all the pupils, regardless of which intervention they experienced. Considering the two groups separately, the correlation only reaches significance in the Wednesday, verbally taught, group but the relevant correlations do not differ significantly between the groups.

It must be considered what it is about the abilities of these pupils whose MCT scores do not improve that caused this outcome. Due to the nature of the relationship of the visual/verbal ratio to MCT gain (Chapter 5, section 5.31), there is a range of visual/verbal ratios among those with negative MCT gains. However, it is striking that while it is possible to be a verbaliser on this scale and score across the range on MCT gain, all the visualisers had negative MCT gains.

These visualisers were split between the two intervention groups so the problem was not a mismatch between teaching and learning style. Such a suggestion is further undermined by the finding that the visual/verbal ratio did not significantly interact with intervention group to affect MCT gain. Given the above argument about the differing ways that a visualiser score could be produced, and particularly the proposal that some identified visualisers might be, in effect, generally unskilled test-takers, it is interesting to consider the discrimination scores of these visualisers. Of the six visualisers, all of whom achieved negative MCT gains, there are two who scored distinctly below the mean discrimination score for all the participants (mean =

0.35; their scores are both 0.18). The idea that they were generally unskilled at deploying any appropriate strategies is encouraged by the observations that one of the two had recognised Special Educational Needs and a classroom assistant employed to help him and both children's MidYIS scores were generally low. It seems reasonable then to conclude that these two individuals might not be encoding in a predominantly visual way so much as failing to encode very much at all. Their identification as visualisers can then be seem as a combination of chance and the fact that more genuinely visual pupils might have scored towards the verbal end of the continuum through an attempt at using a naming and verbal rehearsal strategy.

The other four visualisers can now be considered. Two of them scored close to the mean discrimination score (0.3 and 0.35) while the other two scored considerably above it (0.6 and 0.75). It seems likely then that they were managing to approach the test with a certain degree of competence in strategy deployment and self-monitoring. This idea that they were making reasonable use of the strategies and skills at their disposal is interesting in light of their patterns of scores on the MidYIS vocabulary and non-verbal tests. In all four cases the non-verbal score is higher than the vocabulary score, and, for two of the children, the difference is very large. In some ways this makes sense as it might be expected that thinking visually might be linked to good non-verbal skills and an accompanying lack of verbal knowledge, such as is needed for the vocabulary test. Specifically, it seems unlikely that children with low verbal skills would attempt much naming and verbal rehearsal. This does suggest that the visual/verbal ratio as used is touching on a real aspect of cognitive functioning and a similar finding emerged from Mayer and Massa's (2003) investigation of visual and verbal learners. They found that visualiser-verbaliser scores derived from a questionnaire about habitual thinking styles correlated negatively with two measures of verbal ability.

Furthermore, while these four visualisers have MidYIS non-verbal scores above the sample mean, all the verbalisers with low MCT gains also have low MidYIS scores both for the non-verbal and the vocabulary tests (Chapter 5, section 5.332). Although there is a slight tendency for these verbalisers to score more highly on the vocabulary test than on the non-verbal test, in a reversal of the pattern for the visualisers, the main conclusion that can be drawn is that their scores are generally lower. Of course this would not be expected to extend to more successful verbalisers, such as the Year 10 interviewees and the Year 7 verbalisers with high MCT gains. However, it remains the case that different patterns of achievement or ability seem to apply to the visualisers and verbalisers who performed similarly poorly on the post intervention MCT: whereas the verbalisers seem to be generally low achievers, a sizable number of the visualisers would appear to be skilled non-verbally but struggling verbally.

The generally held theory of learning or cognitive styles is that they are more determined by preferred ways of thinking and should not be seen as directly related to ability (Sternberg & Grigorenko, 2001; Riding, 2001. See Chapter 2, section 2.4). In fact, the visual/verbal ratio's independence from ability has previously been suggested, given that the results as a whole do not correlate with the Mid YIS scores, or with any combination of them. In general this might be the case, but given the detailed examination of the visualisers, it does seem that there might be a tendency, in at least some places along the scale, for it to relate quite distinctly to ability and particularly to the balance of verbal and non-verbal abilities. This adds to the contention being developed that there might be a number of ways of being a ‘visualiser’ or a ‘verbaliser’, and perhaps that these relate to varying profiles of underlying skills and abilities.

Of course the MCT gain is only one of the outcomes of the teaching interventions and any other insights into the effects of the teaching need to be considered. While acknowledging that there is no consistent interaction between the visual/verbal ratio and intervention group, it should be revealing to look at the work produced and classroom behaviour of these visualisers in the two classes. Of the two visualisers in the Monday (visual) group, one was one of the two children with low discrimination scores, which lead to doubts about the sense in which they could be said to be visualisers. The other child, though, had a high discrimination score and, furthermore, was reasonably well observed during the teaching since he was a good attender and quite demanding, though generally enthusiastic, in the classroom. The visualiser he can be most fairly compared with in the verbal class, with similar scores on the MidYIS tests as well as on the recognition test, was also a good attender. However, this child was much less obvious, which might have been partly to do with the style of lessons, but it is not thought to be entirely the result of the teaching approach.

A comparison of the work produced by these students is interesting because they both did most of the more routine questions and wrote down results from the whole class ‘mental starter’ at the beginnings of the lessons. But neither of them managed any insights on the investigations or produced answers to the more involved problems. In neither case did the initial activities seem to lead them anywhere. Among the work from the verbal class, the visualiser child’s work is notable for being particularly empty of words. Although they were not instructed to, many of the other children wrote down definitions and vocabulary as well as the observations and explanations they were encouraged to produce. The visualiser child, in contrast, worked through numerical problems, reasonably accurately and his pages frequently contain only numerals. If this seems to explain why he did not get beyond the introductory work, it is revealing to consider the visualiser in the visual class. He also did the introductory work with relative ease, briskly shading squares and arranging dots in patterns, but he was not able to use these methods to extend his ideas or his skills. Even with assistance, he found it very hard to make use of the empty number line to solve problems, finding it extremely difficult to link the information given with the abstract representation. Similarly, his dot patterns, though neatly drawn and elegant, did not seem to allow him to see any abstract qualities of numbers, so when dot patterns were suggested to him as a way to explain his findings on the Odds and Evens investigation, he looked completely blank.

Comparing these two individuals seems to offer an insight into the finding that the visualiser-verbaliser learning styles did not interact with lesson style. Although, the visualiser in the verbal class seemed to have the expected problems associated with a reluctance to use words to explain, the visualiser in the visual class did not seem able to use his visual preference, or ability, to develop his mathematical understanding or skills. This was despite being given assistance to use abstract representations which could be expected to be useful. Unlike visualisers in other research (e.g.Hegarty & Kozhevnikov, 1999; Pitta et al, 2000), this child's difficulty was not due to overly concrete or pictorial visualisation. Perhaps the real problem, as both children's MidYIS scores suggest, is that they were not visually skilled so much as verbally unskilled. This caused predictable difficulties for the verbally taught child, while the visual thinking of the other child did not seem adequate for the mathematical understanding attempted through visual methods.

6.13 Summary of the experimental outcomes

It appears legitimate to conclude that this research involved a fair test of visual-spatial against verbal approaches to early secondary school mathematics teaching. The two groups taught appear to have been similar enough at the outset and the two teaching approaches seem representative of lesson suggestions and theoretical understanding relating to the differing styles of teaching. Given this, it is reasonable to conclude that a verbal approach can produce limited benefits, in terms of mathematical performance, and that this is not related to individual styles or abilities. There was no evidence that matched teaching and learning styles affected any element of performance or understanding in either class. However, the current research has concentrated on a particular age group with fairly low levels of mathematical achievement, so further research is necessary to see if these findings generalise.

This research found some support for the general idea that at least some of those identified as visualisers might have problems with mathematics. Among the Year 7 pupils, even when those who seemed generally less competent at the recognition test were ignored, a high visual/verbal ratio did seem to indicate a child with difficulties. In addition to the MCT scores, experience in class and work produced backed up such an idea. However, considering the MidYIS ability measures together with class performance suggested that the problem was not caused by visualisation strategies but by general difficulties with abstract ideas and a lack of verbal proficiency. This supports the finding of Pitta (1998) that among her participants a tendency to use visual processing did not indicate numerical achievement. Therefore, there might be some individuals who use predominantly visual methods, perhaps because of other cognitive deficiencies, who have difficulties with mathematics, but this will not extend to other visualisers. Such a conclusion was supported in the present study by the finding of students with high visual/verbal ratios among the more successful Year 10 pupils.

The problem, suggested above, of understanding who the visual/verbal ratio is identifying as visualisers or verbalisers, and why, can be taken as a n indication of the difficulty of assessing people according to the visualiser-verbaliser construct. Although the theoretical backgroud to the concept is adequate and the presumption of visualiser and verbaliser styles appears reasonable, there is no real consensus on how such styles should be assessed (Mayer & Massa, 2003). The current research used a method that aimed to assess general processing style by providing a task to be completed and attempting to identify the predominant style of processing used. The fact that this task allowed both involuntary encoding and consciously controlled strategy use could be expected to make it a more valid measure, but in fact seemed to be adding to confusion about who it was identifying. The suggestion that such concerns produce, that the visualiser-verbaliser construct may not be valid, needs to be properly considered.

6.2 Understanding visual thinking

6.21 The visualiser-verbaliser distinction

Despite the methodological problems with the particular assessment used in this study, a quantity of evidence has been collected about what it means to classify, teach and assess children under the assumptions of a visualiser-verbaliser continuum of cognitive styles. This experience needs to be considered against the continuing discussion of the visualiser-verbaliser distinction.

Recall that the idea of these particular styles of learning is underpinned essentially by the evidence for separable cognitive processes and the dual coding theory of the representation of information. As has been discussed (Chapter 2, section 2.2), it has been found that visual-spatial and verbal processes do appear to be separable, both at a psychological and at a neurological level. The dual task paradigm has been developed and used to build up a wealth of evidence (e.g. Baddeley, 1997) and is now even used to investigate the visual-verbal balance of cognitive resources in other thinking (Trbovich & LeFevre, 2003). The theory of two systems of working memory, the ‘phonological loop’ and the ‘visuo-spatial sketch pad’ (Logie & Baddeley, 1990), which has been developed partly through this work, makes sense of other findings in psychology and is a vital part of many cognitive models (see e.g. Humphreys & Bruce, 1989). The main continuing concern, about the legitimacy of running together visual and spatial processing, mirrors investigations in neurology and so this also provides reassurance of the reality of the proposed processes and their organisation.

The dual coding theory of representing information seems plausible given the psychological and neurological work and it has continued to be tested and found to have explanatory power (Richardson, 2003). Philosophically, it has been argued that the idea of two main forms of representation in human beings makes sense (Phillips, 1983). While the two systems must be linked, one sort of representation will not reduce to the other (Phillips, 1983) and it only creates problems to postulate a unitary deep level of thought where meaning is contained (Kaufmann, 1996; Anderson, 1978)

However, the difficulties begin when these ideas about general human functioning are extrapolated to explain the results of differing human beings performing differing tasks. Postulating specific abilities is one solution, but these lead to other problems, such as designing tasks that really do require the ability under examination (Lohman & Kyllonen, 1983; Chapter 2, section 2.2). The clear alternative to an abilities conception of human performance has come to be seen as styles. Since teachers are more interested in how a child actually processes information, as opposed to how they could optimally do so, ideas of learning, or cognitive, styles are popular with educators (Klein, 2003). Presumably there will be a relationship between a person's cognitive style and their abilities, and this is sometimes discussed (Sternberg & Grigorenko, 2001) and investigated (Mayer & Massa, 2003) but a particular relationship is not central to the idea of styles.

The visualiser-verbaliser style, then, has a good psychological and neurological background. There do appear to be dissociable visual and verbal processes, often using different parts of the brain, and supporting two distinct forms of representation. But does this mean that individuals can be classified in terms of their use of these processes, into visualisers and verbalisers? Logically, it does not and therefore investigation into actual human functioning is required. This has tended to try to establish that some, if not all people have a consistent tendency to use one system of representation over the other for a particular task (e.g. MacLeod et al, 1978; Ford, 1995) and so can be said to have a preference or tendency towards a visual or verbal style. The extension of such processing tendencies across different tasks is then required for the concept of global processing styles to be legitimate. Evidence for such personal consistency is hard to find and when it emerges partly through self-report (e.g. Leutner & Plas, 1998), there is real doubt that it reflects any more than the desire of participants to identify themselves consistently through the constructs being suggested by the research. The likelihood that participants can be led to give certain responses is suggested by the results of the interviews of Year 10 pupils conducted during the present study. Immediately after a test of visual-spatial image manipulation, there was a tendency for all the students to identify themselves as visual thinkers (Chapter 5, section 5.33). However, as has been argued previously (Chapter 4, section 4.11), if the concept of visual and verbal thinkers is to have any utility, it needs to imply fairly general thinking tendencies and the reality of this has been assumed. If this were the only difficulty for the construct then finding definitive evidence to support such a conclusion would become more important. But, as will be argued below, there are other problems with the identification of visualiser and verbaliser styles.

A central question is whether the two possible styles need to be thought of as in opposition to eachother. Riding (e.g. Riding & Calvey, 1981) developed his imager-verbaliser scale specifically to avoid the problems of other visualiser, or imager, scales, where to be a non-visualiser was a purely negative attribute, leading to concerns about validity. The more valid self report scales of habitual processing (see Richardson, 1977, for a review) involve positive and negative statements about behaviour and processes associated with the two sorts of cognitive style, leading to a position on a visualiser-verbaliser continuum. This seems to make sense, until one contemplates actual human behaviour, especially when a person is engaged in a complicated problem, rather than a short test item or experimental task. Considering the children learning during this research, there seem to be times when an individual used both forms of representation (e.g. Gavin's Odds and Evens investigation proof, Chapter 5, section 5.112) and other occasions when a child did not really seem to be making use of either system (e.g. the visualisers, in either visual or verbal lessons). If these children were to continue in these ways, this would presumably put them all in the middle of the visualiser-verbaliser continuum. It must be questioned whether this makes much sense when one is actively exploiting his visual understanding to construct verbal understanding, while the others are failing to represent anything in any form. The idea that there might be no real justification for assuming that the visualiser-verbaliser style is a bi-polar construct is supported by the finding of Leutner and Plas (1998) that in some tests the visualiser and verbaliser scales correlate positively.

As has been previously noted (Chapter 3, section 3.2), it seems possible that the mathematicians popularly considered to be visualisers (Stylianou, 2002) might indeed frequently use elaborate visual processing, but be combining this with verbal reasoning. Such a possibility is suggested by Stylianou's accounts of mathematicians at work, but with notable mathematicians this is generally not checked because once the presence of visual thinking is established, the assumed opposition of visual and verbal processing classifies such people as visualisers, not verbalisers. Furthermore, the psychological evidence for the visual/verbal distinction, with which this discussion began, does not put the sorts of processing in opposition. Instead, because of their partial independence, there is the possibility of simultaneous processing using the two systems, with this being an efficient and effective use of cognitive resources. Klein (2003) makes this point in relation to the adaptation of learning styles ideas to educational practice, arguing that teachers and children should instead aim to develop both sorts of processing.

Yet, even if it is not a necessity for the two sorts of processing to be seen in opposition, it could be that some people, at least, do tend to make consistently more use of one processing style than of the other. Even if the continuum idea is not very illuminating about individuals who fall in the middle, it could have some utility in explaining the rather less balanced thinking of those at either end. This is the sense in which Krutetskii (1976) discussed 'geometric' and 'analytic' styles of mathematical understanding, and he contended that most of the students he studied used a mixture of these styles. For the minority of students who could be described as geometric or analytic 'types', he described the disadvantages of relying so heavily on one type of thinking. For example, considering capable analytic pupils answering maths questions, he states: "An analytic course of solution was used even when it was less rational than a solution by visual-pictorial means" (p.319). Meanwhile, given a potentially misleading diagram, "almost all the [geometric type] pupils took their cue from the drawing, as a result of which they made gross errors" (p.322).

Given such an understanding of the visualiser-verbaliser concept, the tests which oppose visual and verbal styles might be considered interesting because they draw attention to the people who are relying particularly heavily on a certain style of processing. On this interpretation, the visual/verbal ratio used in the present research can only be expected to be properly valid for individuals at the extremes. For these people, there is less concern about distinguishing the conscious application of strategies from fairly uncontrolled encoding tendencies since they should all be in the same style. If they are not, then the person will, quite correctly, end up with a more intermediate score on the scale. This suggests some commonality among the people at either end of these scales, even if the processing styles of those in the middle is more complicated. It is worth questioning whether such a suggestion is at all supported by the present study or by other research.

As has been argued, the visual/verbal ratio did appear to categorise as visualisers a sub-group of ineffectual test-takers, this being a particular problem with the younger and less able children. This difficulty did not seem to occur with the older sample, with the identified visualisers all achieving reasonable discrimination scores on the test. They were also more willing and apparently able to reflect on their own thinking. Therefore it is the interviews of the Year 10 visualisers that will now be briefly considered.

Despite a central tendency to make use of visual strategies and mathematical preferences that usually involved liking geometric or graphical topics and disliking algebraic and numerical areas, there was much variation (see Chapter 5, section 5.322). A particularly obvious distinction is between those whose use of visual strategies manifested itself in diagram drawing and those who used mental images. Such variations between visualisers have been previously reported of course. Kozhevnikov et al (2002) propose that this variation be understood as resulting from the existence of two sorts of visualisers, either high or low in spatial ability. However, this idea has not been advanced by other researchers and the results of the present study do not support it (see Chapter 5, section 5.321). Presmeg (1985; 1992b) drew attention to the types of images reported, with some visualisers having more abstract images, and her idea of a continuum from concrete to abstract has been used previously (Chapter 3, section 3.21) to explain the observations of Pitta (1998) that young children's images of number appear able to benefit some but to mislead others. In both cases, though, the visualisers were not identified by a visualiser-verbaliser scale that could be expected to find only 'unbalanced' visualisers, so these individuals could well be more varied. It would be illuminating to consider, in detail, the images and strategies used by visualisers identified through a visual/verbal scale, such as Riding's or one of the habitual processing style questionnaires.

In general it would appear that people who often use visual thinking might differ in two distinct ways. On the one hand will be the extent to which they are also able to use verbal strategies and their ability to integrate the two styles. This issue of balance should be picked up by assessment scales that oppose the two styles, but this is by no means certain since it assumes that these simple tasks can be performed in the integrated style that might more generally characterise the individual's thought. If this is not possible, the test may identify as distinctly non-verbal visualisers those who could actually integrate a range of styles when given a suitable challenge. A different problem occurs with scales that simply give a rating of 'visual' over 'non-visual', since these make no attempt to discover the person's verbal tendencies. Therefore they potentially run together people who use a range of styles with those who consistently prefer to think visually.

The other way that identified visualisers can vary is in the sort of visual processing they use and while this need not be at all related to the balance of visual and verbal strategies, it is difficult to find conclusive evidence about this. The Year 10 visualisers interviewed during this research, who could be expected to be similarly unbalanced and not verbal, given the method of assessment, still differed in their visual experiences and strategies. This implies an independence of the two aspects of variation, but unfortunately the doubts about the complete validity of the visual/verbal ratio undermine the certainty that these visualisers really were 'non-verbal visualisers'.

However, until it is definitely disproved, it seems reasonable to assume that visualisers will vary even if those visualisers are identified so that they form a homogenous group regarding balance with verbal skills. This undermines any general reinterpretation in terms of visual and verbal abilities, even if the discussion of how using visual strategies need not preclude verbal competence might be suggesting such an interpretation. The range of visual-spatial strategies and styles would seem to be too varied to reduce to a scale from high to low ability. Furthermore, the results of the current study suggest that there is not a simple relationship between preferred cognitive style and visual-spatial or verbal abilities (see Chapter 5, section 5.214), which is the conclusion that other researchers have also drawn (e.g. Sternberg & Zhang, 2001). Therefore, we are left with few certainties over the concept of a visualiser, the idea meaning that an individual uses visual approaches fairly freely. The exact extent, and circumstances, of the use of these visual methods are left uncertain; as is the detailed content of visual ideas and images. It might be wondered whether such a concept has any utility and it clearly does not produce a definite answer to the question, "What does it mean to be an imager?" (Katz, 1983). However, it does allow this study to be considered, in light of other work with visualisers, without continual questions about which subset of possible visualisers are being identified by each method and the resulting confusions and disagreements when conclusions are compared.

6.211 Matching teaching and learning styles

In his critique of the ideas of learning styles and multiple intelligences, Klein (2003) states that "matching instruction to learning style has failed empirically". He goes on to claim that studies which attempt to match teaching and leaning styles do not, in general, report reliable effects. However, these teaching experiments were concerned with 'modality teaching' and seem to have understood learning style rather simplistically as a perceptual preference. For example, an experiment by Riding and Douglas (1993) investigated the learning that seemed to result from presenting information as text and picture compared to as text and text. A problem with such research is that the differing presentations could be acting so as to capitalise on a pupil's style or, alternatively, compensating for deficiencies in thinking. For example, providing a diagram could help more verbal thinkers, who would struggle to imagine or draw their own, or it could be used, as intended, to support visual thinkers by presenting the information suitably. In the case of the Riding and Douglas study, diagrams perhaps fulfilled both needs, since imagers learnt more from the text and picture presentation, but verbalisers performed similarly given either presentation. This difficulty of establishing the function of material presented in a particular way, and the suggestion that the same material might capitalise on one learner's style while compensating another, suggests why learning style experiments might struggle to produce results.

In the present study, it was intended that the teaching styles compared would extend beyond the presentation of material. The lessons were designed to encourage constructive thinking in the particular styles, with appropriate strategies being suggested and activities carried out that relied on visual or verbal methods. The intention was that the lessons should capitalise on students' styles, rather than being compensatory, so increasing the likelihood of revealing interactions between teaching and learning styles. However, no significant interactions were found and reasons for this must be considered.

The central explanation seems likely to be uncertainty over the use to which a learner puts any material, however carefully designed, and, related to this, the extremely complex interactions that seem likely to develop between the teacher, the learner and the classroom activity. Even though the strategies and methods suggested through the two teaching approaches were designed to build on and exercise the related, and not the opposed, learning style there is no guarantee that this always occurred. All the practice with representing numbers through dot patterns did not, generally, develop a visual sense of numbers, which could be utilised in the Odds and Evens investigation. Only two of the visual class made any attempt at this, while in the verbal class one child did begin a visual explanation (Chapter 5, section 5.114). It is interesting to question where this idea of hers appeared from, and consider whether anything in the lessons provoked it. Clearly the idea of encouraging particular styles of learning through teaching approach is undermined by the finding that the verbal teaching could precipitate, in one child, a very visual appreciation of number.

Given the various responses that children might have to any teaching, it is not surprising that straightforward interactions do not seem to occur. Presmeg (1985) comes to a related conclusion over her investigation of the interactions between the habitual mathematical styles of teachers and students. She argues that visualiser pupils and visualiser teachers may, effectively, not be matched since their preferred styles may differ in ways other than visual tendency and also in the type of visual thinking used. This means that a true match between pupil and teacher will only happen very occasionally and will need much more precise characterisations of style to identify it. As has been discussed, the global assessments of general cognitive style seem likely to reveal a range of 'visualisers', and presumably 'verbalisers' (see Leutner & Plas, 1998, for a suggestion of this in their finding that some verbaliser scales are uncorrelated), so it is not surprising if such assessments do not satisfy this requirement.

6.212 Visualisers doing mathematics

It must be remembered, however, that a major reason for examining teaching and learning styles was to attempt to improve the learning of visualisers. If the idea of matching teaching and learning styles has to be abandoned as an unhelpful simplification, given the variety of visual styles and complexity of classroom interactions, what is the solution?

First, it must be considered whether visualisers can be said to have difficulties that need addressing and clearly this depends partly on the method used to identify visualisers. As Campbell et al (1995) found, merely experiencing vivid mental images need have no general effect on mathematical performance, although it might sometimes affect the approach taken on a particular task. Similarly, Pitta (1998) identified a difference in frequency of visual images among her participants but concluded that this did not "provide an indicator of the level of numerical achievement of the children" (p.280). If visualisers are identified through their tendency to use visual methods in mathematics, care must be taken to distinguish a general failure to cope from an actual preference for visual strategies. This is often not done and presents a particular problem if the same mathematics questions are used with participants who perform at quite different mathematical levels (e.g. Lean & Clements, 1981; Hegarty & Kozhevnikov, 1999). Finally, visualisers who are characterised by their use of visual instead of verbal thinking and identified through tests that oppose visual and verbal style, will have slightly different problems. They could be expected to have difficulties integrating visual and verbal strategies, which might leave them disadvantaged. However, the extent of this would seem likely to depend on their actual abilities with visual and verbal methods with, perhaps, a lack of balance only being problematic if verbal skills were particularly poor. Such difficulties are suggested in the present study by the finding that the visualisers, who under-perform on the MCT, have noticeably low MidYIS vocabulary scores. It seems arguable whether the solution to any problems experienced by such visualisers is to try to develop their mathematics through visual thinking, as originally proposed, or whether it would be more beneficial to try to improve their verbal skills, through, for example, help with reading and writing. Certainly for the visualiser discussed earlier (section 6.122), the visual approaches to mathematics did not seem to develop his understanding, as he did not appear able to get beyond immediate visual appearances to the mathematical abstractions.

Even if visual methods are judged by a teacher to be useful in certain circumstances, as when illustrating or introducing a concept, the problem remains of how to make these appropriate to a particular visualiser. Interviews with visualisers carried out by other researchers (Presmeg, 1985; Pitta, 1998) and during the current study, suggest a considerable range of visual thinking. There seems to be variation in the properties of reported visual thinking as well as variation in how it tends to be applied, shown most clearly in some visualisers drawing diagrams, while others work with mental images. Somehow, overlapping these considerations, are the concerns about integrating the visual with the verbal.

Therefore it must be questioned whether classifying learners as visualisers is helpful when considering the difficulties that some children have with mathematics. Not only does it deflect attention from the difficulties which might be associated with using more verbal methods and being a ‘verbaliser’ (see Chapter 5, section 5.33), but it does not seem especially illuminating when examining visual approaches. The problems of identifying visualisers, and the variety of methods attempted, make it difficult to compare directly between studies and theories. Instead of discussing visual methods, the debate becomes one of trying to define who is included in each way of finding visualisers and the consequences of this for any conclusions. If the utility of the visualiser/verbaliser distinction is in doubt in relation to judging research and developing theories, this suggests it will be particularly unhelpful in actual classroom situations. Such is the argument of Klein (2003), backed up by his contention of the empirical failure of learning styles approaches to teaching. The implications for educational practice of these proposals and conclusions will be further considered below. First, though, the alternative conception of visualisation, as a range of methods and strategies present to varying extents in all the material and ideas involved in learning, needs to be examined. If the idea of being a visualiser is to be rejected in favour of an emphasis on actual instances of visualisation, wherever they are found, this conception will have to be illuminating.

6.22 Visualisation

It would seem that visualisation as a concept has the underlying recommendation that it is a much more logical necessity of the distinction between visual-spatial and verbal mental processes. In contrast the idea of visualisers needs this foundation and also assurance that people really do have stable cognitive tendencies. That judgement, clearly, is a matter of degree since there are individual differences in approach to any task and, unless strategies are utilised randomly, there must be some personal tendencies. However, this stability within the individual might be so slight as to make identifying different types unhelpful and this, furthermore, could cause other problems. Such a conclusion has been proposed for the visualiser-verbaliser concept, but it must be questioned whether the alternative understanding of visualisation as process results in a loss of explanatory power. To help establish this, the nature of visualisation processes and instances will be considered together with their relevance for teaching and learning mathematics, and in light of the findings of this and other research. If an understanding based on considering visualisation, and not visualisers, can both explain research finding and help to make recommendations for teaching practice, then it would seem to be worth pursuing.

The first aspect to note about this conception of visualisation is that it encompasses both physical representations, such as number lines, and mental images, such as number forms. This might seem worrying, given traditional arguments over the nature of mental imagery (see e.g. Anderson, 1978), but it avoids difficulties that result from trying to impose a distinction between mental and physical representation. Kaput (1998) warns against being too determined to separate ‘external’ from ‘internal’ representation and there are in fact many indications that this is a false dichotomy. Cultural understandings and established representational forms clearly affect mental images and processes, as when people who use left-to-right writing systems report left to right number forms (Seron et al, 1992), while those who write in Arabic appear to have a right to left SNARC effect (Dehaene et al, 1993). The direction of influence can be in the opposite direction, as Wheatley (1991) implies when he argues that sketches drawn in response to mathematics problems must depend on some sort of prior visual image.

It is next necessary to consider further the nature of the visualisation involved in mathematics. Although mathematics generally involves abstraction away from particular concrete examples towards a general principle, it has been argued (Chapter 1, section 1.31) that visual representations can assist with that abstraction. Examples have been collected of mental images fulfilling an abstract function, with some arguing that they are particularly suitable because they can be vague and vary in their precision and accuracy (e.g. see Gowers’, 2002, p.77-78,description of visualising in more than three dimensions). This idea of abstraction being linked to mental images that do not have clear, photographic attributes might suggest difficulties for physical representations. These have been considered in the present research (Chapter 1, sections 1.411 &1.42; Chapter 5, section 3.12) and the conclusion reached, in line with other writers (e.g. Arcavi, 2003), that physical visual representations, if thoughtfully constructed and used, can fulfil an abstract function in mathematics.

This is supported by the difficulties many of the research participants experienced in constructing their own representations, since these problems appeared to be with abstracting ideas and information. For instance, the difficulty of representing information on an empty number line (Chapter 5, section 5.112) seemed to occur because the children were struggling to see how such an abstract representation could be made to contain the real world relationships described in the written problems.

It could be advocated that this abstract sort of visualisation is in fact more spatial, than visual, in nature. It is tempting to identify ‘true’ visualisation with concrete images and particular pictures, while aligning the more abstract imagery with spatial concepts. Some writers do this explicitly. For example, Wheatley (1991) claims that “spatial ability is at the heart of meaningfulness” and only those able to think in abstract, spatial terms about a range of mathematical concepts have true, flexible understanding. The studies of Hegarty and Kozhevnikov (1999) and Kozhevnikov et al (2002) identified two sorts of visual image and the researchers link the more abstract images to participants’ spatial ability. However, this begins to suggest that abstract, spatial images somehow preclude more literal mental images, despite the fact that it seems possible for an individual to experience both. Furthermore for certain tasks, including some mathematical ones, it would seem beneficial to be able to make use of vivid visual images. This is reflected in the fact that many of the tests for visual-spatial ability require skill in manipulating visual images. The resulting on-going debate about how visual-spatial ability should be most accurately decomposed into distinct visual and spatial skills (Chapter 2, section 2.2) warns against attempting to link spatial functioning with abstraction. Just as visual-spatial ability appears to result from a complicated mixture of some more visual and some more spatial processes; it seems likely that some abstraction may be more spatial than others.

It would seem then that it must be accepted that visualisation can take many forms. As well as representations being both mental and physical, these representations can be more or less abstract and also more or less spatial. The nature of mathematics makes the abstract function particularly important, regardless of whether this is seen as spatial, and this will clearly have implications for the use of visualisation in teaching and learning. As Presmeg (1985; 1986) argues, the over-arching problem is one of avoiding the pitfalls of visualisation, many of which are linked to their concrete and particular functioning, so that the benefits of abstraction and generality can be realised.

First it would seem necessary to review the possible uses of visualisation in learning mathematics to confirm that this sort of thinking really does have a place in teaching the subject. Much of this discussion has been considered previously (Chapter 1, sections 1.31 & 1.411; Chapter 3, section 3.12) but it is now possible to relate previous conclusions to the teaching and learning that occurred during the research. It will be seen how these uses of visual representation succeed because of the nature of mathematical knowledge and because of the needs of human learners. As has been emphasised previously (Chapter 1, section 1.1), the conception of mathematics underlying this research means that these two aspects will be seen as intimately related.

It has been argued that visual representations are able to convey mathematically important information, and this was previously linked to the ultimate physical origins of mathematics (Chapter 1, section 1.31). Although the physical representations used in the visual lessons during this research did not appear to convey information perfectly, there were some notable successes. In both the pilot and main studies, the number line was useful in providing a visible representation of numbers, which allowed a physical sense of greater and smaller to be developed as well as suggesting a way of thinking about fractions and negative numbers. Although the number line might have disadvantages in that it conveys a particular conception of the numerical entities, this can be overcome by providing other representations on other occasions.

A particular advantage of the number line was that it made it possible for the teacher to refer to quite complicated concepts using relatively few words. This advantage was also found with other visual demonstrations, such as the use of coloured cubes to introduce the idea of ratio. The benefit of this can be explained in a number of slightly different ways. It could be argued that by avoiding an excess of words, the danger is lessened of there occurring a form of verbal over-shadowing that is common in education. This is where imperfectly understood words are used by the learner instead of constructing more full concepts, and these words can then become the totality of the child's understanding. Such misuse of words can be seen to underlie criticisms made by a number of educationalists, such as Skemp's (1976) complaints about 'instrumental understanding' and Davis' (1984) observations about "students who placed their full reliance on natural language statements" (p.202). It would be anticipated then that the children experiencing the visual intervention lessons would, as a result of avoiding this verbal over shadowing, hold more full understandings of the concepts covered. However, attempts to discover any changes in understanding through the interventions proved inconclusive.

An alternative conception of the advantages of using few words is that this makes efficient use of cognitive space. Instead of overloading the verbal processes, the use of a visual representation allows some information to be stored visually. As with verbal over-shadowing, this explanation is based on the idea of utilising a range of cognitive representations and processes to expand mathematical understanding. An individual example of the power of using visual processes is found in Gavin's visual proof for the Odds and Evens investigation. His initially visual understanding allowed him to conceive of the solution so that he could find words to describe his understanding. This seems to be making efficient use of a range of cognitive processes and it would be expected that his understanding of the relevant concepts, such as odd and even, would have improved. However, it proved difficult to be sure that this had happened and attempts to indicate the success of the visual teaching by looking for improved understanding throughout the class, and even in Gavin's individual case, were inconclusive.

However, neither was evidence found of the verbal lessons improving understanding. The superiority of the pupils who were taught verbally was limited on the MCT to the questions heavily reliant on words and did not extend to those requiring mathematical definitions. Therefore the visual presentation of mathematical concepts would appear, on the basis of the present research, to be no worse for student understanding than the verbal approach. Given this, it seems reasonable to include in the assessment the fact that introducing mathematical concepts visually with few words has advantages for the teacher. A visual representation gives the children something to look at and provides a focus, while freedom from producing lengthy verbal explanations gives the teacher more opportunity to concentrate on the detail of the introduction or explanation. This includes being able to be careful about the words that are used and being precise about their use so that verbal confusions are less likely.

A specific aspect of mathematics that it was anticipated that visual representations would be able to convey is the idea of mathematical entities as both process and concept. As has been discussed, previously (Section 6.113), there were many occasions during the research when the visual intervention activities seemed successful in linking these two aspects, through requiring a procedure that resulted in a diagrammatic concept. The children were all able to participate in these activities and in some cases they seemed to lead to a marked development in the underlying concept. For example, one child moved from shading squares to listing multiples to solve a ratio problem (Chapter 5, section 5.112). However, there were also occasions when the process did not seem to lead to conceptual understanding, as when another child refused to shade squares and instead reverted to an incorrect conception of ratio based on simple number manipulation (Chapter 5, section 5.112). In the latter case, the procedural activity had become an end in itself instead of being a means to the end of a more complete understanding. However, analogous problems were found during the verbal lessons where describing and naming sometimes took precedent over developing full concepts (Section 6.122; Chapter 5, section 5.113). Neither style of teaching seemed, then, entirely to avoid the problem of procedural elements of mathematics appearing overly important, presumably at the expense of conceptual understanding. Although the visual approach was not a perfect solution, this research produced suggestions that it could be beneficial in this respect, at least some of the time, and did not appear to lead to any more problems than did the verbal approach.

It has been argued previously that visual representations are able to support abstraction, but it must be questioned whether they actually fulfilled this function during the research. Repeatedly it was found that the children could appreciate the abstraction involved in a representation, if it was constructed for them, but struggled to construct their own. However, this observation need not be a criticism of visual representation, but is instead more a result of the difficulties inherent in abstraction. If it really does put such a strain on the mental processes of the learner, it would be expected that using visual processing would reduce this overload through making use of various ways of thinking. On occasions where individual pupils did manage to use the suggested representations, as when iconic symbolism was used for formulae and empty number lines for numerical relationships, these did seem to facilitate reasoning. The visual forms also provided evidence for the teacher that some abstraction was occurring and a shared representation to discuss with the learner.

However, it could be argued that the verbal approach was more successful in many of the mathematical areas mentioned above, since the children often needed less persuading to complete verbal activities. In particular, they seemed much more comfortable writing natural language rules for functions than the visually taught students were when using the iconic formulae. Yet this clearly involves a lower level of abstraction than using any sort of formulae, and, as on other occasions, it seems likely that rather than working more easily than the visual class at an abstract level, they were actually just not abstracting. As has been argued (Section 6.113), the use of language can cover this deficiency, whereas the requirement to produce diagrams makes it clear that the child can not abstract certain information from the context in which it is first presented. The most sensible use of the iconic formula would probably be to allow it to bridge the gap between the natural language descriptions that follow easily from the numbers and the much more abstract, and mathematically powerful, algebraic formulation of the relationship. In certain important ways it is a less abstract conception than true algebra; for example the association of multiplication with visible extent can be maintained ('▲▲▲' for '3 lots of ▲')

without violating principles of algebra (as writing ‘xxx’ would do). However it is clearly a beginning to abstracting away from particular given numbers to general cases and so constitutes a step in the right direction, mathematically. Yet if such ideas as this are to support abstraction, they need to be used by the learners and this is a concern, raised in this research and elsewhere, which will be further addressed below.

First it must be questioned whether the above discussion has provided a justification for advocating the use of visual methods in teaching mathematics. Furthermore, it has been argued that understanding visualising in terms of visualisers leads to problems and it is necessary to assess whether this alternative conception of visualisation is more useful. These two issues are linked in that it seems inevitable that visual representations will appear in some form in any mathematics classroom because of the way that human beings process information. For a striking example, consider the child in the present study who, despite being in the verbal class, began a visual explanation for the Odds and Evens investigation (Chapter 5, section 5.114). Yet the sum total of the experience of this research is that understanding these visualisation episodes is not facilitated by the classification of visualisers and verbalisers. Aside from the technical difficulties experienced in identifying them reliably, the classroom experience suggests the futility of this attempt. In particular, the one child who constructed a helpful visual proof for the odds and evens investigation was not identified as a visualiser by the recognition test. Nor was he always helped by visual representation and activities.

Therefore it can be concluded that visualisation will be used in the mathematics classroom, and has the potential to be helpful for some learners some of the time, so thought should be given to how visual methods can be best used. There is reason to believe that, in general, people are not competent at making use of visual imagery (Antonietti, 1999) and this perhaps extends to other sorts of visualisation. Certainly the negative opinions about visual methods expressed by some educators (e.g. Anghileri, 1999; the teachers interviewed by Moseley, 2003) imply that they do not appreciate the possibilities for visualisation and mathematics. All this means that it is important to consider carefully how and when to use visual methods.

It has been noted elsewhere that children might be reluctant to use physical representations to assist with basic arithmetic. Gray and Pitta (1996) draw attention to the desire that children have to calculate mentally, without physical aids, because that is what they observe others doing. Munn (1998) points out that in Western society the classroom counting aids are not used by adults, so are often considered to be 'babyish' or only for incompetents. At secondary school level, it has been observed (Noss et al, 1997) that diagrams are perceived as ritualistic additions to a task, rather than as a possible means to solve a problem. During this research, occasions were observed when visual representations were either rejected as childish or were treated as an unnecessary diversion and simply ignored. The empty number line approach to representing a problem, and so choosing the correct calculation, was explicitly rejected as a primary school method by some pupils in both the pilot and main study classes and, despite the drawing of dot patterns in earlier lessons, most of the children tackling the Odds and Evens investigation did not attempt diagrams.

It seems likely that there is no simple solution to this problem of the perceived legitimacy of visual methods. Various enthusiasts for a more visual approach to mathematics teaching have urged teachers to be clear about valuing visualisation and to provide visual examples (Arcavi, 2003; Clements & Battista, 1991). However, the experience of this research shows that this is not an immediate solution, since the visual lessons explicitly encouraged and demonstrated visual methods, yet these were frequently avoided by the learners. A further hindrance to the use of visualisation, suggested by the current research, is perhaps that visual representations are actually quite difficult to construct. This could be seen as resulting from the necessity of translating information from a verbal or numerical form into a visual mode, since any sort of transforming of knowledge is known to be more challenging than simple repetition. Alternatively, or perhaps additionally, it has been observed that many apparently straightforward visual representations involve a significant amount of abstraction. Perhaps, when it is suggested that learners be encouraged to construct visual representations, it should be more openly acknowledged that this will prove difficult and challenging, causing some of them to avoid any attempt.

A further concern when using visual methods is with how visual and verbal ideas are balanced and, possibly, integrated. On the one hand, it seems beneficial to be able to use a multiplicity of approaches and it is often stated that the essence of mathematics lies in making links and translating from one form to another (e.g. Sierpinska & Lerman, 1996; see Chapter 1, section 1.23). Contrary to this, though, are observations about 'verbal over-shadowing' on certain tasks and the concern that a form of verbal over-shadowing could occur in mathematics learning if a child relies too heavily on natural language descriptions of mathematical entities. During this research this problem was taken seriously and tackled by the visual teaching trying to avoid using words. Although this sometimes seemed successful, as when the introduction to ratio appeared to convey the relevant concepts more clearly and precisely through using few words, on other occasions trying to avoid verbalising ideas was less helpful. For example, the visual teaching of function rules was quite difficult and, as has been suggested above, would probably have been facilitated by initially using natural language descriptions of the functions, before proceeding to iconic symbolism. This is in fact the way that Davis (1972) advocates using iconic formulae.

Using such approaches, and in general not attempting to avoid any verbal descriptions, is likely to be the way in which many teachers will proceed and this does mean that verbal over-shadowing could occur. Whether this is likely to be a problem requires consideration of the evidence for verbal over-shadowing. Research evidence clearly demonstrates that remembering essentially non-verbal information, such as the exact appearance of a potentially ambiguous drawing (Brandimonte & Gerbino, 1996), is facilitated by avoiding translating the information, inexactly, into words. However, it must be questioned whether much of the content of mathematics is of this form, since the concepts involved are generally broader, concerning more than particular appearances. More worrying for mathematics teaching is the suggestion (Schooler et al, 1993) that describing problem solving interferes with insights. Although this idea is more controversial, it has been noted that during this research there were occasions in the verbal lessons when describing a problem seemed to distract the learner from seeking a true explanation. Therefore it would seem that while true verbal over shadowing is unlikely to be much of a problem when learning mathematics, continually attempting to find words for observations might prove distracting or interfere with developing understanding. This implies that while, on occasions, linking visual and verbal ideas might be helpful, and certainly involves an efficient use of cognitive space, verbal translations should not be considered obligatory as this is sometimes likely to be distracting or to over-whelm the visual representation.

6.23 Summary

The concept of visual and verbal styles of thought is underpinned by evidence, from psychological and neurological investigation, of distinct visual-spatial and verbal cognitive processes. However, it has been argued that for the idea of visualisers and verbalisers to be a valid construct there must be consistency to the thought processes and strategies that individuals use. Furthermore, it would seem that for the construct to be useful, theoretically and, especially, practically, the level of this consistency must be fairly high. Doubts have been expressed about some reports of apparently consistent individual assessment, while the experience and findings of the present study supports the contention that such identification of visualisers and verbalisers is not reasonable. On any occasion there are many reasons for choosing, or not choosing, a visual strategy while there appear to be a range of, often very different, visual-spatial processes. This leads to any assessment of visual tendency producing a very varied group of individuals and slightly different subsets of the population being identified by different measures. Such reasoning also explains the doubts that are sometimes expressed about particular viaulaiser-verbaliser scales (see e.g. Peterson et al, 2003, for criticism of Riding's scale), since if the visualiser-verbaliser continuum is an unwarranted construction then scales can not be expected to measure it accurately or reliably.

However, if, as this suggests, the idea of visualisers should be rejected in favour of a focus on the process of visualisation, this conceptualisation needs to be useful and to have explanatory value. Such a conclusion is supported by considering the results of the present study through the concept of visualisation. In total, this study appears to support the conclusion recently reached by Richardson (1999) that "the distinction between imaginal and verbal coding appears to relate more to optional strategies that could be used within the same subject than to cognitive styles that distinguish between different subjects" (p.112).

6.3 Implications for the classroom

Although this research project did not produce many of the anticipated results and, in particular, failed to find a way of helping children identified as visualisers to be mathematically successful, there are implications to be drawn from it for mathematics teaching. These are evident from the discussion so far, but will now be explicitly considered.

6.31 Teaching and learning styles

Many teachers are interested in the ideas of learning styles (Klein, 2003) but this research strongly suggests that attempting to categorise learners as visualisers and verbalisers, then teach them in their preferred style, is unlikely to be beneficial. Although the experimental interventions did not last for a very long period of time, the study did allow a more complete concentration on particular teaching styles than would be possible in a standard classroom. Yet still no interactions were found between teaching and leaning styles. Various explanations for this finding have been considered, which all involve the idea that the classroom interactions of teacher, teaching material and learner are too complex to be reduced to the simple effect of visual representations helping a visualiser, while verbal representations assist a verbaliser.

Furthermore, it has been argued that classifying learners as visualisers or verbalisers is not useful and could even be harmful. It is difficult to make the rather crude dichotomy representative of human variation and where this is attempted, perhaps by classifying visual images along a continuum of abstraction (Presmeg, 1985; 1992b), this further confuses the idea of a ‘visualiser’ and makes comparisons across the literature difficult. In the present study, problems with the visualiser/verbaliser distinction were suggested by the difficulties experienced in identifying visualisers accurately using the visual/verbal ratio and by the interviews with Year 10 visualisers and verbalisers, which found similarities between the two sets of students as well as differences within them. During the main study, pupils did not always appear to learn in the ways expected of them, given their visual/verbal ratios. Most dramatically, the only pupil who clearly benefited from the visual approach to the odds and evens investigation was not assessed as a visualiser using the visual/verbal ratio and on other occasions did not respond to visual methods. If a visual approach had been reserved for ‘visualisers’, identified either through the visual/verbal ratio or by assessing the pupils visual approach to mathematics, it is likely that this child would have missed out on the opportunity to approach odds and even numbers in the visual manner that proved so fruitful. As Klein argues, educators should to moving “from categorising students to teaching them how to use representations for thinking and learning”.

However, in addition to suggesting the futility of classifying learners according to cognitive style and trying to teach them accordingly, this research points to the importance of awareness of the style used in teaching and assessment. Although there was no interaction between teaching and learning styles, there did seem to be a limited interaction between teaching and assessment styles. This is seen in the finding that the verbal teaching significantly improved performance on the MCT items which depend most heavily on literacy skills. This seems to have occurred through familiarity with words used in a mathematical context, and perhaps because of confidence gained in using words, rather than any improvement in mathematical understanding, since the superiority of the verbal class did not extend to MCT items that needed mathematical definitions or concepts. The visual teaching did not have an analogous effect on performance on MCT items that used diagrams, which further implies the narrow and limited effect of the style of teaching. However, the discovery of any effect of teaching approach on pupil performance on particular styles of test item has implications for the classroom, where test performance is considered important. Depending on the extent to which particular tests rely on verbal skills, it might be worth a teacher emphasising vocabulary and verbal description in the teaching before the test. However, it cannot be known from the results of the present study whether this would be useful to children whose ages and levels of achievement differ from those of this study's participants. Furthermore, there is no certainty over how much verbal teaching is beneficial and there might be a tendency for this emphasis to distract from other valuable aspects of mathematics. Although such a disadvantage was not reflected here in MCT performance, during the verbal intervention lessons it was sometimes felt that the verbal emphasis was interfering with learning.

6.32 Visualisation

Given the finding that verbal teaching led to more MCT gain than visual teaching, it might be thought that this research does not have anything to say in support if visual approaches in the classroom. However, the limited nature of the success of the verbal approach, together with the experience of using visual methods during the intervention, does suggest that visualisation should not be ignored. A fundamental reason to embrace visual methods and make careful use of visual representation in the classroom is that visualisation will happen anyway. This was argued to be based on the processes humans use to think as well as on the nature and history of mathematics, and instances occurred during the interventions when learners produced their own visual mathematics. Since visual representations can hinder thought, as well as helping it, it seems important to make careful use of valuable representations, rather than just leaving children to manage with their own, often inadequate, visualisations. This point has been made frequently by other researchers, especially in relation to mental images (Hegarty and Kozhevnikov,1999; Campbell et al, 1995).

Although, as a whole, the visual lessons did not succeed in the stated aim of improving MCT performance, particular lessons and activities did appear successful if this is measured by considering the reactions of the pupils and the work they produced. These instances of apparent success provide another reason not to dismiss visual methods and it has been argued that there appear to be certain points in learning when a visual approach can be beneficial. An important use of visual representations is as a stepping stone on the way to abstraction, since they allow a movement away from paricular real-world situations but still provide a representation for the learner to observe and consider, which bears some relation to the original information. An example of this, that has been discussed, is the way that number lines are clearly an abstract representation of number, but they preserve and emphasise certain features of the number system in a readily perceived form. An advantage of this use of visualisation to develop abstraction, from the teacher's point of view, is that it can allow the teacher to see if the learner is abstracting information. During the research, the finding that some learners struggled to draw empty number line representations of given problems was taken as an indication that the children were having difficulties abstracting the numerical relationships. Similar difficulties, during the verbal lessons, in choosing the correct calculation suggested that similar problems were occurring in this class, but here the teacher did not have any visual indications of the nature of the problem.

This sort of consideration can also be seen a special case of learners transforming information. Such transformations appear useful in learning whether because they involve a sort of ‘self-explanation’ (Chi et al, 1994), or are themselves an essential component of understanding (e.g.Pyke, 2003). Clearly many visual representations in mathematics will involve transforming knowledge, so they should be useful. On many occasions this will involve transforming information between visual and verbal forms, which has been considered in light of concerns about verbal over-shadowing. It has been argued that this problem is likely to be limited to a tendency for verbal description to distract and seem too important, so, with a little care, visual and verbal methods and approaches should be beneficial when used together.

6.33 Verbalisation

In debate about the appropriate place of visualisation in teaching and learning, it is often forgotten that verbal approaches also have their advantages and disadvantages. This is evident in the present study from the result that although verbal teaching succeeded in some respects in improving mathematics performance, that improvement was limited. Although a verbal style of teaching might be sufficient to improve performance on literacy-dependent test items, it did not appear generally to improve mathematical understanding and this was despite the emphasis on explanation and the encouragement that was given to go beyond mere description.

During this project, there were instances when the verbal approach seemed to support and extend mathematical understanding, which have been mentioned above (Section 6.112). However, there were other occasions when a verbal approach produced clear disadvantages and problems for the learner. Many of these were examples of the perceived need for words and description distracting the learner from the task of properly constructing understanding, but on other occasions the difficulty seemed to result from the narrowness of the verbal conception of a mathematical entity (Chapter 5, section 5.33). For example, during the pilot study interviews, a child who tended to use counting strategies, and did not report visual images, was observed to be misled by spurious number patterns. This pupil gave the answer "47" to the question "What time is it 45 minutes after 2pm?" and explained that this was because you "add the 2 here on to the 45". It would seem that even a mathematically poor, pictorial image of a particular clock face would have helped her to avoid this mistake.

Therefore it would seem important for the mathematics teacher to be aware of the disadvantages, as well as the advantages, of verbal methods and try to promote a balance in approaches. Related to this observation, and to the concerns about the visualiser style categorisation, it would seem appropriate for the teacher to think about verbal approaches and instances of verbal processing instead of categorising a learner as a verbaliser. Not only does such categorising probably involve unwarranted assumptions about cognitive styles, but it risks trapping a learner in an approach that is as narrow and unbalanced as that which results from the equivalent over use of visual methods.

6.34 Understanding

Underlying this research project has been the aim of improving understanding, with visual representations and methods seen as an alternative route to this goal. Although full time teachers have other short-term objectives, it would seem that this aim ultimately provides the foundation for their actions. With this in mind, many educationalists have voiced opinions and beliefs about understanding but it seems easier to describe examples of lack of understanding than to say what is intended by 'understanding'. For example, Skemp (1976) criticises the limited knowledge associated with 'instrumental understanding', while Holt (1982) describes a child with "a headful of scrambled facts and recipes" (p.195).

However, what examples of lack of understanding generally have in common is the observation that facts are stored as disconnected instances, without connections being made and over-arching ideas developed. The successes of some teaching methods in enhancing understanding are explained by their effect of inducing learners to engage in the sorts of mental activity that lead to constructing and connecting knowledge. For example, reciprocal teaching (e.g. Palinscar & Brown, 1989) provides training in cognitive strategies such as questioning and clarifying, while Chi et al (1994) argue that eliciting self-explanations can improve understanding.

These activities also transform knowledge and this is considered to be a key element in any understanding. Furthermore, it has been argued that transforming and translating information are essential features of mathematics (Chapter 1, section 1.23). Therefore encouraging both visual and verbal thinking in the mathematics classroom should be beneficial, since such encouragement will induce constructive cognitive activities while practicing a fundamental mathematical skill. However, it should be recognised that understanding something in a number of different ways will tend to be hard and learners may be reluctant to make the effort. Sfard (1991; 1994) argues that even mathematicians sometimes have difficulty in achieving both operational and structural understanding. Piaget's (1952) investigations into the origins of individual mathematical understanding suggest the difficulty inherent in trying to think of items as both wholes and as parts of another whole. Perhaps this struggle is repeated every time a human tries to grasp an idea in more than one way.

Sometimes learners might find it hard to appreciate why they need to attempt a multiplicity of understandings and teachers trying to provide a range of methods might simply confuse their students. Presmeg (1985) reports this criticism by the students of some of the teachers she studied who used visual methods, but, because they tended to offer a number of alternative solutions to mathematical problems, sometimes provoked confusion. Similarly, during the present research, there was sometimes reluctance on the part of the pupil to approach some mathematics in more than one way. Just producing more than one outcome for a specific problem seemed difficult for many children and this was a particular difficulty with some of the visual activities, such as the dot pattern method of finding factors, which relied on a number of diagrams for each item. However, it has been argued that an equivalent problem in the verbal class was in persuading the children to produce explanations once they had managed a description. In both cases going beyond the immediate, disconnected 'answer to the question' was hard to encourage.

Therefore, although a reasonable teaching aim would be to encourage in learners a range of methods and ideas, some visual and some verbal, with transforming and over-arching understanding to link them, this might prove hard to achieve. The present research shows that neither visual nor verbal methods offer a perfect route to understanding, even for certain individuals, but there is the suggestion that sensitively used and integrated they should be beneficial.

6.35 Summary

Although this study suggests that some benefit might be gained from using a verbal style of teaching, the limitations of such a teaching approach are also conveyed by the results. The verbal teaching only improved performance on mathematics test items that relied heavily on literacy skills and did not appear to have a more general affect on mathematical understanding or competence. Furthermore, it is not known whether this effect of verbal teaching would generalise to children of other ages with differing levels of mathematical achievement.

The total experience of this research has revealed both the problems and the benefits, felt at a classroom level, associated with adopting a verbal teaching style. Similarly, the particular characteristics of visual approaches and methods have been examined and the conclusion reached that both types of teaching are useful.

However, the impression many teachers have of the importance of matching teaching and learning styles was not supported by this study. The quantitative results do not reveal interactions between teaching and learning styles, while the study taken as a whole conveys the difficulty of assigning a cognitive style to an individual. It has been argued that considering previous research together with this study leads to the conclusion that the visualiser-verbaliser construct is not useful and may not be valid. It would appear more sensible for a teacher to understand students' learning in terms of instances of differing processes and attempt to develop this range, rather than limit expectations and opportunities by trying to attach cognitive style labels to individuals. Flexible, transformational thinking is argued by many to be the essence of understanding, and may have particular relevance to mathematics, but those pursuing this goal should realise that sometimes it may be hard to achieve.

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Appendices

Appendix A

Sheet of items to remember for recognition test.

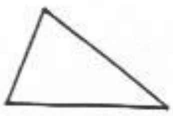
square

clock

circle

quarter

star

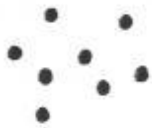


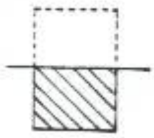
moon

reflection

door

tree





line

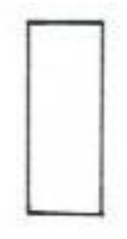
Appendix B

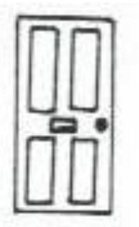
Complete set of 60 recognition test items

These follow in the order presented to the participants.

reflection

triangle





door

flag

rectangle

nine

square

envelope

spots

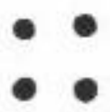
tree

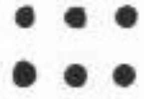
boat

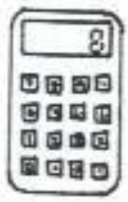
moon

star

pencil

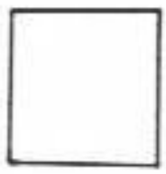






six

line



right-angle

balloon

house

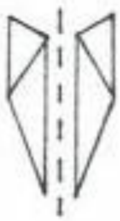
circle

angle

book

clock

calculator



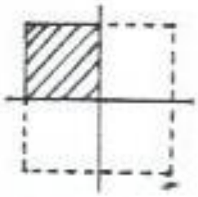
three

half

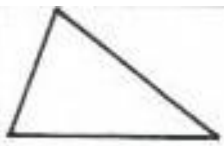
four

five

quarter



fish



Appendix C

Reading test

This was given to pilot study participants to check they could read the words on the item list. For each question, the word from the item list was read out and the children were required to circle this word.

1) squash	square	quarry	squeak
2) rock	close	clock	climb
3) icicle	circuit	cycle	circle
4) quantity	equal	quarter	question
5) star	stall	tart	start
6) soon	mood	pool	moon
7) reflection	action	election	retain
8) floor	doll	door	food
9) trim	tree	see	trot
10) lie	line	fine	link

Appendix D

Tests of mathematics achievement used during the pilot study.

The NFER Yardsticks (Milward, 1981) scheme of classroom tests was used to provide matched items for two tests used before and after the teaching. The tests each consist of 45 items, with 15 items from each of three NFER levels. Within these thirds, five questions cover the four rules of arithmetic, five cover fractions, decimals and percentages and five cover number properties. This content was chosen to reflect the content of the lessons taught.

Test 1

1. What is the missing numeral?

58, 59, 60, ___

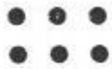
   A. 57 B. 61 C. 65 D. 70

2. Which is the smallest of these numbers?

   {50, 81, 35, 29}

   A. 50 B. 81 C. 35 D. 29

3. What is the answer to 2 × 3?



   A. 4 B. 6 C. 5 D. 7

4. Which numeral shows *one hundred and ninety*?

   A. 190 B. 109 C. 119 D. 901

5. How could you write this amount?

   38

   A. 38 tens
   B. 8 tens + 3 units
   C. 3 units + 8 units
   D. 3 tens + 8 units

6. What is the product?

   $2 \times 1 = \square$

   A. 3 B. 4 C. 1 D. 2

7. What is the difference?

   $14 - 8 = \square$

   A. 5 B. 6 C. 7 D. 8

8. Which numeral goes in the □?

   $8 + 9 = \square$

   A. 16 B. 17 C. 18 D. 19

9. Susan had 9 books. Her father gave her 6 more books. How many books did Susan have then?

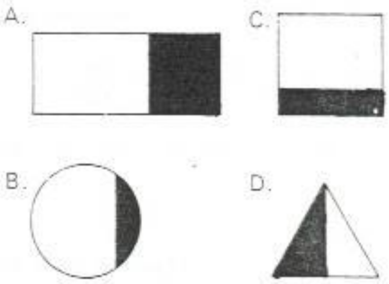
   A. 16 B. 15 C. 14 D. 3

10. What is the sum?

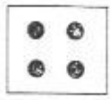
$$\begin{array}{r} 23 \\ +\ 34 \\ \hline \end{array}$$

   A. 58 B. 57 C. 47 D. 67

11. Which region is half-shaded?



12. What is half of the set?

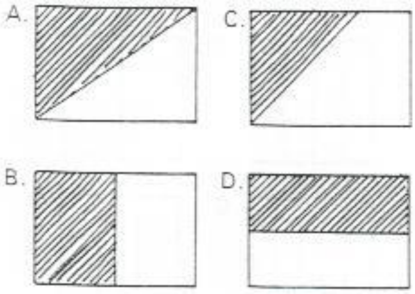


   A. 1 B. 2 C. 3 D. 4

13. What numeral names half of the set?

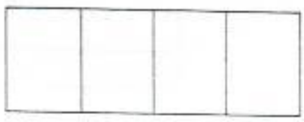
   A. 6 B. 5 C. 4 D. 3

14. Which is *not* half-shaded?

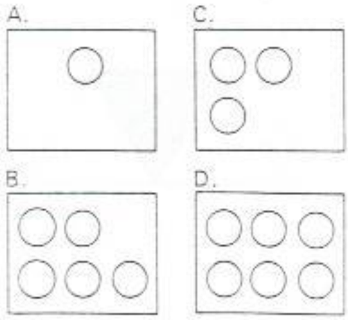


15. How many parts should you shade to show one half?

A. 1
B. 2
C. 3
D. 4



16. Which set has an even number of objects?



17. Which odd number comes just before 17?

A. 16 B. 15 C. 14 D. 13

18. Which number is a multiple of 3?

A. 4 B. 6 C. 7 D. 10

19. Which number is a multiple both of 6 and of 8?

A. 12 B. 8 C. 24 D. 36

20. 28 is the product of which two factors?

A. 4×5 C. 4×7
B. 4×6 D. 4×8

21. What is the difference?

$$\begin{array}{r} 52 \\ -42 \\ \hline \end{array}$$

A. 10 B. 12 C. 14 D. 94

22. What is the sum?

$$\begin{array}{r} 46 \\ +39 \\ \hline \end{array}$$

A. 75 B. 85 C. 86 D. 87

23. What is the product?

$4 \times 6 = \square$

A. 20 B. 24 C. 10 D. 28

24. What is the missing numeral?

$27 \div 3 = \square$

A. 6 B. 7 C. 8 D. 9

25. One box will hold 8 bottles. How many bottles will 4 of these boxes hold?

A. 36 B. 28 C. 32 D. 12

26. Which shaded part shows *one fifth*?

A.

C.

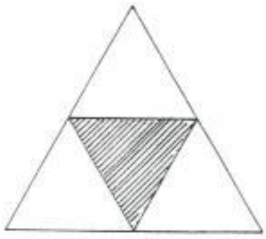
B.

D.


27. Which fraction names the shaded part?

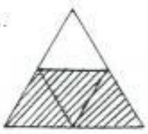
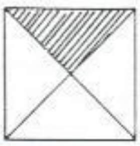
A. $\frac{1}{4}$ B. $\frac{1}{3}$ C. $\frac{1}{2}$ D. $\frac{1}{5}$

28. What part of this region is shaded?



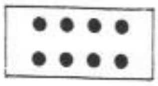
A. $\frac{3}{4}$ B. $\frac{1}{4}$ C. $\frac{1}{3}$ D. $\frac{1}{2}$

29. Which region is one quarter shaded?

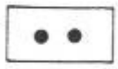
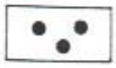
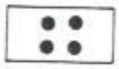
A.

C.

B.

D.


30. Which set contains one quarter of the objects in this set?



A.

C.

B.

D.


31. What number names a multiple of 9 ?

A. 51 B. 45 C. 3 D. 21

32. Which number below is a factor of 8 ?

A. 2 B. 3 C. 5 D. 6

33. Which number is *not* a factor of 12 ?

A. 1 B. 2 C. 3 D. 5

34. Which set names *all* the factors of 12 ?

A. {1, 3, 4, 8, 12} C. {1, 2, 3, 4, 8, 12}
B. {1, 2, 3, 4, 6, 12} D. {1, 2, 3, 4, 6}

35. Which number below is a multiple both of 1 and of 17?

A. 1 B. 7 C. 17 D. 21

36. What is the sum?

$$\begin{array}{r} 768 \\ +217 \\ \hline \end{array}$$

A. 974 B. 975 C. 984 D. 985

37. Which is the correct answer?

$740 - 217 = \square$

A. 423 B. 433 C. 523 D. 533

38. What is the product?

100×37

A. 1,370 C. 7,310
B. 10,037 D. 3,700

39. Which numeral completes the sentence?

$56 \div 8 = \square$

A. 7 B. 12 C. 8 D. 9

40.
Mary has 48 stamps that she wants to put into sets of 6. How many complete sets can she make?

A. 8 B. 42 C. 54 D. 288

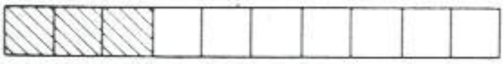
41.
Which fraction below is *greater than* $\frac{1}{4}$?

A. $\frac{1}{5}$ B. $\frac{1}{3}$ C. $\frac{1}{6}$ D. $\frac{1}{7}$

42.
What fraction of the number line is the darkened part?

A. $\frac{4}{3}$ B. $\frac{3}{4}$ C. $\frac{3}{7}$ D. $\frac{1}{2}$

43.
What decimal fraction of the diagram has been shaded?



A. 0·1 B. 0·3 C. 0·5 D. 0·7

44.
What is another way of writing the decimal 0·5?

A. $\frac{5}{100}$ B. $\frac{5}{10}$ C. $\frac{5}{1}$ D. $\frac{0}{5}$

45.
What is another way of writing $12\frac{4}{10}$?

A. 12·4 C. 1·24
B. 12·04 D. 12·410

Test 2

1. What is the missing numeral?

	80	81	82	83	84

A. **81** B. **79** C. **78** D. **70**

2. Which is the largest of these numbers?

{53, 27, 70, 65}

A. **53** B. **27** C. **70** D. **65**

3. What number is one more than 29 ?

A. **30** B. **31** C. **28** D. **20**

4. Which numeral shows *two hundred and fifty-five*?

A. **552** B. **250** C. **255** D. **525**

5. How could you write this amount?

$\boxed{91}$

A. **11 tens + 9 units**
B. **9 tens + 1 unit**
C. **91 tens**
D. **90 tens + 1 unit**

6. What is the product?

$8 \times 2 = \square$

A. **14** B. **10** C. **16** D. **18**

7. What is the difference?

$13 - 5 = \square$

A. **6** B. **7** C. **8** D. **9**

8. Which numeral goes in the $\square$?

$10 + 8 = \square$

A. **19** B. **17** C. **16** D. **18**

9. Tony had 17 toy cars. He gave 4 away to Jackie. How many cars did Tony have left?

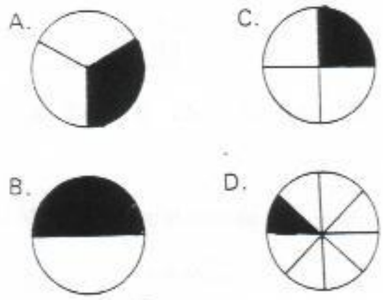
A. **16** B. **13** C. **21** D. **14**

10. Which numeral goes in the $\square$?

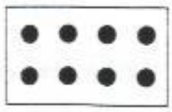
$$\begin{array}{r} 72 \\ +\underline{22} \\ \square \end{array}$$

A. **50** B. **54** C. **90** D. **94**

11. Which region is half-shaded?

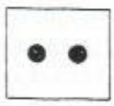


12. What is half of the set?



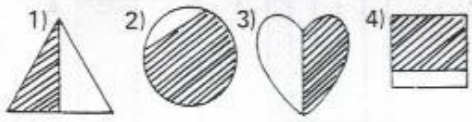
A. **3** B. **4** C. **5** D. **8**

13. What numeral names half of the set?



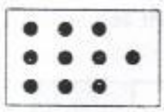
A. **0** B. **1** C. **2** D. **4**

14. Which objects are half-shaded?



A. 1 and 2
B. 2 and 4
C. 1 and 3
D. 3 and 4

15. What is half of the set?

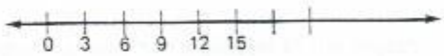


A. 3 B. 4 C. 5 D. 10

16. Which of these is an odd number?

A. 3 B. 2 C. 0 D. 8

17. Which numbers are the next two multiples of 3?



A. 18, 21
B. 15, 18
C. 18, 24
D. 15, 21

18. Which number is a multiple of 4?

A. 6 B. 9 C. 10 D. 12

19. Which number is a multiple of 4 *and* a multiple of 6?

A. 2 B. 18 C. 6 D. 12

20. 30 is the product of which two factors?

A. 5×7
B. 5×8
C. 5×6
D. 5×4

21. Which numeral goes in the □?

$$\begin{array}{r} 88 \\ -45 \\ \hline \end{array}$$

A. 33 B. 34 C. 43 D. 44

22. What is the sum?

$$\begin{array}{r} 18 \\ +17 \\ \hline \end{array}$$

A. 21 B. 25 C. 34 D. 35

23. Which numeral names the product?

$5 \times 4 = \square$

A. 24 B. 20 C. 16 D. 15

24. What numeral goes in the □?

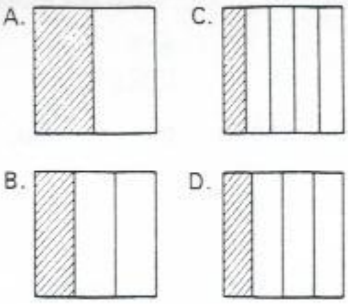
$9 \div 3 = \square$

A. 2 B. 4 C. 3 D. 5

25. Mrs. Knight shared 20 cakes equally among 5 children at a party. How many cakes did each child get?

A. 2 B. 4 C. 5 D. 25

26. Which shaded part shows *one quarter*?



27. Which fraction names the shaded part?

A. $\frac{1}{4}$ B. $\frac{1}{2}$ C. $\frac{1}{3}$ D. $\frac{1}{5}$

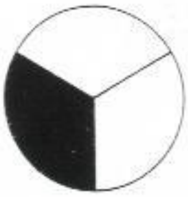
28. Which numeral names the shaded part?

A. $\frac{1}{2}$

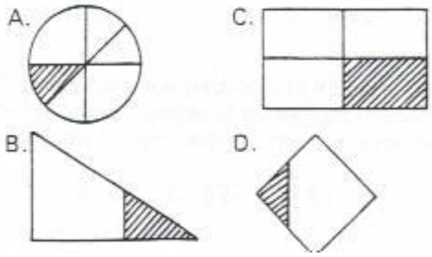
B. $\frac{1}{3}$

C. $\frac{1}{4}$

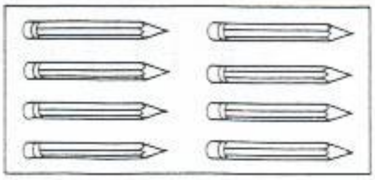
D. $\frac{1}{5}$



29. Which shape has a quarter of the region shaded?



30. How many objects are in one quarter of this set?



A. 4 B. 1 C. 2 D. 3

31. Which number below is a multiple of 7 ?

A. 35 B. 30 C. 25 D. 20

32. Which number below is a factor of 15 ?

A. 30 B. 3 C. 9 D. 10

33. Which number is *not* a factor of 16 ?

A. 6 B. 4 C. 2 D. 1

34. Which set names *all* the factors of 15 ?

A. {1, 3, 5, 10, 15}
B. {3, 5}
C. {1, 3, 5, 15}
D. {1, 3, 5}

35. Which number is a multiple of 2 and of 3 ?

A. 2 B. 3 C. 4 D. 6

36. What is the sum?

$$\begin{array}{r} 119 \\ +230 \\ \hline \end{array}$$

A. **329** B. **339** C. **349** D. **359**

37. What is the difference?

$208 - 172 = \square$

A. **176** B. **36** C. **136** D. **180**

38. What is the missing numeral?

$60 \times 100 = \square$

A. **160** C. **6,000**
B. **6,100** D. **60,100**

39. What is the result?

$54 \div 9$

A. **4** B. **6** C. **5** D. **7**

40. 72 toffee bars are packed into 9 boxes. If there are the same number of toffee bars in each box, how many toffee bars are there in each box?

A. **7** B. **8** C. **63** D. **81**

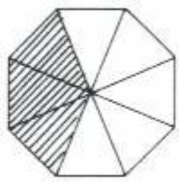
41. Which fraction below is *greater than* $\frac{1}{5}$?

A. $\frac{1}{6}$ B. $\frac{1}{9}$ C. $\frac{1}{4}$ D. $\frac{1}{7}$

42. What part of this region is shaded?

A. $\frac{3}{8}$ C. $\frac{5}{8}$

B. $\frac{3}{5}$ D. $\frac{1}{2}$



43. What fraction of the small squares have been filled in?

A. **0·3** B. **0·6** C. **0·2** D. **0·7**

44. How could you write 7·3 in another way?

A. $3\frac{7}{10}$ B. $\frac{73}{100}$ C. $7\frac{3}{10}$ D. $7\frac{3}{100}$

45. What is another way of writing $4\frac{2}{10}$?

A. **0·42** B. **4·02** C. **4·2** D. **40·2**

Appendix E

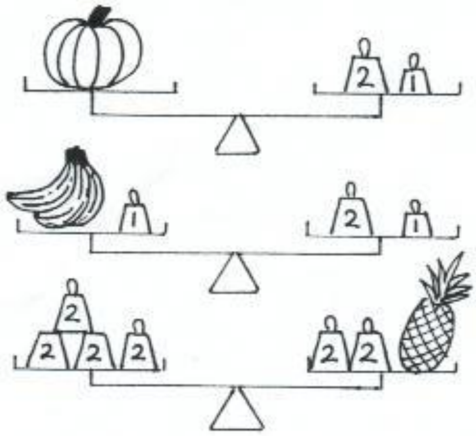
Strategy choice questions used with Year 7 participants

The questions were chosen to be either appropriate for visual methods, verbal methods or no particular method.

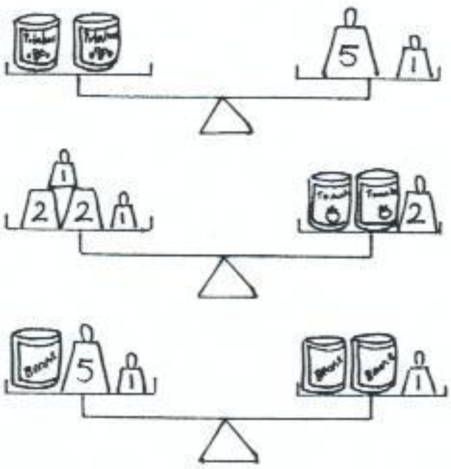
Visual questions

A

Each of these pictures shows a pair of scales. All of them balance. All the weights are in kilograms. Write down the weights of the fruits.



Now find the weights of these tins.



B

What time is it 4 hours before 1pm?

C

A pair of twins on one side of a see-saw are balanced by an adult weighing 12 stone on the other side.
How much does each twin weigh?

L

My recipe uses 24 lemons for 6 people. How many lemons do I need for 2 people?

Verbal questions

H

What time is it 45 minutes after 2pm?

I

I'm thinking of a number. If I double it, I get 6.
What number am I thinking of?

J

I'm thinking of a number. If I add 6 to it, the answer is the same as if I double it and add 1.
What number am I thinking of?

K

My recipe uses 30g of oatmeal for 6 people. How much oatmeal do I need for 2 people?

No particular method questions

D Find the missing number:

$? \times 7 = 322$

E Find the missing number:

$13 \times ? = 143$

F Find the missing number:

$133 \div ? = 19$

G Find the missing number:

$? \div 6 = 129$

M Last night the temperature was 14°C colder than during the day. It was 9°C during the day.
How cold was it last night?

Appendix F

Additional questions for Year 10 participants

These questions were among those used by Presmeg (1985). The first question was originally used by Krutetskii (1976).

N

A train passenger who had travelled half his journey fell asleep. When he woke up, he still had to travel half the distance that he had travelled while sleeping. For what part of the whole journey had he been asleep?

P

A straight path is divided into two unequal sections. The length of the second section is half the length of the first section.
What fraction of the whole path is the first section?

Q

Only four teams took part in a football competition. Each team played against each of the other teams once. How many matches were there in the competition?

S

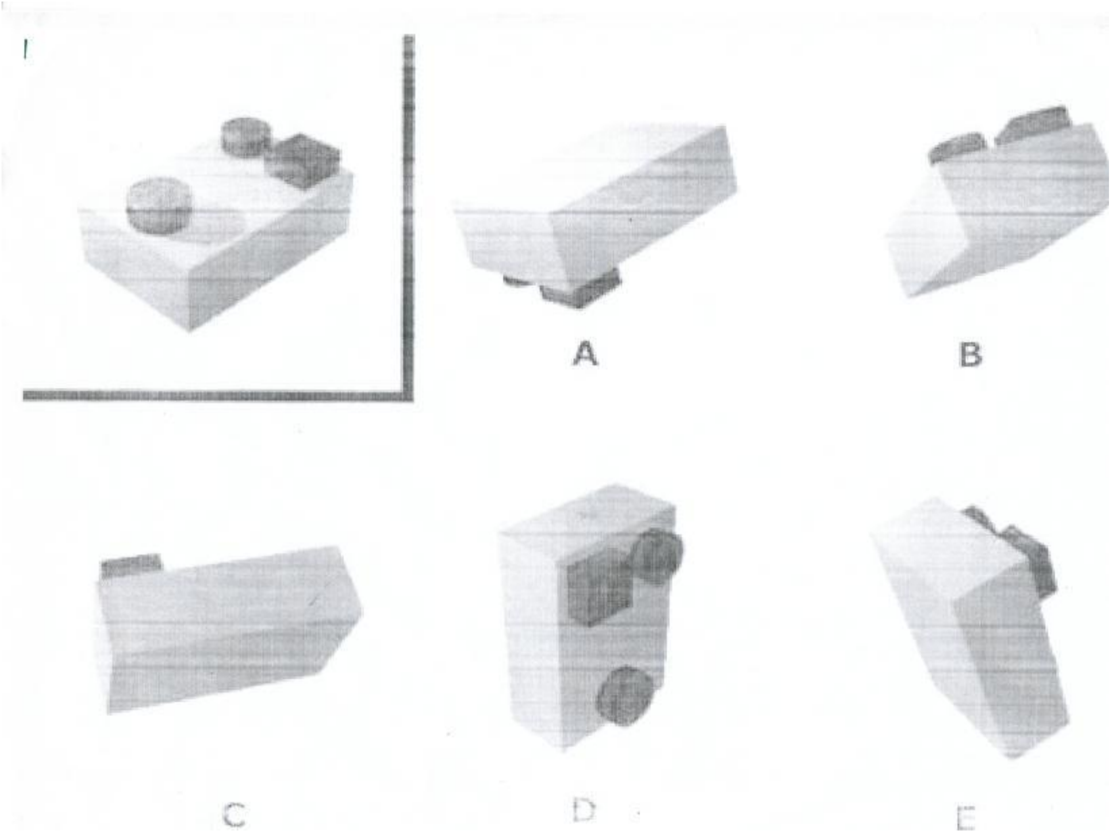
A man planted a tree at each of the two ends of a straight, 25 metres long path. He then planted a tree every 5 metres along the path (along one side only). How many trees were planted along the path altogether?

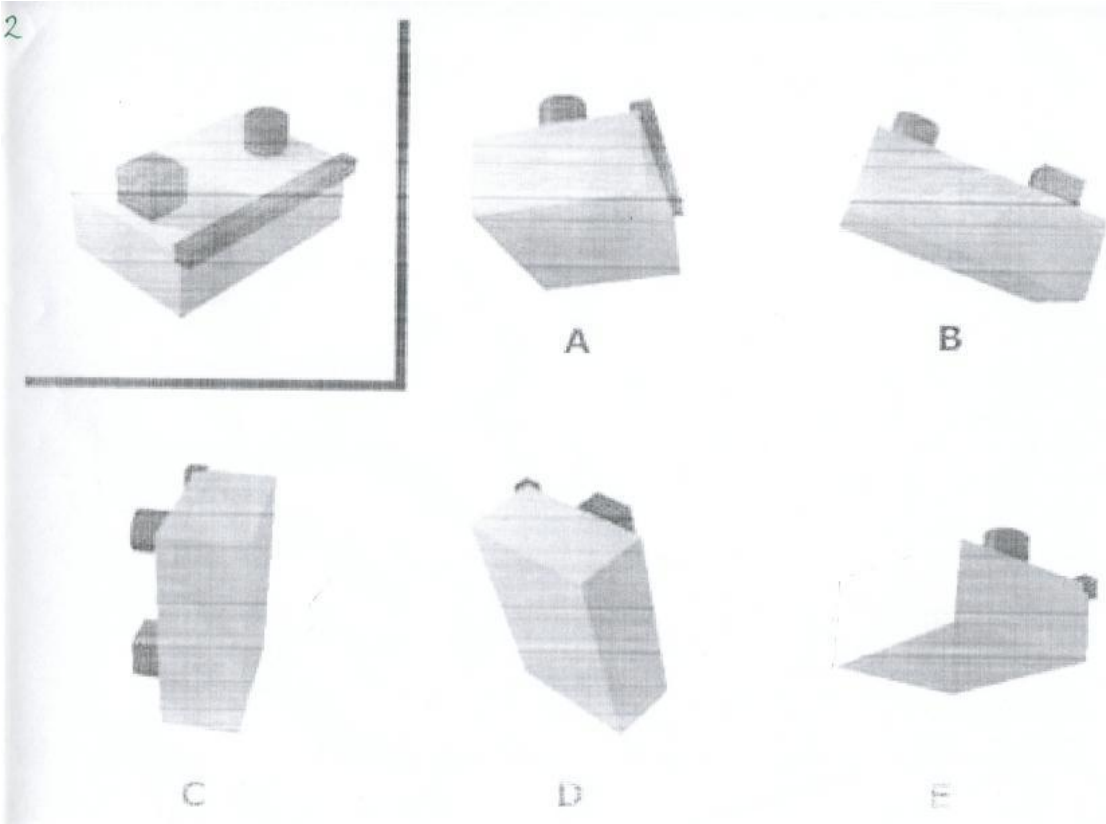
R

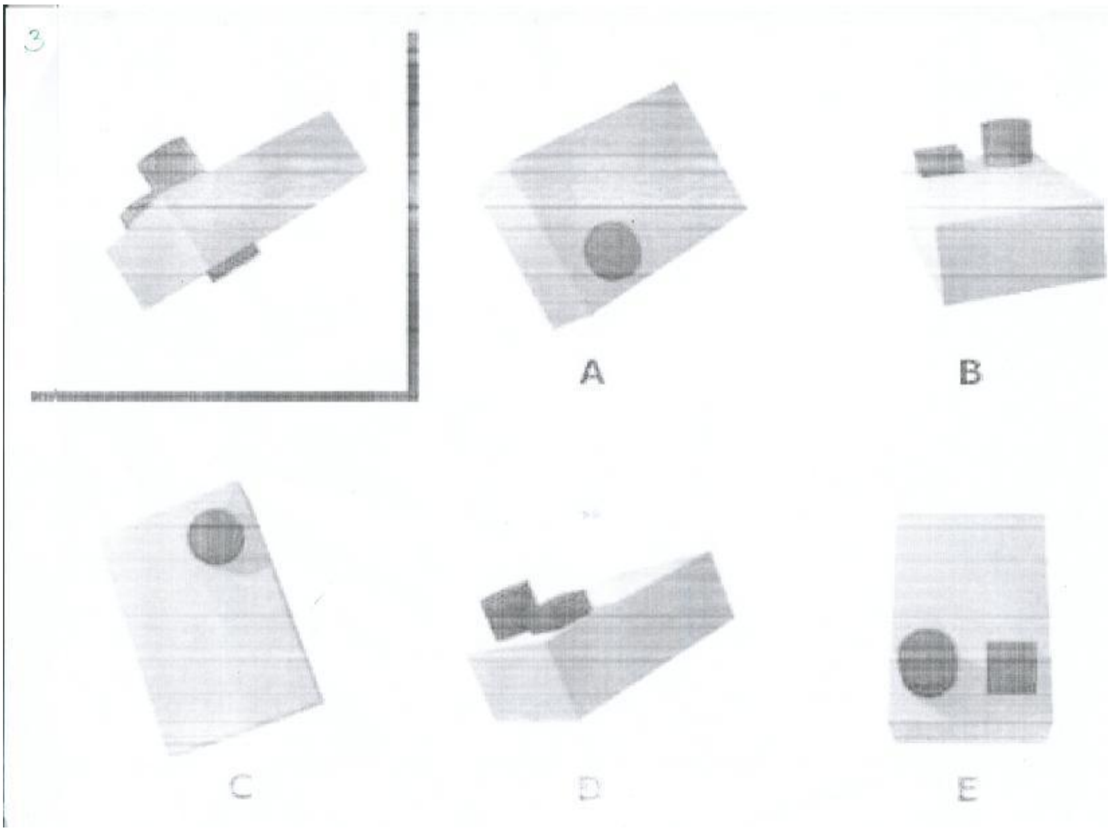
A mother is seven times as old as her daughter. The difference between their ages is 24 years.
How old are they?

Appendix G

Visual-spatial mental rotation items.




2
A
B
C
D
E


3
A
B
C
D
E

Appendix H

Questions to sort as assessment of mathematical understanding

Six friends won £78. They shared out the money equally.

How much did each friend receive?

How many weeks and days is 87 days?

How many years and months is 115 months?

An old drinks machine only accepts 5p pieces.

How many 5p pieces do you need to put in for a drink that costs 65p?

A boy takes 1 hour to walk 3 miles.

How long will he take to walk 12 miles?

What time is it 2 hours after 7:00am?

The librarian went to the bookshelf and took away 35 books. After she had gone there were 47 books left on the shelf.

How many books were there before the librarian came?

Jade has £53 saved. Her uncle gave her £10 more.

How much has she saved altogether?

54% of the class walk to school. 32% of the class catch the bus.

What percentage of the class either walks or catches the bus?

What time is it 3 hours before 9:00pm?

Mark had 24 books. His uncle gave him some more.

Then he had 33 altogether.

How many books did his uncle give him?

30% of pupils at the school have school dinners.

What percentage of the school does not have school dinners?

A snail takes 1 hour to travel 3m.

How far will it travel in 5 hours?

The king shared out his land equally between his 7 daughters. Each daughter got 5 square miles.

How many square miles did the king share out?

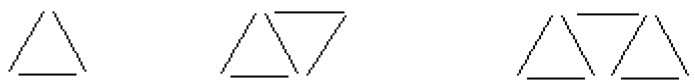
A school lesson lasts for 40 minutes.

How long do 3 lessons last for?

How many days are there in 6 weeks?

A plant measures 14cm. Each day it grows 2.5cm.

Find its height each day for the next 3 days.



3 sticks   5 sticks   7 sticks

How many sticks are needed for the next two triangle patterns?

94, 97, 100, 103….

Find the next two numbers in this sequence.

Jenny's weekly pocket money increases by £1.50 each year. This year she gets £3 per week.

Find how much her pocket money will be for the next 3 years.

On one side of the street the houses are numbered 1, 3, 5, 7, 9.

How do you expect the houses on the other side to be numbered?

Write down the even numbers less than 10.

Think of some numbers that divide by 2 with no remainder.

What do you notice about these numbers?

How far is it from Marsden to Glossop?

How far is it from the Granada services to the Welcome Break services?

Appendix I

Tests of Normality on MCT scores and standardised residuals

	Intervention group	Kolmogorov-Smirnov Statistic	df	Sig.	Shapiro-Wilk Statistic	df	Sig.
MCT gain	Monday	.121	17	.200	.951	17	.475
	Wednesday	.132	17	.200	.950	17	.452
MCT:post intervention	Monday	.117	17	.200	.943	17	.362
	Wednesday	.119	17	.200	.970	17	.815

* This is a lower bound of the true significance.

a Lilliefors Significance Correction

Appendix J

Other assessor's sorting of MCT questions

Type	Question style	Item numbers	Total on test
1	Numerals only or reading of everyday words	8,13,14,16,18,22,32,33,36,38,44	11
2	A mathematical or numerical definition is needed	1,7,10,11,15,17,20,21,25,27,30,31	12
3	Heavy literacy demands involving instructions or definitions embedded in several sentences	2,3,4,5,6,9,12,19,23,24,26,28,29,34,35,37,39,40,41,42,43,45,46	23

Performance on MCT with alternative sorting of questions

	Intervention group	N	Mean	Std. Deviation	Sig. (2-tailed)
Type 1 questions	Monday	17	3.29	1.45	
	Wednesday	19	4.16	2.09	.163
Type 2 questions	Monday	17	5.00	1.97	
	Wednesday	19	6.42	2.67	.081
Type 3 questions	Monday	17	6.59	2.27	
	Wednesday	19	8.74	3.30	.031

Appendix K

Raw error scores and subtractive visual tendency score

Participant	Verbal errors (verb)	Visual errors (vis)	Subtractive score (vis – verb + 6)
Main study 1	6	0	0
2	1	5	10
3	4	4	6
4	1	3	8
5	3	0	3
6	.	.	.
7	4	3	5
8	1	4	9
9	5	2	3
10	.	.	.
11	5	2	3
12	4	1	3
13	0	3	9
14	.	.	.
15	3	3	6
16	4	1	3
17	4	1	3
18	2	7	11
19	5	3	4

20	.	.	.
21	4	0	2
22	4	3	5
23	7	2	1
24	.	.	.
25	7	4	3
26	2	2	6
27	2	2	6
28	6	1	1
29	5	1	2
30	4	2	4
31	1	1	6
32	3	0	3
33	.	.	.
34	7	8	7
35	3	5	8
36	5	8	9
37	3	5	8
38	4	5	7
39	6	2	2
40	6	4	4
41	.	.	.
42	2	5	9
43	.	.	.

44	.	.	.
45	6	2	2
46	7	3	2
47	.	.	.
Pilot study 1	1	2	7
2	4	4	6
3	6	3	3
4	4	2	4
5	4	1	3
6	4	1	3
7	6	3	3
8	.	.	.
9	3	2	5
10	.	.	.
11	3	1	4
12	.	.	.
13	2	1	5
14	2	0	4

Correlations of subtractive score with other measures

		Subtractive visual tendency score
MidYIS vocabulary	Pearson Correlation	-.150
	Sig. (2-tailed)	.396
	N	34
MidYIS non-verbal	Pearson Correlation	-.110
	Sig. (2-tailed)	.542
	N	33
MidYIS maths	Pearson Correlation	.061
	Sig. (2-tailed)	.730
	N	34
MidYIS skills	Pearson Correlation	-.162
	Sig. (2-tailed)	.368
	N	33
Spatial memory test	Pearson Correlation	.155
	Sig. (2-tailed)	.440
	N	27
MCT gain	Pearson Correlation	-.427*
	Sig. (2-tailed)	.017
	N	31
Subtractive visual tendency score	Pearson Correlation	1
	Sig. (2-tailed)	.
	N	34

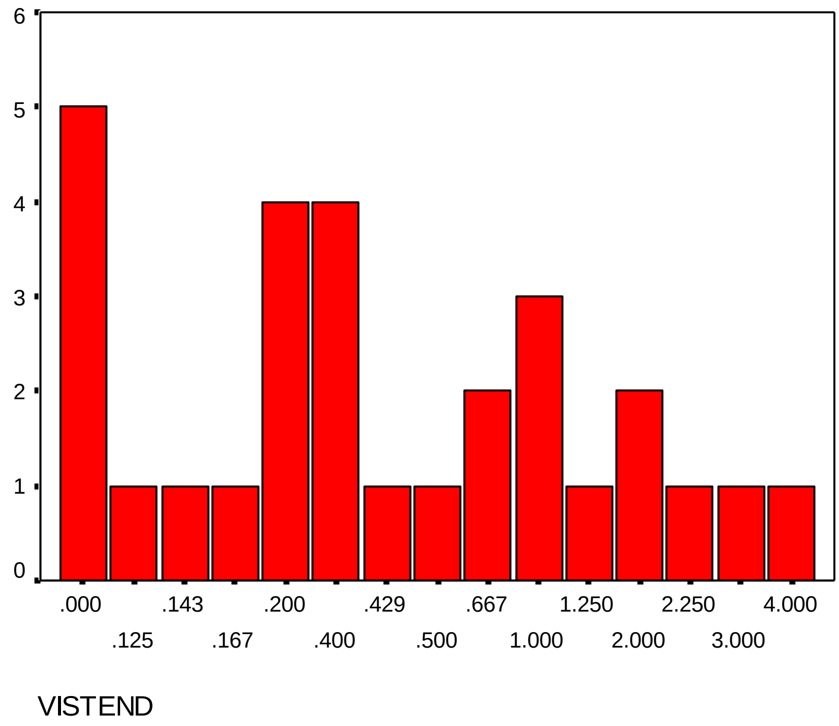
** Correlation is significant at the 0.01 level (2-tailed).

* Correlation is significant at the 0.05 level (2-tailed).

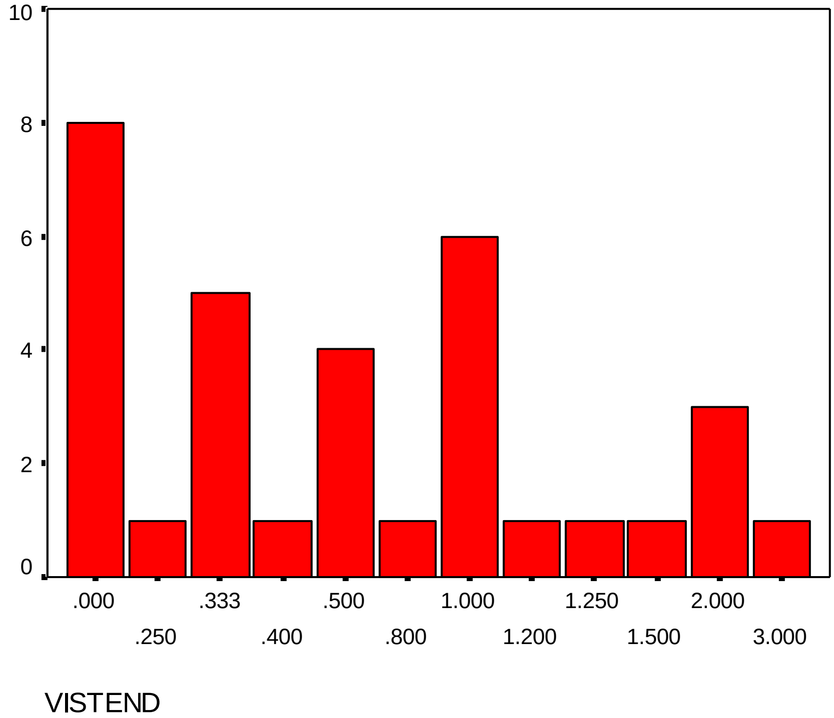
Appendix L

Distributions of visual/verbal ratio scores for additional participants

Year 7 Participants



Year 10 Participants



Appendix M

Correlations of visual/verbal ratio scores with mathematics test marks for Year 10 participants

Correlations

		V/V RATIO	PAP2	PAP1
V/V RATIO	Pearson Correlation	1	-.150	-.328
	Sig. (2-tailed)	.	.406	.077
	N	33	33	30
PAP2	Pearson Correlation		1	.789**
	Sig. (2-tailed)		.	.000
	N			30
PAP1	Pearson Correlation			1
	Sig. (2-tailed)			.
	N			30

** Correlation is significant at the 0.01 level (2-tailed).